Question 3

Over all real numbers find the minimum value of a positive real number y such that

$$y = \sqrt{(x + 6)^2 + 25} + \sqrt{(x - 6)^2 + 121}$$

Simplify y

$$y = \sqrt{x^2 + 12x + 61} + \sqrt{x^2 - 6x + 157}$$

Find the derivative of y

Using function of function where

$$\frac{dy}{dx} = \frac{du}{dx}\ x\ \frac{dy}{du}$$

$$Let\ u = x^2 + 12x + 61$$

$$\frac{du}{dx} = 2x + 12$$

$$Let\ y = u^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{-1}{2}}$$

$$\frac{dy}{dx} = 2x + 12 * \frac{1}{2\sqrt{u}}$$

$$\frac{dy}{dx} = \frac{2x + 12}{2\sqrt{u}}$$

$$\frac{dy}{dx} = \frac{2x + 12}{2\sqrt{x^2 + 12x + 61}}$$

$$\frac{dy}{dx} = \frac{x + 6}{\sqrt{x^2 + 12x + 61}}$$

$$Let\ u = x^2 - 6x + 157$$

$$\frac{du}{dx} = 2x - 6$$

$$Let\ y = u^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{-1}{2}}$$

$$\frac{dy}{dx} = 2x - 6 * \frac{1}{2\sqrt{u}}$$

$$\frac{dy}{dx} = \frac{2x - 6}{2\sqrt{u}}$$

$$\frac{dy}{dx} = \frac{2x - 6}{2\sqrt{x^2 - 12x + 157}}$$

$$\frac{dy}{dx} = \frac{x - 6}{\sqrt{x^2 - 12x + 157}}$$

Therefore:

$$\frac{dy}{dx} = \frac{x + 6}{\sqrt{x^2 + 12x + 61}} + \frac{x - 6}{\sqrt{x^2 - 12x + 157}}$$

$$When\frac{dy}{dx} = 0\ at\ inflexion\ point$$

$$\frac{x + 6}{\sqrt{x^2 + 12x + 61}} + \frac{x - 6}{\sqrt{x^2 - 12x + 157}} = 0$$

$$\frac{x + 6}{\sqrt{x^2 + 12x + 61}} = -\frac{x - 6}{\sqrt{x^2 - 12x + 157}}$$

Square both sides:

$$\left(\frac{x + 6}{\sqrt{x^2 + 12x + 61}}\right)^2 = -\left(\frac{x - 6}{\sqrt{x^2 - 12x + 157}}\right)^2$$

$$\frac{(x + 6)^2}{x^2 + 12x + 61} = \frac{(x - 6)^2}{x^2 - 12x + 157}$$

Simplify the equation:

$$(x + 6)^2x^2 - 12x + 157 = (x - 6)^2x^2 + 12x + 61$$

$$(x^2 + 12x + 36)x^2 - 12x + 157 = (x^2 - 12x + 36)x^2 + 12x + 61$$

$$x^4 - 12x^3 + 157x^2 + 12x^3 - 144x^2 + 1884x + 36x^2 - 432x + 5652$$
$$= x^4 + 12x^3 + 61x^2 - 12x^3 - 144x^2 - 732x + 36x^2 + 432x + 2196$$

$$x^4 + 49x^2 + 1452x + 5652 = x^4 - 47x^2 - 300x + 2196$$

Equate the above equation to 0:

$$x^4 + 49x^2 + 1452x + 5652 - x^4 + 47x^2 + 300x - 2196 = 0$$

Group like terms:

$$x^4 - x^4 + 49x^2 + 47x^2 + 1452x + 300x + 5652 - 2196 = 0$$

$$96x^2 + 1752x + 3456 = 0$$

To find the roots of the above equation divide all terms by 24:

$$4x^2 + 438x + 864 = 0$$

Using Quadratic Formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where b = 73, a = 4 and c = 144

$$\frac{-73 \pm \sqrt{73^2 - 4 * 4 * 144}}{2a}$$

$$\frac{-73 \pm \sqrt{5329 - 2304}}{8}$$

$$\frac{-73 \pm 55}{8}$$

$$\frac{-73 + 55}{8} or \frac{-73 - 55}{8}$$

$$\frac{-18}{8} or \frac{-128}{8}$$

$$\frac{-9}{4} or - 16$$

$Therefore\ x\ = \boldsymbol{\frac{-9}{4}} or - \mathbf{16}$

Substituting the values of x into the first derivative

When x = $\boldsymbol{\frac{-9}{4}}$

$$\frac{\frac{-9}{4} + 6}{\sqrt{\frac{-9}{4}^2 + 12\frac{-9}{4} + 61}} + \frac{\frac{-9}{4} - 6}{\sqrt{\frac{-9}{4}^2 - 12\frac{-9}{4} + 157}} = 0$$

$$\frac{\frac{15}{4}}{\sqrt{\frac{81}{16}+\frac{-108}{4}+61}}+\frac{\frac{-33}{4}}{\sqrt{\frac{81}{16}-\frac{-108}{4}+157}}=0$$

$$\frac{\frac{15}{4}}{\sqrt{\frac{81}{16}+\frac{-108}{4}+61}}+\frac{\frac{-33}{4}}{\sqrt{\frac{81}{16}-\frac{-108}{4}+157}}=0$$

$$\frac{\frac{15}{4}}{\sqrt{\frac{625}{16}}}+\frac{\frac{-33}{4}}{\sqrt{\frac{3025}{16}}}=0$$

$$\frac{\frac{15}{4}}{\frac{25}{4}}+\frac{\frac{-33}{4}}{\frac{55}{4}}=0$$

$$\frac{15}{4}*\frac{4}{25}+\frac{-33}{4}*\frac{4}{55}=0$$

$$\frac{15}{25}+\frac{-33}{55}= 0$$

$$\frac{15}{25}+\frac{-33}{55}= 0$$

$$\frac{825-825}{825}= 0$$

$$\frac{0}{825}= 0$$

Since 0 = 0

x = $\frac{-9}{4}$ satisfies the equation

When x = -16

$$\frac{-16+6}{\sqrt{-16^2+12*-16+61}}+\frac{-16-6}{\sqrt{x^2-12*-16+157}}=0$$

$$\frac{-10}{\sqrt{256 - 192 + 61}} + \frac{-22}{\sqrt{256 + 192 + 157}} = 0$$

$$\frac{-10}{\sqrt{125}} + \frac{-22}{\sqrt{605}} = 0$$

$$\frac{-10}{5\sqrt{5}} + \frac{-22}{11\sqrt{5}} = 0$$

$$\frac{-2}{\sqrt{5}} + \frac{-2}{\sqrt{5}} = 0$$

$$\frac{-4\sqrt{5}}{\sqrt{5}} = 0$$

$\frac{-4\sqrt{5}}{\sqrt{5}}$ *is not equal to 0*

Therefore $\frac{-4\sqrt{5}}{\sqrt{5}}$ *doesn't satisfy the equation*

Substitute $x = \frac{-9}{4}$ in $y = \sqrt{(x + 6)^2 + 25} + \sqrt{(x - 6)^2 + 121}$

$$y = \sqrt{\left(\frac{-9}{4} + 6\right)^2 + 25} + \sqrt{\left(\frac{-9}{4} - 6\right)^2 + 121}$$

$$y = \sqrt{\frac{625}{16}} + \sqrt{\frac{3025}{16}}$$

$$y = \frac{25}{4} + \frac{55}{4}$$

$$y = \frac{80}{4}$$

$$y = 20$$

Therefore:
The minimum value is y = **20** when x is = $\frac{-9}{4}$