Geometric EM

You are given a mixture model with mixture components c_1, c_2 which are linked to geometric distributions with parameters $\theta_{c_1} = 0.2, \theta_{c_2} = 0.6$. You observe the data set

$$\{0, 2, 2, 3\} \ .$$

Assume that the latent variables are i.i.d. and that $P(Y = c_1 | \Theta = \theta^{(0)}) = 0.2$.

a) What is the (marginal) log-likelihood of this data set under the model? Feel free to use calculators.

b) Find the most likely mixture component for each data point.

c) Perform one EM iteration.

d) Compute the marginal log-likelihood of the data with the updated parameters. The new value should be higher than the one computed in the beginning.

Geometric EM

a)

$$\log(0.2 \cdot 0.2 \cdot 0.8^0 + 0.8 \cdot 0.6 \cdot 0.4^0)$$
$$+ 2 \cdot \log(0.2 \cdot 0.2 \cdot 0.8^2 + 0.8 \cdot 0.6 \cdot 0.4^2)$$
$$+ \log(0.2 \cdot 0.2 \cdot 0.8^3 + 0.8 \cdot 0.6 \cdot 0.4^3) = -8.1836793$$

b) $P(Y = c_1 | X = 0, \Theta = \theta^{(0)}) = \dfrac{0.2 \cdot 0.2 \cdot 0.8^0}{0.2 \cdot 0.2 \cdot 0.8^0 + 0.8 \cdot 0.6 \cdot 0.4^0} =$ 0.0769231

$$P(Y = c_1 | X = 2, \Theta = \theta^{(0)}) = \frac{0.2 \cdot 0.2 \cdot 0.8^2}{0.2 \cdot 0.2 \cdot 0.8^2 + 0.8 \cdot 0.6 \cdot 0.4^2} = 0.25$$

$$P(Y = c_1 | X = 3, \Theta = \theta^{(0)}) = \frac{0.2 \cdot 0.2 \cdot 0.8^3}{0.2 \cdot 0.2 \cdot 0.8^3 + 0.8 \cdot 0.6 \cdot 0.4^3} = 0.4$$

Geometric EM

c) **E-step**

$$\mathbb{E}[\sum_i \mathbb{1}\left(Y_i = c_1\right) \mid X = x, \Theta = \theta^{(0)}] = 0.0769231 + 2 \cdot 0.25 + 0.4$$

$$= 0.9769231$$

$$\mathbb{E}[\sum_i \mathbb{1}\left(Y_i = c_2\right) \mid X = x, \Theta = \theta^{(0)}] = 4 - 0.9769231 = 3.0230769$$

$$\mathbb{E}[\sum_i x_i \mathbb{1}\left(Y_i = c_1\right) \mid X = x, \Theta = \theta^{(0)}] = 0.0769231 \cdot 0 + 2 \cdot 0.25 \cdot 2 + 0.4 \cdot 3$$

$$= 2.2$$

$$\mathbb{E}[\sum_i x_i \mathbb{1}\left(Y_i = c_2\right) \mid X = x, \Theta = \theta^{(0)}] = 0.9230769 \cdot 0 + 2 \cdot 0.75 \cdot 2 + 0.6 \cdot 3$$

$$= 4.8$$

Geometric EM

M-step

$$P(Y = c_1 \mid \Theta = \theta^{(1)}) = \frac{0.9769231}{0.9769231 + 3.0230769} = 0.2442308$$

$$P(Y = c_2 \mid \Theta = \theta^{(1)}) = 1 - 0.3231552 = 0.7557692$$

new parameter of c_1 : $\dfrac{0.9769231}{2.2 + 0.9769231} = 0.3075061$

new parameter of c_2 : $\dfrac{3.0230769}{4.8 + 3.0230769} = 0.3864307$

Geometric EM

d)

$$\begin{aligned}
&\log(0.2442308 \cdot 0.3075061 \cdot 0.6924939^0 \\
&+ 0.7557692 \cdot 0.3864307 \cdot 0.6135693^0) \\
&+ 2 \cdot \log(0.2442308 \cdot 0.3075061 \cdot 0.6924939^2 \\
&+ 0.7557692 \cdot 0.3864307 \cdot 0.6135693^2) \\
&+ \log(0.2442308 \cdot 0.3075061 \cdot 0.6924939^3 \\
&+ 0.7557692 \cdot 0.3864307 \cdot 0.6135693^3) = -7.2323861
\end{aligned}$$