THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

A Look into Projective Quadrangle Geometry

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Contents

List of Figures

List of Tables

Chapter 1

Bilinear Form

Fundamental Theorem of Projective Geometry:
We are able to use a transformation to send any four arbitrary non-collinear points of a quadrangle to the four points;

$$a_1 \equiv [1:1:1], \quad a_2 \equiv [-1:-1:1], \quad a_3 \equiv [1:-1:1], \quad a_4 \equiv [-1:1:1].$$

This transformation results in a change of Bilinear form to

$$\mathbf{A} \equiv \begin{bmatrix} af & d & f \\ d & b & g \\ f & g & c \end{bmatrix}, \quad \text{for some } \ a,b,c,d,e,f \in \text{Nat},$$

with inverse

$$\mathbf{B} \equiv \begin{bmatrix} g^2 - bc & dc - fg & fb - dg \\ dc - fg & f^2 - ac & ga - df \\ fb - dg & ga - df & d^2 - ab \end{bmatrix}.$$

We will refer to the quadrangle $\overline{a_1a_2a_3a_4}$ as the standard quadrangle. The standard quadrangle has six sides $\overline{a_ia_j}$ for $i \neq j \in [6]$, which determine the six lines

$$\begin{array}{lll} L_{\{1,2\}} \equiv a_1a_2 \equiv \langle 1:-1:0 \rangle, & L_{\{1,3\}} \equiv a_1a_3 \equiv \langle 1:0:-1 \rangle, & L_{\{1,4\}} \equiv a_1a_4 \equiv \langle 0:1:-1 \rangle \\ L_{\{3,4\}} \equiv a_3a_4 \equiv \langle 1:1:0 \rangle, & L_{\{2,4\}} \equiv a_2a_4 \equiv \langle 1:0:1 \rangle, & L_{\{2,3\}} \equiv a_2a_3 \equiv \langle 0:1:1 \rangle. \end{array}$$

These six lines of the respective sides let us find the diagonal triangle $\overline{d_1d_2d_3}$ of the standard quadrangle,

$$\begin{aligned} d_\alpha &\equiv L_{\{1,2\}}L_{\{3,4\}} \equiv [0:0:1], \\ d_\beta &\equiv L_{\{1,3\}}L_{\{2,4\}} \equiv [0:1:0], \\ d_\gamma &\equiv L_{\{1,4\}}L_{\{2,3\}} \equiv [1:0:0]. \end{aligned}$$

The labeling will become more obvious below. Define

$$D \equiv abc + 2fdg - ag^2 - bf^2 - cd^2.$$

Then indeed we have that

$$det(A) = det \begin{pmatrix} a & d & f \\ d & b & g \\ f & g & c \end{pmatrix} = D, \text{ and}$$
$$det(B) = det \begin{pmatrix} g^2 - bc & dc - fg & fb - dg \\ dc - fg & f^2 - ac & ga - df \\ fb - dg & ga - df & d^2 - ab \end{pmatrix} = -D^2.$$

Also define the variable

$$\begin{aligned} A_1 &= a_1 \cdot a_1 \equiv a + b + c + 2(d + f + g), \quad A_2 = a_2 \cdot a_2 \equiv a + b + c + 2(d - f - g), \\ A_3 &= a_3 \cdot a_3 \equiv a + b + c + 2(-d + f - g), \quad A_4 = a_4 \cdot a_4 \equiv a + b + c + 2(-d - f + g), \end{aligned}$$

in an effort to simplify the expressions in the following theorem.

Theorem 1 (Quadrangle quadrances and spreads) *Using these coordinates described above, the quadrances of the quadrangle are*

$$\begin{aligned} q(a_1, a_2) &= 4\frac{c(a + b + 2d) - (f + g)^2}{A_1 A_2}, \quad & q(a_3, a_4) &= 4\frac{c(a + b - 2d) - (f - g)^2}{A_3 A_4}, \\ q(a_1, a_3) &= 4\frac{b(a + c + 2f) - (d + g)^2}{A_1 A_3}, \quad & q(a_2, a_4) &= 4\frac{b(a + c - 2f) - (d - g)^2}{A_2 A_4}, \\ q(a_1, a_4) &= 4\frac{a(b + c + 2g) - (d + f)^2}{A_1 A_4}, \quad & q(a_2, a_3) &= 4\frac{a(b + c - 2g) - (d - f)^2}{A_2 A_3} \end{aligned}$$

These numbers also satisfy

$$\begin{aligned} 1 - q(a_1, a_2) &= \frac{(c - b - a - 2d)^2}{A_1 A_2}, \quad & 1 - q(a_3, a_4) &= \frac{(c - b - a + 2d)^2}{A_3 A_4}, \\ 1 - q(a_1, a_3) &= \frac{(a - b + c + 2f)^2}{A_1 A_3}, \quad & 1 - q(a_2, a_4) &= \frac{(a - b + c - 2f)^2}{A_2 A_4}, \\ 1 - q(a_1, a_4) &= \frac{(b - a + c + 2g)^2}{A_1 A_4}, \quad & 1 - q(a_2, a_3) &= \frac{(b - a + c - 2g)^2}{A_2 A_3}. \end{aligned}$$

Proof. Computations give you these results. □ ■

Theorem 2 ***Theorem 3 (Side Midpoints)*** *Suppose that p_1 and p_2 are non-null, non-perpendicular points, forming a non-null side $\overline{p_1 p_2}$. Then $\overline{p_1 p_2}$ has a non-null midpoint m precisely when $1 - q(p_1, p_2)$ is a square, and in this case there are exactly two perpendicular midpoints m.*

Proof. *Proof* We suppose that without loss of generality that $p_1 = a_1 \equiv [1 : 1 : 1]$ and $p_2 = a_2 \equiv [-1 : -1 : 1]$ so that by the spreads and quadrances theorem

$$1 - q(p_1, p_2) = \frac{(c - b - a - 2d)^2}{A_1 A_2}$$

By assumption each of the variables A_1 and A_2 are nonzero. An arbitrary point m on the line $L_{\{1,2\}} \equiv \langle 1 : -1 : 0 \rangle$ has the form $m = [x - y : x - y : x + y]$, which is null precisely when

$$(a+b+2d)(x-y)^2 + c(x+y)^2 + 2\,(f+g)\,(x^2-y^2) = 0$$

by the Null point theorem. Assuming that m is non-null, we compute that

$$\begin{aligned} q(p_1,m) &= \frac{4y^2\left(c\,(a+b+d)-(f+g)^2\right)}{A_1\left((a+b+2d)(x-y)^2+c(x+y)^2+2\,(f+g)\,(x^2-y^2)\right)}, \\ q(p_2,m) &= \frac{4x^2\left(c\,(a+b+d)-(f+g)^2\right)}{A_2\left((a+b+2d)(x-y)^2+c(x+y)^2+2\,(f+g)\,(x^2-y^2)\right)} \end{aligned}$$

By assumption $\overline{p_1p_2}$ is non-null, so by the Corollary to the Null points/lines theorem, $c\,(a+b+d)-(f+g)^2 \neq 0$, and so the above expressions are equal precisely when

$$y^2A_2 = x^2A_1$$

has a solution, which occurs precisely when $1 - q(p_1,p_2)$ is a square. In fact if

$$\frac{1}{A_1A_2} = \sigma^2_{\{1,2\}}$$

then the two midpoints are

$$m \equiv \left[1 \pm \sigma_{\{1,2\}}A_1 : 1 \pm \sigma_{\{1,2\}}A_1 : 1 \mp \sigma_{\{1,2\}}A_1\right]$$

and they are perpendicular, since

$$m_+\mathbf{A}m_-^T = 0$$

by computations. □ ■

We denote the pair of midpoints as **opposites**. Where it follows that the dual line M of a midpoint m is incident with the opposite midpoint.

Theorem 4 (Vertex bilines) *Dual*

We want to impose the conditions that all six sides $\overline{a_ia_j}$ for $i \neq j \in [4]$ have midpoints. From the Side midpoint theorem we know this is true precisely when $1 - q(a_i,a_j)$ is a square. This condition is equivalent to having numbers $\sigma_{\{1,2\}}, \sigma_{\{3,4\}}, \sigma_{\{1,3\}}, \sigma_{\{2,4\}}, \sigma_{\{1,4\}}, \sigma_{\{2,3\}}$ in Rat, satisfying the **quadratic relations**

$$\frac{1}{A_1A_2} = \sigma^2_{\{1,2\}}, \quad \frac{1}{A_1A_3} = \sigma^2_{\{1,3\}}, \quad \frac{1}{A_1A_4} = \sigma^2_{\{1,4\}}, \tag{1.1}$$

$$\frac{1}{A_3A_4} = \sigma^2_{\{3,4\}}, \quad \frac{1}{A_2A_4} = \sigma^2_{\{2,4\}}, \quad \frac{1}{A_2A_3} = \sigma^2_{\{2,3\}}. \tag{1.2}$$

Clearly all of $\sigma_{\{1,2\}}, \sigma_{\{3,4\}}, \sigma_{\{1,3\}}, \sigma_{\{2,4\}}, \sigma_{\{1,4\}}, \sigma_{\{2,3\}}$ are nonzero. We can further take the product of these quadratic relations in threes, say $\frac{1}{A_1A_2} = \sigma^2_{\{1,2\}}, \frac{1}{A_1A_3} = \sigma^2_{\{1,3\}}$, and $\frac{1}{A_2A_3} = \sigma^2_{\{2,3\}}$, with possibly changing the sign of any or all of *sigma* values to produce the following **cubic relations**

$$\begin{aligned} \frac{1}{A_1A_2A_3} &= \sigma_{\{1,2\}}\sigma_{\{2,3\}}\sigma_{\{1,3\}}, & \frac{1}{A_1A_2A_4} &= \sigma_{\{1,2\}}\sigma_{\{2,4\}}\sigma_{\{1,4\}}, \\ \frac{1}{A_1A_3A_4} &= \sigma_{\{1,3\}}\sigma_{\{3,4\}}\sigma_{\{1,4\}}, & \frac{1}{A_2A_3A_4} &= \sigma_{\{2,3\}}\sigma_{\{3,4\}}\sigma_{\{2,4\}}. \end{aligned} \tag{1.3}$$

From these relations we get

$$\begin{aligned} A_1 &= \frac{\sigma_{\{2,3\}}}{\sigma_{\{1,2\}}\sigma_{\{1,3\}}} = \frac{\sigma_{\{2,4\}}}{\sigma_{\{1,2\}}\sigma_{\{1,4\}}} = \frac{\sigma_{\{3,4\}}}{\sigma_{\{1,3\}}\sigma_{\{1,4\}}}, \\ A_2 &= \frac{\sigma_{\{1,3\}}}{\sigma_{\{1,2\}}\sigma_{\{2,3\}}} = \frac{\sigma_{\{1,4\}}}{\sigma_{\{1,2\}}\sigma_{\{2,4\}}} = \frac{\sigma_{\{3,4\}}}{\sigma_{\{2,3\}}\sigma_{\{2,4\}}}, \\ A_3 &= \frac{\sigma_{\{1,2\}}}{\sigma_{\{1,3\}}\sigma_{\{2,3\}}} = \frac{\sigma_{\{1,4\}}}{\sigma_{\{1,3\}}\sigma_{\{3,4\}}} = \frac{\sigma_{\{2,4\}}}{\sigma_{\{2,3\}}\sigma_{\{3,4\}}}, \\ A_4 &= \frac{\sigma_{\{1,2\}}}{\sigma_{\{1,4\}}\sigma_{\{2,4\}}} = \frac{\sigma_{\{1,3\}}}{\sigma_{\{1,4\}}\sigma_{\{3,4\}}} = \frac{\sigma_{\{2,3\}}}{\sigma_{\{2,4\}}\sigma_{\{3,4\}}}. \end{aligned}$$

Furthermore these in turn imply the relations

$$\frac{\sigma_{\{2,3\}}}{\sigma_{\{1,3\}}} = \frac{\sigma_{\{2,4\}}}{\sigma_{\{1,4\}}}, \qquad \frac{\sigma_{\{2,3\}}}{\sigma_{\{1,2\}}} = \frac{\sigma_{\{3,4\}}}{\sigma_{\{1,4\}}}, \qquad \frac{\sigma_{\{2,4\}}}{\sigma_{\{1,2\}}} = \frac{\sigma_{\{3,4\}}}{\sigma_{\{1,3\}}}, \tag{1.4}$$

or simply

$$\sigma_{\{1,2\}}\sigma_{\{3,4\}} = \sigma_{\{1,3\}}\sigma_{\{2,4\}} = \sigma_{\{1,4\}}\sigma_{\{2,3\}}.$$

All these realtions have a strong correlation with the symmetries of four objects, namely that described realtion is the division of four objects into three pairs of two, that is the pairing

$$\{\{1,2\},\{3,4\}\}, \ \{\{1,3\},\{2,4\}\}, \ \{\{1,4\},\{2,3\}\}$$

with respect to the order given. We will from now on refer to the three pairings aboves as **pairing** α, β and γ respectfully.

1.1 Midpoints

By the Side midpoint theorem for a side $\overline{a_ia_j}$ where $a_i = [v_i]$ and $a_j = [v_j]$ we are able to normalise v_i and v_j such that $v_i^2 = v_j^2$ giving the midpoints $m_{ij} \equiv [v_i + v_j]$ and $m_{ji} \equiv [v_i - v_j]$ of $\overline{a_ia_j}$, where the ordering is arbitrary. In the end of the proof of the Side midpoints Theorem we see that the midpoints for the side $\overline{a_1a_2}$ are $m \equiv \left[1 \pm \sigma_{\{1,2\}}A_1 : 1 \pm \sigma_{\{1,2\}}A_1 : 1 \mp \sigma_{\{1,2\}}A_1\right]$, but from above these can be rewritten as

$$m \equiv \left[\sigma_{\{1,3\}} \pm \sigma_{\{2,3\}} : \sigma_{\{1,3\}} \pm \sigma_{\{2,3\}} : \sigma_{\{1,3\}} \mp \sigma_{\{2,3\}}\right] = \left[\sigma_{\{1,4\}} \pm \sigma_{\{2,4\}} : \sigma_{\{1,4\}} \pm \sigma_{\{2,4\}} : \sigma_{\{1,4\}} \mp \sigma_{\{2,4\}}\right],$$

or some other combination with respect to the α *sigma relations*.

Theorem 5 (Quadrangle Midpoints) *The side midpoints of the quadrangle $\overline{a_1a_2a_3a_4}$ have the form*

$$\begin{aligned}
m_{12} &\equiv \left[1-\sigma_{\{1,2\}}A_1 : 1-\sigma_{\{1,2\}}A_1 : 1+\sigma_{\{1,2\}}A_1\right]\\
&= \left[\sigma_{\{1,3\}}-\sigma_{\{2,3\}} : \sigma_{\{1,3\}}-\sigma_{\{2,3\}} : \sigma_{\{1,3\}}+\sigma_{\{2,3\}}\right]\\
&= \left[\sigma_{\{1,4\}}-\sigma_{\{2,4\}} : \sigma_{\{1,4\}}-\sigma_{\{2,4\}} : \sigma_{\{1,4\}}+\sigma_{\{2,4\}}\right]\\
m_{21} &\equiv \left[1+\sigma_{\{1,2\}}A_1 : 1+\sigma_{\{1,2\}}A_1 : 1-\sigma_{\{1,2\}}A_1\right]\\
&= \left[\sigma_{\{1,3\}}+\sigma_{\{2,3\}} : \sigma_{\{1,3\}}+\sigma_{\{2,3\}} : \sigma_{\{1,3\}}-\sigma_{\{2,3\}}\right]\\
&= \left[\sigma_{\{1,4\}}+\sigma_{\{2,4\}} : \sigma_{\{1,4\}}+\sigma_{\{2,4\}} : \sigma_{\{1,4\}}-\sigma_{\{2,4\}}\right]
\end{aligned}$$

$$\begin{aligned}
m_{34} &\equiv \left[1-\sigma_{\{3,4\}}A_3 : \sigma_{\{3,4\}}A_3-1 : 1+\sigma_{\{3,4\}}A_3\right]\\
&= \left[\sigma_{\{1,3\}}-\sigma_{\{1,4\}} : \sigma_{\{1,4\}}-\sigma_{\{1,3\}} : \sigma_{\{1,3\}}+\sigma_{\{1,4\}}\right]\\
&= \left[\sigma_{\{2,3\}}-\sigma_{\{2,4\}} : \sigma_{\{2,4\}}-\sigma_{\{2,3\}} : \sigma_{\{2,3\}}+\sigma_{\{2,4\}}\right]\\
m_{43} &\equiv \left[1+\sigma_{\{3,4\}}A_3 : -1-\sigma_{\{3,4\}}A_3 : 1-\sigma_{\{3,4\}}A_3\right]\\
&= \left[\sigma_{\{1,3\}}+\sigma_{\{1,4\}} : -\sigma_{\{1,3\}}-\sigma_{\{1,4\}} : \sigma_{\{1,3\}}-\sigma_{\{1,4\}}\right]\\
&= \left[\sigma_{\{2,3\}}+\sigma_{\{2,4\}} : -\sigma_{\{2,3\}}-\sigma_{\{2,4\}} : \sigma_{\{2,3\}}-\sigma_{\{2,4\}}\right]
\end{aligned}$$

$$\begin{aligned}
m_{13} &\equiv \left[1+\sigma_{\{1,3\}}A_1 : 1-\sigma_{\{1,3\}}A_1 : 1+\sigma_{\{1,3\}}A_1\right]\\
&= \left[\sigma_{\{1,2\}}+\sigma_{\{2,3\}} : \sigma_{\{1,2\}}-\sigma_{\{2,3\}} : \sigma_{\{1,2\}}+\sigma_{\{2,3\}}\right]\\
&= \left[\sigma_{\{1,4\}}+\sigma_{\{3,4\}} : \sigma_{\{1,4\}}-\sigma_{\{3,4\}} : \sigma_{\{1,4\}}+\sigma_{\{3,4\}}\right]\\
m_{31} &\equiv \left[1-\sigma_{\{1,3\}}A_1 : 1+\sigma_{\{1,3\}}A_1 : 1-\sigma_{\{1,3\}}A_1\right]\\
&= \left[\sigma_{\{1,2\}}-\sigma_{\{2,3\}} : \sigma_{\{1,2\}}+\sigma_{\{2,3\}} : \sigma_{\{1,2\}}-\sigma_{\{2,3\}}\right]\\
&= \left[\sigma_{\{1,4\}}-\sigma_{\{3,4\}} : \sigma_{\{1,4\}}+\sigma_{\{3,4\}} : \sigma_{\{1,4\}}-\sigma_{\{3,4\}}\right]
\end{aligned}$$

$$\begin{aligned}
m_{24} &\equiv \left[-1-\sigma_{\{2,4\}}A_2 : \sigma_{\{2,4\}}A_2-1 : 1+\sigma_{\{2,4\}}A_2\right]\\
&= \left[-\sigma_{\{1,2\}}-\sigma_{\{1,4\}} : \sigma_{\{1,4\}}-\sigma_{\{1,2\}} : \sigma_{\{1,2\}}+\sigma_{\{1,4\}}\right]\\
&= \left[-\sigma_{\{2,3\}}-\sigma_{\{3,4\}} : \sigma_{\{3,4\}}-\sigma_{\{2,3\}} : \sigma_{\{2,3\}}+\sigma_{\{3,4\}}\right]\\
m_{42} &\equiv \left[1-\sigma_{\{2,4\}}A_2 : 1+\sigma_{\{2,4\}}A_2 : \sigma_{\{2,4\}}A_2-1\right]\\
&= \left[\sigma_{\{1,2\}}-\sigma_{\{1,4\}} : \sigma_{\{1,4\}}+\sigma_{\{1,2\}} : \sigma_{\{1,4\}}-\sigma_{\{1,2\}}\right]\\
&= \left[\sigma_{\{2,3\}}-\sigma_{\{3,4\}} : \sigma_{\{3,4\}}+\sigma_{\{2,3\}} : \sigma_{\{3,4\}}-\sigma_{\{2,3\}}\right]
\end{aligned}$$

$$\begin{aligned}
m_{14} &\equiv \left[1-\sigma_{\{1,4\}}A_1 : 1+\sigma_{\{1,4\}}A_1 : 1+\sigma_{\{1,4\}}A_1\right]\\
&= \left[\sigma_{\{1,4\}}-\sigma_{\{2,4\}} : \sigma_{\{1,4\}}+\sigma_{\{2,4\}} : \sigma_{\{1,4\}}+\sigma_{\{2,4\}}\right]\\
&= \left[\sigma_{\{1,3\}}-\sigma_{\{3,4\}} : \sigma_{\{1,3\}}+\sigma_{\{3,4\}} : \sigma_{\{1,3\}}+\sigma_{\{3,4\}}\right]\\
m_{41} &\equiv \left[1+\sigma_{\{1,4\}}A_1 : 1-\sigma_{\{1,4\}}A_1 : 1-\sigma_{\{1,4\}}A_1\right]\\
&= \left[\sigma_{\{1,4\}}+\sigma_{\{2,4\}} : \sigma_{\{1,4\}}-\sigma_{\{2,4\}} : \sigma_{\{1,4\}}-\sigma_{\{2,4\}}\right]\\
&= \left[\sigma_{\{1,3\}}+\sigma_{\{3,4\}} : \sigma_{\{1,3\}}-\sigma_{\{3,4\}} : \sigma_{\{1,3\}}-\sigma_{\{3,4\}}\right]
\end{aligned}$$

$$\begin{aligned}
m_{23} &\equiv \left[\sigma_{\{2,3\}}A_2 - 1 : -1 - \sigma_{\{2,3\}}A_2 : 1 + \sigma_{\{2,3\}}A_2\right] \\
&= \left[\sigma_{\{1,3\}} - \sigma_{\{2,3\}} : -\sigma_{\{2,3\}} - \sigma_{\{1,3\}} : \sigma_{\{2,3\}} + \sigma_{\{1,3\}}\right] \\
&= \left[\sigma_{\{3,4\}} - \sigma_{\{2,4\}} : -\sigma_{\{2,4\}} - \sigma_{\{3,4\}} : \sigma_{\{2,4\}} + \sigma_{\{3,4\}}\right] \\
m_{32} &\equiv \left[1 + \sigma_{\{2,3\}}A_2 : 1 - \sigma_{\{2,3\}}A_2 : \sigma_{\{2,3\}}A_2 - 1\right] \\
&= \left[\sigma_{\{2,3\}} + \sigma_{\{1,3\}} : \sigma_{\{2,3\}} - \sigma_{\{1,3\}} : \sigma_{\{1,3\}} - \sigma_{\{2,3\}}\right] \\
&= \left[\sigma_{\{2,4\}} + \sigma_{\{3,4\}} : \sigma_{\{2,4\}} - \sigma_{\{3,4\}} : \sigma_{\{3,4\}} - \sigma_{\{2,4\}}\right].
\end{aligned}$$

Proof. *This is shown through computations and then careful use of the identities described above.* ■

These side midpoints have corresponding *side midlines*, which are precisely the duals to each side midpoint. That is they are given by the matrix multiplications $M_{ij} \equiv \mathbf{A}m_{ij}^T$, which highlights the opposite relations aluuded to above.

A **subtriangle** is one of the natural divisons of the quadrangle $\overline{a_1a_2a_3a_4}$ into distinct triangles triangles $\triangle_4 \equiv \overline{a_1a_2a_3}$, $\triangle_3 \equiv \overline{a_1a_2a_4}$, $\triangle_2 \equiv \overline{a_1a_3a_4}$ and $\triangle_1 \equiv \overline{a_2a_3a_4}$.

1.1.1 Circumlines and Circumcenters

Theorem 6 (Circumlines and circumcenters) *Midpoints m_{ij} for $i \neq j \in \{1,2,3\}$ of the subtriangle $\triangle_4$ are collinear three at a time, lying on four distinct* ***Circumlines*** *C_1^4, C_2^4, C_2^4, and C_3^4. Midlines M_{ij} for $i \neq j \in \{1,2,3\}$ of the subtriangle $\triangle_4$ are con5current three at a time, meeting at four distinct* ***Circumcenters*** *c_1^4, c_2^4, c_3^4 and c_4^4.*

Proof. The following triples of midpoints m_{ij} for $i \neq j \in \{1,2,3\}$ are colinear:

$C_1^4 \equiv \left\langle \sigma_{\{1,2\}} - \sigma_{\{1,3\}} : \sigma_{\{2,3\}} - \sigma_{\{1,2\}} : \sigma_{\{1,3\}} + \sigma_{\{2,3\}}\right\rangle$ through,

$$\begin{aligned}
m_{21} &\equiv \left[\sigma_{\{1,3\}} + \sigma_{\{2,3\}} : \sigma_{\{1,3\}} + \sigma_{\{2,3\}} : \sigma_{\{1,3\}} - \sigma_{\{2,3\}}\right], \\
m_{31} &\equiv \left[\sigma_{\{1,2\}} - \sigma_{\{2,3\}} : \sigma_{\{1,2\}} + \sigma_{\{2,3\}} : \sigma_{\{1,2\}} - \sigma_{\{2,3\}}\right], \\
m_{32} &\equiv \left[\sigma_{\{2,3\}} + \sigma_{\{1,3\}} : \sigma_{\{2,3\}} - \sigma_{\{1,3\}} : \sigma_{\{1,3\}} - \sigma_{\{2,3\}}\right],
\end{aligned}$$

$C_2^4 \equiv \left\langle \sigma_{\{1,2\}} + \sigma_{\{1,3\}} : -\sigma_{\{1,2\}} - \sigma_{\{2,3\}} : -\sigma_{\{1,3\}} - \sigma_{\{2,3\}}\right\rangle$ through,

$$\begin{aligned}
m_{21} &\equiv \left[\sigma_{\{1,3\}} + \sigma_{\{2,3\}} : \sigma_{\{1,3\}} + \sigma_{\{2,3\}} : \sigma_{\{1,3\}} - \sigma_{\{2,3\}}\right], \\
m_{13} &\equiv \left[\sigma_{\{1,2\}} + \sigma_{\{2,3\}} : \sigma_{\{1,2\}} - \sigma_{\{2,3\}} : \sigma_{\{1,2\}} + \sigma_{\{2,3\}}\right], \\
m_{23} &\equiv \left[\sigma_{\{1,3\}} - \sigma_{\{2,3\}} : -\sigma_{\{2,3\}} - \sigma_{\{1,3\}} : \sigma_{\{2,3\}} + \sigma_{\{1,3\}}\right],
\end{aligned}$$

$C_3^4 \equiv \left\langle -\sigma_{\{1,2\}} - \sigma_{\{1,3\}} : \sigma_{\{1,2\}} - \sigma_{\{2,3\}} : \sigma_{\{1,3\}} - \sigma_{\{2,3\}}\right\rangle$ through,

$$\begin{aligned}
m_{12} &\equiv \left[\sigma_{\{1,3\}} - \sigma_{\{2,3\}} : \sigma_{\{1,3\}} - \sigma_{\{2,3\}} : \sigma_{\{1,3\}} + \sigma_{\{2,3\}}\right], \\
m_{31} &\equiv \left[\sigma_{\{1,2\}} - \sigma_{\{2,3\}} : \sigma_{\{1,2\}} + \sigma_{\{2,3\}} : \sigma_{\{1,2\}} - \sigma_{\{2,3\}}\right], \\
m_{23} &\equiv \left[\sigma_{\{1,3\}} - \sigma_{\{2,3\}} : -\sigma_{\{2,3\}} - \sigma_{\{1,3\}} : \sigma_{\{2,3\}} + \sigma_{\{1,3\}}\right],
\end{aligned}$$

$C_4^4 \equiv \left\langle \sigma_{\{1,3\}} - \sigma_{\{1,2\}} : \sigma_{\{1,2\}} + \sigma_{\{2,3\}} : \sigma_{\{2,3\}} - \sigma_{\{1,3\}}\right\rangle$ through,

$$\begin{aligned}
m_{12} &\equiv \left[\sigma_{\{1,3\}} - \sigma_{\{2,3\}} : \sigma_{\{1,3\}} - \sigma_{\{2,3\}} : \sigma_{\{1,3\}} + \sigma_{\{2,3\}}\right], \\
m_{13} &\equiv \left[\sigma_{\{1,2\}} + \sigma_{\{2,3\}} : \sigma_{\{1,2\}} - \sigma_{\{2,3\}} : \sigma_{\{1,2\}} + \sigma_{\{2,3\}}\right], \\
m_{32} &\equiv \left[\sigma_{\{2,3\}} + \sigma_{\{1,3\}} : \sigma_{\{2,3\}} - \sigma_{\{1,3\}} : \sigma_{\{1,3\}} - \sigma_{\{2,3\}}\right].
\end{aligned}$$

This is checked by computing

$$\begin{aligned}
0 &= \det\begin{pmatrix} \sigma_{\{1,3\}}+\sigma_{\{2,3\}} & \sigma_{\{1,3\}}+\sigma_{\{2,3\}} & \sigma_{\{1,3\}}-\sigma_{\{2,3\}} \\ \sigma_{\{1,2\}}-\sigma_{\{2,3\}} & \sigma_{\{1,2\}}+\sigma_{\{2,3\}} & \sigma_{\{1,2\}}-\sigma_{\{2,3\}} \\ \sigma_{\{2,3\}}+\sigma_{\{1,3\}} & \sigma_{\{2,3\}}-\sigma_{\{1,3\}} & \sigma_{\{1,3\}}-\sigma_{\{2,3\}} \end{pmatrix} \\
&= \det\begin{pmatrix} \sigma_{\{1,3\}}+\sigma_{\{2,3\}} & \sigma_{\{1,3\}}+\sigma_{\{2,3\}} & \sigma_{\{1,3\}}-\sigma_{\{2,3\}} \\ \sigma_{\{1,2\}}+\sigma_{\{2,3\}} & \sigma_{\{1,2\}}-\sigma_{\{2,3\}} & \sigma_{\{1,2\}}+\sigma_{\{2,3\}} \\ \sigma_{\{1,3\}}-\sigma_{\{2,3\}} & -\sigma_{\{2,3\}}-\sigma_{\{1,3\}} & \sigma_{\{2,3\}}+\sigma_{\{1,3\}} \end{pmatrix} \\
&= \det\begin{pmatrix} \sigma_{\{1,3\}}-\sigma_{\{2,3\}} & \sigma_{\{1,3\}}-\sigma_{\{2,3\}} & \sigma_{\{1,3\}}+\sigma_{\{2,3\}} \\ \sigma_{\{1,2\}}-\sigma_{\{2,3\}} & \sigma_{\{1,2\}}+\sigma_{\{2,3\}} & \sigma_{\{1,2\}}-\sigma_{\{2,3\}} \\ \sigma_{\{1,3\}}-\sigma_{\{2,3\}} & -\sigma_{\{2,3\}}-\sigma_{\{1,3\}} & \sigma_{\{2,3\}}+\sigma_{\{1,3\}} \end{pmatrix} \\
&= \det\begin{pmatrix} \sigma_{\{1,3\}}-\sigma_{\{2,3\}} & \sigma_{\{1,3\}}-\sigma_{\{2,3\}} & \sigma_{\{1,3\}}+\sigma_{\{2,3\}} \\ \sigma_{\{1,2\}}+\sigma_{\{2,3\}} & \sigma_{\{1,2\}}-\sigma_{\{2,3\}} & \sigma_{\{1,2\}}+\sigma_{\{2,3\}} \\ \sigma_{\{2,3\}}+\sigma_{\{1,3\}} & \sigma_{\{2,3\}}-\sigma_{\{1,3\}} & \sigma_{\{1,3\}}-\sigma_{\{2,3\}} \end{pmatrix}.
\end{aligned}$$

The corresponding meets are

$$\begin{aligned}
& \left[\sigma_{\{1,3\}}+\sigma_{\{2,3\}}:\sigma_{\{1,3\}}+\sigma_{\{2,3\}}:\sigma_{\{1,3\}}-\sigma_{\{2,3\}}\right]\times\left[\sigma_{\{1,2\}}-\sigma_{\{2,3\}}:\sigma_{\{1,2\}}+\sigma_{\{2,3\}}:\sigma_{\{1,2\}}-\sigma_{\{2,3\}}\right] \\
= & \left\langle\sigma_{\{1,2\}}-\sigma_{\{1,3\}}:\sigma_{\{2,3\}}-\sigma_{\{1,2\}}:\sigma_{\{1,3\}}+\sigma_{\{2,3\}}\right\rangle\equiv C_1^4,
\end{aligned}$$

and similarly for the other circumlines. The situation with midlines M_{ij} for $i\neq j\in\{1,2,3\}$ is precisely dual. ■

Theorem 7 (Subtriangle Circumlines of the Quadrangle) *The Circumlines for the Subtriangles $\triangle_3,\triangle_2$, and $\triangle_1$, are as follows;*

The Circumlines for the subtriangle $\triangle_3$:

$C_1^3\equiv\left\langle\sigma_{\{1,2\}}-\sigma_{\{2,4\}}:\sigma_{\{1,4\}}-\sigma_{\{1,2\}}:-\sigma_{\{1,4\}}-\sigma_{\{2,4\}}\right\rangle$ *through*

$$\begin{aligned}
m_{21} &\equiv \left[\sigma_{\{1,4\}}+\sigma_{\{2,4\}}:\sigma_{\{1,4\}}+\sigma_{\{2,4\}}:\sigma_{\{1,4\}}-\sigma_{\{2,4\}}\right], \\
m_{42} &\equiv \left[\sigma_{\{1,2\}}-\sigma_{\{1,4\}}:\sigma_{\{1,2\}}+\sigma_{\{1,4\}}:\sigma_{\{1,4\}}-\sigma_{\{1,2\}}\right], \\
m_{41} &\equiv \left[\sigma_{\{1,4\}}+\sigma_{\{2,4\}}:\sigma_{\{1,4\}}-\sigma_{\{2,4\}}:\sigma_{\{1,4\}}-\sigma_{\{2,4\}}\right],
\end{aligned}$$

$C_2^3\equiv\left\langle\sigma_{\{1,2\}}+\sigma_{\{2,4\}}:-\sigma_{\{1,2\}}-\sigma_{\{1,4\}}:\sigma_{\{1,4\}}+\sigma_{\{2,4\}}\right\rangle$ *through*

$$\begin{aligned}
m_{21} &\equiv \left[\sigma_{\{1,4\}}+\sigma_{\{2,4\}}:\sigma_{\{1,4\}}+\sigma_{\{2,4\}}:\sigma_{\{1,4\}}-\sigma_{\{2,4\}}\right], \\
m_{24} &\equiv \left[-\sigma_{\{1,2\}}-\sigma_{\{1,4\}}:\sigma_{\{1,4\}}-\sigma_{\{1,2\}}:\sigma_{\{1,2\}}+\sigma_{\{1,4\}}\right], \\
m_{14} &\equiv \left[\sigma_{\{1,4\}}-\sigma_{\{2,4\}}:\sigma_{\{1,4\}}+\sigma_{\{2,4\}}:\sigma_{\{1,4\}}+\sigma_{\{2,4\}}\right],
\end{aligned}$$

$C_3^3\equiv\left\langle\sigma_{\{1,2\}}+\sigma_{\{2,4\}}:\sigma_{\{1,4\}}-\sigma_{\{1,2\}}:\sigma_{\{2,4\}}-\sigma_{\{1,4\}}\right\rangle$ *through*

$$\begin{aligned}
m_{12} &\equiv \left[\sigma_{\{1,4\}}-\sigma_{\{2,4\}}:\sigma_{\{1,4\}}-\sigma_{\{2,4\}}:\sigma_{\{1,4\}}+\sigma_{\{2,4\}}\right], \\
m_{42} &\equiv \left[\sigma_{\{1,2\}}-\sigma_{\{1,4\}}:\sigma_{\{1,2\}}+\sigma_{\{1,4\}}:\sigma_{\{1,4\}}-\sigma_{\{1,2\}}\right], \\
m_{14} &\equiv \left[\sigma_{\{1,4\}}-\sigma_{\{2,4\}}:\sigma_{\{1,4\}}+\sigma_{\{2,4\}}:\sigma_{\{1,4\}}+\sigma_{\{2,4\}}\right],
\end{aligned}$$

$C_4^3 \equiv \left\langle \sigma_{\{1,2\}} - \sigma_{\{2,4\}} : -\sigma_{\{1,2\}} - \sigma_{\{1,4\}} : \sigma_{\{1,4\}} - \sigma_{\{2,4\}} \right\rangle$ *through*

$$\begin{aligned} m_{12} &\equiv \left[\sigma_{\{1,4\}} - \sigma_{\{2,4\}} : \sigma_{\{1,4\}} - \sigma_{\{2,4\}} : \sigma_{\{1,4\}} + \sigma_{\{2,4\}}\right], \\ m_{24} &\equiv \left[-\sigma_{\{1,2\}} - \sigma_{\{1,4\}} : \sigma_{\{1,4\}} - \sigma_{\{1,2\}} : \sigma_{\{1,2\}} + \sigma_{\{1,4\}}\right], \\ m_{41} &\equiv \left[\sigma_{\{1,4\}} + \sigma_{\{2,4\}} : \sigma_{\{1,4\}} - \sigma_{\{2,4\}} : \sigma_{\{1,4\}} - \sigma_{\{2,4\}}\right]. \end{aligned}$$

The Circumlines for the subtriangle $\triangle_2$:

$$\begin{aligned} C_1^2 &\equiv \left\langle \sigma_{\{3,4\}} - \sigma_{\{1,3\}} : \sigma_{\{3,4\}} - \sigma_{\{1,4\}} : \sigma_{\{1,3\}} + \sigma_{\{1,4\}} \right\rangle, \ \textit{through } m_{43}, m_{31}, m_{41}, \\ C_2^2 &\equiv \left\langle \sigma_{\{1,3\}} + \sigma_{\{3,4\}} : \sigma_{\{1,4\}} + \sigma_{\{3,4\}} : -\sigma_{\{1,3\}} - \sigma_{\{1,4\}} \right\rangle, \ \textit{through } m_{43}, m_{13}, m_{14}, \\ C_3^2 &\equiv \left\langle \sigma_{\{1,3\}} + \sigma_{\{3,4\}} : \sigma_{\{3,4\}} - \sigma_{\{1,4\}} : \sigma_{\{1,4\}} - \sigma_{\{1,3\}} \right\rangle, \ \textit{through } m_{34}, m_{31}, m_{14}, \\ C_4^2 &\equiv \left\langle \sigma_{\{3,4\}} - \sigma_{\{1,3\}} : \sigma_{\{1,4\}} + \sigma_{\{3,4\}} : \sigma_{\{1,3\}} - \sigma_{\{1,4\}} \right\rangle, \ \textit{through } m_{34}, m_{13}, m_{41}, \end{aligned}$$

The Circumlines for the subtriangle $\triangle_2$:

$$\begin{aligned} C_1^1 &\equiv \left\langle \sigma_{\{2,4\}} - \sigma_{\{3,4\}} : \sigma_{\{2,3\}} - \sigma_{\{3,4\}} : \sigma_{\{2,3\}} + \sigma_{\{2,4\}} \right\rangle, \ \textit{through } m_{43}, m_{32}, m_{42} \\ C_2^1 &\equiv \left\langle \sigma_{\{2,4\}} + \sigma_{\{3,4\}} : \sigma_{\{2,3\}} + \sigma_{\{3,4\}} : \sigma_{\{2,3\}} + \sigma_{\{2,4\}} \right\rangle, \ \textit{through } m_{43}, m_{23}, m_{24}, \\ C_3^1 &\equiv \left\langle \sigma_{\{2,4\}} + \sigma_{\{3,4\}} : \sigma_{\{3,4\}} - \sigma_{\{2,3\}} : \sigma_{\{2,4\}} - \sigma_{\{2,3\}} \right\rangle, \ \textit{through } m_{34}, m_{23}, m_{42}, \\ C_4^1 &\equiv \left\langle \sigma_{\{3,4\}} - \sigma_{\{2,4\}} : \sigma_{\{2,3\}} + \sigma_{\{3,4\}} : \sigma_{\{2,3\}} - \sigma_{\{2,4\}} \right\rangle, \ \textit{through } m_{34}, m_{32}, m_{24}. \end{aligned}$$

Though the labeling may seem arbitrary infact it is a little more subtle than this. Now for any triangle there are four circumlines which are the joins of three distinct midpoints. Therefore by counting we see that each midpoint is incident with two distinct circumlines.

For the quadrangle $\square$ any two subtriangles share exactly one side. Say subtriangle $\triangle_i$ and $\triangle_i$ for $i \neq j \in \{1,2,3,4\}$ share the side $\overline{a_\ell a_k}$ where $\ell \neq k \in \{1,2,3,4\} \setminus \{i,j\}$, that is for subtriangles $\triangle_1$ and $\triangle_2$ they share the side $\overline{a_3 a_4}$, and so on. Thus each midpoint m of the quadrangle $\square$ is associated to four circumlines C, two distinct circumlines from each of the subtriangles that share the side corresponding to the midpoint m. Therefore there is a type of pairing of pairs of circumlines from different subtriangles $P_{ij} \equiv \left\{C_r^i, C_s^i, C_\ell^j, C_k^j\right\}$ for some $r \neq s, \ell \neq k \in \{1,2,3,4\}$, for each midpoint m_{ij}. The set P_{ij} will be called a **pairing** of circumlines.

We will say that circumlines C_k^i and C_ℓ^j are **(midpoint) neighbours** if one of the pairings P_{ij} and P_{ji} induced from the midpoints m_{ij} and m_{ji} contains both C_k^i and C_ℓ^j.

After forcing a label on one set of circumlines (in this case for the subtriangle $\triangle_4$) the aim is to label the rest in such a way so that for each pairing $P_{ij} = \left\{C_r^i, C_s^i, C_\ell^j, C_k^j\right\}$ we have that $\ell = r$ and $k = s$. It turns out there are exactly two such labelings one is giving above;

$$\begin{aligned} P_{12} &\equiv \left\{C_3^3, C_4^3, C_3^4, C_4^4\right\}, & P_{21} &\equiv \left\{C_1^3, C_2^3, C_1^4, C_2^4\right\}, \\ P_{34} &\equiv \left\{C_3^1, C_4^1, C_3^2, C_4^2\right\}, & P_{43} &\equiv \left\{C_1^1, C_2^1, C_1^2, C_2^2\right\}, \\ P_{13} &\equiv \left\{C_2^2, C_4^2, C_2^4, C_4^4\right\}, & P_{31} &\equiv \left\{C_1^2, C_3^2, C_1^4, C_3^4\right\}, \\ P_{24} &\equiv \left\{C_2^1, C_4^1, C_2^3, C_4^3\right\}, & P_{42} &\equiv \left\{C_1^1, C_3^1, C_1^3, C_3^3\right\}, \\ P_{14} &\equiv \left\{C_2^2, C_3^2, C_2^3, C_3^3\right\}, & P_{41} &\equiv \left\{C_1^2, C_4^2, C_1^3, C_4^3\right\}, \\ P_{23} &\equiv \left\{C_2^1, C_3^1, C_2^4, C_3^4\right\}, & P_{32} &\equiv \left\{C_1^1, C_4^1, C_1^4, C_4^4\right\}. \end{aligned}$$

Moreover if we define $\pi_4 \equiv (1)\,, \pi_3 \equiv (12)\,(34)\,, \pi_2 \equiv (13)\,(24)$ and $\pi_1 \equiv (14)\,(23)$, then the other labeling is giving as follows $C^i_{\pi_i(j)}$ for $i, j \in \{1,2,3,4\}$.

Furthermore both of these labelings of the circumlines we get that the elements of the sets $\mathcal{C}_i \equiv \left\{C^1_i, C^2_i, C^3_i, C^4_i\right\}$ for $i = 1,2,3,4$ are neighbours.

Define $\mathcal{C}^i \equiv \left\{C^i_1, C^i_2, C^i_3, C^i_4\right\}$ to be the set of circumlines assosciated with the subtriangle $\triangle_i$, and $\mathcal{C}_i$ as above. Lets consider the circumlines of the the quadrangle $\square$ as vertices and say that two vertices share an edge precisely when the *corresponding circumlines are not midpoint neighbours*. Note that by definition the induced graph is quadpartite with respect to the vertex partition $\cup_i \mathcal{C}^i$. Furthermore as each $\mathcal{C}_i$ contains no neighbours, the vertex quad-partion $\cup_i \mathcal{C}_i$ is consistent with the previous one. Moreover as graphs they are isomorphic with respect to the map $C^i_j \mapsto C^j_i$.

Now each vetex C^i_j for some subtriangle $\triangle_i$ and index j, has exactly two midpoint neighbours in $\mathcal{C}^k$ and $\mathcal{C}_k$ for $k \neq i$ and $k \neq j$ respectfully. Thus each vertex has degree six and so by the Handshanking lemma from Graph Theory there are exactly fourtyeight edges and hence distinct meets of non-neighbouring circumlines. It turns out that these meets are collinear four at a time producing twelve distinct lines, but before we get to that some structure of the circumlines needs to be explored.

Looking at the bipartite subgraph with vertex set $\mathcal{C}_1 \cup \mathcal{C}_2$ the edge set can be worked out by exaiming the pairings P_{ij} for $i \neq j \in [4]$, above. What results is the set,

$$\left\{\left(C^1_1 C^4_2\right), \left(C^4_2 C^2_1\right), \left(C^2_1 C^3_2\right), \left(C^3_2 C^1_1\right), \left(C^1_2 C^4_1\right), \left(C^4_1 C^2_2\right), \left(C^2_2 C^3_1\right), \left(C^3_1 C^1_2\right)\right\},$$

and so the edge set of the bipartite subgraph is the union of distjoint cycles

$$\mathcal{C}_{12} \equiv C^1_1 C^3_2 C^2_1 C^4_2 C^1_1 \text{ and } \mathcal{C}_{21} \equiv C^1_2 C^3_1 C^2_2 C^4_1 C^1_2.$$

This is similarly true for the remaining bipartite subgraphs with respect to the partitions $\mathcal{C}_i$,

$$\begin{array}{ll}
\mathcal{C}_3 \cup \mathcal{C}_4: & \mathcal{C}_{34} \equiv C^1_3 C^3_4 C^2_3 C^4_4 C^1_3 \text{ and } \mathcal{C}_{43} \equiv C^1_4 C^3_3 C^2_4 C^4_3 C^1_4, \\
\mathcal{C}_1 \cup \mathcal{C}_3: & \mathcal{C}_{13} \equiv C^1_1 C^2_3 C^3_1 C^4_3 C^1_1 \text{ and } \mathcal{C}_{31} \equiv C^1_3 C^2_1 C^3_3 C^4_1 C^1_3, \\
\mathcal{C}_2 \cup \mathcal{C}_4: & \mathcal{C}_{24} \equiv C^1_2 C^2_4 C^3_2 C^4_4 C^1_2 \text{ and } \mathcal{C}_{42} \equiv C^1_4 C^2_2 C^3_4 C^4_2 C^1_4, \\
\mathcal{C}_1 \cup \mathcal{C}_4: & \mathcal{C}_{14} \equiv C^1_1 C^2_4 C^4_1 C^3_4 C^1_1 \text{ and } \mathcal{C}_{41} \equiv C^1_4 C^2_1 C^4_4 C^3_1 C^1_4, \\
\mathcal{C}_2 \cup \mathcal{C}_3: & \mathcal{C}_{23} \equiv C^1_2 C^2_3 C^4_2 C^3_3 C^1_2 \text{ and } \mathcal{C}_{32} \equiv C^1_3 C^2_2 C^4_3 C^3_2 C^1_3.
\end{array}$$

There is an anologous structure to the bipartite subgraphs with repect to the partitions $\mathcal{C}^i$, and it is as follows,

$$\begin{array}{ll}
\mathcal{C}^1 \cup \mathcal{C}^2: & \mathcal{C}^{12} \equiv C^1_1 C^2_3 C^1_2 C^2_4 C^1_1 \text{ and } \mathcal{C}^{21} \equiv C^2_1 C^1_3 C^2_2 C^1_4 C^2_1, \\
\mathcal{C}^3 \cup \mathcal{C}^4: & \mathcal{C}^{34} \equiv C^3_1 C^4_3 C^3_2 C^4_4 C^3_1 \text{ and } \mathcal{C}^{43} \equiv C^4_1 C^3_3 C^4_2 C^3_4 C^4_1, \\
\mathcal{C}^1 \cup \mathcal{C}^3: & \mathcal{C}^{13} \equiv C^1_1 C^3_2 C^1_3 C^3_4 C^1_1 \text{ and } \mathcal{C}^{31} \equiv C^3_1 C^1_2 C^3_3 C^1_4 C^3_1, \\
\mathcal{C}^2 \cup \mathcal{C}^4: & \mathcal{C}^{24} \equiv C^2_1 C^4_2 C^2_3 C^4_4 C^2_1 \text{ and } \mathcal{C}^{42} \equiv C^4_1 C^2_2 C^4_3 C^2_4 C^4_1, \\
\mathcal{C}^1 \cup \mathcal{C}^4: & \mathcal{C}^{14} \equiv C^1_1 C^4_2 C^1_4 C^4_3 C^1_1 \text{ and } \mathcal{C}^{41} \equiv C^4_1 C^1_2 C^4_4 C^1_3 C^4_1, \\
\mathcal{C}^2 \cup \mathcal{C}^3: & \mathcal{C}^{23} \equiv C^2_1 C^3_2 C^2_4 C^3_3 C^2_1 \text{ and } \mathcal{C}^{32} \equiv C^3_1 C^2_2 C^3_4 C^2_3 C^3_1.
\end{array}$$

Theorem 8 (Meets of Circumlines) *The meets of non-neighbouring circumlines are collinear four at a time, producing twelve distinct c-lines.*

Proof. Now as the graphs are isomorphic each edge, say $\left(C_r^i C_s^j\right)$, must appear exactly two cycles, $\mathcal{C}_{rs}$ or $\mathcal{C}_{sr}$ and $\mathcal{C}^{ij}$ or $\mathcal{C}^{ji}$, one from each of the respective partitions. For example the edge corresponding to the meet $\left(C_1^1 C_2^3\right)$ is in the cycles $\mathcal{C}_{12}$ and $\mathcal{C}^{13}$.

Without loss of generality since meets are symetric lets assume that the edge $\left(C_r^i C_s^j\right)$ appears in the cycles $\mathcal{C}_{rs}$ and $\mathcal{C}^{ij}$. There are six vertices in the union of these two cycles $\mathcal{C}_{rs} \cup \mathcal{C}^{ij}$, and two remaining edges when removing the vertices C_r^i, C_s^j. In our example the six vertices are

$$C_1^2, C_2^4, C_1^1, C_2^3, C_3^1, C_4^3 \in \mathcal{C}_{12} \cup \mathcal{C}^{13}$$

and the two edges

$$\left(C_1^2 C_2^4\right), \left(C_3^1 C_4^3\right) \in \left(\mathcal{C}_{12} \cup \mathcal{C}^{13}\right) \setminus \left\{C_1^1, C_2^3\right\}.$$

Now computations show us that the three meets corresponding to the edges found above are collinear. That is the line

$$\left\langle \sigma_{\{1,2\}} + \sigma_{\{3,4\}} : \sigma_{\{3,4\}} - \sigma_{\{1,2\}} - \sigma_{\{1,4\}} - \sigma_{\{2,3\}} : \sigma_{\{1,4\}} - \sigma_{\{2,3\}} \right\rangle$$

goes through the points

$$\left(C_1^1 C_2^3\right), \left(C_1^2 C_2^4\right), \text{ and } \left(C_3^1 C_4^3\right).$$

Now this is true for every vertex, and so there is some double counting. Infact the triples produced by picking vertices will overlap four at a time. So the line above is associated with the cycles $\mathcal{C}^{13}, \mathcal{C}^{24}, \mathcal{C}_{12}$ and $\mathcal{C}_{34}$ and so is also incident with the meet

$$\left(C_3^2 C_4^4\right).$$

That is we can group the circumlines into sets of fours which are colinear producing twelve distinct lines

$$C_{\{14,32\}}^{\{13,42\}} \equiv \left\langle \sigma_{\{1,2\}} - \sigma_{\{3,4\}} : \sigma_{\{2,3\}} - \sigma_{\{1,2\}} - \sigma_{\{1,4\}} - \sigma_{\{3,4\}} : \sigma_{\{1,4\}} + \sigma_{\{2,3\}} \right\rangle$$
$$\text{through } \left(C_1^1 C_4^3\right), \left(C_3^1 C_2^3\right), \left(C_1^4 C_4^2\right), \text{ and } \left(C_3^4 C_2^2\right),$$

$$C_{\{21,43\}}^{\{31,42\}} \equiv \left\langle \sigma_{\{1,2\}} + \sigma_{\{3,4\}} : \sigma_{\{1,4\}} - \sigma_{\{1,2\}} + \sigma_{\{2,3\}} + \sigma_{\{3,4\}} : \sigma_{\{2,3\}} - \sigma_{\{1,4\}} \right\rangle$$
$$\text{through } \left(C_2^1 C_1^3\right), \left(C_4^1 C_3^3\right), \left(C_2^2 C_1^4\right), \text{ and } \left(C_4^2 C_3^4\right),$$

$$C_{\{41,23\}}^{\{31,24\}} \equiv \left\langle \sigma_{\{3,4\}} - \sigma_{\{1,2\}} : \sigma_{\{1,2\}} - \sigma_{\{1,4\}} + \sigma_{\{2,3\}} + \sigma_{\{3,4\}} : \sigma_{\{1,4\}} + \sigma_{\{2,3\}} \right\rangle$$
$$\text{through } \left(C_2^1 C_3^3\right), \left(C_4^1 C_1^3\right), \left(C_2^4 C_3^2\right), \text{ and } \left(C_4^4 C_1^2\right),$$

$$C_{\{13,24\}}^{\{12,34\}} \equiv \left\langle \sigma_{\{1,3\}} + \sigma_{\{2,4\}} : \sigma_{\{2,3\}} - \sigma_{\{1,4\}} : \sigma_{\{1,4\}} - \sigma_{\{1,3\}} + \sigma_{\{2,3\}} + \sigma_{\{2,4\}} \right\rangle$$
$$\text{through } \left(C_1^1 C_3^2\right), \left(C_2^1 C_4^2\right), \left(C_1^3 C_3^4\right), \text{ and } \left(C_2^3 C_4^4\right),$$

$$C^{\{12,43\}}_{\{14,23\}} \equiv \left\langle \sigma_{\{2,4\}} - \sigma_{\{1,3\}} : \sigma_{\{2,3\}} + \sigma_{\{1,4\}} : \sigma_{\{1,3\}} - \sigma_{\{1,4\}} + \sigma_{\{2,3\}} + \sigma_{\{2,4\}} \right\rangle$$
$$\text{through } \left(C^1_1 C^2_4\right), \left(C^1_2 C^2_3\right), \left(C^4_1 C^3_2\right), \text{ and } \left(C^4_2 C^3_3\right),$$
$$C^{\{21,43\}}_{\{31,42\}} \equiv \left\langle \sigma_{\{1,3\}} + \sigma_{\{2,4\}} : \sigma_{\{1,4\}} - \sigma_{\{2,3\}} : \sigma_{\{2,4\}} - \sigma_{\{1,4\}} - \sigma_{\{1,3\}} - \sigma_{\{2,3\}} \right\rangle$$
$$\text{through } \left(C^1_3 C^2_1\right), \left(C^1_4 C^2_2\right), \left(C^3_3 C^4_1\right), \text{ and } \left(C^3_4 C^4_2\right),$$
$$C^{\{21,34\}}_{\{41,32\}} \equiv \left\langle \sigma_{\{1,3\}} - \sigma_{\{2,4\}} : \sigma_{\{2,3\}} + \sigma_{\{1,4\}} : \sigma_{\{2,3\}} - \sigma_{\{1,4\}} - \sigma_{\{1,3\}} - \sigma_{\{2,4\}} \right\rangle$$
$$\text{through } \left(C^1_3 C^2_2\right), \left(C^1_4 C^2_1\right), \left(C^4_3 C^3_2\right), \text{ and } \left(C^4_4 C^3_1\right),$$
$$C^{\{14,23\}}_{\{12,43\}} \equiv \left\langle \sigma_{\{3,4\}} - \sigma_{\{1,2\}} - \sigma_{\{1,3\}} - \sigma_{\{2,4\}} : \sigma_{\{1,2\}} + \sigma_{\{3,4\}} : \sigma_{\{1,3\}} - \sigma_{\{2,4\}} \right\rangle$$
$$\text{through } \left(C^1_1 C^4_2\right), \left(C^1_4 C^4_3\right), \left(C^3_1 C^4_3\right), \text{ and } \left(C^3_2 C^4_4\right),$$
$$C^{\{14,32\}}_{\{13,42\}} \equiv \left\langle \sigma_{\{2,4\}} - \sigma_{\{1,2\}} - \sigma_{\{1,3\}} - \sigma_{\{3,4\}} : \sigma_{\{1,2\}} - \sigma_{\{3,4\}} : \sigma_{\{1,3\}} + \sigma_{\{2,4\}} \right\rangle$$
$$\text{through } \left(C^1_1 C^4_3\right), \left(C^1_4 C^4_2\right), \left(C^4_1 C^3_2\right), \text{ and } \left(C^4_2 C^3_3\right),$$
$$C^{\{41,32\}}_{\{21,34\}} \equiv \left\langle \sigma_{\{1,3\}} - \sigma_{\{1,2\}} + \sigma_{\{2,4\}} + \sigma_{\{3,4\}} : \sigma_{\{1,2\}} + \sigma_{\{3,4\}} : \sigma_{\{2,4\}} - \sigma_{\{1,3\}} \right\rangle$$
$$\text{through } \left(C^1_2 C^4_1\right), \left(C^1_3 C^4_4\right), \left(C^4_3 C^3_2\right), \text{ and } \left(C^4_4 C^3_1\right),$$
$$C^{\{41,43\}}_{\{31,24\}} \equiv \left\langle \sigma_{\{1,2\}} - \sigma_{\{1,3\}} + \sigma_{\{2,4\}} + \sigma_{\{3,4\}} : \sigma_{\{3,4\}} - \sigma_{\{1,2\}} : \sigma_{\{1,3\}} + \sigma_{\{2,4\}} \right\rangle$$
$$\text{through } \left(C^1_2 C^4_4\right), \left(C^1_3 C^4_1\right), \left(C^3_3 C^4_1\right), \text{ and } \left(C^3_4 C^4_2\right).$$

These computations are all done in matlab, the github address will be provided at the end. ■

1.1.2

Centroids

Median lines (or just **medians**) D of a Triangle $\triangle$ are the joins of corresponding Midpoints m and Points a. There are six Medians, two passing every Point of the Triangle. The Dual to these are the Median points d of a Triangle, which are the meets of corresponding Midlines M and Dual lines A. Since a Quadrangle has a natural divide into Triangles, it possesses median structures.

Theorem 9 (Subtriangle Medians of the Quadrangle) *The Medians of the Quadrangle $\square = \overline{a_1 a_2 a_3 a_4}$ are given as follows;*

The Medians for the subtriangle $\triangle_1$:

$$\begin{aligned} D^1_{23} &\equiv \left\langle \sigma_{\{2,4\}} + \sigma_{\{3,4\}} : \sigma_{\{3,4\}} : \sigma_{\{2,4\}} \right\rangle, & D^1_{32} &\equiv \left\langle \sigma_{\{2,4\}} - \sigma_{\{3,4\}} : -\sigma_{\{3,4\}} : \sigma_{\{2,4\}} \right\rangle, \\ D^1_{24} &\equiv \left\langle \sigma_{\{3,4\}} : \sigma_{\{2,3\}} + \sigma_{\{3,4\}} : \sigma_{\{2,3\}} \right\rangle, & D^1_{42} &\equiv \left\langle \sigma_{\{3,4\}} : \sigma_{\{3,4\}} - \sigma_{\{2,3\}} : -\sigma_{\{2,3\}} \right\rangle, \\ D^1_{34} &\equiv \left\langle -\sigma_{\{2,4\}} : \sigma_{\{2,3\}} : \sigma_{\{2,3\}} - \sigma_{\{2,4\}} \right\rangle, & D^1_{43} &\equiv \left\langle \sigma_{\{2,4\}} : \sigma_{\{2,3\}} : \sigma_{\{2,3\}} + \sigma_{\{2,4\}} \right\rangle. \end{aligned}$$

The Medians for the subtriangle $\triangle_2$:

$$\begin{aligned}
D^2_{13} &\equiv \left\langle \sigma_{\{3,4\}} : \sigma_{\{1,4\}} + \sigma_{\{3,4\}} : -\sigma_{\{1,4\}} \right\rangle, & D^2_{31} &\equiv \left\langle \sigma_{\{3,4\}} : \sigma_{\{3,4\}} - \sigma_{\{1,4\}} : \sigma_{\{1,4\}} \right\rangle, \\
D^2_{14} &\equiv \left\langle \sigma_{\{1,3\}} + \sigma_{\{3,4\}} : \sigma_{\{3,4\}} : -\sigma_{\{1,3\}} \right\rangle, & D^2_{41} &\equiv \left\langle \sigma_{\{3,4\}} - \sigma_{\{1,3\}} : \sigma_{\{3,4\}} : \sigma_{\{1,3\}} \right\rangle, \\
D^2_{34} &\equiv \left\langle \sigma_{\{1,3\}} : -\sigma_{\{1,4\}} : \sigma_{\{1,4\}} - \sigma_{\{1,3\}} \right\rangle, & D^2_{43} &\equiv \left\langle \sigma_{\{1,3\}} : \sigma_{\{1,4\}} : -\sigma_{\{1,4\}} - \sigma_{\{1,3\}} \right\rangle.
\end{aligned}$$

The Medians for the subtriangle $\triangle_3$:

$$\begin{aligned}
D^3_{12} &\equiv \left\langle \sigma_{\{2,4\}} : \sigma_{\{1,4\}} : \sigma_{\{2,4\}} - \sigma_{\{1,4\}} \right\rangle, & D^3_{21} &\equiv \left\langle \sigma_{\{2,4\}} : -\sigma_{\{1,4\}} : \sigma_{\{2,4\}} + \sigma_{\{1,4\}} \right\rangle, \\
D^3_{14} &\equiv \left\langle \sigma_{\{1,2\}} + \sigma_{\{2,4\}} : -\sigma_{\{1,2\}} : \sigma_{\{2,4\}} \right\rangle, & D^3_{41} &\equiv \left\langle \sigma_{\{2,4\}} - \sigma_{\{1,2\}} : \sigma_{\{1,2\}} : \sigma_{\{2,4\}} \right\rangle, \\
D^3_{24} &\equiv \left\langle \sigma_{\{2,3\}} : -\sigma_{\{2,3\}} - \sigma_{\{3,4\}} : \sigma_{\{3,4\}} \right\rangle, & D^3_{42} &\equiv \left\langle \sigma_{\{2,3\}} : \sigma_{\{3,4\}} - \sigma_{\{2,3\}} : -\sigma_{\{3,4\}} \right\rangle.
\end{aligned}$$

The Medians for the subtriangle $\triangle_4$:

$$\begin{aligned}
D^4_{12} &\equiv \left\langle \sigma_{\{1,3\}} : \sigma_{\{2,3\}} : \sigma_{\{2,3\}} - \sigma_{\{1,3\}} \right\rangle, & D^4_{21} &\equiv \left\langle -\sigma_{\{1,3\}} : \sigma_{\{2,3\}} : \sigma_{\{2,3\}} + \sigma_{\{1,3\}} \right\rangle, \\
D^4_{13} &\equiv \left\langle -\sigma_{\{1,2\}} : \sigma_{\{1,2\}} + \sigma_{\{2,3\}} : \sigma_{\{2,3\}} \right\rangle, & D^4_{31} &\equiv \left\langle \sigma_{\{1,2\}} : \sigma_{\{2,3\}} - \sigma_{\{1,2\}} : \sigma_{\{2,3\}} \right\rangle, \\
D^4_{23} &\equiv \left\langle -\sigma_{\{1,2\}} - \sigma_{\{1,3\}} : \sigma_{\{1,2\}} : \sigma_{\{1,3\}} \right\rangle, & D^4_{32} &\equiv \left\langle \sigma_{\{1,2\}} - \sigma_{\{1,3\}} : -\sigma_{\{1,2\}} : \sigma_{\{1,3\}} \right\rangle.
\end{aligned}$$

Proof. Simple computations will show this. ■

Theorem 10 (Centroids) *The Medians lines D of a Triangle $\triangle = \overline{p_1p_2p_3}$ are concurrent in threes, meeting at four Centroid points g. The* ***Median points*** *d of a Triangle are collinear in threes, joining on four* ***Centroid lines*** *G.*

Proof. By the fundamental theorem of projective geometry we may assume without any loss of generality that $p_1 = a_1 \equiv [1 : 1 : 1]$, $p_2 = a_2 \equiv [-1 : -1 : 1]$, and $p_3 = a_3 \equiv [-1 : 1 : 1]$, and so by the Medians of the Quadrangle Theorem the Median lines of the Triangle $\triangle = \overline{p_1p_2p_3}$ are;

$$\begin{aligned}
D^4_{12} &\equiv \left\langle \sigma_{\{1,3\}} : \sigma_{\{2,3\}} : \sigma_{\{2,3\}} - \sigma_{\{1,3\}} \right\rangle, & D^4_{21} &\equiv \left\langle -\sigma_{\{1,3\}} : \sigma_{\{2,3\}} : \sigma_{\{2,3\}} + \sigma_{\{1,3\}} \right\rangle, \\
D^4_{13} &\equiv \left\langle -\sigma_{\{1,2\}} : \sigma_{\{1,2\}} + \sigma_{\{2,3\}} : \sigma_{\{2,3\}} \right\rangle, & D^4_{31} &\equiv \left\langle \sigma_{\{1,2\}} : \sigma_{\{2,3\}} - \sigma_{\{1,2\}} : \sigma_{\{2,3\}} \right\rangle, \\
D^4_{23} &\equiv \left\langle -\sigma_{\{1,2\}} - \sigma_{\{1,3\}} : \sigma_{\{1,2\}} : \sigma_{\{1,3\}} \right\rangle, & D^4_{32} &\equiv \left\langle \sigma_{\{1,2\}} - \sigma_{\{1,3\}} : -\sigma_{\{1,2\}} : \sigma_{\{1,3\}} \right\rangle.
\end{aligned}$$

The following triples of Medians are concurrent;

$$D^4_{12}, D^4_{13}, D^4_{23},$$

passing through

$$g^4_1 \equiv \left[\begin{array}{c} \sigma_{\{1,2\}}\sigma_{\{1,3\}} - \sigma_{\{1,2\}}\sigma_{\{2,3\}} + \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \\ \sigma_{\{1,2\}}\sigma_{\{1,3\}} - \sigma_{\{1,2\}}\sigma_{\{2,3\}} - \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \sigma_{\{1,2\}}\sigma_{\{1,3\}} + \sigma_{\{1,2\}}\sigma_{\{2,3\}} + \sigma_{\{1,3\}}\sigma_{\{2,3\}} \end{array} \right],$$

$$D^4_{12}, D^4_{31}, D^4_{32},$$

passing through

$$g^4_2 \equiv \left[\begin{array}{c} \sigma_{\{1,2\}}\sigma_{\{1,3\}} - \sigma_{\{1,2\}}\sigma_{\{2,3\}} - \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \\ \sigma_{\{1,2\}}\sigma_{\{1,3\}} - \sigma_{\{1,2\}}\sigma_{\{2,3\}} + \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \sigma_{\{1,2\}}\sigma_{\{1,3\}} + \sigma_{\{1,2\}}\sigma_{\{2,3\}} - \sigma_{\{1,3\}}\sigma_{\{2,3\}} \end{array} \right],$$

$$D^4_{21}, D^4_{13}, D^4_{32},$$

passing through

$$g_3^4 \equiv \left[\begin{array}{c} \sigma_{\{1,2\}}\sigma_{\{1,3\}} + \sigma_{\{1,2\}}\sigma_{\{2,3\}} + \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \\ \sigma_{\{1,2\}}\sigma_{\{1,3\}} + \sigma_{\{1,2\}}\sigma_{\{2,3\}} - \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \sigma_{\{1,2\}}\sigma_{\{1,3\}} - \sigma_{\{1,2\}}\sigma_{\{2,3\}} + \sigma_{\{1,3\}}\sigma_{\{2,3\}} \end{array} \right],$$

$$D_{21}^4, D_{31}^4, D_{23}^4,$$

passing through

$$g_4^4 \equiv \left[\begin{array}{c} \sigma_{\{1,2\}}\sigma_{\{1,3\}} + \sigma_{\{1,2\}}\sigma_{\{2,3\}} - \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \\ \sigma_{\{1,2\}}\sigma_{\{1,3\}} + \sigma_{\{1,2\}}\sigma_{\{2,3\}} + \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \sigma_{\{1,2\}}\sigma_{\{1,3\}} - \sigma_{\{1,2\}}\sigma_{\{2,3\}} - \sigma_{\{1,3\}}\sigma_{\{2,3\}} \end{array} \right].$$

This is checked by computing

$$\begin{aligned} 0 &= \det \begin{pmatrix} \sigma_{\{1,3\}} & \sigma_{\{2,3\}} & \sigma_{\{2,3\}} - \sigma_{\{1,3\}} \\ -\sigma_{\{1,2\}} & \sigma_{\{1,2\}} + \sigma_{\{2,3\}} & \sigma_{\{2,3\}} \\ -\sigma_{\{1,2\}} - \sigma_{\{1,3\}} & \sigma_{\{1,2\}} & \sigma_{\{1,3\}} \end{pmatrix} \\ &= \det \begin{pmatrix} \sigma_{\{1,3\}} & \sigma_{\{2,3\}} & \sigma_{\{2,3\}} - \sigma_{\{1,3\}} \\ \sigma_{\{1,2\}} & \sigma_{\{2,3\}} - \sigma_{\{1,2\}} & \sigma_{\{2,3\}} \\ \sigma_{\{1,2\}} - \sigma_{\{1,3\}} & -\sigma_{\{1,2\}} & \sigma_{\{1,3\}} \end{pmatrix} \\ &= \det \begin{pmatrix} -\sigma_{\{1,3\}} & \sigma_{\{2,3\}} & \sigma_{\{2,3\}} + \sigma_{\{1,3\}} \\ -\sigma_{\{1,2\}} & \sigma_{\{1,2\}} + \sigma_{\{2,3\}} & \sigma_{\{2,3\}} \\ \sigma_{\{1,2\}} - \sigma_{\{1,3\}} & -\sigma_{\{1,2\}} & \sigma_{\{1,3\}} \end{pmatrix} \\ &= \det \begin{pmatrix} -\sigma_{\{1,3\}} & \sigma_{\{2,3\}} & \sigma_{\{2,3\}} + \sigma_{\{1,3\}} \\ \sigma_{\{1,2\}} & \sigma_{\{2,3\}} - \sigma_{\{1,2\}} & \sigma_{\{2,3\}} \\ -\sigma_{\{1,2\}} - \sigma_{\{1,3\}} & \sigma_{\{1,2\}} & \sigma_{\{1,3\}} \end{pmatrix} \end{aligned}$$

The coresponding meets are given above. The situations with the Centroid lines G is precisely dual. ■

Theorem 11 (Subtraingle Centroids of the Quadrangle) *The Subtriangle Centroids of the Quadrangle* $\square = \overline{a_1 a_2 a_3 a_4}$ *are given as follows;*

The Centoids for the Subtriangle $\triangle_1$:

$$g_1^1 \equiv \left[\begin{array}{c} \sigma_{\{2,3\}}\sigma_{\{3,4\}} - \sigma_{\{2,3\}}\sigma_{\{2,4\}} - \sigma_{\{2,4\}}\sigma_{\{3,4\}} : \\ \sigma_{\{2,4\}}\sigma_{\{3,4\}} - \sigma_{\{2,3\}}\sigma_{\{3,4\}} - \sigma_{\{2,3\}}\sigma_{\{2,4\}} : \sigma_{\{2,3\}}\sigma_{\{2,4\}} + \sigma_{\{2,3\}}\sigma_{\{3,4\}} + \sigma_{\{2,4\}}\sigma_{\{3,4\}} \end{array} \right],$$

incident with the Medians $D_{23}^1, D_{24}^1, D_{34}^1$,

$$g_2^1 \equiv \left[\begin{array}{c} \sigma_{\{2,3\}}\sigma_{\{2,4\}} - \sigma_{\{2,3\}}\sigma_{\{3,4\}} - \sigma_{\{2,4\}}\sigma_{\{3,4\}} : \\ \sigma_{\{2,3\}}\sigma_{\{2,4\}} + \sigma_{\{2,3\}}\sigma_{\{3,4\}} + \sigma_{\{2,4\}}\sigma_{\{3,4\}} : \sigma_{\{2,4\}}\sigma_{\{3,4\}} - \sigma_{\{2,3\}}\sigma_{\{3,4\}} - \sigma_{\{2,3\}}\sigma_{\{2,4\}} \end{array} \right],$$

incident with the Medians $D_{23}^1, D_{42}^1, D_{43}^1$,

$$g_3^1 \equiv \left[\begin{array}{c} \sigma_{\{2,3\}}\sigma_{\{2,4\}} + \sigma_{\{2,3\}}\sigma_{\{3,4\}} + \sigma_{\{2,4\}}\sigma_{\{3,4\}} : \\ \sigma_{\{2,3\}}\sigma_{\{2,4\}} - \sigma_{\{2,3\}}\sigma_{\{3,4\}} - \sigma_{\{2,4\}}\sigma_{\{3,4\}} : \sigma_{\{2,3\}}\sigma_{\{3,4\}} - \sigma_{\{2,3\}}\sigma_{\{2,4\}} - \sigma_{\{2,3\}}\sigma_{\{3,4\}} \end{array} \right],$$

incident with the Medians $D_{32}^1, D_{24}^1, D_{43}^1$,

$$g_4^1 \equiv \left[\begin{array}{c} \sigma_{\{2,3\}}\sigma_{\{2,4\}} + \sigma_{\{2,3\}}\sigma_{\{3,4\}} - \sigma_{\{2,4\}}\sigma_{\{3,4\}} : \\ \sigma_{\{2,3\}}\sigma_{\{2,4\}} - \sigma_{\{2,3\}}\sigma_{\{3,4\}} + \sigma_{\{2,4\}}\sigma_{\{3,4\}} : \sigma_{\{2,3\}}\sigma_{\{3,4\}} - \sigma_{\{2,3\}}\sigma_{\{2,4\}} + \sigma_{\{2,3\}}\sigma_{\{3,4\}} \end{array} \right],$$

incident with the Medians $D^1_{32}, D^1_{42}, D^1_{34}$;

The Centroids for the Subtriangle $\triangle_2$:

$$g^2_1 \equiv \left[\begin{array}{c} \sigma_{\{1,3\}}\sigma_{\{1,4\}} + \sigma_{\{1,3\}}\sigma_{\{3,4\}} - \sigma_{\{1,4\}}\sigma_{\{3,4\}} : \\ \sigma_{\{1,3\}}\sigma_{\{1,4\}} - \sigma_{\{1,3\}}\sigma_{\{3,4\}} + \sigma_{\{1,4\}}\sigma_{\{3,4\}} : \sigma_{\{1,3\}}\sigma_{\{1,4\}} + \sigma_{\{1,3\}}\sigma_{\{3,4\}} + \sigma_{\{1,4\}}\sigma_{\{3,4\}} \end{array} \right],$$

incident with the Medians $D^2_{13}, D^2_{14}, D^2_{34}$,

$$g^2_2 \equiv \left[\begin{array}{c} \sigma_{\{1,3\}}\sigma_{\{1,4\}} + \sigma_{\{1,3\}}\sigma_{\{3,4\}} + \sigma_{\{1,4\}}\sigma_{\{3,4\}} : \\ \sigma_{\{1,3\}}\sigma_{\{1,4\}} - \sigma_{\{1,3\}}\sigma_{\{3,4\}} - \sigma_{\{1,4\}}\sigma_{\{3,4\}} : \sigma_{\{1,3\}}\sigma_{\{1,4\}} + \sigma_{\{1,3\}}\sigma_{\{3,4\}} - \sigma_{\{1,4\}}\sigma_{\{3,4\}} \end{array} \right],$$

incident with the Medians $D^2_{13}, D^2_{41}, D^2_{43}$,

$$g^2_3 \equiv \left[\begin{array}{c} \sigma_{\{1,3\}}\sigma_{\{1,4\}} - \sigma_{\{1,3\}}\sigma_{\{3,4\}} - \sigma_{\{1,4\}}\sigma_{\{3,4\}} : \\ \sigma_{\{1,3\}}\sigma_{\{1,4\}} + \sigma_{\{1,3\}}\sigma_{\{3,4\}} + \sigma_{\{1,4\}}\sigma_{\{3,4\}} : \sigma_{\{1,3\}}\sigma_{\{1,4\}} - \sigma_{\{1,3\}}\sigma_{\{3,4\}} + \sigma_{\{1,4\}}\sigma_{\{3,4\}} \end{array} \right],$$

incident with the Medians $D^2_{31}, D^2_{14}, D^2_{43}$,

$$g^2_4 \equiv \left[\begin{array}{c} \sigma_{\{1,3\}}\sigma_{\{1,4\}} - \sigma_{\{1,3\}}\sigma_{\{3,4\}} + \sigma_{\{1,4\}}\sigma_{\{3,4\}} : \\ \sigma_{\{1,3\}}\sigma_{\{1,4\}} + \sigma_{\{1,3\}}\sigma_{\{3,4\}} - \sigma_{\{1,4\}}\sigma_{\{3,4\}} : \sigma_{\{1,3\}}\sigma_{\{1,4\}} - \sigma_{\{1,3\}}\sigma_{\{3,4\}} - \sigma_{\{1,4\}}\sigma_{\{3,4\}} \end{array} \right],$$

incident with the Medians $D^2_{31}, D^2_{41}, D^2_{34}$;

The Centroids for the Subtriangle $\triangle_3$:

$$g^3_1 \equiv \left[\begin{array}{c} \sigma_{\{1,2\}}\sigma_{\{1,4\}} - \sigma_{\{1,2\}}\sigma_{\{2,4\}} - \sigma_{\{1,4\}}\sigma_{\{2,4\}} : \\ \sigma_{\{1,2\}}\sigma_{\{1,4\}} - \sigma_{\{1,2\}}\sigma_{\{2,4\}} + \sigma_{\{1,4\}}\sigma_{\{2,4\}} : \sigma_{\{1,2\}}\sigma_{\{1,4\}} + \sigma_{\{1,2\}}\sigma_{\{2,4\}} + \sigma_{\{1,4\}}\sigma_{\{2,4\}} \end{array} \right],$$

incident with the Medians $D^3_{12}, D^3_{14}, D^3_{24}$,

$$g^3_2 \equiv \left[\begin{array}{c} \sigma_{\{1,2\}}\sigma_{\{1,4\}} - \sigma_{\{1,2\}}\sigma_{\{2,4\}} + \sigma_{\{1,4\}}\sigma_{\{2,4\}} : \\ \sigma_{\{1,2\}}\sigma_{\{1,4\}} - \sigma_{\{1,2\}}\sigma_{\{2,4\}} - \sigma_{\{1,4\}}\sigma_{\{2,4\}} : \sigma_{\{1,2\}}\sigma_{\{1,4\}} + \sigma_{\{1,2\}}\sigma_{\{2,4\}} - \sigma_{\{1,4\}}\sigma_{\{2,4\}} \end{array} \right],$$

incident with the Medians $D^3_{12}, D^3_{41}, D^3_{42}$,

$$g^3_3 \equiv \left[\begin{array}{c} \sigma_{\{1,2\}}\sigma_{\{1,4\}} + \sigma_{\{1,2\}}\sigma_{\{2,4\}} - \sigma_{\{1,4\}}\sigma_{\{2,4\}} : \\ \sigma_{\{1,2\}}\sigma_{\{1,4\}} + \sigma_{\{1,2\}}\sigma_{\{2,4\}} + \sigma_{\{1,4\}}\sigma_{\{2,4\}} : \sigma_{\{1,2\}}\sigma_{\{1,4\}} - \sigma_{\{1,2\}}\sigma_{\{2,4\}} + \sigma_{\{1,4\}}\sigma_{\{2,4\}} \end{array} \right],$$

incident with the Medians $D^3_{21}, D^3_{14}, D^3_{42}$,

$$g^3_4 \equiv \left[\begin{array}{c} \sigma_{\{1,2\}}\sigma_{\{1,4\}} + \sigma_{\{1,2\}}\sigma_{\{2,4\}} + \sigma_{\{1,4\}}\sigma_{\{2,4\}} : \\ \sigma_{\{1,2\}}\sigma_{\{1,4\}} + \sigma_{\{1,2\}}\sigma_{\{2,4\}} - \sigma_{\{1,4\}}\sigma_{\{2,4\}} : \sigma_{\{1,2\}}\sigma_{\{1,4\}} - \sigma_{\{1,2\}}\sigma_{\{2,4\}} - \sigma_{\{1,4\}}\sigma_{\{2,4\}} \end{array} \right],$$

incident with the Medians $D^3_{21}, D^3_{41}, D^3_{24}$;

The Centroids for the Subtriangle $\triangle_4$:

$$g^4_1 \equiv \left[\begin{array}{c} \sigma_{\{1,2\}}\sigma_{\{1,3\}} - \sigma_{\{1,2\}}\sigma_{\{2,3\}} + \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \\ \sigma_{\{1,2\}}\sigma_{\{1,3\}} - \sigma_{\{1,2\}}\sigma_{\{2,3\}} - \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \sigma_{\{1,2\}}\sigma_{\{1,3\}} + \sigma_{\{1,2\}}\sigma_{\{2,3\}} + \sigma_{\{1,3\}}\sigma_{\{2,3\}} \end{array} \right],$$

incident with the Medians $D^4_{12}, D^4_{13}, D^4_{23}$,

$$g^4_2 \equiv \left[\begin{array}{c} \sigma_{\{1,2\}}\sigma_{\{1,3\}} - \sigma_{\{1,2\}}\sigma_{\{2,3\}} - \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \\ \sigma_{\{1,2\}}\sigma_{\{1,3\}} - \sigma_{\{1,2\}}\sigma_{\{2,3\}} + \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \sigma_{\{1,2\}}\sigma_{\{1,3\}} + \sigma_{\{1,2\}}\sigma_{\{2,3\}} - \sigma_{\{1,3\}}\sigma_{\{2,3\}} \end{array} \right],$$

incident with the Medians $D^4_{12}, D^4_{31}, D^4_{32}$,

$$g^4_3 \equiv \left[\begin{array}{c} \sigma_{\{1,2\}}\sigma_{\{1,3\}} + \sigma_{\{1,2\}}\sigma_{\{2,3\}} + \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \\ \sigma_{\{1,2\}}\sigma_{\{1,3\}} + \sigma_{\{1,2\}}\sigma_{\{2,3\}} - \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \sigma_{\{1,2\}}\sigma_{\{1,3\}} - \sigma_{\{1,2\}}\sigma_{\{2,3\}} + \sigma_{\{1,3\}}\sigma_{\{2,3\}} \end{array}\right],$$

incident with the Medians $D^4_{21}, D^4_{13}, D^4_{32}$,

$$g^4_4 \equiv \left[\begin{array}{c} \sigma_{\{1,2\}}\sigma_{\{1,3\}} + \sigma_{\{1,2\}}\sigma_{\{2,3\}} - \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \\ \sigma_{\{1,2\}}\sigma_{\{1,3\}} + \sigma_{\{1,2\}}\sigma_{\{2,3\}} + \sigma_{\{1,3\}}\sigma_{\{2,3\}} : \sigma_{\{1,2\}}\sigma_{\{1,3\}} - \sigma_{\{1,2\}}\sigma_{\{2,3\}} - \sigma_{\{1,3\}}\sigma_{\{2,3\}} \end{array}\right].$$

incident with the Medians $D^4_{21}, D^4_{31}, D^4_{23}$.

Proof. This is worked out computationally. ■

The **set of associated midpoints** (or **assosciated midpoints**) S_g for a Centroid point g of a Triangle $\triangle$, is the set of three distinct midpoints used to construct it. For example the Centroid g^4_1 of the Subtriangle $\triangle_4$ is constructed from the Median lines D^4_{12}, D^4_{13}, and D^4_{23}, and hence $S^4_1 \equiv \{m_{12}, m_{13}, m_{23}\}$ is the set of associated midpoints.

Note that the set of associated midpoints is unique and distinct for every distinct Centroid.

A set $\left\{g^1_{i_1}, g^2_{i_2}, g^3_{i_3}, g^4_{i_4}\right\}$ containing one Centroid from each Subtriangle is said to be **midpoint consistent** if the union of the associated midpoints $S_m = \cup S^k_{i_k}$, conatins exactly one midpoint for every side $\overline{a_i a_j}$ of the Quadrangle.

For example the set

$$S = \left\{g^1_1, g^2_1, g^3_1, g^4_1\right\},$$

is midpoint consistent since the union of assosciated midpoints is

$$S_m = \{m_{12}, m_{34}, m_{13}, m_{24}, m_{14}, m_{23}\}.$$

Theorem 12 (Midpoint consistent sets of Subtriangle Centroids) *In total there are eight distinct sets of Subtriangle Centroids* $S = \left\{g^1_{i_1}, g^2_{i_2}, g^3_{i_3}, g^4_{i_4}\right\}$ *which are midpoint consistent.*

Proof. We prove this by trying to construct a midpoint consistent set of Subtriangle Centroids $S = \left\{g^1_{i_1}, g^2_{i_2}, g^3_{i_3}, g^4_{i_4}\right\}$. First let's choose $i_1 = 1$, that is let g^1_1 be in the set. The Centroid g^1_1 has assosciated midpoints $S^1_1 \equiv \{m_{23}, m_{24}, m_{34}\}$, and so for any other Centroid in S, the midpoints m_{32}, m_{42}, and m_{43} cannot be in their respective set of assosciated midpoints.

So looking at the Centroids g^2_i of the Subtriangle $\triangle_2$ and their assosciated midpoints

$$\begin{array}{ll} g^2_1 : S^2_1 \equiv \{m_{13}, m_{14}, m_{34}\}, & g^2_2 : S^2_2 \equiv \{m_{13}, m_{41}, m_{43}\}, \\ g^2_3 : S^2_3 \equiv \{m_{31}, m_{14}, m_{43}\}, & g^2_4 : S^2_4 \equiv \{m_{31}, m_{41}, m_{34}\}, \end{array}$$

we see that the Centroids g^2_1 and g^2_4 are the only options for S. If we choose $i_2 = 1$, then the set

$$\{m_{34}, m_{13}, m_{24}, m_{14}, m_{23}\} \subseteq S_m,$$

which forces $i_3 = i_4 = 1$, as $S^3_1 \equiv \{m_{12}, m_{14}, m_{24}\}$ and $S^4_1 \equiv \{m_{12}, m_{13}, m_{23}\}$. Else if $i_2 = 4$, then the set

$$\{m_{34}, m_{31}, m_{24}, m_{41}, m_{23}\} \subseteq S_m,$$

which forces $i_3 = i_4 = 4$, as $S_4^3 \equiv \{m_{21}, m_{31}, m_{24}\}$ and $S_4^4 \equiv \{m_{21}, m_{31}, m_{23}\}$. Therefore there are two distinct midpoint consistent sets of Subtriangle Centroids which contain the Centroid g_1^1. The above method can be used for any choice of $i_1 \in [4]$, and hence there are in total exactly eight distinct sets of Subtriangle Centroids which are midpoint consistent, and are given as follows;

$$\begin{array}{cc}
S: & S_m: \\
\{g_1^1, g_1^2, g_1^3, g_1^4\} & \{m_{12}, m_{34}, m_{13}, m_{24}, m_{14}, m_{23}\} \\
\{g_1^1, g_4^2, g_4^3, g_4^4\} & \{m_{21}, m_{34}, m_{31}, m_{24}, m_{41}, m_{23}\} \\
\{g_2^1, g_2^2, g_2^3, g_1^4\} & \{m_{12}, m_{43}, m_{13}, m_{42}, m_{41}, m_{23}\} \\
\{g_2^1, g_3^2, g_2^3, g_4^4\} & \{m_{21}, m_{43}, m_{31}, m_{42}, m_{14}, m_{23}\} \\
\{g_3^1, g_2^2, g_4^3, g_3^4\} & \{m_{21}, m_{43}, m_{13}, m_{24}, m_{41}, m_{32}\} \\
\{g_3^1, g_3^2, g_1^3, g_2^4\} & \{m_{12}, m_{43}, m_{31}, m_{24}, m_{14}, m_{32}\} \\
\{g_4^1, g_1^2, g_3^3, g_3^4\} & \{m_{12}, m_{34}, m_{31}, m_{42}, m_{41}, m_{32}\} \\
\{g_4^1, g_4^2, g_2^3, g_1^4\} & \{m_{21}, m_{34}, m_{13}, m_{42}, m_{14}, m_{32}\}
\end{array}$$

■

A **Bimedian line** $B_{\{ij,k\ell\}}$ is the join of two midpoints m_{ij}, and $m_{k\ell}$ from opposite sides of the Quadrangle $\{a_i, a_j\}$ and $\{a_k, a_\ell\}$, where $[4] = \{i, j, k, \ell\}$.

Theorem 13 (Bimedian Lines of the Quadrangle) *The Bimedian lines $B_{\{ij,k\ell\}}$ of the Quadrangle $\square = \overline{a_1 a_2 a_3 a_4}$ are given as follows;*

The Bimedian lines corresponding the α opposite sides:

$$\begin{aligned}
B_{\{12,34\}} &\equiv \left\langle \sigma_{\{1,3\}} - \sigma_{\{2,4\}} : \sigma_{\{2,3\}} - \sigma_{\{1,4\}} : \sigma_{\{2,3\}} + \sigma_{\{1,4\}} - \sigma_{\{1,3\}} - \sigma_{\{2,4\}} \right\rangle, \\
B_{\{12,43\}} &\equiv \left\langle \sigma_{\{1,3\}} + \sigma_{\{2,4\}} : \sigma_{\{2,3\}} + \sigma_{\{1,4\}} : \sigma_{\{2,3\}} - \sigma_{\{1,4\}} - \sigma_{\{1,3\}} + \sigma_{\{2,4\}} \right\rangle, \\
B_{\{21,34\}} &\equiv \left\langle \sigma_{\{1,3\}} + \sigma_{\{2,4\}} : -\sigma_{\{2,3\}} - \sigma_{\{1,4\}} : \sigma_{\{1,4\}} - \sigma_{\{2,3\}} - \sigma_{\{1,3\}} + \sigma_{\{2,4\}} \right\rangle, \\
B_{\{21,43\}} &\equiv \left\langle \sigma_{\{2,4\}} - \sigma_{\{1,3\}} : \sigma_{\{2,3\}} - \sigma_{\{1,4\}} : \sigma_{\{2,3\}} + \sigma_{\{1,4\}} + \sigma_{\{1,3\}} + \sigma_{\{2,4\}} \right\rangle,
\end{aligned}$$

The Bimedian lines corresponding the β opposite sides:

$$\begin{aligned}
B_{\{13,24\}} &\equiv \left\langle \sigma_{\{3,4\}} - \sigma_{\{1,2\}} : \sigma_{\{1,4\}} + \sigma_{\{3,4\}} + \sigma_{\{2,3\}} + \sigma_{\{1,2\}} : \sigma_{\{1,4\}} - \sigma_{\{2,3\}} \right\rangle, \\
B_{\{13,42\}} &\equiv \left\langle \sigma_{\{3,4\}} + \sigma_{\{1,2\}} : \sigma_{\{1,4\}} - \sigma_{\{2,3\}} + \sigma_{\{3,4\}} - \sigma_{\{1,2\}} : -\sigma_{\{1,4\}} - \sigma_{\{2,3\}} \right\rangle, \\
B_{\{31,24\}} &\equiv \left\langle \sigma_{\{3,4\}} + \sigma_{\{1,2\}} : \sigma_{\{2,3\}} - \sigma_{\{1,4\}} + \sigma_{\{3,4\}} - \sigma_{\{1,2\}} : \sigma_{\{1,4\}} + \sigma_{\{2,3\}} \right\rangle, \\
B_{\{31,42\}} &\equiv \left\langle \sigma_{\{1,2\}} - \sigma_{\{3,4\}} : \sigma_{\{2,3\}} + \sigma_{\{1,4\}} - \sigma_{\{3,4\}} - \sigma_{\{1,2\}} : \sigma_{\{2,3\}} - \sigma_{\{1,4\}} \right\rangle,
\end{aligned}$$

The Bimedian lines corresponding the γ opposite sides:

$$\begin{aligned}
B_{\{14,23\}} &\equiv \left\langle \sigma_{\{1,2\}} + \sigma_{\{3,4\}} + \sigma_{\{1,3\}} + \sigma_{\{2,4\}} : \sigma_{\{3,4\}} - \sigma_{\{1,2\}} : \sigma_{\{2,4\}} - \sigma_{\{1,3\}} \right\rangle, \\
B_{\{14,32\}} &\equiv \left\langle \sigma_{\{1,2\}} - \sigma_{\{3,4\}} + \sigma_{\{2,4\}} - \sigma_{\{1,3\}} : -\sigma_{\{3,4\}} - \sigma_{\{1,2\}} : \sigma_{\{2,4\}} + \sigma_{\{1,3\}} \right\rangle, \\
B_{\{41,23\}} &\equiv \left\langle \sigma_{\{3,4\}} - \sigma_{\{1,2\}} + \sigma_{\{2,4\}} - \sigma_{\{1,3\}} : \sigma_{\{3,4\}} + \sigma_{\{1,2\}} : \sigma_{\{2,4\}} + \sigma_{\{1,3\}} \right\rangle, \\
B_{\{41,32\}} &\equiv \left\langle \sigma_{\{3,4\}} + \sigma_{\{1,2\}} - \sigma_{\{2,4\}} - \sigma_{\{1,3\}} : \sigma_{\{3,4\}} - \sigma_{\{1,2\}} : \sigma_{\{1,3\}} - \sigma_{\{2,4\}} \right\rangle.
\end{aligned}$$

Proof. The Bimedian

$$\begin{aligned}
B_{\{12,34\}} &\equiv \left[1 - \sigma_{\{1,2\}} A_1 : 1 - \sigma_{\{1,2\}} A_1 : 1 + \sigma_{\{1,2\}} A_1\right] \times \left[1 - \sigma_{\{3,4\}} A_3 : \sigma_{\{3,4\}} A_3 - 1 : 1 + \sigma_{\{3,4\}} A_3\right] \\
&=
\end{aligned}$$

■

Theorem 14

Chapter 2

Introduction

Throughout this thesis definitions will be given in bold and italics will be reserved for emphasis.

Universal Geometry Through relatively recent developements in the field of geometry, Norman Wildberger has shown that hyperbolic geometry can be considered as an agebraic projective geometry. In the following pages we will aim to define the fundamental objects of hyperbolic geometry and how they interact with each other.

Projective Geometry Universal Projective geometry is a geometry in the space of lines through the origin of a vector space with a metrical structure given by a symmetric bilinear form.
The complete algebraic nature of Universal geometry implies that we have an algebraic construction of Projective geometry. The focus of this chapter is to introduce the main objects of Universal Projective geometry and define their incidence relations in such a way to induce a complete duality between points and lines in this projective setting. This concept of complete duality is a defining characteristic of Projective geometry.
Universal geometry is given as an algebraic geometry, where the algebraic framework for Universal Projective geometry is given to us through *projective linear algebra*. Projective linear algebra is much like normal linear algebra but vectors and matrices are only defined up to non-zero scalar multiples. In this thesis the convention of writing the usual affine vectors and matrices with round brackets and projective vectors and matrices with square brackets. Hence for a given row vector $v = (1 \;\; 2 \;\; 3)$ we denote the associated projective vector $a = [v]$ as $a = [1 \;\; 2 \;\; 3]$ which by definition also equal to $[-1 \quad -2 \quad -3]$ or to $[2 \;\; 4 \;\; 6]$. We will also use bold labels to denote projective matrices: for for ordinary matrices

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & -1 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

the associated projective matrices are

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 2 & -1 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & -8 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Where infact $\mathbf{A}^{-1} = \mathbf{B}$ in the projective setting, as scalar multiplies can be disregarded. It turns out that addition of projective matrices is not well defined but multiplication is.

It is now important to introduce the main objects of projective geometry in a way that is consistent. A **(projective) point** is a *non-zero* projective row vector a and will be written in either of two ways:

$$a \equiv [x\ y\ z] \equiv [x : y : z].$$

A **(projective) line** is a *non-zero* projective column vector L written as

$$L \equiv \begin{bmatrix} l \\ m \\ n \end{bmatrix} \equiv \langle l : m : n \rangle.$$

For the point $a = [x : y : z]$ and line $L = \langle l : m : n \rangle$ we say they are **incident** precisely when

$$aL \equiv [x\ y\ z] \begin{bmatrix} l \\ m \\ n \end{bmatrix} \equiv 0. \tag{2.1}$$

Three or more lines are **concurrent** precisely when they are all incident with a point a, and dually three or more points are **collinear** precisely when they are all incident with a line L.

The **join** a_1a_2 of distinct points $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ is the line

$$a_1a_2 \equiv [x_1 : y_1 : z_1] \times [x_2 : y_2 : z_2] \equiv \langle y_1z_2 - z_1y_2 : z_1x_2 - x_1z_2 : x_1y_2 - y_1x_2 \rangle.$$

The **meet** L_1L_2 of two distinct points $L_1 \equiv \langle l_1 : m_1 : n_1 \rangle$ and $l_2 \equiv \langle l_2 : m_2 : n_2 \rangle$ is the point

$$L_1L_2 \equiv \langle l_1 : m_1 : n_1 \rangle \times \langle l_2 : m_2 : n_2 \rangle \equiv [m_1n_2 - n_1m_2 : n_1l_2 - l_1n_2 : l_1m_2 - m_1l_2].$$

The cross here is the usual Euclidean cross product which is well defined. This also induces the result that *the join a_1a_2 is a unique line which is incident with the points a_1 and a_2*. Dually *the meet L_1L_2 is a unique point which is incident with the lines L_1 and L_2*.

A **3-proportion** $x : y : z$ is an ordered triple of numbers x, y and z, *not all zero*, with the convention that for any non-zero number λ

$$x : y : z = \lambda x : \lambda y : \lambda z.$$

This is equivalent to saying that

$$x_1 : y_1 : z_1 = x_2 : y_2 : z_2$$

precisely when the following conditions hold

$$x_1 y_2 - x_2 y_1 = 0 \quad y_1 z_2 - y_2 z_1 = 0 \quad z_1 x_2 - z_2 x_1 = 0. \tag{2.2}$$

Now that the notion of a proportion is set up we can define the two main hyperbolic objects. A **(hyperbolic) point** is a 3-proportion $a \equiv [x : y : z]$ enclosed in square brackets. Where a **(hyperbolic) line** is a 3-proportion $L \equiv (l : m : n)$ enclosed in round brackets.

The definitions of points and lines is equivalent to that of projective geometry, where the two types of geometry differ becomes obvious in the notion of duality. The point $a \equiv [x, y, z]$ is **dual** to the line $L \equiv (l : m : n)$ precisely when

$$x : y : z = l : m : n.$$

In this case we say that $a^{\perp} = L$ or $L^{\perp} = a$.
From the definition of points and lines we get that each point is dual to exactly one line, and conversely. This new idea of duality induces the same property that *there is a complete duality in the theory between points and lines*, within this new projective geometry.

Now that we have set up the basic objects of hyperbolic geometry, an important step is to define the incidence of these objects with each other. The following theorems and definitions aim to do exactly that.

The point $a \equiv [x : y : z]$ **lies on** the line $L \equiv (l : m : n)$, or equivalently L **passes through** a, precisely when

$$lx + my - nz = 0.$$

Points $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ are **perpendicular** precisely when

$$x_1 x_2 + y_1 y_2 - z_1 z_2 = 0.$$

This is equivalent to the condition that a_1 is incident with $a_2^{\perp}$, or that a_2 is incident with $a_1^{\perp}$.

Similarly the line $L_1 \equiv (l_1 : m_1 : n_1)$ and $L_2 \equiv (l_2 : m_2 : n_2)$ are **perpendicular** precisely when

$$l_1 l_2 + m_1 m_2 - n_1 n_2 = 0.$$

This is equivalent to the condition that L_1 is incident with $L_2^\perp$, or that L_2 is incident with $L_1^\perp$.

We will denote by $\mathbb{F}^3$ the 3-dimensional space of **vectors** $v \equiv (x, y, z)$. If $v \equiv (x, y, z)$ has coordinates which are not all zero, then let $[v] \equiv [x : y : z]$ denote the (hyperbolic) point, and $(v) \equiv (l : m : n)$ denote the (hyperbolic) line.

Theorem 15 (Joins of points) *If $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ are distinct points, then there is exactly one line L which passes through them both, namely*

$$L \equiv a_1 a_2 \equiv (y_1 z_2 - y_2 z_1 : z_1 x_2 - z_2 x_1 : x_2 y_1 - x_1 y_2).$$

The line $L \equiv a_1 a_2$ is the **join** of the points a_1 and a_2.

Theorem 16 (Meets of lines) *If $L_1 \equiv (l_1 : m_1 : n_1)$ and $L_2 \equiv (l_2 : m_2 : n_2)$ are distinct lines, then there is exactly one point a which lies on both, namely*

$$a \equiv L_1 L_2 \equiv [m_1 n_2 - m_2 n_1 : n_1 l_2 - n_2 l_1 : l_2 m_1 - l_1 m_2].$$

The point $a \equiv L_1 L_2$ is the **meet** of the lines L_1 and L_2. These definitions give the following consequence, for any distinct points a_1 and a_2, and distinct lines L_1 and L_2,

$$(a_1 a_2)^\perp = a_1^\perp a_2 \perp, \quad \text{and} \quad (L_1 L_2)^\perp = L_1^\perp L_2^\perp.$$

Three or more points which lie on a common line are **collinear**. Three or more lines which pass through a common point are **concurrent**.

Theorem 17 (Collinear points) *The points $a_1 \equiv [x_1 : y_1 : z_1]$, $a_2 \equiv [x_2 : y_2 : z_2]$ and $a_3 \equiv [x_3 : y_3 : z_3]$ are collinear precisely when*

$$x_1 y_2 z_3 - x_1 y_3 z_2 + x_2 y_3 z_1 - x_2 y_1 z_3 + x_3 y_1 z_2 - x_3 y_2 z_1 = 0.$$

Theorem 18 (Concurrent lines) *The lines $L_1 \equiv (l_1 : m_1 : n_1)$, $L_2 \equiv (l_2 : m_2 : n_2)$ and $L_3 \equiv (l_3 : m_3 : n_3)$ are concurrent precisely when*

$$l_1 m_2 n_3 - l_1 m_3 n_2 + l_2 m_3 n_1 - l_2 m_1 n_3 + l_3 m_1 n_2 - l_3 m_2 n_1 = 0.$$

Now that the fundamental objects and there interactions are defined will begin to classify these objects. Firstly we say that the point $a \equiv [x : y : z]$ is **null** precisely when it lies on its dual line, that is when

$$x^2 + y^2 - z^2 = 0,$$

and analogously that the line $L \equiv (l : m : n)$ is **null** precisely when it passes through its dual point, that is when

$$l^2 + m^2 - n^2 = 0.$$

The dual of a null point is a null line and conversely. I'll now use the following theorem to further this classification of objects.

Theorem 19 (Line through null points, and Point on null lines) *Any line L passes through at most two null points, and any point a lies on at most two null lines.*

Therefore we have a naturally classification of points and lines. For a non-null point a we say that it is **internal** precisely when it lies on no null lines, and is **external** precisely when it lies on 2 null lines. Whereas a non-null line L is said to be **external** precisely when it passes through no null points and **internal** precisely when it passes through no null points. That is all points and lines are either *internal, null* or *external.* Unlike null points and lines we have that the dual of an internal point is an external line, and the dual of an external point is an internal line and conversely.

We now go onto define the geometric objects of hyperbolic geometry.

A **side** $\overline{a_1a_2}$ is a set $\{a_1, a_2\}$ of two points. A **vertex** $\overline{L_1L_2}$ is a set $\{L_1, L_2\}$ of two lines. From the definition it is clear that

$$\overline{a_1a_2} = \overline{a_2a_1} \text{ and } \overline{L_1L_2} = \overline{L_2L_1}.$$

For a side $\overline{a_1a_2}$ we say that a_1a_2 is the **line** of the side. Whilst for a vertex $\overline{L_1L_2}$ we say that L_1L_2 is the **point** of the vertex.

Much like the fundamental objects, we can continue to classify these new objects. We say that a side $\overline{a_1a_2}$ is a **nil side** precisely when at least one of a_1 or a_2 is a null point. Thus we are able to further classify sides, such as the side $\overline{a_1a_2}$ as a **singly-nil side**, or a **doubly-nil side** respectively, precisely when exactly one of the points a_1 or a_2 are null, or exactly both of the points a_1 and a_2 are null points respectively. Similarly we are able to classify the vertex $\overline{L_1L_2}$ as a **singly-nil vertex** or a **doubly-nil vertex** in a natural way.

Theorem 20 (Perpendicular point) *For any side $\overline{a_1a_2}$ there is a unique point p which is perpendicular to both a_1 and a_2, namely*

$$p \equiv a_1^\perp a_2^\perp = (a_1a_2)^\perp.$$

The point p is the **perpendicular point** of $\overline{a_1a_2}$. It is possible that p may lie on a_1a_2; this occurs precisely when a_1a_2 is a null line.

Theorem 21 (Perpendicular line) *For any vertex $\overline{L_1L_2}$ there is a unique line P which is perpendicular to both L_1 and L_2, namely*

$$P \equiv L_1^\perp L_2^\perp = (L_1L_2)^\perp.$$

The line P is the **perpendicular line** of $\overline{L_1L_2}$. It also may happen that P passes through L_1L_2, which occurs precisely when L_1L_2 is a null point.

As we are in a projective setting, the definition of a (hyperbolic) quadrangle (quadrilateral respectively,) will come from the projective definition of a complete quadrangle (quadrilateral respectively.)

A **quadrangle** $\overline{a_1a_2a_3a_4}$ is a set $\{a_1, a_2, a_3, a_4\}$ of points which has the property that no three are collinear. A **quadrilateral** $\overline{L_1L_2L_3L_4}$ is a set $\{L_1, L_2, L_3, L_4\}$ of lines which has the property that no three are concurrent.

The quadrangle $\square \equiv \overline{a_1a_2a_3a_4}$ has a **dual quadrilateral** $\square^\perp \equiv \overline{a_1^\perp a_2^\perp a_3^\perp a_4^\perp}$ consisting of four **dual lines** of the quadrangle, namely $a_1^\perp, a_2^\perp, a_3^\perp$ and $a_4^\perp$.

The quadrilateral $\lozenge \equiv \overline{L_1L_2L_3L_4}$ has a **dual quadrangle** $\lozenge^\perp \equiv \overline{L_1^\perp L_2^\perp L_3^\perp L_4^\perp}$ consisting of four **dual points** of the quadrilateral, namely $L_1^\perp, L_2^\perp, L_3^\perp$ and $L_4^\perp$.

There are 6 distinct sides of a quadrangle, namely $\overline{a_1a_2}, \overline{a_3a_4}, \overline{a_1a_3}, \overline{a_2a_4}, \overline{a_1a_4}$ and $\overline{a_2a_3}$. We can naturally divide these 6 sides into 3 pairs, $\{\overline{a_1a_2}, \overline{a_3a_4}\}$, $\{\overline{a_1a_3}, \overline{a_2a_4}\}$, and $\{\overline{a_1a_4}, \overline{a_2a_3}\}$. The intersection of these pairs of sides give three new points called the **diagonal points** of the quadrangle.
Similarly there are 6 distinct vertices of a quadrilateral, namely $\overline{L_1L_2}, \overline{L_3L_4}, \overline{L_1L_3}, \overline{L_2L_4}, \overline{L_1L_4}$and $\overline{L_2L_3}$. These too have a natural divide into 3 pairs, $\{\overline{L_1L_2}, \overline{L_3L_4}\}$, $\{\overline{L_1L_3}, \overline{L_2L_4}\}$, and $\{\overline{L_1L_4}, \overline{L_2L_3}\}$. The join of these pairs of vertices give three new lines called the **diagonal lines** of the quadrilateral.

Theorem 22 (Diagonal triangle) *The diagonal points* $d_1 \equiv (a_1a_4)(a_2a_3)$, $d_2 \equiv (a_2a_4)(a_1a_3)$ *and* $d_3 \equiv (a_3a_4)(a_1a_2)$ *of the quadrangle* $\square \equiv \overline{a_1a_2a_3a_4}$ *form the triangle* $\triangle \equiv \overline{d_1d_2d_3}$.

Proof.

- Use join of points and meets of lines to write out each d_i.
- Condition for d_is to be collinear
- a_is non collinear follows d_is non collinear.

■

Theorem 23 (Diagonal trilateral) *The diagonal lines* $D_1 \equiv (L_1L_4)(L_2L_3)$, $D_2 \equiv (L_2L_4)(L_1L_3)$ *and* $D_3 \equiv (L_3L_4)(L_1L_2)$ *of the quadrilateral* $\lozenge \equiv \overline{L_1L_2L_3L_4}$ *form the trilateral* $\nabla \equiv \overline{D_1D_2D_3}$.

Proof. Dual to the previous theorem. ■

2.1 Geogebra Tools

Tools that I've made to be used in GeoGebra for use in exploring Universal Hyperbolic geometry. Firstly I will present a list of all the tool that I have created within GeoGebra (using it's tool creating system), and then I will present how I created them.

- Polar Line (Polar)
- Pole Point (Pole)
- Reflections (Reflect_PiP, Reflect_PiL, Reflect_LiP, Reflect_LiL)
- Midpoints
- Sydpoint
- Smydpoint
- Diagonal Triangle (DiagTri)

Polar Line

For the **Polar** tool I used the pre-installed *Polar or Diameter Line* tool in GeoGebra to create the Polar Line to a Point, by selecting the given point and then the absolute conic, *Polar(*l, c*)*.

Pole Point

The **Pole** tool was created by once again using the pre-installed *Polar or Diameter Line* tool on GeoGebra, and is used for producing a Pole point to a line. Given a line L and the absolute conic c, using *Polar or Diameter Line* on L and c produces a new line L_1 which is perpendicular (in the UHG sense) to L and a diameter of c. The *Intesection of two Objects* tool on L and L_1 gives the point $l_1 = (LL_1)$. Now using *Polar* on l_1 gives the line L_2 perpendicular to L_1 and parallel to L. The point $l = (L_1L_2) = L^{\perp}$ is the pole of L.
Pole is used by selecting the given line L and then the absolute conic c, *Pole(*L, c*)*.

Reflections

The **Reflect_PiP** is a tool to used to find the reflection of a point in a point. Given a reflecting point a, reflecter point b and absolute c. First create the line (ab) through the *Line Between two Points* tool. Then use *Polar or Diameter Line* with (ab) and c to create D_1. If D_1 intersects c (then it will intersect c twice), we are able to construct a cycle quadrangle from one of the intersection points and the points a and b. We do this by choosing one of the intersection points, call it i_1 and then join it to both a and b, giving the lines I_2 and I_1 respectfully. The lines I_2 and I_1 will then intersect c once more, say at i_2 and i_4 respectfully. If we let $I_3 = i_2b$ then we get another null point $i_3 \in \{I_3c\}$. These four points give us the cyclic quadrangle $\overline{i_1i_2i_3i_4}$ whose diagonal points either lie on $B = b^{\perp}$ or are b itself. Finally the reflection of a in b is $a'_1 = I_4(ab) = (i_3i_4)(ab)$.

If D_1 does not intersect c (possible if c is a hyperbola) then we use *Polar or Diameter Line* with D_1 and c to create another diameter D_2 which is perpendicular to D_1. We can then use the same constuction as above to create a_2'.
If both D_1 and D_2 intersect c then clearly $a_1' = a_2'$. Thus to cover all cases and ensure that the tool produces one point we use the inbuilt *If, Then Logic* tool as follows, if a_1' is defined then $a' = a_1'$ otherwise $a' = a_2$.
Since the reflection in the point a is equivalent to the reflection in the line $A = a^{\perp}$ the rest of the tools **Reflect_PiL, Reflect_LiP** and **Reflect_LiL** were created using the *Polar, Pole* and *Reflect_ PiP* tools.