

## CHAPTER NINE

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# Thermodynamic diagrams

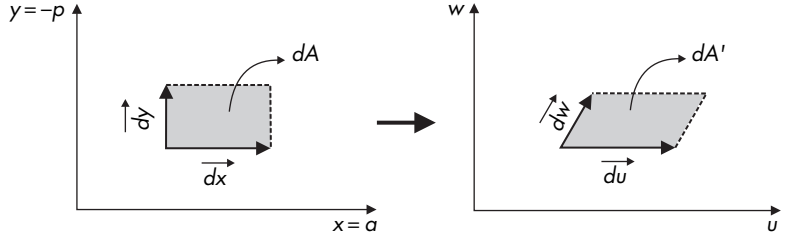
Chapters 1 to 6 laid down the fundamental physical and mathematical concepts pertaining to thermodynamics. While we always kept the discussion close to our atmosphere, it was not until Chapter 7 that, in a more applied mood, we presented how these concepts are applied to yield quantities useful to atmospheric processes. However, even if we understand all the mathematics, we still need an efficient way to present and visualize thermodynamic processes in the atmosphere. Thermodynamic diagrams can do that. Up to this point we made an effort to visualize the thermodynamic process using a  $(p, V)$  or a  $(p, T)$  diagram. However, such diagrams, while simple, may not be very convenient when it comes to their utilization.

Since the purpose of a diagram is efficiently and clearly to display processes and estimate thermodynamic quantities, the following are very desirable in a thermodynamic diagram: (1) for every cyclic process the area should be proportional to work done or energy (area-equivalent transformations), (2) lines should be straight (easy to use), and (3) the angle between adiabats and isotherms should be as large as possible (easy to distinguish). The  $(p, V)$  diagram satisfies the first condition ( $pda = dw$ ), but the angle between isotherms and adiabats is not very large (Figure 4.5(a)). Because of this, while it is used for illustration purposes, this diagram is not used in practice.

### 9.1 Conditions for area-equivalent transformations

When we are constructing a new diagram, in effect, we go from the  $x = a, y = -p$  domain to a new domain characterized by two new coordinates, say  $u$  and  $w$ . These two coordinate systems are shown in Figure 9.1. If we want the first condition to be satisfied then an

**Figure 9.1**  
Area-equivalent  
transformations.



area  $dA = | \vec{dx} \times \vec{dy} |$  in the  $x, y$  domain is mapped onto an area  $dA' = | \vec{du} \times \vec{dw} |$  in the  $u, w$  domain, such that  $dA = \zeta dA'$  ( $\times$  here is the cross product between two vectors and  $\zeta > 0$  is a constant). Since any pair of points  $x$  and  $y$  are transformed to  $u$  and  $w$  we have that  $x = x(u, w)$  and  $y = y(u, w)$ . We can thus write

$$\vec{dx} = \frac{\partial x}{\partial u} \vec{du} + \frac{\partial x}{\partial w} \vec{dw}$$

$$\vec{dy} = \frac{\partial y}{\partial u} \vec{du} + \frac{\partial y}{\partial w} \vec{dw}.$$

Then

$$dA = \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial w} - \frac{\partial x}{\partial w} \frac{\partial y}{\partial u} \right) | \vec{du} \times \vec{dw} |, \quad (9.1)$$

where we have used the facts that

$$\vec{du} \times \vec{du} = \vec{dw} \times \vec{dw} = 0$$

and

$$\vec{du} \times \vec{dw} = - \vec{dw} \times \vec{du}.$$

In equation (9.1)  $| \vec{du} \times \vec{dw} | = dA'$ . Thus,

$$dA = \zeta dA' \quad (9.2)$$

where

$$\zeta = J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \end{vmatrix}$$

is the Jacobian of the coordinate transformation. It follows that the condition for area-equivalence is that the Jacobian should be a constant (if  $J = 1$ , then this is an equal-area transformation). The above approach tells us whether or not the two new coordinates provide an area-equivalent transformation, but both  $u$  and  $w$  must

be specified. If only one is specified, then the following approach provides the other one. Since  $dA = \zeta dA'$  it follows that

$$\oint y dx = \oint -p da = \zeta \oint w du$$

or

$$\oint (p da + \zeta w du) = 0.$$

The above condition is true if  $p da + \zeta w du = dz$  is an exact differential. Thus,  $z = f(a, u)$  and

$$dz(a, u) = \left( \frac{\partial z}{\partial a} \right)_u da + \left( \frac{\partial z}{\partial u} \right)_a du.$$

It follows that the sufficient conditions for an area-equivalent transformation are

$$p = \left( \frac{\partial z}{\partial a} \right)_u$$

and

$$\zeta w = \left( \frac{\partial z}{\partial u} \right)_a.$$

From the above two equations we have

$$\left( \frac{\partial p}{\partial u} \right)_a = \frac{\partial^2 z}{\partial a \partial u}$$

and

$$\zeta \left( \frac{\partial w}{\partial a} \right)_u = \frac{\partial^2 z}{\partial a \partial u}.$$

Thus, if

$$\zeta \left( \frac{\partial w}{\partial a} \right)_u = \left( \frac{\partial p}{\partial u} \right)_a \quad (9.3)$$

then the areas will be equivalent (again here if  $\zeta = 1$  the areas will be equal). From equation (9.3) if we specify  $u$  we can determine what  $w$  should be in order for the transformation to be an area-equivalent transformation.

## 9.2 Examples of thermodynamic diagrams

### 9.2.1 The tephigram

If we suppose that  $u = T$  then

$$\left( \frac{\partial p}{\partial u} \right)_a = \left( \frac{\partial p}{\partial T} \right)_a.$$

Using the ideal gas law, the above leads to

$$\left(\frac{\partial p}{\partial u}\right)_a = \frac{R}{a}.$$

From equation (9.3) we then have

$$\zeta \left(\frac{\partial w}{\partial a}\right)_T = \frac{R}{a}$$

or

$$\zeta \left(\frac{\partial w}{\partial a}\right)_T da = \frac{R}{a} da$$

or after integrating

$$\zeta w = R \ln a + f(T) \quad (9.4)$$

where  $f(T)$  is the constant of integration which we can choose. By manipulating Poisson's equation we can write

$$\frac{T}{\theta} = \left(\frac{RT}{1000a}\right)^k$$

or

$$\ln a = \frac{1}{k}(\ln \theta - \ln T) + \ln T + \ln R - \ln 1000$$

or

$$R \ln a = c_p \ln \theta + f'(T).$$

If we combine the above equation with equation (9.4) and choose  $f(T) = -f'(T)$  and  $\zeta = c_p$ , we arrive at an area-equivalent diagram with coordinates

$$w = \ln \theta \quad (9.5)$$

$$u = T.$$

Obviously, in this diagram the dry adiabats are straight lines and perpendicular to the isotherms (Figure 9.2). The equation for the isobars can be derived from Poisson's equation for  $p = \text{constant}$ :

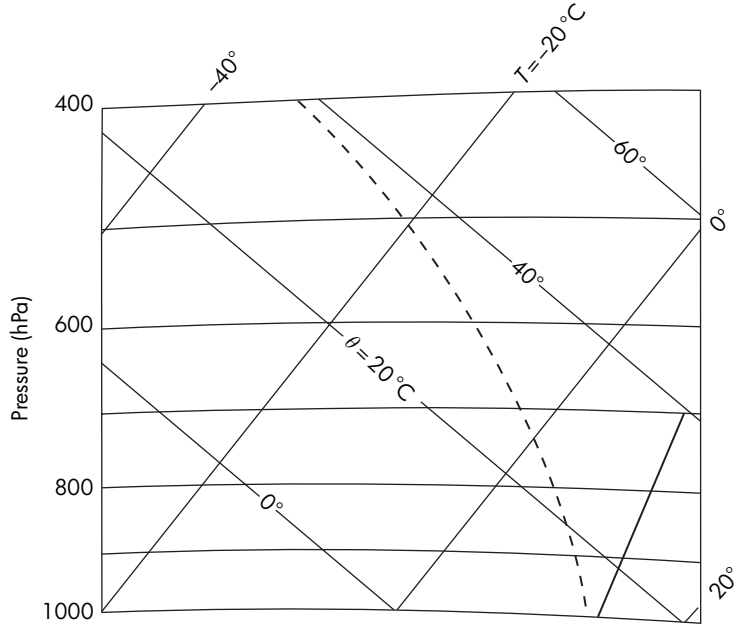
$$\ln \theta = \ln T + \text{constant}.$$

In a coordinate system  $(\ln \theta, T)$  the above equation describes logarithmic curves. As expected, the work done on a unit mass of air in a reversible cycle  $(-q)$  is proportional to the cycle:  $-q = -\oint T ds = -c_p \oint T d \ln \theta = -c_p A_{\text{teph}}$  where  $A_{\text{teph}}$  is the area of the cycle in the tephigram\* (positive if described counterclockwise). If the air is dry,  $c_p = c_{pd}$ .

\* The word "tephigram" originated from the coordinates of this diagram, which are  $T$  and entropy. Entropy was originally denoted as  $\phi$  (phi), but is now denoted by  $S$ .

**Figure 9.2**

Isobars, isotherms, and dry adiabats on a tephigram. Also shown is the  $w_s = 10 \text{ g kg}^{-1}$  saturation mixing ratio line (solid bold line) and the pseudo-adiabat  $\theta_e = 40^\circ \text{C}$  (broken line).



### 9.2.2 The emagram

Let us assume again that  $u = T$ . By taking the logarithm of the equation of state we get

$$\ln a = -\ln p + \ln R + \ln T.$$

Combining this equation with equation (9.4) yields

$$w = -\zeta^{-1} R \ln p + \zeta^{-1} [R \ln R + R \ln T + f(T)].$$

If we now choose  $f(T) = -R \ln R - R \ln T$  and  $\zeta = R$ , we arrive at the area-equivalent diagram with coordinates

$$\begin{aligned} w &= -\ln p \\ u &= T. \end{aligned} \tag{9.6}$$

In this diagram (which is called an emagram – Energy per unit MAss diaGRAM) the isobars are parallel straight lines and perpendicular to the isotherms. The dry adiabats are again derived from Poisson's equation

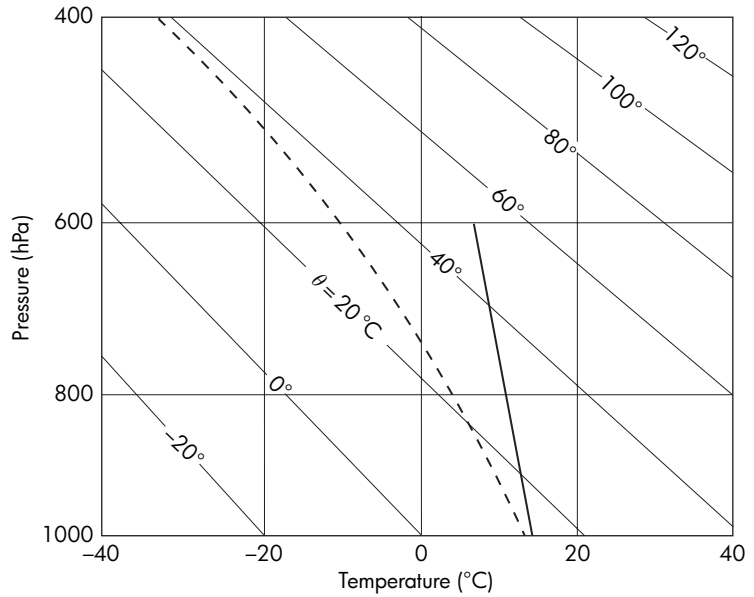
$$\ln \theta = \ln T + k_d \ln 1000 - k_d \ln p$$

or

$$-\ln p = -\frac{1}{k_d} \ln T + \text{constant}.$$

**Figure 9.3**

Same as Figure 9.2 but for an emagram.



Since the coordinates are  $\ln p$  and  $T$ , the above equation is not a straight line. In the emagram the dry adiabats are slightly curved lines (Figure 9.3). Their angle with the isotherms is about  $45^\circ$ . Here, again, the work done on a unit mass of air in a reversible cycle ( $-\oint p da$ ) is proportional to the area of the cycle:

$$\begin{aligned}
 -\oint p da &= \oint [-d(pa) + adp] \\
 &= \oint [-RdT + adp] \\
 &= -\oint RdT + \oint adp \\
 &= \oint adp \\
 &= \oint RT \frac{dp}{p} \\
 &= -R \oint T d(-\ln p) \\
 &= -RA_{\text{em}}
 \end{aligned}$$

where  $A_{\text{em}}$  is the area in the emagram enclosed by the cycle, which is positive if described counterclockwise. If the air is dry,  $R = R_d$ .

### 9.2.3 The skew emagram (skew $(T-\ln p)$ diagram)

Let us now assume that  $u = -\ln p$ . Then

$$\zeta \left( \frac{\partial w}{\partial a} \right)_{\ln p} = - \left( \frac{\partial p}{\partial \ln p} \right)_a$$

or

$$\zeta \left( \frac{\partial w}{\partial a} \right)_{\ln p} = -p$$

or

$$\zeta \left( \frac{\partial w}{\partial a} \right)_{\ln p} da = -p da.$$

Integrating for  $p = \text{constant}$  yields

$$\zeta w = -pa + F(\ln p)$$

or

$$\zeta w = -RT + F(\ln p).$$

We may choose  $F(\ln p) = -\xi \ln p$  where  $\xi$  is a constant. Then

$$\zeta w = -RT - \xi \ln p.$$

If we choose  $\zeta = R$ , we arrive at a new diagram with coordinates

$$w = -T - \xi/R \ln p$$

$$u = -\ln p.$$

Since the sign of an area in a diagram involves only the direction in which a cycle is carried out, the new coordinate system can be written as

$$w = T + \frac{\xi}{R} \ln p \quad (9.7)$$

$$u = -\ln p.$$

This represents a correction (in fact a rotation) to the emagram which provides larger angles between any adiabats and isotherms (Figure 9.4). An isotherm in this diagram has the equation

$$w = \text{constant} + \frac{\xi}{R} \ln p$$

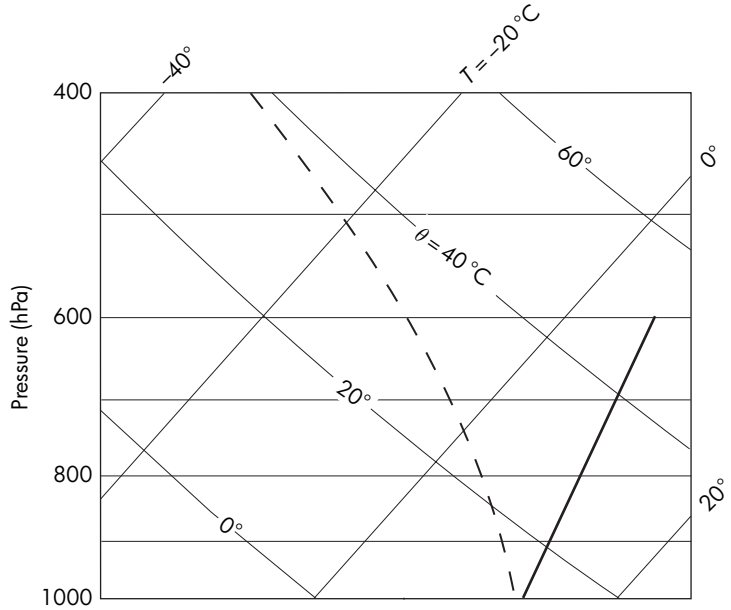
or

$$w = \text{constant} - \frac{\xi}{R} u.$$

In the  $(u, w)$  domain this equation represents a straight line with a slope equal to  $\xi/R$ . Thus, we may choose a  $\xi$  for which the isotherms are perpendicular to the dry adiabats. Note that the equation for dry adiabats is not linear. From Poisson's equation we have that  $\ln \theta = \ln T + k_d \ln 1000 - k_d \ln p$  or  $\ln p = (1/k_d) \ln T + \text{constant}$ .

**Figure 9.4**

Same as figure 9.2 but for a skew ( $T-\ln p$ ) diagram.



While here  $\ln p$  is a coordinate,  $\ln T$  is not and so the dry adiabats are not exactly straight lines. Here again it is easy to show that the work done in a cyclic process is proportional to the area of the cycle. The skew emagram and tephigram are the most commonly used diagrams. Other diagrams have been proposed (see problems 9.1–9.3), but their practical use is limited. A detailed skew  $T-\ln p$  diagram is supplied at the end of the book (Figure A.1).

For a complete description of a thermodynamic process in a diagram, in addition to isotherms, isobars, and adiabats, we also need to define the constant mixing ratio lines and the saturated adiabats (see also Chapter 7). From the definition of mixing ratio we have

$$p = e_{\text{sw}}(T) + \frac{\epsilon e_{\text{sw}}(T)}{w_{\text{sw}}}$$

or

$$p \approx \frac{\epsilon e_{\text{sw}}(T)}{w_{\text{sw}}}. \quad (9.8)$$

For a constant  $w_{\text{sw}}$  the above equation defines a family of curves, which in the  $(T, \ln p)$  domain are approximately (but not quite) straight lines:

$$p \approx \frac{6.11\epsilon}{w_{\text{sw}}} \exp \left( 19.83 - \frac{5417}{T} \right)$$

or

$$\ln p \approx \ln \frac{6.11\epsilon}{w_{\text{sw}}} + 19.83 - \frac{5417}{T}$$



or

$$\ln p \approx A + \frac{B}{T}$$

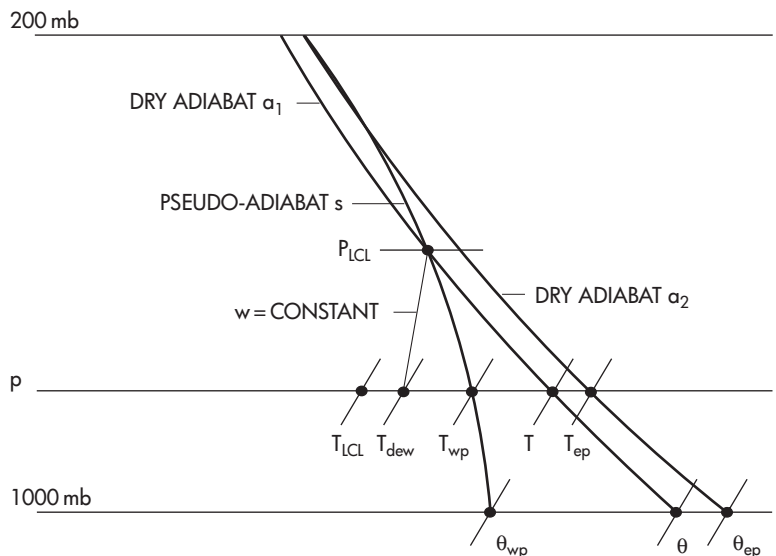
where  $A = \ln(6.11\epsilon/w_{\text{sw}}) + 19.83$  and  $B = -5417$ . The above equation for typical values of  $w_{\text{sw}}$  and  $T$  is nearly a straight line. Since  $w = w_{\text{sw}}(T_{\text{dew}}, p)$ , the dew point of unsaturated air defines the mixing ratio  $w$  at  $(T, p)$ . It follows that the mixing ratio at  $(T, p)$  defines the dew point temperature.

The saturated adiabats are determined numerically from equation (7.63). They can be labeled in terms of the estimated  $\theta_{\text{ep}}$  or in terms of the related (through equation (7.64))  $\theta_{\text{wp}}$ .

### 9.3 Graphical representation of thermodynamic variables in a $T-\ln p$ diagram

Figure 9.5 shows the graphical procedure to obtain all the thermodynamic variables derived analytically in Chapter 7, for a skew  $T-\ln p$  diagram. We start with the state  $(T, p)$  where the air is assumed to be unsaturated and to have a mixing ratio  $w$ . The intersection between the  $w = \text{constant}$  line and the temperature axis gives the dew point temperature. By following the dry adiabat passing through point  $(T, p)$  (line  $a_1$ ) down to 1000 mb we obtain the potential temperature  $\theta$ . By following it up until we intersect the  $w = \text{constant}$  line we arrive at the lifting condensation level ( $p_{\text{LCL}}$ ), where the parcel first becomes saturated and has a temperature  $T_{\text{LCL}}$ . Through point  $(T_{\text{LCL}}, p_{\text{LCL}})$  passes a pseudo-adiabat  $s$

**Figure 9.5**  
Graphical procedure to obtain thermodynamic variables, from a skew  $(T-\ln p)$  diagram, given an initial state  $(T, p)$ .

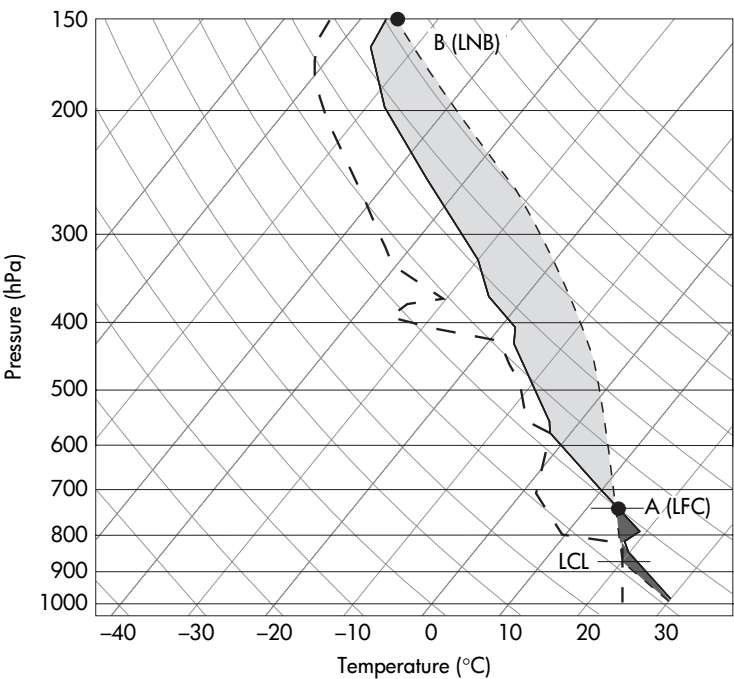


defined by  $\theta_{ep}$ . If we follow this pseudo-adiabat down to the original level  $p$ , we obtain the pseudo-wet-bulb temperature,  $T_{wp}$ , and continuing down to the 1000 mb level we obtain the pseudo-wet-bulb potential temperature,  $\theta_{wp}$  (this also indicates the one-to-one relationship between  $\theta_{wp}$  and  $\theta_{ep}$  hence the labeling of pseudo-adiabats in terms of either  $\theta_{ep}$  or  $\theta_{wp}$ ). If we follow the pseudo-adiabat  $s$  up to a pressure level where we can assume that all vapor has condensed and has fallen out of the parcel (typically this level is taken to be the 200 mb level), and then follow the dry adiabat passing through that point (line  $a_2$ ) down to the original level we obtain the pseudo-equivalent temperature,  $T_{ep}$ . Continuing down to the 1000 mb level yields the pseudo-equivalent potential temperature,  $\theta_{ep}$ . Note that  $T_{ei}$ ,  $T_w$ ,  $\theta_w$  and  $\theta_e$  are not directly estimated in diagrams. As we discussed in Chapter 7, even though they are very close to each other, in actuality  $T_{ei} \neq T_{ep}$ ,  $T_w \neq T_{wp}$ ,  $\theta_w \neq \theta_{wp}$ , and  $\theta_{ei} \neq \theta_{ep}$  (recall equation (7.66)).

9.3.1 Using diagrams in forecasting

Figure 9.6 shows in a skew  $T$ - $\ln p$  diagram a sounding taken at Davenport, IA, on 21 June 1997 at 0000 UTC. The solid line is the vertical profile of environmental temperature and the heavy broken

**Figure 9.6**  
Sounding at Davenport, IA, on 21 June 1997 at 0000 UTC. As explained in the text, the information provided by this diagram would not support the development of deep convection that day. Nevertheless, deep convection did occur later in the day and very severe weather resulted in parts of the Midwest. See text for details.



line to its left is the vertical profile of dew point temperature. The parcels at the surface are unsaturated. The lifting condensation level of these parcels is about 860 mb where  $T_{\text{LCL}} = 20^\circ\text{C}$ . Above the LCL the light broken line is the pseudo-adiabat passing through the point  $(T_{\text{LCL}}, p_{\text{LCL}})$  and below the LCL the light broken line is the dry adiabat passing through point  $(T, p)$  at the surface. The pseudo-adiabat intersects the temperature profile at point  $A$ . Up to this point the parcel is colder than the environment. As such the layer from surface to point  $A$  is stable. This means that the buoyancy in this layer is negative and that work must be done against the buoyancy if parcels originating at the surface are to reach that level. The work needed is proportional to the darker shaded area (negative area). This work or the magnitude of this negative area is often called the *convective inhibition energy* (CINE) given by equation (8.17) for  $i = p_{\text{surface}}$  and  $f = p_A$  (here and in practice we make no distinction between  $T$  and  $T_{\text{virt}}$ ).

$$\text{CINE} = R \int_{p_{\text{surface}}}^{p_A} (T' - T) d \ln p.$$

In our example,  $p_{\text{surface}} \approx 980$  mb and  $p_A \approx 740$  mb. Assuming that the average  $T' - T$  in this layer is about  $-0.8^\circ\text{C}$  and that  $R \approx R_d$ , we find that  $\text{CINE} \approx 64 \text{ J kg}^{-1}$ . If we now recall equation (8.15) we have that

$$\text{CINE} = \int_{p_{\text{surface}}}^{p_A} \frac{dv}{dt} v dt = \frac{1}{2} v_A^2 - \frac{1}{2} v_{\text{surface}}^2$$

where  $v$  is the vertical velocity. Assuming that  $v_A \approx 0$  it follows that if a parcel that originated at the surface were to reach point  $A$ , the *minimum* initial impulse has to be

$$v_{\text{min}} = \sqrt{2 \text{CINE}} \approx 11.4 \text{ m s}^{-1}.$$

Therefore, unless an initial impulse of  $11.4 \text{ m s}^{-1}$  is applied to the parcels they will not make it to point  $A$ . This point is called the level of free convection (LFC). Above this level the parcel is warmer than the environment all the way up to 150 mb. The buoyancy from LFC to 150 mb is positive and now work is done by buoyancy. This work is proportional to the lighter shaded area (positive area). In this layer the parcels that made it to LFC are now free to rise and accelerate. Note that owing to the temperature inversion in the stratosphere, above 150 mb the temperature normally increases with height. As such, there is a point ( $B$  in our example) where the temperatures of the parcel and the environment become equal again. Above this level, which is referred to as the level of neutral buoyancy (LNB), work is done against buoyancy and the parcels

begin to decelerate. The magnitude of the positive area is called the convective available potential energy (CAPE) and is given by

$$\text{CAPE} = -R \int_{\text{LFC}}^{\text{LNB}} (T' - T) d \ln p.$$

Assuming that between LFC and LNB the average difference between the temperature of the parcels and the environment is about  $7^\circ\text{C}$  and that  $R \approx R_d$  we can estimate that  $\text{CAPE} \approx 3200 \text{ J kg}^{-1}$ . As we did above, we can show that in this case the *maximum* velocity a parcel will attain is

$$v_{\text{max}} = \sqrt{2 \text{CAPE}} \approx 80 \text{ m s}^{-1}.$$

Both CINE and CAPE are very useful as they provide information on whether or not convection will occur (CINE) and on how severe a storm might become (CAPE).

In our example CAPE is rather high. Thus, if the parcels make it to LFC, they will then accelerate greatly and deep convection will ensue. The problem in predicting deep convection in this particular example, however, is that the value of CINE is considerable. Such values may inhibit development of convection. In practice, situations of high CAPE values should be monitored to see whether the negative area is removed by warming of the air in the lower levels, or whether upward acceleration is aided by a low level jet or frontogenesis. In our case all of the above happened thereby allowing deep convection and severe weather to develop later in the day in the Midwest.

Together with CAPE and CINE we may calculate the precipitable water from soundings. If we recall the definition of precipitable water in problem 7.19 and the definition of specific humidity in equation (7.2) we have

$$d_w = \frac{1}{\rho_w} \int_0^\infty q \rho dz$$

or using the hydrostatic approximation

$$d_w = \frac{1}{g \rho_w} \int_0^{p_{\text{surface}}} q dp.$$

The above equation indicates that precipitable water can be estimated by integrating the specific humidity profile. The specific humidity profile as a function of pressure can be obtained from the vertical profiles of  $T$  and  $T_d$  using the equations presented in Chapter 7.

### Example

(9.1) An air parcel at 1000 mb has a temperature of  $20^\circ\text{C}$  and a mixing ratio of  $10 \text{ g kg}^{-1}$ . The parcel is lifted to 700 mb by passing

over a mountain. During the ascent 80% of the water vapor that condenses falls out of the parcel. Find the temperature and potential temperature of the air when it returns to 1000 mb on the other side of the mountain.

The parcel is located at the point with coordinates  $T = 20^\circ\text{C}$  and  $p = 1000\text{ mb}$  ( $w = 10\text{ g kg}^{-1}$ ). At this point, the saturation mixing ratio is  $\sim 16\text{ g kg}^{-1}$ . Therefore, the parcel is unsaturated and as it begins to rise it will follow the dry adiabat at that point (which because we are at 1000 mb is  $20^\circ\text{C}$ ). The point at which this dry adiabat intersects the  $w = 10\text{ g kg}^{-1}$  line is at  $p = 910\text{ mb}$  and  $T = T_{\text{LCL}} \approx 12.4^\circ\text{C}$ . This is the lifting condensation level and from that point on the parcel will follow the corresponding pseudo-adiabat ( $\theta_{\text{ep}} = 49.5^\circ\text{C}$ , which in the skew  $T\text{--}\ln p$  diagram is labeled in terms of the related  $\theta_{\text{wp}} = 16.5^\circ\text{C}$ ). At 700 mb the saturation mixing ratio is  $\sim 6.66\text{ g kg}^{-1}$ . Therefore,  $10 - 6.66 = 3.34\text{ g}$  of vapor per kg of dry air have condensed during the saturated ascent from 910 mb to 700 mb. Since 80% of this amount is removed by precipitation, it follows that  $0.668\text{ g kg}^{-1}$  of liquid water remains in the parcel as it begins its descent. This means that initially the descent will be dictated by the same saturated adiabat until all the water evaporates, i.e. until the mixing ratio increases from 6.66 to  $6.66 + 0.668 = 7.328\text{ g kg}^{-1}$ . This will happen at the point where the pseudo-adiabat intersects the  $w = 7.328$  line. This is the point with coordinates  $p \approx 750\text{ mb}$  and  $T \approx 4^\circ\text{C}$ . From this point on, the descent is dictated by the corresponding dry adiabat ( $\theta \approx 28^\circ\text{C}$ ). Thus, on the other side of the mountain at  $p = 1000\text{ mb}$ ,  $T = \theta = 28^\circ\text{C}$ .

### Problems

- (9.1) Show that the Refsdal diagram with coordinates  $u = \ln T$ ,  $w = -T \ln p$  is an area-equivalent diagram and that neither its isobars nor its dry adiabats are straight lines.
- (9.2) Show that the Stüve (or pseudo-adiabatic) diagram with coordinates  $u = T$ ,  $w = -p^{k_d}$  is not an area-equivalent diagram. Show that isobars, isotherms, and dry adiabats are all straight lines.
- (9.3) Using the Jacobian method show that the diagram with coordinates  $u = T$ ,  $w = c_p \ln \theta$  is an equal-area diagram.
- (9.4) Given the sounding shown in the table,

$p(\text{mb})$	$T(^{\circ}\text{C})$
950	22.5
900	18.0
850	15.0
800	16.0
750	12.0
700	7.0
650	4.0
600	-1.5
500	-10.0
400	-20.0

determine using a diagram (1) the mixing ratio, (2) the relative humidity, (3) the potential temperature, (4) the pseudo-wet-bulb temperature, (5) the pseudo-wet-bulb potential temperature. Having determined the above, compute the wet-bulb temperature, the equivalent temperature, the pseudo-equivalent potential temperature, and the pseudo-equivalent temperature. Assume that  $l_v = 2.45 \times 10^6 \text{ J kg}^{-1}$ ,  $c_w = 4218 \text{ J kg}^{-1}$ , and that the dew point at 950 mb is  $15.7^{\circ}\text{C}$ . ( $12.4 \text{ g kg}^{-1}$ ,  $0.68$ ,  $27.5^{\circ}\text{C}$ ,  $18^{\circ}\text{C}$ ,  $20^{\circ}\text{C}$ ,  $19^{\circ}\text{C}$ ,  $51^{\circ}\text{C}$ ,  $66^{\circ}\text{C}$ ,  $60^{\circ}\text{C}$ )

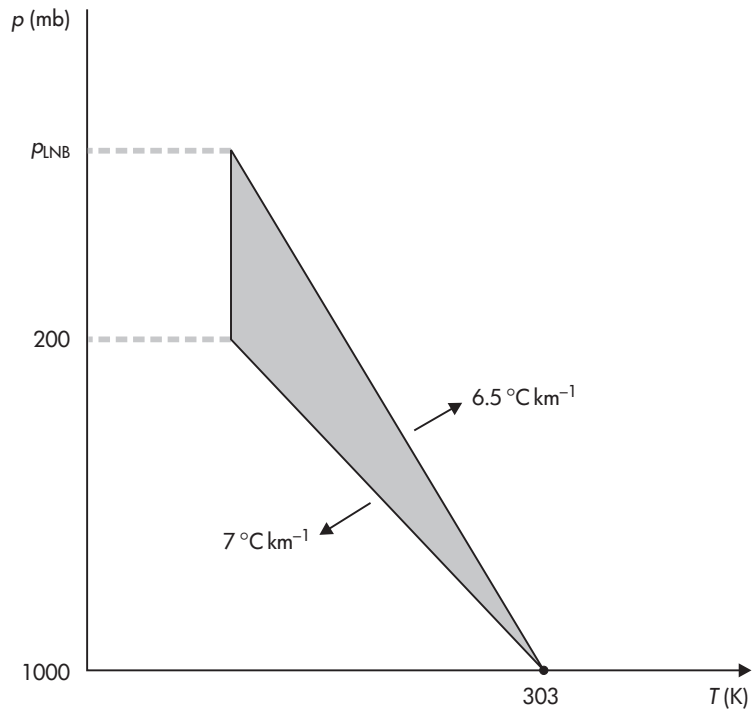
- (9.5) Approximate CAPE if in the previous problem we extend the sounding as follows:

$p(\text{mb})$	$T(^{\circ}\text{C})$
300	-35.0
200	-55.0
150	-60.0

Would you expect deep convection to occur?

- (9.6) Consider the following scenario (Figure 9.7). On a given day, on a skew  $T-\ln p$  diagram the temperature profile from the surface (1000 mb) to 200 mb has a constant lapse rate of  $7^{\circ}\text{C km}^{-1}$ . Above 200 mb the temperature is constant. If the parcels at the surface are saturated, what will the maximum vertical velocity be on that day assuming that the saturated adiabatic ascent can be approximated by the constant lapse rate of  $6.5^{\circ}\text{C km}^{-1}$ ? The temperature at the surface is  $30^{\circ}\text{C}$ . Assume  $R = R_d$  ( $\sim 54 \text{ m s}^{-1}$ ).
- (9.7) A parcel of air with an initial temperature of  $15^{\circ}\text{C}$  and dew point  $0^{\circ}\text{C}$  is lifted adiabatically from 1000 mb. Determine the lifting condensation level and temperature at that level. If the parcel is lifted a further 250 mb what will its final

**Figure 9.7**  
Figure for problem 9.6



temperature be and how much liquid water is produced? ( $\sim 800$  mb,  $\sim -3^\circ\text{C}$ ,  $\sim -22^\circ\text{C}$ ,  $\sim 2.7 \text{ g kg}^{-1}$ )

- (9.8) In the previous problem if all the condensed water falls as precipitation, can you roughly estimate the height of water collected on a rain collector of area  $1 \text{ m}^2$ ? Here you have to make your own assumptions. How does this result compare with the result obtained using the formula in problem 7.19? Can you explain any differences?
- (9.9) An air parcel at 900 mb has a temperature of  $10^\circ\text{C}$  and a mixing ratio of  $5 \text{ g kg}^{-1}$ . The parcel is then lifted to 700 mb by passing a mountain. If in this process 70% of the condensed water is removed by precipitation, what will its temperature, potential temperature, and pseudo-wet-bulb potential temperature be when it returns to its initial level of 900 mb on the other side of the mountain? ( $\sim 14^\circ\text{C}$ ,  $\sim 23.5^\circ\text{C}$ ,  $\sim 11^\circ\text{C}$ )
- (9.10) If a temperature profile in  $^\circ\text{C}$  could be described as  $T(z) = 28 - 8z$  for  $z \geq 1 \text{ km}$  and  $T = 20^\circ\text{C}$  for  $z < 1 \text{ km}$ , and the mixing ratio at the surface is  $16 \text{ g kg}^{-1}$ , estimate using a diagram the level of free convection. ( $\sim 2.4 \text{ km}$ )





## CHAPTER TEN

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# Beyond this book

The previous chapters have presented the basics in atmospheric thermodynamics. As we know, in atmospheric sciences the ultimate goal is to predict as accurately as possible the changes in weather and climate. Thermodynamic processes are crucial in predicting changes in weather patterns. For example, during cloud and precipitation formation vast amounts of heat are exchanged with the environment that affect the atmosphere at many different spatial scales. In this final chapter we will present the basic concepts behind predicting weather changes. This chapter is not meant to treat the issue thoroughly, but only to offer a glimpse of what comes next.

### 10.1 Basic predictive equations in the atmosphere

In the Newtonian framework the state of the system is described exactly by the position and velocity of all its constituents. In the thermodynamical framework the state is defined by the temperature, pressure, and density of all its constituents. In a dynamical system such as the climate system both frameworks apply. Accordingly, a starting point in describing such a system will be to seek a set of equations that combine both the mechanical motion and thermodynamical evolution of the system.

The fundamental equations that govern the motion and evolution of the atmosphere (and for that matter of the oceans and sea ice) are derived from the three basic conservation laws: the conservation of momentum, the conservation of mass, and the conservation of energy. For the atmosphere the equation of state relates temperature, density, and pressure. In summary, from the conservation of momentum we derive the following set of predictive equations of motion (Washington and Parkinson (1986))

$$\frac{du}{dt} - \left( f + u \frac{\tan \phi}{a} \right) v = -\frac{1}{a \cos \phi} \frac{1}{\rho} \frac{\partial p}{\partial \lambda} + F_\lambda \quad (10.1)$$

$$\frac{dv}{dt} + \left( f + u \frac{\tan \phi}{a} \right) u = -\frac{1}{\rho a} \frac{\partial p}{\partial \phi} + F_\phi \quad (10.2)$$

$$\frac{d\omega}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g + F_z \quad (10.3)$$

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \phi} + \omega \frac{\partial}{\partial z}$$

$$u = a \cos \phi \frac{d\lambda}{dt}$$

$$v = a \frac{d\phi}{dt}$$

$$\omega = \frac{dz}{dt}$$

where  $u, v$ , and  $\omega$  are the horizontal and vertical components of the motion,  $\phi$  is the latitude,  $\lambda$  is the longitude,  $a$  is the radius of the earth,  $f$  is the Coriolis force, and  $F$  is the friction force. In the above equations the forces that drive the motion are the local pressure gradients, gravity, Coriolis, and friction.

Equation (10.3) can be approximated by the hydrostatic equation (by assuming that  $d\omega/dt = 0$  and  $F_z = 0$ )

$$g = -\frac{1}{\rho} \frac{\partial p}{\partial z} \quad (10.4)$$

which relates density to pressure.

From the law of conservation of mass we can derive the continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{1}{a \cos \phi} \left[ \frac{\partial}{\partial \lambda} (\rho u) + \frac{\partial}{\partial \phi} (\rho v \cos \phi) \right] - \frac{\partial}{\partial z} (\rho \omega) \quad (10.5)$$

which provides a predictive equation for density. The last two equations in this basic formulation come from thermodynamics. The first law of thermodynamics (which expresses the law of conservation of energy)

$$C_p \frac{dT}{dt} - \frac{1}{\rho} \frac{dp}{dt} = \frac{dQ}{dt}, \quad (10.6)$$

where  $dQ/dt$  is the net heat gain, provides a predictive equation for temperature. The above equations (10.1, 10.2, 10.3 (or 10.4),

10.5, and 10.6) make a system of five equations with six unknowns ( $u, v, \omega, p, \rho, T$ ). The equation of state

$$p = \rho RT \quad (10.7)$$

provides the additional equation that connects pressure, density, and temperature, thus resulting in a system of six equations (called the *primitive equations system*), with six unknowns (assuming of course that  $dQ/dt$ ,  $F_\lambda$ , and  $F_\phi$  are constants and known). Note, however, that this system is not a closed one because  $dQ/dt$ ,  $F_\lambda$ , and  $F_\phi$  must be determined from the other variables. These terms are very important for climate simulations but for short-term weather prediction they are often ignored.

## 10.2 Moisture

The above system of predictive equations was derived without the inclusion of moisture. Even though a model can be derived without moisture in it, including moisture can dramatically improve modeling and prediction. In a manner analogous to the continuity of mass, the changes of moisture must be balanced by the moisture's sources and sinks. An equation of continuity for water vapor mixing ratio can be written as

$$\frac{dw}{dt} = \frac{1}{\rho} M + E \quad (10.8)$$

where  $M$  is the time rate of change of water vapor per unit volume due to condensation or freezing,  $E$  is the time rate of change of water vapor per unit mass due to evaporation from the surface and to horizontal and vertical diffusion of moisture occurring on scales below the resolution of the model, and  $w$  is the mixing ratio. Often the above equation is combined with equation (10.5) to obtain

$$\frac{\partial(\rho\omega)}{\partial t} + \frac{1}{a \cos \phi} \left[ \frac{\partial}{\partial \lambda}(\rho\omega u) + \frac{\partial}{\partial \phi}(\rho\omega v \cos \phi) \right] + \frac{\partial}{\partial z}(\rho\omega w) = M + \rho E. \quad (10.9)$$

When water vapor changes to water (or ice) and vice versa, heat is added to or removed from the atmospheric system. In this case  $dQ = d(lw_s/T)$ .

The above general and simple equations constitute only the starting point for studying and investigating the fascinating dynamics of atmospheric motion. The equations have to be modified and adjusted in order to include the effect of many other factors (for example, radiative processes) and to be applied over a specific range of scales (mesoscale models, for example). Such studies are crucial in the effort to understand the climate and ecology of our planet.