Analysis of Algorithms.

Lab 02

1. Solve the following recurrences. The solution of each exercise includes finding the closed form and then verifying using mathematical induction. The solution of the recurrences can be done by hand and scanned to a PDF file.

1.1. $T(n) = 3T(n/2) + n$, $T(1) = 0$, for all n powers of two.

$n = 2^k$, $S(k) = T(2^k)$

Then,

$$S(k) = 3S(k-1) + 2^k \; ; \quad S(0) = T(1) = 0$$

$$S_h(k) = C \cdot 3^k$$

for $S_p(k) = a \cdot 2^k$

$$a2^k = 3a2^{k-1} + 2^k = \frac{3a}{2} 2^k + 2^k$$

$$a = \frac{3a}{2} + 1 \Rightarrow a - \frac{3a}{2} = 1 \Rightarrow -\frac{a}{2} = 1 \Rightarrow a = -2$$

Then, $S_p(k) = -2 \cdot 2^k$

General Solution:

$$S(k) = C3^k - 2 \cdot 2^k$$

for $S(0) = 0$

$$0 = C \cdot 1 - 2 \cdot 1 \Rightarrow C = 2$$

Then,

$$S(k) = 2 \cdot 3^k - 2 \cdot 2^k$$

n is 2^k , $3^k = (2^k)^{\log_2 3} = n^{\log_2 3}$

Notas

$T(n) = 2n^{\log_2 3} - 2n \quad (n = 2^k)$

Induction:

$k \geq 0$

$$T(2^k) = 2 \cdot 3^k - 2 \cdot 2^k$$

$k = 0 : 2^0 = 1 \qquad T(1) = 0, \; 2 \cdot 3^0 - 2 \cdot 2^0 = 2 - 2 = 0 \;//$

$k - 1:$

$$T(2^{k-1}) = 2 \cdot 3^{k-1} - 2 \cdot 2^{k-1}$$

For k,

$$T(2^k) = 3 + (2^{k-1}) + 2^k$$

$$T(2^k) = 3(2 \cdot 3^{k-1} - 2 \cdot 2^{k-1}) + 2^k = 6 \cdot 3^{k-1} - 6 \cdot 2^{k-1} + 2^k$$

$$T(2^k) = 2 \cdot 3^k - 3 \cdot 2^k + 2^k = 2 \cdot 3^k - 2 \cdot 2^k$$

1.2 $T(n) = 3T(n-1) + 4T(n-2), \; T(0) = 0, \; T(1) = 5$

$T(n) = r^n$

$r^n = 3r^{n-1} + 4r^{n-2} \Rightarrow r^2 = 3r + 4$

$r^2 - 3r - 4 = 0 \Rightarrow (r-4)(r+1) = 0$

$$T(n) = A4^n + B(-1)^n$$

$n = 0 : 0 = A + B \Rightarrow B = -A$

$n = 1 : 5 = 4A + B(-1) = 4A - B$

$5 = AA - (-A) = 5A \Rightarrow A = 1 \quad , B = -1$

$T(n) = A^n - (-1)^n$

For $n \geq 2$

$T(n) = 3T(n-1) + 4T(n-2)$

$3(A^{n-1} - (-1)^{n-1}) + 4(A^{n-2} - (-1)^{n-2})$

$3 \cdot A^{n-1} + A^{n-1} - (-1)^{n-2}(-3(-1) + 4)$

$T(n) = A^n - (-1)^n$ //

1.3 $T(n) = 5T(n-1) - 6T(n-2)$, $T(0) = 0$; $T(1) = 1$

$r^2 = 5r - 6 \Rightarrow r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0$

$T(n) = A2^n + B3^n$

① $n = 0 \quad 0 = A + B \Rightarrow B = -A$

$n = 1 : \quad 1 = 2A + 3B = 2A + 3(-A) \quad A = -1, \; B = 1$

$T(n) = 3^n - 2^n$

$n-1$, $n-2$ $\quad T(n) = 5T(n-1) - 6T(n-2)$

$= 5(3^{n-1} - 2^{n-1}) - 6(3^{n-2} - 2^{n-2})$

$= (5 \cdot 3^{n-1} - 6 \cdot 3^{n-2}) - (5 \cdot 2^{n-1} - 6 \cdot 2^{n-2})$

$= 3^{n-2}(15 - 6) - 2^{n-2}(10 - 6) = 9 \cdot 3^{n-2} - 4 \cdot 2^{n-2}$

$3^n - 2^n$ //

1. A° $x(n) = 2x(n-1) + 4x(n-2)$, $x(0) = 1$, $x(1) = 2$, $n > 1$

①

$$r^2 = 2r + 4 \Rightarrow r^2 - 2r - 4 = 0$$

$$r = \frac{2 \pm \sqrt{4+16}}{2} = \frac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}$$

$$x(n) = A(1+\sqrt{5})^n + B(1-\sqrt{5})^n$$

$n = 0$: $1 = A + B$

$n = 1$

$$2 = A(1+\sqrt{5}) + B(1-\sqrt{5}) = (A+B) + \sqrt{5}(A-B)$$

$$= 1 + \sqrt{5}(A-B) \Rightarrow \sqrt{5}(A-B) = 1 \Rightarrow A - B = \frac{1}{\sqrt{5}}$$

$$A = \frac{(A+B) + (A-B)}{2} = \frac{1 + \frac{1}{\sqrt{5}}}{2} \; ; \; B = \frac{(A+B) - (A-B)}{2} = \frac{1 - 1/\sqrt{5}}{2}$$

$$x(n) = \frac{1 + 1/\sqrt{5}}{2}(1+\sqrt{5})^n + \frac{1 - 1/\sqrt{5}}{2}(1-\sqrt{5})^n$$

Notas

Fibonacci Iterative (C++)

```cpp
#include <iostream>
using namespace std;

int main() {
    int n;
    cout << "How many terms? ";
    cin >> n;

    long long a = 0, b = 1;

    for (int i = 0; i < n; i++) {
        cout << a;
        if (i < n - 1) cout << " ";

        long long c = a + b;
        a = b;
        b = c;
    }

    cout << endl;
    return 0;
}
```

Listing 1: Fibonacci iterative

Fibonacci Recursive (C++)

```cpp
#include <iostream>
using namespace std;

long long fib(int n) {
    if (n == 0) return 0;
    if (n == 1) return 1;
    return fib(n - 1) + fib(n - 2);
}

int main() {
    int n;
    cout << "How many terms? ";
    cin >> n;

    for (int i = 0; i < n; i++) {
        cout << fib(i);
        if (i < n - 1) cout << " ";
    }
```

```
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20      cout << endl;
21      return 0;
22  }
```

Listing 2: Fibonacci recursive