Lecture 5

Linear Equations II

1. Pivoting
2. Gauss-Jordan Elimination
3. Iteration Approaches
   - Jacobi
   - Gauss-Seidel

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Numerical Methods for Engineers and Scientists: An Introduction with Applications Using MATLAB by Amos Gilat and Vish Subramaniam.

Lecture 5 part I

Review Newton and Secant Method

Gauss elimination with pivoting

(Python)

```
# forward elimination
for k in range(n-1):
    # partial pivoting
    i = np.argmax(abs(Aug[k:n,k]))
    big = np.max(abs(Aug[k:n,k]))
    ipr = i + k
    if ipr ~= k
        Aug[[k,ipr],:] = Aug[[ipr,k],:]
    for i in range(k+1,n):
        factor = Aug[i,k]/Aug[k,k]
        Aug[i,k:nb] = Aug[i,k:nb] - factor*Aug[k,k:nb]

# back substitution
x = zeros((n,1));
x[n-1] = Aug[n,nb]/Aug[n,n];
for i in range(n-2,-1,-1)
    x[i]=(Aug[i,nb]-Aug[i,i+1:n]*x[i+1:n])/Aug[i,i];
```

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

```
n = 4
nb = n+1
```

(MATLAB)

```
% forward elimination
for k = 1:n-1
  % partial pivoting
  [big,i]=max(abs(Aug(k:n,k)));
  ipr=i+k-1;
  if ipr~=k
    Aug([k,ipr],:)=Aug([ipr,k],:);
  end
  for i = k+1:n
    factor=Aug(i,k)/Aug(k,k);
    Aug(i,k:nb)=Aug(i,k:nb)-factor*Aug(k,k:nb);
  end
end
% back substitution
x=zeros(n,1);
x(n)=Aug(n,nb)/Aug(n,n);
for i = n-1:-1:1
  x(i)=(Aug(i,nb)-Aug(i,i+1:n)*x(i+1:n))/Aug(i,i);
end
```

Lecture 5 part II

Gauss-Jordan Elimination

Gauss-Jordan Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}$$

Figure 3-18

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{bmatrix} \xrightarrow{\text{Gauss–Jordan procedure}} \begin{bmatrix} 1 & 0 & 0 & 0 & b'_1 \\ 0 & 1 & 0 & 0 & b'_2 \\ 0 & 0 & 1 & 0 & b'_3 \\ 0 & 0 & 0 & 1 & b'_4 \end{bmatrix}$$

Figure 3-19

Example

$$4x_1 - 2x_2 - 3x_3 + 6x_4 = 12$$
$$-6x_1 + 7x_2 + 6.5x_3 - 6x_4 = -6.5$$
$$x_1 + 7.5x_2 + 6.25x_3 + 5.5x_4 = 16$$
$$-12x_1 + 22x_2 + 15.5x_3 - x_4 = 17$$

$$\begin{bmatrix} 4 & -2 & -3 & 6 \\ -6 & 7 & 6.5 & -6 \\ 1 & 7.5 & 6.25 & 5.5 \\ -12 & 22 & 15.5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ -6.5 \\ 16 \\ 17 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & -3 & 6 & 12 \\ -6 & 7 & 6.5 & -6 & -6.5 \\ 1 & 7.5 & 6.25 & 5.5 & 16 \\ -12 & 22 & 15.5 & -1 & 17 \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{4} & \frac{-2}{4} & \frac{-3}{4} & \frac{6}{4} & \frac{12}{4} \\ -6 & 7 & 6.5 & -6 & -6.5 \\ 1 & 7.5 & 6.25 & 5.5 & 16 \\ -12 & 22 & 15.5 & -1 & 17 \end{bmatrix} = \begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 & 3 \\ -6 & 7 & 6.5 & -6 & -6.5 \\ 1 & 7.5 & 6.25 & 5.5 & 16 \\ -12 & 22 & 15.5 & -1 & 17 \end{bmatrix}$$

All the first elements in row 2, 3, and 4 are eliminated:

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 & 3 \\ -6 & 7 & 6.5 & -6 & -6.5 \\ 1 & 7.5 & 6.25 & 5.5 & 16 \\ -12 & 22 & 15.5 & -1 & 17 \end{bmatrix} \begin{matrix} \\ \leftarrow -(-6)[1 \quad -0.5 \quad -0.75 \quad 1.5 \quad 3] \\ \leftarrow -(1)[1 \quad -0.5 \quad -0.75 \quad 1.5 \quad 3] \\ \leftarrow -(-12)[1 \quad -0.5 \quad -0.75 \quad 1.5 \quad 3] \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 & 3 \\ 0 & 4 & 2 & 3 & 11.5 \\ 0 & 8 & 7 & 4 & 13 \\ 0 & 16 & 6.5 & 17 & 53 \end{bmatrix}$$

The next pivot row is the second row, with the second elements as the pivot element. The row is normalized by dividing it by the pivot element:

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 & 3 \\ 0 & \frac{4}{4} & \frac{2}{4} & \frac{3}{4} & \frac{11.5}{4} \\ 0 & 8 & 7 & 4 & 13 \\ 0 & 16 & 6.5 & 17 & 53 \end{bmatrix} = \begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 & 3 \\ 0 & 1 & 0.5 & 0.75 & 2.875 \\ 0 & 8 & 7 & 4 & 13 \\ 0 & 16 & 6.5 & 17 & 53 \end{bmatrix}$$

All the second elements in row 1, 3, and 4 are eliminated:

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 & 3 \\ 0 & 1 & 0.5 & 0.75 & 2.875 \\ 0 & 8 & 7 & 4 & 13 \\ 0 & 16 & 6.5 & 17 & 53 \end{bmatrix} \begin{array}{l} \leftarrow \; -(-0.5)[0 \quad 1 \quad 0.5 \quad 0.75 \quad 2.875] \\ \\ \leftarrow \; -(8)[0 \quad 1 \quad 0.5 \quad 0.75 \quad 2.875] \\ \leftarrow \; -(16)[0 \quad 1 \quad 0.5 \quad 0.75 \quad 2.875] \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -0.5 & 1.875 & 4.4375 \\ 0 & 1 & 0.5 & 0.75 & 2.875 \\ 0 & 0 & 3 & -2 & -10 \\ 0 & 0 & -1.5 & 5 & 7 \end{bmatrix}$$

The next pivot row is the third row, with the third element as the pivot element. The row is normalized by dividing it by the pivot element:

$$\begin{bmatrix} 1 & 0 & -0.5 & 1.875 & 4.4375 \\ 0 & 1 & 0.5 & 0.75 & 2.875 \\ 0 & 0 & \frac{3}{3} & \frac{-2}{3} & \frac{-10}{3} \\ 0 & 0 & -1.5 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.5 & 1.875 & 4.4375 \\ 0 & 1 & 0.5 & 0.75 & 2.875 \\ 0 & 0 & 1 & -0.667 & -3.333 \\ 0 & 0 & -1.5 & 5 & 7 \end{bmatrix}$$

All the third elements in row 1, 2, and 4 are eliminated:

$$\begin{bmatrix} 1 & 0 & -0.5 & 1.875 & 4.4375 \\ 0 & 1 & 0.5 & 0.75 & 2.875 \\ 0 & 0 & 1 & -0.667 & -3.333 \\ 0 & 0 & -1.5 & 5 & 7 \end{bmatrix} \begin{matrix} \leftarrow & -(-0.5)[0 \quad 0 \quad 1 \quad -0.667 \quad -3.333] \\ \leftarrow & -(-0.5)[0 \quad 0 \quad 1 \quad -0.667 \quad -3.333] \\ & \\ \leftarrow & -(1.5)[0 \quad 0 \quad 1 \quad -0.667 \quad -3.333] \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1.5417 & 2.7708 \\ 0 & 1 & 0.5 & 1.0833 & 4.5417 \\ 0 & 0 & 0 & -0.667 & -3.333 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1.5417 & 2.7708 \\ 0 & 1 & 0 & 1.0833 & 4.5417 \\ 0 & 0 & 1 & -0.667 & -3.333 \\ 0 & 0 & 0 & \frac{4}{4} & \frac{2}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1.5417 & 2.7708 \\ 0 & 1 & 0 & 1.0833 & 4.5417 \\ 0 & 0 & 1 & -0.667 & -3.333 \\ 0 & 0 & 0 & 1 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1.5417 & 2.7708 \\ 0 & 1 & 0 & 1.0833 & 4.5417 \\ 0 & 0 & 1 & -0.667 & -3.333 \\ 0 & 0 & 0 & 1 & 0.5 \end{bmatrix} \begin{matrix} \leftarrow -(1.5417)\begin{bmatrix} 0 & 0 & 0 & 1 & 0.5 \end{bmatrix} \\ \leftarrow -(1.0833)\begin{bmatrix} 0 & 0 & 0 & 1 & 0.5 \end{bmatrix} \\ \leftarrow -(-0.667)\begin{bmatrix} 0 & 0 & 0 & 1 & 0.5 \end{bmatrix} \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0.5 \end{bmatrix}$$

The solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 0.5 \end{bmatrix}$$

- Application of Gauss-Jordan method?

Lecture 5 part III

Iteration Approaches

- Jacobi
- Gauss-Seidel

Iteration Method

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3$$
$$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4$$

Write the equations in an explicit form →

$$x_1 = [b_1 - (a_{12}x_2 + a_{13}x_3 + a_{14}x_4)]/a_{11}$$
$$x_2 = [b_2 - (a_{21}x_1 + a_{23}x_3 + a_{24}x_4)]/a_{22}$$
$$x_3 = [b_3 - (a_{31}x_1 + a_{32}x_2 + a_{34}x_4)]/a_{33}$$
$$x_4 = [b_4 - (a_{41}x_1 + a_{42}x_2 + a_{43}x_3)]/a_{44}$$

$$x_i = \frac{1}{a_{ii}}\left[b_i - \left(\sum_{\substack{j=1 \\ j \neq i}}^{j=n} a_{ij}x_j \right) \right], i = 1, 2, \ldots, n$$

time iterative

Jacobi Iteration Method

$$x_i^{(2)} = \frac{1}{a_{ii}}\left[b_i - \left(\sum_{\substack{j=1\\ j\neq i}}^{j=n} a_{ij}x_j^{(1)}\right)\right], i = 1, 2, \ldots, n$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \left(\sum_{\substack{j=1\\ j\neq i}}^{j=n} a_{ij}x_j^{(k)}\right)\right], i = 1, 2, \ldots, n$$

$$\left|\frac{x_i^{(k+1)} - x_i^{(k)}}{x_i^{(k)}}\right| < \varepsilon, i = 1, 2, \ldots, n$$

$$x_1 = [b_1 - (a_{12}x_2 + a_{13}x_3 + a_{14}x_4)]/a_{11}$$
$$x_2 = [b_2 - (a_{21}x_1 + a_{23}x_3 + a_{24}x_4)]/a_{22}$$
$$x_3 = [b_3 - (a_{31}x_1 + a_{32}x_2 + a_{34}x_4)]/a_{33}$$
$$x_4 = [b_4 - (a_{41}x_1 + a_{42}x_2 + a_{43}x_3)]/a_{44}$$

Initial

Step 1

Step 2

…

Step n-1

Step n

Gauss-Seidel Iterative Method

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - \left(\sum_{j=2}^{j=n} a_{1j} x_j^{(k)}\right)\right]$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \left(\sum_{j=1}^{j=i-1} a_{ij} x_j^{(k+1)} + \sum_{j=i+1}^{j=n} a_{ij} x_j^{(k)}\right)\right], i = 2, 3, \dots, n-1$$

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left[b_n - \left(\sum_{j=1}^{j=n-1} a_{nj} x_j^{(k+1)}\right)\right]$$

$$x_1 = [b_1 - (a_{12}x_2 + a_{13}x_3 + a_{14}x_4)]/a_{11}$$

$$x_2 = [b_2 - (a_{21}x_1 + a_{23}x_3 + a_{24}x_4)]/a_{22}$$

$$x_3 = [b_3 - (a_{31}x_1 + a_{32}x_2 + a_{34}x_4)]/a_{33}$$

$$x_4 = [b_4 - (a_{41}x_1 + a_{42}x_2 + a_{43}x_3)]/a_{44}$$