Manifolds and Bundles

An Introduction to Topology

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1 Ready, Set, Topology

Definition: A topology τ on a set X is a subset of the power set of X such that the following conditions hold:

1. $\emptyset, X \in \tau$.
2. Arbitrary unions of sets in τ are in X. i.e. if $U_\alpha \in \tau$, for all α in some indexing set A then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
3. Finite intersections of sets in τ are in X. i.e. if $V_1, ..., V_N \in \tau$ then $\bigcap_{i=1}^{N} V_i \in \tau$.

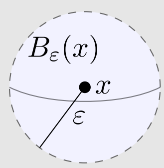
Remark: The elements of τ are referred to as the open sets of X (w.r.t τ). Furthermore the pair (X, τ) is referred to as a topological space whenever τ is topology on X.

Abuse of notation: If the topology τ for a given set X is assumed to be known most people will refer to X as a topological space.

Examples:

1. Any set X admits a so called trivial topology $\tau = \{\emptyset, X\}$. This clearly satisfies the conditions in the definition.
2. $X = \{a, b, c\}$, $\tau = \big\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\big\}$. It is easy to check that τ is a topology on the set X.
3. $X = \mathbb{R}^n$ and τ is the smallest topology containing all sets $B_\varepsilon(x)$, $\forall \epsilon > 0$ and $x \in \mathbb{R}^n$ where

$$B_\varepsilon(x) = \left\{ y \in \mathbb{R}^n : \underbrace{\sqrt{\sum_{i=1}^{n} (y_i - x_i)^2} < \varepsilon}_{\text{dist}(x,y) < \varepsilon} \right\}$$



This is called the open ball of radius ε centred at x. And this topology τ is referred to as the standard topology on $\mathbb{R}^n$.

Definition: Let (X, τ_x) and (Y, τ_y) be topological spaces. Then a function $f : X \to Y$ is said to be continuous if the image of every open set in X is an open set in Y. i.e. whenever $U \in \tau_x \implies f(U) \in \tau_y$.

Definition: A function $f : X \to Y$ is called a homeomorphism whenever f is continuous and has a continuous inverse. These are the structure preserving maps in topology, in other words we will view two topological spaces to be "the same" (homeomorphic) if there exists such a map.

2 What the heck is a manifold?

Before we actually get to defining what a manifold is we need to get through a few more definitions:

Definition: A topological space (X, τ) is called Hausdorff or T_2 if $\forall x, y \in X$, $\exists U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

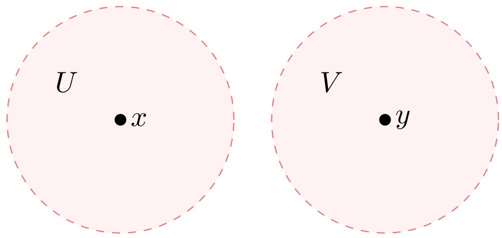


Figure 1: A visual describing a Hausdorff (separable) space

Definition: A topological space (X, τ) is called second-countable if there exists a countable collection of sets $\{U_i \in \tau : i \in \mathbb{N}\}$ such that any open set $U \in \tau$ can be written as the union of U_i's.

Definition: A topological space (X, τ) is called locally euclidean if there exists a fixed positive integer n such that $\forall x \in X$, $\exists U \in \tau$ such that $x \in U$ and U is homeomorphic to some (connected) subset of $\mathbb{R}^n$.

Remark: n is fixed for all the open sets, i.e. if we have that $U, V \in \tau$ and U is homeomorphic to $\mathbb{R}^n$ and V is homeomorphic to $\mathbb{R}^m$ and $n \neq m$ then (X, τ) is not locally euclidean.

Definition A *topological manifold* $(\mathcal{M}, \tau)$ or simply $\mathcal{M}$ is a locally euclidean topological space that is Hausdorff and second-countable.

From the second countability condition we know that there exists a countable set $\{U_\alpha\}$ such that they cover the manifold i.e. $\mathcal{M} = \bigcup_\alpha U_\alpha$. Each of these open neighbourhoods are homeomorphic to $\mathbb{R}^n$ by a homeomorphism φ_α. The tuple $(U_\alpha, \varphi_\alpha)$ is called a chart and the collection of all charts is called an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$.

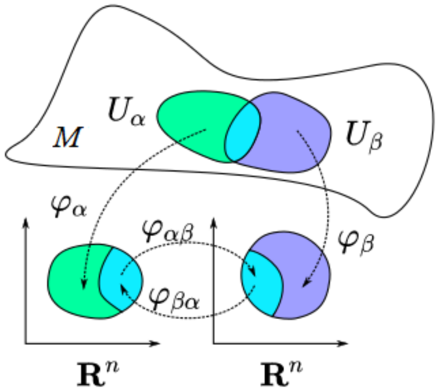


Figure 2: A graphic showing a manifold with local neighbourhoods U_α and U_β and their associated chart maps and transition functions. (source: Wikipedia)

If two neighbourhoods U_α and U_β overlap, then we can define the transition function (see Figure 2):

$$\varphi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha \left(U_\alpha \cap U_\beta\right) \to \varphi_\beta \left(U_\alpha \cap U_\beta\right)$$

If all transition functions are $C^k(\mathcal{M})$ then we say that $\mathcal{M}$ has a $C^k(\mathcal{M})$ compatible atlas. Furthermore if a manifold is equipped with a $C^\infty(\mathcal{M})$ atlas the manifold is said to be *smooth*.

Example: Since we can embed the sphere S^2 in $\mathbb{R}^3$ and $\mathbb{R}^3$ has a standard topology this induces a topology on the sphere. Our open neighbourhoods are going to be six hemispheres $U_i^\pm$ covering the sphere from top, bottom, left, right, front and back .

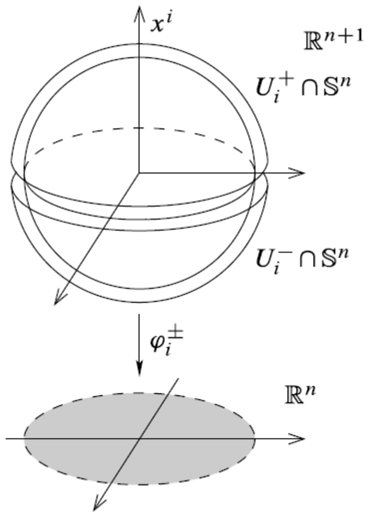


Figure 3: A diagram depicting a generalization of the above described example embedding the n dimensional sphere S^n in $\mathbb{R}^{n+1}$ and using $2(n+1)$ charts. (source: Lee's Introduction To Smooth Manifolds)

3 You're Bundled up now...

We can construct bigger manifolds by taking Cartesian products of other manifolds.

Example: if we take the Cartesian product of the line $\mathbb{R}$ and the circle S^1, we obtain the cylinder which is also a topological manifold.

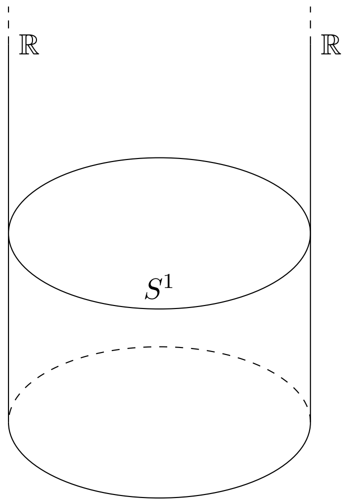


Figure 4: The cylinder $\mathbb{R} \times S^1$ can be thought of as attaching a line to each part of the circle or vice versa.

But what if we wanted to construct something like the Mobius band which locally looks the product of an interval with S^1 but is globally does not look like a product of two manifolds. For this we need the notion of bundles.

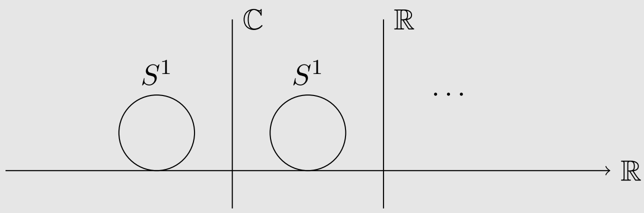
Figure 5: The Mobius band (source: Wikipedia)

Definition: A bundle is a triple (E, π, M) usually written as $E \xrightarrow{\pi} M$, where E, M are sets (sometimes manifolds) and $\pi : E \to M$ is a surjective map. E is called the total space, M is called the base space and π is the projection map.

Let $p \in M$ then the *fibre* at p is all the elements in E that project down to p by π in other words $F_p = \text{preim}_\pi(p) = \{x \in E : \pi(x) = p\}$.

Examples:

1. The following is a bundle where the fibres are different sets, if we choose all the fibres to be the same we call the bundle a *fibre bundle*.



2. The cylinder is a fibre bundle where the base space is S^1 and the fibres are $\mathbb{R}$ as we mentioned before.

3. We can construct the Mobius band by considering a bundle where the base space is S^1 and the fibres are intervals (say $[0, 1]$) but we twist the fibres along S^1 such that when we reach back to the original point the first line is rotated $180°$ with respect to the last. (See Figure 5)