# Board Questions

Seventh Session, Oct 17, 2016

## 1 Chain Rule for Entropy

Prove the chain rule for entropy, namely that $H(X,Y) = H(X|Y) + H(Y)$.

## 2 Codes

The following are three (binary symbol) codes $C, D, E$ for the random variable $X$, with $\mathcal{X} = \{a, b, c, d\}$:

| $x$ | $P(X = x)$ | $C(x)$ | $D(x)$ | $E(x)$ |
|---|---|---|---|---|
| $a$ | $1/2$ | 0 | 0 | 0 |
| $b$ | $1/4$ | 10 | 010 | 01 |
| $c$ | $1/8$ | 110 | 01 | 011 |
| $d$ | $1/8$ | 111 | 10 | 111 |

These codes can be used to encode strings of symbols by concatenation . For instance, The encoding of string "adba" under code $E$ is

$$E(adba) = E(a)E(d)E(b)E(a) = 0\ 111\ 01\ 0 = 0111010$$

1. What is the encoding of *adba* under codes $C$ and $D$?
2. What is the decoding of 0100100 under code $D$? Is it unique?
3. What is the decoding of 001111 under code $E$? Is it unique? What happens if you learn that the next bit is 1 (so you have to decode 0011111 under $E$)?
4. Can you prove that arbitrary concatenations of codewords of $C$ are uniquely decodable? What about concatenations of codewords of $E$?
5. Which of the above codes is the most convenient to work with? Why?

## 3 Code Length

The *average code length* of a binary symbol code is defined as follows. Let $\ell(s)$ denote the length of a string $s \in \{0,1\}^*$. The (average) length of a code $C$ for a source $X$ is defined as

$$\ell_C(X) := \mathbb{E}[\ell(C(X))] = \sum_{x \in \mathrm{supp}(X)} P(X = x)\ell(C(x))\,.$$

1. Compute $\ell_C(X)$, $\ell_D(X)$, $\ell_E(X)$ for the codes of the previous section.
2. Compute the entropy $H(X)$ for the distribution $P_X$ above and compare both the obtained values and the way you have obtained them.

In the Information Theory course, we will prove Shannon's source-coding theorem:

**Theorem 1.** *Let $P_X$ be a distribution and $\ell_{\min}(X) := \min_C \ell_C(X)$ the minimal average codeword length among all uniquely decodable codes. Then,*

$$H(X) \leq \ell_{\min}(X) \leq H(X) + 1\,.$$

In other words, the Shannon entropy pretty much determines the optimal average codeword length.

## 4 Optimal Codes

1. Show that code $C$ from Section 2 is optimal in terms of average coding length.
2. Construct an optimal symbol code for the following distribution:

| $y$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ |
|---|---|---|---|---|---|---|---|
| $P(Y = y)$ | $1/4$ | $1/4$ | $1/8$ | $1/8$ | $1/8$ | $1/16$ | $1/16$ |

   **Hint:** Should symbols with high probability to occur receive long or short codewords?

3. Prove that the code you found is optimal!
4. Look up on the internet what Huffman coding is and use it to find an optimal binary symbol code for the following distribution:

| $z$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $P(Z = z)$ | 0.25 | 0.25 | 0.2 | 0.15 | 0.15 |