

CHAPTER SEVEN

Moist air

Our atmosphere is basically a two-component system. One component is dry air and the other is water existing in vapor and possibly one of the condensed phases (liquid water or ice). According to Dalton's law, in a mixture of ideal gases each gas can be assumed to behave as if the other gases were absent. As such, in a mixture of dry air, water vapor, and a condensed phase, the "water" system (water vapor + condensed phase) can be treated as being independent of the dry air. In this case, the concepts developed in the previous chapter (for the one-component heterogeneous system "water") are valid for the two-component heterogeneous system "dry air + water". We will call the system consisting of dry air and water vapor "moist air" and it can be unsaturated or saturated with water vapor. Since liquid water is absent, moist air is a two-component system with one phase present. As such, according to equation (6.1) we need three state variables to specify the system's state. Usually these variables are taken to be pressure, temperature, and a new variable (to be defined soon) called *mixing ratio*. If the condensed phase is present and in equilibrium with the vapor phase, then two variables are needed (typically temperature and pressure).

For clarity in our notation we will use the subscripts d , w , v to indicate dry air, liquid water, and water vapor, respectively. The only exception will be for the vapor pressure, which we will denote simply as e . Variables with no subscripts will correspond to a mixture of dry air and water vapor.

The following table gives various systems, their type, and the variables needed to specify their state (see equation 6.1).

System	Type	Variables needed to specify system's state
Dry air and water vapor	Heterogeneous – two components, one phase	3 (p, T mixing ratio)
Dry air, water vapor, and liquid water	Heterogeneous – two components, two phases	2 (p, T)
Dry air, water vapor, liquid water, and ice	Heterogeneous – two components, three phases	1 (T)
Liquid water and water vapor	Heterogeneous – one component, two phases	1 (T)
Liquid water, water vapor, and ice	Heterogeneous – one component, three phases	0
Water vapor	Homogeneous – one component, one phase	2 (p, T)
Dry air	Homogeneous – one component, one phase	2 (p, T)

7.1 Measures and description of moist air

7.1.1 Humidity variables

In a sample of moist air, dry air and water vapor have the same temperature T and occupy the same volume V . Thus, for water vapor

$$eV = m_v R_v T$$

or

$$e = \rho_v R_v T$$

where e is the vapor pressure, ρ_v is the density of water vapor, and R_v is the specific gas constant for water vapor. Since the molecular weight of water $M_v = 18.01 \text{ g mol}^{-1}$, the value of R_v is equal to $R^*/M_v = 461.5 \text{ J kg}^{-1} \text{ K}^{-1}$. Since $R^* = R_d M_d$ it follows that $R_d M_d = R_v M_v$. Thus,

$$\epsilon = \frac{R_d}{R_v} = \frac{M_v}{M_d} = 0.622. \quad (7.1)$$

We define the specific humidity, q , and mixing ratio, w , as

$$q = \frac{\rho_v}{\rho} = \frac{m_v}{m}$$

and

$$w = \frac{\rho_v}{\rho_d} = \frac{m_v}{m_d} \quad (7.2)$$

where $m = m_d + m_v$ is the total mass of the mixture (ρ_v is also called the absolute humidity). Since $\rho_v = eM_v/R^*T$, $\rho_d = p_dM_d/R^*T$, and $p_d = p - e$ it follows that

$$w = \epsilon \frac{e}{p - e}. \quad (7.3)$$

At saturation the *saturation* mixing ratio is

$$w_s = \epsilon \frac{e_s}{p - e_s} \quad (7.4)$$

where e_s is either the equilibrium (saturation) vapor pressure with respect to liquid water (e_{sw}) or with respect to ice (e_{si}). In general $e_s, e \ll p$ which reduces the above two equations to

$$w \approx \epsilon \frac{e}{p}, \quad w_s \approx \epsilon \frac{e_s}{p}. \quad (7.5)$$

Since $1/q = (m_d + m_v)/m_v = (m_d/m_v) + 1$ it follows that

$$\frac{1}{q} = \frac{1}{w} + 1$$

or

$$w = \frac{q}{1 - q}, \quad q = \frac{w}{1 + w}. \quad (7.6)$$

In the atmosphere both w and q are very small ($w, q \ll 1$). For this reason we can always assume that $w \approx q$.

The relative humidity, r , is defined as

$$r = \frac{m_v}{m_{vs}}$$

where m_v is the mass of water vapor in a sample of moist air of volume V and m_{vs} is the mass of water vapor the sample would have had if it were saturated. Because of the ideal gas law we can write r as

$$r = \frac{e}{e_s}. \quad (7.7)$$

Using (7.5) it follows that approximately

$$r \approx \frac{w}{w_s}.$$

7.1.2 Mean molecular weight of moist air and other quantities

According to equation (3.13) we have that the mean molecular weight of moist air is

$$\bar{M} = \frac{m_d + m_v}{\frac{m_d}{M_d} + \frac{m_v}{M_v}}.$$

We can manipulate the above equation by writing it as

$$\frac{1}{\bar{M}} = \left(\frac{m_d}{M_d} + \frac{m_v}{M_v} \right) \left(\frac{1}{m_d + m_v} \right)$$

or

$$\frac{1}{\bar{M}} = \frac{1}{M_d} \left[\frac{1}{m_d + m_v} \left(m_d + \frac{m_v M_d}{M_v} \right) \right]$$

or

$$\frac{1}{\bar{M}} = \frac{1}{M_d} \frac{m_d}{m_d + m_v} \left(1 + \frac{m_v/m_d}{M_v/M_d} \right)$$

or

$$\frac{1}{\bar{M}} = \frac{1}{M_d} \frac{1}{1 + w} \left(1 + \frac{w}{\epsilon} \right)$$

or

$$\frac{1}{\bar{M}} = \frac{1}{M_d} \left(\frac{1}{1 + w} + \frac{q}{\epsilon} \right).$$

Since

$$\frac{1}{1 + w} = \frac{q}{w}$$

and

$$w = \frac{q}{1 - q},$$

then

$$\frac{1}{\bar{M}} = \frac{1}{M_d} \left(1 - q + \frac{q}{\epsilon} \right)$$

or

$$\frac{1}{\bar{M}} = \frac{1}{M_d} \left[1 + \left(\frac{1}{\epsilon} - 1 \right) q \right]$$

or

$$\frac{1}{\bar{M}} = \frac{1}{M_d} (1 + 0.61q).$$

Then for moist air the equation of state is

$$\begin{aligned} pa = R_{\text{moist}} T &= \frac{R^*}{\bar{M}} T = \frac{R^*}{M_d} (1 + 0.61q) T, \\ &= R_d (1 + 0.61q) T. \end{aligned} \quad (7.8)$$

The above equation defines the *virtual temperature*,

$$T_{\text{virt}} = (1 + 0.61q) T, \quad (7.9)$$

which can be interpreted as the temperature of dry air having the same values of p and a as the moist air. Otherwise stated, virtual temperature is the temperature that air of a given pressure and volume (or density) would have if the air were completely free of water vapor. Since q in reality is always greater than zero, T_{virt} is always greater than T . The above equation also gives the gas constant of the mixture:

$$R = (1 + 0.61q)R_d. \quad (7.10)$$

Similarly, we can define the specific heat capacities of moist air. Suppose that a sample of such air receives, at a constant pressure, an amount of heat δQ which increases its temperature by dT . Some of this amount (δQ_d) is absorbed by the dry air and some (δQ_v) by the water vapor. In this case we can write that

$$\delta Q = \delta Q_d + \delta Q_v$$

or

$$\delta Q = m_d \delta q_d + m_v \delta q_v$$

or

$$\delta q = \frac{m_d}{m_d + m_v} \delta q_d + \frac{m_v}{m_d + m_v} \delta q_v$$

or

$$\delta q = (1 - q) \delta q_d + q \delta q_v.$$

Recalling the definition of specific heat capacities we then have

$$c_p = (1 - q) \frac{\delta q_d}{dT} + q \frac{\delta q_v}{dT}$$

or

$$c_p = (1 - q)c_{pd} + q c_{pv}$$

or

$$c_p = c_{pd} \left[1 + \left(\frac{c_{pv}}{c_{pd}} - 1 \right) q \right]$$

or

$$c_p = c_{pd}(1 + 0.87q) \approx c_{pd}(1 + 0.87w). \quad (7.11)$$

Similarly

$$c_v = (1 - q)c_{vd} + q c_{vv}$$

which leads to

$$c_v = c_{vd}(1 + 0.97q) \approx c_{vd}(1 + 0.97w). \quad (7.12)$$

From equations (7.11) and (7.12) we also have

$$\gamma = \frac{c_p}{c_v} = \frac{c_{pd}(1 + 0.87q)}{c_{vd}(1 + 0.97q)},$$

which because $q \ll 1$ can be approximated with

$$\gamma \approx \gamma_d(1 + 0.87q)(1 - 0.97q)$$

or after ignoring the second-order term

$$\gamma \approx \gamma_d(1 - 0.1q), \quad (7.13)$$

where $\gamma_d = 1.4$. Similarly,

$$\begin{aligned} k &= \frac{\gamma - 1}{\gamma} = \frac{R}{c_p} = \frac{R_d}{c_{pd}} \frac{(1 + 0.61q)}{(1 + 0.87q)} \\ &\approx k_d(1 + 0.61q)(1 - 0.87q) \\ &\approx k_d(1 - 0.26q), \end{aligned} \quad (7.14)$$

where $k_d = 0.286$. Note that similar expressions in the presence of liquid water cannot be derived because liquid water + vapor does not constitute an ideal gas.

7.2 Processes in the atmosphere

In this section we will discuss scenarios that are relevant to basic processes in the atmosphere. To make the discussion flow effectively we will start with processes happening at a constant pressure level (for example, at the surface). After that we will proceed with processes involving ascent, first of unsaturated air and then of saturated air. Finally, we will discuss processes that involve mixing of two air masses in the horizontal and in the vertical. The laws and formulas describing each process will be derived analytically. Subsequently, each process will be described visually in a (p, T) diagram. As we will see later, better diagrams can be designed. However, for now the simple and familiar (p, T) diagram will be used for illustrating the various processes. In the following presentation when a condensed phase is present, we will assume that it is liquid water. The same theory holds when the condensed phase is ice as long as the appropriate substitutions are made (for example e_{si} for e_{sw} , l_s for l_v , c_i for c_w , etc.).

7.2.1 Isobaric cooling – dew and frost temperatures

Consider a parcel of *unsaturated* moist air and assume it is a closed system. In this case q and w remain constant. If this parcel begins to cool at a constant pressure the vapor pressure e will also remain constant as long as condensation does not occur (recall that $e = wp/(w + \epsilon)$). However, the equilibrium vapor pressure will not remain constant. As we saw in the last chapter both e_{sw} and e_{si} are

strongly dependent on T , and as T decreases so does e_s . Since e_s decreases it follows that the relative humidity increases and saturation is approached. The temperature at which saturation is reached is called the *dew point temperature*, T_{dew} , if saturation is reached with respect to liquid water or the *frost point temperature*, T_f , if saturation is reached with respect to ice.

These temperatures derive their name from the fact that if the parcel cools isobarically *below* these temperatures droplets (or ice) will form. However, keep in mind that for condensation (or sublimation) to happen at temperatures just below saturation point either condensation nuclei or solid surfaces must be present. Spontaneous condensation in the absence of nuclei or surfaces needs extremely high relative humidities – supersaturation – which cannot easily occur in the atmosphere.

Isobaric cooling can happen naturally because of radiation cooling of the surface or the air itself. In this case the ground can lose enough heat to cool below the temperature at which atmospheric water vapor condenses. If this happens at temperatures greater than 0°C water vapor in the layer very close to the surface condenses on the surface and becomes visible as dew. If it happens at temperatures less than 0°C the product is frost. If the cooling is very strong or if the air itself cools by radiation, the layers of air above the surface may also cool below their saturation point leading to the formation of radiation fog. Alternatively, isobaric cooling can be achieved when air moves in the horizontal ($p \approx \text{constant}$) and travels over colder regions. Then saturation may occur as the air cools by heat conduction to the surface. This leads to advection fogs.

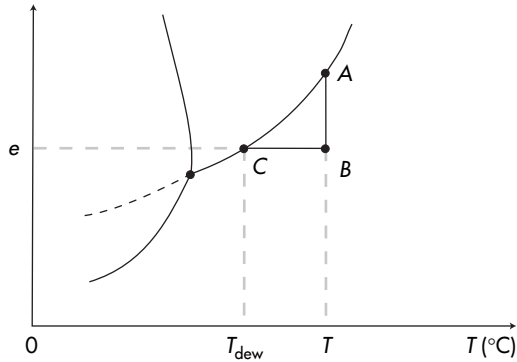
At T_{dew} vapor pressure becomes the equilibrium (saturation) vapor pressure (and to a very good approximation the mixing ratio becomes the saturation mixing ratio). As such if e is the partial vapor pressure at T , the dew point must satisfy the equation

$$e_{\text{sw}}(T_{\text{dew}}) = e. \quad (7.15)$$

This is shown graphically in Figure 7.1 where point B is assumed to have a temperature T and vapor pressure e . To approach saturation isobarically we must draw a line from B perpendicular to the pressure axis. Point C corresponds to a temperature where the water vapor pressure is its equilibrium vapor pressure. This point thus defines T_{dew} . In this diagram $AT = e_{\text{sw}}(T)$ and $BT = CT_{\text{dew}} = e = e_{\text{sw}}(T_{\text{dew}})$. Obviously, since the equilibrium vapor pressure curve is known the relative humidity and dew point temperature of any point in the diagram can easily be found ($r = TB/TA$, $T_{\text{dew}} = OT - BC$). If only the temperature is known then we need to know T_{dew} in order to estimate the relative humidity and the absolute humidity. From the figure it follows that neither relative humidity nor the *dew point spread* (or dew

Figure 7.1

Graphical definition of the dew point temperature, T_{dew} . Clearly, $e_{\text{sw}}(T_{\text{dew}}) = e(T)$.



point depression), $T - T_{\text{dew}}$, are direct measures of the actual water vapor amount. Both describe how far we are from the equilibrium curve but similar values for r or $T - T_{\text{dew}}$ correspond to different e and thus to different ρ_v (absolute humidity). Note that, depending on where a given point in the (e, T) diagram is, T_{dew} (or T_f) may or may not be defined. For example, in Figure 7.2, given point P only T_{dew} can be defined. For point P' , $T'_{\text{dew}} = T'_f$ = triple point temperature. For point P'' and in the presence of supercooled water, both T''_f and T''_{dew} can be defined with $T''_f > T''_{\text{dew}}$. If supercooled water is not present, T''_{dew} cannot be defined.

It can be shown that the dew point temperature of a rising parcel decreases. If we denote as $T_{\text{dew},0}$ and $T_{\text{dew},z}$ the dew point temperatures at the surface and at some higher level z , respectively, we have that

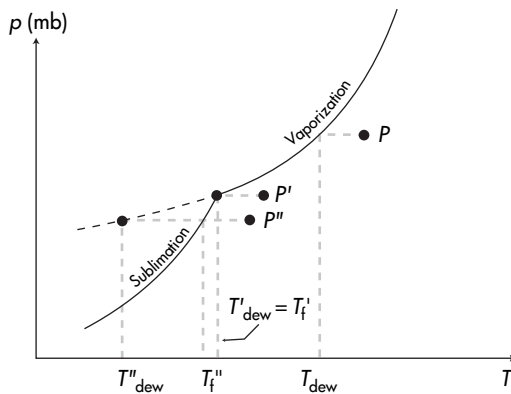
$$e_{\text{sw}}(T_{\text{dew},0}) = e_0 = R_v \rho_v T_0$$

$$e_{\text{sw}}(T_{\text{dew},z}) = e_z = R_v \rho_v T_z.$$

As a parcel rises and expands, both its temperature and vapor density decrease (because the vapor is distributed over a larger

Figure 7.2

Depending on where a given state is in a (p, T) diagram, T_{dew} or T_f may or may not be defined. For example, for state P only T_{dew} can be defined.



volume). Thus $e_0 > e_z$ and $e_{\text{sw}}(T_{\text{dew},0}) > e_{\text{sw}}(T_{\text{dew},z})$. Then from equation (6.17), it follows that $T_{\text{dew},z} < T_{\text{dew},0}$.

Estimating T_{dew}

Since $e = e_{\text{sw}}(T_{\text{dew}})$ the C-C equation can be written as

$$\frac{de}{dT_{\text{dew}}} = \frac{l_v e}{R_v T_{\text{dew}}^2}$$

or

$$\frac{de_{\text{sw}}}{e_{\text{sw}}} = \frac{l_v dT}{R_v T^2}.$$

Considering $l_v \approx \text{constant}$ and integrating from T_{dew} (where $e_{\text{sw}}(T_{\text{dew}}) = e$) to T (where $e_{\text{sw}}(T) = e_{\text{sw}}$) we obtain

$$\ln \frac{e_{\text{sw}}}{e} = \frac{l_v}{R_v} \left(\frac{T - T_{\text{dew}}}{TT_{\text{dew}}} \right)$$

or

$$-\ln r = \frac{l_v}{R_v} \left(\frac{T - T_{\text{dew}}}{TT_{\text{dew}}} \right)$$

or

$$T - T_{\text{dew}} = -R_v T T_{\text{dew}} \ln r / l_v. \quad (7.16)$$

Equation (7.16) provides T_{dew} from temperature and relative humidity. Using $l_v \approx 2.501 \times 10^6 \text{ J kg}^{-1}$ and $R_v = 461.51 \text{ J kg}^{-1} \text{ K}^{-1}$, the above equation becomes

$$T - T_{\text{dew}} = -1.845 \times 10^{-4} T T_{\text{dew}} \ln r.$$

Following the same procedure we can obtain (for $l_s = 2.8345 \text{ J kg}^{-1}$)

$$T - T_f = -1.628 \times 10^{-4} T T_f \ln r.$$

Estimating the dew point temperature can be visualized in the following (p, T) diagram (Figure 7.3) where pressure decreases along the y-axis.

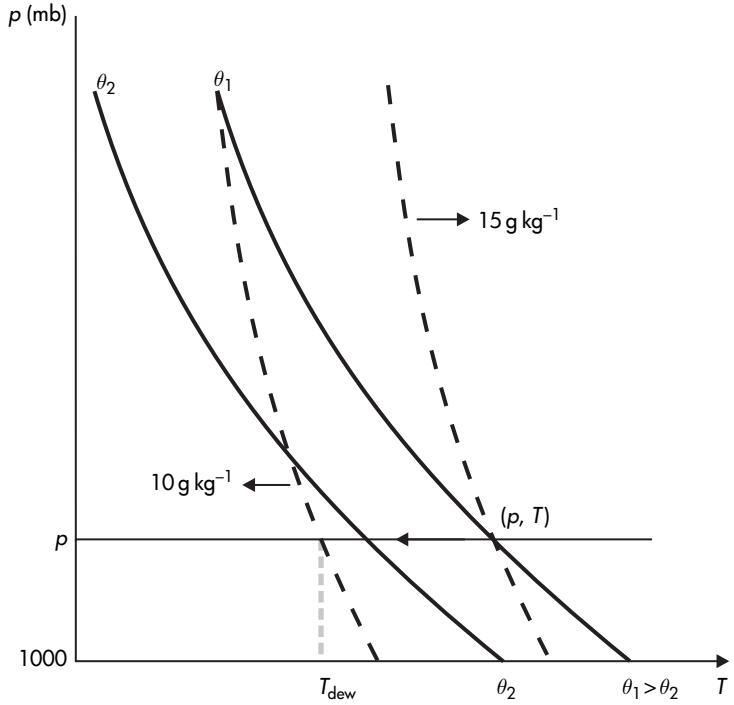
In this diagram a couple of dry adiabatic (solid) and a couple of saturation mixing ratio (broken) lines are shown. The dry adiabats can be calculated from Poisson's equation

$$\ln \theta = \ln T + k_d \ln 1000 - k_d \ln p$$

by setting $\ln \theta = \text{constant}$:

$$\ln p = \frac{1}{k_d} \ln T + \text{constant}.$$

Figure 7.3
Illustration of the
estimation procedure of
the dew point
temperature.



In a (p, T) coordinate system, this is not a straight line. The saturation mixing ratio lines can be calculated from equations (7.5) and (6.17)

$$p \approx \frac{\epsilon e_{\text{sw}}(T)}{w_{\text{sw}}}$$

or

$$p \approx \frac{6.11\epsilon}{w_{\text{sw}}} \exp\left(19.83 - \frac{5417}{T}\right)$$

or

$$\ln p \approx \ln \frac{6.11\epsilon}{w_{\text{sw}}} + 19.83 - \frac{5417}{T}$$

or for some $w_{\text{sw}} = \text{constant}$

$$\ln p \approx A + \frac{B}{T}$$

where $A = \ln 6.11\epsilon/w_{\text{sw}} + 19.83$ and $B = -5417$. In a (p, T) diagram this is not a straight line either. For this reason both the dry adiabats and the saturation mixing ratio lines are depicted as nonlinear curves. Since the purpose here is to illustrate the process graphically, these curves are realistic but by no means exact. Let us now assume that an air parcel is at a pressure level p and has a temperature T . Let us further assume that its mixing ratio is 10 g kg^{-1} . The parcel is not saturated because at this (p, T) state

the saturation mixing ratio is 15 g kg^{-1} . In order to estimate the dew point, we then have to move, as the arrow shows, to the left (lower temperatures) (but remain at the same pressure level) until we “meet” the 10 g kg^{-1} saturation mixing ratio line. At that temperature, the mixing ratio of the parcel is equal to the saturation mixing ratio and thus the parcel becomes saturated. That temperature is then the parcel’s dew point temperature. Note that if from (p, T) we follow the dry adiabat θ_1 down to 1000 mb we obtain the potential temperature of the parcel, which will be greater than its temperature, T .

7.2.2 Adiabatic isobaric processes – wet-bulb temperatures

Let us now suppose that our system is unsaturated moist air and liquid water and that the mixture has a temperature T and a combined water mixing ratio $w_t = (m_v + m_w)/m_d$. Since this is an unstable situation the system will tend to approach equilibrium. This will be achieved by water evaporating. If we assume that the system is closed and the process is taking place adiabatically (i.e. the process is also isenthalpic), then the liquid water must draw heat for the evaporation from the unsaturated moist air. Since no other processes are taking place, this decrease in temperature must be related to the change in the mixing ratio of moist air due to evaporating liquid water. Note that, since at any time during this process the system differs slightly from saturation, the process is spontaneous and irreversible. If there is enough liquid water, saturation will be reached. The temperature at saturation is called the *isobaric wet-bulb temperature*, T_w . An important atmospheric process described by this type of transformation is cooling by evaporation of rain at a given level. The system consists of a certain air mass plus water that evaporates into it from the rain falling through it. If instead of water we had ice present, then we would define the isobaric ice-bulb temperature, T_i . Note that T_w is not the same as T_{dew} since the two cooling processes are different. Now, if we follow the reverse procedure, we could define the temperature which air would achieve if all its vapor were to condense in an adiabatic isobaric process and with the produced water being withdrawn. This limiting temperature is called the isobaric equivalent temperature, T_{ei} . Note, however, that the definition of this temperature is *not* a “legal” one. The reason is that we assume that the reverse of an irreversible process can happen, which is not possible.

Derivation of the wet-bulb temperature

Consider a closed heterogeneous system consisting of dry air, water vapor, and liquid water. If this system undergoes a change, then the change in enthalpy is

$$\Delta H = H_f - H_i$$

where the subscripts i and f refer to the initial and final states, respectively. We can write the above equation in terms of the three components

$$\Delta H = (H_{df} + H_{vf} + H_{wf}) - (H_{di} + H_{vi} + H_{wi})$$

or

$$\Delta H = (m_d h_{df} + m_{vf} h_{vf} + m_{wf} h_{wf}) - (m_d h_{di} + m_{vi} h_{vi} + m_{wi} h_{wi}). \quad (7.17)$$

If we recall the definition of l_v ($l_v = h_v - h_w$) and consider that the total amount of water $m_t = m_v + m_w$ is conserved (i.e. $m_{vi} + m_{wi} = m_{vf} + m_{wf} = m_t$), then the above equation becomes

$$\Delta H = m_d(h_{df} - h_{di}) + m_t(h_{wf} - h_{wi}) + m_{vf}l_{vf} - m_{vi}l_{vi}. \quad (7.18)$$

If we now assume that in the range of meteorological interest c_{pd} and c_w are nearly independent of temperature, then we can write (recall equation (4.18)) that

$$h_{df} - h_{di} = c_{pd}(T_f - T_i)$$

$$h_{wf} - h_{wi} = c_w(T_f - T_i).$$

In addition, if T_i is not greatly different from T_f we can assume that

$$l_{vi} = l_{vf} = l_v.$$

Then equation (7.18) becomes

$$\Delta H = m_d c_{pd}(T_f - T_i) + m_t c_w(T_f - T_i) + (m_{vf} - m_{vi})l_v$$

or

$$\Delta H/m_d = \left(c_{pd} + \frac{m_t}{m_d} c_w \right) (T_f - T_i) + \left(\frac{m_{vf} - m_{vi}}{m_d} \right) l_v$$

or

$$\Delta H/m_d = (c_{pd} + w_t c_w)(T_f - T_i) + (w_f - w_i)l_v$$

or

$$\Delta H/m_d = (c_{pd} + w_t c_w)\Delta T + l_v \Delta w. \quad (7.19)$$

Note that w_t (the total water mixing ratio) is a different constant for each system and varies only with the concentration of liquid water. As such it does not depend on temperature. Since we have also assumed that $c_{pd}, c_w, l_v \neq f(T)$ we can write the above equation in differential form:

$$dH/m_d = (c_{pd} + w_t c_w)dT + l_v dw. \quad (7.20)$$

Equation (7.20) is a general expression for the change in enthalpy of a closed heterogeneous system of dry air, water vapor, and liquid water during any process. If we restrict ourselves to isenthalpic processes only, then equation (7.20) becomes

$$(c_{pd} + w_t c_w)(T_i - T_f) = (w_f - w_i)l_v. \quad (7.21)$$

Equation (7.21) indicates that in an adiabatic isobaric process, where moist air cools from evaporating liquid water, the change in temperature relates to the change in the mixing ratio. It also tells us that T_f depends on w_t with T_f becoming minimum when $w_t \rightarrow 0$. Since $m_t = m_v + m_w$ cannot be zero (if it were neither the mixing ratio nor temperature would change), this condition requires that $m_d \rightarrow \infty$. Of course this cannot be true but to a good approximation $m_d \gg m_t$. We define this limiting temperature as the wet-bulb temperature, T_w , which is determined by setting in equation (7.21) $T_i = T, T_f = T_w, w_i = w, w_f = w_{sw}$, and $w_t = 0$:

$$c_{pd}(T - T_w) = (w_{sw} - w)l_v$$

or

$$T_w + \frac{l_v}{c_{pd}}w_{sw} = T + \frac{l_v}{c_{pd}}w \quad (7.22)$$

where w_{sw} is the saturation mixing ratio over liquid water at temperature T_w . As expected, since $w_{sw} > w, T_w < T$.

Estimation of the isobaric equivalent temperature

In order to estimate this temperature (T_{ei}), the only thing we have to do is to set in equation (7.21) $w_i = w_t = w$ (the mixing ratio of the moist air), $w_f = 0$, $T_i = T$, and $T_f = T_{ei}$. Then we obtain

$$T_{ei} = T + \frac{l_v w}{c_{pd} + w c_w}. \quad (7.23)$$

Obviously $T_{ei} > T$. Also, since for typical values of w (around 10 g kg^{-1}) $l_v/(c_{pd} + w c_w) \approx 2500$ it follows that $T_{ei} > T_{virt}$. Neither the wet bulb nor the isobaric equivalent temperatures is estimated graphically. They are calculated from equations (7.22) and (7.23), respectively.

Relation between wet-bulb and dew point temperatures

If we employ the general approximation $w \approx \epsilon e/p$ we can write equation (7.21) as

$$c_p(T_i - T_f) = \frac{\epsilon}{p}(e_f - e_i)l_v$$

where $c_p = c_{pd} + w_t c_w$. Setting $T_i = T$, $T_f = T_w$, and $e_f = e_{sw}(T_w)$ yields

$$e_i = e_{sw}(T_w) - \frac{p c_p}{l_v \epsilon} (T - T_w).$$

From the definition of dew point temperature we have that $e_i = e_{sw}(T_{dew})$. Then, the above equation becomes

$$e_{sw}(T_{dew}) = e_{sw}(T_w) - \frac{p c_p}{l_v \epsilon} (T - T_w).$$

Since $T - T_w > 0$ it follows that $e_{sw}(T_{dew}) < e_{sw}(T_w)$. Then, from equation (6.17) it follows that

$$T_{dew} < T_w.$$

Are you keeping up with all these temperatures? Up to now we have discussed five different temperatures: T , temperature; T_{virt} , virtual temperature; T_{dew} , dew point temperature; T_w , isobaric wet-bulb temperature; and T_{ei} , isobaric equivalent temperature. They are related as follows:

$$T_{dew} < T_w < T < T_{virt} < T_{ei}. \quad (7.24)$$

Brace yourself. There are more!

7.2.3 Adiabatic expansion (or compression) of unsaturated moist air

Recall Poisson's equation for an adiabatic expansion or compression of an ideal gas from p, T to p', T'

$$T' = T \left(\frac{p'}{p} \right)^k$$

which for $p' = 1000$ mb defined the potential temperature, θ ,

$$\theta = T \left(\frac{1000}{p} \right)^k. \quad (7.25)$$

Poisson's equation is valid for any ideal gas. As such it is valid for moist air provided that the appropriate k is used and that condensation does not take place. We can then define the potential temperature of unsaturated moist air as

$$\theta_m = T \left(\frac{1000}{p} \right)^{k_d(1-0.26q)}. \quad (7.26)$$

If we substitute T from equation (7.25) applied to dry air into equation (7.26) we obtain (assuming that $p_d \approx p$)

$$\theta_m = \theta_d \left(\frac{1000}{p} \right)^{-k_d 0.26 q}$$

or

$$\theta_m = \theta_d \left(\frac{1000}{p} \right)^{-0.07 q}. \quad (7.27)$$

As we have already discussed, in our atmosphere $q \ll 1$ which indicates that to a very good approximation

$$\theta_m \approx \theta_d.$$

As in the case of virtual temperature, here again we can define the *virtual potential temperature*, θ_{virt} , by substituting T_{virt} for T in equation (7.25):

$$\theta_{\text{virt}} = T_{\text{virt}} \left(\frac{1000}{p} \right)^{k_d}. \quad (7.28)$$

This will be the temperature dry air will acquire if it were to expand or compress from a level (T_{virt}, p) to the 1000 mb level. Since $T_{\text{virt}} > T$ it follows that $\theta_{\text{virt}} > \theta$.

Note that even though q and w are very small (of the order of 0.01) we should not jump to the conclusion that all variables or parameters of unsaturated moist air are for all practical purposes the same as those of dry air. If we consider a typical value of $q = w = 0.01$ we find that differences in virtual temperature of the order of 2–3 °C are not uncommon. Such differences are often not negligible. On the other hand, as we saw above, differences in moist potential temperature are indeed negligible (of the order of 0.1 °C). Consequently, from now on the symbol θ will be used to indicate the potential temperature of both dry and unsaturated moist air. Similarly to the dry adiabatic lapse rate we may define here the moist (unsaturated) adiabatic lapse rate as

$$\Gamma_m = \frac{g}{c_p} = \frac{g}{c_{pd}(1 + 0.87w)}$$

or

$$\Gamma_m = \frac{\Gamma_d}{1 + 0.87w} \approx \Gamma_d(1 - 0.87w). \quad (7.29)$$

7.2.4 Reaching saturation by adiabatic ascent

As an unsaturated moist air parcel rises adiabatically, it cools. During the ascent (and as long as no condensation occurs) its q or w remains the same but the equilibrium vapor pressure decreases. At the same time the vapor pressure e also decreases. This can be

easily seen by recalling that $e = wp/(w + \epsilon)$. As w and ϵ remain constant e/p remains constant. Since during the ascent p decreases, e must decrease if e/p must remain constant. Accordingly, when unsaturated moist air ascends adiabatically, its relative humidity ($r = e/e_s$) increases only if e decreases at a slower rate than e_s . If this condition is not met clouds will not form in ascent but they could still form in descent. However, in our experience, clouds never do form as a result of descending air, so this condition is apparently satisfied. But what is the mathematics and physics behind it? If we log-differentiate equation (7.7) we get

$$d \ln r = d \ln e - d \ln e_{sw}. \quad (7.30)$$

For moist air, Poisson's equation tells us that $Tp^{\frac{1-\gamma}{\gamma}} = \text{constant}$. Since e/p remains constant, it follows that $Te^{\frac{1-\gamma}{\gamma}} = \text{constant}$. Then, again by log-differentiating, we obtain

$$d \ln T = \frac{\gamma - 1}{\gamma} d \ln e. \quad (7.31)$$

Combining (7.30) and (7.31) with the C-C equation yields

$$d \ln r = \frac{\gamma}{\gamma - 1} d \ln T - \frac{l_v}{R_v T^2} dT. \quad (7.32)$$

The first term on the right-hand side of the above equation accounts for the change in relative humidity due to the decrease in e while the second accounts for the change in relative humidity due to the temperature decrease. As one term is positive and the other negative, the net result can be either positive or negative. This indicates that adiabatic expansion can lead to an increase or to a decrease in r , which in turn implies that the opposite (i.e. adiabatic compression) can also lead to a decrease or an increase in r . This opens the road to bizarre scenarios where clouds can actually form on descent. If we rewrite equation (7.32) as

$$\frac{dr}{dT} = \frac{r}{T} \left(\frac{\gamma}{\gamma - 1} - \frac{l_v}{R_v T} \right),$$

we see that if

$$\frac{\gamma}{\gamma - 1} > \frac{l_v}{R_v T}, \quad (7.33)$$

then $dr < 0$ when $dT < 0$ and $dr > 0$ when $dT > 0$, which would indicate that clouds can form on descent. Assuming the typical values for $\gamma \approx 1.4$, $l_v = 2.5 \times 10^6 \text{ J kg}^{-1}$, and $R_v = 461.5 \text{ J kg}^{-1} \text{ K}^{-1}$ we find that inequality (7.33) is satisfied only when $T \geq 1550 \text{ K}$. Obviously, such a situation does not exist on our planet and clouds can only form on ascent. However, all it takes is an exotic planet whose atmosphere has a condensable gas with smaller l_v and all sorts of things may happen.

Now that we have settled this issue, we can expect that ascending moist unsaturated air will eventually reach a relative humidity of 100%. Immediately after that condensation will take place in order to sustain equilibrium conditions. The level at which saturation of ascending unsaturated moist parcels is achieved is called the *lifting condensation level* (LCL). The temperature at this level, T_{LCL} , is called the *saturation temperature*. By integrating equation (7.32) from an initial state T, T_{dew}, r to the final state $T_{\text{LCL}} = T_{\text{dew,LCL}}, r_{\text{LCL}} = 1$ we obtain

$$-\ln r = \frac{\gamma}{\gamma - 1} \ln \frac{T_{\text{LCL}}}{T} + \frac{l_v}{R_v} \left(\frac{1}{T_{\text{LCL}}} - \frac{1}{T} \right). \quad (7.34)$$

Since $r = e/e_{\text{sw}}(T)$ and $e = e_{\text{sw}}(T_{\text{dew}})$ we have from the C-C equation

$$e_{\text{sw}}(T_{\text{dew}}) = 6.11 \exp \left(\frac{l_v}{273R_v} - \frac{l_v}{R_v T_{\text{dew}}} \right)$$

and

$$e_{\text{sw}}(T) = 6.11 \exp \left(\frac{l_v}{273R_v} - \frac{l_v}{R_v T} \right).$$

Dividing the above two equations and taking logs yields

$$-\ln r = \frac{l_v}{R_v} \left(\frac{T - T_{\text{dew}}}{TT_{\text{dew}}} \right).$$

Accordingly, equation (7.34) becomes

$$\frac{T - T_{\text{dew}}}{TT_{\text{dew}}} = A \ln \frac{T_{\text{LCL}}}{T} + \left(\frac{1}{T_{\text{LCL}}} - \frac{1}{T} \right) \quad (7.35)$$

where

$$A = \left(\frac{\gamma}{\gamma - 1} \right) \bigg/ \left(\frac{l_v}{R_v} \right).$$

Equation (7.35) can be solved numerically to provide the saturation temperature T_{LCL} from T and T_{dew} . If you would rather not solve this equation you may use the approximation provided by Bolton (1980)

$$T_{\text{LCL}} = \frac{1}{\frac{1}{T-55} - \frac{\ln r}{2840}} + 55, \quad (7.36)$$

where T is in K, $r = w/w_{\text{sw}}$, $w_{\text{sw}} = \epsilon[e_{\text{sw}}/(p - e_{\text{sw}})]$, $e_{\text{sw}} = 6.11 \exp(19.83 - 5417/T)$, and $T - T_{\text{dew}} = -R_v TT_{\text{dew}} \ln r/l_v$.

A parcel that rises from an initial level where the temperature is T to a level where the temperature is T_{LCL} has expanded. Therefore, the vapor density at LCL is smaller than that at the initial level. From the ideal gas law it then follows that $e(T_{\text{LCL}}) < e(T)$. Because

$T_{\text{LCL}} = T_{\text{dew,LCL}}$ and $e(T) = e_{\text{sw}}(T_{\text{dew}})$ this inequality is equivalent to $e_{\text{sw}}(T_{\text{LCL}}) < e_{\text{sw}}(T_{\text{dew}})$. Then from equation (6.17) it follows that $T_{\text{LCL}} < T_{\text{dew}}$. Thus, relation (7.24) can be extended to include yet one more temperature:

$$T_{\text{LCL}} < T_{\text{dew}} < T_{\text{w}} < T < T_{\text{v}} < T_{\text{ei}}. \quad (7.37)$$

Estimating the height of LCL

The height of LCL for a given parcel (z_{LCL}) depends only on its temperature and relative humidity. At z_{LCL} the parcel's temperature equals its dew point: $T_{\text{LCL}} = T_{\text{dew,LCL}}$. As the parcel is lifted from a reference level z_0 its temperature decreases according to

$$T(z) = T_0 - \Gamma_{\text{d}}(z - z_0) \quad (7.38)$$

where T_0 is the temperature of the parcel of z_0 and we have assumed that $\Gamma_{\text{m}} \approx \Gamma_{\text{d}}$. Similarly, its dew point temperature decreases according to

$$T_{\text{dew}}(z) = T_{\text{dew},0} - \Gamma_{\text{dew}}(z - z_0). \quad (7.39)$$

Here both Γ_{d} and Γ_{dew} are considered constant. From equations (7.38) and (7.39) we have that

$$T_{\text{LCL}} = T_0 - \Gamma_{\text{d}}(z_{\text{LCL}} - z_0)$$

$$T_{\text{dew,LCL}} = T_{\text{dew},0} - \Gamma_{\text{dew}}(z_{\text{LCL}} - z_0)$$

or since $T_{\text{LCL}} = T_{\text{dew,LCL}}$,

$$z_{\text{LCL}} - z_0 = \frac{T_0 - T_{\text{dew},0}}{\Gamma_{\text{d}} - \Gamma_{\text{dew}}}. \quad (7.40)$$

In the above equation we know everything except for Γ_{dew} . In order to estimate Γ_{dew} we proceed as follows. The equation that defines dew point is $e = e_{\text{sw}}(T_{\text{dew}})$. Differentiation with respect to z gives

$$\frac{de_{\text{sw}}(T_{\text{dew}})}{dz} = \frac{de}{dz}$$

or

$$\frac{de_{\text{sw}}(T_{\text{dew}})}{dT_{\text{dew}}} \frac{dT_{\text{dew}}}{dz} = \frac{de}{dz}. \quad (7.41)$$

Now we know that $e = wp/(w + \epsilon)$. If we assume that w remains constant until condensation occurs, we have

$$\frac{de}{dz} = \frac{w}{w + \epsilon} \frac{dp}{dz}$$

or

$$\frac{de}{dz} = \frac{e}{p} \frac{dp}{dz} = \frac{e_{\text{sw}}(T_{\text{dew}})}{p} \frac{dp}{dz}. \quad (7.42)$$

Combining equations (7.41) and (7.42) gives

$$\frac{1}{e_{\text{sw}}(T_{\text{dew}})} \frac{de_{\text{sw}}(T_{\text{dew}})}{dT_{\text{dew}}} \frac{dT_{\text{dew}}}{dz} = \frac{1}{p} \frac{dp}{dz}.$$

After considering the C-C equation we can reduce the above to

$$\frac{l_v}{R_v T_{\text{dew}}^2} \frac{dT_{\text{dew}}}{dz} = \frac{1}{p} \frac{dp}{dz}. \quad (7.43)$$

Since the ascent is assumed to be adiabatic, we have from Poisson's equation that

$$d \ln T = \frac{\gamma - 1}{\gamma} d \ln p$$

or

$$\frac{dT}{T dz} = \frac{\gamma - 1}{\gamma} \frac{dp}{p dz}. \quad (7.44)$$

From equations (7.43) and (7.44) it follows that

$$\begin{aligned} \frac{l_v}{R_v T_{\text{dew}}^2} \frac{dT_{\text{dew}}}{dz} &= \frac{\gamma}{\gamma - 1} \frac{dT}{T dz} \\ &= \frac{\gamma}{(\gamma - 1)T} \left(-\frac{g}{c_{pd}} \right) \end{aligned}$$

or assuming $\gamma \approx \gamma_d$

$$\frac{l_v}{R_v T_{\text{dew}}^2} \frac{dT_{\text{dew}}}{dz} = -\frac{g}{R_d T}$$

which then gives

$$\Gamma_{\text{dew}} = -\frac{dT_{\text{dew}}}{dz} = \frac{g}{\epsilon l_v} \frac{T_{\text{dew}}^2}{T}.$$

The value of the right-hand side in the atmosphere ranges from about 1.7 to 1.9 °C km⁻¹. Therefore we can assume that on the average T_{dew} decreases with height at the rate of

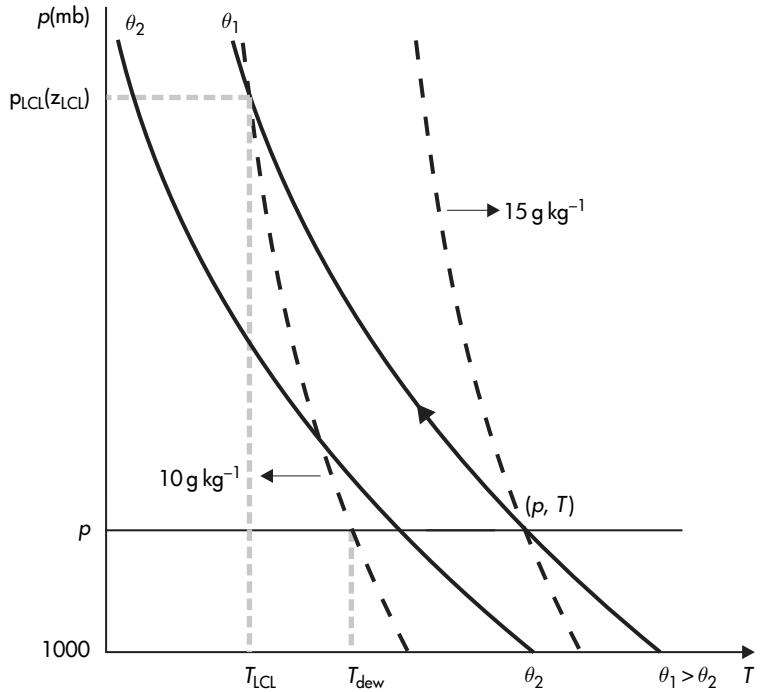
$$\Gamma_{\text{dew}} = 1.8 \text{ °C km}^{-1}.$$

By dropping the subscript 0 in equation (7.40) it follows that the height of LCL is given by the approximate relationship

$$z_{\text{LCL}} - z \approx \frac{T - T_{\text{dew}}}{8}. \quad (7.45)$$

Equations (7.35) or (7.36) and (7.45) give the temperature and height (in km) of the level where clouds begin to form from the temperature and dew point temperature at some reference level z . Reaching saturation via adiabatic ascent and the estimation of T_{LCL}

Figure 7.4
Illustration of the
estimation procedure of
the lifting condensation
level and saturation
temperature.



and the height of LCL can be visualized in Figure 7.4. The setup in this figure is the same as in Figure 7.3. Since the parcel is initially unsaturated, as it ascends, it follows the dry adiabat passing from its initial state (p, T) (recall that $\theta_m \approx \theta_d$). As it rises (follow the arrow) it cools and approaches the $w_{sw} = 10 \text{ g kg}^{-1}$ line, which it intersects at a pressure level (height) $p_{LCL}(z_{LCL})$. At that point, the temperature is T_{LCL} . As the figure illustrates, $T_{LCL} < T_{dew}$.

7.2.5 Saturated ascent

Once saturation is reached, further ascent results in a further increase in relative humidity, which means that the vapor pressure becomes greater than the equilibrium vapor pressure. At this point, the system will return to equilibrium after the “extra” water vapor condenses and water droplets (or ice depending on the temperature) form on condensation nuclei. From this point on two extreme possibilities may be considered. (1) The condensation products remain in the parcel. This is a reversible process because if we reverse the ascent the products will evaporate. It is also adiabatic as we assume that no heat is exchanged between the parcel and the environment. Since it is reversible and adiabatic it is also isentropic. (2) All condensation products fall out and the parcel consists always

of dry air plus saturated water vapor. This makes the parcel an open system. Obviously, this process is neither reversible nor adiabatic and thus it is not isentropic. We will call this process a pseudo-adiabatic process. In reality these two extreme cases may never exist as some condensation products are likely to remain suspended in the parcel. However, because of the conditions imposed the extreme cases are easier to treat. Nevertheless, the study of the extreme cases offers useful insights and approximations for the real cases in between.

Reversible saturated adiabatic process

Since the process is assumed to be reversible, the heterogeneous system of dry air, water vapor, and liquid water (or ice if it is too cold) must be at equilibrium at all times. In this case the total entropy of the system, S , is conserved:

$$S = S_d + S_v + S_w = \text{constant}.$$

If we divide all the terms in the above equation by the mass of dry air we arrive at the following expression

$$\frac{S}{m_d} = s_d + \frac{m_v}{m_d} s_v + \frac{m_w}{m_d} s_w.$$

Recalling that the parcel is saturated ($w = w_{sw}$) and the definition of the total water mixing ratio ($w_t = (m_v + m_w)/m_d$), the above equation can be written as

$$\frac{S}{m_d} = s_d + w_{sw} s_v + (w_t - w_{sw}) s_w. \quad (7.46)$$

For a reversible process

$$dS = \frac{\delta Q}{T} = \frac{dU}{T} + \frac{pdV}{T} = \frac{1}{T} \left[\left(\frac{\partial U}{\partial T} \right)_v dT + \left(\frac{\partial U}{\partial V} \right)_T dV \right] + \frac{p}{T} dV. \quad (7.47)$$

In addition,

$$dS = \left(\frac{\partial S}{\partial T} \right)_v dT + \left(\frac{\partial S}{\partial V} \right)_T dV. \quad (7.48)$$

From equations (7.47) and (7.48) it follows that

$$\left(\frac{\partial S}{\partial T} \right)_v = \frac{1}{T} \left(\frac{\partial U}{\partial T} \right)_v \quad (7.49)$$

and

$$\left(\frac{\partial S}{\partial V} \right)_T = \frac{1}{T} \left[\left(\frac{\partial U}{\partial V} \right)_T + p \right]. \quad (7.50)$$

For liquid water and water vapor in *equilibrium* at temperature T and pressure p equation (7.50) gives

$$T \frac{S_v - S_w}{V_v - V_w} = \frac{U_v - U_w}{V_v - V_w} + p$$

or

$$T(S_v - S_w) = U_v - U_w + p(V_v - V_w).$$

If we use equation (6.3) the above equation becomes

$$T(S_v - S_w) = H_v - H_w = L_v$$

or

$$T(s_v - s_w) = l_v.$$

Substituting s_v from the above equation into equation (7.46) results in

$$\frac{S}{m_d} = s_d + w_t s_w + \frac{l_v w_{sw}}{T}. \quad (7.51)$$

Now recall that

$$s_d = c_{pd} \ln T - R_d \ln p_d + \text{constant} \quad [p = p_d + e_{sw}(T)]$$

and, from equation (5.9),

$$s_w = c_w \ln T + \text{constant}.$$

Accordingly, equation (7.51) becomes

$$\frac{S}{m_d} = (c_{pd} + w_t c_w) \ln T - R_d \ln p_d + \frac{l_v w_{sw}}{T} + \text{constant}. \quad (7.52)$$

Since here S and m_d are conserved, it follows that

$$(c_{pd} + w_t c_w) \ln T - R_d \ln p_d + \frac{l_v w_{sw}}{T} = \text{constant}'. \quad (7.53)$$

As previously, here we can assume that $w_t, c_{pd}, c_w \neq f(T)$. However, w_{sw} through equations (7.5) and (6.17) is a function of T . If we assume that l_v is a function of T (and that may be a good idea here as the parcel may rise to very cold temperatures), then we can write equation (7.53) in differential form as follows:

$$(c_{pd} + w_t c_w) d \ln T - R_d d \ln p_d + d \left(\frac{l_v w_{sw}}{T} \right) = 0. \quad (7.54)$$

Equations (7.53) and (7.54) describe reversible saturated adiabatic processes. Note again that here T and p correspond to *saturation*. If we now define θ' as

$$\theta' = T \left(\frac{1000}{p_d} \right)^{R_d / (c_{pd} + w_t c_w)} \quad (7.55)$$

it follows that

$$(c_{pd} + w_t c_w) d \ln \theta' = (c_{pd} + w_t c_w) d \ln T - R_d d \ln p_d. \quad (7.56)$$

Combining (7.56) and (7.54) gives

$$(c_{pd} + w_t c_w) d \ln \theta' = -d \left(\frac{l_v w_{sw}}{T} \right)$$

or

$$d \ln \theta' = -d \left[\frac{l_v w_{sw}}{(c_{pd} + w_t c_w) T} \right].$$

Integrating the above equation results in

$$\theta' \exp \left[\frac{l_v w_{sw}}{(c_{pd} + w_t c_w) T} \right] = \text{constant}. \quad (7.57)$$

The above equation defines the family of curves that describe reversible saturated adiabatic processes. We can evaluate equation (7.57) at a state where all the vapor has condensed ($w_{sw} = 0$). This defines a new temperature, the equivalent potential temperature θ_e given by

$$\theta_e = \theta' \exp \left[\frac{l_v w_{sw}}{(c_{pd} + w_t c_w) T} \right] = \text{constant}. \quad (7.58)$$

Although θ_e is defined for saturated air it can also be defined for unsaturated air at temperature T , pressure p , and mixing ratio w by considering it to be lifted to the LCL. At LCL we have

$$\theta' = T_{LCL} \left(\frac{1000}{p_d} \right)^{R_d / (c_{pd} + w_t c_w)}$$

(equation (7.55)) and thus

$$\theta_e = T_{LCL} \left(\frac{1000}{p_d} \right)^{R_d / (c_{pd} + w_t c_w)} \exp \left[\frac{l_v (T_{LCL}) w}{(c_{pd} + w_t c_w) T_{LCL}} \right]. \quad (7.59)$$

Note that in the above two expressions $p_d = p_{LCL} - e_{sw}(T_{LCL})$. Another way to express θ_e for unsaturated air at T, p, w is given by Emanuel (1994):

$$\begin{aligned} \theta_e = T \left(\frac{1000}{p_d} \right)^{R_d / (c_{pd} + w_t c_w)} & r^{-w R_v / (c_{pd} + w_t c_w)} \\ & \times \exp [l_v w / (c_{pd} + w_t c_w) T] \end{aligned} \quad (7.60)$$

where r is the relative humidity. Here $p_d = p - e(T)$. When the initial mixing ratio is zero, both (7.59) and (7.60) give (as expected) $\theta_e = \theta$, the potential temperature.

Recall that the moist potential temperature θ_m (7.26) is

$$\theta_m = T \left(\frac{1000}{p} \right)^{R_d/c_{pd}(1-0.26w)}.$$

Comparison between $\theta_m (\approx \theta)$ and θ' indicates that θ' is only approximately equal to θ . Given the definition of θ' , it follows from (7.60) that the equivalent potential temperature is *approximately* equal to the potential temperature a parcel would have if it were lifted to very low pressure so that all its water vapor condensed ($w = 0$).

Pseudo-adiabatic processes

The description of reversible saturated adiabatic processes depends on the value of w_t . The problem with this dependence is that although w_{sw} can be determined at a given state, (p, T) , w_t cannot. The liquid water mixing ratio can take on any value and as such at a given (p, T) point there may exist an infinite number of reversible saturated adiabats. If we were to describe such a process in the (p, T) domain this would present a great inconvenience. This problem can be dealt with by defining a new process called a *pseudo-adiabatic process*. In such a process we do away with liquid water by assuming that it is removed as soon as it is produced. Obviously, our system is now an open system and this process is not reversible. However, we can think of the whole process as a two-step process. During the first step we have a reversible saturated adiabatic process, with condensation of a mass of water (dm_w). During the second step the water produced is leaving the system. For the first step the entropy may be defined as previously but with the liquid water term omitted. Accordingly, we set in equation (7.52) $w_t = w_{sw}$ and we obtain

$$\frac{S}{m_d} = (c_{pd} + w_{sw}c_w) \ln T - R_d \ln p_d + \left(\frac{l_v w_{sw}}{T} \right) + \text{constant}. \quad (7.61)$$

Since in this stage S and m_d are conserved we can write the above equation in differential form

$$d[(c_{pd} + w_{sw}c_w) \ln T] - R_d d \ln p_d + d \left(\frac{l_v w_{sw}}{T} \right) = 0. \quad (7.62)$$

After water forms it is removed immediately, thereby decreasing the entropy of the system. However, this process has no effect on the values of T and p . As such equation (7.62) describes the variation of p and T in a pseudo-adiabatic process. Equation (7.62) is very similar to equation (7.54) but here w_{sw} depends on temperature whereas w_t in equation (7.54) does not. This makes equation (7.62) very difficult to solve analytically. However, numerical techniques

can be employed to define, similarly to the case of the equivalent potential temperature, a pseudo-equivalent potential temperature, θ_{ep} . As is shown in Bolton (1980), within 0.3°C , θ_{ep} is given by

$$\theta_{\text{ep}} = T \left(\frac{1000}{p} \right)^{0.285(1-0.28w)} \exp \left[w(1 + 0.81w) \left(\frac{3376}{T_{\text{LCL}}} - 2.54 \right) \right] \quad (7.63)$$

where (T, p, w) is any state of the parcel saturated or not (w here is dimensionless and T, T_{LCL} are in K).

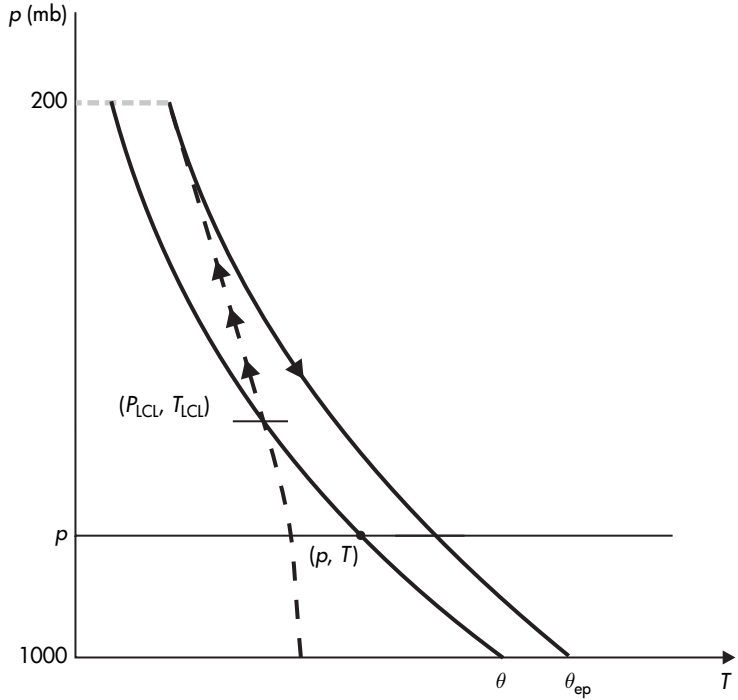
The pseudo-equivalent potential temperature can be interpreted as the actual temperature achieved by the parcel under the following thermodynamic processes: (a) pseudo-adiabatic expansion to zero (or to a very low) pressure level at which we can assume that all water vapor has condensed and fallen out, and (b) subsequent dry adiabatic descent to 1000 mb. Note that according to the definition of θ_e (the condensation products do not leave the parcel) no similar meaning can be attached to θ_{ep} . Nevertheless, the differences in the temperature of a parcel in a reversible saturated adiabatic and in a pseudo-adiabatic process are not great and are often neglected as processes in the atmosphere lie somewhere in between true reversible saturated adiabatic processes and pseudo-adiabatic processes. This can be easily verified if we assume that $w_{\text{sw}} \approx w_t \ll c_{\text{pd}}$ (i.e. $c_{\text{pd}} + w_{\text{sw}}c_w \approx c_{\text{pd}}$) and $p_d \gg e_{\text{ws}}$ (i.e. $p \approx p_d$). In this case equations (7.54) and (7.62) become identical: $c_{\text{pd}}d \ln T - R_d d \ln p + d(w_{\text{sw}}l_v/T) = 0$.

Graphically a saturated ascent can be described with the following diagram (Figure 7.5). At LCL the parcel is saturated. Now, as it continues to rise, it follows a path (depicted by the broken line) dictated by the saturated adiabat passing through $(p_{\text{LCL}}, T_{\text{LCL}})$ (in practice by the pseudo-adiabat, i.e. from equation (7.62) or (7.63)). At some very low pressure level (say, 200 mb), we may assume that all vapor has condensed and the condensation products have fallen out. At that limit, the pseudo-adiabat approaches a dry adiabat (solid line). This dry adiabat corresponds to θ_{ep} (in other words if followed down to 1000 mb, it gives the pseudo-equivalent potential temperature).

7.2.6 A few more temperatures

The dry adiabatic descent from zero pressure level to the 1000 mb level defined the pseudo-equivalent potential temperature, θ_{ep} . At the original (p, T) level it defines the pseudo-equivalent temperature, T_{ep} (Figure 7.6). Since the descent is dry adiabatic we have that

Figure 7.5
Illustration of the
estimation procedure of
the pseudo-equivalent
potential temperature.



$$T_{\text{ep}} = \theta_{\text{ep}} \left(\frac{p}{1000} \right)^{k_d}.$$

Since $T = \theta (p/1000)^{k_d}$ it follows that

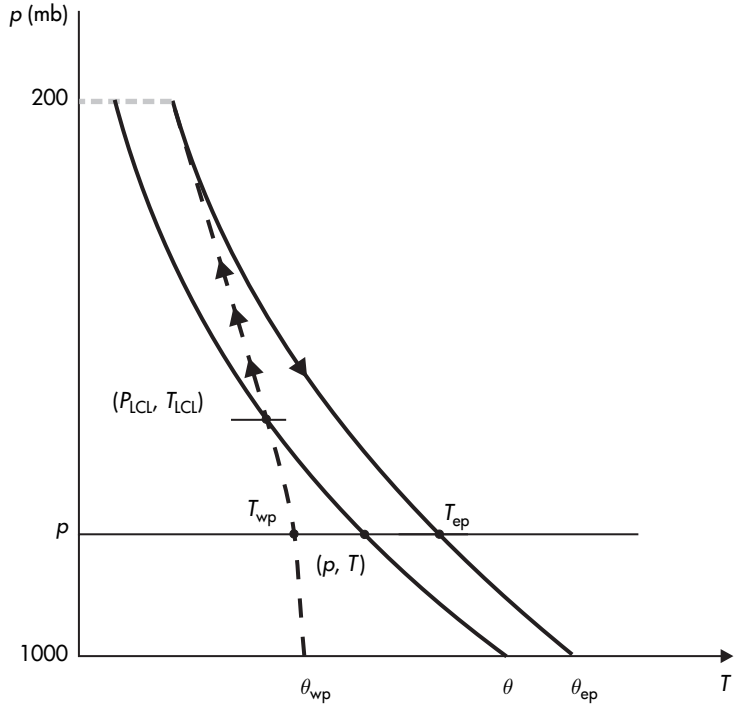
$$T_{\text{ep}} = T \frac{\theta_{\text{ep}}}{\theta}. \quad (7.64)$$

During the process by which T_{ep} is defined, much more water vapor condenses than it would have if it condensed during an adiabatic isobaric process at an initial level below LCL. Therefore (recall the definition of T_{ei}), if all that vapor were to condense at the initial level it would result in a temperature greater than T_{ei} . Thus, $T_{\text{ep}} > T_{\text{ei}}$, which makes T_{ep} the highest temperature.

Many more temperatures can be defined but I will spare you this! Just keep in mind that at a given state, if the air is unsaturated, a dry adiabat can be followed to 1000 mb to define a corresponding potential temperature. As such from the state (T_{ei}, p) the isobaric equivalent potential temperature, θ_{ei} , can be defined. Because $T_{\text{ep}} > T_{\text{ei}}$ it follows that $\theta_{\text{ep}} > \theta_{\text{ei}}$. It also follows that $\theta_{\text{ep}} > \theta$.

Similarly, if the air is saturated, a pseudo-adiabat can be followed to either the original pressure level or to the 1000 mb level to define yet more temperatures. For example, and recalling that

Figure 7.6
Illustration of the
estimation procedure of
 T_{ep} , T_{wp} , and θ_{wp} .



T_w is attained at saturation in an adiabatic process, following the pseudo-adiabat corresponding to point (T_{LCL}, p_{LCL}) to (T, p) level defines the pseudo-wet-bulb temperature, T_{wp} , and continuing down to 1000 mb defines the pseudo-wet-bulb potential temperature θ_{wp} (see Figure 7.6). Obviously, since both θ_{ep} and θ_{wp} are defined by the same curve they must be related and conserved in a pseudo-adiabatic process. θ_{wp} can be estimated from equation (7.63) by setting $T = T_{LCL} = \theta_{wp}$ and $w = w' = w_s(p = 1000 \text{ mb}, T = \theta_{wp})$, i.e.

$$\theta_{ep} = \theta_{wp} \exp \left[w'(1 + 0.81w') \left(\frac{3376}{\theta_{wp}} - 2.54 \right) \right]. \quad (7.65)$$

Similarly, following the pseudo-adiabat passing through point (T_w, p) to 1000 mb defines the isobaric wet-bulb potential temperature θ_w . Here we should note that T_{wp} and T_w (or θ_{wp} and θ_w) are not exactly the same. T_w and θ_w correspond to a true saturated adiabatic process whereas T_{wp} and θ_{wp} correspond to a pseudo-adiabatic process. Since water remains around in a true saturated adiabatic process it turns out that $T_{wp} < T_w$ and $\theta_{wp} < \theta_w$.

It is interesting to note here that if at the initial point (p, T) (see Figure 7.6), the amount of vapor content (w) is higher, then the dry

adiabatic ascent will last a shorter time and therefore the LCL will be found at higher pressure and temperature. The pseudo-adiabat passing through this point will be of a greater temperature than before. This means that higher θ_{wp} (or θ_{ep}) values would correspond to higher moisture (in accordance with 7.63).

Here is then the final breakdown of all the temperatures and potential temperatures and their relationships:

$$T_{LCL} < T_{dew} < T_{wp} < T_w < T < T_{virt} < T_{ei} < T_{ep} \quad (7.66)$$

$$\theta_{wp} < \theta_w < \theta(\approx \theta_m) < \theta_{virt} < \theta_{ei} < \theta_{ep}.$$

7.2.7 Saturated adiabatic lapse rate

Since condensation of water vapor releases heat during a reversible saturated adiabatic ascent, the cooling of the ascending air slows down. As such the corresponding lapse rate should be smaller than that of the dry or moist unsaturated adiabatic ascent.

We could derive an expression for this saturated lapse rate, Γ_s , by starting with equation (7.54)

$$c_p \frac{dT}{T} - R_d \frac{dp_d}{p_d} + d \left(\frac{l_v w_{sw}}{T} \right) = 0$$

where $c_p = c_{pd} + w_t c_w$. Multiplying all terms of the above equation by T/c_p gives

$$dT - \frac{R_d T}{c_p p_d} d(p - e_{sw}) + \frac{1}{c_p} d(l_v w_{sw}) - \frac{l_v w_{sw}}{c_p T} dT = 0,$$

where we have considered that $p = p_d + e_{sw}$. The second term, using the C-C equation in the form $de_{sw}/dT = e_{sw} l_v / R_v T^2$ and the fact that $R_d e_{sw} / R_v p_d = w_{sw}$, can be written as

$$- \frac{R_d T}{c_p p_d} dp + \frac{w_{sw} l_v}{c_p T} dT.$$

Substituting this into the previous equation, dividing all terms by dz , assuming that $dp \approx dp_d$, and using the hydrostatic approximation, leads to

$$\frac{dT}{dz} = -\frac{g}{c_p} - \frac{1}{c_p} \frac{d}{dz} (l_v w_{sw}). \quad (7.67)$$

Note that since in the above equation $c_p \neq c_{pd}$, the term $-g/c_p$ is not exactly the dry adiabatic lapse rate. By performing the differentiation on the right-hand side we can write equation (7.67) as

$$\frac{dT}{dz} = -\frac{g}{c_p} - \frac{w_{sw}}{c_p} \frac{dl_v}{dz} - \frac{l_v}{c_p} \frac{dw_{sw}}{dz}$$

or

$$\frac{dT}{dz} = -\frac{g}{c_p} - \frac{w_{sw}}{c_p} \frac{dl_v}{dT} \frac{dT}{dz} - \frac{l_v}{c_p} \frac{dw_{sw}}{dz}$$

or

$$\left(1 + \frac{w_{sw}}{c_p} \frac{dl_v}{dT}\right) \frac{dT}{dz} = -\frac{g}{c_p} - \frac{l_v}{c_p} \frac{dw_{sw}}{dz}. \quad (7.68)$$

Using equation (6.12) the above equation becomes

$$\left[1 - \frac{w_{sw}(c_w - c_{pv})}{c_p}\right] \frac{dT}{dz} = -\frac{g}{c_p} - \frac{l_v}{c_p} \frac{dw_{sw}}{dz}.$$

Typical values for c_w , c_{pv} , c_p , and w_{sw} are $4220 \text{ J kg}^{-1} \text{ K}^{-1}$, $1850 \text{ J kg}^{-1} \text{ K}^{-1}$, $1050 \text{ J kg}^{-1} \text{ K}^{-1}$, and 0.01, respectively. Such values make the term $w_{sw}(c_w - c_{pv})/c_p$ of the order of 0.02. As such, it can be neglected to arrive at the following approximation

$$\frac{dT}{dz} = -\frac{g}{c_p} - \frac{l_v}{c_p} \frac{dw_{sw}}{dz}. \quad (7.69)$$

Now all we have to do is to find an expression for dw_{sw}/dz . Log-differentiating equation (7.5) leads to

$$\begin{aligned} \frac{1}{w_{sw}} \frac{dw_{sw}}{dz} &= -\frac{1}{p} \frac{dp}{dz} + \frac{1}{e_{sw}} \frac{de_{sw}}{dz} \\ &= -\frac{1}{p} \frac{dp}{dz} + \frac{1}{e_{sw}} \frac{de_{sw}}{dT} \frac{dT}{dz} \end{aligned}$$

where p is the total pressure of the parcel or the surroundings. If we again use the hydrostatic approximation and the C-C equation the above equation becomes

$$\frac{1}{w_{sw}} \frac{dw_{sw}}{dz} = \frac{g}{RT} + \frac{l_v}{R_v T^2} \frac{dT}{dz}. \quad (7.70)$$

Combining equations (7.70) and (7.69) leads to

$$\Gamma_s = -\frac{dT}{dz} = \frac{g}{c_p} \frac{1 + l_v w_{sw}/RT}{1 + l_v^2 w_{sw}/c_p R_v T^2} \quad (7.71)$$

where $c_p = c_{pd} + w_t c_w$ and R is the gas constant for the surroundings. In other thermodynamics books you may find some slight differences in the formula giving Γ_s . Most of the differences are due to certain assumptions and approximations. For typical values of saturation mixing ratio and temperature in the lower troposphere, Γ_s ranges around 5°C km^{-1} , which is half the dry adiabatic rate.

7.3 Other processes of interest

7.3.1 Adiabatic isobaric mixing

Let us consider two air masses of temperatures T_1 and T_2 and humidities q_1 and q_2 , which are mixing at the same pressure. This corresponds to a horizontal mixing. If we assume the mixing to be an adiabatic process, then it is also isenthalpic. If we assume that no condensation takes place, then we can write for the change in enthalpy of the whole system

$$\Delta H = m_1 \Delta h_1 + m_2 \Delta h_2 = 0 \quad (7.72)$$

where

$$\begin{aligned} \Delta h_1 &= c_{p1} \Delta T = c_{p1} (T - T_1) \\ \Delta h_2 &= c_{p2} \Delta T = c_{p2} (T - T_2) \end{aligned}$$

and T is the final temperature of the mixture. Considering equation (7.11), equation (7.72) becomes

$$m_1 c_{pd} (1 + 0.87 q_1) (T - T_1) + m_2 c_{pd} (1 + 0.87 q_2) (T - T_2) = 0.$$

Solving for T we get

$$T = \frac{(m_1 T_1 + m_2 T_2) + 0.87 (m_1 q_1 T_1 + m_2 q_2 T_2)}{m + 0.87 m_v}. \quad (7.73)$$

In the above expression $m = m_1 + m_2$ is the total mass and $m_v = m_1 q_1 + m_2 q_2$ is the total water vapor mass. Since there is no condensation, m_v remains constant. If the final specific humidity is q , then the water vapor mass is qm . It follows that

$$q = \frac{m_1 q_1 + m_2 q_2}{m} \quad (7.74)$$

i.e. the final q is the weighted average of q_1 and q_2 . Using equation (7.74) we can rewrite equation (7.73) as

$$T = \frac{(m_1 T_1 + m_2 T_2) + 0.87 (m_1 q_1 T_1 + m_2 q_2 T_2)}{m(1 + 0.87 q)}. \quad (7.75)$$

If we neglect all vapor terms the above equation becomes

$$T \approx \frac{m_1 T_1 + m_2 T_2}{m} \quad (7.76)$$

indicating that, approximately, the temperature of the mixture is also the weighted average of the initial temperatures. One can

show that the potential temperature and the vapor pressure of the mixture will also, approximately, be the weighted averages of the initial values (see problem 7.1):

$$\theta \approx \frac{m_1\theta_1 + m_2\theta_2}{m} \quad (7.77)$$

$$e \approx \frac{m_1e_1 + m_2e_2}{m}. \quad (7.78)$$

Equations (7.76) and (7.78) indicate that in a vapor pressure–temperature diagram (Figure 7.7) if the two masses correspond to points A_1 and A_2 then the mixture will correspond to the point A which will lie on the straight line connecting A_1 and A_2 . From the similarity of triangles AA'_1A_1 and $A_2A'_2A$ it follows that point A will be found at a distance such that $A_1A/AA_2 = AA'_1/A_2A'_2 = (T_1 - T)/(T - T_2) = m_2/m_1$ (here equation (7.76) was used). The thin solid lines in Figure 7.3 represent the equal relative humidity curves corresponding to points A_1, A_2 , and A . Because of their curvature it can easily be seen that the final relative humidity will be greater than the weighted average. This can easily be visualized if we consider the simple case where the air masses have the same relative humidity, i.e. points A_1 and A_2 lie on the same equal relative humidity curve (Figure 7.8). Then point A will lie to the left of the curve indicating that the final relative humidity is greater than the weighted average, $(mr + mr)/2m = r$. We conclude that isobaric adiabatic mixing increases the relative humidity. This is a very interesting result because it implies that two air masses that are initially unsaturated (but close to saturation) may result in a mixture that is supersaturated. In this case condensation will occur. This mechanism can produce mixing fogs and other important processes in the atmosphere.

Figure 7.7
When two initially unsaturated air masses (A_1 at r_1 and A_2 at r_2) mix isobarically, the final product (A) may be saturated.

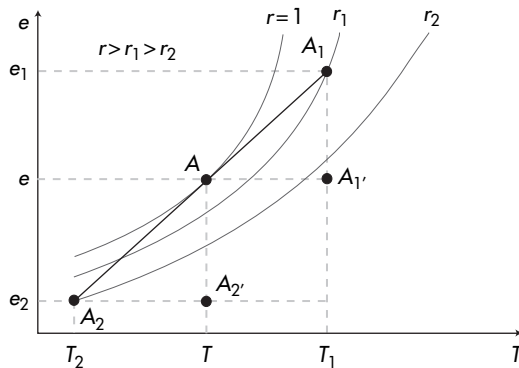
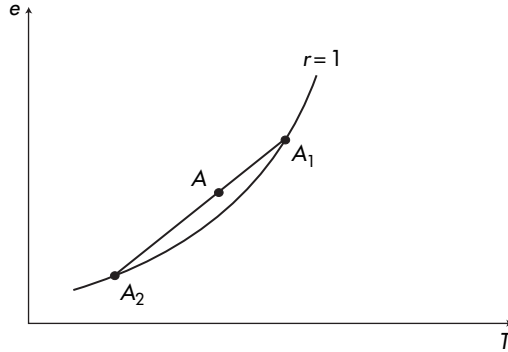


Figure 7.8

When two saturated ($r = 1$) air masses (A_1 and A_2) mix isobarically the final product (A) is supersaturated.



7.3.2 Vertical mixing

Let us now look at vertical mixing occurring because of turbulent and/or convective processes. Such a process is much more complicated than the previous one because now the vertical variation of T and p come into the picture. However, we can simplify the treatment of vertical mixing by (1) considering two isolated air masses m_1 and m_2 at pressure levels p_1 and p_2 ($p_1 > p_2$) and temperatures T_1 and T_2 , which are taken to a pressure level p where they mix isobarically, and (2) redistributing the mixture in the layer $\Delta p = p_1 - p_2$. The excursions to level p can be considered as adiabatic ascent and descent (during which the specific humidities q_1 and q_2 remain constant) resulting in new temperatures for the air masses:

$$T'_1 = T_1 \left(\frac{p}{p_1} \right)^k$$

$$T'_2 = T_2 \left(\frac{p}{p_2} \right)^k.$$

Once at level p the air masses mix isobarically. From what we discussed in the previous section it follows that in this case

$$\begin{aligned} T &\approx \frac{m_1 T'_1 + m_2 T'_2}{m} \\ q &\approx \frac{m_1 q_1 + m_2 q_2}{m} \end{aligned} \quad (7.79)$$

and

$$\theta \approx \frac{m_1 \theta_1 + m_2 \theta_2}{m}$$

where θ_1 and θ_2 are the initial potential temperature values, which remain constant during the adiabatic ascent and descent. The subsequent redistribution of the mixture in the layer Δp

will preserve θ assuming that it consists of adiabatic ascents and descents. It follows that when the layer is well mixed, θ will not vary with height. We can generalize equation (7.79) by considering mixing of n air masses and employing the general expression for the weighted average. For example,

$$\bar{\theta} = \frac{\int_0^m \theta dm}{m},$$

$$\bar{q} = \frac{\int_0^m q dm}{m},$$

$$\bar{w} = \frac{\int_0^m w dm}{m}.$$

Using the hydrostatic approximation we can express the above equations as

$$\bar{x} = \frac{\int_0^z x \rho dz}{\int_0^z \rho dz} = - \frac{\int_{p_1}^{p_2} x dp}{p_1 - p_2}$$

where x is θ , or q , or w .

The temperature variation with pressure can then be expressed according to

$$T = \bar{\theta} \left(\frac{p}{1000} \right)^k. \quad (7.80)$$

If in the initial distribution (before mixing) there is a level ($p, T < \bar{T}$), where $r = \bar{r}$, then this level is defined as the mixing condensation level.

7.3.3 Freezing inside a cloud

As air rises and cools and clouds begin to form, it is often the case that the developing cloud would reach the freezing level. From that point on, freezing of cloud droplets (or rain drops) would occur and water vapor and ice will co-exist. Does this freezing affect the further development of the cloud? We can get an answer to this question by making the following simple assumptions:

- 1) Initially the air is saturated with respect to water at temperature T .
- 2) As water freezes latent heat is released into the environment.
- 3) Once water freezes, water vapor is supersaturated with respect to ice. Deposition is then initiated releasing more latent heat. This continues until the vapor pressure becomes equal to the saturation vapor pressure over ice.
- 4) The whole process occurs at a constant pressure level and it is adiabatic. Since it is adiabatic and isobaric, it is also isenthalpic.

Accordingly, we can divide this process into three “treatable” steps:

Step 1: Water freezes at a constant temperature T . In this case we have a change of phase at constant pressure and the change in enthalpy is (see Chapter 6)

$$\Delta H_1 = -\delta Q_{p=\text{constant}} = -L_f = -l_f m_w$$

where m_w is the amount of water that freezes. The minus sign is due to the fact that heat is released. Dividing the above equation by the mass of dry air yields

$$\frac{\Delta H_1}{m_d} = -l_f w_w(T) \quad (7.81)$$

Step 2: Vapor condenses on ice at a constant temperature T until the water vapor pressure become the saturation vapor pressure over ice at a temperature $T' > T$. Like in Step 1, the change in enthalpy during this step will be

$$\Delta H_2 = -l_s(m_w - m_i)$$

or

$$\frac{\Delta H_2}{m_d} = -l_s[w_w(T) - w_i(T')] \quad (7.82)$$

Step 3: The whole system is warmed from T to T' . The change in enthalpy during this term will be (recall 4.18)

$$\Delta H_3 = C_p \Delta T$$

or

$$\frac{\Delta H_3}{m_d} = c_{pd} \Delta T \quad (7.83)$$

where it is assumed that $C_p/m_d \approx c_{pd}$. Since the whole process is isenthalpic, it follows that

$$-l_f w_w(T) - l_s[w_w(T) - w_i(T')] + c_{pd} \Delta T = 0$$

or

$$\Delta T = \frac{l_f w_w(T) + l_s[w_w(T) - w_i(T')]}{c_{pd}}$$

or using (7.5)

$$\Delta T = \frac{\epsilon l_f e_{sw}(T) + \epsilon l_s[e_{sw}(T) - e_{si}(T')]}{p c_{pd}}.$$

In the above equation and in the range of typical conditions in our atmosphere, the right-hand side is positive. As such the effect of freezing is to increase the temperature inside the developing cloud. Typically, this temperature increase is of the order of 3 K, a significant warming during cloud development.

Examples

(7.1) The initial state of a parcel of air is $p = 1000$ mb, $T = 30^\circ\text{C}$, $w = 14$ g kg⁻¹. Find (a) r , (b) T_{virt} , (c) T_{dew} , (d) T_w , (e) T_{ei} , (f) T_{LCL} , (g) z_{LCL} , (h) θ_{ep} , (i) T_{ep} , (j) θ_{wp} , (k) θ_{virt} .

- (a) From equation (6.17) we estimate that $e_{\text{sw}}(T) = 43.1$ mb. From equation (7.3) we then find that $e(T) \approx 22.0$ mb. Thus, $r \approx 0.51$, $w_{\text{sw}} \approx 28$ g kg⁻¹.
- (b) $T_{\text{virt}} = (1 + 0.61q)T = \left[1 + 0.61\left(\frac{w}{1+w}\right)\right]T = 305.55$ K.
- (c) From equation (7.16) we obtain $T_{\text{dew}} = 291.75$ K.
- (d) From equation (7.22) we have

$$T_w = T + \frac{l_v}{c_{pd}}(w - w_{\text{sw}})$$

where $l_v(T = 303 \text{ K}) = 2.43 \times 10^6$ J kg⁻¹. Now recall that w_{sw} in the above formula is the saturation mixing ratio at T_w . Making use of the approximations in equations (7.5) and (6.17) we have

$$T_w = T + \frac{l_v \epsilon}{c_{pd} p} \left[e(T) - 6.11 \exp\left(19.83 - \frac{5417}{T_w}\right) \right]$$

Numerical solution of the above equation gives $T_w \approx 295.4$ K.

- (e) From equation (7.23) for $c_w = 4218$ J kg⁻¹ we find that $T_{\text{ei}} \approx 335$ K.
- (f) From equation (7.36) it follows that $T_{\text{LCL}} \approx 289$ K.
- (g) From equation (7.45) and assuming that the initial level is $z = 0$ we find that $z_{\text{LCL}} = 1.4$ km.
- (h) Using equation (7.63) with $T = 303$ K, $p = 1000$ mb, $w = 0.014$ g kg⁻¹, $T_{\text{LCL}} = 289$ K, we obtain that $\theta_{\text{ep}} \approx 345$ K.
- (i) From equation (7.64) it follows (since $T = \theta$) that $T_{\text{ep}} = \theta_{\text{ep}} = 345$ K.
- (j) Considering that $w' = w_s(1000 \text{ mb}, T = \theta_{\text{wp}}) = (\epsilon/p)6.11 \exp[19.83 - (5417/\theta_{\text{wp}})]$, numerical solution of equation (7.65) yields $\theta_{\text{wp}} \approx 295$ K.
- (k) From equation (7.28) and since $p = 1000$ mb it follows that $\theta_{\text{virt}} = T_{\text{virt}} = 305.55$ K.

Note that all estimated temperatures and potential temperatures are consistent with equation (7.66).

- (7.2) The temperature of saturated air at 1000 mb is 10°C as it begins to drop through radiation loss. As a result, some water vapor condenses and radiation fog forms. During the formation of fog, $12 \times 10^3 \text{ J kg}^{-1}$, of heat is lost to the surroundings. Find the final temperature and the decrease in vapor pressure.

The process is non-adiabatic and thus non-isenthalpic. From equation (7.20) we have

$$dh = c_p dT + l_v dw$$

where $c_p = c_{pd} + w_t c_w$ and we have assumed that $dH/m_d \approx dh$. During the process the air remains in a state of saturation, thus w corresponds to saturation. Also, the process is isobaric, so $dh = \delta q$ and we can write the above equation as

$$\delta q = c_p dT + l_v dw_{sw}. \quad (7.84)$$

Using the approximate relationship $w_{sw} \approx \epsilon e_{sw}/p$, we find that $dw_{sw} \approx (\epsilon/p) de_{sw}$ or, using the C-C equation,

$$dw_{sw} = \frac{\epsilon}{p} \frac{l_v e_{sw}}{R_v T^2} dT. \quad (7.85)$$

By substituting (7.82) into (7.81) we obtain

$$\delta q = \left(c_p + \frac{\epsilon l_v^2 e_{sw}}{p R_v T^2} \right) dT.$$

In the above equation, c_p is to a good approximation independent of temperature. Also, for small temperature changes, l_v is nearly independent of temperature. The saturation vapor pressure, however, is dependent on T (via equation (6.17)). This makes the integration of the above equation very difficult. One way to overcome this difficulty is to assume that e_{sw}/T^2 is nearly constant for small temperature differences and then treat the differentials as differences. In this case, for $T = 10^\circ\text{C}$, $e_{sw} = 12.16 \text{ mb}$, $l_v(10^\circ\text{C}) = 2.4774 \times 10^6 \text{ J kg}^{-1}$, $\delta q = -12 \times 10^3 \text{ J kg}^{-1}$, and $c_p \approx c_{pd} = 1005 \text{ J kg}^{-1} \text{ K}^{-1}$, we find that $\Delta T = -5.3^\circ\text{C}$. Thus, the final temperature is 4.7°C . Note that this result can be improved if we do not assume that $c_p \approx c_{pd}$. However, this will require the value of w_t which is not known. We could of course assume that $w_t \approx w_{sw}(10^\circ\text{C})$ and consider $c_p \approx c_{pd} + w_{sw} c_w$. In this case, we find that the difference in the result is small ($\sim 0.066^\circ\text{C}$). In order to find the decrease in vapor

pressure we combine equation (7.81) with $dw_{\text{sw}} \approx (\epsilon/p)de_{\text{sw}}$ to arrive at

$$\delta q = c_p dT + \frac{l_v \epsilon}{p} de_{\text{sw}}.$$

Then incorporating the C-C equation gives

$$\delta q = \left(\frac{c_p R_v T^2}{l_v e_{\text{sw}}} + \frac{l_v \epsilon}{p} \right) de_{\text{sw}}.$$

Here again if we treat the differentials as differences and use the values mentioned above we find that $\Delta e_{\text{sw}} = -4.326$ mb.

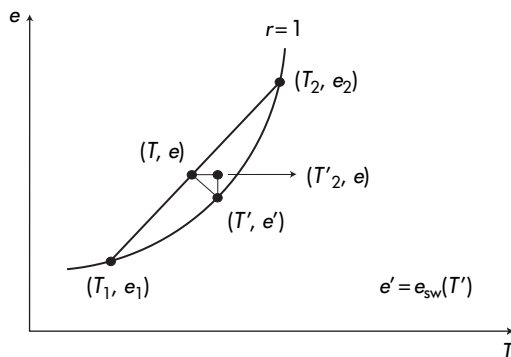
- (7.3) Two equal air masses, both at 1000 mb, mix adiabatically. Their initial temperatures and mixing ratios are $T_1 = 293.8$ K, $w_1 = 16.3$ g kg⁻¹, $T_2 = 266.4$ K, and $w_2 = 1.3$ g kg⁻¹. First show that the mixture is supersaturated. Then find the final temperature, the final mixing ratio, the amount of liquid water content per unit mass of air, and the amount of liquid water per unit volume of air.

As the two air masses mix isobarically, the initial temperature of the mixture will be (equation (7.75)) $T = 280.2$ K. Since $q_1 = 16.0386$ g kg⁻¹ and $q_2 = 1.2983$ g kg⁻¹ we find that, for the mixture, $q = 8.6685$ g kg⁻¹ (equation (7.74)), $w = q/(1 - q) = 8.7443$ g kg⁻¹, and $e = wp/(\epsilon + w) = 13.86$ mb (using the approximate equations (7.76) and (7.78) results in $T = 280.7$ K and $e = 14.1$ mb).

For $T = 280.2$ K equation (6.17) yields $e_{\text{sw}} = 10.05$ mb. This result indicates that initially the mixture is supersaturated ($e > e_{\text{sw}}$). Thus, immediately after mixing the excess vapor will condense so that the mixture reaches the equilibrium state (T', e') . This step, graphically shown in Figure 7.9, will result in a decrease in vapor pressure and an increase in temperature (remember the mixing is adiabatic, so the

Figure 7.9

Diagram for example 7.3. When two initially saturated air masses mix isobarically, the final product is supersaturated (T, e) . In order for the mixture to reach the equilibrium state (T', e') , the excess vapor will condense. The amount of vapor that condenses is proportional to the difference $e - e'$.



heat gain due to condensation remains in the mixture). The increase in temperature is denoted by the line connecting points (T, e) and (T', e) and the decrease in water vapor pressure by the line connecting points (T', e') and (T', e) . Since the process from (T, e) to (T', e') is isobaric and adiabatic it is isenthalpic. As such, equation (7.21) applies:

$$(c_{pd} + w_t c_w)(T - T') = (w' - w)l_v$$

or

$$(c_{pd} + w_t c_w)(T - T') = (e' - e)l_v \frac{\epsilon}{p}$$

or

$$(c_{pd} + w_t c_w)(T - T') = \left[6.11 \exp \left(19.83 - \frac{5417}{T'} \right) - e \right] l_v \frac{\epsilon}{p}$$

where, assuming that the mixture is a closed system, $w_t = w = 8.7443 \text{ g kg}^{-1}$. Numerical solution of the above equation, with $c_w = 4218 \text{ J kg}^{-1} \text{ K}^{-1}$ and $l_v = 2.4774 \times 10^6 \text{ J kg}^{-1}$, gives $T' \approx 282.8 \text{ K}$. This corresponds to $e' = e_{sw}(T') = 12 \text{ mb}$ and hence to $w' = 7.55 \text{ g kg}^{-1}$.

The amount of liquid water produced per unit mass of air is given by the difference $q - q' \approx w - w' \approx \frac{\epsilon}{p}(e - e') \approx 1.16 \text{ g kg}^{-1}$. This amount must be equal to the decrease of water vapor per unit mass, $-dm_v$, which is proportional to the decrease in water vapor pressure, $-de_{sw}$. From the ideal gas law we know that the mass of water vapor per unit volume at temperature T is $e_{sw}/R_v T$ and its variation with temperature is

$$de_{sw}/dT = de_{sw}/R_v T - (e_{sw}/R_v T^2)dT.$$

Using the C-C equation we can verify that the second term on the right-hand side of the above equation is much smaller than the first term. As such, the amount of water vapor lost is approximately equal to $de_{sw}/R_v T$. It follows that the amount of liquid water *per volume* is equal to $de_{sw}/R_v T$, or equal to

$$\frac{e - e'}{R_v T} \approx 1.44 \text{ g m}^{-3}.$$

- (7.4) The following problem (Salby, 1996) is a classic example, demonstrating the effect of the downslope wind called Chinook wind (Chinook in American Indian means “snow-eater”). Moist air from the Pacific is forced to rise over the continental divide. On the west side, the surface lies at 800 mb where the temperature is 293 K and the mixing ratio is 15 g kg^{-1} . If the summit lies at 600 mb and assuming that any condensation products fall out,

find the surface temperature and relative humidity at a place on the leeward side at 830 mb.

For $T = 293 \text{ K}$ equation (6.17) gives $e_{\text{sw}} = 23.4 \text{ mb}$. From equation (7.4) we then find that the saturation mixing ratio at state (T, p, w) : $(293 \text{ K}, 800 \text{ mb}, 15 \text{ g kg}^{-1})$ is 18.74 g kg^{-1} . Since $w < w_s$, the parcel is initially unsaturated. As the air begins to rise over the mountain it cools and its relative humidity increases. From equation (7.36) we find that, for $r = 15/18.74 = 0.8$, $T_{\text{LCL}} = 288.7 \text{ K}$.

The ascent to the level where the air first becomes saturated is described by Poisson's equation

$$p_{\text{LCL}} = p \left(\frac{T_{\text{LCL}}}{T} \right)^{1/k}$$

where $k = k_d(1 - 0.26w) = 0.2849$. It follows that $p_{\text{LCL}} \approx 760 \text{ mb}$. At this point the air is saturated. As it continues to rise condensation takes place and all condensation products fall out. As such, the ascent above p_{LCL} is described by a pseudo-adiabatic process. Therefore, θ_{ep} from 760 to 600 mb is conserved. From equation (7.63) we find, for $T = T_{\text{LCL}} = 288.7 \text{ K}$, $p = p_{\text{LCL}} = 760 \text{ mb}$, and $w = w_{\text{sw}}(T_{\text{LCL}}) = 15 \text{ g kg}^{-1}$, that $\theta_{\text{ep}} \approx 359 \text{ K}$. Evaluating the same equation at $p' = 600 \text{ mb}$ gives

$$359 = T' \left(\frac{1000}{p'} \right)^{0.285(1-0.28w')} \times \exp \left[w'(1 + 0.81w') \left(\frac{3376}{T_{\text{LCL}}} - 2.54 \right) \right] \quad (7.86)$$

where T' , p' , and w' is the state of the air at the summit. In the above equation we know p' but T' and w' are unknowns. However, w' can be expressed as a function of T' alone by combining equations (7.5) and (6.17):

$$w' = w_{\text{sw}}(T') \approx \frac{\epsilon e_{\text{sw}}(T')}{p'} = \frac{6.11\epsilon}{p'} \exp \left(19.83 - \frac{5417}{T'} \right).$$

Thus, in effect, equation (7.83) has only one unknown, T' . Solving this equation numerically gives $T' \approx 279 \text{ K}$. It follows that at the summit $w' \approx 9.6 \text{ g kg}^{-1}$. From that point on the air descends adiabatically with the mixing ratio remaining constant (9.6 g kg^{-1}). From Poisson's equation we then find that at 830 mb, $T \approx 306 \text{ K}$. At this temperature $w_{\text{sw}} \approx 38.5 \text{ g kg}^{-1}$ and at $p = 830 \text{ mb}$ the relative humidity is $r = 0.25$. Thus the air on the leeward side is warmer and drier than that on the west side. Note that we assume

that the parcel from 760 to 600 mb remains saturated with respect to *liquid* water. This is a safe assumption because $T_{\text{LCL}} = 288.7\text{ K}$ and the vertical distance between 760 and 600 mb in a standard atmosphere is about 1.5 km. So, we would expect that the temperature at the summit would be above 0°C .

(7.5) What is the maximum rate of precipitation when saturated air rises?

Consider a saturated parcel of air rising. For simplicity, we assume that this parcel is a unit mass. As the parcel rises water vapor condenses. If we assume that all condensed products (rain or snow) fall out we can estimate the upper limit for the rate of precipitation. In order to achieve this we consider equation (7.54) that describes reversible saturated adiabatic processes, which we simplify by assuming that $l_v = \text{constant}$ and $c_{\text{pd}} + w_t c_w \approx c_{\text{pd}}$.

$$c_{\text{pd}} d \ln T - R_d d \ln p_d + l_v d \left(\frac{w_{\text{sw}}}{T} \right) = 0$$

or

$$c_{\text{pd}} d \ln T - R_d d \ln p_d + l_v \frac{dw_{\text{sw}}}{T} - \frac{l_v w_{\text{sw}}}{T^2} dT = 0$$

or by multiplying all terms by T

$$c_{\text{pd}} dT - T R_d d \ln p_d + l_v dw_{\text{sw}} - \frac{l_v w_{\text{sw}}}{T} dT = 0$$

or

$$\left(c_{\text{pd}} - \frac{l_v w_{\text{sw}}}{T} \right) dT - \frac{p_d}{\rho} d \ln p_d + l_v dw_{\text{sw}} = 0$$

or using the hydrostatic approximation

$$\left(c_{\text{pd}} - \frac{l_v w_{\text{sw}}}{T} \right) dT + g dz + l_v dw_{\text{sw}} = 0.$$

For typical conditions in the atmosphere $l_v w_{\text{sw}}/T$ is very small compared to c_{pd} . We can thus neglect it to arrive at

$$c_{\text{pd}} dT + g dz + l_v dw_{\text{sw}} = 0$$

or

$$-\frac{l_v dw_{\text{sw}}}{dz} = c_{\text{pd}} \frac{dT}{dz} + g$$

or

$$-\frac{l_v}{c_{\text{pd}}} \frac{dw_{\text{sw}}}{dz} = \frac{dT}{dz} + \frac{g}{c_{\text{pd}}}$$

Now, recall that $\Gamma_d = \frac{g}{c_{pd}}$ and that $-\frac{dT}{dz} = \Gamma_s$. Thus, the above equation becomes

$$-\frac{l_v}{c_{pd}} \frac{dw_{sw}}{dz} = \Gamma_d - \Gamma_s$$

or

$$-\frac{dw_{sw}}{dz} = \frac{c_{pd}(\Gamma_d - \Gamma_s)}{l_v}$$

or

$$-\frac{dw_{sw}}{dt} \frac{dt}{dz} = \frac{c_{pd}(\Gamma_d - \Gamma_s)}{l_v}$$

where $\frac{dz}{dt}$ is the rate of ascent or speed of the parcel, U . It follows that

$$-\frac{dw_{sw}}{dt} = \frac{c_{pd}(\Gamma_d - \Gamma_s)}{l_v} U.$$

The above equation gives the mass of condensed water *per unit mass of air and unit time*. If we multiply it by the mass of air, m , we can obtain an expression for the mass of water condensed and precipitated per unit time

$$P = \frac{c_{pd}(\Gamma_d - \Gamma_s)}{l_v} U m$$

or

$$P = \frac{c_{pd}(\Gamma_d - \Gamma_s)}{l_v} U \rho V$$

or

$$P = \frac{c_{pd}(\Gamma_d - \Gamma_s)}{l_v} U \rho A \Delta z$$

where A is the area and Δz the thickness of the rising volume. Thus, per unit area and unit time

$$P = \frac{c_{pd}(\Gamma_d - \Gamma_s)}{l_v} U \rho \Delta z.$$

In the above equation, P is given (in MKS units) in $\text{kg m}^{-2} \text{s}^{-1}$. In practice, the amount of condensed water and precipitation per unit area is expressed by the depth of the collected water in millimeters (mm). Considering that the density is about 10^3 kg m^{-3} , then 1 kg m^{-2} is equivalent to 1 mm of precipitated water. If, in addition, we consider that $1 \text{ h} = 3600 \text{ s}$, then we can express the above equation in mm h^{-1} , which is the unit for the rate of precipitation used in practice

$$P = 3600 \frac{c_{pd}(\Gamma_d - \Gamma_s)}{l_v} U \rho \Delta z.$$

Problems

- (7.1) Show that when two air masses mix adiabatically and isobarically without condensation taking place, the final potential temperature and final vapor pressure are the weighted averages of the initial values (make necessary assumptions).
- (7.2) Show that

$$\frac{p_{\text{LCL}}}{e_{\text{sw}}(T_{\text{LCL}})} = \frac{p}{e_{\text{sw}}(T_{\text{dew}})}.$$

Then use equation (6.17) to derive a relationship between T_{LCL} , T_{dew} , p , and p_{LCL} .

- (7.3) An air mass has a temperature of 20°C at 970 mb pressure with a mixing ratio of 5 g kg^{-1} . After some time this mass has acquired a temperature of 10°C and a pressure of 750 mb. Assume that no condensation or mixing between the air mass and the environment takes place and calculate the initial and final values of vapor pressure, relative humidity, and dew point temperature. (7.74 mb, 5.98 mb, 0.33, 0.49, 276.5 K, 272.8 K)
- (7.4) Show that evaporation of dm grams of water into m_{d} grams of dry air (under constant temperature) requires an amount of heat absorbed given by

$$\delta Q = m_{\text{d}} l_{\text{v}} dw.$$

- (7.5) A Chinook wind blowing at 800 mb has a temperature of 38°C and a mixing ratio of 4 g kg^{-1} . Is this the same as the air at 1000 mb on the windward side of the mountains having a temperature of 294.5 K and a mixing ratio of 10 g kg^{-1} or is it the same as the air at 800 mb with 278 K and 5 g kg^{-1} ? (It is the same as the air at 1000 mb)
- (7.6) Show that if an isothermal layer of temperature T_0 contained between isobars p_1 and p_2 mixes vertically, the final temperature distribution is given by

$$T(p) = [T_0/(1 - k)] (p_1^{1-k} - p_2^{1-k}) p^k / (p_1 - p_2).$$

- (7.7) What will be the lowest possible temperature to which very dry air at $T = 40^\circ\text{C}$ can be cooled by evaporation? Consider $p = 1000\text{ mb}$ and l_{v} constant and equal to $2.4 \times 10^6\text{ J kg}^{-1}$. ($\sim 288\text{ K}$)
- (7.8) When it is cold outside our breath creates mixing clouds. Is it possible to create a mixing cloud when the temperature of our breath is less than the air temperature? Assume that the temperature of breath is 30°C . (Hint: Plot e_{sw} as a function of T using equation (6.17) and make assumptions about this curve for high temperatures. Then use the theory discussed in section 7.3.1.)

- (7.9) On a summer day in Milwaukee the maximum temperature is predicted to reach 35°C with dew point around 28°C . What will be the lowest temperature to which the air can be cooled by evaporation in this case? Does the answer make sense? Elaborate on the difference between this answer and the answer in problem 7.7. Consider l_v constant and equal to $2.4 \times 10^6 \text{ J kg}^{-1}$. ($\sim 303 \text{ K}$)
- (7.10) Moist air moves over land where the temperature is 20°C and the relative humidity is 70%. Through contact with the ground the air cools. At what temperature will fog form? Consider l_v constant and equal to $2.45 \times 10^6 \text{ J kg}^{-1}$. (287.5 K)
- (7.11) The main cabin in an airplane is maintained at a pressure of 900 mb and at a temperature of 25°C . If it suddenly decompresses adiabatically to a pressure of 500 mb, what should the relative humidity be to avoid cloud formation in the cabin? (6.29%)
- (7.12) Outside air has a temperature of -15°C and a relative humidity of 0.6. If the air indoors is 25°C , what is the relative humidity inside the room (assuming that the air inside is only heated, not humidified)? If the room has a volume of 100 m^3 how much water vapor must be supplied so that the relative humidity rises to 50% and what will then be the total mixing ratio? If the temperature change due to evaporating water is neglected, what amount of heat must be added for this to happen? Assume l_v constant and equal to $2.5 \times 10^6 \text{ J kg}^{-1}$. (3.6%, 1.08 kg, 8.97 g kg^{-1} , $2.7 \times 10^6 \text{ J}$)
- (7.13) Which of the following quantities are conserved in (a) reversible adiabatic unsaturated transformation, (b) reversible adiabatic saturated transformation, (c) adiabatic unsaturated isobaric transformation, (d) adiabatic saturated isobaric transformation?

$$w, q, w_t, q_t, e, e_{\text{sw}}, w_{\text{sw}}, r, p_{\text{LCL}}, T_{\text{LCL}}, T_w, \theta, \theta_{\text{virt}}, \theta_e, \theta_w, s, h.$$

- (7.14) A refrigerator of volume 2 m^3 is initially filled with air at temperature 303 K and with a relative humidity of 50%. At what temperature condensation will form on the walls? Assuming that the desired final temperature is 275 K , how much water vapor must condense to achieve the final temperature? What is the total amount of heat that will be given away to the surroundings during the whole process? Assume $l_v = 2.5 \times 10^6 \text{ J kg}^{-1}$, $c_w = 4218 \text{ J kg}^{-1} \text{ K}^{-1}$, $\rho_d = 1.293 \text{ kg m}^{-3}$. ($\sim 19^\circ\text{C}$, 19.8 g, 117 368 J)
- (7.15) If the temperature inside a room is 25°C and the temperature outside is -10°C calculate the maximum relative humidity that can be accommodated inside the room without windows fogging when (a) the interior part of the

windows is not thermally isolated (i.e. heat is transferred freely) from the exterior part, and (b) it is thermally isolated. Assume $l_v = 2.5 \times 10^6 \text{ J kg}^{-1}$, $c_w = 4218 \text{ J kg}^{-1} \text{ K}^{-1}$. (0.32, 1.0)

- (7.16) If the wet-bulb temperature of a parcel at 1000 mb is 15.6°C and its mixing ratio is 6 g kg^{-1} , find the relative humidity of the parcel after it is lifted dry adiabatically to 900 mb. Assume $l_v = 2.47 \times 10^6 \text{ J kg}^{-1}$. (36.14%)
- (7.17) Raindrops at 10°C evaporate into air of temperature 20°C . If the saturated mixing ratio of air at 10°C is 8 g kg^{-1} , what is the mixing ratio of the environmental air? Make assumptions and consider $l_v = 2.47 \times 10^6 \text{ J kg}^{-1}$. (3.93 g kg^{-1})
- (7.18) If water vapor comprises 1% of the volume of air (i.e. it accounts for 1% of the molecules) what is the virtual temperature correction? ($T_{\text{virt}} = 1.0038 T$)
- (7.19) If all the water vapor in a column of air extending from the surface to the troposphere were to condense and fall to the surface, it would occupy the lower d_w of the height of the column. In this case, we can write that

$$\rho_w d_w = \int_0^\infty \rho_v dz$$

where ρ_w and ρ_v are the densities of liquid water and water vapor, respectively. Show that the *upper* limit for d_w is given by

$$d_w \approx \frac{T_0 e_{s0}}{\rho_w l_v \Gamma}$$

where Γ is the lapse rate, which is assumed to be constant, T_0 is the surface temperature, and e_{s0} is the saturation vapor pressure with respect to water at T_0 . Get an estimate of d_w by assuming realistic values for T_0 and Γ . What do you think of the estimated d_w ? Is it realistic? [d_w is often called precipitable water.]

- (7.20) Show that the mixing ratio of a parcel of air is given exactly by

$$w = \frac{r w_s}{1 + (1 - r) \frac{w_s}{\epsilon}}.$$

- (7.21) In *Meteorology* Book I, Aristotle says: “Both frost and dew are found when the sky is clear and there is no wind.” What do you think? Is he right or wrong? Explain why.