## Sets

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### Introduction

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**Sets** are a mathematical representations of a collection of objects. We usually denote a set with curly brackets and write the elements of the set separated by commas for example

$$S = \{1, 2, 3\} \tag{S.1}$$

This is the set $S$ containing the three numbers 1,2,3. To denote that $x$ is an a element of a set $A$ we write $x \in A$, for example in the set given above $2 \in S$. Note that sets don't count repeated elements, if one wants repeated elements then one can use tuples to give the elements an ID for example

$$S' = \{(1, \text{apple}), (2, \text{orange}), (3, \text{apple})\} \tag{S.2}$$

The above set contains represents a set containing two apples and one orange, we've given an ID to each of the elements and thus we can have in some sense repeated elements but of course they are distinct since they are given unique IDs. Two sets are equal if they contain the same elements, the order of the elements written in a representation of a set has no significance so the set $\{3, 1, 2\}$ is equal to the set $S$ in equation $(S.1)$.

If $A, B$ are sets and all the elements of $A$ are also elements of $B$, we say that $B$ is a *subset* of $A$, denoted as $B \subseteq A$ (or one can say $A$ is a *superset* of $B$). One the most important sets is the set with no elements or *the empty set* usually denoted $\emptyset$ or $\{\}$, notice that this set is a subset of any set since all of its elements (there are non) are contained in any other set. Any set is of course a subset of itself $A \subseteq A$ since $A$ contains all the elements contained in $A$. We say a subset $B$ is a proper subset of $A$ if it is a subset of a $A$ and not equal to $A$ and we denote it as $B \subset A$.

### Set builder

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To define a set one can list all the elements of that set, this can be quite cumbersome especially if the set in question is large or infinite. To avoid this, one can use *set builder notation*. In set builder notation one can use a super set along with some conditions that specify a subset of elements that satisfy the condition. The following examples should clarify:

1. Let $S = \{\text{Alice}, \text{Bob}, \text{Alan}, \text{Bryan}\}$, if we want the subset of names contained in this set that start with the letter A we can denote it as follows:

$$S' = \{x \in S \mid x \text{ Starts with the letter A}\} = \{\text{Alice}, \text{Alan}\}$$

2. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers, we can write the set of even natural numbers $E$ as

$$E = \{x \in \mathbb{N} \mid x \text{ is even}\} = \{2, 4, 6, \ldots\}$$

### Basic Operations on Sets

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Given some sets one might want to do some operations on those sets, here are a few basic ones:

1. **Union of sets**: if $X, Y$ are sets one can consider the union $X \cup Y$, this is the set that contain all the elements that are in $X$ or in $Y$ without double counting. For example:

$$\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$$

2. **Intersection of sets**: if $X, Y$ are sets then the intersection $X \cap Y$ is the set of elements which are both in $X$ and in $Y$. For example:

$$\{1,2,3\} \cap \{3,4,5\} = \{3\}$$

3. **Power set**: the power set of a set $X$ is the set of all subsets of the set $X$ and is denoted $\mathcal{P}(X) = \{A \mid A \subseteq X\}$. For example if $X = \{1,2\}$

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$$

   Note that 1 is not the same as $\{1\}$ which is the set containing 1.

4. **Cartesian product**: if $X, Y$ are sets, the cartesian product $X \times Y$ is the set of all tuples of elements in $X$ and $Y$. In set builder notation we can write this as $X \times Y = \{(x,y) \mid x \in X,\ y \in Y\}$. For example take $X = \{1,2\}$ and $Y = \{a,b\}$ then

$$X \times Y = \{(1,a), (1,b), (2,a), (2,b)\}$$

5. **Complement of a subset**: if $B \subseteq A$ we can define the complement of $B$ denoted $\overline{B}$ or $B^c$ is the set of all elements in $A$ that are not in $B$. In set builder we can write $\overline{B} = \{x \in A \mid x \notin B\}$. For example take $A = \{1,2,3,4,5\}$ and $B = \{1,2\}$

$$\overline{B} = \{3,4,5\}$$

   Notice the $B \cup \overline{B} = A$, this works in the general setting as well.

## The Axiom of infinity

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The axiom of infinity is an axiom of ZFC set theory in which one can build an infinite set recursively. For example one can construct the so called Von-Neumann ordinals which defines the natural numbers in terms of sets recursively in the following way.

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\begin{aligned}
&0 = \emptyset \\
&1 = \{\emptyset\} = \{0\} \\
&2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\
&3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} \\
&\vdots
\end{aligned}
$$

One can use the recursive rule $A_{n+1} = A_n \cup \{A_n\}$ with $A_0 = \emptyset$ to obtain the above sets. The set of natural numbers are contained in the limiting behaviour of this sequence. One can define the inductive limit of the given rule to be the set $S$ satisfying

$$\forall X \in S, \quad X \cup \{X\} \in S$$

and stating that $\emptyset \in S$ as a base case.

## Equivalence Relation

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An equivalence relation $\sim$ on a set $X$ is binary relation satisfying the following conditions:

1. Reflexivity: for any element $x \in X$, $x \sim x$.
2. Symmetry: for any two elements $x, y \in X$ if $x \sim y$ then we also have $y \sim x$.
3. Transitivity: for any elements $x, y, z \in X$ if $x \sim y$ and $y \sim z$ then we also have $x \sim z$.