CHAPTER 13

Complex Numbers and Functions. Complex Differentiation

The transition from "real calculus" to "complex calculus" starts with a discussion of ***complex numbers*** and their geometric representation in the ***complex plane***. We then progress to ***analytic functions*** in Sec. 13.3. We desire functions to be analytic because these are the "useful functions" in the sense that they are differentiable in some domain and operations of complex analysis can be applied to them. The most important equations are therefore the Cauchy–Riemann equations in Sec. 13.4 because they allow a test of analyticity of such functions. Moreover, we show how the Cauchy–Riemann equations are related to the important ***Laplace equation***.

The remaining sections of the chapter are devoted to elementary complex functions (exponential, trigonometric, hyperbolic, and logarithmic functions). These generalize the familiar real functions of calculus. Detailed knowledge of them is an absolute necessity in practical work, just as that of their real counterparts is in calculus.

Prerequisite: Elementary calculus.
References and Answers to Problems: App. 1 Part D, App. 2.

13.1 Complex Numbers and Their Geometric Representation

The material in this section will most likely be familiar to the student and serve as a review.

Equations without *real* solutions, such as $x^2 = -1$ or $x^2 - 10x + 40 = 0$, were observed early in history and led to the introduction of complex numbers.[1] By definition, a **complex number** z is an ordered pair (x, y) of real numbers x and y, written

$$z = (x, y).$$

[1]First to use complex numbers for this purpose was the Italian mathematician GIROLAMO CARDANO (1501–1576), who found the formula for solving cubic equations. The term "complex number" was introduced by CARL FRIEDRICH GAUSS (see the footnote in Sec. 5.4), who also paved the way for a general use of complex numbers.

x is called the **real part** and y the **imaginary part** of z, written

$$x = \operatorname{Re} z, \qquad y = \operatorname{Im} z.$$

By definition, two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

(0, 1) is called the **imaginary unit** and is denoted by i,

$$i = (0, 1). \tag{1}$$

Addition, Multiplication. Notation $z = x + iy$

Addition of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \quad y_1 + y_2). \tag{2}$$

Multiplication is defined by

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, \quad x_1 y_2 + x_2 y_1). \tag{3}$$

These two definitions imply that

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$

and

$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0)$$

as for real numbers x_1, x_2. Hence the complex numbers "***extend***" the real numbers. We can thus write

$$(x, 0) = x. \qquad \text{Similarly,} \qquad (0, y) = iy$$

because by (1), and the definition of multiplication, we have

$$iy = (0, 1)y = (0, 1)(y, 0) = (0 \cdot y - 1 \cdot 0, \quad 0 \cdot 0 + 1 \cdot y) = (0, y).$$

Together we have, by addition, $(x, y) = (x, 0) + (0, y) = x + iy$.

In practice, complex numbers $z = (x, y)$ are written

$$z = x + iy \tag{4}$$

or $z = x + yi$, e.g., $17 + 4i$ (instead of $i4$).

Electrical engineers often write j instead of i because they need i for the current.

If $x = 0$, then $z = iy$ and is called **pure imaginary**. Also, (1) and (3) give

$$i^2 = -1 \tag{5}$$

because, by the definition of multiplication, $i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1$.

For **addition** the standard notation (4) gives [see (2)]

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

For **multiplication** the standard notation gives the following very simple recipe. Multiply each term by each other term and use $i^2 = -1$ when it occurs [see (3)]:

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).\end{aligned}$$

This agrees with (3). And it shows that $x + iy$ is a more practical notation for complex numbers than (x, y).

If you know vectors, you see that (2) is vector addition, whereas the multiplication (3) has no counterpart in the usual vector algebra.

EXAMPLE 1 Real Part, Imaginary Part, Sum and Product of Complex Numbers

Let $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$. Then $\text{Re}\, z_1 = 8$, $\text{Im}\, z_1 = 3$, $\text{Re}\, z_2 = 9$, $\text{Im}\, z_2 = -2$ and

$$z_1 + z_2 = (8 + 3i) + (9 - 2i) = 17 + i,$$

$$z_1z_2 = (8 + 3i)(9 - 2i) = 72 + 6 + i(-16 + 27) = 78 + 11i.$$ ■

Subtraction, Division

Subtraction and **division** are defined as the inverse operations of addition and multiplication, respectively. Thus the **difference** $z = z_1 - z_2$ is the complex number z for which $z_1 = z + z_2$. Hence by (2),

(6) $$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

The **quotient** $z = z_1/z_2$ $(z_2 \neq 0)$ is the complex number z for which $z_1 = zz_2$. If we equate the real and the imaginary parts on both sides of this equation, setting $z = x + iy$, we obtain $x_1 = x_2x - y_2y$, $y_1 = y_2x + x_2y$. The solution is

(7*) $$z = \frac{z_1}{z_2} = x + iy, \qquad x = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \qquad y = \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

The ***practical rule*** used to get this is by multiplying numerator and denominator of z_1/z_2 by $x_2 - iy_2$ and simplifying:

(7) $$z = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

EXAMPLE 2 Difference and Quotient of Complex Numbers

For $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$ we get $z_1 - z_2 = (8 + 3i) - (9 - 2i) = -1 + 5i$ and

$$\frac{z_1}{z_2} = \frac{8 + 3i}{9 - 2i} = \frac{(8 + 3i)(9 + 2i)}{(9 - 2i)(9 + 2i)} = \frac{66 + 43i}{81 + 4} = \frac{66}{85} + \frac{43}{85}i.$$

Check the division by multiplication to get $8 + 3i$. ■

Complex numbers satisfy the same commutative, associative, and distributive laws as real numbers (see the problem set).

Complex Plane

So far we discussed the algebraic manipulation of complex numbers. Consider the geometric representation of complex numbers, which is of great practical importance. We choose two perpendicular coordinate axes, the horizontal x-axis, called the **real axis**, and the vertical y-axis, called the **imaginary axis**. On both axes we choose the same unit of length (Fig. 318). This is called a **Cartesian coordinate system**.

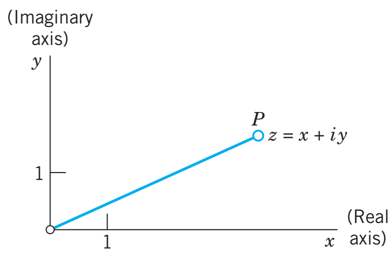

Fig. 318. The complex plane

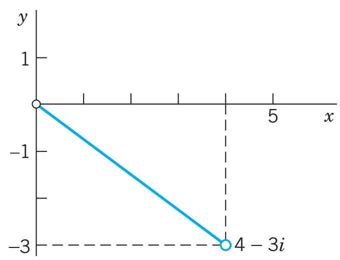

Fig. 319. The number $4 - 3i$ in the complex plane

We now plot a given complex number $z = (x, y) = x + iy$ as the point P with coordinates x, y. The xy-plane in which the complex numbers are represented in this way is called the **complex plane**.[2] Figure 319 shows an example.

Instead of saying "the point represented by z in the complex plane" we say briefly and simply "***the point z in the complex plane***." This will cause no misunderstanding.

Addition and subtraction can now be visualized as illustrated in Figs. 320 and 321.

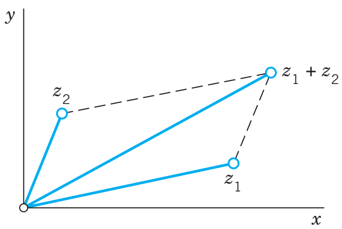

Fig. 320. Addition of complex numbers

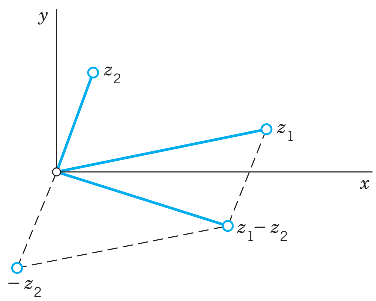

Fig. 321. Subtraction of complex numbers

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[2]Sometimes called the **Argand diagram**, after the French mathematician JEAN ROBERT ARGAND (1768–1822), born in Geneva and later librarian in Paris. His paper on the complex plane appeared in 1806, nine years after a similar memoir by the Norwegian mathematician CASPAR WESSEL (1745–1818), a surveyor of the Danish Academy of Science.

Complex Conjugate Numbers

The complex conjugate $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\bar{z} = x - iy.$$

It is obtained geometrically by reflecting the point z in the real axis. Figure 322 shows this for $z = 5 + 2i$ and its conjugate $\bar{z} = 5 - 2i$.

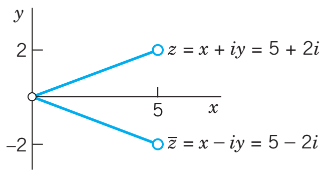

Fig. 322. Complex conjugate numbers

The complex conjugate is important because it permits us to switch from complex to real. Indeed, by multiplication, $z\bar{z} = x^2 + y^2$ (verify!). By addition and subtraction, $z + \bar{z} = 2x$, $z - \bar{z} = 2iy$. We thus obtain for the real part x and the imaginary part y (not iy!) of $z = x + iy$ the important formulas

$$\text{(8)} \qquad \operatorname{Re} z = x = \tfrac{1}{2}(z + \bar{z}), \qquad \operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z}).$$

If z is real, $z = x$, then $\bar{z} = z$ by the definition of $\bar{z}$, and conversely. Working with conjugates is easy, since we have

$$\text{(9)} \qquad \begin{aligned} \overline{(z_1 + z_2)} &= \bar{z}_1 + \bar{z}_2, & \overline{(z_1 - z_2)} &= \bar{z}_1 - \bar{z}_2, \\ \overline{(z_1 z_2)} &= \bar{z}_1 \bar{z}_2, & \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\bar{z}_1}{\bar{z}_2}. \end{aligned}$$

EXAMPLE 3 Illustration of (8) and (9)

Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$. Then by (8),

$$\operatorname{Im} z_1 = \frac{1}{2i}[(4 + 3i) - (4 - 3i)] = \frac{3i + 3i}{2i} = 3.$$

Also, the multiplication formula in (9) is verified by

$$\overline{(z_1 z_2)} = \overline{(4 + 3i)(2 + 5i)} = \overline{(-7 + 26i)} = -7 - 26i,$$

$$\bar{z}_1 \bar{z}_2 = (4 - 3i)(2 - 5i) = -7 - 26i.$$

■

PROBLEM SET 13.1

1. **Powers of *i*.** Show that $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \cdots$ and $1/i = -i, 1/i^2 = -1, 1/i^3 = i, \cdots$.
2. **Rotation.** Multiplication by i is geometrically a counterclockwise rotation through $\pi/2$ (90°). Verify this by graphing z and iz and the angle of rotation for $z = 1 + i, z = -1 + 2i, z = 4 - 3i$.
3. **Division.** Verify the calculation in (7). Apply (7) to $(26 - 18i)/(6 - 2i)$.

4. Law for conjugates. Verify (9) for $z_1 = -11 + 10i$, $z_2 = -1 + 4i$.

5. Pure imaginary number. Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.

6. Multiplication. If the product of two complex numbers is zero, show that at least one factor must be zero.

7. Laws of addition and multiplication. Derive the following laws for complex numbers from the corresponding laws for real numbers.

$$z_1 + z_2 = z_2 + z_1,\ z_1z_2 = z_2z_1 \quad (\textit{Commutative laws})$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (\textit{Associative laws})$$

$$(z_1z_2)z_3 = z_1(z_2z_3)$$

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \quad (\textit{Distributive law})$$

$$0 + z = z + 0 = z,$$

$$z + (-z) = (-z) + z = 0, \qquad z \cdot 1 = z.$$

8–15 COMPLEX ARITHMETIC

Let $z_1 = -2 + 11i$, $z_2 = 2 - i$. Showing the details of your work, find, in the form $x + iy$:

8. z_1z_2, $\overline{(z_1z_2)}$

9. $\text{Re}\,(z_1^2)$, $(\text{Re}\,z_1)^2$

10. $\text{Re}\,(1/z_2^2)$, $1/\text{Re}\,(z_2^2)$

11. $(z_1 - z_2)^2/16$, $(z_1/4 - z_2/4)^2$

12. z_1/z_2, z_2/z_1

13. $(z_1 + z_2)(z_1 - z_2)$, $z_1^2 - z_2^2$

14. $\bar{z}_1/\bar{z}_2$, $\overline{(z_1/z_2)}$

15. $4\,(z_1 + z_2)/(z_1 - z_2)$

16–20 Let $z = x + iy$. Showing details, find, in terms of x and y:

16. $\text{Im}\,(1/z)$, $\text{Im}\,(1/z^2)$

17. $\text{Re}\,z^4 - (\text{Re}\,z^2)^2$

18. $\text{Re}\,[(1 + i)^{16}z^2]$

19. $\text{Re}\,(z/\bar{z})$, $\text{Im}\,(z/\bar{z})$

20. $\text{Im}\,(1/\bar{z}^2)$

13.2 Polar Form of Complex Numbers. Powers and Roots

We gain further insight into the arithmetic operations of complex numbers if, in addition to the xy-coordinates in the complex plane, we also employ the usual polar coordinates r, θ defined by

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

We see that then $z = x + iy$ takes the so-called **polar form**

$$z = r(\cos\theta + i\sin\theta). \tag{2}$$

r is called the **absolute value** or **modulus** of z and is denoted by $|z|$. Hence

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}. \tag{3}$$

Geometrically, $|z|$ is the distance of the point z from the origin (Fig. 323). Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2 (Fig. 324).

θ is called the **argument** of z and is denoted by arg z. Thus $\theta = \arg z$ and (Fig. 323)

$$\tan\theta = \frac{y}{x} \qquad (z \neq 0). \tag{4}$$

Geometrically, θ is the directed angle from the positive x-axis to OP in Fig. 323. Here, as in calculus, all ***angles are measured in radians and positive in the counterclockwise sense***.

For $z = 0$ this angle θ is undefined. (Why?) For a given $z \neq 0$ it is determined only up to integer multiples of 2π since cosine and sine are periodic with period 2π. But one often wants to specify a unique value of arg z of a given $z \neq 0$. For this reason one defines the **principal value** Arg z (with capital A!) of arg z by the double inequality

(5) $$-\pi < \operatorname{Arg} z \leqq \pi.$$

Then we have Arg $z = 0$ for positive real $z = x$, which is practical, and Arg $z = \pi$ (not $-\pi$!) for negative real z, e.g., for $z = -4$. The principal value (5) will be important in connection with roots, the complex logarithm (Sec. 13.7), and certain integrals. Obviously, for a given $z \neq 0$, the other values of arg z are arg $z = \operatorname{Arg} z \pm 2n\pi$ $(n = \pm 1, \pm 2, \cdots)$.

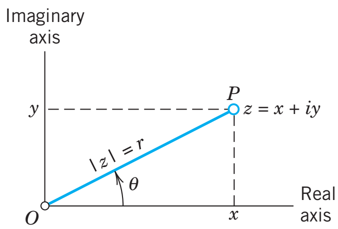

Fig. 323. Complex plane, polar form of a complex number

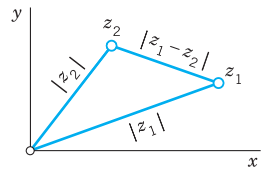

Fig. 324. Distance between two points in the complex plane

EXAMPLE 1 **Polar Form of Complex Numbers. Principal Value Arg z**

$z = 1 + i$ (Fig. 325) has the polar form $z = \sqrt{2}\,(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$. Hence we obtain

$$|z| = \sqrt{2}, \quad \arg z = \tfrac{1}{4}\pi \pm 2n\pi \ (n = 0, 1, \cdots), \qquad \text{and} \qquad \operatorname{Arg} z = \tfrac{1}{4}\pi \quad \text{(the principal value).}$$

Similarly, $z = 3 + 3\sqrt{3}i = 6\,(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$, $|z| = 6$, and $\operatorname{Arg} z = \frac{1}{3}\pi$. ■

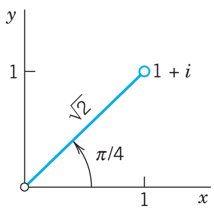

Fig. 325. Example 1

CAUTION! In using (4), we must pay attention to the quadrant in which z lies, since $\tan \theta$ has period π, so that the arguments of z and $-z$ have the same tangent. *Example:* for $\theta_1 = \arg(1 + i)$ and $\theta_2 = \arg(-1 - i)$ we have $\tan \theta_1 = \tan \theta_2 = 1$.

Triangle Inequality

Inequalities such as $x_1 < x_2$ make sense for *real* numbers, but not in complex because *there is no natural way of ordering complex numbers*. However, inequalities between absolute values (which are real!), such as $|z_1| < |z_2|$ (meaning that z_1 is closer to the origin than z_2) are of great importance. The daily bread of the complex analyst is the **triangle inequality**

(6) $$|z_1 + z_2| \leqq |z_1| + |z_2| \qquad \text{(Fig. 326)}$$

which we shall use quite frequently. This inequality follows by noting that the three points 0, z_1, and $z_1 + z_2$ are the vertices of a triangle (Fig. 326) with sides $|z_1|$, $|z_2|$, and $|z_1 + z_2|$, and one side cannot exceed the sum of the other two sides. A formal proof is left to the reader (Prob. 33). (The triangle degenerates if z_1 and z_2 lie on the same straight line through the origin.)

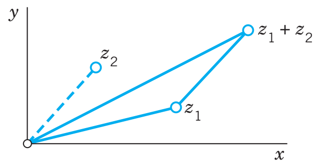

Fig. 326. Triangle inequality

By induction we obtain from (6) the **generalized triangle inequality**

$$|z_1 + z_2 + \cdots + z_n| \leqq |z_1| + |z_2| + \cdots + |z_n|; \tag{6*}$$

that is, *the absolute value of a sum cannot exceed the sum of the absolute values of the terms.*

EXAMPLE 2 Triangle Inequality

If $z_1 = 1 + i$ and $z_2 = -2 + 3i$, then (sketch a figure!)

$$|z_1 + z_2| = |-1 + 4i| = \sqrt{17} = 4.123 < \sqrt{2} + \sqrt{13} = 5.020.$$ ■

Multiplication and Division in Polar Form

This will give us a "geometrical" understanding of multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Multiplication. By (3) in Sec. 13.1 the product is at first

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)].$$

The addition rules for the sine and cosine [(6) in App. A3.1] now yield

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \tag{7}$$

Taking absolute values on both sides of (7), we see that *the absolute value of a product equals the* ***product*** *of the absolute values of the factors,*

$$|z_1 z_2| = |z_1||z_2|. \tag{8}$$

Taking arguments in (7) shows that *the argument of a product equals the* ***sum*** *of the arguments of the factors,*

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \qquad \text{(up to multiples of } 2\pi). \tag{9}$$

Division. We have $z_1 = (z_1/z_2)z_2$. Hence $|z_1| = |(z_1/z_2)z_2| = |z_1/z_2||z_2|$ and by division by $|z_2|$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \neq 0). \tag{10}$$

Similarly, $\arg z_1 = \arg [(z_1/z_2)z_2] = \arg (z_1/z_2) + \arg z_2$ and by subtraction of $\arg z_2$

$$\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2 \qquad \text{(up to multiples of } 2\pi). \tag{11}$$

Combining (10) and (11) we also have the analog of (7),

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]. \tag{12}$$

To comprehend this formula, note that it is the polar form of a complex number of absolute value r_1/r_2 and argument $\theta_1 - \theta_2$. But these are the absolute value and argument of z_1/z_2, as we can see from (10), (11), and the polar forms of z_1 and z_2.

EXAMPLE 3 Illustration of Formulas (8)–(11)

Let $z_1 = -2 + 2i$ and $z_2 = 3i$. Then $z_1 z_2 = -6 - 6i$, $z_1/z_2 = \frac{2}{3} + (\frac{2}{3})i$. Hence (make a sketch)

$$|z_1 z_2| = 6\sqrt{2} = 3\sqrt{8} = |z_1||z_2|, \qquad |z_1/z_2| = 2\sqrt{2}/3 = |z_1|/|z_2|,$$

and for the arguments we obtain $\operatorname{Arg} z_1 = 3\pi/4$, $\operatorname{Arg} z_2 = \pi/2$,

$$\operatorname{Arg}(z_1 z_2) = -\frac{3\pi}{4} = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 - 2\pi, \qquad \operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \frac{\pi}{4} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2.$$ ■

EXAMPLE 4 Integer Powers of *z*. De Moivre's Formula

From (8) and (9) with $z_1 = z_2 = z$ we obtain by induction for $n = 0, 1, 2, \cdots$

$$z^n = r^n (\cos n\theta + i \sin n\theta). \tag{13}$$

Similarly, (12) with $z_1 = 1$ and $z_2 = z^n$ gives (13) for $n = -1, -2, \cdots$. For $|z| = r = 1$, formula (13) becomes **De Moivre's formula**[3]

$$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta. \tag{13*}$$

We can use this to express $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$. For instance, for $n = 2$ we have on the left $\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$. Taking the real and imaginary parts on both sides of (13*) with $n = 2$ gives the familiar formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \qquad \sin 2\theta = 2\cos\theta\sin\theta.$$

This shows that *complex* methods often simplify the derivation of *real* formulas. Try $n = 3$. ■

Roots

If $z = w^n$ $(n = 1, 2, \cdots)$, then to each value of w there corresponds *one* value of z. We shall immediately see that, conversely, to a given $z \neq 0$ there correspond precisely n distinct values of w. Each of these values is called an ***n*th root** of z, and we write

[3]ABRAHAM DE MOIVRE (1667–1754), French mathematician, who pioneered the use of complex numbers in trigonometry and also contributed to probability theory (see Sec. 24.8).

$$w = \sqrt[n]{z}. \tag{14}$$

Hence this symbol is ***multivalued***, namely, *n-valued.* The n values of $\sqrt[n]{z}$ can be obtained as follows. We write z and w in polar form

$$z = r(\cos\theta + i\sin\theta) \qquad \text{and} \qquad w = R(\cos\phi + i\sin\phi).$$

Then the equation $w^n = z$ becomes, by De Moivre's formula (with ϕ instead of θ),

$$w^n = R^n(\cos n\phi + i\sin n\phi) = z = r(\cos\theta + i\sin\theta).$$

The absolute values on both sides must be equal; thus, $R^n = r$, so that $R = \sqrt[n]{r}$, where $\sqrt[n]{r}$ is positive real (an absolute value must be nonnegative!) and thus uniquely determined. Equating the arguments $n\phi$ and θ and recalling that θ is determined only up to integer multiples of 2π, we obtain

$$n\phi = \theta + 2k\pi, \qquad \text{thus} \qquad \phi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

where k is an integer. For $k = 0, 1, \cdots, n-1$ we get n *distinct* values of w. Further integers of k would give values already obtained. For instance, $k = n$ gives $2k\pi/n = 2\pi$, hence the w corresponding to $k = 0$, etc. Consequently, $\sqrt[n]{z}$, for $z \neq 0$, has the n distinct values

$$\boxed{\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)} \tag{15}$$

where $k = 0, 1, \cdots, n-1$. These n values lie on a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute the vertices of a regular polygon of n sides. The value of $\sqrt[n]{z}$ obtained by taking the principal value of arg z and $k = 0$ in (15) is called the **principal value** of $w = \sqrt[n]{z}$.

Taking $z = 1$ in (15), we have $|z| = r = 1$ and Arg $z = 0$. Then (15) gives

$$\boxed{\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},} \qquad k = 0, 1, \cdots, n-1. \tag{16}$$

These n values are called the ***n*th roots of unity**. They lie on the circle of radius 1 and center 0, briefly called the **unit circle** (and used quite frequently!). Figures 327–329 show $\sqrt[3]{1} = 1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$, $\sqrt[4]{1} = \pm 1, \pm i$, and $\sqrt[5]{1}$.

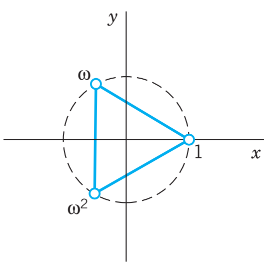

Fig. 327. $\sqrt[3]{1}$

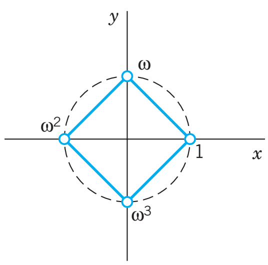

Fig. 328. $\sqrt[4]{1}$

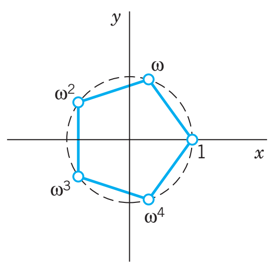

Fig. 329. $\sqrt[5]{1}$

If ω denotes the value corresponding to $k = 1$ in (16), then the n values of $\sqrt[n]{1}$ can be written as

$$1, \omega, \omega^2, \cdots, \omega^{n-1}.$$

More generally, if w_1 is any nth root of an arbitrary complex number $z\ (\neq 0)$, then the n values of $\sqrt[n]{z}$ in (15) are

$$w_1, \quad w_1\omega, \quad w_1\omega^2, \quad \cdots, \quad w_1\omega^{n-1} \tag{17}$$

because multiplying w_1 by ω^k corresponds to increasing the argument of w_1 by $2k\pi/n$. Formula (17) motivates the introduction of roots of unity and shows their usefulness.

PROBLEM SET 13.2

1–8 POLAR FORM

Represent in polar form and graph in the complex plane as in Fig. 325. Do these problems very carefully because polar forms will be needed frequently. Show the details.

1. $1 + i$

2. $-4 + 4i$

3. $2i, \quad -2i$

4. -5

5. $\dfrac{\sqrt{2} + i/3}{-\sqrt{8} - 2i/3}$

6. $\dfrac{\sqrt{3} - 10i}{-\frac{1}{2}\sqrt{3} + 5i}$

7. $1 + \frac{1}{2}\pi i$

8. $\dfrac{-4 + 19i}{2 + 5i}$

9–14 PRINCIPAL ARGUMENT

Determine the principal value of the argument and graph it as in Fig. 325.

9. $-1 + i$

10. $-5, \quad -5 - i, \quad -5 + i$

11. $3 \pm 4i$

12. $-\pi - \pi i$

13. $(1 + i)^{20}$

14. $-1 + 0.1i, \quad -1 - 0.1i$

15–18 CONVERSION TO $x + iy$

Graph in the complex plane and represent in the form $x + iy$:

15. $3\,(\cos \frac{1}{2}\pi - i \sin \frac{1}{2}\pi)$

16. $6\,(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$

17. $\sqrt{8}\,(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$

18. $\sqrt{50}\,(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi)$

ROOTS

19. CAS PROJECT. Roots of Unity and Their Graphs. Write a program for calculating these roots and for graphing them as points on the unit circle. Apply the program to $z^n = 1$ with $n = 2, 3, \cdots, 10$. Then extend the program to one for arbitrary roots, using an idea near the end of the text, and apply the program to examples of your choice.

20. TEAM PROJECT. Square Root. (a) Show that $w = \sqrt{z}$ has the values

$$\begin{aligned} w_1 &= \sqrt{r}\left[\cos\frac{\theta}{2} + i \sin\frac{\theta}{2}\right], \\ w_2 &= \sqrt{r}\left[\cos\left(\frac{\theta}{2} + \pi\right) + i \sin\left(\frac{\theta}{2} + \pi\right)\right] \\ &= -w_1. \end{aligned} \tag{18}$$

(b) Obtain from (18) the often more practical formula

$$\sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z| + x)} + (\operatorname{sign} y)\, i \sqrt{\tfrac{1}{2}(|z| + x)}\right] \tag{19}$$

where sign $y = 1$ if $y \geqq 0$, sign $y = -1$ if $y < 0$, and all square roots of positive numbers are taken with positive sign. *Hint:* Use (10) in App. A3.1 with $x = \theta/2$.

(c) Find the square roots of $-14i$, $-9 - 40i$, and $1 + \sqrt{48}i$ by both (18) and (19) and comment on the work involved.

(d) Do some further examples of your own and apply a method of checking your results.

21–27 ROOTS

Find and graph all roots in the complex plane.

21. $\sqrt[3]{1 + i}$

22. $\sqrt[3]{3 + 4i}$

23. $\sqrt[3]{216}$

24. $\sqrt[4]{-4}$

25. $\sqrt[4]{i}$

26. $\sqrt[8]{1}$

27. $\sqrt[5]{-1}$

28–31 EQUATIONS

Solve and graph the solutions. Show details.

28. $z^2 - (6 - 2i)z + 17 - 6i = 0$

29. $z^2 + z + 1 - i = 0$

30. $z^4 + 324 = 0$. Using the solutions, factor $z^4 + 324$ into quadratic factors with *real* coefficients.

31. $z^4 - 6iz^2 + 16 = 0$

32–35 INEQUALITIES AND EQUALITY

32. Triangle inequality. Verify (6) for $z_1 = 3 + i$, $z_2 = -2 + 4i$

33. Triangle inequality. Prove (6).

34. Re and Im. Prove $|\text{Re } z| \leqq |z|$, $|\text{Im } z| \leqq |z|$.

35. Parallelogram equality. Prove and explain the name

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,(|z_1|^2 + |z_2|^2).$$

13.3 Derivative. Analytic Function

Just as the study of calculus or real analysis required concepts such as domain, neighborhood, function, limit, continuity, derivative, etc., so does the study of complex analysis. Since the functions live in the complex plane, the concepts are slightly more difficult or *different* from those in real analysis. This section can be seen as a reference section where many of the concepts needed for the rest of Part D are introduced.

Circles and Disks. Half-Planes

The **unit circle** $|z| = 1$ (Fig. 330) has already occurred in Sec. 13.2. Figure 331 shows a general circle of radius ρ and center a. Its equation is

$$|z - a| = \rho$$

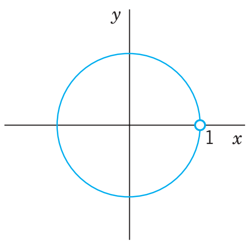

Fig. 330. Unit circle

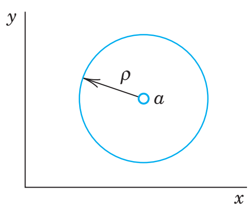

Fig. 331. Circle in the complex plane

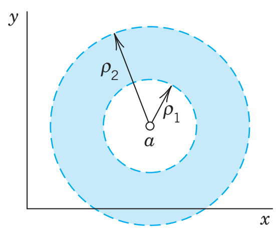

Fig. 332. Annulus in the complex plane

because it is the set of all z whose distance $|z - a|$ from the center a equals ρ. Accordingly, its interior ("**open circular disk**") is given by $|z - a| < \rho$, its interior plus the circle itself ("**closed circular disk**") by $|z - a| \leqq \rho$, and its exterior by $|z - a| > \rho$. As an example, sketch this for $a = 1 + i$ and $\rho = 2$, to make sure that you understand these inequalities.

An open circular disk $|z - a| < \rho$ is also called a **neighborhood** of a or, more precisely, a ρ-*neighborhood* of a. And a has infinitely many of them, one for each value of ρ (> 0), and a is a point of each of them, by definition!

In modern literature *any set* containing a ρ-neighborhood of a is also called a *neighborhood* of a.

Figure 332 shows an **open annulus** (circular ring) $\rho_1 < |z - a| < \rho_2$, which we shall need later. This is the set of all z whose distance $|z - a|$ from a is greater than ρ_1 but less than ρ_2. Similarly, the **closed annulus** $\rho_1 \leqq |z - a| \leqq \rho_2$ includes the two circles.

Half-Planes. By the (open) *upper* **half-plane** we mean the set of all points $z = x + iy$ such that $y > 0$. Similarly, the condition $y < 0$ defines the *lower half-plane*, $x > 0$ the *right half-plane*, and $x < 0$ the *left half-plane*.

For Reference: Concepts on Sets in the Complex Plane

To our discussion of special sets let us add some general concepts related to sets that we shall need throughout Chaps. 13–18; keep in mind that you can find them here.

By a **point set** in the complex plane we mean any sort of collection of finitely many or infinitely many points. Examples are the solutions of a quadratic equation, the points of a line, the points in the interior of a circle as well as the sets discussed just before.

A set S is called **open** if every point of S has a neighborhood consisting entirely of points that belong to S. For example, the points in the interior of a circle or a square form an open set, and so do the points of the right half-plane $\text{Re}\, z = x > 0$.

A set S is called **connected** if any two of its points can be joined by a chain of finitely many straight-line segments all of whose points belong to S. An open and connected set is called a **domain.** Thus an open disk and an open annulus are domains. An open square with a diagonal removed is not a domain since this set is not connected. (Why?)

The **complement** of a set S in the complex plane is the set of all points of the complex plane that ***do not belong*** to S. A set S is called **closed** if its complement is open. For example, the points on and inside the unit circle form a closed set ("closed unit disk") since its complement $|z| > 1$ is open.

A **boundary point** of a set S is a point every neighborhood of which contains both points that belong to S and points that do not belong to S. For example, the boundary points of an annulus are the points on the two bounding circles. Clearly, if a set S is open, then no boundary point belongs to S; if S is closed, then every boundary point belongs to S. The set of all boundary points of a set S is called the **boundary** of S.

A **region** is a set consisting of a domain plus, perhaps, some or all of its boundary points. WARNING! "Domain" is the *modern* term for an open connected set. Nevertheless, some authors still call a domain a "region" and others make no distinction between the two terms.

Complex Function

Complex analysis is concerned with complex functions that are differentiable in some domain. Hence we should first say what we mean by a complex function and then define the concepts of limit and derivative in complex. This discussion will be similar to that in calculus. Nevertheless it needs great attention because it will show interesting basic differences between real and complex calculus.

Recall from calculus that a *real* function f defined on a set S of real numbers (usually an interval) is a rule that assigns to every x in S a real number $f(x)$, called the *value* of f at x. Now in complex, S is a set of *complex* numbers. And a **function** f defined on S is a rule that assigns to every z in S a complex number w, called the *value* of f at z. We write

$$w = f(z).$$

Here z varies in S and is called a **complex variable**. The set S is called the *domain of definition* of f or, briefly, the **domain** of f. (In most cases S will be open and connected, thus a domain as defined just before.)

Example: $w = f(z) = z^2 + 3z$ is a complex function defined for all z; that is, its domain S is the whole complex plane.

The set of all values of a function f is called the **range** *of* f.

w is complex, and we write $w = u + iv$, where u and v are the real and imaginary parts, respectively. Now w depends on $z = x + iy$. Hence u becomes a real function of x and y, and so does v. We may thus write

$$w = f(z) = u(x, y) + iv(x, y).$$

This shows that a *complex* function $f(z)$ is equivalent to a *pair* of *real* functions $u(x, y)$ and $v(x, y)$, each depending on the two real variables x and y.

EXAMPLE 1 **Function of a Complex Variable**

Let $w = f(z) = z^2 + 3z$. Find u and v and calculate the value of f at $z = 1 + 3i$.

Solution. $u = \operatorname{Re} f(z) = x^2 - y^2 + 3x$ and $v = 2xy + 3y$. Also,

$$f(1 + 3i) = (1 + 3i)^2 + 3(1 + 3i) = 1 - 9 + 6i + 3 + 9i = -5 + 15i.$$

This shows that $u(1, 3) = -5$ and $v(1, 3) = 15$. Check this by using the expressions for u and v. ■

EXAMPLE 2 **Function of a Complex Variable**

Let $w = f(z) = 2iz + 6\bar{z}$. Find u and v and the value of f at $z = \frac{1}{2} + 4i$.

Solution. $f(z) = 2i(x + iy) + 6(x - iy)$ gives $u(x, y) = 6x - 2y$ and $v(x, y) = 2x - 6y$. Also,

$$f(\tfrac{1}{2} + 4i) = 2i(\tfrac{1}{2} + 4i) + 6(\tfrac{1}{2} - 4i) = i - 8 + 3 - 24i = -5 - 23i.$$

Check this as in Example 1. ■

Remarks on Notation and Terminology

1. Strictly speaking, $f(z)$ denotes the value of f at z, but it is a convenient abuse of language to talk about *the function* $f(z)$ (instead of *the function* f), thereby exhibiting the notation for the independent variable.

2. We assume all functions to be ***single-valued relations***, as usual: to each z in S there corresponds but *one* value $w = f(z)$ (but, of course, several z may give the same value $w = f(z)$, just as in calculus). Accordingly, we shall *not use* the term "multivalued function" (used in some books on complex analysis) for a multivalued relation, in which to a z there corresponds more than one w.

Limit, Continuity

A function $f(z)$ is said to have the **limit** l as z approaches a point z_0, written

$$\lim_{z \to z_0} f(z) = l, \tag{1}$$

if f is defined in a neighborhood of z_0 (except perhaps at z_0 itself) and if the values of f are "close" to l for all z "close" to z_0; in precise terms, if for every positive real ϵ we can find a positive real δ such that for all $z \neq z_0$ in the disk $|z - z_0| < \delta$ (Fig. 333) we have

$$|f(z) - l| < \epsilon; \tag{2}$$

geometrically, if for every $z \neq z_0$ in that δ-disk the value of f lies in the disk (2).

Formally, this definition is similar to that in calculus, but there is a big difference. Whereas in the real case, x can approach an x_0 only along the real line, here, by definition,

z may approach z_0 ***from any direction*** in the complex plane. This will be quite essential in what follows.

If a limit exists, it is unique. (See Team Project 24.)

A function $f(z)$ is said to be **continuous** at $z = z_0$ if $f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0). \tag{3}$$

Note that by definition of a limit this implies that $f(z)$ is defined in some neighborhood of z_0.

$f(z)$ is said to be *continuous in a domain* if it is continuous at each point of this domain.

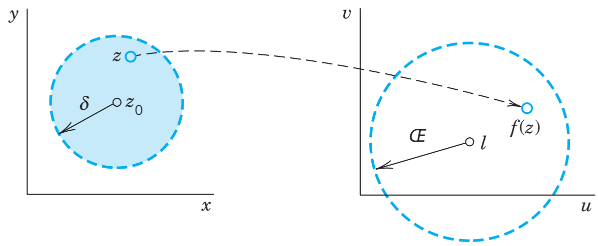

Fig. 333. Limit

Derivative

The **derivative** of a complex function f at a point z_0 is written $f'(z_0)$ and is defined by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{4}$$

provided this limit exists. Then f is said to be **differentiable** at z_0. If we write $\Delta z = z - z_0$, we have $z = z_0 + \Delta z$ and (4) takes the form

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{4'}$$

Now comes an ***important point***. Remember that, by the definition of limit, $f(z)$ is defined in a neighborhood of z_0 and z in (4′) may approach z_0 from any direction in the complex plane. Hence differentiability at z_0 means that, along whatever path z approaches z_0, the quotient in (4′) always approaches a certain value and all these values are equal. This is important and should be kept in mind.

EXAMPLE 3 Differentiability. Derivative

The function $f(z) = z^2$ is differentiable for all z and has the derivative $f'(z) = 2z$ because

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\,\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z. \qquad \blacksquare$$

The ***differentiation rules*** *are the same as in real calculus,* since their proofs are literally the same. Thus for any differentiable functions f and g and constant c we have

$$(cf)' = cf', \quad (f+g)' = f' + g', \quad (fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

as well as the chain rule and the power rule $(z^n)' = nz^{n-1}$ (n integer).

Also, if $f(z)$ is differentiable at z_0, it is continuous at z_0. (See Team Project 24.)

EXAMPLE 4 **$\bar{z}$ not Differentiable**

It may come as a surprise that there are many complex functions that do not have a derivative at any point. For instance, $f(z) = \bar{z} = x - iy$ is such a function. To see this, we write $\Delta z = \Delta x + i\,\Delta y$ and obtain

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y}. \tag{5}$$

If $\Delta y = 0$, this is $+1$. If $\Delta x = 0$, this is -1. Thus (5) approaches $+1$ along path I in Fig. 334 but -1 along path II. Hence, by definition, the limit of (5) as $\Delta z \to 0$ does not exist at any z. ■

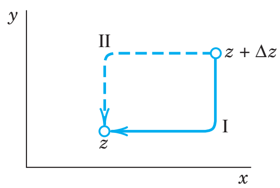

Fig. 334. Paths in (5)

Surprising as Example 4 may be, it merely illustrates that differentiability of a *complex* function is a rather severe requirement.

The idea of proof (approach of z from different directions) is basic and will be used again as the crucial argument in the next section.

Analytic Functions

Complex analysis is concerned with the theory and application of "analytic functions," that is, functions that are differentiable in some domain, so that we can do "calculus in complex." The definition is as follows.

DEFINITION

Analyticity

A function $f(z)$ is said to be *analytic in a domain D* if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be *analytic at a point* $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0.

Also, by an **analytic function** we mean a function that is analytic in *some* domain.

Hence analyticity of $f(z)$ at z_0 means that $f(z)$ has a derivative at every point in some neighborhood of z_0 (including z_0 itself since, by definition, z_0 is a point of all its neighborhoods). This concept is *motivated* by the fact that it is of no practical interest if a function is differentiable merely at a single point z_0 but not throughout some neighborhood of z_0. Team Project 24 gives an example.

A more modern term for *analytic in D* is *holomorphic in D*.

EXAMPLE 5 Polynomials, Rational Functions

The nonnegative integer powers $1, z, z^2, \cdots$ are analytic in the entire complex plane, and so are **polynomials**, that is, functions of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$$

where $c_0, \cdots, c_n$ are complex constants.

The quotient of two polynomials $g(z)$ and $h(z)$,

$$f(z) = \frac{g(z)}{h(z)},$$

is called a **rational function**. This f is analytic except at the points where $h(z) = 0$; here we assume that common factors of g and h have been canceled.

Many further analytic functions will be considered in the next sections and chapters. ■

The concepts discussed in this section extend familiar concepts of calculus. Most important is the concept of an analytic function, the exclusive concern of complex analysis. Although many simple functions are not analytic, the large variety of remaining functions will yield a most beautiful branch of mathematics that is very useful in engineering and physics.

PROBLEM SET 13.3

1–8 REGIONS OF PRACTICAL INTEREST

Determine and sketch or graph the sets in the complex plane given by

1. $|z + 1 - 5i| \leqq \frac{3}{2}$
2. $0 < |z| < 1$
3. $\pi < |z - 4 + 2i| < 3\pi$
4. $-\pi < \text{Im } z < \pi$
5. $|\arg z| < \frac{1}{4}\pi$
6. $\text{Re}\,(1/z) < 1$
7. $\text{Re } z \geqq -1$
8. $|z + i| \geqq |z - i|$
9. **WRITING PROJECT. Sets in the Complex Plane.** Write a report by formulating the corresponding portions of the text in your own words and illustrating them with examples of your own.

COMPLEX FUNCTIONS AND THEIR DERIVATIVES

10–12 Function Values. Find Re f, and Im f and their values at the given point z.

10. $f(z) = 5z^2 - 12z + 3 + 2i$ at $4 - 3i$
11. $f(z) = 1/(1 - z)$ at $1 - i$
12. $f(z) = (z - 2)/(z + 2)$ at $8i$
13. **CAS PROJECT. Graphing Functions.** Find and graph Re f, Im f, and $|f|$ as surfaces over the z-plane. Also graph the two families of curves $\text{Re}\,f(z) = \text{const}$ and $\text{Im}\,f(z) = \text{const}$ in the same figure, and the curves $|f(z)| = \text{const}$ in another figure, where **(a)** $f(z) = z^2$, **(b)** $f(z) = 1/z$, (c) $f(z) = z^4$.

14–17 Continuity. Find out, and give reason, whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function f is equal to:

14. $(\text{Re } z^2)/|z|$
15. $|z|^2 \text{ Im } (1/z)$
16. $(\text{Im } z^2)/|z|^2$
17. $(\text{Re } z)/(1 - |z|)$

18–23 Differentiation. Find the value of the derivative of

18. $(z - i)/(z + i)$ at i
19. $(z - 4i)^8$ at $= 3 + 4i$
20. $(1.5z + 2i)/(3iz - 4)$ at any z. Explain the result.
21. $i(1 - z)^n$ at 0
22. $(iz^3 + 3z^2)^3$ at $2i$
23. $z^3/(z + i)^3$ at i
24. **TEAM PROJECT. Limit, Continuity, Derivative**
   (a) Limit. Prove that (1) is equivalent to the pair of relations
   $$\lim_{z \to z_0} \text{Re}\,f(z) = \text{Re}\,l, \qquad \lim_{z \to z_0} \text{Im}\,f(z) = \text{Im}\,l.$$
   (b) Limit. If $\lim_{z \to z_0} f(x)$ exists, show that this limit is unique.
   (c) Continuity. If $z_1, z_2, \cdots$ are complex numbers for which $\lim_{n \to \infty} z_n = a$, and if $f(z)$ is continuous at $z = a$, show that $\lim_{n \to \infty} f(z_n) = f(a)$.

(d) Continuity. If $f(z)$ is differentiable at z_0, show that $f(z)$ is continuous at z_0.

(e) Differentiability. Show that $f(z) = \text{Re}\, z = x$ is not differentiable at any z. Can you find other such functions?

(f) Differentiability. Show that $f(z) = |z|^2$ is differentiable only at $z = 0$; hence it is nowhere analytic.

25. WRITING PROJECT. Comparison with Calculus. Summarize the second part of this section beginning with *Complex Function*, and indicate what is conceptually analogous to calculus and what is not.

13.4 Cauchy–Riemann Equations. Laplace's Equation

As we saw in the last section, to do complex analysis (i.e., "calculus in the complex") on any complex function, we require that function to be *analytic on some domain* that is differentiable in that domain.

The Cauchy–Riemann equations are the most important equations in this chapter and one of the pillars on which complex analysis rests. They provide a criterion (a test) for the analyticity of a complex function

$$w = f(z) = u(x, y) + iv(x, y).$$

Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two **Cauchy–Riemann equations**[4]

$$\text{(1)} \qquad u_x = v_y, \qquad u_y = -v_x$$

everywhere in D; here $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$ (and similarly for v) are the usual notations for partial derivatives. The precise formulation of this statement is given in Theorems 1 and 2.

Example: $f(z) = z^2 = x^2 - y^2 + 2ixy$ is analytic for all z (see Example 3 in Sec. 13.3), and $u = x^2 - y^2$ and $v = 2xy$ satisfy (1), namely, $u_x = 2x = v_y$ as well as $u_y = -2y = -v_x$. More examples will follow.

THEOREM 1

Cauchy–Riemann Equations

Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then, at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy–Riemann equations (1).

Hence, if $f(z)$ is analytic in a domain D, those partial derivatives exist and satisfy (1) *at all points of D.*

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[4] The French mathematician AUGUSTIN-LOUIS CAUCHY (see Sec. 2.5) and the German mathematicians BERNHARD RIEMANN (1826–1866) and KARL WEIERSTRASS (1815–1897; see also Sec. 15.5) are the founders of complex analysis. Riemann received his Ph.D. (in 1851) under Gauss (Sec. 5.4) at Göttingen, where he also taught until he died, when he was only 39 years old. He introduced the concept of the integral as it is used in basic calculus courses, and made important contributions to differential equations, number theory, and mathematical physics. He also developed the so-called Riemannian geometry, which is the mathematical foundation of Einstein's theory of relativity; see Ref. [GenRef9] in App. 1.

PROOF By assumption, the derivative $f'(z)$ at z exists. It is given by

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}. \tag{2}$$

The idea of the proof is very simple. By the definition of a limit in complex (Sec. 13.3), we can let Δz approach zero along any path in a neighborhood of z. Thus we may choose the two paths I and II in Fig. 335 and equate the results. By comparing the real parts we shall obtain the first Cauchy–Riemann equation and by comparing the imaginary parts the second. The technical details are as follows.

We write $\Delta z = \Delta x + i\,\Delta y$. Then $z + \Delta z = x + \Delta x + i(y + \Delta y)$, and in terms of u and v the derivative in (2) becomes

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\,\Delta y}. \tag{3}$$

We first choose path I in Fig. 335. Thus we let $\Delta y \to 0$ first and then $\Delta x \to 0$. After Δy is zero, $\Delta z = \Delta x$. Then (3) becomes, if we first write the two u-terms and then the two v-terms,

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}.$$

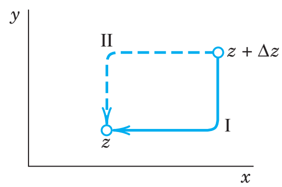

Fig. 335. Paths in (2)

Since $f'(z)$ exists, the two real limits on the right exist. By definition, they are the partial derivatives of u and v with respect to x. Hence the derivative $f'(z)$ of $f(z)$ can be written

$$\boxed{f'(z) = u_x + iv_x.} \tag{4}$$

Similarly, if we choose path II in Fig. 335, we let $\Delta x \to 0$ first and then $\Delta y \to 0$. After Δx is zero, $\Delta z = i\,\Delta y$, so that from (3) we now obtain

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\,\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\,\Delta y}.$$

Since $f'(z)$ exists, the limits on the right exist and give the partial derivatives of u and v with respect to y; noting that $1/i = -i$, we thus obtain

$$\boxed{f'(z) = -iu_y + v_y.} \tag{5}$$

The existence of the derivative $f'(z)$ thus implies the existence of the four partial derivatives in (4) and (5). By equating the real parts u_x and v_y in (4) and (5) we obtain the first

Cauchy–Riemann equation (1). Equating the imaginary parts gives the other. This proves the first statement of the theorem and implies the second because of the definition of analyticity. ■

Formulas (4) and (5) are also quite practical for calculating derivatives $f'(z)$, as we shall see.

EXAMPLE 1 Cauchy–Riemann Equations

$f(z) = z^2$ is analytic for all z. It follows that the Cauchy–Riemann equations must be satisfied (as we have verified above).

For $f(z) = \bar{z} = x - iy$ we have $u = x$, $v = -y$ and see that the second Cauchy–Riemann equation is satisfied, $u_y = -v_x = 0$, but the first is not: $u_x = 1 \neq v_y = -1$. We conclude that $f(z) = \bar{z}$ is not analytic, confirming Example 4 of Sec. 13.3. Note the savings in calculation! ■

The Cauchy–Riemann equations are fundamental because they are not only necessary but also sufficient for a function to be analytic. More precisely, the following theorem holds.

THEOREM 2 Cauchy–Riemann Equations

*If two real-valued continuous functions $u(x, y)$ and $v(x, y)$ of two real variables x and y have **continuous** first partial derivatives that satisfy the Cauchy–Riemann equations in some domain D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.*

The proof is more involved than that of Theorem 1 and we leave it optional (see App. 4).

Theorems 1 and 2 are of great practical importance, since, by using the Cauchy–Riemann equations, we can now easily find out whether or not a given complex function is analytic.

EXAMPLE 2 Cauchy–Riemann Equations. Exponential Function

Is $f(z) = u(x, y) + iv(x, y) = e^x(\cos y + i \sin y)$ analytic?

Solution. We have $u = e^x \cos y$, $v = e^x \sin y$ and by differentiation

$$u_x = e^x \cos y, \qquad v_y = e^x \cos y$$

$$u_y = -e^x \sin y, \qquad v_x = e^x \sin y.$$

We see that the Cauchy–Riemann equations are satisfied and conclude that $f(z)$ is analytic for all z. ($f(z)$ will be the complex analog of e^x known from calculus.) ■

EXAMPLE 3 An Analytic Function of Constant Absolute Value Is Constant

The Cauchy–Riemann equations also help in deriving general properties of analytic functions.

For instance, show that if $f(z)$ is analytic in a domain D and $|f(z)| = k = \text{const}$ in D, then $f(z) = \text{const}$ in D. (We shall make crucial use of this in Sec. 18.6 in the proof of Theorem 3.)

Solution. By assumption, $|f|^2 = |u + iv|^2 = u^2 + v^2 = k^2$. By differentiation,

$$uu_x + vv_x = 0,$$

$$uu_y + vv_y = 0.$$

Now use $v_x = -u_y$ in the first equation and $v_y = u_x$ in the second, to get

$$\text{(6)} \qquad \begin{aligned} &\text{(a)} \quad uu_x - vu_y = 0, \\ &\text{(b)} \quad uu_y - vu_x = 0. \end{aligned}$$

To get rid of u_y, multiply (6a) by u and (6b) by v and add. Similarly, to eliminate u_x, multiply (6a) by $-v$ and (6b) by u and add. This yields

$$(u^2 + v^2)u_x = 0,$$
$$(u^2 + v^2)u_y = 0.$$

If $k^2 = u^2 + v^2 = 0$, then $u = v = 0$; hence $f = 0$. If $k^2 = u^2 + v^2 \neq 0$, then $u_x = u_y = 0$. Hence, by the Cauchy–Riemann equations, also $u_x = v_y = 0$. Together this implies $u = \text{const}$ and $v = \text{const}$; hence $f = \text{const}$. ■

We mention that, if we use the polar form $z = r(\cos\theta + i\sin\theta)$ and set $f(z) = u(r, \theta) + iv(r, \theta)$, then the **Cauchy–Riemann equations** are (Prob. 1)

(7)
$$\begin{aligned} u_r &= \frac{1}{r} v_\theta, \\ v_r &= -\frac{1}{r} u_\theta \end{aligned} \qquad (r > 0).$$

Laplace's Equation. Harmonic Functions

The great importance of complex analysis in engineering mathematics results mainly from the fact that both the real part and the imaginary part of an analytic function satisfy Laplace's equation, the most important PDE of physics. It occurs in gravitation, electrostatics, fluid flow, heat conduction, and other applications (see Chaps. 12 and 18).

THEOREM 3

Laplace's Equation

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then both u and v satisfy **Laplace's equation**

(8) $$\nabla^2 u = u_{xx} + u_{yy} = 0$$

(∇^2 read "nabla squared") *and*

(9) $$\nabla^2 v = v_{xx} + v_{yy} = 0,$$

in D and have continuous second partial derivatives in D.

PROOF Differentiating $u_x = v_y$ with respect to x and $u_y = -v_x$ with respect to y, we have

(10) $$u_{xx} = v_{yx}, \qquad u_{yy} = -v_{xy}.$$

Now the derivative of an analytic function is itself analytic, as we shall prove later (in Sec. 14.4). This implies that u and v have continuous partial derivatives of all orders; in particular, the mixed second derivatives are equal: $v_{yx} = v_{xy}$. By adding (10) we thus obtain (8). Similarly, (9) is obtained by differentiating $u_x = v_y$ with respect to y and $u_y = -v_x$ with respect to x and subtracting, using $u_{xy} = u_{yx}$. ■

Solutions of Laplace's equation having ***continuous*** second-order partial derivatives are called **harmonic functions** and their theory is called **potential theory** (see also Sec. 12.11). Hence the real and imaginary parts of an analytic function are harmonic functions.

If two harmonic functions u and v satisfy the Cauchy–Riemann equations in a domain D, they are the real and imaginary parts of an analytic function f in D. Then v is said to be a **harmonic conjugate function** of u in D. (Of course, this has absolutely nothing to do with the use of "conjugate" for $\bar{z}$.)

EXAMPLE 4 How to Find a Harmonic Conjugate Function by the Cauchy–Riemann Equations

Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function v of u.

Solution. $\nabla^2 u = 0$ by direct calculation. Now $u_x = 2x$ and $u_y = -2y - 1$. Hence because of the Cauchy–Riemann equations a conjugate v of u must satisfy

$$v_y = u_x = 2x, \qquad v_x = -u_y = 2y + 1.$$

Integrating the first equation with respect to y and differentiating the result with respect to x, we obtain

$$v = 2xy + h(x), \qquad v_x = 2y + \frac{dh}{dx}.$$

A comparison with the second equation shows that $dh/dx = 1$. This gives $h(x) = x + c$. Hence $v = 2xy + x + c$ (c any real constant) is the most general harmonic conjugate of the given u. The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic. \qquad \blacksquare$$

Example 4 illustrates that *a conjugate of a given harmonic function is uniquely determined up to an arbitrary real additive constant.*

The Cauchy–Riemann equations are the most important equations in this chapter. Their relation to Laplace's equation opens a wide range of engineering and physical applications, as shown in Chap. 18.

PROBLEM SET 13.4

1. **Cauchy–Riemann equations in polar form.** Derive (7) from (1).

2–11 CAUCHY–RIEMANN EQUATIONS

Are the following functions analytic? Use (1) or (7).

2. $f(z) = iz\bar{z}$
3. $f(z) = e^{-2x}(\cos 2y - i \sin 2y)$
4. $f(z) = e^x(\cos y - i \sin y)$
5. $f(z) = \text{Re}\,(z^2) - i\,\text{Im}\,(z^2)$
6. $f(z) = 1/(z - z^5)$
7. $f(z) = i/z^8$
8. $f(z) = \text{Arg}\, 2\pi z$
9. $f(z) = 3\pi^2/(z^3 + 4\pi^2 z)$
10. $f(z) = \ln |z| + i\,\text{Arg}\, z$
11. $f(z) = \cos x \cosh y - i \sin x \sinh y$

12–19 HARMONIC FUNCTIONS

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

12. $u = x^2 + y^2$
13. $u = xy$
14. $v = xy$
15. $u = x/(x^2 + y^2)$
16. $u = \sin x \cosh y$
17. $v = (2x + 1)y$
18. $u = x^3 - 3xy^2$
19. $v = e^x \sin 2y$
20. **Laplace's equation.** Give the details of the derivative of (9).

21–24

Determine a and b so that the given function is harmonic and find a harmonic conjugate.

21. $u = e^{\pi x} \cos av$
22. $u = \cos ax \cosh 2y$
23. $u = ax^3 + bxy$
24. $u = \cosh ax \cos y$
25. **CAS PROJECT. Equipotential Lines.** Write a program for graphing equipotential lines $u = \text{const}$ of a harmonic function u and of its conjugate v on the same axes. Apply the program to **(a)** $u = x^2 - y^2$, $v = 2xy$, **(b)** $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$.
26. Apply the program in Prob. 25 to $u = e^x \cos y$, $v = e^x \sin y$ and to an example of your own.

27. Harmonic conjugate. Show that if u is harmonic and v is a harmonic conjugate of u, then u is a harmonic conjugate of $-v$.

28. Illustrate Prob. 27 by an example.

29. Two further formulas for the derivative. Formulas (4), (5), and (11) (below) are needed from time to time. Derive

$$f'(z) = u_x - iu_y, \qquad f'(z) = v_y + iv_x. \tag{11}$$

30. TEAM PROJECT. Conditions for $f(z) = $ const. Let $f(z)$ be analytic. Prove that each of the following conditions is sufficient for $f(z) = $ const.

(a) $\operatorname{Re} f(z) = $ const

(b) $\operatorname{Im} f(z) = $ const

(c) $f'(z) = 0$

(d) $|f(z)| = $ const (see Example 3)

13.5 Exponential Function

In the remaining sections of this chapter we discuss the basic elementary complex functions, the exponential function, trigonometric functions, logarithm, and so on. They will be counterparts to the familiar functions of calculus, to which they reduce when $z = x$ is real. They are indispensable throughout applications, and some of them have interesting properties not shared by their real counterparts.

We begin with one of the most important analytic functions, the complex **exponential function**

$$e^z, \qquad \text{also written} \qquad \exp z.$$

The definition of e^z in terms of the real functions e^x, $\cos y$, and $\sin y$ is

$$e^z = e^x(\cos y + i \sin y). \tag{1}$$

This definition is motivated by the fact the e^z ***extends*** the real exponential function e^x of calculus in a natural fashion. Namely:

(A) $e^z = e^x$ for real $z = x$ because $\cos y = 1$ and $\sin y = 0$ when $y = 0$.

(B) e^z is analytic for all z. (Proved in Example 2 of Sec. 13.4.)

(C) The derivative of e^z is e^z, that is,

$$(e^z)' = e^z. \tag{2}$$

This follows from (4) in Sec. 13.4,

$$(e^z)' = (e^x \cos y)_x + i(e^x \sin y)_x = e^x \cos y + ie^x \sin y = e^z.$$

REMARK. This definition provides for a relatively simple discussion. We could define e^z by the familiar series $1 + x + x^2/2! + x^3/3! + \cdots$ with x replaced by z, but we would then have to discuss complex series at this very early stage. (We will show the connection in Sec. 15.4.)

Further Properties. A function $f(z)$ that is analytic for all z is called an **entire function**. Thus, e^z is entire. Just as in calculus the ***functional relation***

$$e^{z_1+z_2} = e^{z_1}e^{z_2} \tag{3}$$

holds for any $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Indeed, by (1),

$$e^{z_1}e^{z_2} = e^{x_1}(\cos y_1 + i \sin y_1)\, e^{x_2}(\cos y_2 + i \sin y_2).$$

Since $e^{x_1}e^{x_2} = e^{x_1+x_2}$ for these *real* functions, by an application of the addition formulas for the cosine and sine functions (similar to that in Sec. 13.2) we see that

$$e^{z_1}e^{z_2} = e^{x_1+x_2}[\cos (y_1 + y_2) + i \sin (y_1 + y_2)] = e^{z_1+z_2}$$

as asserted. An interesting special case of (3) is $z_1 = x$, $z_2 = iy$; then

$$e^z = e^x e^{iy}. \tag{4}$$

Furthermore, for $z = iy$ we have from (1) the so-called **Euler formula**

$$\boxed{e^{iy} = \cos y + i \sin y.} \tag{5}$$

Hence the **polar form** of a complex number, $z = r(\cos \theta + i \sin \theta)$, may now be written

$$\boxed{z = re^{i\theta}.} \tag{6}$$

From (5) we obtain

$$\boxed{e^{2\pi i} = 1} \tag{7}$$

as well as the important formulas (verify!)

$$e^{\pi i/2} = i, \qquad e^{\pi i} = -1, \qquad e^{-\pi i/2} = -i, \qquad e^{-\pi i} = -1. \tag{8}$$

Another consequence of (5) is

$$|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1. \tag{9}$$

That is, for pure imaginary exponents, the exponential function has absolute value 1, a result you should remember. From (9) and (1),

$$|e^z| = e^x. \qquad \text{Hence} \qquad \arg e^z = y \pm 2n\pi \quad (n = 0, 1, 2, \cdots), \tag{10}$$

since $|e^z| = e^x$ shows that (1) is actually e^z in polar form.

From $|e^z| = e^x \neq 0$ in (10) we see that

$$\boxed{e^x \neq 0} \qquad \text{for all } z. \tag{11}$$

So here we have an entire function that never vanishes, in contrast to (nonconstant) polynomials, which are also entire (Example 5 in Sec. 13.3) but always have a zero, as is proved in algebra.

Periodicity of e^z with period $2\pi i$,

$$e^{z+2\pi i} = e^z \qquad \text{for all } z \tag{12}$$

is a basic property that follows from (1) and the periodicity of $\cos y$ and $\sin y$. Hence all the values that $w = e^z$ can assume are already assumed in the horizontal strip of width 2π

$$-\pi < y \leqq \pi \qquad \text{(Fig. 336).} \tag{13}$$

This infinite strip is called a **fundamental region** of e^z.

EXAMPLE 1 **Function Values. Solution of Equations**

Computation of values from (1) provides no problem. For instance,

$$e^{1.4-0.6i} = e^{1.4}(\cos 0.6 - i \sin 0.6) = 4.055(0.8253 - 0.5646i) = 3.347 - 2.289i$$
$$|e^{1.4-1.6i}| = e^{1.4} = 4.055, \qquad \text{Arg } e^{1.4-0.6i} = -0.6.$$

To illustrate (3), take the product of

$$e^{2+i} = e^2(\cos 1 + i \sin 1) \qquad \text{and} \qquad e^{4-i} = e^4(\cos 1 - i \sin 1)$$

and verify that it equals $e^2e^4(\cos^2 1 + \sin^2 1) = e^6 = e^{(2+i)+(4-i)}$.

To solve the equation $e^z = 3 + 4i$, note first that $|e^z| = e^x = 5, x = \ln 5 = 1.609$ is the real part of all solutions. Now, since $e^x = 5$,

$$e^x \cos y = 3, \qquad e^x \sin y = 4, \qquad \cos y = 0.6, \qquad \sin y = 0.8, \qquad y = 0.927.$$

Ans. $z = 1.609 + 0.927i \pm 2n\pi i$ $(n = 0, 1, 2, \cdots)$. These are infinitely many solutions (due to the periodicity of e^z). They lie on the vertical line $x = 1.609$ at a distance 2π from their neighbors.

To summarize: many properties of $e^z = \exp z$ parallel those of e^x; an exception is the periodicity of e^z with $2\pi i$, which suggested the concept of a fundamental region. Keep in mind that e^z is an *entire function.* (Do you still remember what that means?)

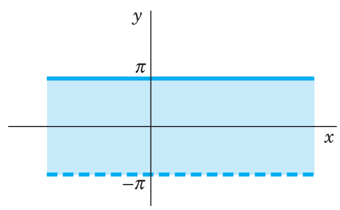

Fig. 336. Fundamental region of the exponential function e^z in the z-plane

PROBLEM SET 13.5

1. e^z is entire. Prove this.

2–7 **Function Values.** Find e^z in the form $u + iv$ and $|e^z|$ if z equals

2. $3 + 4i$

3. $2\pi i(1 + i)$

4. $0.6 - 1.8i$

5. $2 + 3\pi i$

6. $11\pi i/2$

7. $\sqrt{2} + \frac{1}{2}\pi i$

8–13 **Polar Form.** Write in exponential form (6):

8. $\sqrt[n]{z}$

9. $4 + 3i$

10. $\sqrt{i}, \quad \sqrt{-i}$

11. -6.3

12. $1/(1 - z)$

13. $1 + i$

14–17 **Real and Imaginary Parts.** Find Re and Im of

14. $e^{-\pi z}$

15. $\exp(z^2)$

16. $e^{1/z}$

17. $\exp(z^3)$

18. TEAM PROJECT. Further Properties of the Exponential Function. (a) Analyticity. Show that e^z is entire. What about $e^{1/z}$? $e^{\bar{z}}$? $e^x(\cos ky + i \sin ky)$? (Use the Cauchy–Riemann equations.)

(b) Special values. Find all z such that (i) e^z is real, (ii) $|e^{-z}| < 1$, (iii) $e^{\bar{z}} = \overline{e^z}$.

(c) Harmonic function. Show that $u = e^{xy} \cos(x^2/2 - y^2/2)$ is harmonic and find a conjugate.

(d) Uniqueness. It is interesting that $f(z) = e^z$ is uniquely determined by the two properties $f(x + i0) = e^x$ and $f'(z) = f(z)$, where f is assumed to be entire. Prove this using the Cauchy–Riemann equations.

19–22 **Equations.** Find all solutions and graph some of them in the complex plane.

19. $e^z = 1$

20. $e^z = 4 + 3i$

21. $e^z = 0$

22. $e^z = -2$

13.6 Trigonometric and Hyperbolic Functions. Euler's Formula

Just as we extended the real e^x to the complex e^z in Sec. 13.5, we now want to extend the familiar *real* trigonometric functions to *complex trigonometric functions*. We can do this by the use of the Euler formulas (Sec. 13.5)

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x.$$

By addition and subtraction we obtain for the *real* cosine and sine

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

This suggests the following definitions for complex values $z = x + iy$:

$$\textbf{(1)} \qquad \cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

It is quite remarkable that here in complex, functions come together that are unrelated in real. This is not an isolated incident but is typical of the general situation and shows the advantage of working in complex.

Furthermore, as in calculus we define

$$(2) \qquad \tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}$$

and

$$(3) \qquad \sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}.$$

Since e^z is entire, $\cos z$ and $\sin z$ are entire functions. $\tan z$ and $\sec z$ are not entire; they are analytic except at the points where $\cos z$ is zero; and $\cot z$ and $\csc z$ are analytic except where $\sin z$ is zero. Formulas for the derivatives follow readily from $(e^z)' = e^z$ and (1)–(3); as in calculus,

$$(4) \qquad (\cos z)' = -\sin z, \qquad (\sin z)' = \cos z, \qquad (\tan z)' = \sec^2 z,$$

etc. Equation (1) also shows that **Euler's formula *is valid in complex***:

$$\textbf{(5)} \qquad \boxed{e^{iz} = \cos z + i \sin z} \qquad \text{for all } z.$$

The real and imaginary parts of $\cos z$ and $\sin z$ are needed in computing values, and they also help in displaying properties of our functions. We illustrate this with a typical example.

EXAMPLE 1 Real and Imaginary Parts. Absolute Value. Periodicity

Show that

$$\textbf{(6)} \qquad \begin{aligned} &\textbf{(a)} \quad \cos z = \cos x \cosh y - i \sin x \sinh y \\ &\textbf{(b)} \quad \sin z = \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

and

$$(7) \qquad \begin{aligned} &\textbf{(a)} \quad |\cos z|^2 = \cos^2 x + \sinh^2 y \\ &\textbf{(b)} \quad |\sin z|^2 = \sin^2 x + \sinh^2 y \end{aligned}$$

and give some applications of these formulas.

Solution. From (1),

$$\begin{aligned} \cos z &= \tfrac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) \\ &= \tfrac{1}{2}e^{-y}(\cos x + i \sin x) + \tfrac{1}{2}e^{y}(\cos x - i \sin x) \\ &= \tfrac{1}{2}(e^{y} + e^{-y}) \cos x - \tfrac{1}{2}i(e^{y} - e^{-y}) \sin x. \end{aligned}$$

This yields (6a) since, as is known from calculus,

$$(8) \qquad \cosh y = \tfrac{1}{2}(e^{y} + e^{-y}), \qquad \sinh y = \tfrac{1}{2}(e^{y} - e^{-y});$$

(6b) is obtained similarly. From (6a) and $\cosh^2 y = 1 + \sinh^2 y$ we obtain

$$|\cos z|^2 = (\cos^2 x)(1 + \sinh^2 y) + \sin^2 x \sinh^2 y.$$

Since $\sin^2 x + \cos^2 x = 1$, this gives (7a), and (7b) is obtained similarly.

For instance, $\cos(2 + 3i) = \cos 2 \cosh 3 - i \sin 2 \sinh 3 = -4.190 - 9.109i$.

From (6) we see that $\sin z$ and $\cos z$ are ***periodic with period*** 2π, just as in real. Periodicity of $\tan z$ and $\cot z$ with period π now follows.

Formula (7) points to an essential difference between the real and the complex cosine and sine; whereas $|\cos x| \leqq 1$ and $|\sin x| \leqq 1$, the complex cosine and sine functions are ***no longer bounded*** but approach infinity in absolute value as $y \to \infty$, since then $\sinh y \to \infty$ in (7). ■

EXAMPLE 2 Solutions of Equations. Zeros of cos *z* and sin *z*

Solve (a) $\cos z = 5$ (which has no real solution!), (b) $\cos z = 0$, (c) $\sin z = 0$.

Solution. (a) $e^{2iz} - 10e^{iz} + 1 = 0$ from (1) by multiplication by e^{iz}. This is a quadratic equation in e^{iz}, with solutions (rounded off to 3 decimals)

$$e^{iz} = e^{-y+ix} = 5 \pm \sqrt{25 - 1} = 9.899 \quad \text{and} \quad 0.101.$$

Thus $e^{-y} = 9.899$ or 0.101, $e^{ix} = 1$, $y = \pm 2.292$, $x = 2n\pi$. *Ans.* $z = \pm 2n\pi \pm 2.292i$ $(n = 0, 1, 2, \cdots)$.

Can you obtain this from (6a)?

(b) $\cos x = 0$, $\sinh y = 0$ by (7a), $y = 0$. *Ans.* $z = \pm\frac{1}{2}(2n + 1)\pi$ $(n = 0, 1, 2, \cdots)$.

(c) $\sin x = 0$, $\sinh y = 0$ by (7b), *Ans.* $z = \pm n\pi$ $(n = 0, 1, 2, \cdots)$.

Hence the only zeros of $\cos z$ and $\sin z$ are those of the real cosine and sine functions. ■

General formulas *for the real trigonometric functions continue to hold for complex values.* This follows immediately from the definitions. We mention in particular the addition rules

$$\begin{aligned} \cos(z_1 \pm z_2) &= \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \\ \sin(z_1 \pm z_2) &= \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1 \end{aligned} \tag{9}$$

and the formula

$$\cos^2 z + \sin^2 z = 1. \tag{10}$$

Some further useful formulas are included in the problem set.

Hyperbolic Functions

The complex **hyperbolic cosine** and **sine** are defined by the formulas

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}), \qquad \sinh z = \tfrac{1}{2}(e^z - e^{-z}). \tag{11}$$

This is suggested by the familiar definitions for a real variable [see (8)]. These functions are entire, with derivatives

$$(\cosh z)' = \sinh z, \qquad (\sinh z)' = \cosh z, \tag{12}$$

as in calculus. The other hyperbolic functions are defined by

$$\begin{aligned} \tanh z &= \frac{\sinh z}{\cosh z}, \qquad & \coth z &= \frac{\cosh z}{\sinh z}, \\ \operatorname{sech} z &= \frac{1}{\cosh z}, \qquad & \operatorname{csch} z &= \frac{1}{\sinh z}. \end{aligned} \tag{13}$$

Complex Trigonometric and Hyperbolic Functions Are Related. If in (11), we replace z by iz and then use (1), we obtain

$$\cosh iz = \cos z, \qquad \sinh iz = i \sin z. \tag{14}$$

Similarly, if in (1) we replace z by iz and then use (11), we obtain conversely

$$\cos iz = \cosh z, \qquad \sin iz = i \sinh z. \tag{15}$$

Here we have another case of *unrelated* real functions that have *related* complex analogs, pointing again to the advantage of working in complex in order to get both a more unified formalism and a deeper understanding of special functions. This is one of the main reasons for the importance of complex analysis to the engineer and physicist.

PROBLEM SET 13.6

1–4 FORMULAS FOR HYPERBOLIC FUNCTIONS

Show that

1. $\cosh z = \cosh x \cos y + i \sinh x \sin y$

$\sinh z = \sinh x \cos y + i \cosh x \sin y.$

2. $\cosh (z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

$\sinh (z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$

3. $\cosh^2 z - \sinh^2 z = 1, \quad \cosh^2 z + \sinh^2 z = \cosh 2z$

4. Entire Functions. Prove that $\cos z$, $\sin z$, $\cosh z$, and $\sinh z$ are entire.

5. Harmonic Functions. Verify by differentiation that $\text{Im} \cos z$ and $\text{Re} \sin z$ are harmonic.

6–12 Function Values. Find, in the form $u + iv$,

6. $\sin 2\pi i$

7. $\cos i, \quad \sin i$

8. $\cos \pi i, \quad \cosh \pi i$

9. $\cosh (-1 + 2i), \quad \cos (-2 - i)$

10. $\sinh (3 + 4i), \quad \cosh (3 + 4i)$

11. $\sin \pi i, \quad \cos (\frac{1}{2}\pi - \pi i)$

12. $\cos \frac{1}{2}\pi i, \quad \cos [\frac{1}{2}\pi(1 + i)]$

13–15 Equations and Inequalities. Using the definitions, prove:

13. $\cos z$ is even, $\cos(-z) = \cos z$, and $\sin z$ is odd, $\sin(-z) = -\sin z$.

14. $|\sinh y| \leqq |\cos z| \leqq \cosh y, |\sinh y| \leqq |\sin z| \leqq \cosh y$. Conclude that the complex cosine and sine are not bounded in the whole complex plane.

15. $\sin z_1 \cos z_2 = \frac{1}{2}[\sin(z_1 + z_2) + \sin(z_1 - z_2)]$

16–19 Equations. Find all solutions.

16. $\sin z = 100$

17. $\cosh z = 0$

18. $\cosh z = -1$

19. $\sinh z = 0$

20. Re tan z and Im tan z. Show that

$$\text{Re} \tan z = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y},$$

$$\text{Im} \tan z = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y}.$$

13.7 Logarithm. General Power. Principal Value

We finally introduce the *complex logarithm*, which is more complicated than the real logarithm (which it includes as a special case) and historically puzzled mathematicians for some time (so if you first get puzzled—which need not happen!—be patient and work through this section with extra care).

The **natural logarithm** of $z = x + iy$ is denoted by $\ln z$ (sometimes also by $\log z$) and is defined as the inverse of the exponential function; that is, $w = \ln z$ is defined for $z \neq 0$ by the relation

$$e^w = z.$$

(Note that $z = 0$ is impossible, since $e^w \neq 0$ for all w; see Sec. 13.5.) If we set $w = u + iv$ and $z = re^{i\theta}$, this becomes

$$e^w = e^{u+iv} = re^{i\theta}.$$

Now, from Sec. 13.5, we know that e^{u+iv} has the absolute value e^u and the argument v. These must be equal to the absolute value and argument on the right:

$$e^u = r, \qquad v = \theta.$$

$e^u = r$ gives $u = \ln r$, where $\ln r$ is the familiar *real* natural logarithm of the positive number $r = |z|$. Hence $w = u + iv = \ln z$ is given by

$$\boxed{\ln z = \ln r + i\theta} \qquad (r = |z| > 0, \quad \theta = \arg z). \tag{1}$$

Now comes an important point (without analog in real calculus). Since the argument of z is determined only up to integer multiples of 2π, ***the complex natural logarithm*** $\ln z$ ($z \neq 0$) ***is infinitely many-valued.***

The value of $\ln z$ corresponding to the principal value $\operatorname{Arg} z$ (see Sec. 13.2) is denoted by $\operatorname{Ln} z$ (Ln with capital L) and is called the **principal value** of $\ln z$. Thus

$$\boxed{\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z} \qquad (z \neq 0). \tag{2}$$

The uniqueness of $\operatorname{Arg} z$ for given z ($\neq 0$) implies that $\operatorname{Ln} z$ is single-valued, that is, a function in the usual sense. Since the other values of $\arg z$ differ by integer multiples of 2π, the other values of $\ln z$ are given by

$$\boxed{\ln z = \operatorname{Ln} z \pm 2n\pi i} \qquad (n = 1, 2, \cdots). \tag{3}$$

They all have the same real part, and their imaginary parts differ by integer multiples of 2π.

If z is positive real, then $\operatorname{Arg} z = 0$, and $\operatorname{Ln} z$ becomes identical with the real natural logarithm known from calculus. If z is negative real (so that the natural logarithm of calculus is not defined!), then $\operatorname{Arg} z = \pi$ and

$$\operatorname{Ln} z = \ln |z| + \pi i \qquad (z \text{ negative real}).$$

From (1) and $e^{\ln r} = r$ for positive real r we obtain

$$e^{\ln z} = z \tag{4a}$$

as expected, but since $\arg (e^z) = y \pm 2n\pi$ is multivalued, so is

$$\ln (e^z) = z \pm 2n\pi i, \qquad n = 0, 1, \cdots. \tag{4b}$$

EXAMPLE 1 Natural Logarithm. Principal Value

$$\begin{aligned}
\ln 1 &= 0, \pm 2\pi i, \pm 4\pi i, \cdots & \operatorname{Ln} 1 &= 0\\
\ln 4 &= 1.386294 \pm 2n\pi i & \operatorname{Ln} 4 &= 1.386294\\
\ln (-1) &= \pm \pi i, \pm 3\pi i, \pm 5\pi i, \cdots & \operatorname{Ln} (-1) &= \pi i\\
\ln (-4) &= 1.386294 \pm (2n + 1)\pi i & \operatorname{Ln} (-4) &= 1.386294 + \pi i\\
\ln i &= \pi i/2, -3\pi/2, 5\pi i/2, \cdots & \operatorname{Ln} i &= \pi i/2\\
\ln 4i &= 1.386294 + \pi i/2 \pm 2n\pi i & \operatorname{Ln} 4i &= 1.386294 + \pi i/2\\
\ln (-4i) &= 1.386294 - \pi i/2 \pm 2n\pi i & \operatorname{Ln} (-4i) &= 1.386294 - \pi i/2\\
\ln (3 - 4i) &= \ln 5 + i \arg (3 - 4i) & \operatorname{Ln} (3 - 4i) &= 1.609438 - 0.927295i\\
&= 1.609438 - 0.927295i \pm 2n\pi i & &\text{(Fig. 337)}
\end{aligned}$$

■

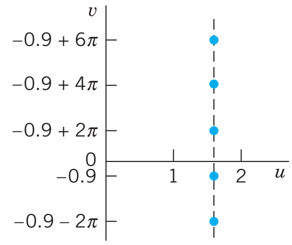

Fig. 337. Some values of ln (3 − 4i) in Example 1

The familiar relations for the natural logarithm continue to hold for complex values, that is,

$$\text{(5)} \qquad \text{(a)} \quad \ln(z_1 z_2) = \ln z_1 + \ln z_2, \qquad \text{(b)} \quad \ln(z_1/z_2) = \ln z_1 - \ln z_2$$

but these relations are to be understood in the sense that each value of one side is also contained among the values of the other side; see the next example.

EXAMPLE 2 Illustration of the Functional Relation (5) in Complex

Let

$$z_1 = z_2 = e^{\pi i} = -1.$$

If we take the principal values

$$\operatorname{Ln} z_1 = \operatorname{Ln} z_2 = \pi i,$$

then (5a) holds provided we write $\ln(z_1 z_2) = \ln 1 = 2\pi i$; however, it is not true for the principal value, $\operatorname{Ln}(z_1 z_2) = \operatorname{Ln} 1 = 0$. ■

THEOREM 1 Analyticity of the Logarithm

For every $n = 0, \pm 1, \pm 2, \cdots$ *formula* (3) *defines a function, which is analytic, except at* 0 *and on the negative real axis, and has the derivative*

$$\text{(6)} \qquad (\ln z)' = \frac{1}{z} \qquad (z \text{ not } 0 \text{ or negative real}).$$

PROOF We show that the Cauchy–Riemann equations are satisfied. From (1)–(3) we have

$$\ln z = \ln r + i(\theta + c) = \frac{1}{2}\ln(x^2 + y^2) + i\left(\arctan\frac{y}{x} + c\right)$$

where the constant c is a multiple of 2π. By differentiation,

$$u_x = \frac{x}{x^2 + y^2} = v_y = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x}$$

$$u_y = \frac{y}{x^2 + y^2} = -v_x = -\frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right).$$

Hence the Cauchy–Riemann equations hold. [Confirm this by using these equations in polar form, which we did not use since we proved them only in the problems (to Sec. 13.4).] Formula (4) in Sec. 13.4 now gives (6),

$$(\ln z)' = u_x + iv_x = \frac{x}{x^2+y^2} + i\frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{x-iy}{x^2+y^2} = \frac{1}{z}.$$

Each of the infinitely many functions in (3) is called a **branch** of the logarithm. The negative real axis is known as a **branch cut** and is usually graphed as shown in Fig. 338. The branch for $n = 0$ is called the **principal branch** of $\ln z$.

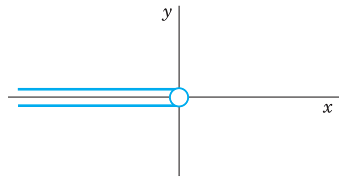

Fig. 338. Branch cut for $\ln z$

General Powers

General powers of a complex number $z = x + iy$ are defined by the formula

(7) $$z^c = e^{c \ln z} \qquad (c \text{ complex}, z \neq 0).$$

Since $\ln z$ is infinitely many-valued, z^c will, in general, be multivalued. The particular value

$$z^c = e^{c \operatorname{Ln} z}$$

is called the **principal value** *of* z^c.

If $c = n = 1, 2, \cdots$, then z^n is single-valued and identical with the usual nth power of z. If $c = -1, -2, \cdots$, the situation is similar.

If $c = 1/n$, where $n = 2, 3, \cdots$, then

$$z^c = \sqrt[n]{z} = e^{(1/n)\ln z} \qquad (z \neq 0),$$

the exponent is determined up to multiples of $2\pi i/n$ and we obtain the n distinct values of the nth root, in agreement with the result in Sec. 13.2. If $c = p/q$, the quotient of two positive integers, the situation is similar, and z^c has only finitely many distinct values. However, if c is real irrational or genuinely complex, then z^c is infinitely many-valued.

EXAMPLE 3 General Power

$$i^i = e^{i \ln i} = \exp(i \ln i) = \exp\left[i\left(\frac{\pi}{2}i \pm 2n\pi i\right)\right] = e^{-(\pi/2)\mp 2n\pi}.$$

All these values are real, and the principal value ($n = 0$) is $e^{-\pi/2}$.

Similarly, by direct calculation and multiplying out in the exponent,

$$\begin{aligned}(1+i)^{2-i} &= \exp[(2-i)\ln(1+i)] = \exp[(2-i)\{\ln\sqrt{2} + \tfrac{1}{4}\pi i \pm 2n\pi i\}] \\ &= 2e^{\pi/4 \pm 2n\pi}[\sin(\tfrac{1}{2}\ln 2) + i\cos(\tfrac{1}{2}\ln 2)].\end{aligned}$$

It is a ***convention*** that for real positive $z = x$ the expression z^c means $e^{c \ln x}$ where $\ln x$ is the elementary real natural logarithm (that is, the principal value $\operatorname{Ln} z$ $(z = x > 0)$ in the sense of our definition). Also, if $z = e$, the base of the natural logarithm, $z^c = e^c$ is *conventionally* regarded as the unique value obtained from (1) in Sec. 13.5.

From (7) we see that for any complex number a,

$$a^z = e^{z \ln a}. \tag{8}$$

We have now introduced the complex functions needed in practical work, some of them (e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$) entire (Sec. 13.5), some of them ($\tan z$, $\cot z$, $\tanh z$, $\coth z$) analytic except at certain points, and one of them ($\ln z$) splitting up into infinitely many functions, each analytic except at 0 and on the negative real axis.

For the **inverse trigonometric** and **hyperbolic functions** see the problem set.

PROBLEM SET 13.7

1–4 VERIFICATIONS IN THE TEXT

1. Verify the computations in Example 1.
2. Verify (5) for $z_1 = -i$ and $z_2 = -1$.
3. Prove analyticity of $\operatorname{Ln} z$ by means of the Cauchy–Riemann equations in polar form (Sec. 13.4).
4. Prove (4a) and (4b).

COMPLEX NATURAL LOGARITHM *ln z*

5–11 Principal Value Ln z. Find $\operatorname{Ln} z$ when z equals

5. -11
6. $4 + 4i$
7. $4 - 4i$
8. $1 \pm i$
9. $0.6 + 0.8i$
10. $-15 \pm 0.1i$
11. ei

12–16 All Values of ln z. Find all values and graph some of them in the complex plane.

12. $\ln e$
13. $\ln 1$
14. $\ln(-7)$
15. $\ln(e^i)$
16. $\ln(4 + 3i)$
17. Show that the set of values of $\ln(i^2)$ differs from the set of values of $2 \ln i$.

18–21 Equations. Solve for z.

18. $\ln z = -\pi i/2$
19. $\ln z = 4 - 3i$
20. $\ln z = e - \pi i$
21. $\ln z = 0.6 + 0.4i$

22–28 General Powers. Find the principal value. Show details.

22. $(2i)^{2i}$
23. $(1 + i)^{1-i}$
24. $(1 - i)^{1+i}$
25. $(-3)^{3-i}$
26. $(i)^{i/2}$
27. $(-1)^{2-i}$
28. $(3 + 4i)^{1/3}$
29. How can you find the answer to Prob. 24 from the answer to Prob. 23?
30. **TEAM PROJECT. Inverse Trigonometric and Hyperbolic Functions.** By definition, the **inverse sine** $w = \arcsin z$ is the relation such that $\sin w = z$. The **inverse cosine** $w = \arccos z$ is the relation such that $\cos w = z$. The **inverse tangent**, **inverse cotangent**, **inverse hyperbolic sine**, etc., are defined and denoted in a similar fashion. (Note that all these relations are ***multivalued***.) Using $\sin w = (e^{iw} - e^{-iw})/(2i)$ and similar representations of $\cos w$, etc., show that

    (a) $\arccos z = -i \ln(z + \sqrt{z^2 - 1})$

    (b) $\arcsin z = -i \ln(iz + \sqrt{1 - z^2})$

    (c) $\operatorname{arccosh} z = \ln(z + \sqrt{z^2 - 1})$

    (d) $\operatorname{arcsinh} z = \ln(z + \sqrt{z^2 + 1})$

    (e) $\arctan z = \dfrac{i}{2} \ln \dfrac{i + z}{i - z}$

    (f) $\operatorname{arctanh} z = \dfrac{1}{2} \ln \dfrac{1 + z}{1 - z}$

    (g) Show that $w = \arcsin z$ is infinitely many-valued, and if w_1 is one of these values, the others are of the form $w_1 \pm 2n\pi$ and $\pi - w_1 \pm 2n\pi$, $n = 0, 1, \cdots$. (The *principal value of* $w = u + iv = \arcsin z$ is defined to be the value for which $-\pi/2 \leqq u \leqq \pi/2$ if $v \geqq 0$ and $-\pi/2 < u < \pi/2$ if $v < 0$.)

CHAPTER 13 REVIEW QUESTIONS AND PROBLEMS

1. Divide $15 + 23i$ by $-3 + 7i$. Check the result by multiplication.

2. What happens to a quotient if you take the complex conjugates of the two numbers? If you take the absolute values of the numbers?

3. Write the two numbers in Prob. 1 in polar form. Find the principal values of their arguments.

4. State the definition of the derivative from memory. Explain the big difference from that in calculus.

5. What is an analytic function of a complex variable?

6. Can a function be differentiable at a point without being analytic there? If yes, give an example.

7. State the Cauchy–Riemann equations. Why are they of basic importance?

8. Discuss how e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ are related.

9. $\ln z$ is more complicated than $\ln x$. Explain. Give examples.

10. How are general powers defined? Give an example. Convert it to the form $x + iy$.

11–16 **Complex Numbers.** Find, in the form $x + iy$, showing details,

11. $(2 + 3i)^2$ **12.** $(1 - i)^{10}$

13. $1/(4 + 3i)$ **14.** $\sqrt{i}$

15. $(1 + i)/(1 - i)$ **16.** $e^{\pi i/2}$, $e^{-\pi i/2}$

17–20 **Polar Form.** Represent in polar form, with the principal argument.

17. $-4 - 4i$ **18.** $12 + i$, $12 - i$

19. $-15i$ **20.** $0.6 + 0.8i$

21–24 **Roots.** Find and graph all values of:

21. $\sqrt{81}$ **22.** $\sqrt{-32i}$

23. $\sqrt[4]{-1}$ **24.** $\sqrt[3]{1}$

25–30 **Analytic Functions.** Find $f(z) = u(x, y) + iv(x, y)$ with u or v as given. Check by the Cauchy–Riemann equations for analyticity.

25 $u = xy$ **26.** $v = y/(x^2 + y^2)$

27. $v = -e^{-2x} \sin 2y$ **28.** $u = \cos 3x \cosh 3y$

29. $u = \exp(-(x^2 - y^2)/2) \cos xy$

30. $v = \cos 2x \sinh 2y$

31–35 **Special Function Values.** Find the value of:

31. $\cos(3 - i)$ **32.** $\text{Ln}(0.6 + 0.8i)$

33. $\tan i$

34. $\sinh(1 + \pi i)$, $\sin(1 + \pi i)$

35. $\cosh(\pi + \pi i)$

SUMMARY OF CHAPTER 13
Complex Numbers and Functions. Complex Differentiation

For arithmetic operations with **complex numbers**

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta), \tag{1}$$

$r = |z| = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and for their representation in the complex plane, see Secs. 13.1 and 13.2.

A complex function $f(z) = u(x, y) + iv(x, y)$ is **analytic** in a domain D if it has a **derivative** (Sec. 13.3)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{2}$$

everywhere in D. Also, $f(z)$ is *analytic at a point* $z = z_0$ if it has a derivative in a neighborhood of z_0 (not merely at z_0 itself).

If $f(z)$ is analytic in D, then $u(x, y)$ and $v(x, y)$ satisfy the (very important!) **Cauchy–Riemann equations** (Sec. 13.4)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3}$$

everywhere in D. Then u and v also satisfy **Laplace's equation**

$$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0 \tag{4}$$

everywhere in D. If $u(x, y)$ and $v(x, y)$ are continuous and have *continuous* partial derivatives in D that satisfy (3) in D, then $f(z) = u(x, y) + iv(x, y)$ is analytic in D. See Sec. 13.4. (More on Laplace's equation and complex analysis follows in Chap. 18.)

The complex **exponential function** (Sec. 13.5)

$$e^z = \exp z = e^x (\cos y + i \sin y) \tag{5}$$

reduces to e^x if $z = x$ ($y = 0$). It is periodic with $2\pi i$ and has the derivative e^z.

The **trigonometric functions** are (Sec. 13.6)

$$\begin{aligned} \cos z &= \tfrac{1}{2}(e^{iz} + e^{-iz}) = \cos x \cosh y - i \sin x \sinh y \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) = \sin x \cosh y + i \cos x \sinh y \end{aligned} \tag{6}$$

and, furthermore,

$$\tan z = (\sin z)/\cos z, \qquad \cot z = 1/\tan z, \quad \text{etc.}$$

The **hyperbolic functions** are (Sec. 13.6)

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}) = \cos iz, \qquad \sinh z = \tfrac{1}{2}(e^z - e^{-z}) = -i \sin iz \tag{7}$$

etc. The functions (5)–(7) are **entire**, that is, analytic everywhere in the complex plane.

The **natural logarithm** is (Sec. 13.7)

$$\ln z = \ln|z| + i \arg z = \ln|z| + i \operatorname{Arg} z \pm 2n\pi i \tag{8}$$

where $z \neq 0$ and $n = 0, 1, \cdots$. Arg z is the **principal value** of arg z, that is, $-\pi < \operatorname{Arg} z \leqq \pi$. We see that $\ln z$ is infinitely many-valued. Taking $n = 0$ gives the **principal value** Ln z of ln z; thus $\operatorname{Ln} z = \ln|z| + i \operatorname{Arg} z$.

General powers are defined by (Sec. 13.7)

$$z^c = e^{c \ln z} \qquad (c \text{ complex}, z \neq 0). \tag{9}$$