

Sufficient statistics

You are given a data set $x = x_1^n$ of n i.i.d. geometrically distributed observations. Show that $\sum_{i=1}^n$ is a sufficient statistic for the geometric distribution.

Sufficient statistics

We will use the factorisation theorem for this exercise.

$$P(X_1^n | \Theta = \theta) = \prod_{i=1}^n P(X_i = x_i | \Theta = \theta) = \theta^n (1 - \theta)^{\sum_{i=1}^n x_i}$$

By letting $g(\theta, \sum_{i=1}^n x_i) = \theta^n (1 - \theta)^{\sum_{i=1}^n x_i}$ and $h(x, \sum_{i=1}^n x_i) = \frac{1}{c}$ the result follows from the factorisation theorem. Here, c is the number of sequences of n geometric draws whose sum is equal to $\sum_{i=1}^n x_i$.

Coin Tosses

A coin is taken from a box containing three coins, which give heads with probability $p = 1/3, 1/2$, and $2/3$. The mystery coin is tossed 80 times, resulting in 49 heads and 31 tails.

- a) What is the likelihood of this data for each type of coin? Which coin gives the maximum likelihood?
- b) Now suppose that we have a single coin with unknown probability p of landing heads. Find the likelihood and log likelihood functions given the same data. What is the maximum likelihood estimate for p ?

Coin Tosses

- a) The data x is 49 heads in 80 tosses. We have three hypotheses: the coin has probability $p = 1/3, p = 1/2, p = 2/3$. So the likelihood function $L_x : p \mapsto P(X = x | P = p)$ takes 3 values:

$$L_x\left(\frac{1}{3}\right) = \binom{80}{49} \left(\frac{1}{3}\right)^{49} \left(\frac{2}{3}\right)^{31} = 6.24 \cdot 10^{-7}$$

$$L_x\left(\frac{1}{2}\right) = \binom{80}{49} \left(\frac{1}{2}\right)^{49} \left(\frac{1}{2}\right)^{31} = 0.024$$

$$L_x\left(\frac{2}{3}\right) = \binom{80}{49} \left(\frac{2}{3}\right)^{49} \left(\frac{1}{3}\right)^{31} = 0.082$$

Thus the maximum likelihood is achieved under $p = 2/3$.

Coin Tosses

- b) We already know that the MLE for the Bernoulli (and binomial) distribution is $\frac{k}{n}$ where k is the number of successes and n is the total number of Bernoulli trials. Thus we get

$$p^* = \frac{49}{80} = 0.6125$$

as the MLE in the present example.

Dice

There are five fair dice each with a different number of sides: 4,6,8,12,20. Jon picks one of them uniformly at random rolls it and reports a 13.

- a) Compute the posterior probability for each die to have generated this outcome.
- b) Compute the posterior probabilities if the result had been a 5 instead. *Hint: Drawing a table may help here. And please do use a calculator!*

Dice

- a) For all but the 20-sided die the likelihood and hence the posterior probability is 0. By Bayes' rule this means that the posterior probability of the 20-sided die is 1. The numerator and the denominator in Bayes' rule take on the same value in this example.

Dice

We use a table to display the solution. We identify the dice by their number of sides.

Die	P(Die)	P(5—Die)	P(5—Die) P(Die)	P(Die—5)
4	$\frac{1}{5}$	0	0	0
6	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{30}$	0.392
8	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{40}$	0.294
12	$\frac{1}{5}$	$\frac{1}{12}$	$\frac{1}{60}$	0.196
20	$\frac{1}{5}$	$\frac{1}{20}$	$\frac{1}{100}$	0.118
total	1		0.085	1

Geometric MLE

For an i.i.d. data set $x = x_1^n$ find the MLE for the geometric distribution:

$$P(X = x) = (1 - \theta)^x \theta$$

Geometric MLE

The likelihood function is

$$L_x(\theta) = \prod_{i=1}^n \theta(1 - \theta)^{x_i} = \theta^n (1 - \theta)^{\sum_{i=1}^n x_i}$$

and thus the log-likelihood is

$$\mathcal{L}_x(\theta) = n \log(\theta) + \sum_{i=1}^n x_i \log(1 - \theta) .$$

Geometric MLE

In the next step we compute the score function.

$$\begin{aligned}\frac{d}{d\theta}\mathcal{L}_x(\theta) &= \frac{d}{d\theta}n\log(\theta) + \frac{d}{d\theta}\sum_{i=1}^n x_i \log(1 - \theta) \\ &= \frac{n}{\theta} - \frac{\sum_{i=1}^n x_i}{1 - \theta}\end{aligned}$$

Geometric MLE

Setting this to 0 gives

$$0 = \frac{n}{\theta} - \frac{\sum_{i=1}^n x_i}{1 - \theta} \quad \Leftrightarrow$$

$$\frac{n}{\theta} = \frac{\sum_{i=1}^n x_i}{1 - \theta} \quad \Leftrightarrow$$

$$n - n\theta = \theta \sum_{i=1}^n x_i \quad \Leftrightarrow$$

$$\theta = \frac{n}{n + \sum_{i=1}^n x_i}$$