

# Theory Assignment 3 – Basic Probability, Computing and Statistics

Fall 2015, Master of Logic, University of Amsterdam

Submission deadline: Monday, September 21th, 2015, 9 a.m.

**Cooperation** Cooperation among students for both theory and programming exercises is strongly encouraged. However, after this discussion phase, every student writes down and submits his/her own individual solution.

**Guidelines** Starred exercises are relatively easy exercises for you to practice. No points are awarded for them. You may pick and choose **2 exercises from exercise type I**, as well as **1 from exercise types II and III** from your submission, i.e. you need to submit **a total of 4 exercises** to be able to get all points. Numbered exercises with an exclamation mark are supposed to be a bit harder and you may challenge yourself by trying to solve them.

In the directory of your private url there is folder called ‘theory\_submissions’. Please upload your submission there. Your submission should be a PDF-document (use a scanner for handwritten documents!) entitled *AssignmentX\_yourStudentNumber.pdf*, where  $X$  is the number of the assignment and *yourStudentNumber* is your student number. If your submission does not comply with this format, we will deduct 1 point. For each day that your submission is late, we deduct 2 points. N.B.: If multiple files are submitted for a single assignment before the deadline, the latest version will be graded.

If you have any question about the homework or if you need help, do not hesitate to contact [Thomas](#).

## Exercises

### Type I [2 exercises: 2 points per exercise]

1. Assume that 4 balls are randomly picked one after another from an urn of 20 balls, labeled from 1 to 20, without returning them. Let  $X$  stand for the largest label selected. What is  $P_X(X \geq 16)$ ?
2. Let  $X$  be a random variable taking any of the values 0, 5 or  $-5$  with probabilities  $P_X(X = -5) = .3, P_X(X = 0) = .3, P_X(X = 5) = .4$ , respectively. What is  $E_{P_X}[X^2]$ ?
3. Compute the probability mass function of the number of tails for 7 subsequent coin tosses.
4. A peddler is trying to sell her goods to two different clients. She estimates her first meeting to yield a profit with probability .4, and her second, independently, with probability .7. Any profit made is equally likely to be either from selling her luxury goods, which bring a profit of \$1000, her generic goods, bringing in a profit of \$500. Compute the probability mass function of  $X$ , where  $X$  is the total value of her profits.
5. Let  $E[X] = 5$  and  $var(X) = 7$ , compute
  - (i)  $E[(2 + X)^2]$
  - (ii)  $var(4 + 3X)$

- 6! Indicate the maximal number of people you can invite to your party so that the probability of any of them having the same birthday as you is less than  $\frac{1}{2}$ . Assume that birthdays are uniformly distributed and that we do not care about a persons birth year.

**Type II [1 exercise: 3 points per exercise]**

1. Calculate the variance of a loaded 6-sided die that has a probability of  $\frac{1}{6}$  for all odd numbered sides, and  $\mathbb{P}(2) = .1, \mathbb{P}(4) = .1, \mathbb{P}(6) = .3$ .
2. A jar contains  $N$  Euro and  $M$  GBP coins. Coins are taken out randomly up to the first draw of a GBP coin. If each drawn coin is put back before picking a new one, what is the probability that
  - (i) exactly  $n$  draws are needed?
  - (ii) at least  $k$  draws are needed?
3. You make a bet with a friend to the effect that he has to pay you an amount  $L$  should an event  $M$  happen within a year. If you estimate  $M$  to happen with probability  $q$  within this period, what should you charge him to enter the bet for an expected profit of 10% percent of  $L$ ?
4. Consider a group of  $n$  randomly chosen students and let  $E_{i,j}$  denote the event that students  $i$  and  $j$  have the same birthday,  $i \neq j$ . Under the assumption that the students' birthdays are uniformly distributed throughout the same year, compute
  - (i)  $\mathbb{P}(E_{c,d}|E_{a,b})$
  - (ii)  $\mathbb{P}(E_{a,c}|E_{a,b})$
  - (iii)  $\mathbb{P}(E_{b,c}|E_{a,b} \cap E_{a,c})$
5. Let  $X$  and  $Y$  be two random variables with joint distribution  $P_{XY}$ . Show that for arbitrary  $a, b \in \mathbb{R}$  it holds that  $E_{XY}[aX + bY] = aE_X[X] + bE_Y[Y]$ .

**Type III [1 exercise: 3 points per exercise]**

1. For a RV  $X \sim \text{binom}(n, \theta)$  with  $n \in \mathbb{N}, n > 0$  and  $\theta \in [0, 1]$  calculate
  - (a) the expectation  $E[X]$  and
  - (b) the variance  $\text{var}(X)$ .

Notice that the computation should be general, i.e. neither  $n$  nor  $\theta$  should be fixed.

2. Assume that a number of independent trials, each with a probability of success of  $p$ ,  $0 < p < 1$ , are performed until  $q$  successes are registered. Let  $X$  be equal to the number of trials required, then

$$P(X = n) = \binom{n-1}{q-1} p^q (1-p)^{n-q} \quad n = q, q+1, \dots$$

Any RV  $X$  whose pmf is given by the above is said to be a *negative binomial RV* with parameters  $(q, p)$ . Compute the expectation and variance of this RV with parameters  $(q, p)$ .