National Taiwan University - Calculus 1 for Class 01-12

Worksheet 2 : The Precise Definition of a Limit

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Deriving the Precise Definition of a Limit.

The intuitive definition of a limit, $\lim_{x\to a} f(x) = L$, says that

we can make the values of $f(x)$ **arbitrarily** *close to L by restricting x to be sufficiently close to a but not equal to a.*

Verifying the above statement is a back and forth, offensive and defensive process. When one claims $\lim_{x\to a} f(x) = L$, he/she is responsible to answer the following challenges:

How close to a does x have to be so that $|f(x) - L| < 0.1$?

How close to a does x have to be so that $|f(x) - L| < 0.01$? And so on...Thus he/she must once and for all answer:

How close to a does x have to be so that $|f(x) - L| < \epsilon$, where ϵ is an arbitrarily small positive number?

In general, given smaller ϵ, one may need to restrict x closer to a. Hence, the above answer (the distance between x and a) depends on ϵ. Now we conclude that

$$\lim_{x\to a} f(x) = L \textbf{ means that}$$

for any number $\epsilon > 0$ there is a $\delta > 0$ (depending on ϵ) such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

The last statement is the **precise definition** of $\lim_{x\to a} f(x) = L$. In this worksheet we will have hands-on experiences of using this definition.

Exercise 1.

$$\text{Consider } f(x) = \begin{cases} \dfrac{2x^2 - x - 1}{x - 1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$$

- Simplify $f(x)$, $f(x) =$ <u>$2x+1$</u> for $x \neq 1$. Guess the limit, $\lim_{x\to 1} f(x) =$ <u>3</u>.
- When you wrote down $\lim_{x\to 1} f(x) = L$, you have made a very *strong* statement! You claimed that

  We can make $f(x)$ **arbitrarily** *close to L by restricting x to be sufficiently close to 1 but not equal to 1.*

  One may ask you to show him :

  How close to 1 does x have to be so that $|f(x) - L| < 0.1$, 0.01, 0.001...?

  Starting from the goal inequality $|f(x) - L|$, derive an equivalent inequality regarding $|x - 1|$.

  If $0<|x-1|<k$
  $|2x+1-3|<0.1$
  $\Rightarrow -0.1<2x-2<0.1$
  $\Rightarrow 0.95<x<1.05$
  $\Rightarrow 0<|x-1|=0.05$
  $\Rightarrow k=0.05$

  If $0<|x-1|<m$
  $|2x+1-3|<0.01$
  $\Rightarrow -0.01<2x-2<0.01$
  $\Rightarrow 0.995<x<1.005$
  $\Rightarrow -0.005<x-1<0.005$
  $\Rightarrow 0<|x-1|<0.005$
  $\Rightarrow m=0.005$

  If $0<|x-1|<\delta$
  $|2x+1-3|<\varepsilon$
  $\Rightarrow -\varepsilon<2x-2<\varepsilon$
  $\Rightarrow 1-\varepsilon/2<x<1+\varepsilon/2$
  $\Rightarrow -\varepsilon/2<x-1<\varepsilon/2$
  $\Rightarrow 0<|x-1|<\varepsilon/2$
  $\Rightarrow \delta=\varepsilon/2$

  Fill in the blank.

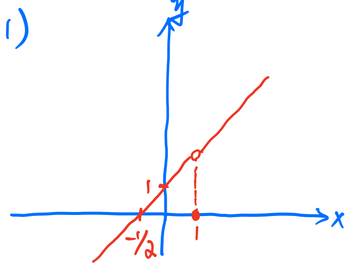
  If $0 < |x - 1| <$ <u>0.05</u>, $|f(x) - L| < 0.1$.

  If $0 < |x - 1| <$ <u>0.005</u>, $|f(x) - L| < 0.01$.

  If $0 < |x - 1| <$ <u>$\varepsilon/2$</u>, $|f(x) - L| < \epsilon$, where ϵ is any positive number.

- What happens at $x = 1$? Obviously, $|f(1) - L| = |0 - L| > 0.1$. Does this violate the statement $\lim_{x\to 1} f(x) = L$?

1)



2) No, this consequence doesn't violate $\lim_{x\to 1} f(x) = L$

Via definition of limit, $\lim_{x\to 1} f(x)$ only need to consider function's behavior near $x=1$.

- For any $\epsilon > 0$, we find $\delta =$ $\underline{\ \epsilon/2\ }$ such that if $0 < |x - 1| < \delta$, then $|f(x) - L| < \epsilon$. This proves that indeed $\lim_{x\to 1} f(x) = L$.

Exercise 2.

(a) Show that $\lim_{x\to 0} \sqrt[3]{x} = 0$.

To prove the limit is 0, for any $\epsilon > 0$, we need to find a $\delta > 0$ such that if $0 < |x - 0| < \delta$ then $|\sqrt[3]{x} - 0| < \epsilon$.

- Starting from the desired inequality $|\sqrt[3]{x} - 0| < \epsilon$, derive an inequality for $|x - 0|$.

Given $|\sqrt[3]{x} - 0| < \varepsilon$

$\Rightarrow -\varepsilon < \sqrt[3]{x} < \varepsilon$

$\Rightarrow -\varepsilon^3 < x < \varepsilon^3$

$\Rightarrow 0 < |x - 0| < \varepsilon^3$

- For a given ϵ, find such δ and show that if $0 < |x - 0| < \delta$ then $|\sqrt[3]{x} - 0| < \epsilon$.

<pf.> Given ε, Choose $\delta = \varepsilon^3$

Suppose $0 < |x - 0| < \delta$

Check $|\sqrt[3]{x} - 0| < \varepsilon$

$\Rightarrow -\varepsilon^3 < x < \varepsilon^3$

$\Rightarrow 0 < |x - 0| < \varepsilon^3 = \delta$

(b) Imitating the precise definition of a limit, write down precise definitions of one-sided limits.

- $\lim_{x\to a^+} f(x) = L \Leftrightarrow$ $\forall\, \varepsilon > 0,\ \delta > 0$

if $0 < x - a < \delta$,

then $|f(x) - L| < \varepsilon$

- $\lim_{x\to a^-} f(x) = L \Leftrightarrow$ $\forall\, \varepsilon > 0,\ \delta > 0$

if $0 < -(x - a) < \delta$

then $|f(x) - L| < \varepsilon$

Precise Definition of an Infinite Limit.

The intuitive definition of an infinite limit, $\lim_{x\to a} f(x) = \infty$, says that

we can make the values of $f(x)$ **arbitrarily** *large by restricting x to be sufficiently close to a but not equal to a.*

Again, when one claims $\lim_{x\to a} f(x) = \infty$, he/she is responsible to answer the following challenges:

How close to a does x have to be so that $f(x) > 100$?

How close to a does x have to be so that $f(x) > 1000$? And so on...Thus he/she must once and for all answer:

How close to a does x have to be so that $f(x) > N$, where N is an arbitrarily large positive number?

Hence we conclude that

$$\lim_{x\to a} f(x) = \infty \ \textbf{means that}$$

for any number $N > 0$ there is a $\delta > 0$ (depending on N) such that if $0 < |x - a| < \delta$ then $f(x) > N$.

The last statement is the **precise definition** of $\lim_{x\to a} f(x) = \infty$.

Exercise 3.

(a) Imitating the above definition, write down precise definitions of other limits regarding infinity.

- $\lim_{x\to a^+} f(x) = -\infty \Leftrightarrow$ $\forall N<0, \ \delta>0$
  if $0 < x-a < \delta$, $f(x)<N$
- $\lim_{x\to \infty} f(x) = L \Leftrightarrow$ $\forall \varepsilon>0, \ N>0$
  if $x>N$, $|f(x)-L|<\varepsilon$
- $\lim_{x\to -\infty} f(x) = \infty \Leftrightarrow$ $\forall N>0, \ n<0$
  if $x<n$, $f(x)>N$

(b) Show that $\lim_{x\to 0^+} \ln x = -\infty$.

To prove that the limit is negative infinity, for any negative $N < 0$, we need to find a $\delta > 0$ such that if $0 < x - 0 < \delta$ then $\ln x < N$.

- Starting from the desired inequality $\ln x < N$, derive an inequality for $x - 0$.

  $\ln x<N$
  $\Rightarrow x<e^N$
  $\Rightarrow 0<x-0<e^N$

- For a given $N < 0$, find such δ and show that if $0 < x - 0 < \delta$ then $\ln x < N$.

  Given $N<0$, choose $\delta = e^N$
  suppose $0<x-0<\delta$
  check $\ln x<N$
  $\Rightarrow 0<x<e^N$ (x must greater than 0)
  $\Rightarrow 0<x-0<e^N$

Exercise 4 (Optional).

We can use the precise definition of a limit to prove limit laws and corollaries about limits. Try to prove the following statements.

(a) Prove that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then $\lim_{x\to a}(f(x)+g(x)) = L+M$.
(Hint: For any $\epsilon > 0$ we need to find a $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|(f(x)+g(x)) - (L+M)| < \epsilon$. However, we have $|(f(x)+g(x)) - (L+M)| \leq |f(x)-L| + |g(x)-M|$. Moreover, since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, we can let $|f(x)-L| < \epsilon/2$ and $|g(x)-M| < \epsilon/2$ if x is sufficiently close to a.)

$\forall \varepsilon>0$, there is $\delta>0$

$\begin{cases} \text{if } 0<|x-a|<\delta_1, & |f(x)-L|<\varepsilon/2 \\ \text{if } 0<|x-a|<\delta_2, & |g(x)-M|<\varepsilon/2 \end{cases}$

make $\delta=\min(\delta_1,\delta_2)$, then: $\rightarrow$ if $0<|x-a|<\delta$,

$|f(x)-L| + |g(x)-M| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\because |(f(x)+g(x)) - (L+M)| \leq |f(x)-L| + |g(x)-M|$

$\therefore |(f(x)+g(x)) - (L+M)| < \varepsilon$

Thus, $\lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

$= L+M$

(b) Prove that if $\lim_{x\to a} f(x) = 0$ and $|g(x)| < M$ for all x where $M > 0$ is a constant then $\lim_{x\to a} f(x)g(x) = 0$.

(c) Prove that if $\lim_{x\to a} f(x) = L > 0$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a} f(x)g(x) = \infty$.

(d) Prove that if $f(x) < g(x)$ for all $x \neq a$ and the limits $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.
(Hint: If $\lim_{x\to a} f(x) > \lim_{x\to a} g(x)$ can you derive a contradiction?)

$f(x) < g(x)$, where $x \neq a$

and $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist.

via definition of limit, $x \to a$, but $x \neq a$

and from the topic, $f(x) < g(x)$, when $x \neq a$

thus, $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$