

# Coin Tosses

A coin is taken from a box containing three coins, which give heads with probability  $p = 1/3, 1/2$ , and  $2/3$ . The mystery coin is tossed 80 times, resulting in 49 heads and 31 tails.

- a) What is the likelihood of this data for each type of coin? Which coin gives the maximum likelihood?
- b) Now suppose that we have a single coin with unknown probability  $p$  of landing heads. Find the likelihood and log likelihood functions given the same data. What is the maximum likelihood estimate for  $p$ ?

# Coin Tosses

- a) The data  $x$  is 49 heads in 80 tosses. We have three hypotheses: the coin has probability  $p = 1/3, p = 1/2, p = 2/3$ . So the likelihood function  $L_x : p \mapsto P(X = x | P = p)$  takes 3 values:

$$L_x\left(\frac{1}{3}\right) = \binom{80}{49} \left(\frac{1}{3}\right)^{49} \left(\frac{2}{3}\right)^{31} = 6.24 \cdot 10^{-7}$$

$$L_x\left(\frac{1}{2}\right) = \binom{80}{49} \left(\frac{1}{2}\right)^{49} \left(\frac{1}{2}\right)^{31} = 0.024$$

$$L_x\left(\frac{2}{3}\right) = \binom{80}{49} \left(\frac{2}{3}\right)^{49} \left(\frac{1}{3}\right)^{31} = 0.082$$

Thus the maximum likelihood is achieved under  $p = 2/3$ .

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- b) We already know that the MLE for the Bernoulli (and binomial) distribution is  $\frac{k}{n}$  where  $k$  is the number of successes and  $n$  is the total number of Bernoulli trials. Thus we get

$$p^* = \frac{49}{80} = 0.6125$$

as the MLE in the present example.

# Geometric MLE

For an i.i.d. data set  $x = x_1^n$  find the MLE for the geometric distribution:

$$P(X = x) = (1 - \theta)^x \theta$$

# Geometric MLE

The likelihood function is

$$L_x(\theta) = \prod_{i=1}^n \theta(1 - \theta)^{x_i} = \theta^n (1 - \theta)^{\sum_{i=1}^n x_i}$$

and thus the log-likelihood is

$$\mathcal{L}_x(\theta) = n \log(\theta) + \sum_{i=1}^n x_i \log(1 - \theta) .$$

# Geometric MLE

In the next step we compute the score function.

$$\begin{aligned}\frac{d}{d\theta}\mathcal{L}_x(\theta) &= \frac{d}{d\theta}n\log(\theta) + \frac{d}{d\theta}\sum_{i=1}^n x_i \log(1 - \theta) \\ &= \frac{n}{\theta} - \frac{\sum_{i=1}^n x_i}{1 - \theta}\end{aligned}$$

# Geometric MLE

Setting this to 0 gives

$$0 = \frac{n}{\theta} - \frac{\sum_{i=1}^n x_i}{1 - \theta} \quad \Leftrightarrow$$

$$\frac{n}{\theta} = \frac{\sum_{i=1}^n x_i}{1 - \theta} \quad \Leftrightarrow$$

$$n - n\theta = \theta \sum_{i=1}^n x_i \quad \Leftrightarrow$$

$$\theta = \frac{n}{n + \sum_{i=1}^n x_i}$$