Histograms of Images

- Histograms in general are frequency distributions, and histograms of images describe the frequency of the intensity values that occur in an image.
- A histogram h for a grayscale image I with intensity values in the range $I(x, y) \in [0, K-1]$ would contain exactly K entries,
- h(i) = the number of pixels in I with the intensity value I for all $0 \leq i < K$.
- More formally stated, $h(i) = \text{card}\{(x, y) \mid I(x, y) = i\}$

Histograms …

- Since a histogram encodes no information about where each of its individual entries originated in the image, histograms contain no information about the spatial arrangement of pixels in the image.

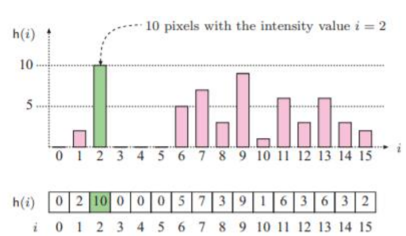


Figure Histogram vector for an image with $K = 16$ possible intensity values. The indices of the vector element $i = 0 \ldots 15$ represent intensity values. The value of 10 at index 2 means that the image contains 10 pixels of intensity value 2.

Histograms …

- Is it possible to reconstruct an image using only its histogram?

Figure Three very different images with identical histograms.

Given the loss of spatial information, it is not possible to reconstruct an image using its histogram

Image Acquisition

Exposure

A histogram where a large span of the intensity range at one end is largely unused while the other end is crowded with high-value peaks is representative of an improperly exposed image.

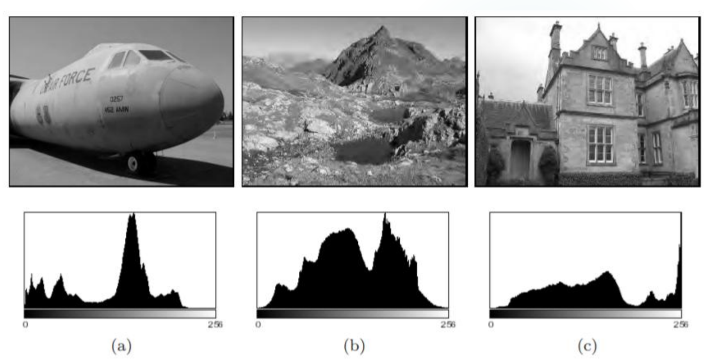


Figure Exposure errors are readily apparent in histograms. Underexposed (a), properly exposed (b), and overexposed (c) photographs.

Image Acquisition …

Contrast

Contrast is understood as the range of intensity values effectively used within a given image, that is the difference between the image's maximum and minimum pixel values. A full-contrast image makes effective use of the entire range of available intensity values from a = 0…K − 1.

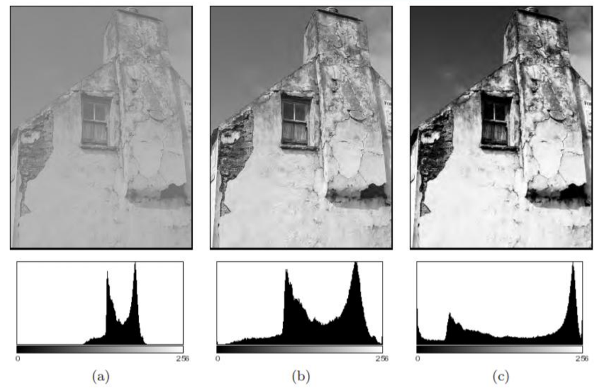


Figure How changes in contrast affect a histogram: low contrast (a), normal contrast (b), high contrast (c).

Image Defects

Saturation

Illumination outside of the sensor's contrast range, arising for example from glossy highlights and especially dark parts of the scene, cannot be captured and is lost. The result is a histogram that is saturated at one or both ends of its range. The illumination values lying outside of the sensor's range are mapped to its minimum or maximum values and appear on the histogram as significant spikes at the tail ends.

Histogram Binning

Normally histograms are computed in order to visualize the image's distribution on the screen. When an image uses a larger range of values, for instance 16- and 32-bit or floating-point images, then the growing number of necessary histogram entries makes this no longer practical. Instead let a given entry in the histogram represent a range of intensity values. This technique is often referred to as "binning" since you can visualize it as collecting a range of pixel values in a container such as a bin or bucket. In a binned histogram of size B, each bin h(j) contains the number of image elements having values within the interval $a_j \leq a < a_{j+1}$, and therefore

$$\mathsf{h}(j) = \mathrm{card}\,\{(u,v) \mid a_j \leq I(u,v) < a_{j+1}\}, \quad \text{for } 0 \leq j < B.$$

Histogram Binning …

Typically the range of possible values in B is divided into bins of equal size $k_B = K/B$ such that the starting value of the interval j is

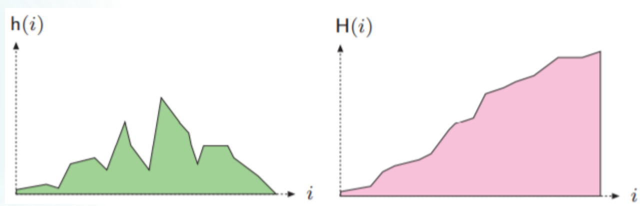
$$a_j = j \cdot \frac{K}{B} = j \cdot k_B.$$

As an index to the appropriate histogram bin h(j), we require an integer value

$$j = \left\lfloor \frac{I(u,v) \cdot B}{K} \right\rfloor$$

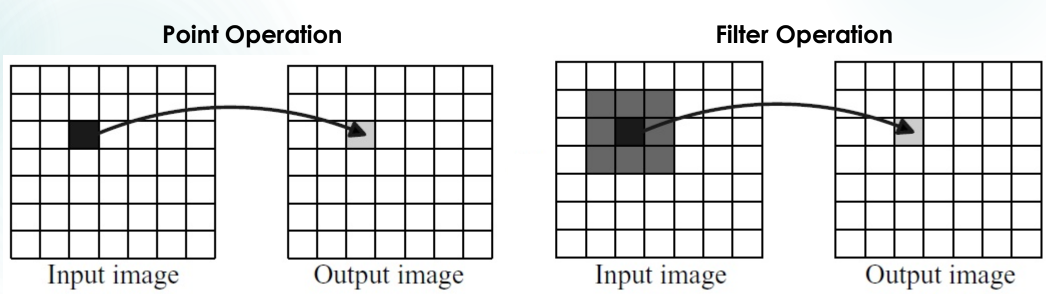
Cumulative Histogram

- The cumulative histogram H is defined as: $\mathsf{H}(i) = \sum_{j=0}^{i} \mathsf{h}(j) \quad \text{for } 0 \leq i < K.$
- Alternatively, we can define H recursively $\mathsf{H}(i) = \begin{cases} \mathsf{h}(0) & \text{for } i = 0 \\ \mathsf{H}(i-1) + \mathsf{h}(i) & \text{for } 0 < i < K. \end{cases}$



Point Operations

Point operations perform a modification of the pixel values without changing the size, geometry, or local structure of the image. The original pixel values are mapped to the new values by a function f(a), $a' \leftarrow f(a)$



Modifying Image Intensity

- **Contrast and Brightness**

Increasing the image's contrast by 50% (i. e., by the factor 1.5) or raising the brightness by 10 units can be expressed by the mapping functions:

$$f_{\text{contr}}(a) = a \cdot 1.5 \qquad \text{and} \qquad f_{\text{bright}}(a) = a + 10$$

Limiting the Results by Clamping

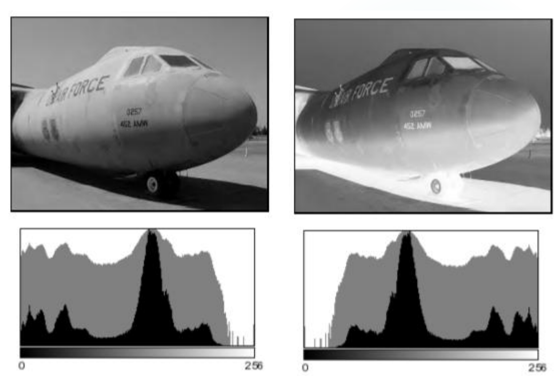
if (a > 255) a = 255;

if (a < 0) a = 0;

Modifying Image Intensity …

Inverting Images

$$f_{\text{invert}}(a) = -a + a_{\max} = a_{\max} - a.$$

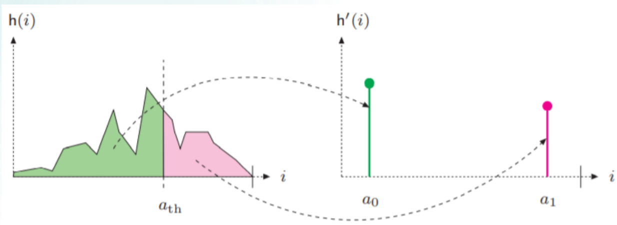


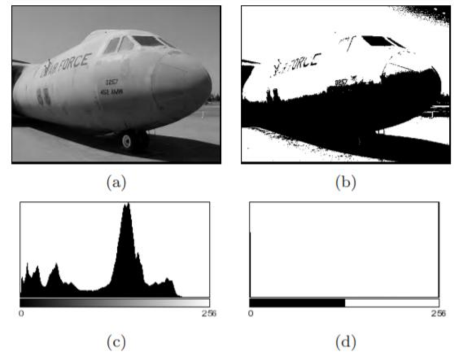
Modifying Image Intensity …

Threshold Operation

$$f_{\text{threshold}}(a) = \begin{cases} a_0 & \text{for } a < a_{\text{th}} \\ a_1 & \text{for } a \geq a_{\text{th}} \end{cases}$$

with $0 < a_{\text{th}} \leq a_{\max}$.

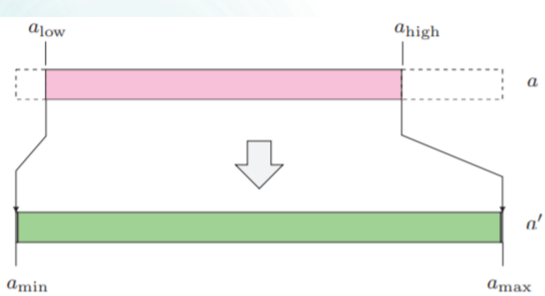




Automatic Contrast Adjustment

Is to modify the pixels such that the available range of values is fully covered. This is done by mapping the current darkest and brightest pixels to the lowest and highest available intensity values, respectively, and linearly distributing the intermediate values.

The mapping function for the auto-contrast operation is thus,



$$f_{\text{ac}}(a) = a_{\min} + (a - a_{\text{low}}) \cdot \frac{a_{\max} - a_{\min}}{a_{\text{high}} - a_{\text{low}}}$$

Modified Auto-Contrast Adjustment

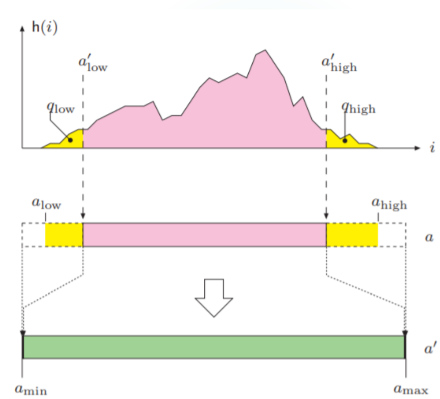
In practice, the mapping function in auto-contrast adjustment could be strongly influenced by only a few extreme (low or high) pixel values, which may not be representative of the main image content. This can be avoided to a large extent by "saturating" a fixed percentage of pixels at the upper and lower ends of the target intensity range.

Modified Auto-Contrast Adjustment …

$$f_{\mathrm{mac}}(a) = \begin{cases} a_{\min} & \text{for } a \leq a'_{\mathrm{low}} \\ a_{\min} + (a - a'_{\mathrm{low}}) \cdot \dfrac{a_{\max} - a_{\min}}{a'_{\mathrm{high}} - a'_{\mathrm{low}}} & \text{for } a'_{\mathrm{low}} < a < a'_{\mathrm{high}} \\ a_{\max} & \text{for } a \geq a'_{\mathrm{high}}. \end{cases}$$

$$\begin{aligned} a'_{\mathrm{low}} &= \min\{\, i \mid \mathsf{H}(i) \geq M \cdot N \cdot q_{\mathrm{low}} \}, \\ a'_{\mathrm{high}} &= \max\{\, i \mid \mathsf{H}(i) \leq M \cdot N \cdot (1 - q_{\mathrm{high}}) \}, \end{aligned}$$

where $0 \leq q_{\mathrm{low}}, q_{\mathrm{high}} \leq 1$, $q_{\mathrm{low}} + q_{\mathrm{high}} \leq 1$



Histogram Equalization

The goal of histogram equalization is to find and apply a point operation such that the histogram of the modified image approximates a uniform distribution.

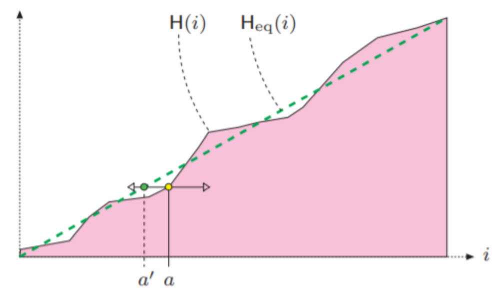
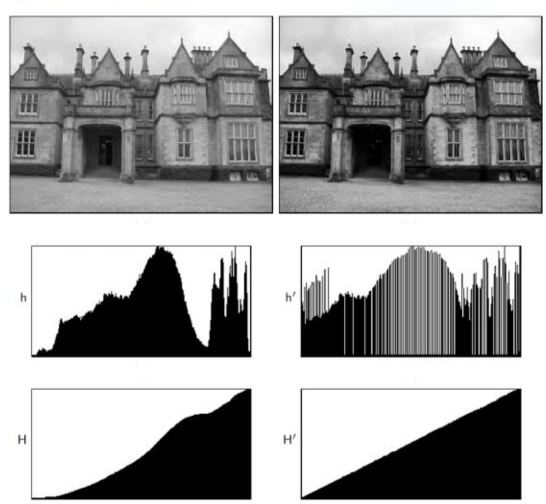


Figure Histogram equalization on the cumulative histogram. A suitable point operation $a' \leftarrow f_{\text{eq}}(a)$ shifts each histogram line from its original position a to a' (left or right) such that the resulting cumulative histogram H_{eq} is approximately linear.

Histogram Equalization …

The desired point operation $f_{\text{eq}}()$ is simply obtained from the cumulative histogram H of the original image as

$$f_{\text{eq}}(a) = \left\lfloor \mathsf{H}(a) \cdot \frac{K-1}{MN} \right\rfloor$$



Principle of Histogram Specification

The goal of histogram specification is to modify a given image I_A by some point operation such that its distribution function P_A matches a reference distribution P_R as closely as possible.

$$a' = \mathsf{P}_R^{-1}(\mathsf{P}_A(a))$$

$$f_{\mathrm{hs}}(a) = a' = \mathsf{P}_R^{-1}(\mathsf{P}_A(a))$$

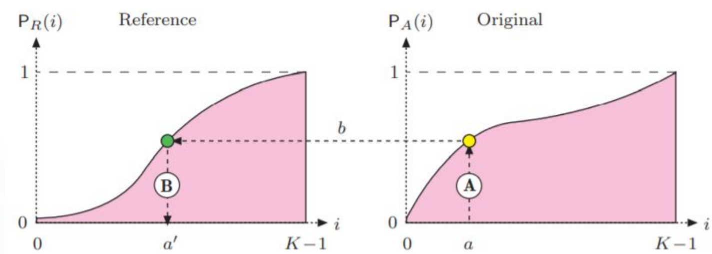


Figure Principle of histogram specification. Given is the reference distribution P_R (left) and the distribution function for the original image P_A (right). The result is the mapping function $f_{\mathrm{hs}}: a \to a'$ for a point operation, which replaces each pixel a in the original image I_A by a modified value a'. The process has two main steps: A For each pixel value a, determine $b = \mathsf{P}_A(a)$ from the right distribution function. B a' is then found by inverting the left distribution function as $a' = \mathsf{P}_R^{-1}(b)$. In summary, the result is $f_{\mathrm{hs}}(a) = a' = \mathsf{P}_R^{-1}(\mathsf{P}_A(a))$.

Adjusting to a Piecewise Linear Distribution

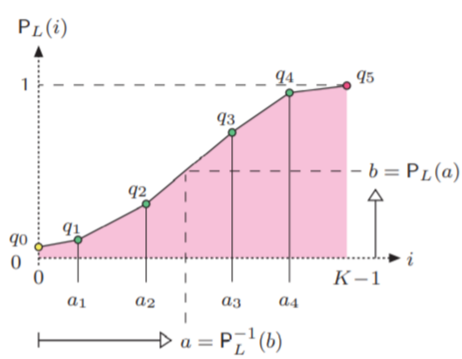


Figure Piecewise linear reference distribution. The function $\mathsf{P}_L(i)$ is specified by $N = 5$ control points $\langle 0, q_0\rangle, \langle a_1, q_1\rangle, \ldots \langle a_4, q_4\rangle$, with $a_k < a_{k+1}$ and $q_k < q_{k+1}$. The final point q_5 is fixed at $\langle K-1, 1\rangle$.

$$\mathsf{P}_L(i) = \begin{cases} q_m + (i - a_m)\cdot\dfrac{(q_{m+1} - q_m)}{(a_{m+1} - a_m)} & \text{for } 0 \le i < K-1 \\ 1 & \text{for } i = K-1, \end{cases}$$

Histogram Matching

If we want to adjust one image to the histogram of another image, the reference distribution function PR(i) is not continuous and thus, in general, cannot be inverted.

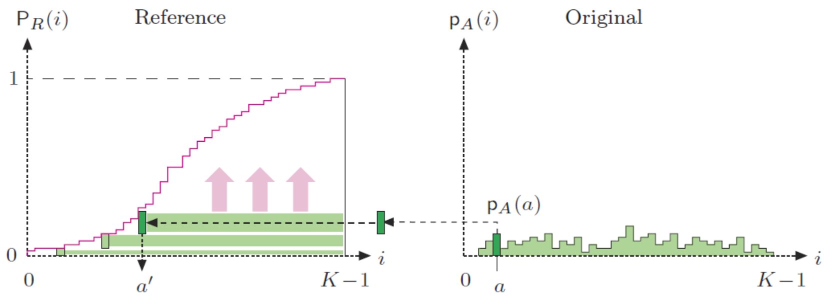


Figure Discrete histogram specification. The reference distribution P_R (left) is "filled" layer by layer from bottom to top and from right to left. For every possible intensity value a (starting from $a = 0$), the associated probability $\mathsf{p}_A(a)$ is added as a horizontal bar to a stack accumulated 'under" the reference distribution P_R. The bar with thickness $\mathsf{p}_A(a)$ is drawn from right to left down to the position a', where the reference distribution P_R is reached. This value a' is the one which a should be mapped to by the function $f_{\mathrm{hs}}(a)$.

Example: Histogram Equalization

Suppose that a 64×64, 8-level image has the gray-level distribution as shown in the table given below:

Gray-level	0	1	2	3	4	5	6	7
Frequency	790	1023	850	656	329	245	122	81

With regard to the gray-level distributions answer the following:

i. Describe the *histogram equalisation* method used for digital image enhancement.

ii. Draw the equalised histogram of these gray-levels showing your work in detail.

iii. State briefly how the final image will differ from the input image by considering their histograms.

Histogram Equalization …

Step 1: Obtain $p_r(\mathrm{r})$ from input image.

Step 2: Obtain values of s = T(r).

Step 3: Obtain G(z) from specified pdf

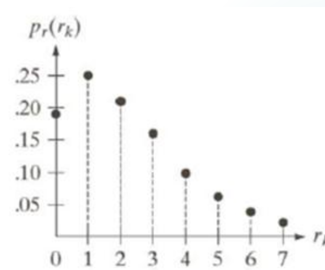
Step 4: Obtain inverse transformation, z = $G^{-1}(s)$. Since z is obtained from s, this is an s-to-z mapping

Step 5: obtain pixel image by equalizing input image (to uniform PDF); for each pixel with value s, obtain z = $G^{-1}(s)$ to get output image pixel value.

Histogram Equalization …

Gray-level	frequency	$p_r(r_k) = \frac{n_k}{M.N}$
0	790	0.19
1	1023	0.25
2	850	0.21
3	656	0.16
4	329	0.08
5	245	0.06
6	122	0.03
7	81	0.02

where M.N = 4096



Histogram Equalization …

(ii). Histogram equalisation

$$\bullet s_0 = T(r_0) = 7 \cdot \sum_{j=0}^{0} p_r(r_j) = 7 \cdot p_r(0) = 1.33 \rightarrow 1$$

$$\bullet s_1 = T(r_1) = 7 \cdot \sum_{j=0}^{1} p_r(r_j) = 7 \cdot (p_r(0) + p_r(1)) = 3.08 \rightarrow 3$$

$$\bullet s_2 = T(r_2) = 7 \cdot \sum_{j=0}^{2} p_r(r_j) = 7 \cdot (p_r(0) + p_r(1) + p_r(2)) = 4.55 \rightarrow 5$$

$$s_3 = 5.67 \rightarrow 6,\ s_4 = 6.23 \rightarrow 6, s_5 = 6.65 \rightarrow 7, s_6 = 6.86 \rightarrow 7, s_7 = 7.00 \rightarrow 7$$

Final Transform

Histogram Equalization …

$r_0 \rightarrow s_0 = 1 \Rightarrow 790$ pixels map to 1

$r_1 \rightarrow s_1 = 3 \Rightarrow 1023$ pixels map to 3

$r_2 \rightarrow s_2 = 5 \Rightarrow 850$ pixels map to 5

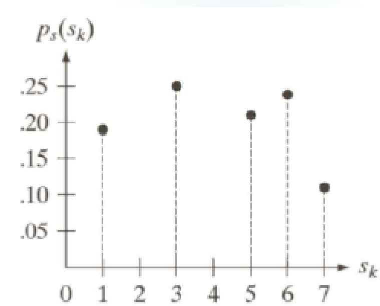
$r_3 \rightarrow s_3 = 6 \Rightarrow 656 + 329 = 985$ pixels map to 6

$r_4 \rightarrow s_4 = 6 \Rightarrow 656 + 329 = 985$ pixels map to 6

$r_5 \rightarrow s_5 = 7 \Rightarrow 245 + 122 + 81 = 458$ pixels map to 7

$r_6 \rightarrow s_6 = 7 \Rightarrow 245 + 122 + 81 = 458$ pixels map to 7

$r_7 \rightarrow s_7 = 7 \Rightarrow 245 + 122 + 81 = 458$ pixels map to 7



The original image is dark image whereas, the final image is of high contrast.

Histogram Matching …

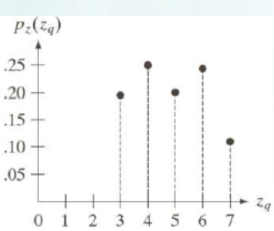
Use the *histogram matching* technique to obtain an enhanced image by using the target probability density function as given in the table below, and draw the histogram of the enhanced image with same gray-level range.

z_k	0	$^1/_7$	$^2/_7$	$^3/_7$	$^4/_7$	$^5/_7$	$^6/_7$	1
$p(z_k)$	0	0	0	0.15	0.2	0.3	0.2	0.15

Histogram Matching …

z_k	$p_z(z_k)$	z_q	G(z)
0	0	0	0
1	0	0	0
2	0	0	0
3	0.15	0.19	1
4	0.2	0.25	2
5	0.3	0.21	5
6	0.2	0.24	6
7	0.15	0.11	7

s_k	z_q
1	3
3	4
5	5
6	6
7	7



- first obtain scaled histogram equalized values:
  $s_0 = 1;\ s_1 = 3;\ s_2 = 5;\ s_3 = 6;\ s_4 = 6;\ s_{5,6,7} = 7$
- next compute and round all values of transformation G:
  $G(z_0) = 0 \to 0;\ G(z_1) = 0 \to 0;\ G(z_2) = 0 \to 0;$
  $G(z_3) = 1.05 \to 1;\ G(z_4) = 2.45 \to 2;\ G(z_5) = 4.55 \to 5;$
  $G(z_6) = 5.95 \to 6;\ G(z_7) = 7 \to 7$
- need to find smallest value of z_q so that $G(z_q)$ is closest to s_k; do this for all s_k to create required mapping:
  $s_1 \to z_3;\ s_3 \to z_4;\ s_5 \to z_5;\ s_6 \to z_6;\ s_7 \to z_7$

Alpha blending

Is a simple method for transparently overlaying two images, I_{BG} and I_{FG}. The background image I_{BG} is covered by the foreground image I_{FG}, whose transparency is controlled by the value α in the form:

$$I'(u,v) \leftarrow \alpha \cdot I_{\mathrm{BG}}(u,v) + (1-\alpha) \cdot I_{\mathrm{FG}}(u,v)$$