Measure Theory

1. Sigma algebra

Given a set Ω a σ-algebra on Ω is a set $\Sigma \subseteq \mathcal{P}(\Omega)$ satisfying the following rules

1. The empty set and the whole set is in Σ: $\emptyset, \Omega \in \Sigma$.
2. The countable union of subsets $A_n \in \Sigma$: if $A_n \in \Sigma$ then $\bigcup\limits_{n=1}^{\infty} A_n \in \Sigma$.
3. Closed under compliments: if $A \in \Sigma$ then $\overline{A} \in \Sigma$.

1.1 Borel σ-algebra

Let (X, τ) be a topological space, a Borel σ-algebra on the space X is the smallest σ-algebra Σ containing the topology τ on the set X.

1.2 Examples of σ-algebras

1. The trivial σ-algebra on any set Ω is the one containing only the empty set and the whole space; $\Sigma = \{\emptyset, \Omega\}$.
2. The Borel σ-algebra on $\Omega = \mathbb{R}^n$ with the standard topology given by the basis of open balls, is a sigma algebra that will be of interest.

2. Measurable Space

A measurable space (Ω, Σ) is a set Ω along with a σ-Algebra Σ on Ω. One would be interested in studying the functions that preserve the structure of a measurable space, these are called measurable functions.

2.1 Measurable functions

Let (Ω, Σ) and (Ξ, Λ) be two measurable spaces. A function $f : \Omega \longrightarrow \Xi$ is called a measurable function if for every set S in Λ, the inverse image of S under f is in Σ:

$$\forall S \in \Lambda \quad f^{-1}(S) \in \Sigma$$

Where $f^{-1}(S) = \{x \in \Omega \mid f(x) \in S\}$.

2.2 Measure

Let (Ω, Σ) be a measurable space, a measure μ is a function $\mu : \Sigma \longrightarrow \mathbb{R}$ such that

1. The measure of the empty set is 0: $\mu(\emptyset) = 0$.
2. Positivity: $\forall S \in \Sigma,\ \mu(S) \geq 0$.
3. Countable additivity: if $\{A_n\}_n$ are a countable collection of disjoint sets $A_n \in \Sigma$ then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu\left(A_n\right)$.

   A ***measure space*** (not to be confused with measurable space) is the tuple (Ω, Σ, μ) of a set along with a σ-algebra and a measure on the σ-algebra. One very special example of a measure is called a *Lebesgue measure*.

2.2.1 The Lebesgue measure

A Lebesgue λ measure on $(\mathbb{R}^n, \Sigma)$ where Σ is the Borel σ-algebra with respect to the standard topology on $\mathbb{R}^n$, is defined as the measure satisfying the following.

1. Translation invariance: If $X \in \Sigma$ is a measurable set, and $\mathbf{a} \in \mathbb{R}^n$, the measure of the translated set $X + \mathbf{a}$ is the same as the measure of X. $\lambda(X + \mathbf{a}) = \lambda(X)$
2. Complete: The measure of any subset of $S \subseteq N \in \Sigma$ where $\lambda(N) = 0$ is also 0. $\lambda(S) = 0$.

The Lebesgue measure λ is the unique measure satisfying the above two properties. An important formula for calculating the measure of rectangular sets comes from the above properties. The measure of the n-dimensional hypercube is given by the usual formula for the hypervolume

$$\mu\left([a_1, b_1] \times [a_2, b_2] \cdots \times [a_n, b_n]\right) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$$

2.2.2 The Haar measure