Coin Tosses

A coin is taken from a box containing three coins, which give heads with probability $p = 1/3, 1/2$, and $2/3$. The mystery coin is tossed 80 times, resulting in 49 heads and 31 tails.

a) What is the likelihood of this data for each type on coin? Which coin gives the maximum likelihood?

b) Now suppose that we have a single coin with unknown probability p of landing heads. Find the likelihood and log likelihood functions given the same data. What is the maximum likelihood estimate for p?

Coin Tosses

a) The data x is 49 heads in 80 tosses. We have three hypotheses: the coin has probability $p = 1/3, p = 1/2, p = 2/3$. So the likelihood function $L_x : p \mapsto P(X = x | P = p)$ takes 3 values:

$$L_x(\frac{1}{3}) = \binom{80}{49} \left(\frac{1}{3}\right)^{49} \left(\frac{2}{3}\right)^{31} = 6.24 \cdot 10^{-7}$$

$$L_x(\frac{1}{2}) = \binom{80}{49} \left(\frac{1}{2}\right)^{49} \left(\frac{1}{2}\right)^{31} = 0.024$$

$$L_x(\frac{2}{3}) = \binom{80}{49} \left(\frac{2}{3}\right)^{49} \left(\frac{1}{3}\right)^{31} = 0.082$$

Thus the maximum likelihood is achieved under $p = 2/3$.

Coin Tosses

b) We already know that the MLE for the Bernoulli (and binomial) distribution is $\frac{k}{n}$ where k is the number of successes and n is the total number of Bernoulli trials. Thus we get

$$p^* = \frac{49}{80} = 0.6125$$

as the MLE in the present example.

Geometric MLE

For an i.i.d. data set $x = x_1^n$ find the MLE for the geometric distribution:

$$P(X = x) = (1 - \theta)^x \theta$$

Geometric MLE

The likelihood function is

$$L_x(\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i} = \theta^n (1-\theta)^{\sum_{i=1}^n x_i}$$

and thus the log-likelihood is

$$\mathcal{L}_x(\theta) = n \log(\theta) + \sum_{i=1}^{n} x_i \log(1-\theta) \ .$$

Geometric MLE

In the next step we compute the score function.

$$\frac{d}{d\theta}\mathcal{L}_x(\theta) = \frac{d}{d\theta} n\log(\theta) + \frac{d}{d\theta}\sum_{i=1}^{n} x_i \log(1-\theta)$$

$$= \frac{n}{\theta} - \frac{\sum_{i=1}^{n} x_i}{1-\theta}$$

Geometric MLE

Setting this to 0 gives

$$
\begin{aligned}
0 = \frac{n}{\theta} - \frac{\sum_{i=1}^{n} x_i}{1-\theta} \qquad & \Leftrightarrow \\
\frac{n}{\theta} = \frac{\sum_{i=1}^{n} x_i}{1-\theta} \qquad & \Leftrightarrow \\
n - n\theta = \theta \sum_{i=1}^{n} x_i \qquad & \Leftrightarrow \\
\theta = \frac{n}{n + \sum_{i=1}^{n} x_i} \qquad &
\end{aligned}
$$

Geometric MLE, variant 2

For an i.i.d. data set $x = x_1^n$ find the MLE for the geometric distribution:

$$P(X = x) = (1 - \theta)^{x-1}\theta$$

Geometric MLE, variant 2

The likelihood function is

$$L_x(\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i - 1} = \theta^n (1-\theta)^{\left(\sum_{i=1}^{n} x_i\right) - n}$$

and thus the log-likelihood is

$$\mathcal{L}_x(\theta) = n \log(\theta) + \left(\left(\sum_{i=1}^{n} x_i \right) - n \right) \log(1-\theta) \ .$$

Geometric MLE, variant 2

In the next step we compute the score function.

$$\frac{d}{d\theta}\mathcal{L}_x(\theta) = \frac{d}{d\theta} n \log(\theta) + \frac{d}{d\theta}\left((\sum_{i=1}^{n} x_i) - n \right) \log(1-\theta)$$

$$= \frac{n}{\theta} - \frac{(\sum_{i=1}^{n} x_i) - n}{1-\theta}$$

Geometric MLE, variant 2

Setting this to 0 gives

$$
\begin{aligned}
0 &= \frac{n}{\theta} - \frac{((\sum_{i=1}^{n} x_i) - n)}{1-\theta} && \Leftrightarrow \\
\frac{n}{\theta} &= \frac{((\sum_{i=1}^{n} x_i) - n)}{1-\theta} && \Leftrightarrow \\
n - n\theta &= (\theta \sum_{i=1}^{n} x_i) - n\theta && \Leftrightarrow \\
\theta &= \frac{n}{\sum_{i=1}^{n} x_i}
\end{aligned}
$$