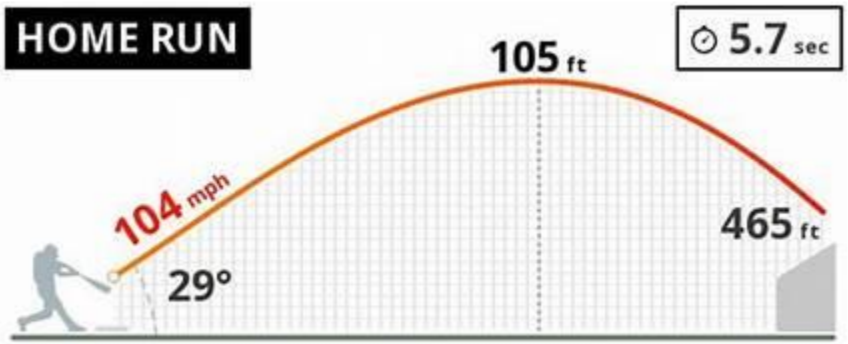
Chapter 4

Motion in two and three dimensions



4.2 Position and Displacement (位置與位移---以向量表示)

Position ~位置以向量表示:

- The position of a particle can be described by a position vector, with respect to a reference origin.

$$\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}$$

Displacement ~位移以向量表示:

- The displacement of a particle is the change of the position vector during a certain time.

$$\Delta\vec{r} = (x_2\hat{\mathrm{i}} + y_2\hat{\mathrm{j}} + z_2\hat{\mathrm{k}}) - (x_1\hat{\mathrm{i}} + y_1\hat{\mathrm{j}} + z_1\hat{\mathrm{k}})$$

$$\Delta\vec{r} = (x_2 - x_1)\hat{\mathrm{i}} + (y_2 - y_1)\hat{\mathrm{j}} + (z_2 - z_1)\hat{\mathrm{k}},$$

Example, two-dimensional motion:

A rabbit runs across a parking lot on which a set of coordinate axes has, strangely enough, been drawn. The coordinates (meters) of the rabbit's position as functions of time t (seconds) are given by

$$x = -0.31t^2 + 7.2t + 28 \tag{4-5}$$

and

$$y = 0.22t^2 - 9.1t + 30. \tag{4-6}$$

(a) At $t = 15$ s, what is the rabbit's position vector $\vec{r}$ in unit-vector notation and in magnitude-angle notation?

KEY IDEA

The x and y coordinates of the rabbit's position, as given by Eqs. 4-5 and 4-6, are the scalar components of the rabbit's position vector $\vec{r}$.

Calculations: We can write

$$\vec{r}(t) = x(t)\hat{\mathrm{i}} + y(t)\hat{\mathrm{j}}. \tag{4-7}$$

(We write $\vec{r}(t)$ rather than $\vec{r}$ because the components are functions of t, and thus $\vec{r}$ is also.)

At $t = 15$ s, the scalar components are

$$x = (-0.31)(15)^2 + (7.2)(15) + 28 = 66 \text{ m}$$

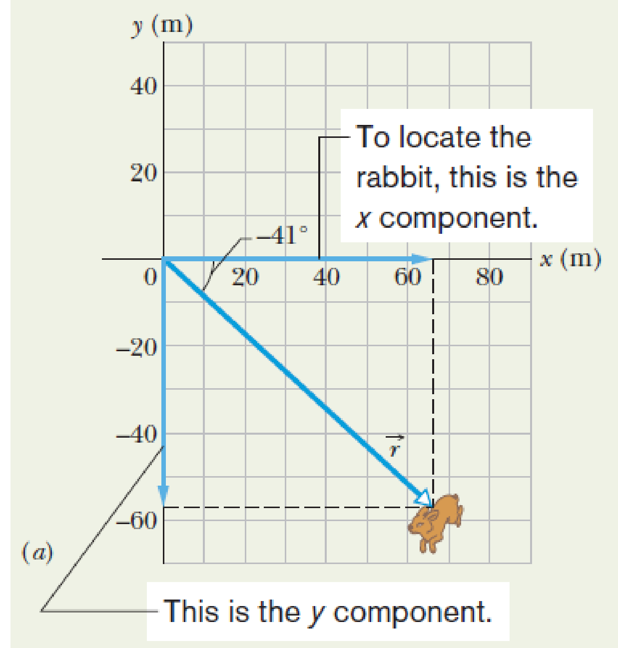
and $$y = (0.22)(15)^2 - (9.1)(15) + 30 = -57 \text{ m},$$

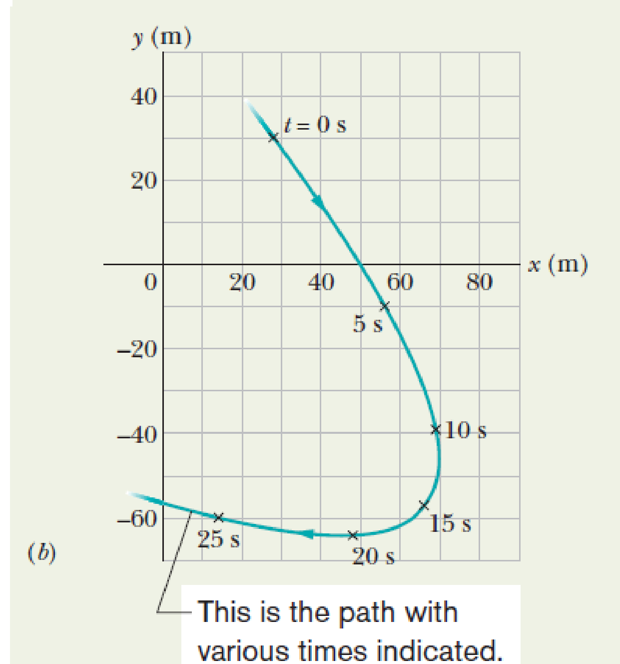
so $$\vec{r} = (66 \text{ m})\hat{\mathrm{i}} - (57 \text{ m})\hat{\mathrm{j}}, \qquad \text{(Answer)}$$

which is drawn in Fig. 4-2*a*. To get the magnitude and angle of $\vec{r}$, we use Eq. 3-6:

$$r = \sqrt{x^2 + y^2} = \sqrt{(66 \text{ m})^2 + (-57 \text{ m})^2}$$
$$= 87 \text{ m}, \qquad \text{(Answer)}$$

and $$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-57 \text{ m}}{66 \text{ m}}\right) = -41°. \qquad \text{(Answer)}$$



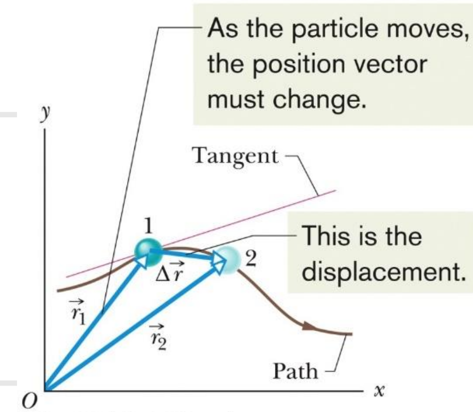


4.3 Average Velocity and Instantaneous Velocity

If a particle moves through a displacement of Δr in Δt time, then the average velocity is:

$$\text{average velocity} = \frac{\text{displacement}}{\text{time interval}},$$

$$\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t}.$$



Copyright © 2014 John Wiley & Sons, Inc. All rights reserved.

In the limit that the Δt time shrinks to a single point in time, the average velocity is approaches instantaneous velocity (瞬時速度) . This velocity is the derivative of displacement with respect to time.

$$\vec{v} = \frac{d\vec{r}}{dt}.$$

$$\vec{v} = \frac{d}{dt}(x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}) = \frac{dx}{dt}\hat{\mathrm{i}} + \frac{dy}{dt}\hat{\mathrm{j}} + \frac{dz}{dt}\hat{\mathrm{k}} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}},$$

The direction of the instantaneous velocity $\vec{v}$ of a particle is always tangent to the particle's path at the particle's position.

Example, two-dimensional velocity

For the rabbit in the preceding Sample Problem, find the velocity $\vec{v}$ at time $t = 15$ s.

KEY IDEA

We can find $\vec{v}$ by taking derivatives of the components of the rabbit's position vector.

Calculations: Applying the v_x part of Eq. 4-12 to Eq. 4-5, we find the x component of $\vec{v}$ to be

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(-0.31t^2 + 7.2t + 28)$$
$$= -0.62t + 7.2. \qquad (4\text{-}13)$$

At $t = 15$ s, this gives $v_x = -2.1$ m/s. Similarly, applying the v_y part of Eq. 4-12 to Eq. 4-6, we find

$$v_y = \frac{dy}{dt} = \frac{d}{dt}(0.22t^2 - 9.1t + 30)$$
$$= 0.44t - 9.1. \qquad (4\text{-}14)$$

At $t = 15$ s, this gives $v_y = -2.5$ m/s. Equation 4-11 then yields

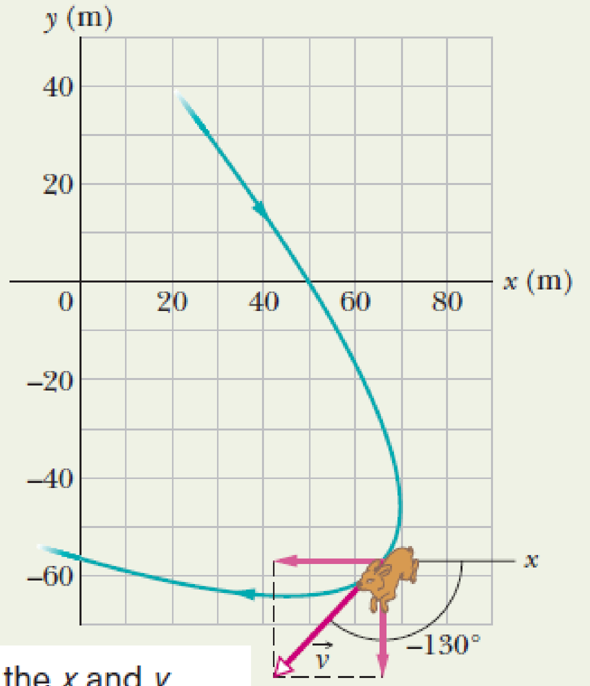
$$\vec{v} = (-2.1 \text{ m/s})\hat{i} + (-2.5 \text{ m/s})\hat{j}, \qquad \text{(Answer)}$$

which is shown in Fig. 4-5, tangent to the rabbit's path and in the direction the rabbit is running at $t = 15$ s.

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(-2.1 \text{ m/s})^2 + (-2.5 \text{ m/s})^2}$$
$$= 3.3 \text{ m/s} \qquad \text{(Answer)}$$

and
$$\theta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\left(\frac{-2.5 \text{ m/s}}{-2.1 \text{ m/s}}\right)$$
$$= \tan^{-1} 1.19 = -130°. \qquad \text{(Answer)}$$

Check: Is the angle $-130°$ or $-130° + 180° = 50°$?



These are the x and y components of the vector at this instant.

4.4 Average and Instantaneous Accelerations (平均加速度&瞬時加速度)

Following the same definition as in average velocity,

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{time interval}},$$

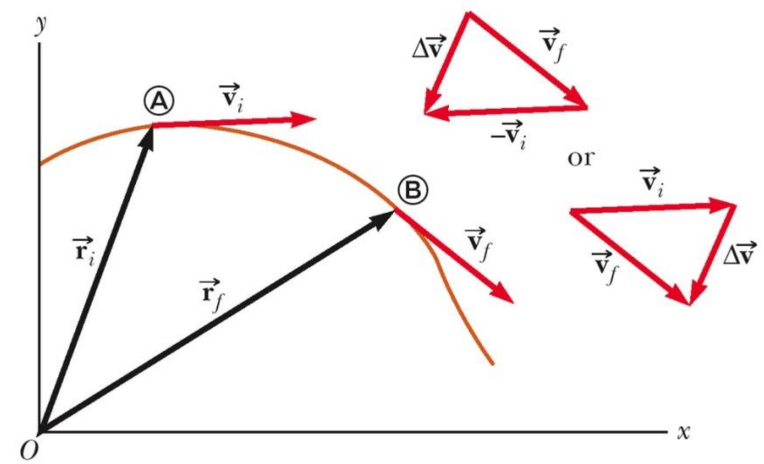
$$\vec{a}_{\text{avg}} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}.$$

If we shrink Δt to zero, then the average acceleration value approaches to the instant acceleration value, which is the derivative of velocity with respect to time:

$$\vec{a} = \frac{d\vec{v}}{dt}.$$

$$\vec{a} = \frac{d}{dt}(v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}}) = \frac{dv_x}{dt}\hat{\mathrm{i}} + \frac{dv_y}{dt}\hat{\mathrm{j}} + \frac{dv_z}{dt}\hat{\mathrm{k}}$$

$$= a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}$$



© 2006 Brooks/Cole - Thomson

Kinematic Equations, Component Equations

- $\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t$ becomes
  - $v_{xf} = v_{xi} + a_x t$ and
  - $v_{yf} = v_{yi} + a_y t$
- $\vec{\mathbf{r}}_f = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t + \frac{1}{2}\vec{\mathbf{a}}t^2$ becomes
  - $x_f = x_i + v_{xi}\, t + 1/2\ a_x t^2$ and
  - $y_f = y_i + v_{yi}\, t + 1/2\ a_y t^2$

4.4 Two-dimensional rabbit run …acceleration problem

For the rabbit in the preceding two Sample Problems, find the acceleration $\vec{a}$ at time $t = 15$ s.

KEY IDEA

We can find $\vec{a}$ by taking derivatives of the rabbit's velocity components.

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(-0.62t + 7.2) = -0.62 \text{ m/s}^2.$$

Similarly, applying the a_y part of Eq. 4-18 to Eq. 4-14 yields the y component as

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}(0.44t - 9.1) = 0.44 \text{ m/s}^2.$$

$$\vec{a} = (-0.62 \text{ m/s}^2)\hat{\text{i}} + (0.44 \text{ m/s}^2)\hat{\text{j}}, \quad \text{(Answer)}$$

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-0.62 \text{ m/s}^2)^2 + (0.44 \text{ m/s}^2)^2}$$
$$= 0.76 \text{ m/s}^2. \quad \text{(Answer)}$$

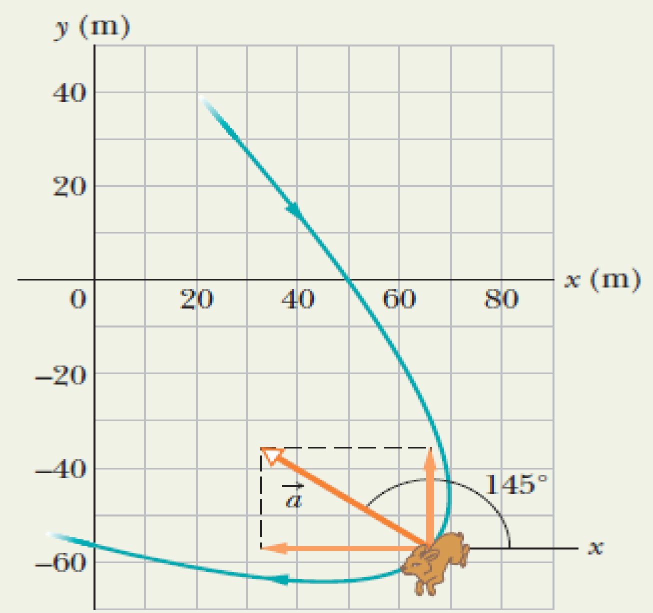
For the angle we have

$$\theta = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\left(\frac{0.44 \text{ m/s}^2}{-0.62 \text{ m/s}^2}\right) = -35°.$$

However, this angle, which is the one displayed on a calculator, indicates that $\vec{a}$ is directed to the right and downward in Fig. 4-7. Yet, we know from the components that $\vec{a}$ must be directed to the left and upward. To find the other angle that has the same tangent as $-35°$ but is not displayed on a calculator, we add 180°:

$$-35° + 180° = 145°. \quad \text{(Answer)}$$

This *is* consistent with the components of $\vec{a}$ because it gives a vector that is to the left and upward. Note that $\vec{a}$ has the same magnitude and direction throughout the rabbit's run because the acceleration is constant.



These are the x and y components of the vector at this instant.

4.5 Projectile motion

A particle moves in a vertical plane, with the only acceleration equal to the free fall acceleration, *g*.

Examples in sports:

Tennis

Baseball

Football

Lacrosse

Racquetball

Soccer……….

© 2006 Brooks/Cole - Thomson

In projectile motion, the horizontal motion and the vertical motion are independent of each other; that is, neither motion affects the other.

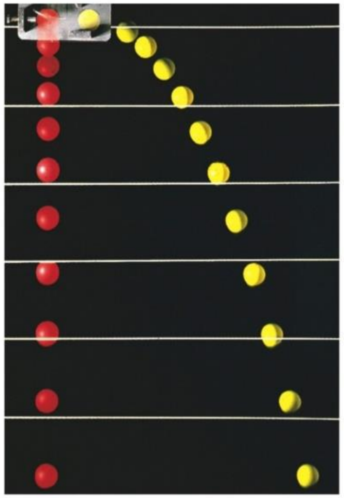
~Therefore we can decompose two-dimensional motion into 2 one-dimensional problems:

$$\boxed{\vec{v}_0 = v_{0x}\hat{\mathrm{i}} + v_{0y}\hat{\mathrm{j}}.}$$

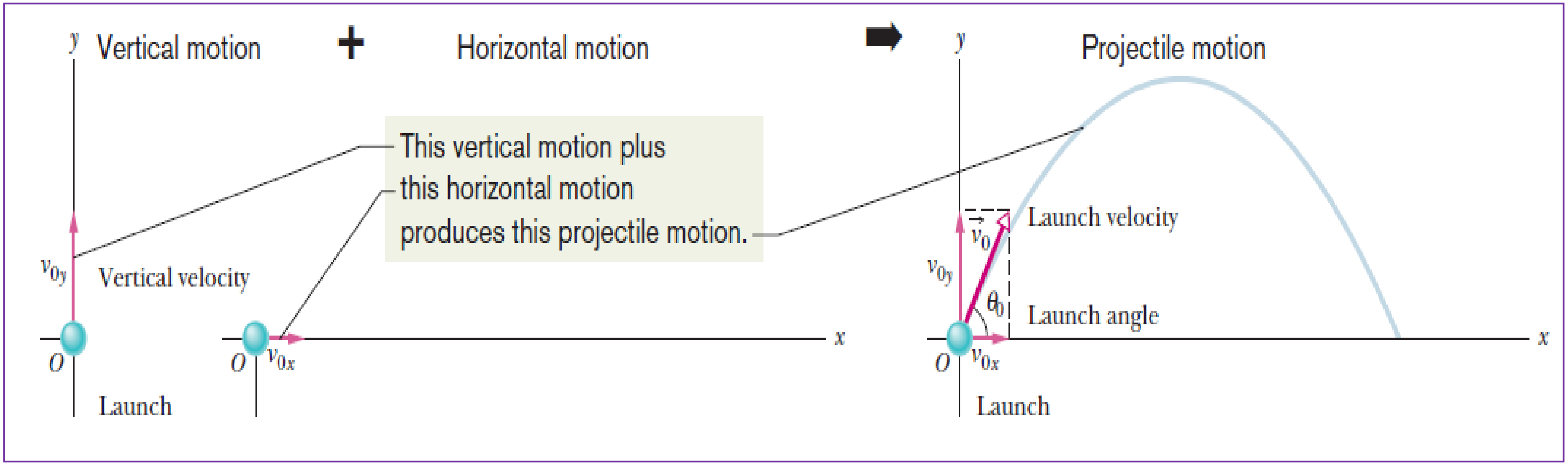
The initial velocity of the projectile is:

Here,

$$\boxed{v_{0x} = v_0 \cos\theta_0 \quad \text{and} \quad v_{0y} = v_0 \sin\theta_0.}$$



Richard Megna/Fundamental Photographs



4.6: Projectile motion analyzed, assuming no external forces other than the weight:

Horizontal Motion: no acceleration

$$x - x_0 = v_{0x}t.$$

$$x - x_0 = (v_0 \cos \theta_0)t.$$

Vertical Motion; acceleration = g

$$\begin{aligned} y - y_0 &= v_{0y}t - \tfrac{1}{2}gt^2 \\ &= (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2, \end{aligned}$$

$$v_y = v_0 \sin \theta_0 - gt$$

$$v_y^2 = (v_0 \sin \theta_0)^2 - 2g(y - y_0).$$

$$\boxed{y = (\tan \theta_0)x - \frac{gx^2}{2(v_0 \cos \theta_0)^2}}$$

Eliminate time, t:

This is in the form of $y = ax - bx^2$ which is the standard form of a parabola~為標準之拋物線方程式!

4.6: Horizontal Range, assuming no external forces:

The horizontal range of a projectile is the horizontal distance when it returns to its launching height (symmetric trajectory)

The distance equations in the x- and y- directions respectively:

$$R = (v_0 \cos\theta_0)t$$

$$0 = (v_0 \sin\theta_0)t - \tfrac{1}{2}gt^2.$$

Eliminating t:

$$R = \frac{2v_0^2}{g}\sin\theta_0\cos\theta_0. \quad = \frac{v_0^2}{g}\sin 2\theta_0.$$

(This is valid only for symmetric trajectory)

The horizontal range R is maximum for a launch angle of 45°.

Height of a Projectile, equation for *h*:

- The maximum height of the projectile can be found in terms of the initial velocity vector:

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g}$$

- This equation is valid only for symmetric motion

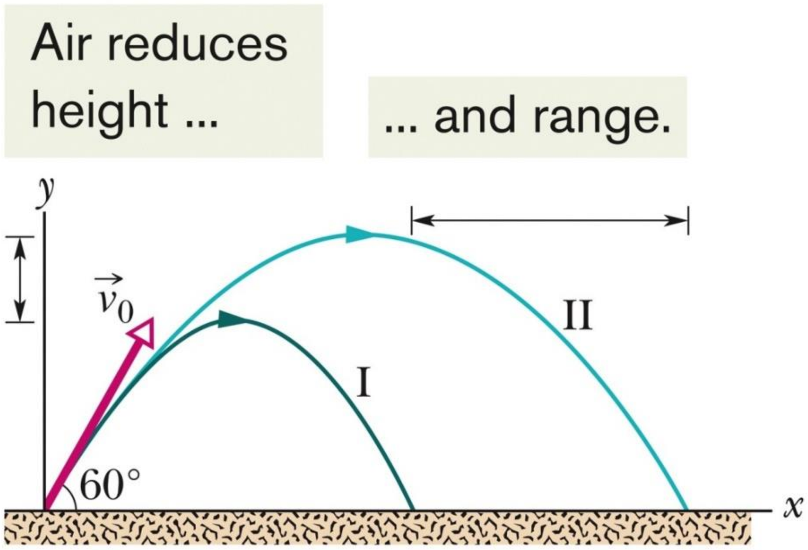
- In these calculations we assume air resistance is negligible
- In many situations this is a poor assumption:

Table 4-1 **Two Fly Balls**[a]

	Path I (Air)	Path II (Vacuum)
Range	98.5 m	177 m
Maximum height	53.0 m	76.8 m
Time of flight	6.6 s	7.9 s

[a]See Fig. 4-13. The launch angle is 60° and the launch speed is 44.7 m/s.

Copyright © 2014 John Wiley & Sons, Inc. All rights reserved.



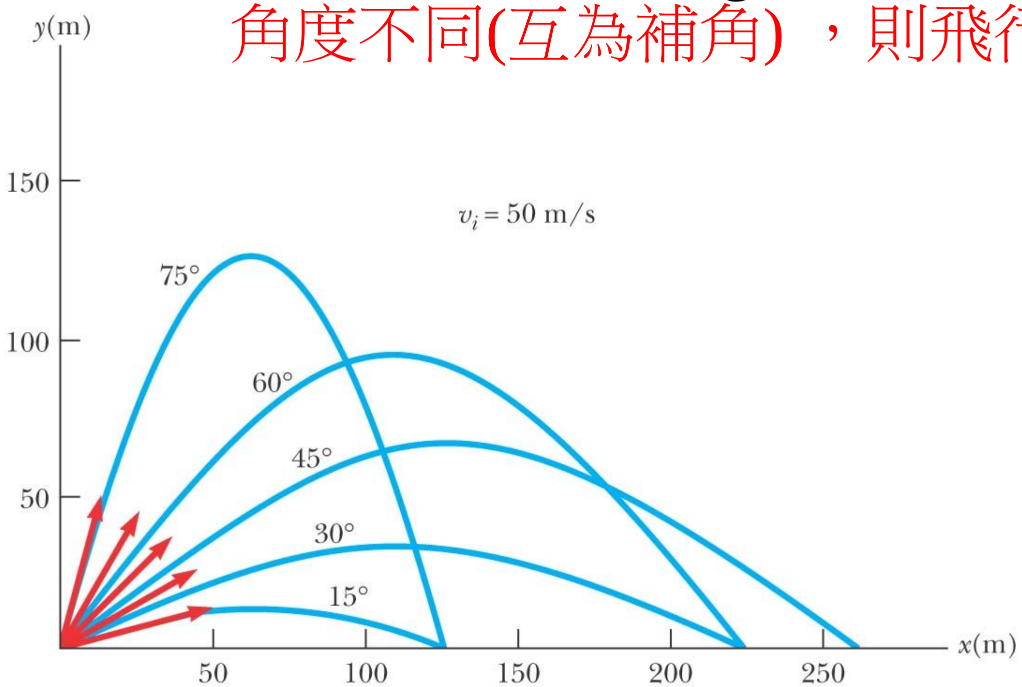
Copyright © 2014 John Wiley & Sons, Inc. All rights reserved.

Table 4-1

Figure 4-13

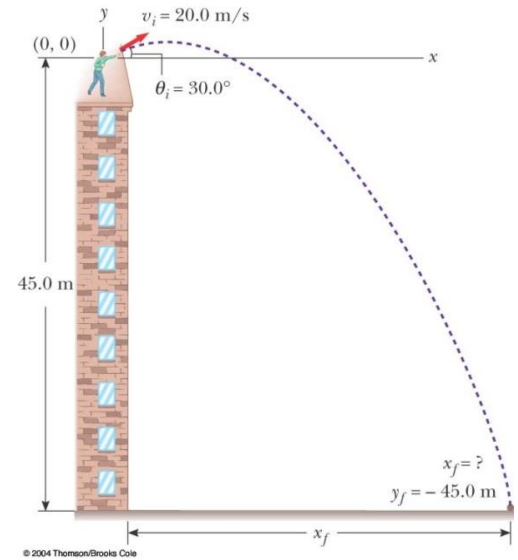
More About the Range of a Projectile

- The maximum range occurs at θ_i = 45º (最遠射程之角度)
- The effect of Air Resistance: launch angle for maximum range <45º ，射程縮短~由computer計算彈道
- Complementary angles will produce the same range
  - The maximum height will be different for the two angles: 角度不同(互為補角)，則*h*不同。
  - The times of the flight will be different for the two angles: 角度不同(互為補角) ，則飛行時間也不同。

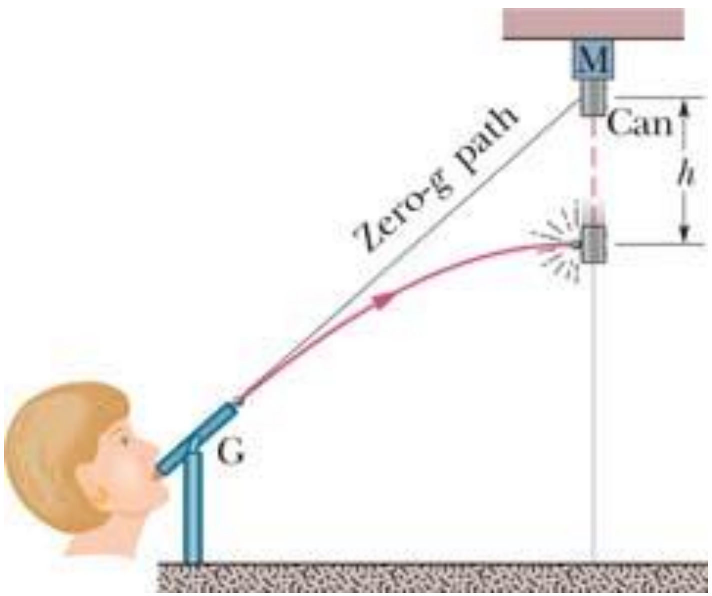


© 2004 Thomson/Brooks Cole

Non-Symmetric Projectile Motion非對稱之拋射運動:



一有趣之實驗:



~小球永遠可擊中下落之鐵罐!?

4.7: Uniform Circular Motion

The speed of the particle is constant

(速率固定)

+

A particle travels around a circle/circular arc

→

Uniform circular motion

As the direction of the velocity of the particle changes, there is an acceleration!!!

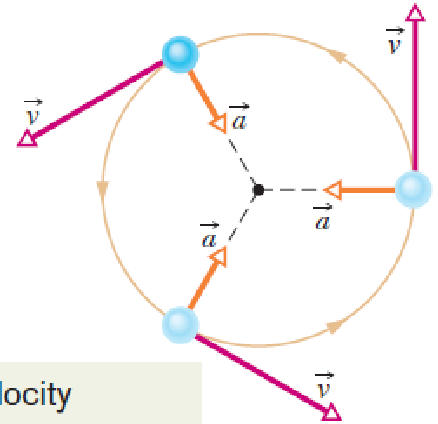
CENTRIPETAL (center-seeking) ACCELERATION

$$a = \frac{v^2}{r} \qquad \text{(centripetal acceleration),}$$

Here v is the speed of the particle and r is the radius of the circle.

- Velocity and acceleration have:
  - Constant magnitude
  - (加速度不固定，但恆垂直於速度 ***v***)
  - Changing direction

The acceleration vector always points toward the center.



The velocity vector is always tangent to the path.

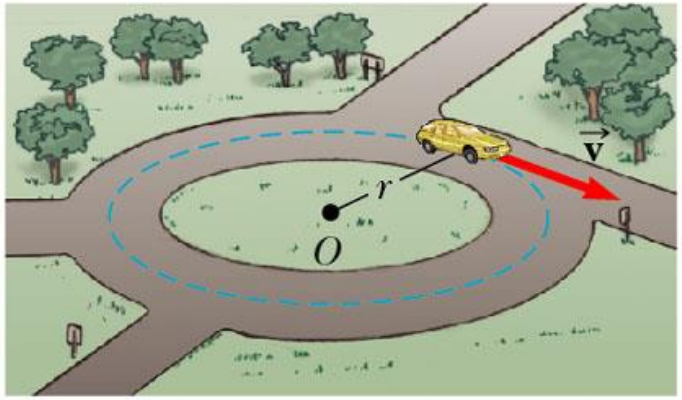
4.7: Centripetal acceleration, proof of $a = v^2/r$

由圖知:

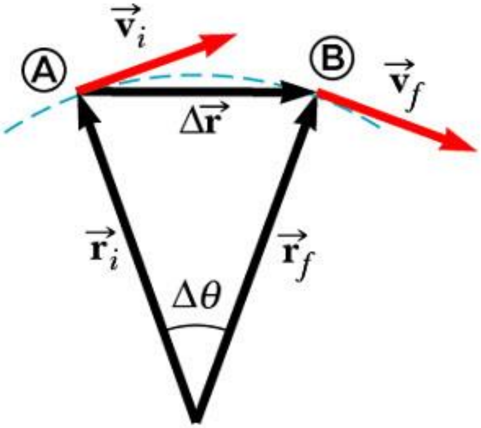
$$\frac{\Delta v}{v} = \frac{\Delta r}{r} \approx \frac{v\Delta t}{r} \qquad \therefore \bar{a} = \frac{\Delta v}{\Delta t} = \frac{v^2}{r}, \text{當} \Delta t \to 0$$

$$\Rightarrow a = \frac{v^2}{r}$$

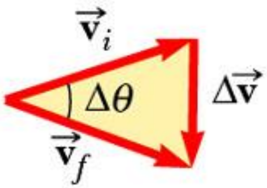
方向: 指向圓心



(a)



(b)



(c)

© 2006 Brooks/Cole - Thomson

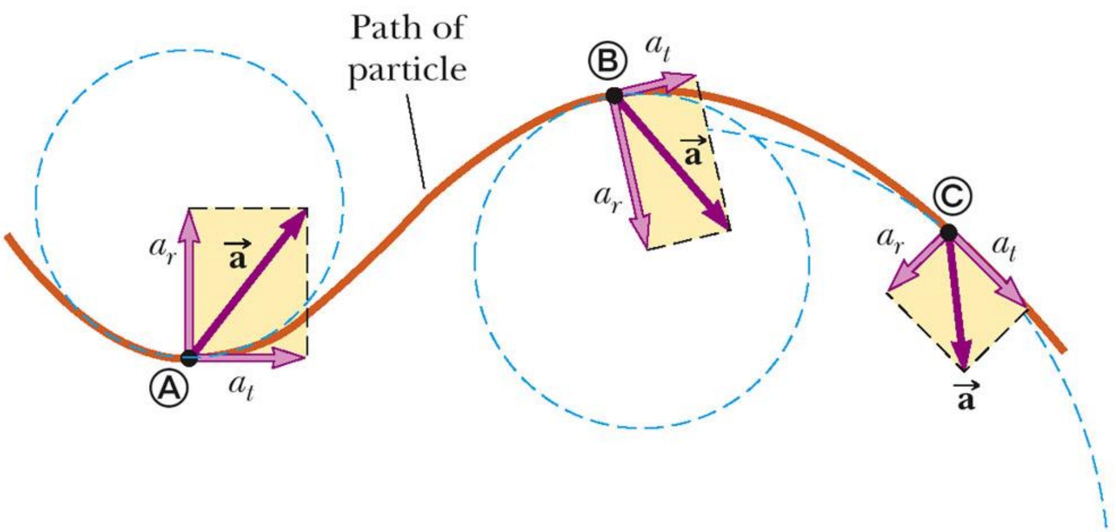
- Acceleration is called **centripetal acceleration** (向心加速度)
    - Means “center seeking”
    - Directed radially inward

$$a = \frac{v^2}{r}$$ **Eq. (4-34)**

- The **period of revolution** is:
    - The time it takes for the particle go around the closed path exactly once

$$T = \frac{2\pi r}{v}$$ **Eq. (4-35)**

Total Acceleration(總加速度)



© 2006 Brooks/Cole - Thomson

- The tangential acceleration causes the change in the speed of the particle (切線加速度:改變速度大小)
- The radial acceleration comes from a change in the direction of the velocity vector (向心加速度:改變速度方向)

Sample problem, top gun pilots

"Top gun" pilots have long worried about taking a turn too tightly. As a pilot's body undergoes centripetal acceleration, with the head toward the center of curvature, the blood pressure in the brain decreases, leading to loss of brain function.

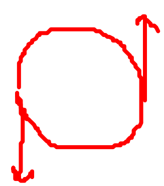
There are several warning signs. When the centripetal acceleration is $2g$ or $3g$, the pilot feels heavy. At about $4g$, the pilot's vision switches to black and white and narrows to "tunnel vision." If that acceleration is sustained or increased, vision ceases and, soon after, the pilot is unconscious—a condition known as g-LOC for "g-induced loss of consciousness."

What is the magnitude of the acceleration, in g units, of a pilot whose aircraft enters a horizontal circular turn with a velocity of $\vec{v}_i = (400\hat{i} + 500\hat{j})$ m/s and 24.0 s later leaves the turn with a velocity of $\vec{v}_f = (-400\hat{i} - 500\hat{j})$ m/s?

KEY IDEAS

We assume the turn is made with uniform circular motion.
Then the pilot's acceleration is centripetal and has magnitude *a given by* $a = v^2/R$.

Also, the time required to complete a full circle is the period given by $T = 2\pi R/v$



Calculations:

Because we do not know radius *R, let's solve for R* from the period equation for *R and substitute into the acceleration eqn.*

$$\boxed{a = \frac{2\pi v}{T}.}$$

Speed *v here is the (constant) magnitude of the velocity* during the turning.

$$v = \sqrt{(400\text{ m/s})^2 + (500\text{ m/s})^2} = 640.31\text{ m/s}.$$

To find the period *T of the motion, first note that the final* velocity is the reverse of the initial velocity. This means the aircraft leaves on the opposite side of the circle from the initial point and must have completed half a circle in the given 24.0 s. Thus a full circle would have taken *T 48.0 s.*

Substituting these values into our equation for *a*, *we find*

$$a = \frac{2\pi(640.31\text{ m/s})}{48.0\text{ s}} = 83.81\text{ m/s}^2 \approx 8.6g. \qquad \text{(Answer)}$$

4.8: Relative motion in one-dimension

The velocity of a particle depends on the reference frame of whoever is observing the velocity.

- Suppose Alex (A) is at the origin of frame A (as in Fig. 4-18), watching car P (the "particle") speed past.
- Suppose Barbara (B) is at the origin of frame B, and is driving along the highway at constant speed, also watching car P. Suppose that they both measure the position of the car at a given moment. Then:

$$x_{PA} = x_{PB} + x_{BA}.$$

where x_{PA} is the position of P as measured by A. Consequently,

$$v_{PA} = v_{PB} + v_{BA}.$$

Also,

$$\frac{d}{dt}(v_{PA}) = \frac{d}{dt}(v_{PB}) + \cancel{\frac{d}{dt}(v_{BA})}.$$

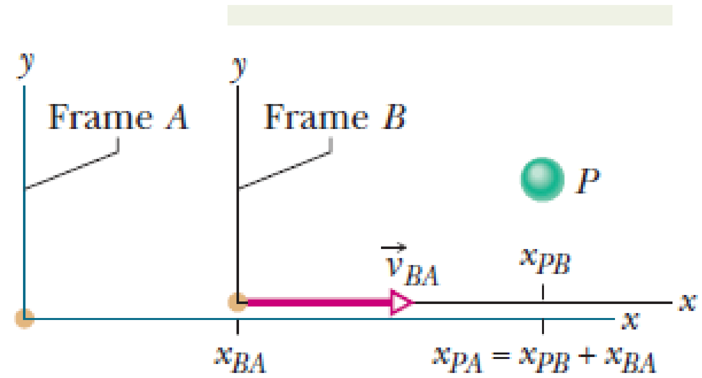


Fig. 4-18 Alex (frame A) and Barbara (frame B) watch car P, as both B and P move at different velocities along the common x axis of the two frames. At the instant shown, x_{BA} is the coordinate of B in the A frame. Also, P is at coordinate x_{PB} in the B frame and coordinate $x_{PA} = x_{PB} + x_{BA}$ in the A frame.

Since v_{BA} *is constant, the last term is zero and we have* accelerations (for non-accelerating reference frames, $a_{BA} = 0$)

$$\boldsymbol{a}_{PA} = \boldsymbol{a}_{PB}.$$

4.9: Relative motion in two-dimensions

A and B, the two observers, are watching P, the moving particle, from their origins of reference. B moves at a constant velocity with respect to A, while the corresponding axes of the two frames remain parallel. $\mathbf{r}_{PA}$ refers to the position of P as observed by A, and so on. From the situation, it is concluded:

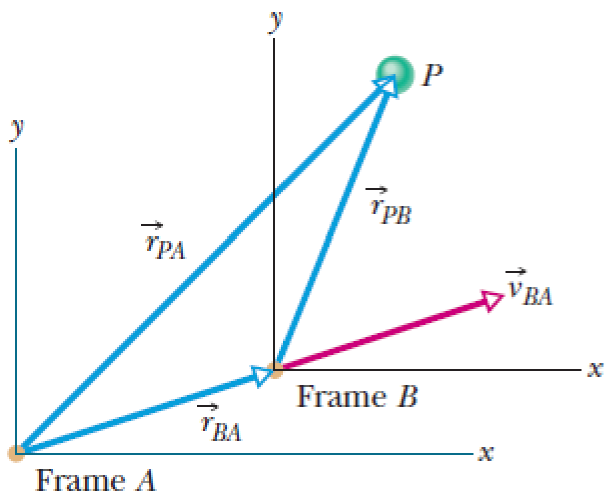


Fig. 4-19 Frame B has the constant two-dimensional velocity $\vec{v}_{BA}$ relative to frame A. The position vector of B relative to A is $\vec{r}_{BA}$. The position vectors of particle P are $\vec{r}_{PA}$ relative to A and $\vec{r}_{PB}$ relative to B.

$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}.$$

$$\downarrow$$

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}.$$

$$\downarrow$$

$$\vec{a}_{PA} = \vec{a}_{PB}.$$

CH 4習題:
7, 30, 31, 35, 41, and 66

[工程師向你展示手機加速度感測器如何工作，有影片輔助說明]