Chapter 12

Equilibrium and Elasticity
平衡與彈性


小烏來瀑布．風動石

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12-1 Equilibrium

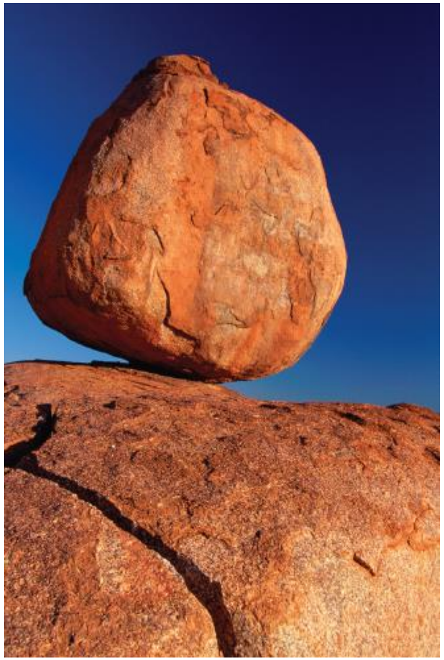
Learning Objectives

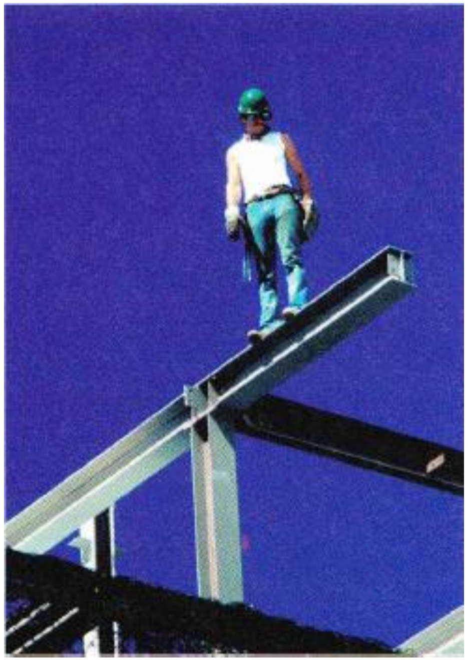
12.01 Distinguish between equilibrium and static equilibrium.

12.02 Specify the four conditions for static equilibrium.

12.03 Explain center of gravity and how it relates to center of mass.

12.04 For a given distribution of particles, calculate the coordinates of the center of gravity and the center of mass.





12-1 Equilibrium

- We often want objects to be stable despite forces acting on them
- Consider a book resting on a table, a puck sliding with constant velocity, a rotating ceiling fan, a rolling bicycle wheel with constant velocity
- These objects have the characteristics that:

1. The linear momentum of the center of mass is constant
2. The angular momentum about the center of mass, or any other point, is constant

12-1 Equilibrium

- Such objects are in **equilibrium** (平衡條件)

$$\boxed{\vec{P} = \text{a constant} \quad \text{and} \quad \vec{L} = \text{a constant.}}$$ Eq. (12-1)

- In this chapter we are largely concerned with objects that are not moving at all; $P = L = 0$
- These objects are in **static equilibrium**: (靜態平衡)

- The only one of the examples from the previous page in static equilibrium is the book at rest on the table:

12-1 Equilibrium

- As discussed in 8-3, if a body returns to static equilibrium after a slight displacement, it is in *stable* static equilibrium
- If a small displacement ends equilibrium, it is *unstable*
- Despite appearances, this rock is in stable static equilibrium, otherwise it would topple at the slightest gust of wind

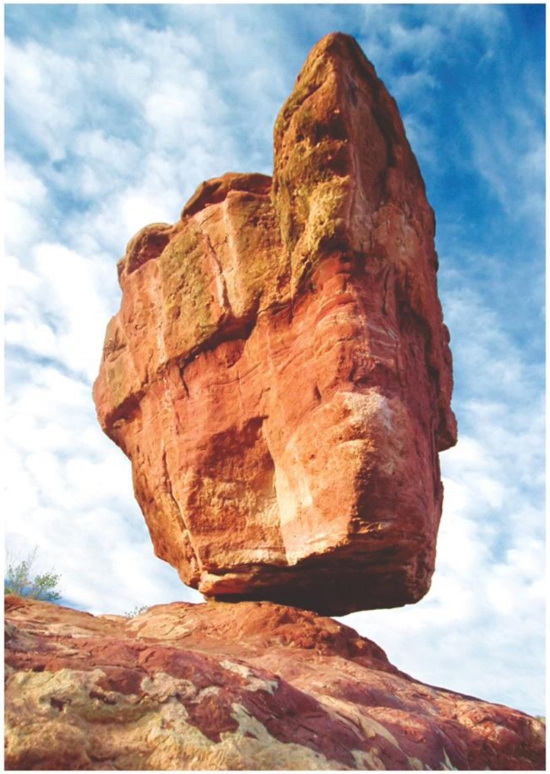


Figure 12-1

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12-1 Equilibrium

- In part (a) of the figure, we have unstable equilibrium
- A small force to the right results in (b)
- In (c) equilibrium is stable, but push the domino so it passes the position shown in (a) and it falls
- The block in (d) is even more stable

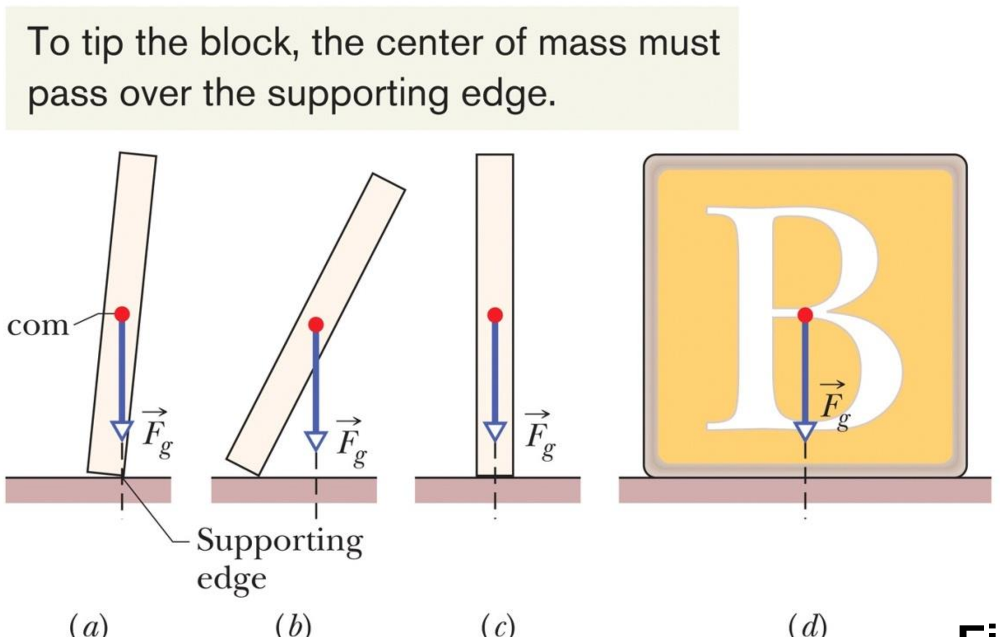


Figure 12-2

12-1 Equilibrium

- Requirements for equilibrium are given by Newton's second law, in linear and rotational form

$$\vec{F}_{\text{net}} = 0 \quad \text{(balance of forces).}$$ Eq. (12-3)

$$\vec{\tau}_{\text{net}} = 0 \quad \text{(balance of torques).}$$ Eq. (12-5)

- Therefore we have for equilibrium:

1. The vector sum of all the external forces that act on the body must be zero.
2. The vector sum of all external torques that act on the body, measured about *any* possible point, must also be zero.

12-1 Equilibrium

- We often simplify matters by considering forces only in the *xy* plane, giving:

$$F_{\mathrm{net},x} = 0 \quad \text{(balance of forces)},$$ **Eq. (12-7)**

$$F_{\mathrm{net},y} = 0 \quad \text{(balance of forces)},$$ **Eq. (12-8)**

$$\tau_{\mathrm{net},z} = 0 \quad \text{(balance of torques)}.$$ **Eq. (12-9)**

- Note that for static equilibrium we have the additional requirements that:

$$\boxed{\vec{F}_{net} = 0}$$ 1. 合力 = 0

$$\boxed{\vec{\tau}_{net} = 0}$$ 2. 合力 矩= 0

3. The linear momentum of the body ***P*** must be zero

4. The angular momentum of the body ***L*** must be zero.

The center of Gravity (cog，重心)

~cog:

The gravitational force $\vec{F}_g$ on a body effectively acts at a single point known as the center of gravity of the body.

~The center of gravity coincides with the center of mass. (because $\vec{g}$ changes very little.)

~*Proof*:

Consider the extended object of mass M shown in fig.a. In fig.a we also show the i-th element of mass m_i. The gravitational force on m_i is equal to $m_i\vec{g}_i$ where $\vec{g}_i$ is the acceleration of gravity in the vicinity of m_i. The torque τ_i on m_i is equal to $F_{gi}x_i$. The net torque

$$\tau_{net} = \sum_i \tau_i = \sum_i F_{gi}x_i \quad \text{(eqs.1)}$$

Consider now fig.b in which we have replaced the forces F_{gi} by the net gravitational force F_g acting at the center of gravity. The net torque τ_{net} is equal to : $\tau_{net} = x_{cog}F_g = x_{cog}\sum_i F_{gi}$ (eqs.2)

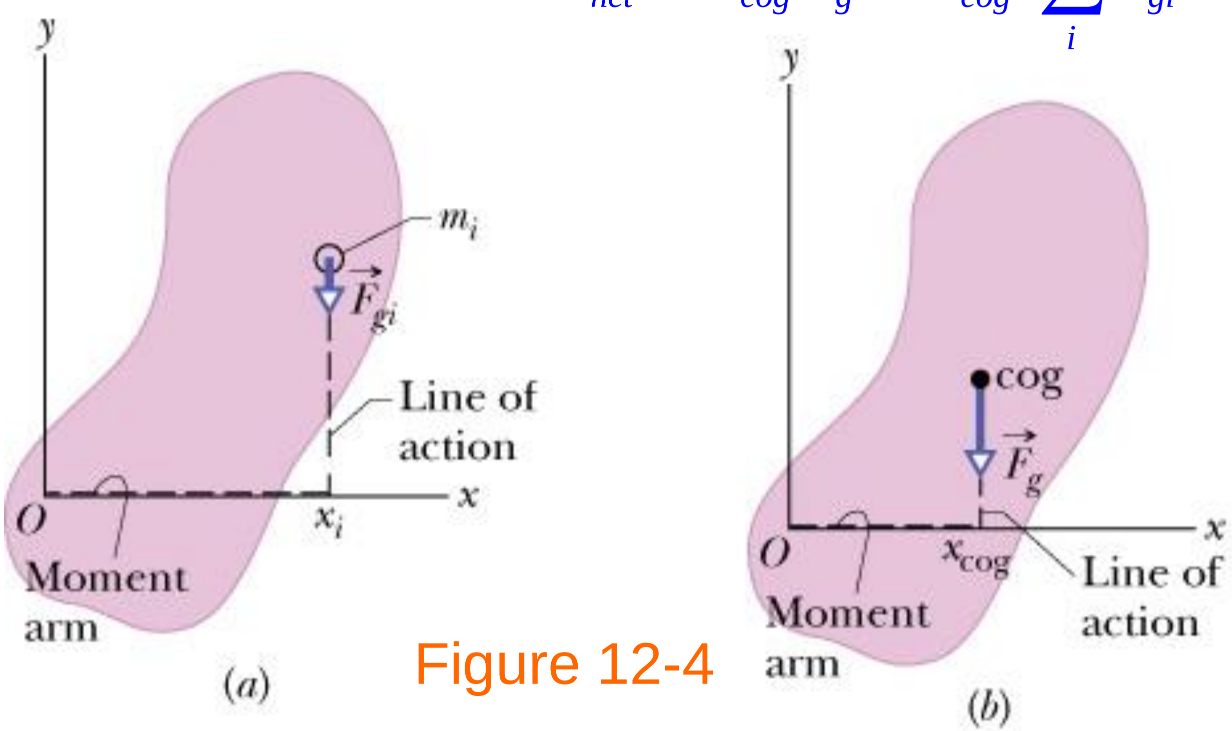


Figure 12-4

If we compare equation 1 with equation 2 we get: $x_{cog} \sum_i F_{gi} = \sum_i F_{gi} x_i$

We substitute $m_i g_i$ for F_{gi} and we have: $x_{cog} \sum_i m_i g_i = \sum_i m_i g_i x_i$

If we set $g_i = g$ for all the elements $\rightarrow$ $x_{cog} = \dfrac{\sum_i m_i x_i}{\sum_i m_i} = x_{com}$

12-2 Some Examples of Static Equilibrium

Learning Objectives

12.05 Apply the force and torque conditions for static equilibrium.

12.06 Identify that a wise choice about the placement of the origin (about which to calculate torques) can simplify the calculations by eliminating one or more unknown forces from the torque equation.

Statics Problem Recipe (解題技巧)

1. Draw a force diagram. (Label the axes)

2. Choose a **convenient** origin O. (or a pivot, 選適當支點) A good choice is to

   ~have one of the unknown forces acting at O

3. Sign of the torque $\boldsymbol{\tau}$ for each force:

   − If the force induces clockwise (CW) rotation

   + If the force induces counter-clockwise (CCW) rotation

4. Equilibrium conditions:

$$\boxed{F_{net,x} = 0 \qquad F_{net,y} = 0 \qquad \tau_{net,z} = 0}$$

5. Make sure that:

numbers of unknowns = number of equations

12-2 Sample Problems of Static Equilibrium:

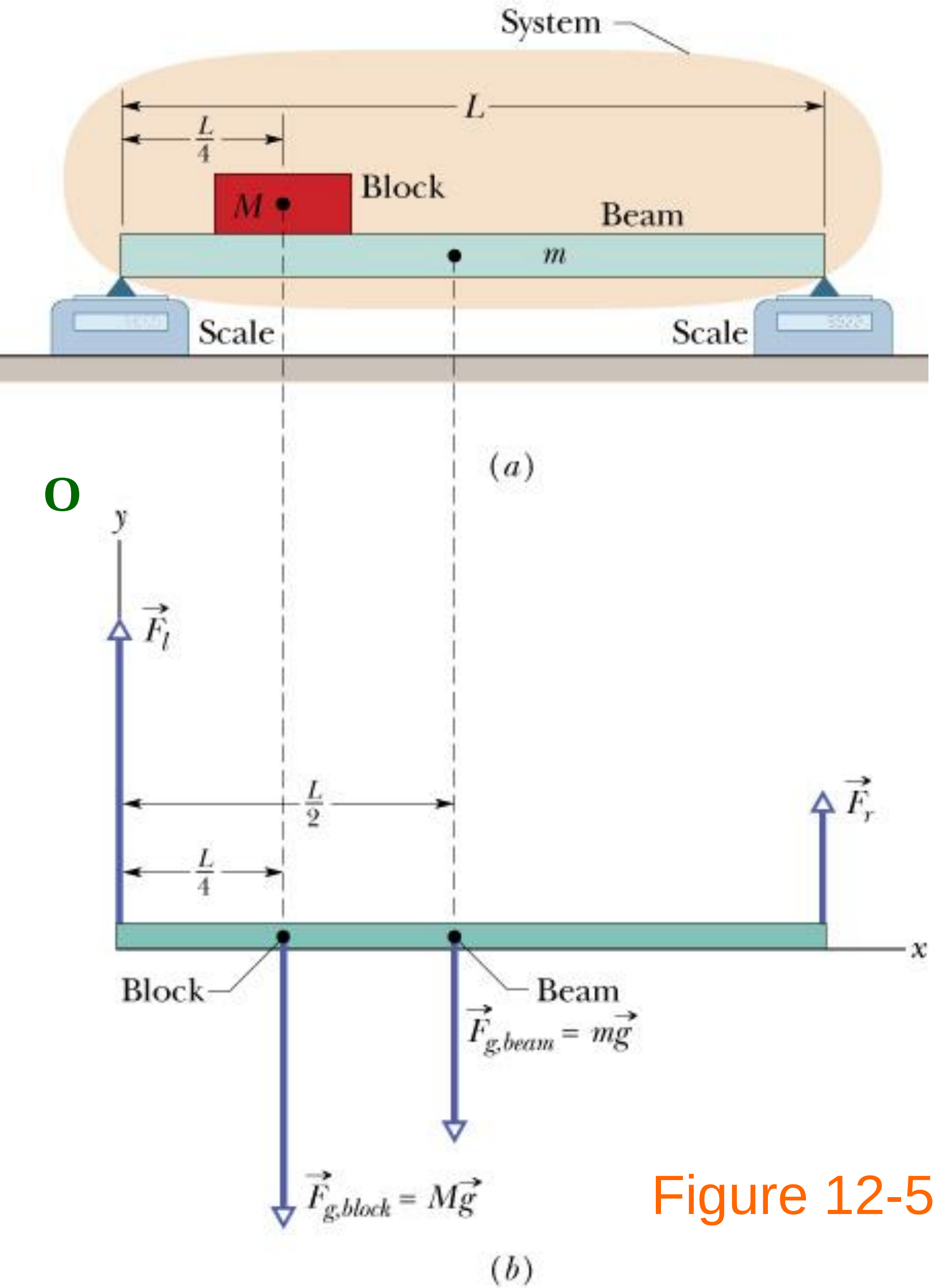


Figure 12-5

Sample Problem 12.01

A uniform beam of length **L** and mass m = 1.8 kg is at rest on two scales.

A uniform block of mass M = 2.7 kg is a rest on the beam at a distance L/4 from its left end.

Calculate the scales readings

$F_{net,y} = F_l + F_r - M_g - m_g = 0$ (eqs.1)

We choose to calculate the torque with respect to an axis through the left end of the bean (point O).

$$\tau_{net,z} = -\left(\frac{L}{4}\right)(mg) - \left(\frac{L}{2}\right)(Mg) + (L)(F_r) = 0 \quad \text{(eqs.2)}$$

From equation 2 we get: $F_r = \dfrac{Mg}{4} + \dfrac{mg}{2} = \dfrac{2.7 \times 9.8}{4} + \dfrac{1.8 \times 9.8}{2} = 15.44 \approx 15 \text{ N}$

We solve quation 1 for $F_\ell \rightarrow F_\ell = Mg + mg - F_r = (2.7+1.8) \times 9.8 - 15.44 = 28.66 \text{ N}$

$F_\ell \approx 29 \text{ N}$

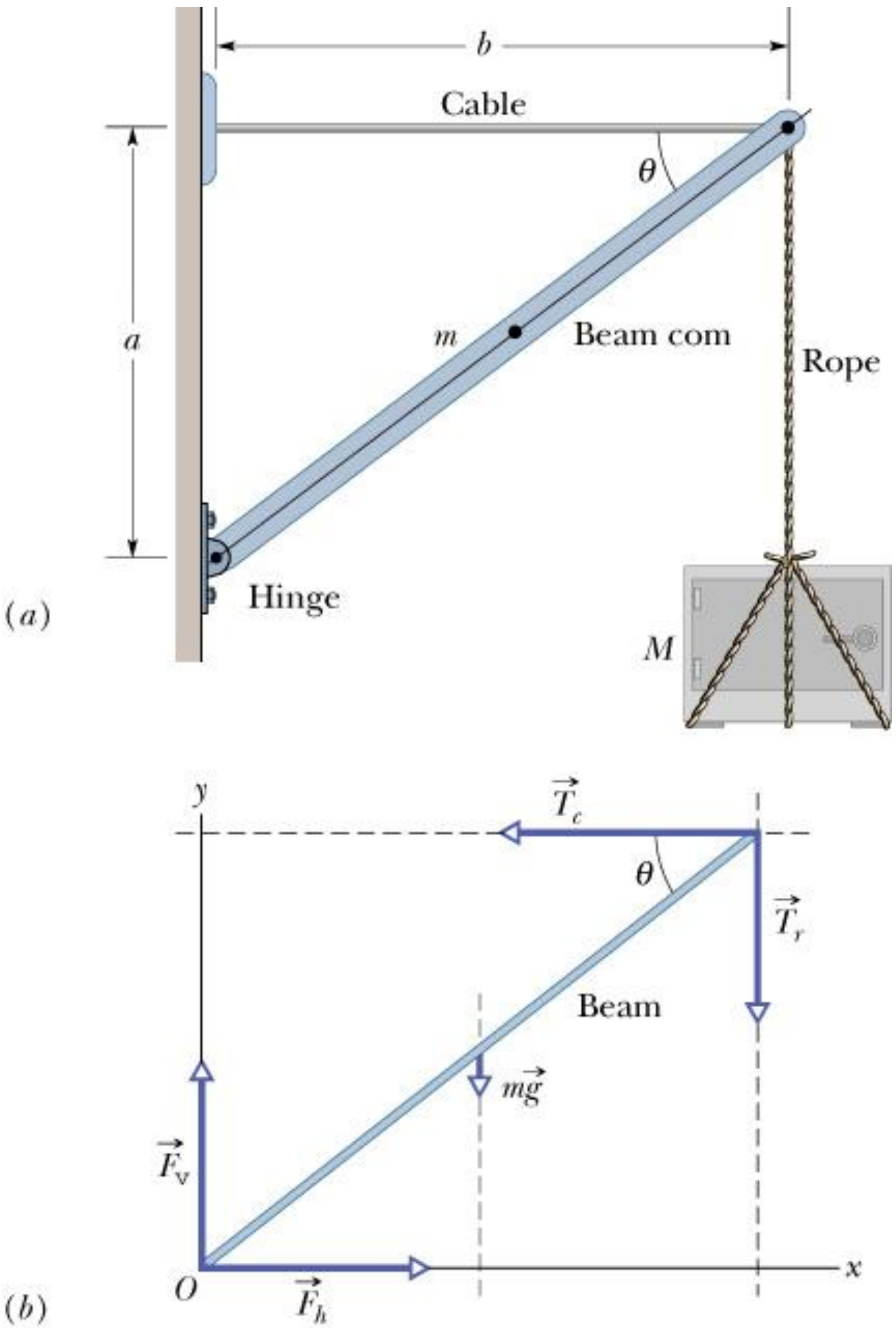


Figure 12-6

Sample Problem 12.02:

A safe of mass M = 430 kg hangs by a rope from a boom with dimensions a = 1.9 m and b = 2.5 m.

The beam of the boom has mass m = 85 kg

Find the tension T_c in the cable and the magnitude of the net force F exerted on the beam by the hinge.

We calculate the net torque about an axis normal to the page that passes through point O.

$$\tau_{net,z} = (a)(T_c) - (b)(T_r) - \left(\frac{b}{2}\right)(mg) = 0 \rightarrow \qquad T_r = Mg,$$

$$T_c = \frac{gb\left(M + \frac{m}{2}\right)}{a} = \frac{9.8 \times 2.5(430 + 85/2)}{1.9} \approx 6100 \text{ N}$$

$$F_{net,x} = F_h - T_c = 0 \rightarrow F_h = T_c = 6093 \text{ N}$$

$$F_{net,y} = F_v - mg - T_r = 0 \rightarrow F_v = mg + T_r = g(m + M) = 9.8 \times (85 + 430) = 5047 \text{ N}$$

$$F = \sqrt{F_h^2 + F_v^2} = \sqrt{(6093)^2 + (5047)^2} \approx 7900 \text{ N}$$

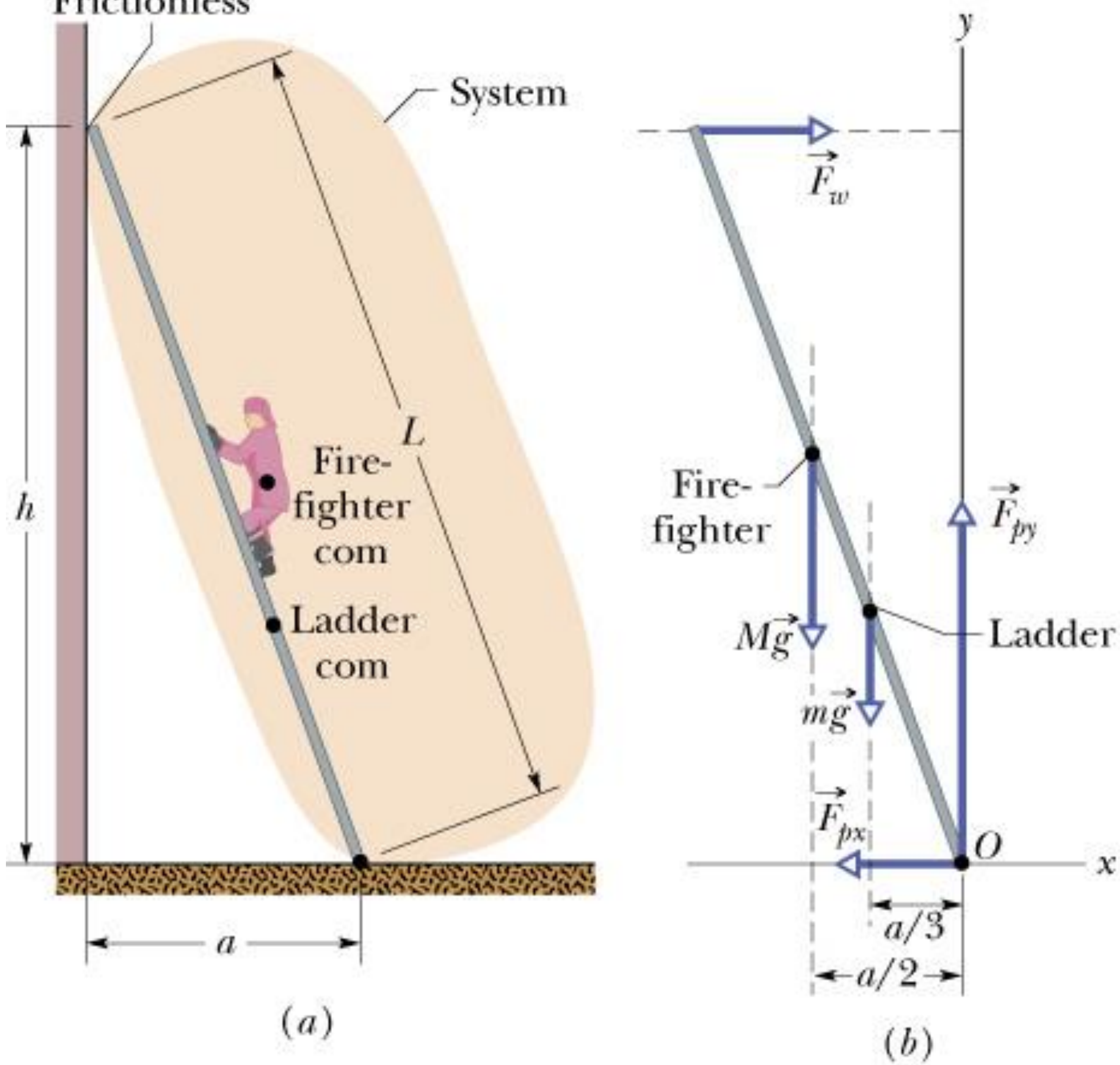


Figure 12-7

Sample Problem 12.03:

A ladder of length L = 12 m and mass $\boldsymbol{m}$ = 45 kg leans against a frictionless wall. The ladder's upper end is at a height h = 9.3 m above the pavement on which the lower end rests.

The com of the ladder is $L/3$ from the lower end. **A** firefighter of mass M = 72 kg climbs half way up the ladder Find the forces exerted on the ladder by the wall and pavement. Distance

$$a = \sqrt{L^2 - h^2} = 7.58 \text{ m}$$

We take torques about an axis through point O.

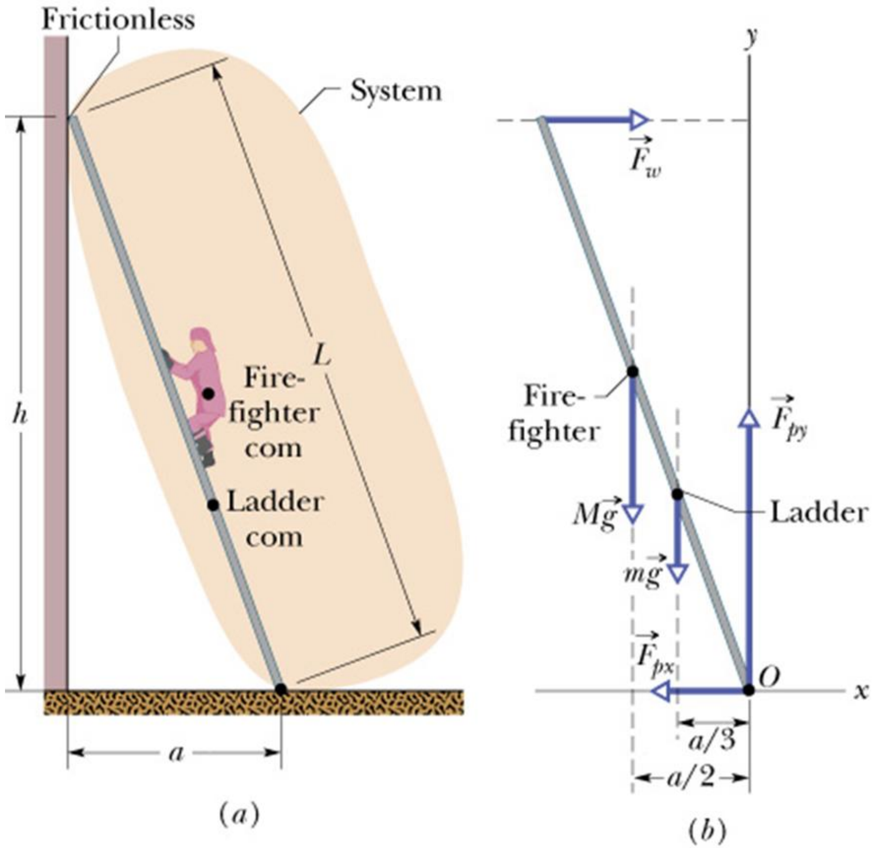
$$\tau_{net,z} = -(h)(F_w) + \left(\frac{a}{3}\right)(mg) + \left(\frac{a}{2}\right)(Mg) = 0$$

(選擇較多"未知"受力之點作為pivot!)

$$\rightarrow F_w = \frac{ga\left(\frac{M}{2}+\frac{m}{3}\right)}{h} = \frac{9.8\times 7.58\times(72/2+45/3)}{9.3} = 407\text{ N} \approx 410\text{ N}$$

$$F_{net,x} = F_w - F_{px} = 0 \rightarrow F_{px} = F_w = 410\text{ N}$$

$$F_{net,y} = F_{py} - Mg - mg = 0 \rightarrow F_{py} = Mg + mg = 9.8\times(72+45) = 1146.6\text{ N} \approx 1100\text{ N}$$



12-2 Some Examples of Static Equilibrium

Example 12.04 Leaning Tower of Pisa

- $R = 9.8$ m, $h = 60$ m, $\theta = 5.5°$
- Model: supported by 2 forces, at left and right edges
- No lean: $F_{NR} = ½\, mg$
- Lean shifts com by
  - $d = ½\, h \tan \theta$

$-(R+d)(mg) + (2R)(F'_{NR}) = 0$

- New force:
  - $F'_{NR} = (R + d)mg\ /\ 2R$
- Force increases by 30%
  - $F'_{NR}/F_{NR} = 1.29$

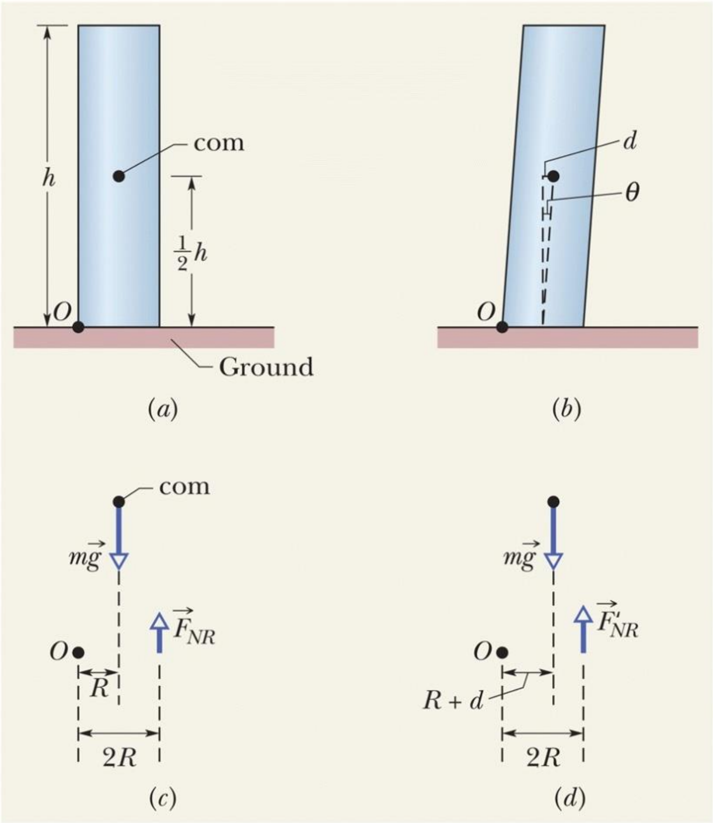


Figure 12-8

12-3 Elasticity

Learning Objectives

12.07 Explain what an indeterminate situation is.

12.08 For tension and compression, apply the equation that relates stress to strain and Young's modulus.

12.09 Distinguish between yield strength and ultimate strength.

12.10 For shearing, apply the equation that relates stress to strain and the shear modulus.

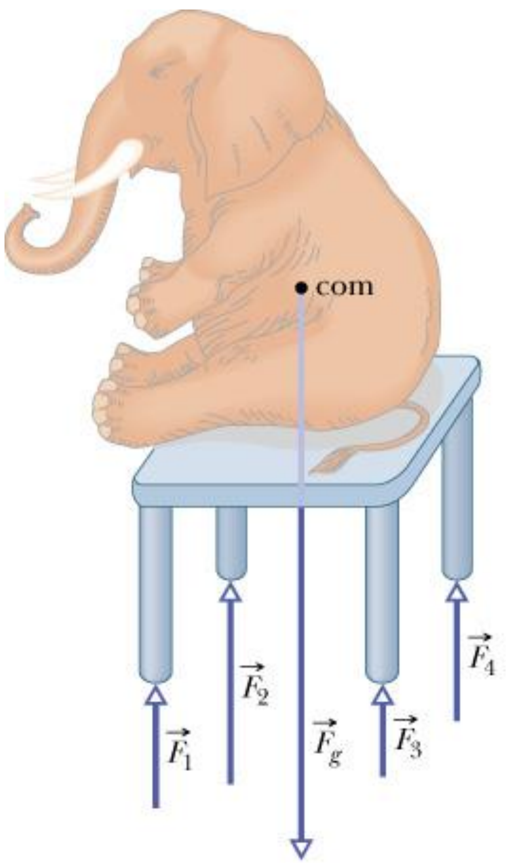
12.11 For hydraulic stress, apply the equation that relates fluid pressure to strain and the bulk modulus.

Indeterminate Structures. (不定結構)

For the problems in this chapter we has the following three equations at our disposal:

$F_{net,x} = 0$ $F_{net,y} = 0$ $\tau_{net,z} = 0$

Figure 12-9



If the problem has more than three unknowns we cannot solve it!

We can solve a statics problem for a table with three legs but not for one with four legs. Problems like these are called **indeterminate**

An example is given in the figure.

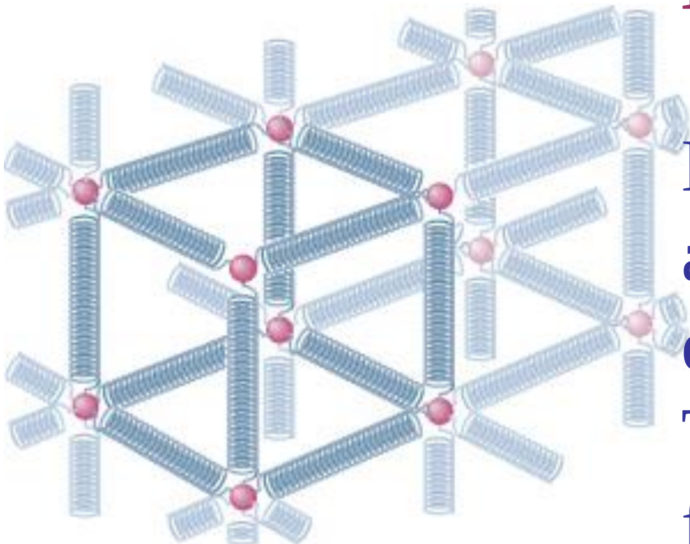
~How can we calculate the values of these forces?

=>Using some knowledge of elasticity(彈性)!

(四支桌腳之長度不可能完全一樣!)

Figure 12-10

Elasticity:



Metallic solids consist of a large number of atoms positioned on a regular three-dimensional lattice as shown in the figure. The lattice is repetition of a pattern (in the figure this pattern is a cube)

~The atoms are held together by interatomic forces that can be modeled as tiny springs.

~Try to change the interatomic distance, → the resulting force is proportional to the atom displacement from the equilibrium position (Hook's Law).

~The spring constants are large and thus the lattice is remarkably rigid

Impurities ←→ Elasticity ?

Three ways in which a solid might change its dimensions under the action of external deforming forces.:

1. In fig.*a* the cylinder is stretched by forces acting along the cylinder axis.
2. In fig.*b* the cylinder is deformed by forces perpendicular to its axis.
3. In fig.*c* a solid placed in a fluid under high pressure is compressed uniformly on all sides.

Stress (應力)≡deforming force per unit area.(Unit: N/m^2) **[與壓力同單位]**

~Three deformation types of stress :

1. tensile/compressive (拉伸/壓縮)for fig.*a*.
2. shearing (剪應力)for fig.*b*.
3. hydraulic (液體應力)for fig.*c*.

Stress is related to strain via the equation:

$$\textbf{stress} = \textbf{modulus} \times \textbf{stain (應變)} \qquad (12\text{-}22)$$

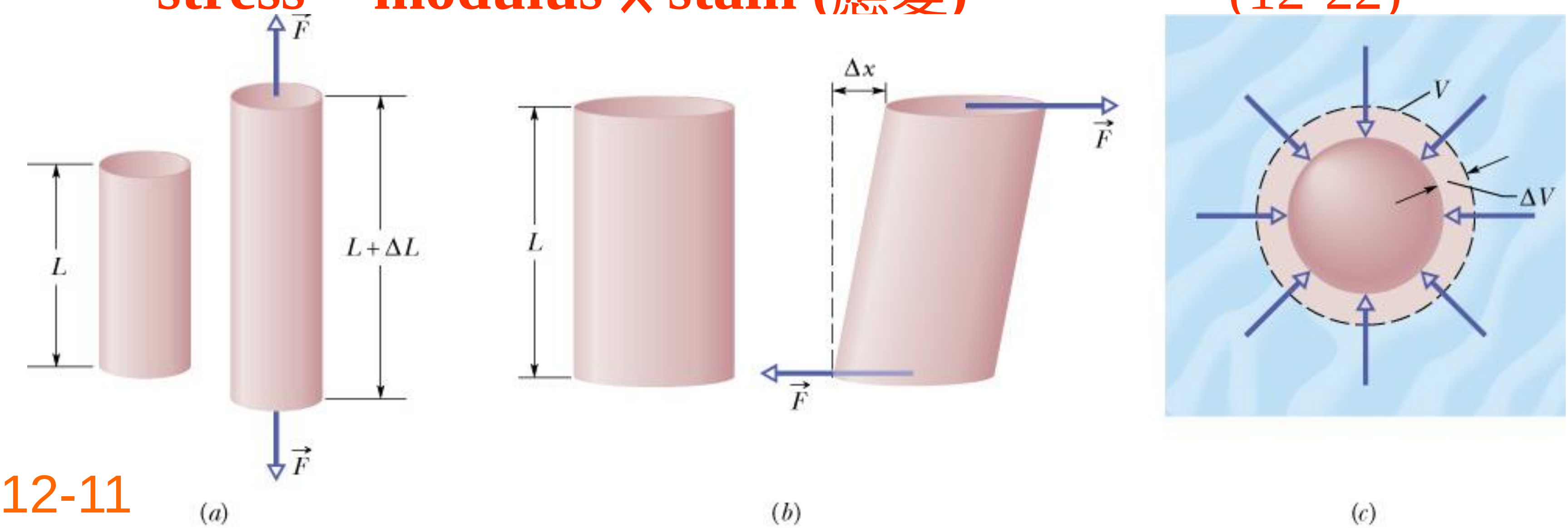


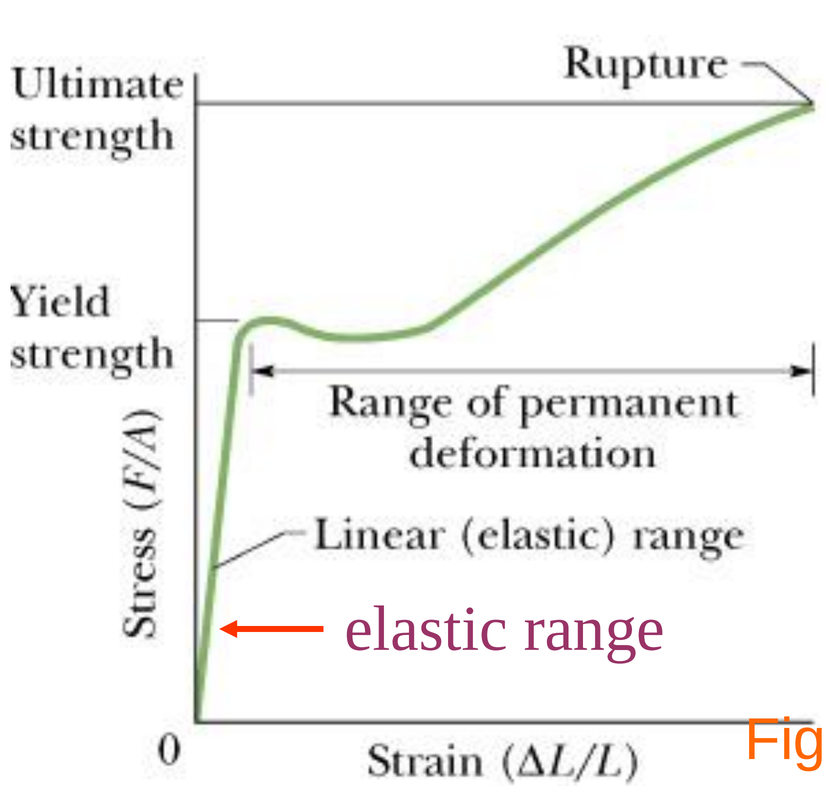
Figure 12-11

Tensile stress = F/A ;where A is the solid area

Strain (應變，symbol S) = $\Delta L/L$; where ΔL is he change in the length L of the cylindrical solid. (dimensionless, 無單位)

~The yield strength S_y (屈服強度): the cylinder becomes permanently deformed.

~The ultimate strength S_u(極限強度): the cylinder breaks.



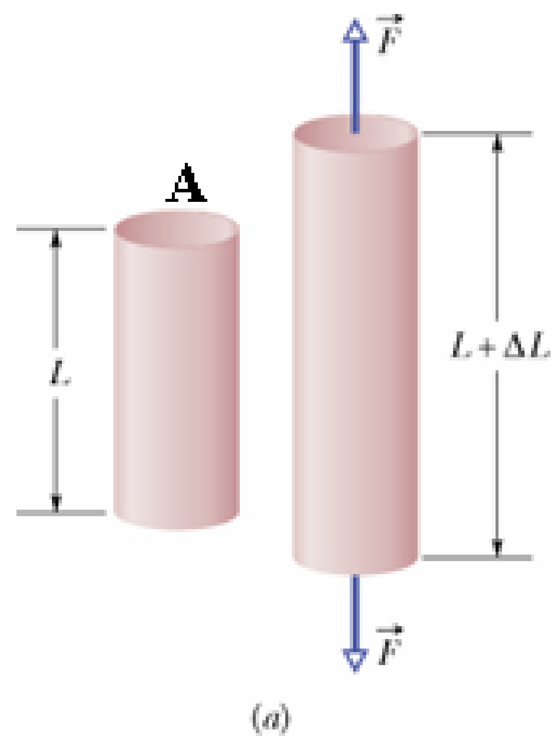


Figure 12-13

For stresses below S_y (elastic range) stress and strain are connected via the equation

$$\frac{F}{A} = E\frac{\Delta L}{L} \qquad (12\text{-}23)$$

The constant E (modulus) is known as: Young's modulus (楊氏係數)

Note:

Young's modulus is almost the same for tensile and compression The ultimate strength S_u maybe different

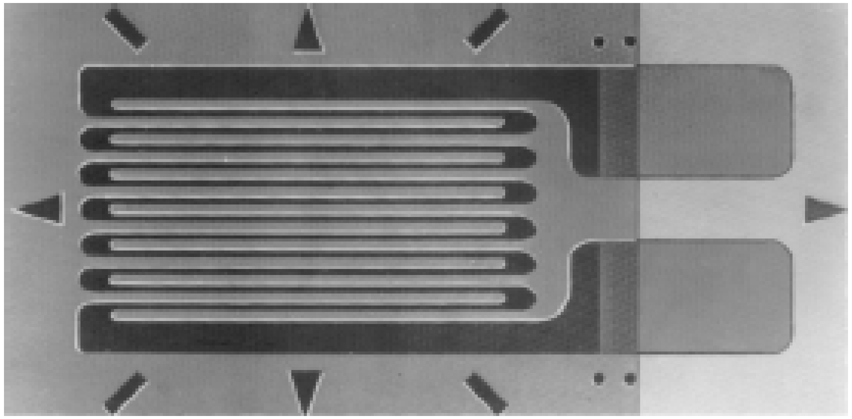


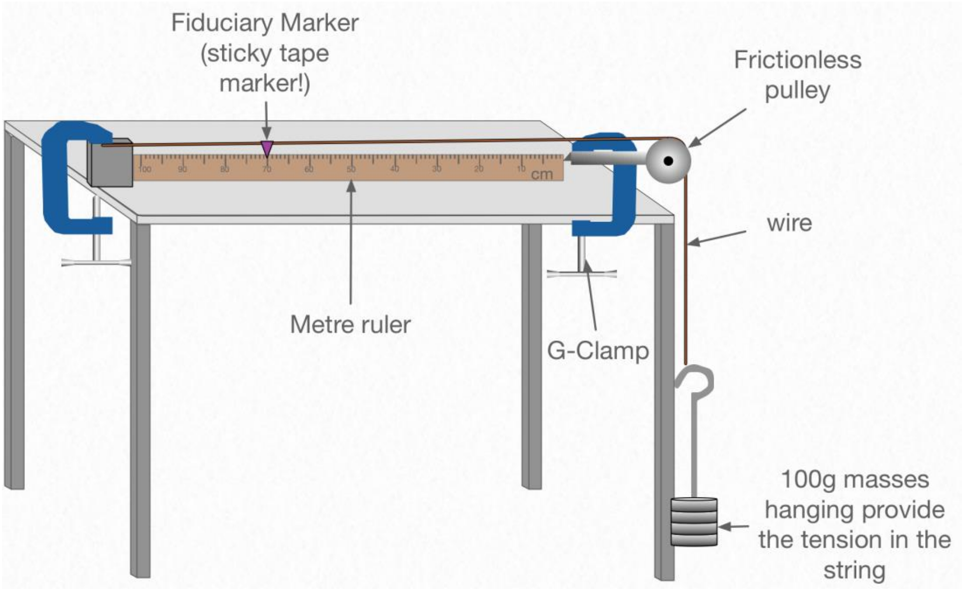
Fig. 12-14 A strain gage of overall dimensions 9.8 mm by 4.6 mm. The gage is fastened with adhesive to the object whose strain is to be measured; it experiences the same strain as the object. The electrical resistance of the gage varies with the strain, permitting strains up to 3% to be measured. Courtesy Vishay Micro-Measurements Group, Inc., Raleigh, NC, USA.

TABLE 12-1

Some Elastic Properties of Selected Materials of Engineering Interest

Material	Density ρ (kg/m^3)	Young's Modulus E (10^9 N/m^2)	Ultimate Strength S_u (10^6 N/m^2)	Yield Strength S_y (10^6 N/m^2)
Steel[a]	7860	200	400	250
Aluminum	2710	70	110	95
Glass	2190	65	50[b]	—
Concrete[c]	2320	30	40[b]	—
Wood[d]	525	13	50[b]	—
Bone	1900	9[b]	170[b]	—
Polystyrene	1050	3	48	—

[a]Structural steel (ASTM-A36).
[b]In compression.
[c]High strength.
[d]Douglas fir.



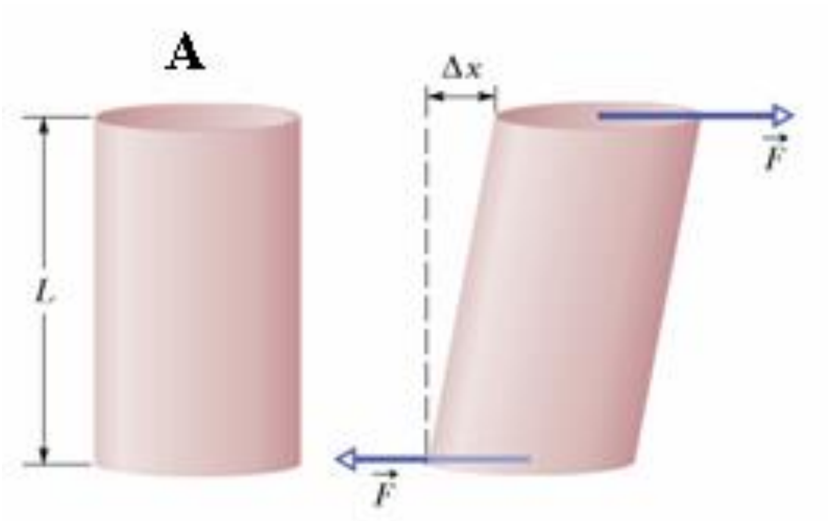
資料來源：維基百科
參見：

物質(substance)	楊氏模數/楊格係數/彈性模數 (Young’s modulus)單位為10^9帕(GPa)
橡膠(rubber)	0.002~0.008
人類軟骨(Human cartilage)	0.024
人類脊椎(Human vertebra)	0.088(壓縮狀態下測得)
	0.17(拉緊狀態下測得)
骨骼中的膠原蛋白(collagen in bone)	0.6
人類韌帶(Human tendon)	0.6
與生長方向垂直木材(Wood, across the grain)	1
尼龍(nylon)	2~6
蜘蛛絲(spider silk)	4
人類大腿骨(Human femur)	9.4(壓縮狀態下測得)
	16(拉緊狀態下測得)
與生長方向平行木材(Wood, along the grain)	10~15
磚塊/磚頭(bricks)	14~20
混凝土(concrete)	20~30(壓縮狀態下測得)
大理岩/大理石(marble)	50~60
鋁(aluminum)	70
鑄鐵/生鐵(cast iron)	100~120
銅/自然銅/紫銅(copper)	120
熟鐵/熱鐵/煅鐵/熟鋼/焊接鋼(wrought iron)	190
鋼/圓鋼銼(steel)	200
鑽石(diamond)	1200

Shearing: In the case of shearing deformation strain is defined as the dimensionless ratio $\frac{\Delta x}{L}$. The stress/strain equation has the form:

$$\frac{F}{A} = G\frac{\Delta x}{L} \qquad (12\text{-}24)$$

The constant G is known as the shear modulus 剪力模數



$$\boxed{\frac{F}{A} = G\frac{\Delta x}{L}}$$

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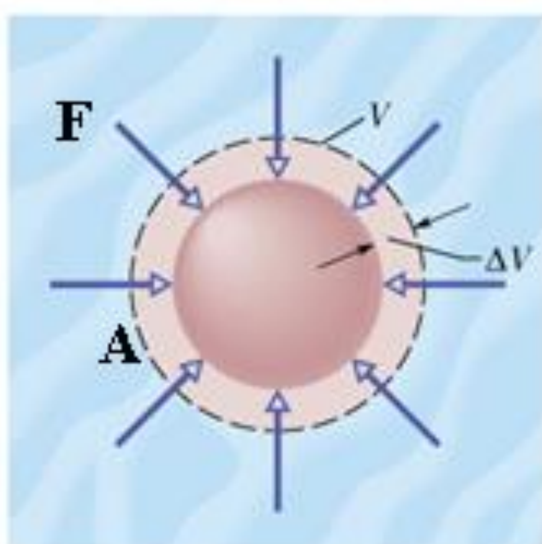
Hyrdaulic Stress. The stress is this case is the pressure $p = \frac{F}{A}$ the surrounding fluid exerts on the immersed object. Here A is the area of the object. In this case strain is defined as the dimensionless ratio $\frac{\Delta V}{V}$ where V is the volume of the object and ΔV the change in the volume due to the fluid pressure. The stress/strain equation has the form: $p = B\frac{\Delta V}{V}$ The constant B is known as the bulk modulus of the material

體積模數



(c)

$$\boxed{p = B\frac{\Delta V}{V}} \qquad (12\text{-}25)$$

Sample Problem 12-5

A steel rod has a radius R of 9.5 mm and a length L of 81 cm. A 62 kN force $\vec{F}$ stretches it along its length. What are the stress on the rod and the elongation and strain of the rod?

Sample Problem 12-6

A table has three legs that are 1.00 m in length and a fourth leg that is longer by $d = 0.50$ mm, so that the table wobbles slightly. A steel cylinder with mass $M = 290$ kg is placed on the table (which has a mass much less than M) so that all four legs are compressed but unbuckled and the table is level but no longer wobbles. The legs are wooden cylinders with cross-sectional area $A = 1.0\ \text{cm}^2$; Young's modulus is $E = 1.3 \times 10^{10}\ \text{N/m}^2$. What are the magnitudes of the forces on the legs from the floor?

12-3 Elasticity

Example Balancing a wobbly table [Indeterminate Structures. (不定結構)]

- Three legs of 1.00 m, a fourth longer by 0.50 mm
- Compressed by M = 290 kg so all four legs are compressed but not buckled and the table does not wobble
- Legs are wooden cylinders with area A = 1.0 sq cm
- $E = 1.3 \times 10^{10}$ N/m^2
- The 3 shorter legs must compress the same amount, the longer leg compresses more
- Write length comparison, use the stress-strain equation, and approximate all legs to be length L

$$\frac{F_4 L}{AE} = \frac{F_3 L}{AE} + d.$$

Eq. (12-27) $(\Delta L_4 = \Delta L_3 + d\ )$

12-3 Elasticity

Example Balancing a wobbly table (continued)

- Get a second equation by balancing forces

$$3F_3 + F_4 - Mg = 0, \quad \text{Eq. (12-28)}$$

- Solve the simultaneous equations to find
  - $F_3 = 550$ N
  - $F_4 = 1200$ N
- Each short leg is compressed by 0.42 mm, and the long leg is compressed by 0.92 mm

12 Summary

Static Equilibrium

$$\vec{F}_{net} = 0 \quad \text{(balance of forces).}$$

Eq. (12-3)

$$\vec{\tau}_{net} = 0 \quad \text{(balance of torques).}$$

Eq. (12-5)

Center of Gravity

- If the gravitational acceleration is the same for all elements of the body, the cog is at the com.

Elastic Moduli

- Three elastic moduli
- Strain: fractional length change
- Stress: force per unit area

$$\text{stress} = \text{modulus} \times \text{strain}.$$

Eq. (12-22)

Tension and Compression

- *E* is Young's modulus

$$\frac{F}{A} = E\,\frac{\Delta L}{L}.$$

Eq. (12-23)

12 Summary

Shearing

- *G* is the shear modulus

$$\frac{F}{A} = G\,\frac{\Delta x}{L}.$$

Eq. (12-24)

Hydraulic Stress

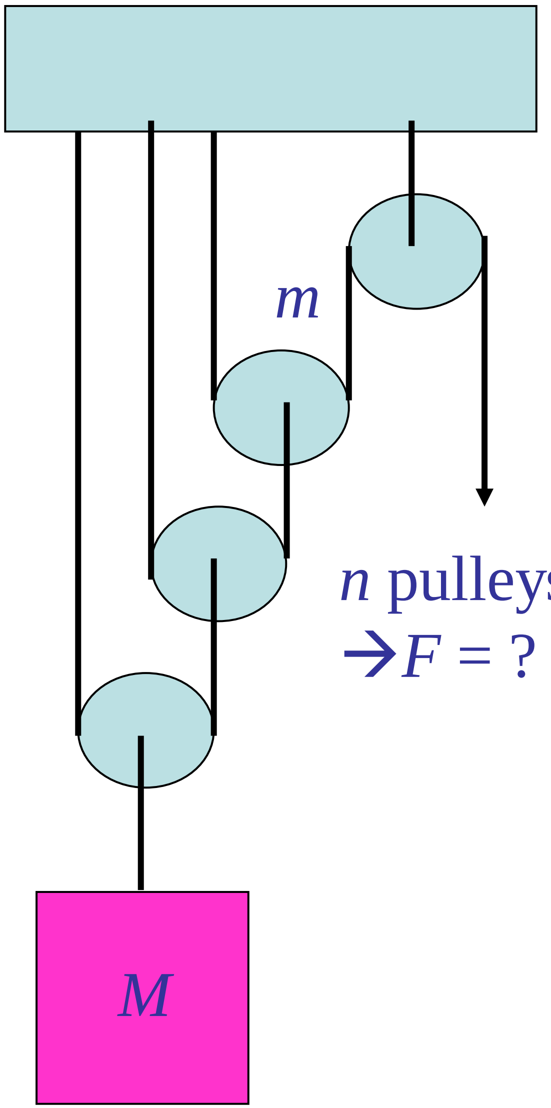
- *B* is the bulk modulus

$$p = B\,\frac{\Delta V}{V}.$$

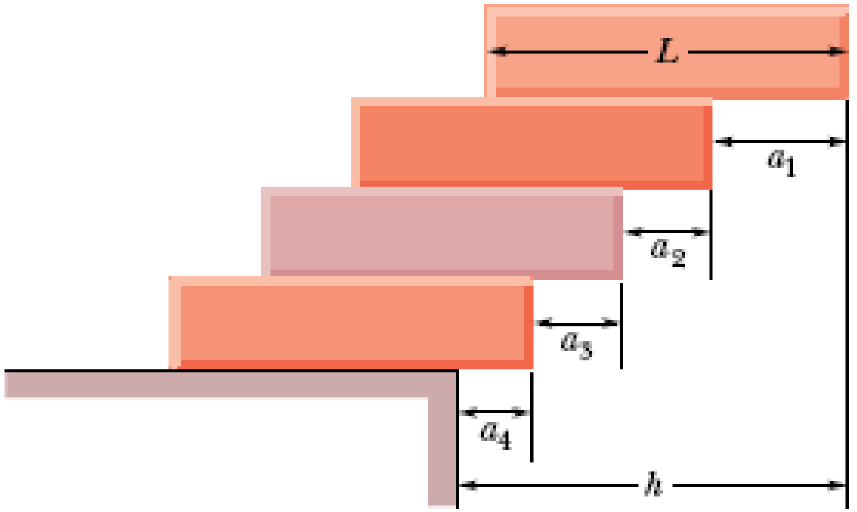
Eq. (12-25)

習題:
2, 8, 13, 22, 23, 33, 45, and 50

Series in Physics: (Problems 2 & 8)

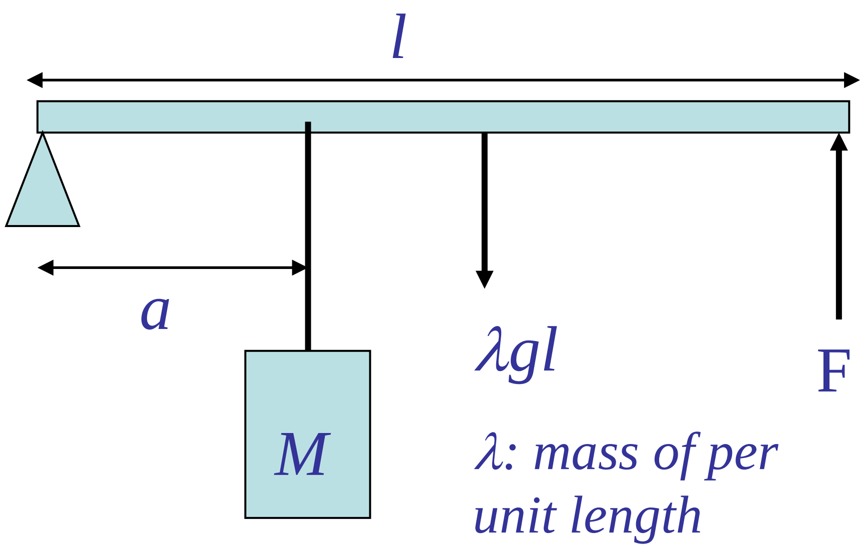


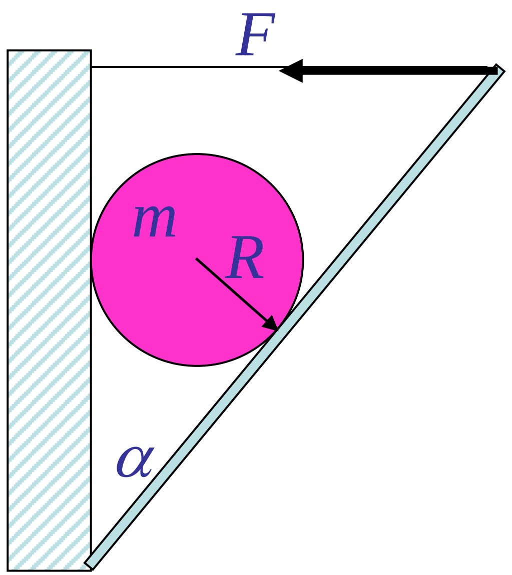
n pulleys
→ F = ?



n bricks
→ h = ?

Extremes in Physics:





=> Find the length l for the minimum F (F_{min}) and F_{min}?

=> Find the angle α for the minimum F (F_{min}) and F_{min}?