

Geometric EM

You are given a mixture model with mixture components c_1, c_2 which are linked to geometric distributions with parameters $\theta_{c_1} = 0.2, \theta_{c_2} = 0.6$. You observe the data set

$$\{0, 2, 2, 3\} .$$

Assume that the latent variables are i.i.d. and that that $P(Y = c_1 | \Theta = \theta^{(0)}) = 0.2$.

- a) What is the (marginal) log-likelihood of this data set under the model? Feel free to use calculators.
- b) Find the most likely mixture component for each data point.
- c) Perform one EM iteration.
- d) Compute the marginal log-likelihood of the data with the updated parameters. The new value should be higher than the one computed in the beginning.

Geometric EM

a)

$$\begin{aligned} & \log(0.2 \cdot 0.2 \cdot 0.8^0 + 0.8 \cdot 0.6 \cdot 0.4) \\ & + 2 \cdot \log(0.2 \cdot 0.2 \cdot 0.8^2 + 0.8 \cdot 0.6 \cdot 0.4^2) \\ & + \log(0.2 \cdot 0.2 \cdot 0.8^3 + 0.8 \cdot 0.6 \cdot 0.4^3) = -8.1836793 \end{aligned}$$

$$\text{b) } P(Y = c_1 | X = 0, \Theta = \theta^{(0)}) = \frac{0.2 \cdot 0.2 \cdot 0.8^0}{0.2 \cdot 0.2 \cdot 0.8^0 + 0.8 \cdot 0.6 \cdot 0.4^0} = 0.0769231$$

$$P(Y = c_1 | X = 2, \Theta = \theta^{(0)}) = \frac{0.2 \cdot 0.2 \cdot 0.8^2}{0.2 \cdot 0.2 \cdot 0.8^2 + 0.8 \cdot 0.6 \cdot 0.4^2} = 0.25$$

$$P(Y = c_1 | X = 3, \Theta = \theta^{(0)}) = \frac{0.2 \cdot 0.2 \cdot 0.8^3}{0.2 \cdot 0.2 \cdot 0.8^3 + 0.8 \cdot 0.6 \cdot 0.4^3} = 0.4$$

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c) E-step

$$\begin{aligned}\mathbb{E}\left[\sum_i \mathbb{1}(Y_i = c_1) \mid X = x, \Theta = \theta^{(0)}\right] &= 0.0769231 + 2 \cdot 0.25 + 0.4 \\ &= 0.9769231\end{aligned}$$

$$\mathbb{E}\left[\sum_i \mathbb{1}(Y_i = c_2) \mid X = x, \Theta = \theta^{(0)}\right] = 4 - 0.9769231 = 3.0230769$$

$$\begin{aligned}\mathbb{E}\left[\sum_i x_i \mathbb{1}(Y_i = c_1) \mid X = x, \Theta = \theta^{(0)}\right] &= 0.0769231 \cdot 0 + 2 \cdot 0.25 \cdot 2 + 0.4 \cdot 3 \\ &= 2.2\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[\sum_i x_i \mathbb{1}(Y_i = c_2) \mid X = x, \Theta = \theta^{(0)}\right] &= 0.9230769 \cdot 0 + 2 \cdot 0.75 \cdot 2 + 0.6 \cdot 3 \\ &= 4.8\end{aligned}$$

Geometric EM

M-step

$$P(Y = c_1 \mid \Theta = \theta^{(1)}) = \frac{0.9769231}{0.9769231 + 3.0230769} = 0.3231552$$

$$P(Y = c_2 \mid \Theta = \theta^{(1)}) = 1 - 0.3231552 = 0.6768448$$

$$\text{new parameter of } c_1 : \frac{0.9769231}{2.2 + 0.9769231} = 0.3075061$$

$$\text{new parameter of } c_2 : \frac{3.0230769}{4.8 + 3.0230769} = 0.3864307$$

Geometric EM

d)

$$\begin{aligned} & \log(0.3231552 \cdot 0.3075061 \cdot 0.6924939^0 \\ & + 0.6768448 \cdot 0.3864307 \cdot 0.6135693) \\ & + 2 \cdot \log(0.3231552 \cdot 0.3075061 \cdot 0.6924939^2 \\ & + 0.6768448 \cdot 0.3864307 \cdot 0.6135693^2) \\ & + \log(0.3231552 \cdot 0.3075061 \cdot 0.6924939^3 \\ & + 0.6768448 \cdot 0.3864307 \cdot 0.6135693^3) = -7.2364299 \end{aligned}$$