CHAPTER EIGHT

Vertical stability in the atmosphere

Over large scales the atmosphere is very nearly hydrostatic, which means that the pressure gradient force balances the force of gravity. Because the net force in the vertical is zero, the atmosphere over those scales exhibits slow vertical motions of constant speed, i.e. there is no upward acceleration of air. However, on smaller scales hydrostatic equilibrium may be invalidated. In this case, accelerated motion leads to convection which embraces many phenomena observed in the atmosphere, from the structure of planetary boundary layers to the dynamics of hurricanes. In this chapter we will investigate the conditions that allow such accelerating motions. In particular, we will examine the fate of a parcel at equilibrium with its environment when it is subjected to a small perturbation. We will make the following assumptions: (1) the environment is in hydrostatic equilibrium; (2) the parcel does not mix with its surroundings; (3) the parcel's movement does not disturb the environment; (4) the process is adiabatic; and (5) at a given level the pressure of the environment and the pressure of the parcel are equal. In problem 8.1 you will be asked to elaborate on the validity of the above assumptions.

8.1 The equation of motion for a parcel

Because the environment is in hydrostatic equilibrium the following equation holds:

$$\frac{dp}{dz} = -\rho g.$$

For the parcel, the above equation is not applicable because as the parcel is displaced (upwards or downwards) it has some acceleration (d^2z/dt^2). Assuming that gravity and the pressure gradient force are acting on the parcel, Newton's second law dictates that per unit volume (primes identify the parcel)

$$\rho' \frac{d^2 z}{dt^2} = -\rho' g - \frac{dp'}{dz}$$

or

$$\ddot{z} = -g - a' \frac{dp'}{dz}.$$

Because of assumption (5) above, $dp'/dz = dp/dz$. Thus, the above equation can be written as

$$\ddot{z} = -g - a' \left(-\frac{g}{a} \right)$$

or

$$\ddot{z} = g \left(\frac{a' - a}{a} \right)$$

or

$$\ddot{z} = g \left(\frac{\rho - \rho'}{\rho'} \right). \tag{8.1}$$

The right-hand side gives the force per unit mass acting on the parcel due to the combination of gravity and pressure gradient and is called the *buoyancy* of the parcel. Using the definition of virtual temperature and assumption (5), we can write the ideal gas law for both environment and parcel as:

$$p = \rho R_{\mathrm{d}} T_{\mathrm{virt}}$$

and

$$p = \rho' R_{\mathrm{d}} T'_{\mathrm{virt}}. \tag{8.2}$$

The reason for adopting these expressions rather than $p = \rho R T$ and $p = \rho' R' T'$ is that this way we deal with T_{virt} and T'_{virt} rather than R, R', T, and T'. This makes the analysis more straightforward. Combining equations (8.1) and (8.2) yields

$$\ddot{z} = g \left(\frac{T'_{\mathrm{virt}} - T_{\mathrm{virt}}}{T_{\mathrm{virt}}} \right). \tag{8.3}$$

What we are interested in here is to investigate the effect on the motion, described by equation (8.3), of small displacements ($z \ll 1$) from an original equilibrium level. If, for simplicity, we take this level to be the $z = 0$ level where the temperature is $T_{\mathrm{virt},0}$ and express both T_{virt} and T'_{virt} in terms of Taylor's series we get

$$T_{\mathrm{virt}} = T_{\mathrm{virt},0} + \frac{dT_{\mathrm{virt}}}{dz} z + \frac{1}{2} \frac{d^2 T_{\mathrm{virt}}}{dz^2} z^2 + \cdots$$

and

$$T'_{\mathrm{virt}} = T_{\mathrm{virt},0} + \frac{dT'_{\mathrm{virt}}}{dz} z + \frac{1}{2} \frac{d^2 T'_{\mathrm{virt}}}{dz^2} z^2 + \cdots$$

If we now neglect terms of order higher than the first and define the environmental virtual temperature lapse rate as $-dT_{\mathrm{virt}}/dz = \mathit{\Gamma}_{\mathrm{virt}}$ and the parcel's virtual temperature lapse rate as $-dT'_{\mathrm{virt}}/dz = \mathit{\Gamma}'_{\mathrm{virt}}$ we arrive at

$$\ddot{z} = \frac{g(\mathit{\Gamma}_{\mathrm{virt}} - \mathit{\Gamma}'_{\mathrm{virt}})z}{T_{\mathrm{virt},0} - \mathit{\Gamma}_{\mathrm{virt}} z}. \tag{8.4}$$

In the above equation we can manipulate the term $1/(T_{\mathrm{virt},0} - \mathit{\Gamma}_{\mathrm{virt}} z)$ to arrive at

$$\begin{aligned}\frac{1}{T_{\mathrm{virt},0} - \mathit{\Gamma}_{\mathrm{virt}} z} &= \frac{1}{T_{\mathrm{virt},0}} \frac{1}{1 - \frac{\mathit{\Gamma}_{\mathrm{virt}} z}{T_{\mathrm{virt},0}}} \\ &\approx \frac{1}{T_{\mathrm{virt},0}} \left(1 + \frac{\mathit{\Gamma}_{\mathrm{virt}} z}{T_{\mathrm{virt},0}}\right) \quad \text{because} \quad \frac{\mathit{\Gamma}_{\mathrm{virt}} z}{T_{\mathrm{virt},0}} \ll 1.\end{aligned}$$

Then equation (8.4) becomes

$$\ddot{z} = \frac{g}{T_{\mathrm{virt},0}} \left(1 + \frac{\mathit{\Gamma}_{\mathrm{virt}} z}{T_{\mathrm{virt},0}}\right) (\mathit{\Gamma}_{\mathrm{virt}} - \mathit{\Gamma}'_{\mathrm{virt}})\, z$$

or by eliminating terms involving z^2

$$\ddot{z} = \frac{g}{T_{\mathrm{virt},0}} (\mathit{\Gamma}_{\mathrm{virt}} - \mathit{\Gamma}'_{\mathrm{virt}})\, z$$

or

$$\ddot{z} + \frac{g}{T_{\mathrm{virt},0}} (\mathit{\Gamma}'_{\mathrm{virt}} - \mathit{\Gamma}_{\mathrm{virt}})\, z = 0. \tag{8.5}$$

8.2 Stability analysis and conditions

The solution of the differential equation (8.5) depends on the constants. Three possibilities exist.

(1) $\mathit{\Gamma}'_{\mathrm{virt}} - \mathit{\Gamma}_{\mathrm{virt}} > 0$
In this case equation (8.5) takes the form $\ddot{z} + \lambda^2 z = 0$ and has the solution

$$z(t) = A \sin \lambda t + B \cos \lambda t$$

where the oscillatory components are characterized by

$$\lambda = \sqrt{\frac{g}{T_{\mathrm{virt},0}} (\mathit{\Gamma}'_{\mathrm{virt}} - \mathit{\Gamma}_{\mathrm{virt}})} > 0.$$

This is the so-called Brunt–Väisälä frequency. Since we assumed that the initial level is $z = 0$, it follows that in this case $B = 0$ and as such $z(t) = A \sin \lambda t$. This indicates that the parcel will oscillate in time about its original position with a period $\tau = 2\pi/\lambda$. This represents a stable case as the parcel does not leave the original level.

(2) $\Gamma'_{\text{virt}} - \Gamma_{\text{virt}} < 0$
In this case equation (8.5) takes the form $\ddot{z} - \lambda^2 z = 0$ and has the solution

$$z(t) = Ae^{\lambda t} + Be^{-\lambda t}$$

where now

$$\lambda = \sqrt{\frac{g}{T_{\text{virt},0}}(\Gamma_{\text{virt}} - \Gamma'_{\text{virt}})} > 0.$$

Since at $t = 0, z(0) = 0$ it follows that $A+B = 0$. This indicates that $A = -B \neq 0$ (the possibility of $A = B = 0$ leads to the trivial solution $z(t) = 0$ which we ignore). Since $A \neq 0$ it then follows that as $t \to \infty$, a parcel's displacement grows exponentially. Note that since for $t \to \infty$, $dz/dt = A\lambda e^{\lambda t}$, the parcel's motion is an accelerating motion. This is the unstable situation where the parcel leaves the original level and never returns.

(3) $\Gamma'_{\text{virt}} - \Gamma_{\text{virt}} = 0$
In this case equation (8.5) becomes $\ddot{z} = 0$ and has the linear solution

$$z(t) = At + B$$

indicating that the displacement grows linearly with time. The parcel's motion is now one of a constant speed ($dz/dt = A$). This is the neutral case where the parcel leaves the original level and never returns. Thus, the only difference between neutral and unstable situations is whether or not the motion is an accelerating one.

The physical meaning of the above is that as long as the parcel's lapse rate is greater than that of the environment and the parcel is forced upwards (downwards), it will become colder (warmer) than the environment and it will sink (rise) back to the original level. If the parcel's lapse rate is less than that of the environment and the parcel is forced upwards (downwards) it will become warmer (colder) than the environment and it will continue to rise (sink) thus moving away from the original level. The difference between the two lapse rates is the net force acting on the parcel. If the net force is zero the motion is one of a constant speed. Otherwise the net force acts in the direction of the initial impulse if $\Gamma'_{\text{virt}} - \Gamma_{\text{virt}} < 0$ or in the opposite direction if $\Gamma'_{\text{virt}} - \Gamma_{\text{virt}} > 0$.

Let us now examine in detail what happens when a parcel is displaced in a layer, which is characterized by a virtual temperature lapse rate Γ_{virt}. From equation (7.9) we have that for an *unsaturated* parcel $T'_{\text{virt}} = (1 + 0.61w')T'$, where w' is the initial mixing ratio, which because of assumption (2) remains constant. Thus,

$$\frac{dT'_{\text{virt}}}{dz} = (1 + 0.61w')\frac{dT'}{dz}$$

or

$$\Gamma'_{\mathrm{virt}} = (1 + 0.61w')\Gamma_{\mathrm{m}}$$

or using equation (7.29)

$$\Gamma'_{\mathrm{virt}} = (1 + 0.61w')(1 - 0.87w')\Gamma_{\mathrm{d}}$$

or

$$\Gamma'_{\mathrm{virt}} \approx (1 - 0.26w')\Gamma_{\mathrm{d}}.$$

The term $0.26w'$ in the above relation is rather small and therefore we may neglect it. Thus, for *unsaturated* parcels

$$\Gamma'_{\mathrm{virt}} \approx \Gamma_{\mathrm{d}}. \tag{8.6}$$

Note that for the environment, w is not constant but a function of z. As such, for the environment we have

$$\frac{dT_{\mathrm{virt}}}{dz} = (1 + 0.61w)\frac{dT}{dz} + 0.61T\frac{dw}{dz}$$

or

$$\Gamma_{\mathrm{virt}} = (1 + 0.61w)\Gamma - 0.61T\frac{dw}{dz}. \tag{8.7}$$

The second term on the right-hand side of equation (8.7) is not negligible and therefore we cannot arrive at a relation similar to equation (8.6) for Γ_{virt} and Γ (the environmental lapse rate). This is why we have to use the virtual temperature in our stability analysis rather than the actual temperature. If Γ_{virt} were approximately equal to Γ, then we could use T rather than T_{virt} in our stability analysis. Because this is not the case, it is more appropriate to use virtual temperatures.

We can then state the condition for stability for an *unsaturated* parcel in a layer with Γ_{virt} as follows:

$$\begin{array}{lll} \text{if} & \Gamma_{\mathrm{virt}} > \Gamma_{\mathrm{d}} & \text{layer is unstable,} \\ \text{if} & \Gamma_{\mathrm{virt}} = \Gamma_{\mathrm{d}} & \text{layer is neutral,} \\ \text{if} & \Gamma_{\mathrm{virt}} < \Gamma_{\mathrm{d}} & \text{layer is stable.} \end{array} \tag{8.8}$$

For *saturated* parcels the mixing ratio decreases with height. Proceeding as above we have

$$T'_{\mathrm{virt}} = (1 + 0.61w')T'$$

or

$$\frac{dT'_{\mathrm{virt}}}{dz} = (1 + 0.61w')\frac{dT'}{dz} + 0.61T'\frac{dw'}{dz}$$

or

$$\Gamma'_{\mathrm{virt}} = (1 + 0.61w')\Gamma_{\mathrm{s}} - 0.61T'\frac{dw'}{dz}.$$

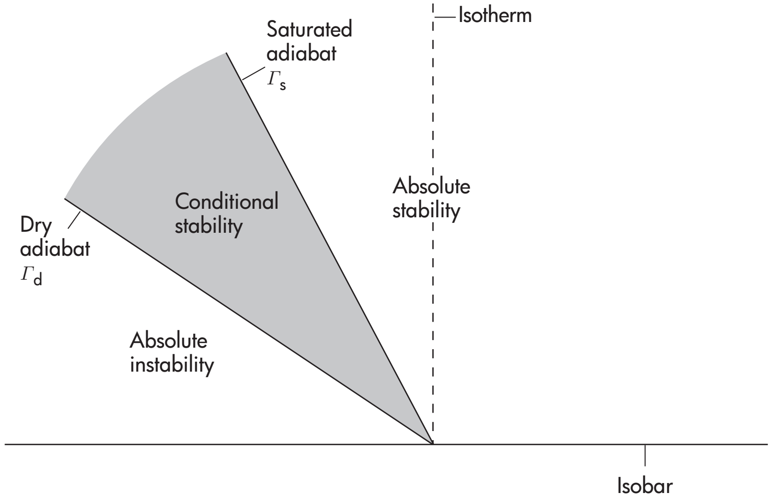

Figure 8.1 Relative position of fundamental lines and regions of stability.

In this case the second term of the right-hand side is much smaller than the first term. Thus, we can approximate the above equation as

$$\Gamma'_{\text{virt}} \approx \Gamma_{\text{s}}.$$

Accordingly, the condition for stability for a *saturated* parcel in a layer with Γ_{virt} can be stated as

$$\begin{array}{lll} \text{if} & \Gamma_{\text{virt}} > \Gamma_{\text{s}} & \text{layer is unstable,} \\ \text{if} & \Gamma_{\text{virt}} = \Gamma_{\text{s}} & \text{layer is neutral,} \\ \text{if} & \Gamma_{\text{virt}} < \Gamma_{\text{s}} & \text{layer is stable.} \end{array} \tag{8.9}$$

Since $\Gamma_{\text{d}} = 9.8\,^\circ\text{C km}^{-1}$ and $\Gamma_{\text{s}} \approx 5\,^\circ\text{C km}^{-1}$, conditions (8.8) and (8.9) can be combined in one as follows:

$$\begin{array}{lll} \text{if} & \Gamma_{\text{virt}} > \Gamma_{\text{d}} & \text{layer is absolutely unstable,} \\ \text{if} & \Gamma_{\text{s}} < \Gamma_{\text{virt}} < \Gamma_{\text{d}} & \text{layer is conditionally unstable,} \\ \text{if} & \Gamma_{\text{virt}} < \Gamma_{\text{s}} & \text{layer is absolutely stable.} \end{array} \tag{8.10}$$

The word “absolutely” indicates that the stability criterion holds for any type (saturated or unsaturated) of parcel. The term “conditionally unstable” means that the layer is stable for displacement of unsaturated parcels and unstable for saturated parcels (Figure 8.1). Now recall that for unsaturated environments equation (7.28) applies:

$$\theta_{\text{virt}} = T_{\text{virt}} \left(\frac{1000}{p} \right)^{k_{\text{d}}}.$$

Logarithmic differentiation of the above equation gives

$$\frac{1}{\theta_{\text{virt}}} \frac{d\theta_{\text{virt}}}{dz} = \frac{1}{T_{\text{virt}}} \frac{dT_{\text{virt}}}{dz} - \frac{k_{\text{d}}}{p} \frac{dp}{dz}$$

or

$$\frac{1}{\theta_{\text{virt}}}\frac{d\theta_{\text{virt}}}{dz} = -\frac{1}{T_{\text{virt}}}\Gamma_{\text{virt}} - \frac{k_{\text{d}}}{p}\left(-\frac{p}{R_{\text{d}}T_{\text{virt}}}g\right)$$

or

$$\frac{1}{\theta_{\text{virt}}}\frac{d\theta_{\text{virt}}}{dz} = -\frac{\Gamma_{\text{virt}}}{T_{\text{virt}}} + \frac{1}{T_{\text{virt}}}\left(\frac{g}{c_{p\text{d}}}\right)$$

or

$$\frac{1}{\theta_{\text{virt}}}\frac{d\theta_{\text{virt}}}{dz} = \frac{1}{T_{\text{virt}}}\left(\Gamma_{\text{d}} - \Gamma_{\text{virt}}\right). \tag{8.11}$$

Combining (8.8) with (8.11) yields an alternative way to express stability conditions for unsaturated parcels:

$$\begin{aligned}
&\text{if} \quad \frac{d\theta_{\text{virt}}}{dz} > 0 \quad \text{layer is stable,}\\
&\text{if} \quad \frac{d\theta_{\text{virt}}}{dz} = 0 \quad \text{layer is neutral,}\\
&\text{if} \quad \frac{d\theta_{\text{virt}}}{dz} < 0 \quad \text{layer is unstable.}
\end{aligned} \tag{8.12}$$

For a saturated parcel similar conditions can be applied if we substitute θ_{virt} by θ_{e} (or in practice θ_{ep}) which is invariant along saturated adiabats (or pseudoadiabats)

$$\begin{aligned}
&\text{if} \quad \frac{d\theta_{\text{e}}}{dz} > 0 \quad \text{layer is stable,}\\
&\text{if} \quad \frac{d\theta_{\text{e}}}{dz} = 0 \quad \text{layer is neutral,}\\
&\text{if} \quad \frac{d\theta_{\text{e}}}{dz} < 0 \quad \text{layer is unstable.}
\end{aligned} \tag{8.13}$$

The above set of criteria applies to the stability as a parcel rises in a motionless layer. If this is not true (for example, when entire layers are lifted or lowered), then the stability of the layer may be affected. Here is why it happens.

Let us consider a stable layer in hydrostatic equilibrium whose bottom has a pressure p_1 and whose top has a pressure p_2. Let us further assume that the difference in pressure, Δp, between top and bottom remains constant during the lifting or sinking. If the whole process is an unsaturated adiabatic process, then equation (8.11) applies. Since in this process θ_{virt} is conserved then the difference in θ_{virt} between top and bottom is conserved. It follows that

$$\frac{1}{\theta_{\text{virt}}}\frac{d\theta_{\text{virt}}}{dz} = \frac{1}{\theta_{\text{virt}}}\frac{d\theta_{\text{virt}}}{dp}(-\rho g) = \frac{1}{T_{\text{virt}}}(\Gamma_{\text{d}} - \Gamma_{\text{virt}})$$

or

$$\frac{1}{\theta_{\mathrm{virt}}}\frac{d\theta_{\mathrm{virt}}}{dp} = -\frac{R_{\mathrm{d}}}{pg}(\Gamma_{\mathrm{d}} - \Gamma_{\mathrm{virt}}) = \text{constant}$$

or

$$R_{\mathrm{d}}(\Gamma_{\mathrm{d}} - \Gamma_{\mathrm{virt}}) = \text{constant} \times g \times p.$$

During lifting pressure decreases, and according to the above equation $\Gamma_{\mathrm{virt}} \to \Gamma_{\mathrm{d}}$. Since the layer is initially stable this tendency will cause the layer's lapse rate to approach the region of instability (recall Figure 8.1). During sinking the opposite is true. As such, lifting of a layer tends always to destabilize the layer and sinking of a layer tends always to stabilize the layer.

If the lifting process is strong enough to cause the layer to saturate, then the picture is quite different. In this case the stability changes depend on the way saturation is reached, which in turn depends on the vertical distribution of moisture and the vertical structure of temperature among other factors.

For example, let us consider an initially stable unsaturated layer at level 1 in which the temperature profile is BT and in which $d\theta_e/dp < 0$ (i.e. the bottom of the layer has more moisture than the top of the layer). As Figure 8.2 illustrates when the layer is lifted, the bottom of the layer (B) and the top of the layer (T)

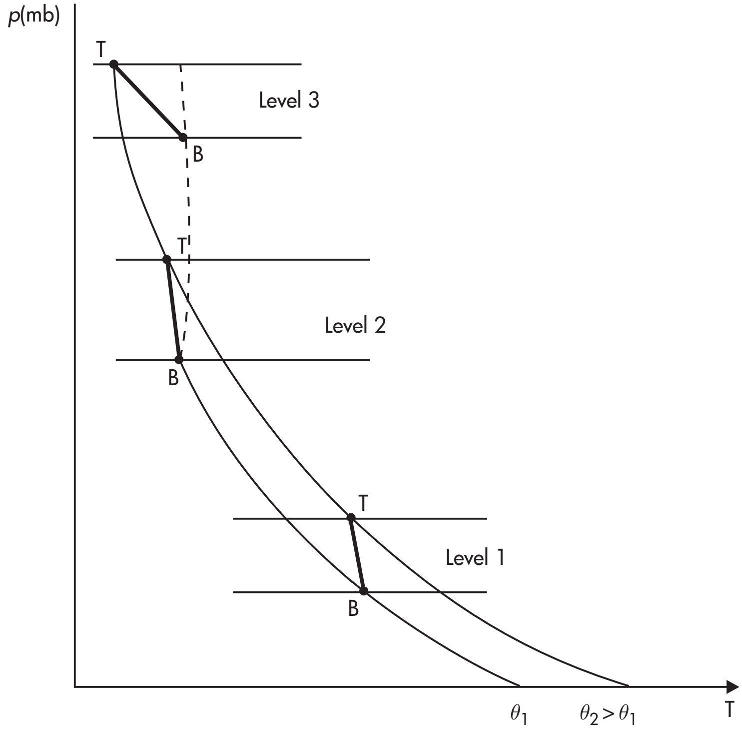

Figure 8.2
Graphical representation of convective instability.

will cool at the dry adiabatic rate (solid lines). However, the bottom will reach saturation sooner than the top. Thus, at level 2 the bottom will start cooling at the moist adiabatic rate (broken line) whereas the top will continue cooling at the dry adiabatic rate. Because of that, at position 3, independently of whether or not the top has reached saturation, the temperature profile will be rotated to the left in relation to the initial profile at level 1. This counterclockwise rotation causes an initially stable situation to become less stable (see Figure 8.1). Thus, when $d\theta_e/dp < 0$ the initially stable unsaturated layer is *potentially* or *convectively unstable.*
If in the same layer $d\theta_e/dz > 0$ (i.e. the top has more moisture than the bottom), then the top will reach saturation first and it will begin to cool at the moist rate while the bottom will continue cooling at the dry rate. In this case the initial temperature profile BT turns clockwise (Figure 8.3), which means that the initially stable layer will tend to become even more stable. Thus, when $d\theta_e/dz > 0$ the layer is *potentially* or *convectively stable.* If in the same layer $d\theta_e/dz = 0$ (i.e. all levels of the layer have the same moisture), then both top and bottom will reach saturation along the same moist adiabat. In this case the initially stable layer is called *potentiallly* or *convectively neutral.*

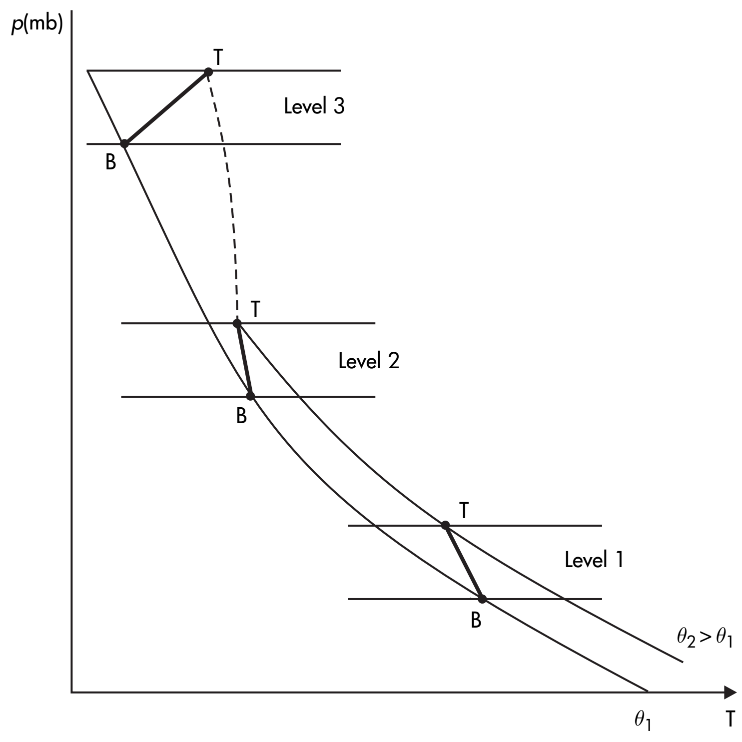

Figure 8.3
Graphical representation of convective stability.

If the whole layer becomes saturated, the stability criteria for the layer translate to the following

if $\dfrac{d\theta_e}{dz} > 0$ the saturated layer will be stable with respect to saturated parcel processes,

if $\dfrac{d\theta_e}{dz} = 0$ the saturated layer will be neutral with respect to saturated parcel processes,

if $\dfrac{d\theta_e}{dz} < 0$ the saturated layer will be unstable with respect to saturated parcel processes. (8.14)

where $d\theta_e/dz$ refers to the initial profile of θ_e (or in practice θ_{ep}) in the layer, which initially is not saturated. In the atmosphere, a potentially unstable layer tends to form cumulus-type clouds and often convective precipitation. On the other hand, a potentially stable layer produces stratiform clouds with light (if any) precipitation.

8.3 Other factors affecting stability

In our initial assumptions we accepted that the parcel does not mix with the environment, that the process is adiabatic, and that the environment is not disturbed (i.e. there are no compensating vertical motions in the environment as a parcel or a layer rises in it). When these assumptions are violated they affect the temperature lapse rate of the parcel and so the stability criteria derived previously do not apply. A detailed treatment of these subjects is beyond the scope of this book. Interested readers can consult Iribarne and Godson (1973).

Examples

(8.1) The table below provides pressure, temperature, and dew point temperature measurements at various pressure levels. (1) Investigate the stability conditions of parcels rising in each layer. (2) If each layer were lifted *en masse* until it became saturated would it become convectively stable, neutral, or unstable? (Assume that l_v is independent of T and equal to $2.45 \times 10^6\,\mathrm{J\,kg^{-1}}$.)

First, we need to establish whether or not at a given level the parcels are saturated. The temperature is at all levels greater than the dew point temperature except at 850 mb. Accordingly, for the stability of the parcels, conditions (8.12) apply for all layers except for the layer 850–800 mb.

p (mb)	T (°C)	T_{dew} (°C)
1000	30	23
950	27	21
900	23	20
850	20	20
800	18	10
750	15	5
700	10	2
650	5	0

For the layer 850–800 mb, conditions (8.13) apply. For the convective stability of each layer conditions (8.14) apply. Thus, in order to answer the questions we need to calculate θ_{virt} and θ_{ep} at each level. In order to estimate θ_{virt} and θ_{ep} (recall equations (7.28) and (7.63)) we need to find at each level w and T_{LCL}. The mixing ratio can be estimated by combining equations (7.3), (7.7), (7.16), and (6.17). T_{LCL} can be estimated from equation (7.36). After plenty of calculations we arrive at the following table; here e_{s} is in mb.

	p	T	T_{dew}	w	T_{virt}	θ_{virt}	θ_{ep}	r	e_{s}	T_{LCL}
	1000	303	296	0.0182	306.4	306.4	357.4	0.66	43.1	294.3
Layer 1										
	950	300	294	0.0169	303.0	307.5	355.1	0.70	36.0	292.6
Layer 2										
	900	296	293	0.0166	299.0	308.1	355.0	0.83	28.2	292.3
Layer 3										
	850	293	293	0.0176	296.1	310.2	360.2	1.00	23.4	293.0
Layer 4										
	800	291	283	0.0099	292.8	312.1	340.1	0.61	20.6	281.3
Layer 5										
	750	288	278	0.0149	290.6	315.5	361.7	0.51	17.0	276.0
Layer 6										
	700	283	275	0.0063	284.1	314.6	333.3	0.58	12.2	273.4
Layer 7										
	650	278	273	0.0058	279.0	315.6	332.9	0.70	8.6	272.0

(1) According to the results in the above table and conditions (8.12) and (8.13), we find that for parcels rising in each of the layers, layer 1 is stable ($d\theta_{\text{virt}}/dz > 0$), layer 2 is stable ($d\theta_{\text{virt}}/dz > 0$), layer 3 is stable

($d\theta_{\mathrm{virt}}/dz > 0$), layer 4 is unstable ($d\theta_{\mathrm{ep}}/dz < 0$), layer 5 is stable ($d\theta_{\mathrm{virt}}/dz > 0$), layer 6 is unstable ($d\theta_{\mathrm{virt}}/dz < 0$), and layer 7 is stable ($d\theta_{\mathrm{virt}}/dz > 0$).

(2) Similarly, we find that layer 1 is convectively unstable ($d\theta_{\mathrm{ep}}/dz < 0$), layer 2 is convectively neutral ($d\theta_{\mathrm{ep}}/dz \approx 0$), layer 3 is convectively stable ($d\theta_{\mathrm{ep}}/dz > 0$), layer 4 is convectively unstable, ($d\theta_{\mathrm{ep}}/dz < 0$), layer 5 is convectively stable ($d\theta_{\mathrm{ep}}/dz > 0$), layer 6 is convectively unstable ($d\theta_{\mathrm{ep}}/dz < 0$), and layer 7 is approximately convectively neutral ($d\theta_{\mathrm{ep}}/dz \approx 0$).

(8.2) In an atmospheric layer the virtual temperature is constant and equal to 10 °C. If an initial fluctuation causes the parcel to rise adiabatically, calculate the energy per unit mass that has to be given to the parcel in order for it to rise 1 km above the bottom of the layer before it begins to sink.

When a parcel rises in the atmosphere, a certain amount of work is performed by or against the buoyancy force depending on whether the motion is along or against the direction of the buoyancy force. If the buoyancy force is directed downwards (negative buoyancy) a certain amount of work is done against buoyancy and if the buoyancy force is directed upwards (positive buoyancy) a certain amount of work is done by the buoyancy force. In the above problem since the layer is isothermal the layer is stable. Thus, the buoyancy on the parcel is negative and a parcel forced to rise will have to return to the initial level. But before it begins its return, it will reach a maximum height which depends on how strong the initial impulse was. Recall from Chapter 4 that

$$\delta W = F dz = madz = m\ddot{z}dz.$$

Thus, the work done, W, when a parcel is forced to rise from a level i to a level f is

$$W = \int_i^f m\ddot{z}dz. \tag{8.15}$$

Using equation (8.3), the above equation becomes

$$W = \int_i^f mg \left[\frac{T'_{\mathrm{virt}}(z) - T_{\mathrm{virt}}(z)}{T_{\mathrm{virt}}(z)}\right] dz \tag{8.16}$$

or

$$W = gm \int_i^f \frac{T'_{\mathrm{virt}}(z)}{T_{\mathrm{virt}}(z)} dz - gm \int_i^f dz$$

or, assuming that the ascent is dry adiabatic,

$$W = gm \int_i^f \frac{(T_{\mathrm{virt},0} - \Gamma_d z)}{T_{\mathrm{virt}}(z)} dz - gm(f - i).$$

Since $T_{\mathrm{virt}}(z) = T_{\mathrm{virt},0}$ and assuming that the bottom of the layer is at $z = 0$ (i.e. $i = 0\,\mathrm{m}$ and $f = 1000\,\mathrm{m}$), we find that the work done per unit mass is equal to $-169.7\,\mathrm{J\,kg^{-1}}$, i.e. work is done against the buoyancy force. Since the ascent is assumed to be adiabatic, $\delta Q = 0$. Then from the first law we obtain

$$du = 169.7\,\mathrm{J\,kg^{-1}}.$$

This is the amount of energy per unit mass that must be given to the parcel for the above process to take place.

Note that using the hydrostatic approximation, equation (8.16) can be expressed as

$$W = \int_i^f -\frac{m}{\rho}\left(\frac{T'_{\mathrm{virt}} - T_{\mathrm{virt}}}{T_{\mathrm{virt}}}\right) dp$$

or using

$$p = \rho R_d T_{\mathrm{virt}}$$

$$W = -R_d m \int_i^f (T'_{\mathrm{virt}} - T_{\mathrm{virt}}) \frac{dp}{p}$$

or

$$W = -R_{\mathrm{d}} m \int_i^f (T'_{\mathrm{virt}} - T_{\mathrm{virt}}) d \ln p. \qquad (8.17)$$

In a $(T - lnp)$ diagram the above equation is proportional to the area enclosed between the vertical profile of the environment and the vertical profile of the parcel's virtual temperature. Equation (8.16) or (8.17) gives the *maximum* work done by the buoyancy and when divided by the total mass defines (remember we are always dealing with adiabatic processes, i.e. $\delta Q = 0$) what is called the *convective available potential energy* (CAPE) and the *convective inhibition energy* (CINE). More on CAPE and CINE and their applications will be discussed in the next chapter.

Problems

(8.1) Elaborate on the five assumptions made in derivation of the stability conditions of a parcel that is subject to a small displacement from equilibrium.

(8.2) If the virtual temperature profile in a layer is $T_{virt}(z) = T_{\mathrm{virt},0}a/(a+z)$, where $a > 0$ and $T_{\mathrm{virt},0} > a\Gamma_{\mathrm{d}}$, derive the vertical profile of the virtual potential temperature. What is the condition for stability for unsaturated parcels in this layer? (Unstable for $z < z_{\mathrm{c}}$, stable for $z > z_{\mathrm{c}}$, and neutral for $z = z_{\mathrm{c}}$, where $z_{\mathrm{c}} = \sqrt{(aT_{\mathrm{virt},0}/\Gamma_{\mathrm{d}})} - a$)

(8.3) If the profile of virtual potential temperature in a layer is

$$\frac{\theta_{\mathrm{virt}}}{\theta_{\mathrm{virt},0}} = e^{\frac{z}{a_1}} - \frac{z}{a_2},$$

what is the condition for stability in the layer for unsaturated parcels? (Stable when $z > a_1 \ln(a_1/a_2)$, unstable when $z < a_1 \ln(a_1/a_2)$)

(8.4) An atmospheric layer is isothermal. A dry parcel is subjected to an upward displacement and begins to oscillate about its original level. Plot the oscillation period as a function of the temperature of the layer. What do you observe?

(8.5) If we define the geopotential ϕ according to $d\phi = gdz$ with $\phi(0) = 0$ show that under hydrostatic conditions

$$\Delta z = \left(\frac{R_{\mathrm{d}}}{g} \ln \frac{p_1}{p_2}\right) \overline{T}_{\mathrm{virt}},$$

where Δz is the thickness of a layer bounded by p_1 and p_2 ($p_1 > p_2$) and $\overline{T}_{\mathrm{virt}}$ is the mean virtual temperature of the layer. Using the above relationship show that if a layer is lifted *en masse* while the mean virtual temperature remains constant, then the upper level pressure change, dp_2, is related to the lower level change, dp_1, via the equation

$$\frac{dp_1}{p_1} = \frac{dp_2}{p_2}.$$

(8.6) Show that for a constant virtual temperature lapse rate atmosphere ($T_{\mathrm{virt}}(z) = T_{\mathrm{virt}} - \Gamma_{\mathrm{virt}} z$)

$$p = p_0 \left[1 - \frac{\Gamma_{\mathrm{virt}} z}{T_{\mathrm{virt},0}}\right]^{g/R_{\mathrm{d}}\Gamma_{\mathrm{virt}}}.$$

(8.7) Show that for an atmosphere where $T_{\mathrm{virt}}(z) = T_{\mathrm{virt},0}$

$$p = p_0 e^{-gz/R_{\mathrm{d}} T_{\mathrm{virt},0}}$$

(8.8) From the following data determine the stability and convective stability of each layer.

p(mb)	T_{virt}(°C)	T_{dew}(°C)
1000	30.0	22.0
950	25.0	21.0
900	18.5	18.0
850	16.5	15.0
800	20.0	10.0
750	10.0	5.0
700	−5.0	−10.0
650	−10.0	−15.0
600	−20.0	−30.0

(8.9) In an unstable layer of air extending from the ground to the pressure level of 900 mb, the virtual temperature decreases with height at a rate of 25 °C km^{-1}. A parcel of air at the surface is given an initial velocity of 1 m s^{-1}. If the virtual temperature at the surface is 7 °C, and assuming that the parcel is and remains dry, find its position and velocity after 1 min. (81 m, 2.1 m s^{-1})

(8.10) Sea-breezes and lake-breezes are established when the air over the land becomes warmer than the air over the water. The warmer air over land rises thus destroying the pressure stratification and resulting in horizontal pressure gradients between land and water. The height of the breeze is the altitude where the horizontal pressure gradient vanishes. Assuming that the vertical virtual temperature profiles over land and water are isothermal and the same, show that this height is given by

$$h = \frac{R_{\mathrm{d}} \ln(p_{\mathrm{w}}/p_{\mathrm{l}})}{g(\frac{1}{T_{\mathrm{virt,w}}} - \frac{1}{T_{\mathrm{virt,l}}})}$$

where the subscripts l and w denote the conditions over land and water, respectively.

(8.11) Consider problems 8.6, 8.7, and 8.10. How would the results change if these problems were stated in terms of the temperature profiles rather than virtual temperature profiles?