The Nonlinear Equatorial Kelvin Wave

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ABSTRACT

Using the method of strained coordinates, a uniformly valid approximation to the nonlinear equatorial Kelvin wave is derived. It is shown that nonlinear effects are negligible for the Kelvin waves associated with the Gulf of Guinea upwelling. The Kelvin waves involved in El Niño, however, are significantly distorted both in shape and speed. The leading edge is smoothed and expanded rather than steepened, but the trailing edge will form sharp fronts and eventually break.

1. Introduction

The importance of equatorially trapped waves in the dynamics of the equatorial ocean is summarized in the review papers by Moore and Philander (1977) and Wunsch (1978). The Kelvin wave is of special significance because, first, it is the lowest meridional mode and, second, theory (McCreary, 1976; Moore *et al.*, 1978; O'Brien *et al.*, 1978) suggests that it plays a major role in El Niño and in similar upwelling in the Gulf of Guinea. For all classes of equatorial waves one would like to understand the effects of nonlinearity on them, but almost all previous analytic work has been linear. The author is presently embarked on a thorough investigation of equatorial solitary waves, and a preliminary report has already appeared (Boyd, 1978). For the Kelvin wave, however, solitary waves are impossible and the qualitative nonlinear behavior is very different from other types of equatorial waves.

The magic word is *nondispersive*: all Kelvin waves with the same equivalent depth have the same phase speed, regardless of the zonal wavenumber. In linear theory then, a Kelvin wave packet will propagate without change of shape. It is well-known, however, that in other nondispersive wave systems—shallow water waves on a nonrotating earth are the most familiar example—nonlinearity will drastically distort the packet, perhaps to the point of breaking. The goal of the present work is to derive a general, uniformly valid small-amplitude theory for nonlinear equatorial Kelvin waves and then to apply this theory to a couple of particular cases.

The principal mathematical tool is Lighthill's method of strained coordinates, which is a singular perturbation technique. The underlying physics for the nonlinear Kelvin wave is so similar to that of the one-dimensional advection equation (the inviscid Burger's equation) that coordinate straining for the former can be done by inspection of its use upon the latter. Consequently, Section 2 discusses the one-dimensional advection equation in some detail, using it to illustrate the method of strained coordinates and other preliminary results. Most of this section is not new, although the ideas have been published only in scattered places. Rather surprisingly, however, it has apparently never been noticed previously that, first, the lowest order strained coordinates approximation is the exact answer for the one-dimensional advection equation (though not, unfortunately, for the Kelvin wave) and, second, that the implicit equation one must solve for either the one-dimensional advection equation or Kelvin wave to obtain the wave in explicit form is the Kepler equation of celestial mechanics for which an analytic solution is known.

The third section deals with the Kelvin wave itself. By introducing "sum" and "difference" variables and using the algebra of the "raising" and "lowering" operators for the Hermite functions, the first-order *regular* perturbation expansion is derived. This approximation is then made uniform in x and t by straining the coordinates, which follows almost trivially from the results for the one-dimensional advection equation. Section 4 applies the nonlinear theory to the Kelvin waves associated with El Niño and the Gulf of Guinea upwelling.

Before continuing, it is appropriate to note some relevant previous work. First, Bennett (1973) studied nonlinear *coastal* (as opposed to equatorial) Kelvin waves. For this simpler problem, strained coordinates perturbation theory is unnecessary. Bennett derived the exact solution via the method of characteristics and found qualitative behavior similar to that for the equatorial Kelvin waves here. The two species of Kelvin wave are related by more

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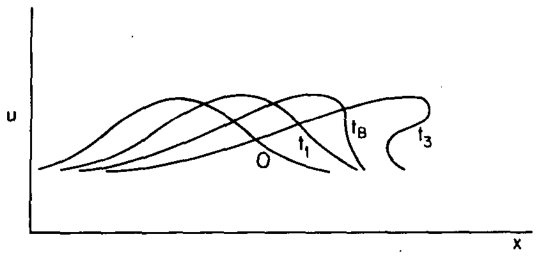

FIG. 1. Successive profiles of a nondispersive wave at various times. The wave breaks at $t = t_B$ (after Whitham, 1974).

than name and lack of dispersion: when the equatorial Kelvin wave strikes the west coast of a continent, it excites coastal Kelvin waves that propagate north and south hugging the shoreline (Moore, 1968).

Second, Hurlburt *et al.* (1975) and Adamec and O'Brien (1978) have *numerically* modeled nonlinear equatorial dynamics including Kelvin waves. In the latter paper, Adamec and O'Brien observed that equatorial Kelvin waves steepened and broke, which cannot be explained on the basis of existing linear theories.

Third, Ripa (1979) has examined nonlinear Kelvin waves from the viewpoint of resonant triad interactions and noted the possibility of steepening and breaking. Although he did not derive analytical solutions as done here, this resonant triad perspective is helpful in interpreting the perturbative solution. Both Ripa (1979) and Adamec and O'Brien (1978) will be referenced again in later sections.

2. Strained coordinates and the one-dimensional advection equation

a. The one-dimensional advection equation

The one-dimensional advection equation is

$$u_t + (c + u)u_x = 0, \tag{2.1}$$

where u is the zonal velocity, c the linear phase speed (a constant), and the subscripts denote differentiation with respect to the subscripted variable. As shown in Whitham (1974), this equation has an exact solution in implicit form. To derive it, we pretend for the moment that it is $[c + u(x,t)]$ rather than $u(x,t)$ which is the particle velocity. Eq. (2.1) can then be interpreted in terms of the total, particle-following velocity as simply

$$\frac{du}{dt} = 0. \tag{2.2}$$

This means that if we trace a curve C in the (x,t) plane which moves with the particle velocity $(c + u)$, i.e., a curve C, such that

$$\frac{dx}{dt} = c + u(x,t), \tag{2.3}$$

then $u(x,t)$ will be constant on C. Eq. (2.3) then implies that C will be a straight line with the slope $(c + u)$. Thus, the general solution of (2.1) depends on construction of a family of straight lines in the (x,t) plane, each with the slope $(c + u)$ corresponding to the value of u on that particular line.

To illustrate this, consider the general initial value problem

$$u(x, t = 0) = Q(x), \quad -\infty < x < \infty. \tag{2.4}$$

Now if one of the curves C intersects $t = 0$ at $x = \xi$, then $u(x,t) = Q(\xi)$ on the whole curve. The corresponding slope of the curve is then $[c + Q(\xi)]$ and the equation of the curve is simply

$$x = \xi + [c + Q(\xi)]t. \tag{2.5}$$

Allowing ξ to vary, we obtain the general solution

$$u(x,t) = Q(\xi), \tag{2.6}$$

where $\xi(x,t)$ is defined implicitly by (2.5). When $Q(\xi)$ is sufficiently simple, it may be possible to solve (2.5) for $\xi(x,t)$ analytically, and substituting it into (2.6) then gives $u(x,t)$ explicitly. In any case, however, it is easy to obtain $u(x,t)$ graphically by drawing a set of straight lines in the (x,t) plane which each satisfy (2.5) for a given $Q(x)$ and different values of ξ.

Several features of this general solution deserve comment. First, as shown in (2.5), the nonlinear wave cannot be characterized by a single phase speed. Instead, each portion of the wave moves along at its own speed with the crest [$Q(\xi)$ maximum] traveling fastest and the trough [$Q(\xi)$ minimum] moving most slowly. Second, because the crest is traveling fastest, it will eventually overtake lower and more slowly moving portions of the wave, causing the wave to "break." The distortion that develops in the compressive part of the wave [where $u(x,t)$ is a *decreasing* function of x] is qualitatively identical to that one observes in a water wave advancing toward a beach, and the end result is the same. This is schematically illustrated in Fig. 1, which shows successive profiles of u. Breaking begins at $t = t_B$ in Fig. 1 where the profile of u first develops an infinite slope. The profile at $t = t_3$ illustrates the fact that the "post-breaking" solution given by (2.5) and (2.6) is triple-valued. In reality, (2.1) is no longer an adequate description of the wave after breaking, and some sort of shock wave theory is needed based on going deeper into the underlying physics of the phenomenon for which (2.1) has been used as a first approximation.

The actual time of breaking t_B can be derived as in Whitham (1974). If one regards $\xi(x,t)$ as a known function implicitly defined by (2.5), then explicit differentiation of (2.6) yields

$$u_t = Q'(\xi)\xi_t, \tag{2.7a}$$

$$u_x = Q'(\xi)\xi_x, \tag{2.7b}$$

where the prime denotes differentiation. By explicit differentiation of (2.5),

$$\xi_t = \frac{-[c + Q(\xi)]}{[1 + Q'(\xi)t]}, \tag{2.8a}$$

$$\xi_x = \frac{1}{[1 + Q'(\xi)t]}, \tag{2.8b}$$

therefore,

$$u_t = \frac{-Q'(\xi)[c + Q(\xi)]}{[1 + Q'(\xi)t]}, \tag{2.9a}$$

$$u_x = \frac{Q'(\xi)}{[1 + Q'(\xi)t]}. \tag{2.9b}$$

From (2.9), it is obvious that (2.5) and (2.6) indeed satisfy (2.1) and (2.4). It is also clear that u_t and u_x are infinite whenever $[1 + Q'(\xi)t] = 0$. Breaking, therefore, occurs on that characteristic $\xi = \xi_B$ for which $Q'(\xi)$ has its largest *negative* maximum with the time of first breaking given explicitly by

$$t_B = \frac{-1}{Q'(\xi_B)}. \tag{2.10}$$

Third, the large distortions of the wave profile that develop in Fig. 1 positively cry out for a singular perturbation method. If one defines

$$\epsilon = \pm\max|Q(x)|, \tag{2.11}$$

where the sign is chosen for convenience, and where $Q(x)$ is measured in the nondimensional units defined by (3.2) and (3.3), then if $\epsilon \ll 1$, one can solve (2.1) by applying an ordinary perturbation expansion in ϵ. Because the distortion is increasing linearly with t, however, the difference between the perturbed and unperturbed solutions may become O(1) long before the wave actually breaks. Therefore, instead of being uniformly accurate, the ordinary perturbation series is useful only for a small fraction of the total interval between $t = 0$ and the time of breaking.

For *dispersive* waves, the method of multiple scales, alias "two-timing", is a powerful tool for circumventing the nonuniformity of an ordinary perturbation expansion. However, Kelvin waves and the solutions of (2.1) are both *nondispersive*, and this makes all the difference in the world. In particular, two-timing adjusts for the nonlinearity by correcting the linear phase speed by a *constant*, independent of x and t, which depends on ϵ alone. Since the exact solution (2.5) and (2.6) shows that the nondispersive, nonlinear wave cannot be characterized by a single phase speed, the method of multiple scales simply will not work. We need a phase speed correction which is a function of x and t as well as of ϵ, and the alternative algorithm described in the next subsection provides this.

b. The method of strained coordinates for nondispersive waves

The basic idea of this method is to introduce a transformation of coordinates such that the perturbative solution, given as a function of the *new* coordinates, is a uniformly valid approximation to the exact answer. This transformation will itself be carried out perturbatively, order by order, and at each order we will choose the transformation or "straining" functions so as to preserve the uniform accuracy of the perturbative approximation. Inspecting the exact solution (2.5), we see that one way to preserve this, at least to lowest order, is to transform from x to the characteristic curve label ξ since the exact solution is an explicit function of ξ for all time, not merely for $t \approx O(1)$. For the one-dimensional advection equation (2.1), this transformation from x to ξ is precisely what the method of strained coordinates does.

There are two different ways, however, of implementing this idea. One can either transform the differential equation first, leaving a lot of undetermined transformation or "straining" functions floating around, and then perturbatively solve the transformed differential equation, as done by Lighthill (1949), who invented this technique, or one can apply perturbation theory first and then rewrite the ordinary perturbation expansion in terms of the new coordinates, as independently suggested by Pritulo (1962) and Martin (1967). Crocco (1972) notes that "the standard procedure [Lighthill's] can become discouragingly cumbersome and confusing", whereas with Pritulo's method, despite its being "all but ignored by the occidental experts", the "sometimes discouraging complication of the transformed equations can be entirely bypassed." Because of its simplicity and despite the fact that it is still somewhat out of fashion, Pritulo's "post-perturbative transformation" method will be adopted here.

Let

$$s = x - ct, \tag{2.12}$$

$$u = \epsilon[u^0 + \epsilon u^1 + \ldots], \tag{2.13}$$

and make the replacement

$$Q(x) \to \epsilon Q(x) \tag{2.14}$$

in the initial condition (2.4). Applying ordinary perturbation theory to (2.1) gives

$$u^0 = Q(s), \tag{2.15a}$$

$$u^1 = -tQ(s)Q'(s), \tag{2.15b}$$

where the prime denotes differentiation. We now introduce the *implicit* transformation from s to ξ, i.e.,

$$s(\xi,t) = \xi + \epsilon s_1(\xi,t) + \epsilon^2 s_2(\xi,t) + \ldots. \tag{2.16}$$

A similar transformation could be introduced for t if need be, but since there is wave propagation in one direction only, both here and for the Kelvin wave, it is only necessary to strain a single coordinate. We want to choose the straining functions $s_1(\xi,t)$, etc., in such a way that the expansion of $u(x,t)$ in terms of ϵ and the *strained* coordinates, i.e.,

$$u(x,t) = \epsilon U^0(\xi,t) + \epsilon^2 U^1(\xi,t) + \ldots, \quad (2.17)$$

is uniformly valid. Introducing (2.12) into (2.9) and expanding the result in powers of ϵ gives

$$U^0(\xi,t) = u^0(\xi,t), \quad (2.18a)$$

$$U^1(\xi,t) = u^1(\xi,t) + s_1(\xi,t)\frac{\partial u^0(\xi,t)}{\partial s}, \quad (2.18b)$$

or using (2.10) and (2.11),

$$U^0(\xi,t) = Q(\xi), \quad (2.19a)$$

$$U^1(\xi,t) = -tQ(\xi)Q'(\xi) + s_1(\xi,t)Q'(\xi). \quad (2.19b)$$

The higher order extensions (more strained coordinates and higher powers in ϵ) of the general expressions (2.18) are given by Crocco (1972).

It is the linear growth of u^1 with time, as indicated in (2.15b), that quickly destroys the usefulness of the ordinary perturbation expansion. In (2.19b), we must therefore choose s_1 so as to cancel out this linear dependence of the first-order solution on time, which implies

$$s_1(\xi,t) = tQ(\xi) + p(\xi,t), \quad (2.20)$$

where $p(\xi,t)$ is a bounded, O(1), but otherwise arbitrary function. This arbitrariness represented by $p(\xi,t)$ is a general feature of the method of strained coordinates and one may choose $p(\xi,t)$ to be whatever is convenient, so long as it does not destroy the uniform validity of the perturbation series. Here, the obvious choice is $p(\xi,t) = 0$ which makes $U^1 = 0$. Letting

$$\epsilon Q \rightarrow Q \quad (2.21)$$

so as to be consistent with the original specification of the initial condition (2.4), and replacing s by $x - ct$, the first-order strained coordinates solution becomes

$$x = \xi + [c + Q(\xi)]t, \quad (2.22a)$$

$$u(x,t) = Q(\xi). \quad (2.22b)$$

Comparing this with (2.5) and (2.6), one sees that Eqs. (2.22a,b), in fact, are the *exact* solution: all higher order corrections, as can be easily verified explicitly, are, in fact, zero.

It is a source of some amazement to me that no one, to my knowledge, has previously called attention to the fact that the lowest order method of strained coordinates solves the one-dimensional advection equation exactly, but it is a compelling argument for applying strained coordinates to fluid waves. In more complicated problems, such as the Kelvin wave, it is too much to expect that the method will give the *exact* answer at lowest order, but the source of nonuniformity is the same—nonlinear advection—so one would expect that strained coordinates would be very effective at providing a uniform approximation.

One drawback of the method is that it usually generates *implicit* rather than *explicit* solutions. As can be seen from the exact answer to (2.1), however, this annoying implicitness is not a weakness of the perturbative algorithm but is rather inherent in the nature of nonlinear, nondispersive waves: the method of strained coordinates mirrors the underlying physics of the problem as closely as any perturbation scheme could hope to.

c. *The signaling problem*

For the Kelvin wave, it is of interest to study not only the familiar initial value problem but also the boundary value problem where $u(x,t)$ is specified at a single value of x for all values of t—what in acoustics is known as the signaling problem. As background for the Kelvin case, it is useful to write down the solution of the signaling problem for the one-dimensional advection equation. As for the initial value case, both Whitham's line of reasoning and the lowest order method of strained coordinates give the exact answer.

Let

$$u(x = 0, t) = R(-ct), \quad 0 \leq t < \infty. \quad (2.23)$$

(Note the minus sign, which has been inserted to make the signaling formulas parallel those for the initial value problem as closely as possible.) The ordinary perturbative solutions are

$$u^0 = R(s), \quad (2.24)$$

$$u^1 = -xR(s)R'(s)/c. \quad (2.25)$$

Straining s so as to remove the nonuniformity in x now gives

$$x - ct = \xi + (\xi + ct)R(\xi)/c, \quad (2.26)$$

$$u = R(\xi). \quad (2.27)$$

Eq. (2.26) is identical with (2.5) except for the substitution of R for Q and the extra term $\xi R(\xi)/c$ on the right-hand side.

Since $R(\xi) \ll 1$ *uniformly* in (x,t), it follows that one can replace $(\xi + ct)$ by x in (2.26) to approximate (2.26) as

$$x - ct = \xi + xR(\xi)/c, \quad (2.28)$$

with a uniformly small error, which is no worse than that usually inherent in the lowest order method of strained coordinates. As shown in the next subsection, Eq. (2.28), which also displays the secular behavior in x more clearly, is often easier to solve. It is only for this special case of the one-dimensional advection equation that (2.28) is *inferior* to (2.26) because for (2.1), Eq. (2.26) is exact while Eq. (2.28) is only approximate. By reasoning similar to that used to derive t_B for the initial value problem, one can show that the breaking distance [from (2.28)] is

$$x_B = \frac{-c}{R'(\xi_B)}, \tag{2.29}$$

where ξ_B is the value of ξ for which $R(\xi)$ has its negative maximum.

d. Periodic initial or boundary conditions

If the initial condition is

$$u(x, t = 0) = \epsilon \sin(kx), \tag{2.30}$$

then ξ is determined from

$$x = \xi + [c + \epsilon \sin(k\xi)]t. \tag{2.31}$$

If one rescales according to

$$\psi = k\xi, \tag{2.32}$$

$$e = -\epsilon kt, \tag{2.33}$$

$$\theta = k(x - ct), \tag{2.34}$$

then (2.8) becomes

$$\psi = \theta + e \sin\psi, \tag{2.35}$$

which is known in celestial mechanics as Kepler's equation.[1] The advantage of this transformation is that we can use the known analytic solution of Kepler's equation to solve (2.28) for all $t < t_B$, i.e.,

$$u(x,t) = \epsilon \sin(\psi), \tag{2.36}$$

$$\psi(x,t) = \theta + 2 \sum_{n=1}^{\infty} \frac{J_n(ne) \sin(n\theta)}{n}. \tag{2.37}$$

The wave breaks[2] when $|e| = 1$, which is equivalent to

$$t_B = \frac{1}{k\epsilon}. \tag{2.38}$$

Note that the breaking time is proportional to $(k\epsilon)^{-1}$ and not merely to ϵ^{-1} alone: If $k \ll 1$, as it is for one of the Kelvin wave applications, then the breaking time may be very long compared to ϵ^{-1}.

For the signaling problem, life is more complicated because Eq. (2.26), the *exact* equation for $\xi(x,t)$, cannot be recast into Kepler's equation. If one uses the approximate but uniformly valid alternative (2.28), however, then the signaling problem

$$u(x = 0, t) = -\epsilon \sin(kct) \tag{2.39}$$

is solved by

$$u(x,t) = \epsilon \sin(\psi), \tag{2.40}$$

where ψ is still given by (2.34) but now

$$e = -\frac{\epsilon kx}{c}. \tag{2.41}$$

As before, the wave breaks when $|e| = 1$, which implies that the breaking distance is

$$x_B = \frac{c}{k\epsilon}. \tag{2.42}$$

3. The nonlinear Kelvin wave

a. Perturbation theory

Let a be the radius of the earth, Ω the frequency of the earth's rotation, and H the depth of the layer of homogeneous, incompressible fluid. Defining

$$E = \frac{4\Omega^2 a^2}{gH}, \tag{3.1}$$

it is convenient to nondimensionalize using

$$L = E^{-1/4}a \qquad \text{(length scale)}, \tag{3.2}$$

$$T = E^{1/4}(2\Omega)^{-1} \qquad \text{(time scale)}. \tag{3.3}$$

The nonlinear shallow-water wave equations on the equatorial beta-plane are then

$$u_t + uu_x + vu_y - yv + \phi_x = 0, \tag{3.4}$$

$$v_t + uv_x + vv_y + yu + \phi_y = 0, \tag{3.5}$$

$$\phi_t + u_x + (u\phi)_x + v_y + (v\phi)_y = 0. \tag{3.6}$$

The first step in Pritulo's procedure is to solve these equations via regular perturbation theory. We shall see that this is the hard part; the second step, straining coordinates, is so similar to that for the one-dimensional advection equation that it can be done by inspection.

The lowest order solution is simply the linear Kelvin wave

$$u^0(x,y,t) = e^{(-1/2)y^2}Q(x - t), \tag{3.7a}$$

$$v^0(x,y,t) = 0, \tag{3.7b}$$

$$\phi^0(x,y,t) = u^0(x,y,t), \tag{3.7c}$$

where $Q(x - t)$ is determined by the initial or

[1] In this astronomical context, e is the eccentricity of the orbit and ψ and θ are the mean and eccentric anomalies. After submission of the first draft of this work, I found that Platzman (1964) had independently also discovered this relationship between Kepler's equation and periodic solutions of the advection equation. He gives extensive references and also explicitly derives the Bessel function series solution. Blumen (1979), which led me to Platzman's work, also refers to other, closely related Bessel function series which have been used in other models of frontogenesis, so series like (2.37) have a long history and a broad usefulness.

[2] This breaking condition is derived in Platzman (1964). It is obvious to celestial mechanics since e (eccentricity) = 1 marks the transition from closed periodic, elliptical orbits to unbounded, hyperbolic trajectories.

boundary conditions, depending on whether we are solving the initial value or signalling problem. The set of equations which determines the first-order solution is

$$u_t^1 - yv^1 + \phi_x^1 = -u^0 u_x^0, \tag{3.8}$$

$$v_t^1 + yu^1 + \phi_y^1 = 0, \tag{3.9}$$

$$\phi_t^1 + u_x^1 + v_y^1 = -2u^0 u_x^0, \tag{3.10}$$

after simplification by using the facts that $u^0 = \phi^0$ and $v^0 = 0$.

To solve (3.8)–(3.10) efficiently, it is necessary to introduce a couple of tricks. The homogeneous solutions of (3.8)–(3.10) are characterized by the fact that the north-south velocity for a given mode is directly proportional to a Hermite function, i.e.,

$$v = p(x,t)e^{-(1/2)y^2}H_n(y), \tag{3.11}$$

where H_n is the nth Hermite polynomial, whose degree is used to label the latitudinal structure of the free modes. Unfortunately, u and ϕ for a given mode are generally equal to the sum of *two* Hermite functions. However, it can be shown that for a given mode

$$S = (\phi + u) = r(x,t)e^{(-1/2)y^2}H_{n+1}(y), \tag{3.12}$$

$$D = (\phi - u) = q(x,t)e^{(-1/2)y^2}H_{n-1}(y). \tag{3.13}$$

The first trick will be to introduce these "sum" and "difference" variables in place of ϕ and u because the fact that each is proportional to a single Hermite function will greatly simplify the algebra. [This device was introduced by Gill and Clarke (1974).] The first-order equations become (dropping the superscript 1's)

$$S_t + S_x + \left[\frac{\partial}{\partial y} - y\right]v = F, \tag{3.14}$$

$$v_t + \frac{1}{2}\left[\frac{\partial}{\partial y} + y\right]S + \frac{1}{2}\left[\frac{\partial}{\partial y} - y\right]D = 0, \tag{3.15}$$

$$D_t - D_x + \left[\frac{\partial}{\partial y} + y\right]v = G, \tag{3.16}$$

where

$$F = -3u^0 u_x^0, \tag{3.17}$$

$$G = -u^0 u_x^0. \tag{3.18}$$

The second trick is to recognize that the linear operators

$$R = \frac{\partial}{\partial y} - y, \tag{3.19}$$

$$L = \frac{\partial}{\partial y} + y, \tag{3.20}$$

which appear in square brackets in (3.14)–(3.16) are, in fact, the "raising" and "lowering" operators for the Hermite functions, i.e.,

$$R[H_n(y)e^{(-1/2)y^2}] = -H_{n+1}(y)e^{(-1/2)y^2}, \tag{3.21}$$

$$L[H_n(y)e^{(-1/2)y^2}] = 2nH_{n-1}(y)e^{(-1/2)y^2}. \tag{3.22}$$

Since all the explicit y dependence of the first-order equations is expressed in terms of the raising and lowering operators alone and since the Hermite functions give the latitudinal structure of the free modes and are a complete, orthogonal set of basis functions, the next step is obvious: expand both sides of (3.14)–(3.16) in terms of Hermite functions. This yields

$$S = e^{(-1/2)y^2}\sum_{n=0}^{\infty} S_n(x,t)H_n(y), \tag{3.23a}$$

$$v = e^{(-1/2)y^2}\sum_{n=0}^{\infty} v_n(x,t)H_n(y), \tag{3.23b}$$

$$D = e^{(-1/2)y^2}\sum_{n=0}^{\infty} D_n(x,t)H_n(y), \tag{3.23c}$$

$$F = e^{(-1/2)y^2}\sum_{n=0}^{\infty} F_n(x,t)H_n(y), \tag{3.24a}$$

$$G = e^{(-1/2)y^2}\sum_{n=0}^{\infty} G_n(x,t)H_n(y). \tag{3.24b}$$

Substituting these expansions into (3.14)–(3.16) and using the properties of the raising and lowering operators, one finds that

$$S_{0t} + S_{0x} = F_0, \tag{3.25}$$

plus the pair of equations

$$S_{1t} + S_{1x} - v_0 = F_1, \tag{3.26a}$$

$$v_{0t} + S_1 = 0, \tag{3.26b}$$

plus the coupled triplets of equations, one for each value of $n \geq 1$

$$(S_{n+1})_t + (S_{n+1})_x - v_n = F_{n+1}, \tag{3.27a}$$

$$v_{nt} + (n+1)S_{n+1} - \tfrac{1}{2}D_{n-1} = 0, \tag{3.27b}$$

$$(D_{n-1})_t - (D_{n-1})_x - 2nv_n = G_{n-1}. \tag{3.27c}$$

The reason that one finds only a single equation for S_0 and just a pair involving v_0 versus a set of three equations involving v_n for $n \geq 1$ is simply a reflection of the well-known fact that there is only one mode with no meridional velocity (the Kelvin wave) and only two modes with v proportional to H_0 (the mixed Rossby-gravity wave plus a westerly gravity wave). For greater values of n, each triplet (3.27) has three homogeneous solutions corresponding to an easterly Rossby wave plus easterly and westerly gravity waves. This direct correspondence between (3.25) to (3.27) and all the allowable free modes is one of the benefits of approaching the shallow water equations as a set: the usual procedure of reducing the three equations down to a single equation for v has the double-barreled disadvantage of (i) leaving out the Kelvin wave entirely and (ii) introducing a third spurious mode for $n = 0$, which one must then argue away by showing that it is not a solution of the original set of equations.

Since the Kelvin wave is symmetric about the equator, as are the Hermite polynomials of even degree while those of odd degree are antisymmetric about the equator, it follows that all the odd-degree Hermite expansion coefficients of F and G are zero. This implies that one need not worry about (3.26) or about those triplets (3.27) for which n is even. Explicitly, by using the known expansion for $[\exp(-(1/2)y^2)]^2 = \exp(-y^2)$, one finds

$$F_n = \begin{cases} -3Q(x-t)Q'(x-t)(2/3)^{1/2}\dfrac{(-1)^{n/2}}{(n/2)!12^{n/2}} & (n \text{ even}) \\ 0 & (n \text{ odd}), \end{cases} \tag{3.28}$$

$$G_n = F_n/3 \qquad (\text{all } n). \tag{3.29}$$

Since the F_n, G_n, and the linear Kelvin wave itself are all functions of $(x - t)$ only, it is convenient to transform the coordinates into a frame of reference moving with the phase speed of the zeroth-order solution by defining

$$s = x - t. \tag{3.30}$$

In the new coordinates, the equations become

$$S_{0t} = F_0(s) \tag{3.31}$$

and the triplet sets

$$(S_{n+1})_t - v_n = F_{n+1}(s), \tag{3.32a}$$

$$v_{nt} - v_{ns} + (n+1)S_{n+1} - \tfrac{1}{2}D_{n-1} = 0, \tag{3.32b}$$

$$(D_{n-1})_t - 2(D_{n-1})_s + 2nv_n = G_{n-1}(s). \tag{3.32c}$$

The general solution to (3.31) and (3.32) will consist of a particular solution plus an appropriate combination of the free modes which satisfy the homogeneous forms of (3.31) and (3.32). The homogeneous solution is given explicitly by

$$S_0(s,t) = A_K(s), \tag{3.33a}$$

$$S_{n+1}(s,t) = \int_{-\infty}^{\infty} dkA_R(k)\exp[iks - i\omega_R(k)t] + \int_{-\infty}^{\infty} dkA_{WG}(k)\exp[iks - i\omega_{WG}(k)t] + \int_{-\infty}^{\infty} dkA_{EG}(k)\exp[iks - i\omega_{EG}(k)t], \tag{3.33b}$$

$$v_n(s,t) = -i\left\{\int_{-\infty}^{\infty} dk\omega_R(k)A_R(k)\exp[iks - i\omega_R(k)t] + \int_{-\infty}^{\infty} dk\omega_{WG}(k)A_{WG}(k)\exp[iks - i\omega_{WG}(k)t] + \int_{-\infty}^{\infty} dk\omega_{EG}(k)A_{EG}(k)\exp[iks - i\omega_{EG}(k)t]\right\}, \tag{3.33c}$$

$$D_{n-1}(s,t) = -2n\left\{\int_{-\infty}^{\infty} dk\frac{\omega_R(k)A_R(k)}{\omega_R(k)+2k}\exp[iks - i\omega_R(k)t] + \int_{-\infty}^{\infty} dk\frac{\omega_{WG}(k)A_{WG}(k)}{\omega_{WG}(k)+2k}\exp[iks - i\omega_{WG}(k)t] + \int_{-\infty}^{\infty} dk\frac{\omega_{EG}(k)A_{EG}(k)}{\omega_{EG}(k)+2k}\exp[iks - i\omega_{EG}(k)t]\right\}, \tag{3.33d}$$

where ω_R, ω_{WG} and ω_{EG} are the three roots of the dispersion relation

$$\omega^3 + 3k\omega^2 + (2k^2 - 2n - 1)\omega - (2n+2)k = 0, \tag{3.34}$$

physically corresponding to Rossby waves, westward propagating gravity waves, and eastward propagating gravity waves. The amplitudes $A_R(k)$, $A_{WG}(k)$ and $A_{EG}(k)$ (which are different for each n) are chosen (with the help of inverse Fourier transformation) so that the sum of the homogeneous solution (3.33) plus the particular solution (3.35) satisfies the initial conditions and $A_K(s)$ similarly. Since these free modes are either sinusoidal in time ($n \geq 1$) or independent of time (S_0), as is explicit in (3.33), their contribution to the first-order equations cannot cause the perturbation expansion to be nonuniform and one need not worry about them further. It is the particular solution that will require straining the coordinates and, in this respect, (3.31) and (3.32) are radically different.

The triple set (3.32) has a particular solution which is a function of s only and is explicitly

$$v_n(s) = -F_{n+1}(s), \tag{3.35a}$$

$$D_{n-1}(s) = -\int_0^s ds'[\tfrac{1}{2}G_{n-1}(s') + nF_{n+1}(s')], \tag{3.35b}$$

$$S_{n+1}(s) = -(n+1)^{-1}\int_0^s ds'[\tfrac{1}{4}G_{n-1}(s') + \tfrac{1}{2}nF_{n+1}(s') + (F_{n+1}(s'))_{ss}]. \tag{3.35c}$$

In contrast,

$$S_0(s,t) = F_0(s)t + (x-t)B(s), \tag{3.36}$$

where $B(s)$ is an arbitrary function of s. A factor of s ($=x-t$) has been explicitly displayed in the term which is a function of s only so that one can see that the only way that one can remove the "secular" (linearly increasing and unbounded) behavior in t is to set $B(s) = F_0(s)$, which generates secular behavior in x. It is this single equation (3.31) that keeps the regular perturbation expansion from being uniformly accurate in x and t. Physically, this distinction between (3.31) and (3.32) is easy to understand because the other free modes [the solutions of the homogeneous form of (3.32) as given by (3.33b)–(3.34)] all have phase speeds different from that of the Kelvin wave. Consequently, a forcing, such as that given by $F(s)$ and $G(s)$ here, which is traveling at the phase speed of the Kelvin wave, will resonantly force just that one mode—the Kelvin wave itself.

An alternative approach to seeing the nature of the solutions of (3.31) is to use (3.12), (3.17), (3.28) and (3.30) to rewrite (3.31) as

$$u_t^0 + u_x^0 + (3/2)^{1/2}u^0u_x^0 = 0. \tag{3.37}$$

Except for the factor of $(3/2)^{1/2} = 1.2247$ (which could be eliminated by rescaling u), this equation is identical to (2.1) for the special case $c = 1$. From the results presented in the previous section, one has immediately the lowest order[3] strained coordinate solutions:

Initial value problem

$$u(x, t=0) = Q(x)e^{(-1/2)y^2}, \tag{3.38}$$

$$u(x,t) = Q(\xi)e^{(-1/2)y^2}, \tag{3.39a}$$

where $\xi(x,t)$ is determined by solving

$$x = \xi + [1 + (3/2)^{1/2}Q(\xi)]t. \tag{3.39b}$$

Boundary value [signaling] problem

$$u(x=0, t) = R(-t)e^{(-1/2)y^2}, \tag{3.40}$$

$$u(x,t) = R(\xi)e^{(-1/2)y^2}, \tag{3.41a}$$

where $\xi(x,t)$ is found from

$$x = \xi + [1 + (3/2)^{1/2}R(\xi)]t + (3/2)^{1/2}\xi R(\xi). \tag{3.41b}$$

The first-order solutions (which will not be considered further in this work) are identical, to this order of approximation, with (3.33) and (3.35) except that, first, s is replaced by ξ and, second, $S_0 = 0$. Section 4 illustrates these solutions (3.39) and (3.41) by applying them to some oceanographic phenomena.

b. *Resonant triads and coastal reflections*

From a resonant triad viewpoint, as considered by Ripa (1979), every Kelvin wave is resonantly interacting with every other Kelvin wave and *not* resonantly interacting with any other species of equatorial wave. This mutual resonance of the Kelvin waves among themselves justifies the use of strained coordinates since an approximation, which represents the Kelvin wave as the sum of a triad of sine functions in x, must inevitably fail long before the wave actually breaks. The *nonresonance* of the Kelvin wave with other types of equatorial waves, however, is even more important.

First, as shown above, it implies that the actual straining of coordinates for the Kelvin wave is the same as that for the one-dimensional advection equation, which simplifies life enormously. Second, and more significant, this nonresonance justifies the initial condition of a pure Kelvin wave assumed in (3.7), which in and of itself is highly unrealistic. In general, however, resonant effects dominate nonresonant effects, and, in the present case, this means that the mutual resonance of the Kelvin waves among themselves is what primarily determines their nonlinear evolution. To a first approximation, the Kelvin wave will steepen and break independent of whatever Rossby or gravity waves may be present, provided only that the amplitude of these other waves is small.

It is this independence of other waves that justifies the rather naive determination of the significance of nonlinearity for Kelvin waves in the Gulf of Guinea, which is presented in the next section. When a Kelvin wave strikes the west coast of a continent, Rossby and gravity waves are generated by reflec-

[3] By the "lowest order" here and "first order" below (3.41b), I refer to U^0 and U^1 in the notation of Section 2b, i.e., the perturbative solutions *after* straining. It is one of the peculiarities of singular perturbation theory that U^0, although "lowest order", is not equal to u^0, which is the linear wave.

tion. Thus, the response to a low-frequency, periodic forcing in a bounded ocean will inevitably consist of a complicated mixture of many different species of equatorial waves. As pointed out to the author by S. G. H. Philander, there may be little physical point to regarding such a soup of multiple reflections as "waves" if the forcing frequency is low enough. It is always possible *mathematically*, however, to expand the total solution as a Hermite function series and examine individual coefficients. In particular, the evolution of the coefficient of $H_0(y)$ in the expansion of the sum variable S, i.e., the Kelvin wave, is still largely independent of all the other coefficients, provided that the others are small. It follows that assuming an initial condition of a pure Kelvin wave as in (3.7) will, to lowest order, lead to a correct analysis of the effects of nonlinearity on the Kelvin wave even in situations where the Kelvin wave by itself cannot possibly be the complete solution to the stated problem, as is true of the Gulf of Guinea.

A similar argument can extend the present theory to the much more realistic case of a *baroclinic* ocean. The Kelvin waves of different vertical modes will not resonantly interact with another, so the steepening and breaking of the Kelvin wave of the first baroclinic mode—which is what will actually be discussed in the oceanographic examples of the next section—will proceed independently of what is happening in other vertical modes. Thus, the barotropic theory presented here can be directly applied to a stably stratified ocean simply by replacing the fluid depth H by the appropriate "equivalent depth". The only other possibility is that there may be resonant triad interactions between different vertical modes, possibly involving Kelvin waves—but here, this sleeping tiger will be left asleep.

4. Applications to El Niño and the Gulf of Guinea

a. Background

At the present time (December 1978), there are no direct observations of Kelvin waves in the oceans. However, Kelvin waves play a major role in McCreary's (1976) theory of El Niño and Moore *et al.*'s (1978) explanation of similar upwelling events in the Gulf of Guinea. While the waves themselves have not been seen, the wind stresses that excite them and the upwelling that occurs when the Kelvin waves reach the eastern boundary are both known observables, and this makes it possible to calculate what the Kelvin waves should be. Because of their great practical importance, the rest of this section will concentrate on these El Niño and Gulf of Guinea Kelvin waves.

The following parameters will be used: $H = 40$ cm [equivalent depth of the first baroclinic mode]; E [defined in (3.1)] = 214 000; the nondimensional unit of length [defined by (3.2)] $L = 295$ km; the nondimensional unit of time [defined by (3.3)] $T = 1.71$ days; and the phase speed of the Kelvin wave is, therefore, 2.00 m s^{-1}. Following McCreary (1978, private communication), the maximum amplitude of the wave is taken to be ~1/3 (nondimensional) for all three oceans (Pacific, Atlantic, Indian), which implies a maximum zonal velocity of 0.66 m s^{-1}. These estimates, however, should be regarded as no more than educated guesses in the absence of direct observational verification.

In McCreary's (1976) model of El Niño, the Kelvin wave was excited by a "sudden event"—in this case, a rapid change in the wind stress applied to the Pacific. At first it was thought (Moore *et al.*, 1978) that the Gulf of Guinea upwelling was an almost identical (though weaker) "sudden event", but more recent observational and theoretical evidence (Moore and McCreary, private communication, 1978) suggests that the phenomenon is periodic with an annual period. Consequently, the next two subsections will treat periodically forced and impulsively forced Kelvin waves in turn. For both cases, the "signaling" problem will be examined with $x = 0$ corresponding to the ocean's western boundary where the wave is excited. The "signaling" solution (3.39) will be used to follow the wave as nonlinearity modifies its shape and speed as it propagates eastward towards the other coast.

b. Periodic forcing: The Gulf of Guinea

The general periodic Kelvin wave solution is given by

$$u(x,y,t) = \epsilon \sin(k\xi)e^{(-1/2)y^2}, \tag{4.1a}$$

$$\phi(x,y,t) = u(x,y,t), \tag{4.1b}$$

$$v(x,y,t) = 0, \tag{4.1c}$$

where ξ is determined by

$$k(x - t) = k\xi + \epsilon k(3/2)^{1/2}x \sin(k\xi), \tag{4.2}$$

using the approximation made in (2.28). Except for the y dependence and the factor of $(3/2)^{1/2}$ in (4.2), this is the same as the solution of the one-dimensional advection equation given in Section 2d. For a wave of annual period, using the parameter values given above, one has

$$k = 0.030, \tag{4.3}$$

$$\epsilon = 1/3. \tag{4.4}$$

One then finds the wave first breaks at a distance from the western boundary of the ocean given by [from (2.42)]

$$x_B = \frac{(2/3)^{1/2}}{k|\epsilon|}, \tag{4.5}$$

$$= 24\ 000\ \text{km}. \tag{4.6}$$

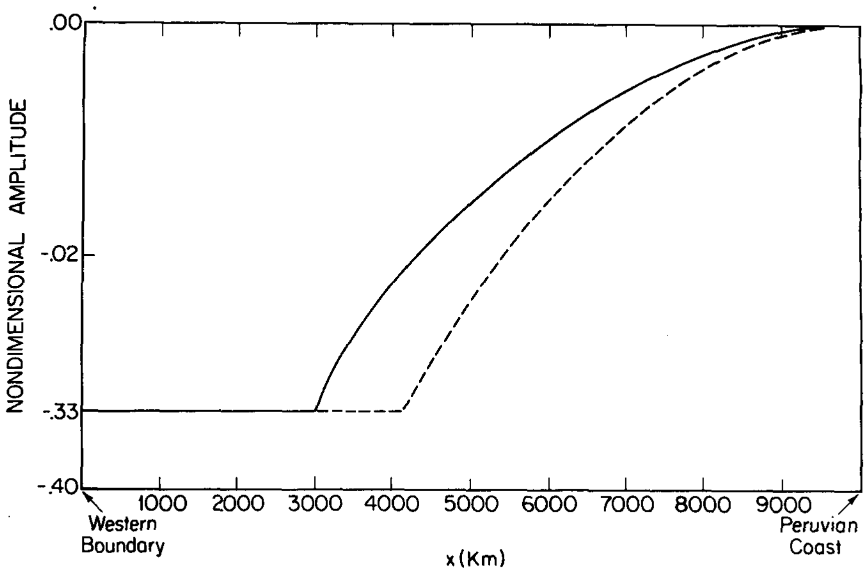

FIG. 2. The nondimensional thermocline thickness of the El Niño Kelvin wave at the moment it strikes the Peruvian coast in linear (dashed line, McCreary, 1976) and nonlinear (solid line) theory.

Since no ocean is this wide, it follows that Kelvin waves with an annual period are unlikely to break anywhere on the globe.

For the Atlantic, in which a Kelvin wave of annual period is thought to play a major role in the Gulf of Guinea upwelling described by Moore *et al.* (1978), one may take the width of the ocean as 5000 km. Ignoring the (nonsecular) first-order terms involving higher modes which are given by (3.33), the *ordinary* perturbation series to first order is

$$u(x,y,t) = \epsilon e^{(-1/2)y^2} \sin[k(x-t)] \times \{1 - (3/2)^{1/2}\epsilon kx \cos[k(x-t)]\}, \quad (4.7)$$

where the maximum value of the correction term (in braces) is only 0.208 at the eastern boundary of the Atlantic.

For the Gulf of Guinea Kelvin wave, then, ordinary perturbation theory is perfectly adequate, and using it, McCreary and Moore (private communication, 1978) have already independently confirmed that nonlinear effects are small. In both the formula for x_B [Eq. (4.5)], and in Eq. (4.7), however, a factor of k appears explicitly, so shorter period waves will be more strongly affected. An Atlantic Kelvin wave with a two-month period and the same amplitude assumed above would break before reaching the Gulf of Guinea.

c. *Impulsive forcing and El Niño*

In McCreary's (1976) model of El Niño, the boundary forcing of the Kelvin wave is assumed to be

$$u(x=0,y,t) = \begin{cases} \epsilon e^{(-1/2)y^2}(t/t_0)^2, & 0 < t \leq t \\ \epsilon e^{(-1/2)y^2}, & t \geq t_0, \end{cases} \quad (4.8)$$

when a zonal wind stress is driving the motion, where $t_0 = 3 \times 10^6$ s and $\epsilon = -1/3$.[4] The approximate solution is

$$u(x,y,t) = \begin{cases} \epsilon e^{(-1/2)y^2}\xi^2/t_0^2, \\ \quad 0 < t < t_0 + [1 - (3/2)^{1/2}\epsilon]x \\ \epsilon e^{(-1/2)y^2}, \\ \quad t \geq t_0 + [1 - (3/2)^{1/2}]x, \end{cases} \quad (4.9a)$$

$$\xi = \frac{2(x-t)}{1 + [1 + 4(3/2)^{1/2}\epsilon x(x-t)/t_0^2]^{1/2}} . \quad (4.9b)$$

Because the wave is strictly a wave of *depression* ($\epsilon < 0$)—it makes the warm, upper layer thinner by raising the thermocline—the sloping portion of the wave is horizontally expanded by nonlinearity as the wave propagates. In linear theory, upper and lower limits of the quadratically varying portion of the wave would be separated by 5780 km; according to (4.9a) these limits are separated by 7000 km—an increase of 21% when that part of the wave generated at $t = 0$ strikes the Peruvian Coast. The linear and nonlinear predictions are compared in Fig. 2. The travel time of the flat portion of the trough from one coast to the other has been increased by twice as much, about 40%.

Thus, nonlinearity significantly alters both the shape and phase speed of equatorial Kelvin wave, even though the idealized wave is of positive definite slope, which precludes the formation of sharp fronts

[4] The sign of ϵ has been changed so that, both here and in the preceding subsection, the Kelvin wave makes the thermocline rise, as consistent with the upwelling theories. Note that $u(x, y,t)$ as given in (4.1a) $= -\epsilon \sin(kt)$ at $x = 0$.

and breaking. In reality, however, the trailing edge of the Kelvin wave packet, instead of being flat as assumed here, probably has negative slope. From (4.8) and (4.9), one can easily see that a wave of the same amplitude and shape as our leading edge, but of opposite sign, will break 8000 km from the western boundary—before reaching the Peruvian coast. This strongly suggests the *trailing* edge of the El Niño Kelvin wave will break also. Although their model is too complex to allow a quantitative comparison, such trailing edge breaking has been observed by Adamec and O'Brien (1978)[5] in their nonlinear numerical calculations.

5. Summary

Kelvin waves play a major role both in El Niño and the related upwelling in the Gulf of Guinea, but nonlinearity is important only for the former. For the waves that propagate into the Gulf of Guinea, nonlinear effects are very small and can be calculated using ordinary perturbation theory. For the Kelvin wave excited during El Niño, however, it is a different story: a 40% reduction in phase speed, a 20% expansion of the leading edge of the wave, and frontogenesis and breaking in the trailing edge. To analyze such large nonlinear effects, the strained coordinates theory presented here is essential.

In view of current observational knowledge of Kelvin waves (essentially none!), the theory's simplicity is a great virtue. For example, one can determine from inspection of (4.5) that a periodic Kelvin wave of the same amplitude as assumed here will break in crossing the Atlantic if its period is two months or less and in crossing the Pacific if its period is four months or less. As other species of Kelvin waves are identified, each can be compared against the theory presented here to determine the role of nonlinearity in its propagation.

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[5] Their model is actually of the Gulf of Guinea upwelling, but because it is impulsively forced, its basic physics is identical with the El Niño model discussed here. In particular, their Fig. 10 is a beautiful illustration of the leading edge elongation and trailing edge compression predicted by the strained coordinates theory.

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