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**Deriving the Precise Definition of a Limit.**

The intuitive definition of a limit,  $\lim_{x \rightarrow a} f(x) = L$ , says that

*we can make the values of  $f(x)$  arbitrarily close to  $L$  by restricting  $x$  to be sufficiently close to  $a$  but not equal to  $a$ .*

Verifying the above statement is a back and forth, offensive and defensive process. When one claims  $\lim_{x \rightarrow a} f(x) = L$ , he/she is responsible to answer the following challenges:

*How close to  $a$  does  $x$  have to be so that  $|f(x) - L| < 0.1$  ?*

*How close to  $a$  does  $x$  have to be so that  $|f(x) - L| < 0.01$  ?* And so on... Thus he/she must once and for all answer:

*How close to  $a$  does  $x$  have to be so that  $|f(x) - L| < \epsilon$ , where  $\epsilon$  is an arbitrarily small positive number?*

In general, given smaller  $\epsilon$ , one may need to restrict  $x$  closer to  $a$ . Hence, the above answer (the distance between  $x$  and  $a$ ) depends on  $\epsilon$ . Now we conclude that

$$\lim_{x \rightarrow a} f(x) = L \text{ means that}$$

**for any number  $\epsilon > 0$  there is a  $\delta > 0$  (depending on  $\epsilon$ ) such that if  $0 < |x - a| < \delta$  then  $|f(x) - L| < \epsilon$ .**

The last statement is the **precise definition** of  $\lim_{x \rightarrow a} f(x) = L$ . In this worksheet we will have hands-on experiences of using this definition.

**Exercise 1.**

$$\text{Consider } f(x) = \begin{cases} \frac{2x^2 - x - 1}{x - 1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$$

- Simplify  $f(x)$ ,  $f(x) = \underline{\hspace{2cm}}$  for  $x \neq 1$ . Guess the limit,  $\lim_{x \rightarrow 1} f(x) = \underline{\hspace{2cm}}$ .
- When you wrote down  $\lim_{x \rightarrow 1} f(x) = L$ , you have made a very *strong* statement! You claimed that *We can make  $f(x)$  arbitrarily close to  $L$  by restricting  $x$  to be sufficiently close to 1 but not equal to 1.*

One may ask you to show him :

How close to 1 does  $x$  have to be so that  $|f(x) - L| < 0.1, 0.01, 0.001 \dots$ ?

Starting from the goal inequality  $|f(x) - L|$ , derive an equivalent inequality regarding  $|x - 1|$ .

Fill in the blank.

$$\text{If } 0 < |x - 1| < \underline{\hspace{2cm}}, \quad |f(x) - L| < 0.1.$$

$$\text{If } 0 < |x - 1| < \underline{\hspace{2cm}}, \quad |f(x) - L| < 0.01.$$

$$\text{If } 0 < |x - 1| < \underline{\hspace{2cm}}, \quad |f(x) - L| < \epsilon, \text{ where } \epsilon \text{ is any positive number.}$$

- What happens at  $x = 1$ ? Obviously,  $|f(1) - L| = |0 - L| > 0.1$ . Does this violate the statement  $\lim_{x \rightarrow 1} f(x) = L$ ?

- For any  $\epsilon > 0$ , we find  $\delta = \underline{\hspace{2cm}}$  such that if  $0 < |x - 1| < \delta$ , then  $|f(x) - L| < \epsilon$ . This proves that indeed  $\lim_{x \rightarrow 1} f(x) = L$ .

**Exercise 2.**

- (a) Show that  $\lim_{x \rightarrow 0} \sqrt[3]{x} = 0$ .

To prove the limit is 0, for any  $\epsilon > 0$ , we need to find a  $\delta > 0$  such that if  $0 < |x - 0| < \delta$  then  $|\sqrt[3]{x} - 0| < \epsilon$ .

- Starting from the desired inequality  $|\sqrt[3]{x} - 0| < \epsilon$ , derive an inequality for  $|x - 0|$ .

- For a given  $\epsilon$ , find such  $\delta$  and show that if  $0 < |x - 0| < \delta$  then  $|\sqrt[3]{x} - 0| < \epsilon$ .

- (b) Imitating the precise definition of a limit, write down precise definitions of one-sided limits.

- $\lim_{x \rightarrow a^+} f(x) = L \Leftrightarrow$

- $\lim_{x \rightarrow a^-} f(x) = L \Leftrightarrow$

### Precise Definition of an Infinite Limit.

The intuitive definition of an infinite limit,  $\lim_{x \rightarrow a} f(x) = \infty$ , says that

*we can make the values of  $f(x)$  arbitrarily large by restricting  $x$  to be sufficiently close to  $a$  but not equal to  $a$ .*

Again, when one claims  $\lim_{x \rightarrow a} f(x) = \infty$ , he/she is responsible to answer the following challenges:

*How close to  $a$  does  $x$  have to be so that  $f(x) > 100$  ?*

*How close to  $a$  does  $x$  have to be so that  $f(x) > 1000$  ?* And so on...Thus he/she must once and for all answer:

*How close to  $a$  does  $x$  have to be so that  $f(x) > N$  , where  $N$  is an arbitrarily large positive number?*

Hence we conclude that

$$\lim_{x \rightarrow a} f(x) = \infty \text{ means that}$$

**for any number  $N > 0$  there is a  $\delta > 0$  (depending on  $N$ ) such that if  $0 < |x - a| < \delta$  then  $f(x) > N$ .**

The last statement is the **precise definition** of  $\lim_{x \rightarrow a} f(x) = \infty$ .

### Exercise 3.

(a) Imitating the above definition, write down precise definitions of other limits regarding infinity.

- $\lim_{x \rightarrow a^+} f(x) = -\infty \Leftrightarrow$

- $\lim_{x \rightarrow \infty} f(x) = L \Leftrightarrow$

- $\lim_{x \rightarrow -\infty} f(x) = \infty \Leftrightarrow$

(b) Show that  $\lim_{x \rightarrow 0^+} \ln x = -\infty$ .

To prove that the limit is negative infinity, for any negative  $N < 0$ , we need to find a  $\delta > 0$  such that if  $0 < x - 0 < \delta$  then  $\ln x < N$ .

- Starting from the desired inequality  $\ln x < N$ , derive an inequality for  $x - 0$ .

- For a given  $N < 0$ , find such  $\delta$  and show that if  $0 < x - 0 < \delta$  then  $\ln x < N$ .

**Exercise 4 (Optional).**

We can use the precise definition of a limit to prove limit laws and corollaries about limits. Try to prove the following statements.

- (a) Prove that if  $\lim_{x \rightarrow a} f(x) = L$  and  $\lim_{x \rightarrow a} g(x) = M$  then  $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M$ .  
(Hint: For any  $\epsilon > 0$  we need to find a  $\delta > 0$  such that if  $0 < |x - a| < \delta$  then  $|(f(x) + g(x)) - (L + M)| < \epsilon$ . However, we have  $|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M|$ . Moreover, since  $\lim_{x \rightarrow a} f(x) = L$  and  $\lim_{x \rightarrow a} g(x) = M$ , we can let  $|f(x) - L| < \epsilon/2$  and  $|g(x) - M| < \epsilon/2$  if  $x$  is sufficiently close to  $a$ .)

- (b) Prove that if  $\lim_{x \rightarrow a} f(x) = 0$  and  $|g(x)| < M$  for all  $x$  where  $M > 0$  is a constant then  $\lim_{x \rightarrow a} f(x)g(x) = 0$ .

- (c) Prove that if  $\lim_{x \rightarrow a} f(x) = L > 0$  and  $\lim_{x \rightarrow a} g(x) = \infty$  then  $\lim_{x \rightarrow a} f(x)g(x) = \infty$ .

- (d) Prove that if  $f(x) < g(x)$  for all  $x \neq a$  and the limits  $\lim_{x \rightarrow a} f(x)$ ,  $\lim_{x \rightarrow a} g(x)$  exist, then  $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$ .  
(Hint: If  $\lim_{x \rightarrow a} f(x) > \lim_{x \rightarrow a} g(x)$  can you derive a contradiction?)