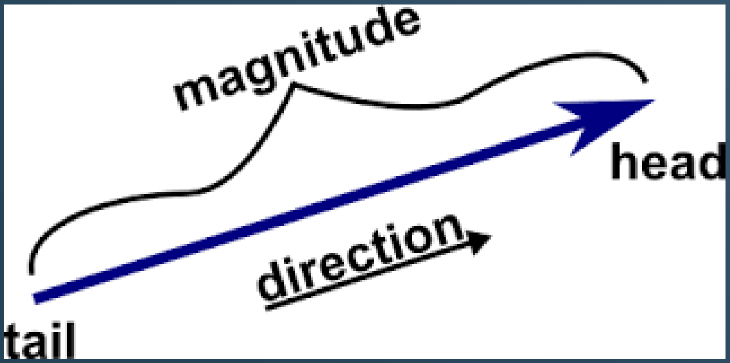
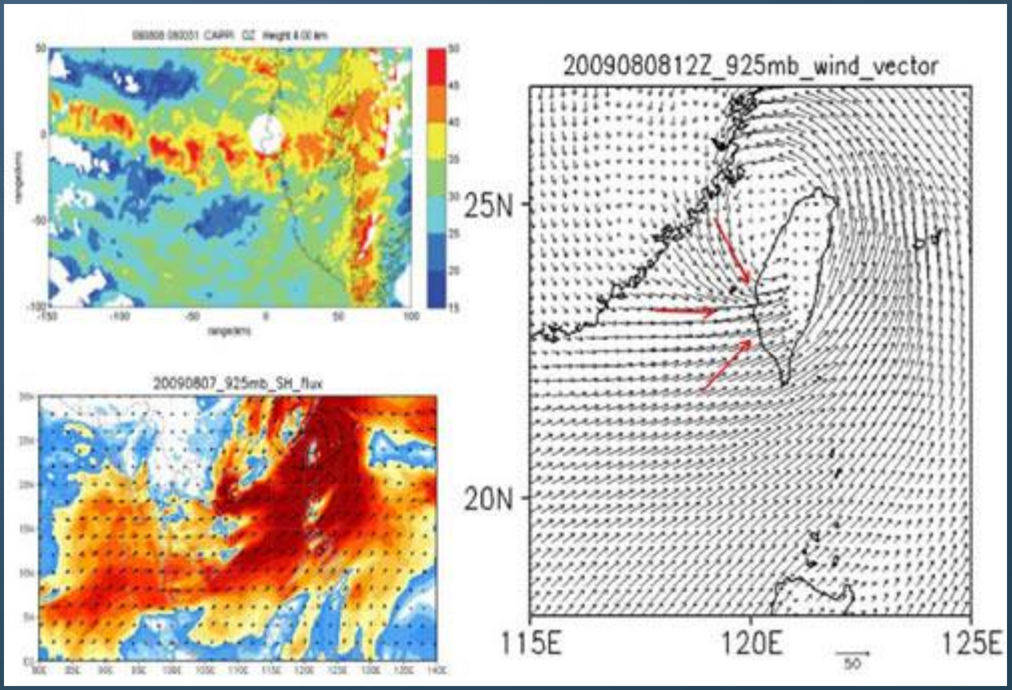
Chapter 3

Vectors





3-1 Vectors and Their Components

Learning Objectives

3.01 Add vectors by drawing them in head-to-tail arrangements, applying the commutative and associative laws.

3.02 Subtract a vector from a second one.

3.03 Calculate the components of a vector on a given coordinate system, showing them in a drawing.

3.04 Given the components of a vector, draw the vector and determine its magnitude and orientation.

3.05 Convert angle measures between degrees and radians.

3-1 Vectors and Their Components

- Physics deals with quantities that have both size and direction
- A **vector** is a mathematical object with size and direction
- A **vector quantity** is a quantity that can be represented by a vector
  - Examples: position, velocity, acceleration
  - Vectors have their own rules for manipulation
- A **scalar** is a quantity that does not have a direction
  - Examples: time, temperature, energy, mass
  - Scalars are manipulated with ordinary algebra

3-1 Vectors and Their Components

- The simplest example is a **displacement vector**
- If a particle changes position from A to B, we represent this by a vector arrow pointing from A to B

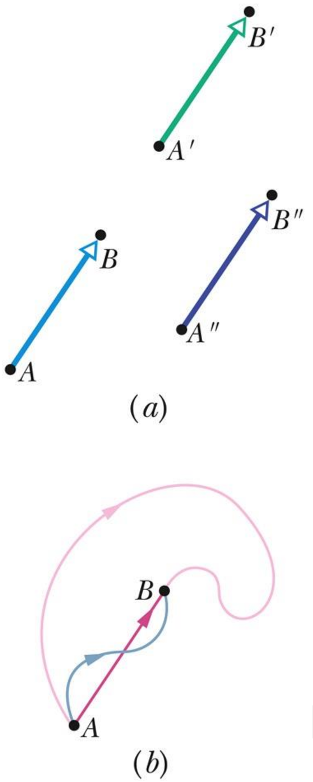


Figure 3-1

- In (a) we see that all three arrows have the same magnitude and direction: they are identical displacement vectors.
- In (b) we see that all three paths correspond to the same displacement vector. The vector tells us nothing about the actual path that was taken between A and B.

3-1 Vectors and Their Components

- The **vector sum**, or **resultant**
  - Is the result of performing vector addition
  - Represents the net displacement of two or more displacement vectors

$$\vec{s} = \vec{a} + \vec{b},$$

Eq. (3-1)

  - Can be added graphically as shown:

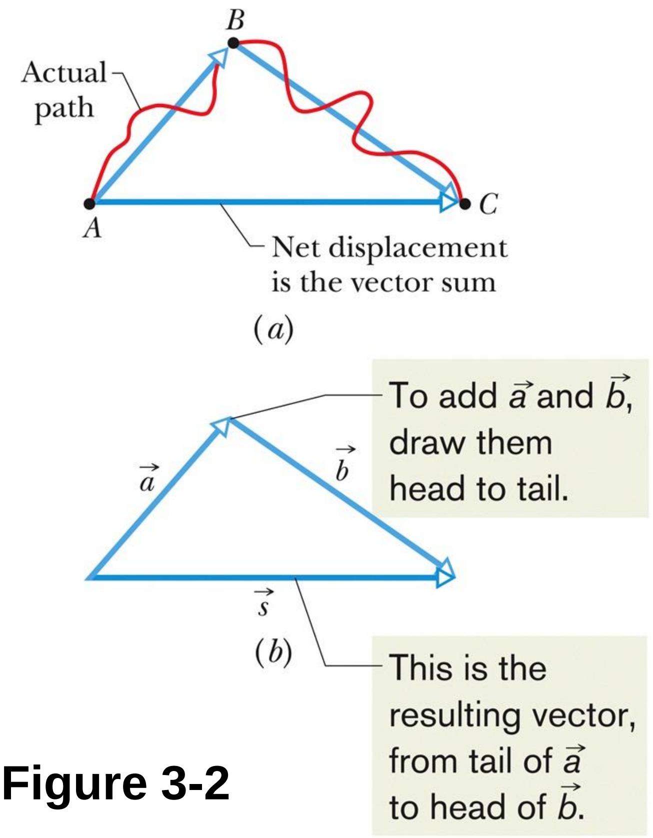


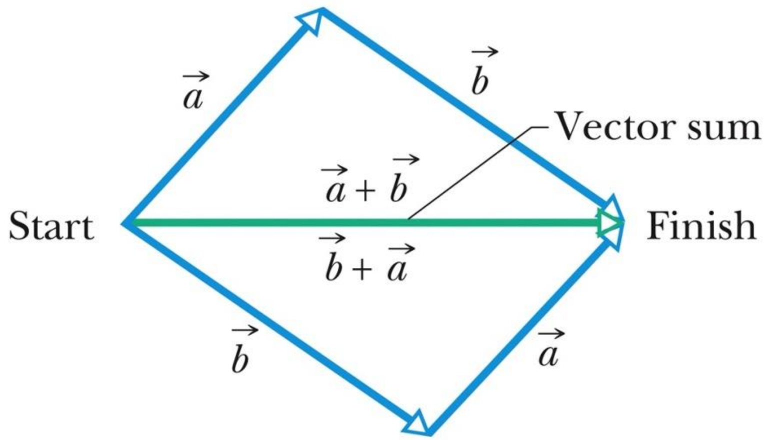
Figure 3-2

3-1 Vectors and Their Components

- Vector addition is **commutative**
  - We can add vectors in any order

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{(commutative law).}$$

Eq. (3-2)



You get the same vector result for either order of adding vectors.

Figure (3-3)

3-1 Vectors and Their Components

- Vector addition is **associative**
  - We can group vector addition however we like

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad \text{(associative law).}$$

Eq. (3-3)

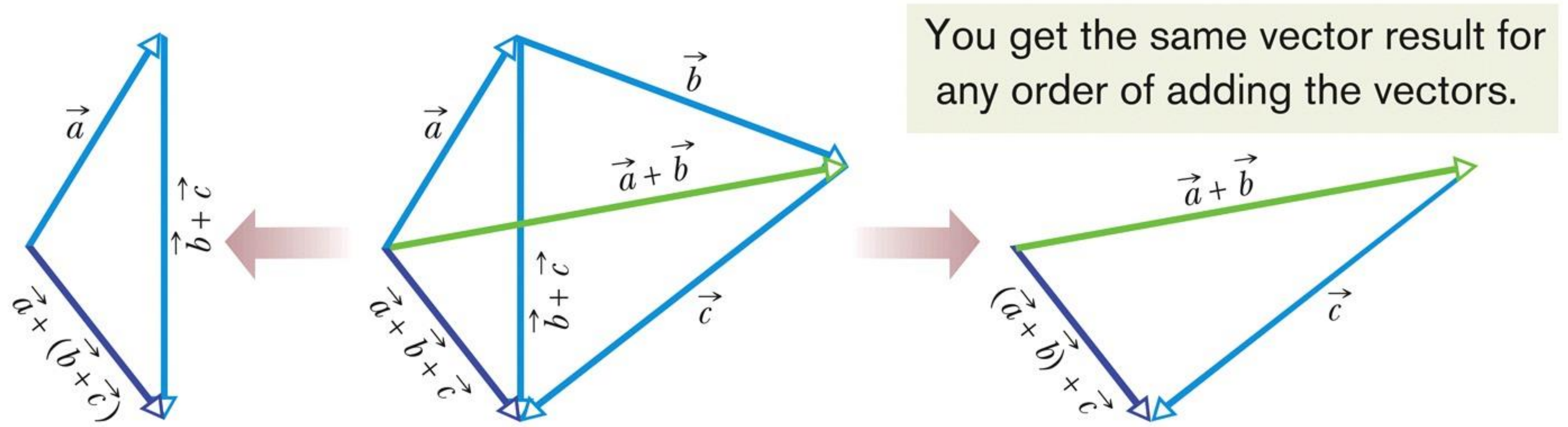


Figure (3-4)

3-1 Vectors and Their Components

- A negative sign reverses vector direction

$$\vec{b} + (-\vec{b}) = 0.$$

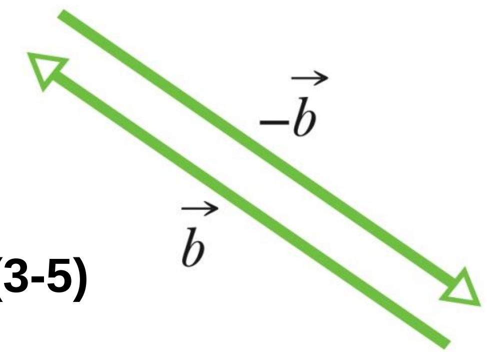


Figure (3-5)

- We use this to define vector subtraction

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

Eq. (3-4)

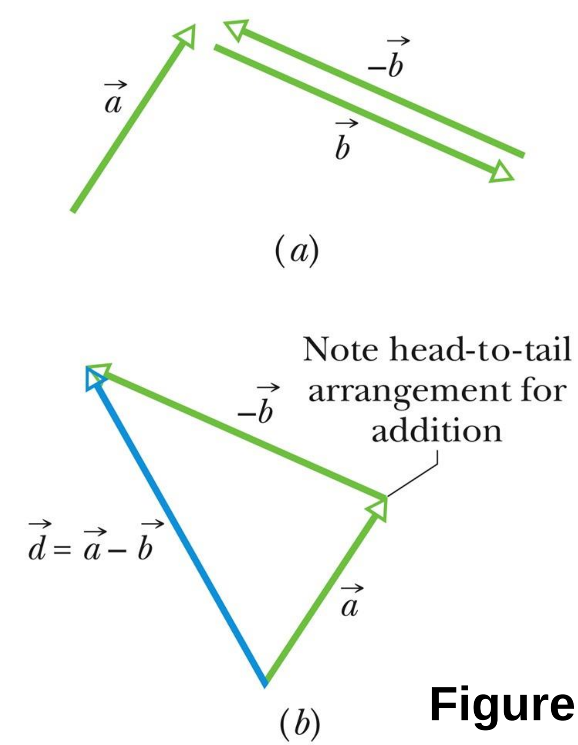


Figure (3-6)

3-1 Vectors and Their Components

- Rather than using a graphical method, vectors can be added by **components**
  - A component is the projection of a vector on an axis
- The process of finding components is called **resolving the vector**
- The components of a vector can be positive or negative.
- They are unchanged if the vector is shifted in any direction (but not rotated).

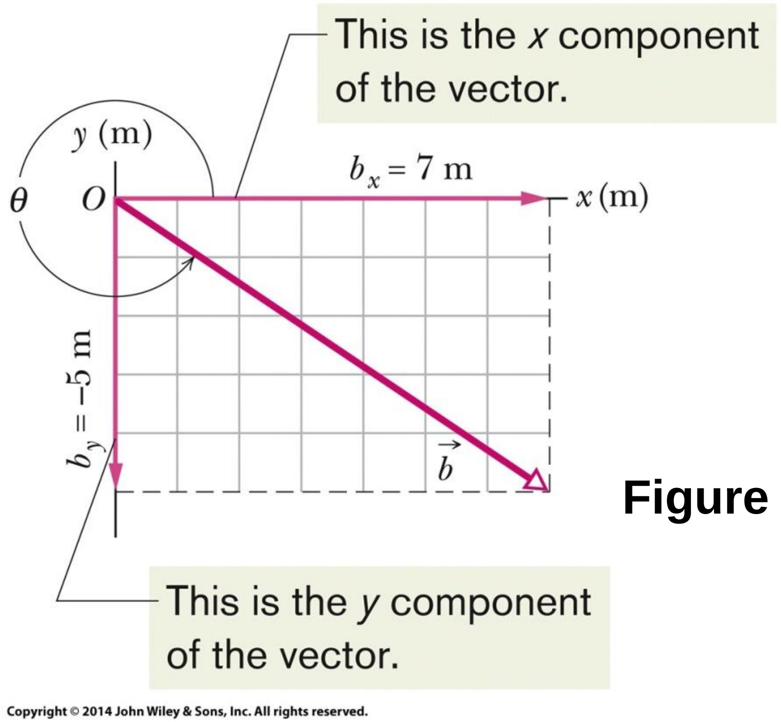


Figure (3-8)

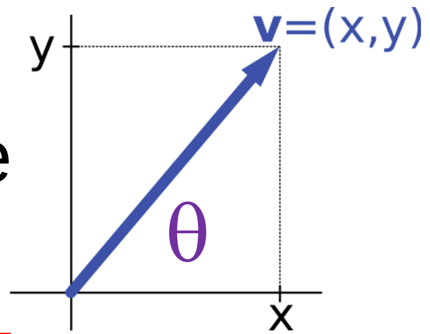
3-1 Vectors and Their Components

- Components in two dimensions can be found by:

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta,$$

Eq. (3-5)

- Where θ is the angle the vector makes with the positive x axis, and a is the vector length
- The length and angle can also be found if the components are known



$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x}$$

Eq. (3-6)

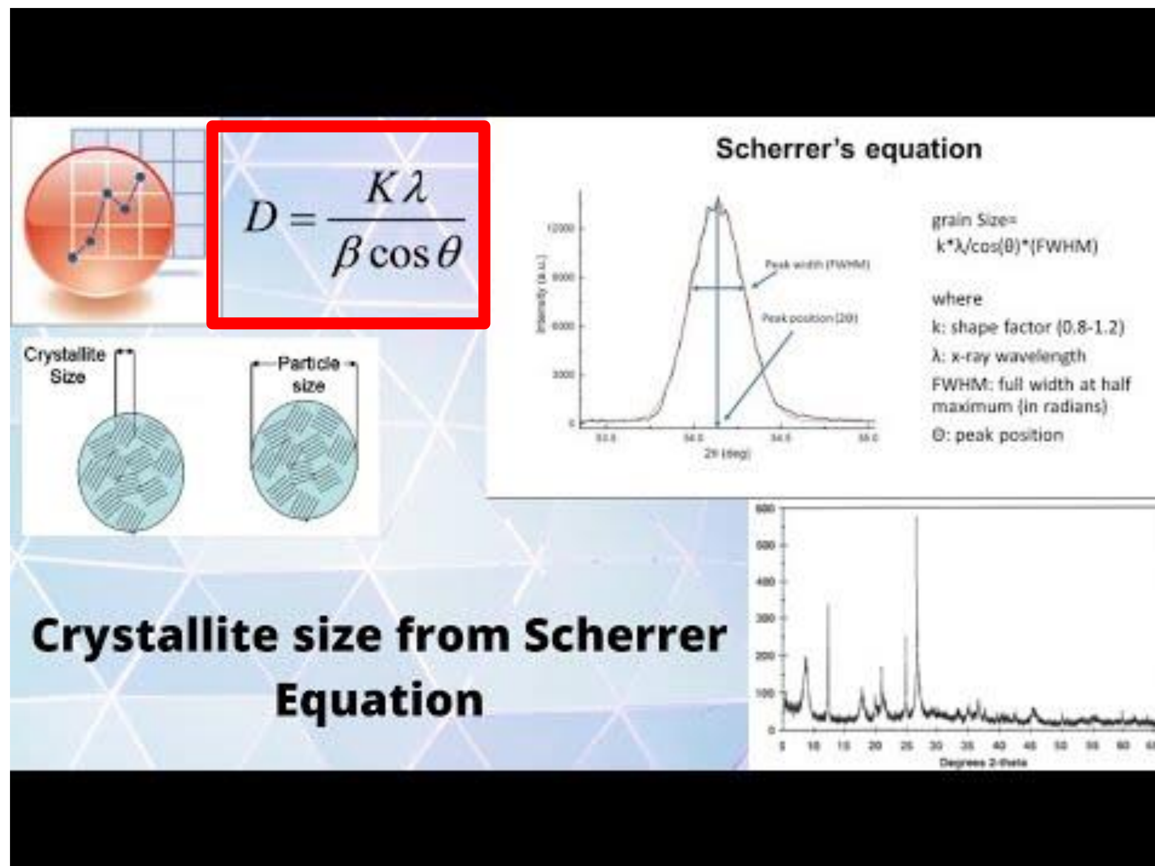
- Therefore, components fully define a vector

3-1 Vectors and Their Components

- Angles may be measured in degrees or radians
- Recall that a full circle is 360˚, or 2π rad

$$40^\circ \, \frac{2\pi \text{ rad}}{360^\circ} = 0.70 \text{ rad.}$$

奈米結晶粒徑大小之估計:
Scherrer equation
~β (半高寬, FWHM):
should be in unit of rad!



3-2 Unit Vectors, Adding Vectors by Components

Learning Objectives

3.06 Convert a vector between magnitude-angle and unit-vector notations.

3.07 Add and subtract vectors in magnitude-angle notation and in unit-vector notation.

3.08 Identify that, for a given vector, rotating the coordinate system about the origin can change the vector's components, but not the vector itself.

3-2 Unit Vectors, Adding Vectors by Components

- A **unit vector**
  - Has magnitude 1
  - Has a particular direction
  - Lacks both dimension and unit
  - Is labeled with a hat: ^

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} \qquad \textbf{Eq. (3-7)}$$

$$\vec{b} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}}. \qquad \textbf{Eq. (3-8)}$$

- We use a **right-handed coordinate system**
  - Remains right-handed when rotated

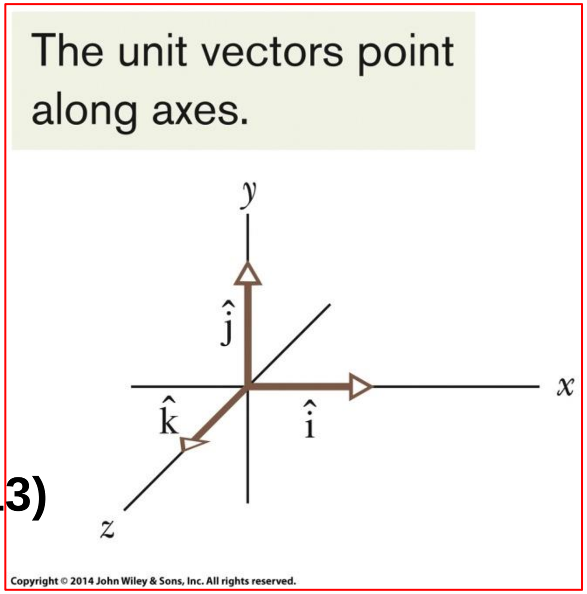


Figure (3-13)

3-2 Unit Vectors, Adding Vectors by Components

- The quantities $a_x\mathbf{i}$ and $a_y\mathbf{j}$ are **vector components**

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}}$$ **Eq. (3-7)**

$$\vec{b} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}}.$$ **Eq. (3-8)**

- The quantities a_x and a_y alone are **scalar components**
  - Or just "components" as before
- Vectors can be added using components

Eq. (3-9) $$\vec{r} = \vec{a} + \vec{b}, \quad \longrightarrow$$

$$r_x = a_x + b_x$$ **Eq. (3-10)**

$$r_y = a_y + b_y$$ **Eq. (3-11)**

$$r_z = a_z + b_z.$$ **Eq. (3-12)**

3-2 Unit Vectors, Adding Vectors by Components

- Vectors are independent of the coordinate system used to measure them
- We can rotate the coordinate system, without rotating the vector, and the vector remains the same

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_x'^2 + a_y'^2}$$ **Eq. (3-14)**

$$\theta = \theta' + \phi.$$ **Eq. (3-15)**

- All such coordinate systems are equally valid

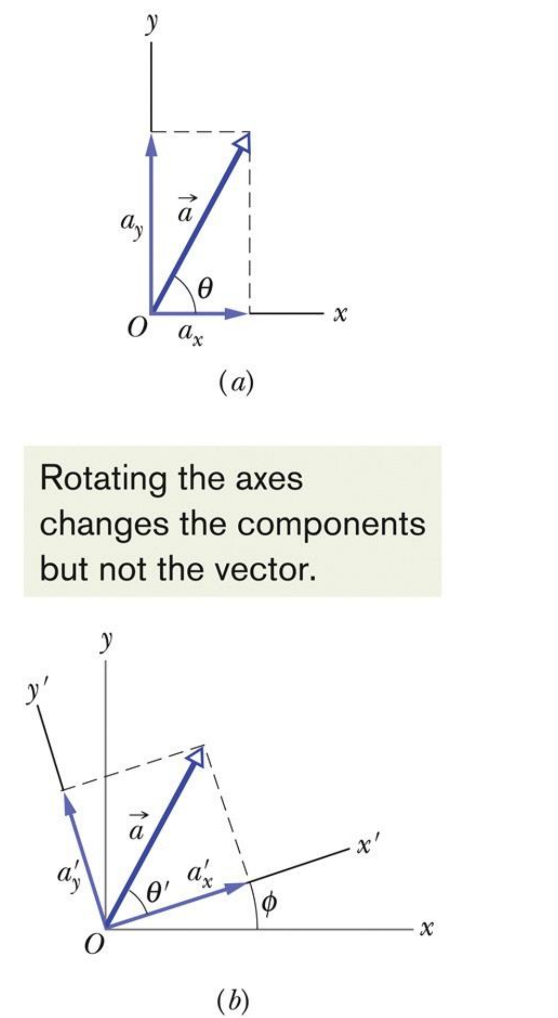


Figure (3-15)

3-3 Multiplying Vectors

Learning Objectives

3.09 Multiply vectors by scalars.

3.10 Identify that multiplying a vector by a scalar gives a vector, the dot product gives a scalar, and the cross product gives a perpendicular vector.

3.11 Find the dot product of two vectors.

3.12 Find the angle between two vectors by taking their dot product.

3.13 Given two vectors, use the dot product to find out how much of one vector lies along the other.

3.14 Find the cross product of two vectors.

3.15 Use the right-hand rule to find the direction of the resultant vector.

3.16 In nested products, start with the innermost product and work outward.

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3-3 Multiplying Vectors

- Multiplying a vector ***z*** by a scalar *c*
  - Results in a new vector
  - Its magnitude is the magnitude of vector ***z*** times |*c*|
  - Its direction is the same as vector ***z***, or opposite if *c* is negative
  - To achieve this, we can simply multiply each of the components of vector ***z*** by *c*
- To divide a vector by a scalar we multiply by 1/*c*

Example Multiply vector ***z*** by 5

- $\boldsymbol{z} = -3\,\mathbf{i} + 5\,\mathbf{j}$
- $5\,\boldsymbol{z} = -15\,\mathbf{i} + 25\,\mathbf{j}$

3-3 Multiplying Vectors

- Multiplying two vectors: the **scalar product**
  - Also called the **dot product**
  - Results in a scalar, where *a* and *b* are magnitudes and *φ* is the angle between the directions of the two vectors:

$$\boxed{\vec{a} \cdot \vec{b} = ab \cos \phi,}$$

Eq. (3-20)

- The commutative law applies, and we can do the dot product in component form

$$\vec{a} \cdot \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$

Eq. (3-22)

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}.$$

$$\boxed{\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z.}$$

Eq. (3-23)

3-3 Multiplying Vectors

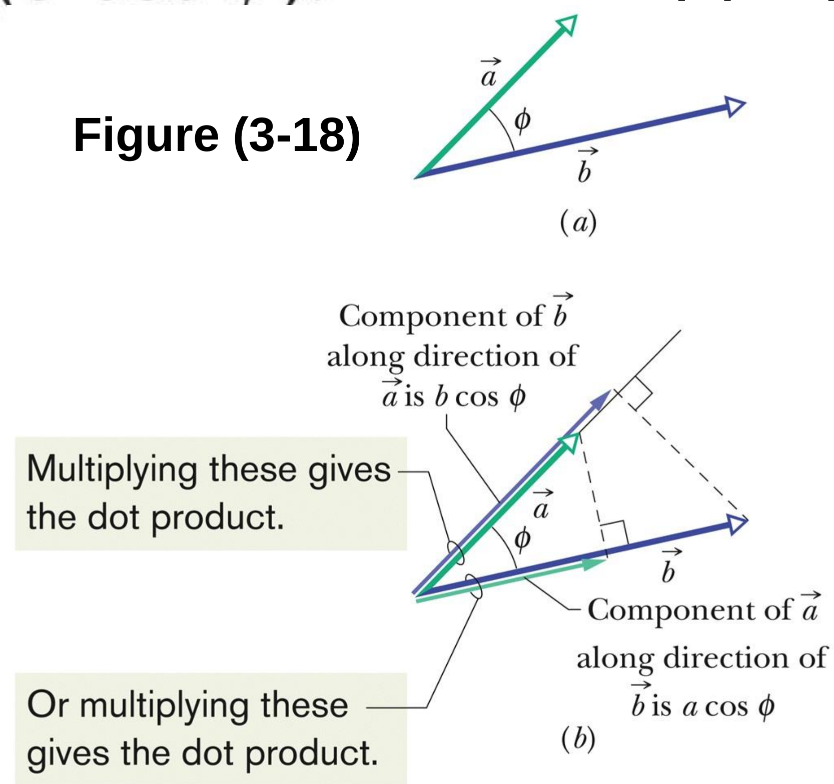
- A dot product is: the product of the magnitude of one vector times the scalar component of the other vector in the direction of the first vector

$$\vec{a} \cdot \vec{b} = (a \cos \phi)(b) = (a)(b \cos \phi).$$

Eq. (3-21)

- Either projection of one vector onto the other can be used
- To multiply a vector by the projection, multiply the magnitudes

Figure (3-18)



3-3 Multiplying Vectors

- Multiplying two vectors: the **vector product**
  - The **cross product** of two vectors with magnitudes *a* & *b*, separated by angle *φ*, produces a vector with magnitude:

$$\boxed{c = ab \sin \phi,}$$

Eq. (3-24)

  - And a direction perpendicular to both original vectors
- Direction is determined by the **right-hand rule**
- Place vectors tail-to-tail, sweep fingers from the first to the second, and thumb points in the direction of the resultant vector

If $\vec{a}$ and $\vec{b}$ are parallel or antiparallel, $\vec{a} \times \vec{b} = 0$. The magnitude of $\vec{a} \times \vec{b}$, which can be written as $|\vec{a} \times \vec{b}|$, is maximum when $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

3-3 Multiplying Vectors

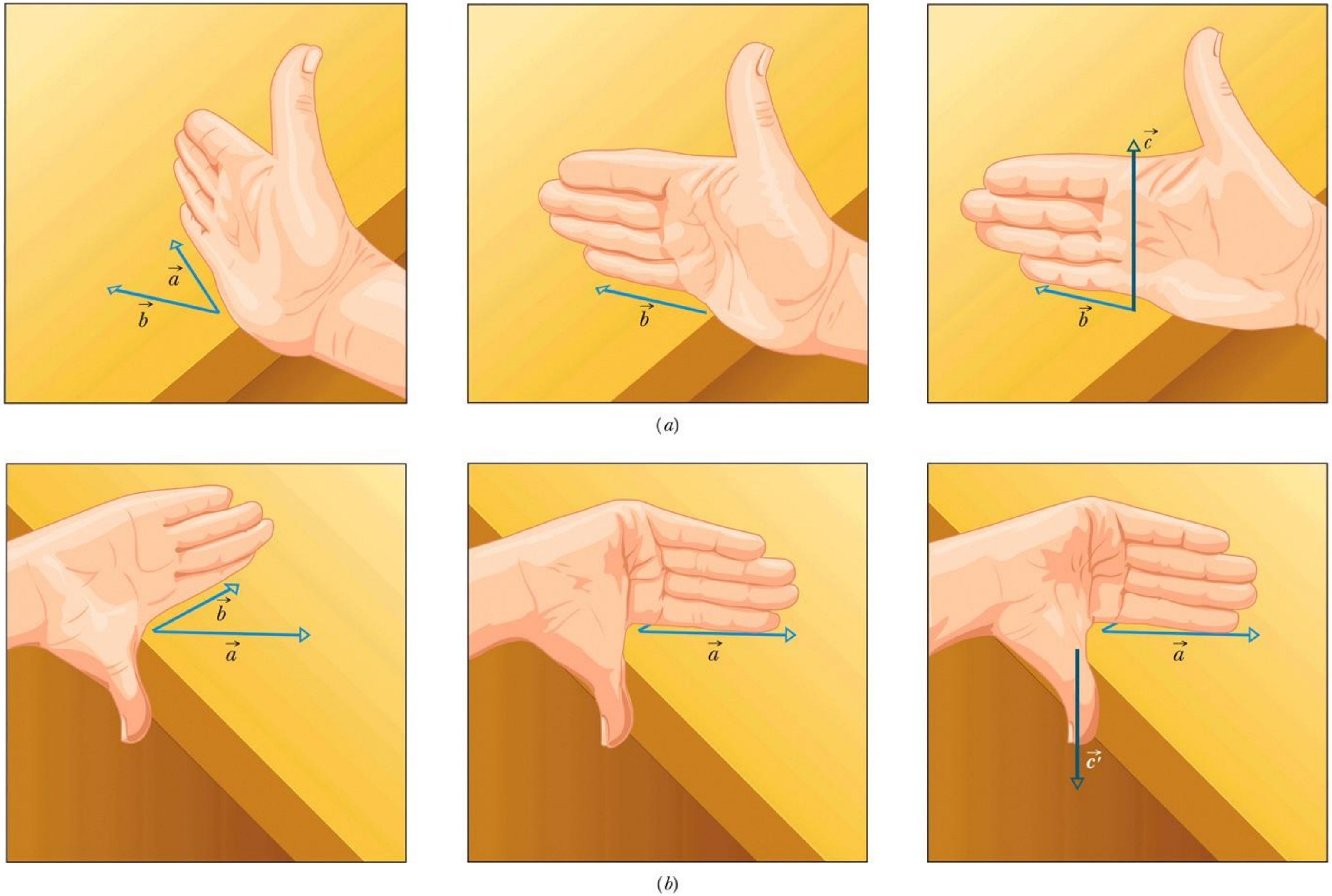


Figure (3-19)

The upper shows vector *a* cross vector *b*, the lower shows vector *b* cross vector *a*

3-3 Multiplying Vectors

- The cross product is not commutative

$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}).$$

Eq. (3-25)

- To evaluate, we distribute over components:

$$\vec{a} \times \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \times (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$

Eq. (3-26)

$$a_x\hat{\mathrm{i}} \times b_x\hat{\mathrm{i}} = a_x b_x(\hat{\mathrm{i}} \times \hat{\mathrm{i}}) = 0,$$

$$a_x\hat{\mathrm{i}} \times b_y\hat{\mathrm{j}} = a_x b_y(\hat{\mathrm{i}} \times \hat{\mathrm{j}}) = a_x b_y\hat{\mathrm{k}}.$$

- Therefore, by expanding (3-26):

$$\boxed{\vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\hat{\mathrm{i}} + (a_z b_x - b_z a_x)\hat{\mathrm{j}} + (a_x b_y - b_x a_y)\hat{\mathrm{k}}.}$$

Eq. (3-27)

The Vector Product $\vec{c} = \vec{a} \times \vec{b}$ in terms of Vector Components

$$\vec{a} = a_x\hat{i} + a_y i + a_z\hat{k}, \vec{b} = b_x\hat{i} + b_y i + b_z k$$

The vector components of vector $\vec{c}$ are given by the equations:

$$c_x = a_y b_z - a_z b_y \quad , \quad c_y = a_z b_x - a_x b_z \quad , \quad c_z = a_x b_y - a_y b_x$$

Note: Those familiar with the use of determinants can use the expression

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Note: The order of the two vectors in the cross product is important

$$\vec{b} \times \vec{a} = -\left(\vec{a} \times \vec{b}\right)$$

3 Summary

Scalars and Vectors

- Scalars have magnitude only
- Vectors have magnitude and direction
- Both have units!

Adding Geometrically

- Obeys commutative and associative laws

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$ **Eq. (3-2)**

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}).$$ **Eq. (3-3)**

Vector Components

- Given by

$$a_x = a \cos\theta \quad \text{and} \quad a_y = a \sin\theta,$$ **Eq. (3-5)**

- Related back by

$$a = \sqrt{a_x^2 + a_y^2} \qquad \text{and} \quad \tan\theta = \frac{a_y}{a_x}$$ **Eq. (3-6)**

Unit Vector Notation

- We can write vectors in terms of unit vectors

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}},$$ **Eq. (3-7)**

3 Summary

Adding by Components

- Add component-by-component

$$r_x = a_x + b_x$$

$$r_y = a_y + b_y$$

Eqs. (3-10) - (3-12) $$r_z = a_z + b_z.$$

Scalar Times a Vector

- Product is a new vector
- Magnitude is multiplied by scalar
- Direction is same or opposite

Scalar Product

- Dot product

$$\vec{a} \cdot \vec{b} = ab \cos \phi,$$

Eq. (3-20)

$$\vec{a} \cdot \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$

Eq. (3-22)

Cross Product

- Produces a new vector in perpendicular direction
- Direction determined by right-hand rule

$$c = ab \sin \phi,$$

Eq. (3-24)

補充:

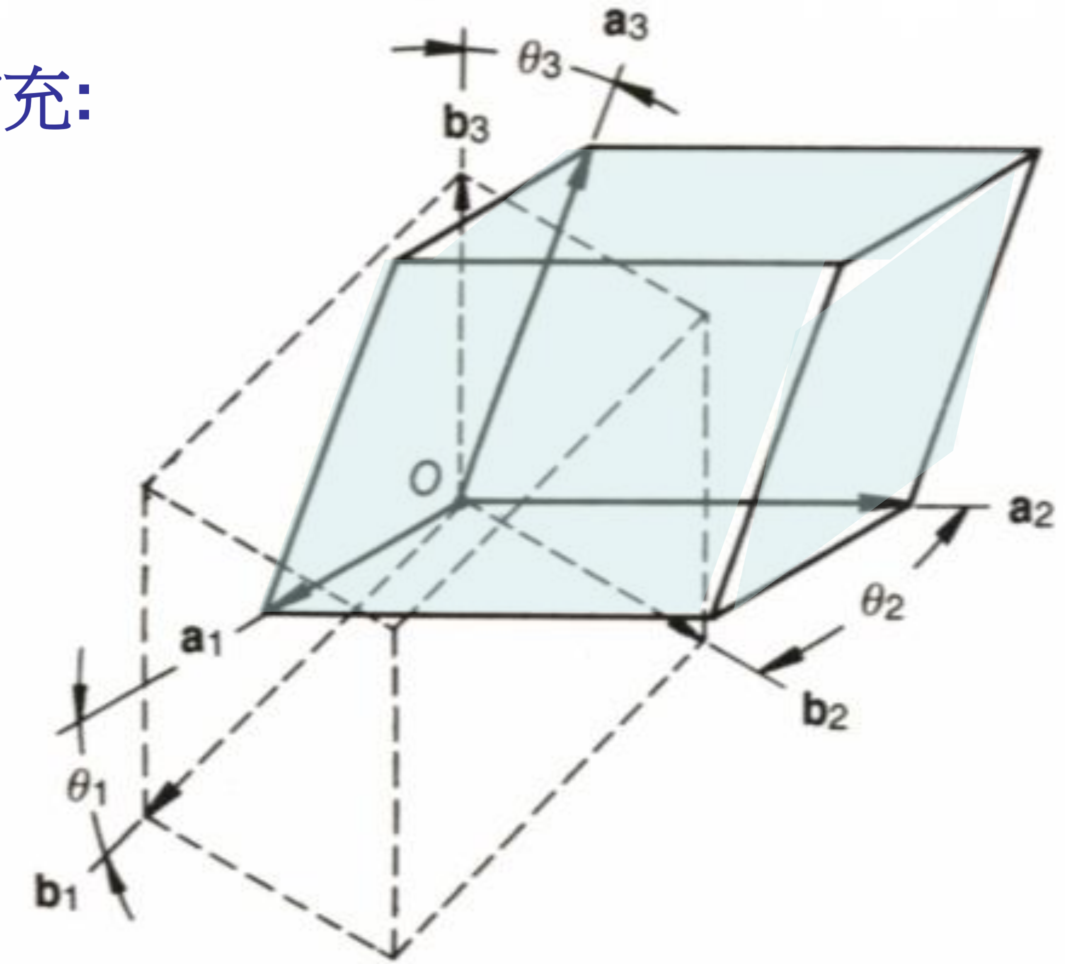


圖 3.1

實際晶格與倒置晶格座標關係示意圖。

- 設V為實際晶格之基本晶胞體積：

$$\boxed{V=(\vec{a}_1\times\vec{a}_2)\bullet\vec{a}_3=(\vec{a}_2\times\vec{a}_3)\bullet\vec{a}_1=(\vec{a}_3\times\vec{a}_1)\bullet\vec{a}_2}$$

補充:
Products of Vector

~Scalar or dot product：

$$\vec{A}\cdot\vec{B} \equiv AB\cos\theta_{AB}$$

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$$

~ Vector or cross product：

$$\vec{A}\times\vec{B} \equiv \hat{a}_n\left|AB\sin\theta_{AB}\right|$$

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C} \quad ($$

補充: ~ Product of three vectors

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}\times(\vec{A}\cdot\vec{C})-\vec{C}\times(\vec{A}\cdot\vec{B}) \qquad ([illegible],\text{"}back-cab\text{"}\ rule)$$

補充: ~metric coefficient (尺規係數)

為表示一長度之微量 ，必須以座標之微量乘上一”metric coefficient” $\boxed{h_i \cdot d\ell_i}$

$ex:$ 二維之極座標 $:(u_1, u_2) = (r, \phi),$

在 $\hat{a}_r$ 方向之微量長度 $d\ell_r = dr \Rightarrow h_r = 1$

在 $\hat{a}_\phi$ 方向之微量長度 $d\ell_\phi = r d\phi \Rightarrow h_\phi = r$

$\Rightarrow \therefore$ 長度之微量:

$$d\vec{\ell} = \hat{a}_{u1}(h_1 du_1) + \hat{a}_{u2}(h_2 du_2) + \hat{a}_{u3}(h_3 du_3)$$

$$d\ell = \left[(h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2\right]^{1/2}$$

體積之微量:

$$dv = h_1 h_2 h_3 du_1 du_2 du_3$$

面積之微量：平行於 $\hat{a}_{u1}$ 之微量面積向量

$$dS_1 = dl_2 dl_3 = h_2 h_3 du_2 du_3$$

同理， ds_2, ds_3 參考 (

補充:Spherical Coordinate（球座標)

$$(u_1, u_2, u_3) = (R, \theta, \phi)$$

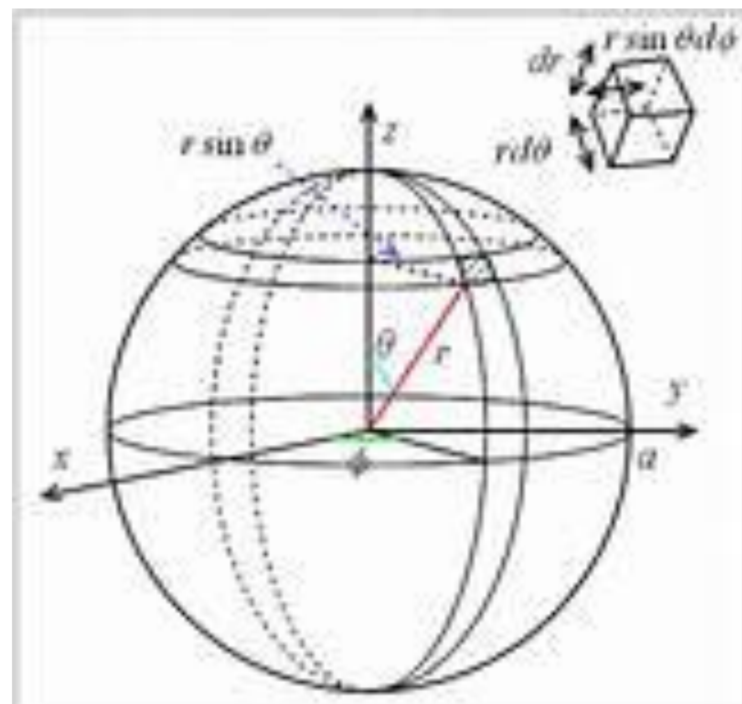
$$\vec{A} = \hat{a}_R A_R + \hat{a}_\theta A_\theta + \hat{a}_\phi A_\phi$$

$$d\vec{l} = \hat{a}_R dR + \hat{a}_\theta R d\theta + \hat{a}_\phi R \sin\theta d\phi$$

$$(h_1 = 1,\ h_2 = R,\ h_3 = R\sin\theta)$$

$$dv = R^2 \sin\theta dR d\theta d\phi$$

ex：求球體積（半徑r）？



補充: Integrals Containing Vector Functions：

1. 體積分： $\int_V \vec{F}dv$ (向量) *or* $\int_V f dv$ (純量)

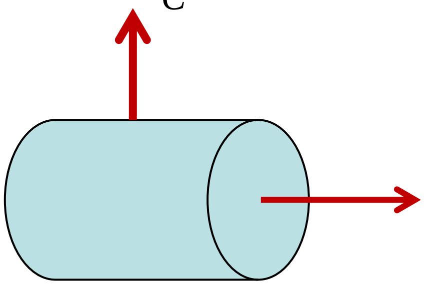
2. 面積分： $\int_S \vec{A}\cdot d\vec{S}$ (純量)

3. 線積分： $\int_C V d\vec{l}$ (向量) *or* $\int_C \vec{F}\cdot d\vec{l}$ (純量)

$$\int_C V d\vec{l} = \int_C V(x,y,z)\left[\hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz\right]$$

$$= \hat{a}_x \int_C v(x,y,z)dx + \hat{a}_y \int_C v(x,y,z)dy + \hat{a}_z \int_C v(x,y,z)dz$$

面積向量 $\hat{a}_n$方向之決定：



補充: Gradient of a Scalar Field

$$\sim grad \quad V = \nabla V \equiv \hat{a}_n \frac{dV}{dn}$$ (dn~微量長度)

V : a scalar function of space

$d\hat{n}$: 垂直於等值（常數）面　　的方向 $\Rightarrow$ 亦即為變化最大之方向

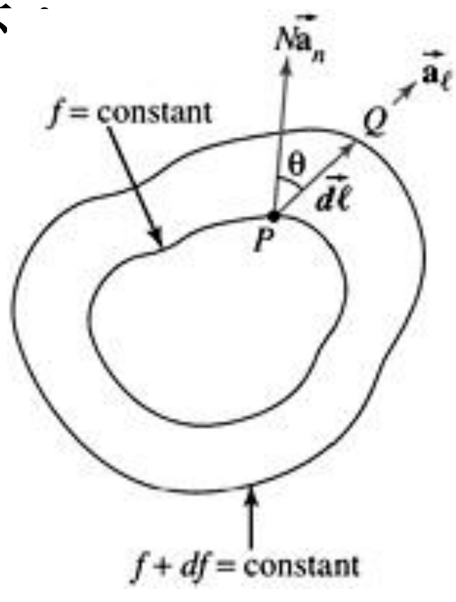
$\hat{a}_n$: $\bar{n}$方向上之單位向量,

~ 沿著路徑 dl 上， V的微量變化 dV之表示

$$\frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\alpha$$

$$= \frac{dV}{dn}\hat{a}_n \cdot \hat{a}_l = (\nabla V)\cdot \hat{a}_l$$

$$\therefore dV = (\nabla V)\cdot dl\,\hat{a}_l = \nabla V \cdot d\vec{l}$$



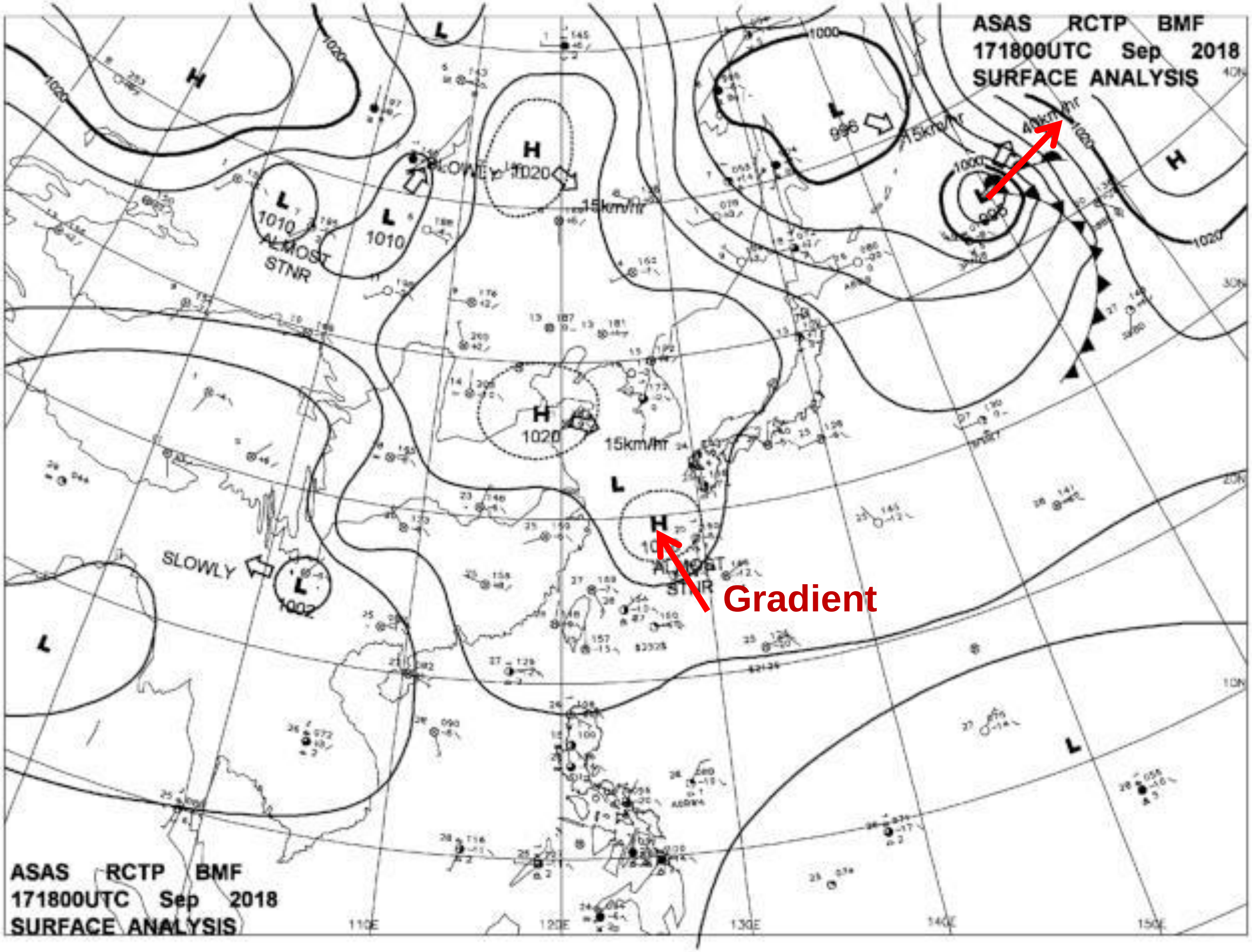
直角座標系：$\nabla V = \hat{a}_x \dfrac{\partial V}{\partial x} + \hat{a}_y \dfrac{\partial V}{\partial y} + \hat{a}_z \dfrac{\partial V}{\partial z}$

∴"∇"視為一微分運算的符號 "向量"(或寫為$\vec{\nabla}$)

$$\nabla \equiv \hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}$$

其他座標系：

$$\nabla \equiv \hat{a}_{u_1} \frac{\partial}{h_1 \partial u_1} + \hat{a}_{u_2} \frac{\partial}{h_2 \partial u_2} + \hat{a}_{u_3} \frac{\partial}{h_3 \partial u_3}$$


ASAS RCTP BMF
171800UTC Sep 2018
SURFACE ANALYSIS
Gradient
40km/hr
15km/hr
15km/hr
15km/hr
SLOWLY
SLOWLY
ALMOST STNR
ALMOST STNR
H
L
1020
1010
1010
1002
996
1000
110E
120E
130E
140E
150E
10N
20N
30N
40N

CH3 習題:

14, 17, and 18

14. A wheel with a ….

17. Rock faults are…

18. Vectors **a** and **b**….