basics - math

basics

Jan 08, 2025

CONTENTS

Argomenti.

- **Calcolo**
  - Calcolo multivariabile e calcolo vettoriale in spazi euclidei 2D e 3D
  - Algebra lineare e multilineare su spazi con prodotto interno
  - Calcolo lineare e multilineare su spazi con prodotto interno
- **Geometria**
  - Geometria differenziale
- **Calcolo delle variazioni** (qui e/o nella scuola superiore?)
- **Calcolo complesso**
  - Analisi complessa
  - **Teoria delle trasformate**: Fourier, Laplace

Part I

Multivariable Calculus

CHAPTER

ONE

INTRODUCTION TO MULTI-VARIABLE CALCULUS

1.1 Function

1.2 Limit

1.3 Derivatives

1.4 Integrals

1.5 Theorems

1.5.1 Green's lemma

$$\begin{aligned}
\int_S \frac{\partial F}{\partial y} dx dy &= -\oint_{\partial S} F dx \\
\int_S \frac{\partial G}{\partial x} dx dy &= \quad \oint_{\partial S} G dy
\end{aligned}$$

Proof for simple domains.

In a simple domain in x, so that the closed contour ∂S is delimited by the curves $y = Y_1(x)$, $y = Y_2(x) > Y_1(x)$, for $x \in [x_1, x_2]$,

$$\begin{aligned}
\int_S \frac{\partial F}{\partial y} dx dy &= \int_{x=x_1}^{x_2} \int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial F}{\partial y} dy\, dx = \\
&= \int_{x=x_1}^{x_2} \left[F(x, Y_2(x)) - F(x, Y_1(x)) \right] dx = \\
&= -\int_{x=x_1}^{x_2} F(x, Y_1(x)) - \int_{x=x_2}^{x_1} F(x, Y_2(x)) dx = \\
&= -\oint_{\partial S} F(x, y) dx
\end{aligned}$$

In a simple domain in y, so that the closed contour ∂S is delimited by the curves $x = X_1(y)$, $x = X_2(y) > X_1(y)$ for $y \in [y_1, y_2]$,

$$\begin{aligned}\int_S \frac{\partial G}{\partial x} dx dy &= \int_{y=y_1}^{y_2} \int_{x=X_1(y)}^{X_2(y)} \frac{\partial G}{\partial x} dx\, dy = \\ &= \int_{y=y_1}^{y_2} \left[G(X_2(y), y) - G(X_1(y), y) \right] dy = \\ &= \int_{y=y_1}^{y_2} G(X_1(y), y) dy + \int_{y=y_2}^{y_1} G(X_2(y), y) dy = \\ &= \oint_{\partial S} G(x, y) dy\end{aligned}$$

Part II

Differential Geometry

CHAPTER

TWO

INTRODUCTION TO DIFFERENTIAL GEOMETRY

2.1 Differential geometry in E^3

2.1.1 Curves

Parametric representation of curve in 3-dimensional (Euclidean) space E^3

$$\vec{r}(q)$$

Differential, $d\vec{r}$.

$$d\vec{r}(q) = \vec{r}\,'(q)\,dq \ .$$

Arc-length parameter, s. So that $ds = |d\vec{r}(s)|$ and thus

$$|d\vec{r}(s)| = |\vec{r}\,'(s)|\,|ds| \qquad \rightarrow \qquad |\vec{r}\,'(s)| = 1 \qquad \rightarrow \qquad \vec{r}\,'(s) = \hat{t}(s) \ .$$

Frenet basis. Using arc-length parameter, Frenet basis is naturally defined as the set $\{\hat{t}, \hat{n}, \hat{b}\}$:

- tangent unit vector, $\hat{t}(s) = \vec{r}\,'(s)$,
- normal unit vector, $\hat{r}''(s) = \hat{t}'(s) =: \kappa(s)\,\hat{n}(s)$, with $\kappa(s)$ local curvature
- binormal unit vector, $\hat{b}(s) = \hat{t}(s) \times \hat{n}(s)$

Using a general parameter, t, with some abuse of notation $\vec{r}(t) = \vec{r}(s(t))$ and indicating $\dot{()} = \frac{d}{dt}$,

- $\dot{\vec{r}} = \frac{ds}{dt}\frac{d\vec{r}}{ds} = \dot{s}\hat{t}$
- $\ddot{\vec{r}} = \dfrac{d}{dt}\dot{\vec{r}} = \dfrac{d}{dt}\left(\dot{s}\hat{t}\right) = \ddot{s}\hat{t} + \dfrac{ds}{dt}\dfrac{d}{ds}\hat{t} = \ddot{s}\hat{t} + \dot{s}^2\kappa\,\hat{n}$

Osculator circle. Circle with $R(s) = \frac{1}{\kappa(s)}$, in plane orthogonal to $\hat{b}(s)$, passing through $\vec{r}(s)$, and thus center in $\vec{r}_C(s) = \vec{r}(s) + \hat{n}R(s)$. Its parametric representation using its arc-length parameter p, with $\vec{r}(p=0) = \vec{r}(s)$ reads

$$\vec{r}(p) = \vec{r}_C(s) + R(s)\left[-\cos\left(\frac{p}{R(s)}\right)\hat{n}(s) + \sin\left(\frac{p}{R(s)}\right)\hat{t}(s)\right] \ .$$

Its first and second order derivatives w.r.t. the arc-length p evaluated in $p=0$, i.e. $\vec{r} = \vec{r}(s)$ read:

- first derivative in $p=0$,

  $$\hat{t}(p)\big|_{p=0} = \vec{r}\,'(p)\big|_{p=0} = \left[\sin\left(\frac{p}{R(s)}\right)\hat{n}(s) + \cos\left(\frac{p}{R(s)}\right)\hat{t}(s)\right]\Bigg|_{p=0} = \hat{t}(s) \ ,$$

  i.e. the osculator circle has the same tangent as the curve in the point.

- second derivative in $p = 0$,

$$\kappa(p)\hat{n}(p)\big|_{p=0} = \vec{r}''(p)\big|_{p=0} = \frac{1}{R(s)}\left[\cos\left(\frac{p}{R(s)}\right)\hat{n}(s) - \sin\left(\frac{p}{R(s)}\right)\hat{t}(s)\right]\Bigg|_{p=0} = \frac{1}{R(s)}\hat{n}(s) = \kappa(s)\hat{n}(s)\ ,$$

i.e. the osculator circle has the same normal vector and curvature as the curve in the point.

2.1.2 Surfaces

$$\vec{r}(q^1, q^2)$$
$$d\vec{r} = \frac{\partial \vec{r}}{\partial q^1}\,dq^1 + \frac{\partial \vec{r}}{\partial q^2}\,dq^2 = \vec{b}_1\,dq^1 + \vec{b}_2\,dq^2$$

A third vector $\vec{b}_3 := \hat{n}$ can be defined so that $|\hat{n}| = 1$ and $\hat{n}\cdot\vec{b}_i = 0$, $i = 1:2$. For $i = 1:2$, $k = 1:2$

$$\frac{\partial \vec{b}_i}{\partial q^j} = \Gamma^k_{ij}\vec{b}_k = \Gamma^1_{ij}\vec{b}_1 + \Gamma^2_{ij}\vec{b}_2 + \Gamma^3_{ij}\vec{b}_3$$

so that

$$\Gamma^k_{ij} = \vec{b}^k \cdot \frac{\partial \vec{b}_i}{\partial q^j}$$

Normal vector.

$$\vec{n}(q^1, q^2) = \frac{\partial \vec{r}}{\partial q^1}(q^1, q^2) \times \frac{\partial \vec{r}}{\partial q^2}(q^1, q^2) = \vec{b}_1(q^1, q^2) \times \vec{b}_2(q^1, q^2)$$

Tangent plane.

$$(\vec{r} - \vec{r}(q^1, q^2)) \cdot \vec{n}(q^1, q^2) = 0$$

Length of elementary segment.

$$\begin{aligned}
|d\vec{r}|^2 &= d\vec{r}\cdot d\vec{r} = \\
&= \left(\vec{b}_1\,dq^1 + \vec{b}_2\,dq^2\right)\cdot\left(\vec{b}_1\,dq^1 + \vec{b}_2\,dq^2\right) = \\
&= g_{11}\,dq^1\,dq^1 + g_{12}\,dq^1\,dq^2 + g_{21}\,dq^2\,dq^1 + g_{22}\,dq^2\,dq^2 = g_{ij}\,dq^i\,dq^j
\end{aligned}$$

Second order approximation.

$$\begin{aligned}
\vec{r}(q^1 + dq^1, q^2 + dq^2) &= \vec{r}(q_1, q_2) + \frac{\partial \vec{r}}{\partial q^i}\,dq^i + \frac{\partial^2 \vec{r}}{\partial q^i \partial q^j}\,dq^i\,dq^j = \\
&= \vec{r}(q_1, q_2) + \vec{b}_i\,dq^i + \vec{b}_k\Gamma^k_{ij}\,dq^i\,dq^j + \hat{n}\,\Gamma^3_{ij}\,dq^i\,dq^j
\end{aligned}$$

so that

$$\begin{aligned}
\left[\vec{r}(q^1 + dq^1, q^2 + dq^2) - \vec{r}(q^1, q^2)\right]\cdot\hat{n} &= \Gamma^3_{ij}\,dq^i\,dq^j = \\
&= \hat{n}\cdot\frac{\partial^2 \vec{r}}{\partial q^i \partial q^j}\,dq^i\,dq^j = \\
&= \hat{n}\cdot\frac{\partial^2 \vec{r}}{\partial q^i \partial q^j}\,\vec{b}^i\cdot\vec{b}_k dq^k\,\vec{b}^j\cdot\vec{b}_l dq^l = \\
&= \underbrace{dq^k\vec{b}_k}_{d\vec{r}}\cdot\left[\hat{n}\cdot\frac{\partial^2 \vec{r}}{\partial q^i \partial q^j}\vec{b}^i\otimes\vec{b}^j\right]\cdot\underbrace{dq^l\vec{b}_l}_{d\vec{r}}
\end{aligned}$$

Part III

Vector and Tensor Algebra and Calculus

CHAPTER

THREE

TENSOR ALGEBRA

3.1 Basis

Definition 1 (Basis)

Definition 2 (Reciprocal basis)

In a inner product space, the reciprocal basis of a given basis $\{\vec{b}_a\}_{a=1:d}$ is the set of vectors $\{\vec{b}_b\}_{b=1:d}$, s.t.

$$\vec{b}^b \cdot \vec{b}_a = \delta_a^b \; .$$

3.2 Exterior algebra

$$\wedge$$

3.3 Exterior product

Generalization of the vector product

CHAPTER

FOUR

TENSOR CALCULUS IN EUCLIDEAN SPACES

This section deals with tensor calculus in Euclidean space or on manifolds embedded in Euclidean spaces, focusing on d-dimensional spaces with $d \leq 3$, with *inner product*.

This section may rely on results of *differential geometry*.

4.1 Coordinates

A set of parameters $\{q^a\}_{a=1:d}$ to represent vector (or point) in space,

$$\vec{r}(q^a)$$

if $\vec{r} \in E^d$, $a = 1 : d$.

In E^3,

- **Coordinate lines**, 2-parameter family of lines, keeping 2 coordinates constant. As an example, coordinate lines with constant q^2, q^3

$$\vec{r}_1(q^1) = \vec{r}(q^1, \bar{q}^2, \bar{q}^3) \ .$$

- **Coordinate surfaces,** 1-parameter family of surfaces, keeping 1 coordinate constant. As an example, coordinate surfaces with constant q^1,

$$\vec{r}_{23}(q^2, q^3) = \vec{r}(\bar{q}^1, q^2, q^3) \ .$$

Definition 3 (Regular parametrization)

If $\frac{\partial \vec{r}}{\partial q^a} \neq 0$.

4.1.1 Natural basis

Definition 4 (Natural basis)

Vectors of natural basis

$$\vec{b}_a := \frac{\partial \vec{r}}{\partial q^a}$$

Definition 5 (Christoffel symbols)

4.2 Fields

Function of the points in space $F : E^d \to V^r$, being V^r a space of tensors of order r.

4.3 Differential operators

4.3.1 Directional derivative

$$F(\vec{r}) = F\left(\vec{r}\left(q^a\right)\right) = f(q^a)$$

$$f(q^a + \beta \Delta q^a) = F(\vec{r}(q^a + \beta \Delta q^a))$$

$$\vec{r}(q^a) + \alpha \vec{v} = \vec{r}(q^a + \beta \Delta q^a) \sim \vec{r}(q^a) + \frac{\partial \vec{r}}{\partial q^b} \beta \Delta q^b$$

$$\alpha \vec{v} \sim \beta \frac{\partial \vec{r}}{\partial q^b}(q^a)\, \Delta q^b = \beta \vec{b}_b(q^a) \Delta q^b \qquad \rightarrow \qquad \Delta q^b = \frac{\alpha}{\beta} \vec{b}^b(q^a) \cdot \vec{v}$$

The directional derivative for an arbitrary vector $\vec{v} \in V$

$$\left.\frac{d}{d\alpha} F(\vec{r} + \alpha \vec{v})\right|_{\alpha=0}$$

is evaluated as the limit for $\alpha \to 0$ of the incremental ratio

$$\begin{aligned}
\frac{F(\vec{r} + \alpha \vec{v}) - F(\vec{r})}{\alpha} &\sim \frac{f(q^a + \beta \Delta q^a) - f(q^a)}{\alpha} = \\
&\sim \frac{1}{\alpha} \frac{\partial f}{\partial q^b}(q^a) \beta \Delta q^b = \\
&\sim \vec{v} \cdot \vec{b}^b(q^a) \frac{\partial f}{\partial q^b}(q^a) = \\
&= \vec{v} \cdot \nabla F(\vec{r})
\end{aligned}$$

4.3.2 Gradient

The gradient is the differential operator is the first-order differential operator appearing in the definition of the directional derivative, $\nabla F(\vec{r})$. It takes a tensor field $F(\vec{r})$ of order r and gives a tensor field $\nabla F(\vec{r})$ of order $r+1$. Given a set of coordinates $\{q^a\}_{a=1:d}$, the gradient can be written using the reciprocal basis of the natural basis as

$$\nabla F(\vec{r}) = \vec{b}^b(\vec{r}) \frac{\partial F}{\partial q^b}(\vec{r})$$

Examples. $\dots$

4.3.3 Divergence

Divergence opearator is a first-order differential operator that can be defined as the contraction of the first two indices of the gradient,

$$\nabla \cdot F = C_1^2 \left(\nabla F \right) \ .$$

It takes a tensor field $F(\vec{r})$ of order $r \ge 1$ and gives a tensor field $\nabla \cdot F(\vec{r})$ of order $r - 1 \ge 0$.

4.3.4 Laplacian

Laplacian operator is second-order differential operator that can be defined as the divergence of the gradient,

$$\Delta F = \nabla^2 F = \nabla \cdot \nabla F \ .$$

4.3.5 Curl

4.4 Integrals in E^d, $d \le 3$

4.4.1 Line integrals

Density

Integrals

$$\int_{\vec{r} \in \gamma} F(\vec{r})$$

represent the summation of contributions $F(\vec{r})$ over elementary segments of path γ, whose dimension is $|d\vec{r}|$, i.e. implicitly means

$$\int_{\vec{r} \in \gamma} F(\vec{r}) = \int_{\vec{r} \in \gamma} F(\vec{r}) \, |d\vec{r}| \ .$$

Given a regular parametrization of the curve $\vec{r}(q^1)$ (with increasing q^1 so that $|dq^1| = dq^1$), and the differential $d\vec{r} = \vec{r}'(q^1) \, dq^1$, the integral can be written as an integral in the parameter q^1

$$\int_{q=q_a^1}^{q_b^1} F(\vec{r}(q^1)) \, |\vec{r}'(q^1)| \, dq^1 \ ,$$

with $\vec{r}(q_a^1)$, $\vec{r}(q_b^1)$ the extreme points of path γ.

Work

Integrals

$$\int_{\vec{r} \in \gamma} F(\vec{r}) \cdot \hat{t}(\vec{r})$$

implicitly mean

$$\int_{\vec{r} \in \gamma} F(\vec{r}) \cdot \hat{t}(\vec{r}) = \int_{\vec{r} \in \gamma} F(\vec{r}) \cdot \hat{t}(\vec{r}) |d\vec{r}| = \int_{\vec{r} \in \gamma} F(\vec{r}) \cdot d\vec{r} \ ,$$

as $\hat{t} = \frac{d\vec{r}}{|d\vec{r}|}$. Given a regular parametrization of the curve $\vec{r}(q^1)$ (with increasing q^1 so that $|dq^1| = dq^1$), and the differential $d\vec{r} = \vec{r}'(q^1)\, dq^1$, the integral can be written as an integral in the parameter q^1

$$\int_{q^1=q^1_a}^{q^1_b} F(\vec{r}(q^1)) \cdot \vec{r}'(q^1)\, dq^1$$

4.4.2 Surface integrals

Given two coordinates $q^1,\ q^2$ describing a surface, $\vec{r}(q^1, q^2)$ the elementary surface with unit normal reads

$$\hat{n}\, dS = d\vec{r}_1 \times d\vec{r}_2 = \frac{\partial \vec{r}}{\partial q^1} \times \frac{\partial \vec{r}}{\partial q^2}\, dq^1\, dq^2 \ ,$$

and the elementary surface thus reads

$$|dS| = |\hat{n} dS| = \left| \frac{\partial \vec{r}}{\partial q^1} \times \frac{\partial \vec{r}}{\partial q^2}\, dq^1\, dq^2 \right|$$

Density

Integrals

$$\int_{\vec{r} \in S} F(\vec{r})$$

implicitly mean

$$\int_{\vec{r} \in S} F(\vec{r}) = \int_{\vec{r} \in S} F(\vec{r}) |dS| \ .$$

Given regular parametrization of the surface, $\vec{r}(q^1,\ q^2),\ (q^1, q^2) \in Q^{12}$, the integral can be written as the multi-dimensional integral in coordinates $q^1,\ q^2$,

$$\int_{\vec{r} \in S} F(\vec{r}) = \int_{(q^1, q^2) \in Q^{12}} F(\vec{r}(q^1, q^2)) \left| \frac{\partial \vec{r}}{\partial q^1} \times \frac{\partial \vec{r}}{\partial q^2}\, dq^1\, dq^2 \right|$$

Flux

Integrals

$$\int_{\vec{r} \in S} \hat{n}(\vec{r}) \cdot F(\vec{r})$$

implicitly mean

$$\int_{\vec{r} \in S} \hat{n}(\vec{r}) \cdot F(\vec{r}) = \int_{\vec{r} \in S} \hat{n}(\vec{r}) \cdot F(\vec{r}) |dS|$$

Given regular parametrization of the surface, $\vec{r}(q^1,\ q^2),\ (q^1, q^2) \in Q^{12}$, the integral can be written as the multi-dimensional integral in coordinates $q^1,\ q^2$,

$$\int_{\vec{r} \in S} \hat{n}(\vec{r}) \cdot F(\vec{r}) = \int_{(q^1, q^2) \in Q^{12}} \frac{\partial \vec{r}}{\partial q^1} \times \frac{\partial \vec{r}}{\partial q^2} \cdot F(\vec{r}(q^1, q^2))\, dq^1\, dq^2$$

4.4.3 Volume

$$dV = \frac{\partial \vec{r}}{\partial q^1} \cdot \frac{\partial \vec{r}}{\partial q^2} \times \frac{\partial \vec{r}}{\partial q^3} \, dq^1 \, dq^2 \, dq^3 \; .$$

Density

Integrals

$$\int_{\vec{r} \in V} F(\vec{r})$$

implicitly mean

$$\int_{\vec{r} \in V} F(\vec{r}) = \int_{\vec{r} \in V} F(\vec{r}) \, |dV| \; .$$

Given regular parametrization of the volume, $\vec{r}(q^1,\, q^2,\, q^3),\ (q^1, q^2, q^3) \in Q$, the integral can be written as the multi-dimensional integral in coordinates $q^1,\, q^2,\, q^3$,

$$\int_{\vec{r} \in V} F(\vec{r}) |dV| = \int_{(q^1, q^2, q^3) \in Q} F(\vec{r}(q^1, q^2, q^3)) \left| \frac{\partial \vec{r}}{\partial q^1} \cdot \frac{\partial \vec{r}}{\partial q^2} \times \frac{\partial \vec{r}}{\partial q^3} \, dq^1 \, dq^2 \, dq^3 \right| \; .$$

Part IV

Functional Analysis

CHAPTER

FIVE

INTRODUCTION TO FUNCTIONAL ANALYSIS

- Lebesgue integral
- L^p, H^p function spaces
- Banach and Hilbert spaces

CHAPTER

SIX

DISTRIBUTIONS (OR GENERALIZED FUNCTIONS)

…

6.1 Dirac's delta

Dirac's delta $\delta(x)$ is a distribution, or generalized function, with the following properties

1.
$$\int_D \delta(x-x_0)\,dx = 1 \quad \text{if } x_0 \in D$$

2.
$$\int_D f(x)\delta(x-x_0)\,dx \quad \text{if } x_0 \in D$$

for $\forall f(x)$ "regular" **todo** *what does regular mean?*

6.1.1 Dirac's delta in terms of regular functions

Approximations …

$$\delta(x) \sim r_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon} & x \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \\ 0 & \text{otherwise} \end{cases}$$

as

1. Unitariety

$$\int_{x=-\infty}^{\infty} r_\varepsilon(x-x_0)\,dx = \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} \frac{1}{\varepsilon}\,dx = 1 \ ,$$

   for $\forall\varepsilon$;

2. Shift property, using mean-value theorem of continuous functions

$$\int_{x=-\infty}^{\infty} r_\varepsilon(x-x_0)f(x)\,dx = \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} \frac{1}{\varepsilon}f(x)\,dx = \frac{1}{\varepsilon}\varepsilon f(\xi) \ ,$$

   with $\xi \in [x_0-\frac{\varepsilon}{2}, x_0+\frac{\varepsilon}{2}]$, for the mean value theorem. As $\varepsilon \to 0$, $\xi \to x_0$, and thus

$$\int_{x=-\infty}^{\infty} r_\varepsilon(x-x_0)f(x)\,dx \to f(x_0)$$

$$\delta(x) \sim t_\varepsilon(x) = \begin{cases} \frac{2}{\varepsilon}\left(1 - \frac{2|x|}{\varepsilon}\right) & x \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \\ 0 & \text{otherwise} \end{cases}$$

as

1. Unitariety

$$\int_{x=-\infty}^{\infty} t_\varepsilon(x - x_0)\, dx = \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} \frac{2}{\varepsilon}\left(1 - \frac{2|x|}{\varepsilon}\right)\, dx = \frac{1}{2}\varepsilon\frac{2}{\varepsilon} = 1\ ,$$

   for $\forall\varepsilon$;

2. Shift property, using mean-value integration scheme in $x \in [x_0 - \frac{\varepsilon}{2}, x_0]$, $x \in [x_0, x_0 + \frac{\varepsilon}{2}]$ (**todo** *why?*)

$$\begin{aligned}\int_{x=-\infty}^{\infty} t_\varepsilon(x - x_0) f(x)\, dx &= \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} \frac{2}{\varepsilon}\left(1 - \frac{2|x-x_0|}{\varepsilon}\right) f(x)\, dx = \\ &= \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0} \frac{2}{\varepsilon}\left(1 - \frac{2|x-x_0|}{\varepsilon}\right) f(x)\, dx + \int_{x=x_0}^{x_0+\frac{\varepsilon}{2}} \frac{2}{\varepsilon}\left(1 - \frac{2|x-x_0|}{\varepsilon}\right) f(x)\, dx = \\ &= \frac{\varepsilon}{2}\frac{2}{\varepsilon}\left(1 - \frac{2}{\varepsilon}\frac{\varepsilon}{4}\right) f\left(x_0 - \frac{\varepsilon}{4}\right)\, dx + \frac{\varepsilon}{2}\frac{2}{\varepsilon}\left(1 - \frac{2}{\varepsilon}\frac{\varepsilon}{4}\right) f\left(x_0 + \frac{\varepsilon}{4}\right)\, dx = \\ &= \frac{1}{2} f\left(x_0 - \frac{\varepsilon}{4}\right) + \frac{1}{2} f\left(x_0 + \frac{\varepsilon}{4}\right)\end{aligned}$$

   As $\varepsilon \rightarrow 0$

$$\int_{x=-\infty}^{\infty} t_\varepsilon(x - x_0) f(x)\, dx \rightarrow f(x_0)$$

Approximation 1. For $\alpha \rightarrow +\infty$,

$$\varphi_\alpha(x) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} \sim \delta(x)$$

Fourier transform of $\varphi_\alpha(x)$ reads

$$\begin{aligned}\mathcal{F}\{\varphi_\alpha(x)\}(k) &= \int_{x=-\infty}^{+\infty} \varphi_\alpha(x) e^{-ikx}\, dx = \\ &= \int_{x=-\infty}^{+\infty} \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} e^{-ikx}\, dx = \\ &= \sqrt{\frac{\alpha}{\pi}} \int_{x=-\infty}^{+\infty} e^{-\alpha\left(x + i\frac{k}{2\alpha}\right)^2}\, dx\, e^{-\frac{k^2}{4\alpha}} = \\ &= \sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{\pi}{\alpha}} e^{-\frac{k^2}{4\alpha}} = e^{-\frac{k^2}{4\alpha}}\ ,\end{aligned}$$

for $\alpha \rightarrow +\infty$,

$$\mathcal{F}\{\varphi_\alpha(x)\}(k) \rightarrow 1$$

and thus $\varphi_\alpha(x) \rightarrow \delta(x)$ for $\alpha \rightarrow +\infty$.

Approximation 2. For $a \rightarrow +\infty$

$$\frac{1}{2\pi} \int_{k=-2\pi a}^{2\pi a} e^{ikx}\, dk = \int_{y=-a}^{+a} e^{i2\pi yx}\, dy \sim \delta(x)$$

or

$$\delta(x) \sim \frac{1}{2\pi}\int_{k=-2\pi a}^{2\pi a} e^{ikx}\,dk = \frac{1}{2\pi}\left(\int_{k=-2\pi a}^{0} e^{ikx}\,dk + \int_{0}^{k=2\pi a} e^{ikx}\,dk\right) = \frac{1}{2\pi}\int_{k=0}^{2\pi a}\left(e^{ikx}+e^{ikx}\right)\,dx = \frac{1}{\pi}\int_{x=0}^{2\pi a}\cos(kx)\,dk$$
$$= \int_{y=-a}^{+a} e^{i2\pi yx}\,dy = \dots = \int_{y=0}^{a}\left(e^{i2\pi yx}+e^{i2\pi yx}\right)dy = 2\int_{y=0}^{a}\cos(2\pi yx)\,dy\ .$$

Approximation 3. For $a \to +\infty$

$$\frac{\sin(2\pi xa)}{\pi x} \sim \delta(x)$$

Directly follows from integral of approximation 2,

$$\int_{y=-a}^{+a} e^{i2\pi yx}\,dy = \frac{1}{i2\pi x}\left.e^{i2\pi yx}\right|_{y=-a}^{+a} = \frac{1}{\pi x}\frac{e^{i2\pi ax}-e^{-i2\pi ax}}{2i} = \frac{\sin(2\pi xa)}{\pi x}$$

Approximation 4. For $x \in [-\pi, \pi]$, and $N \to +\infty$

$$\frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx} = \frac{1}{2\pi}\frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)} \sim \delta(x)$$

Integral $I = \int_{-\infty}^{+\infty} e^{-\alpha x^2}\,dx$

$$\begin{aligned}
I^2 &= \int_{x=-\infty}^{+\infty} e^{-\alpha x^2}\,dx \int_{y=-\infty}^{+\infty} e^{-\alpha y^2}\,dy = \\
&= \int_{x=-\infty}^{+\infty}\int_{y=-\infty}^{+\infty} e^{-\alpha(x^2+y^2)}\,dx\,dy = \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{+\infty} e^{-\alpha r^2}\,r\,dr\,d\theta = \\
&= 2\pi\frac{1}{2\alpha}\int_{r=0}^{+\infty} e^{-\alpha r^2}\,d\left(\alpha r^2\right) = \\
&= \frac{\pi}{\alpha}\left[-e^{\alpha r^2}\right]\Big|_{r=0}^{+\infty} = \frac{\pi}{\alpha}\ .
\end{aligned}$$

Part V

Complex Calculus

CHAPTER

SEVEN

COMPLEX ANALYSIS

7.1 Complex functions, $f : \mathbb{C} \rightarrow \mathbb{C}$

A complex function f of complex variable $z = x + iy$, $f : \mathbb{C} \rightarrow \mathbb{C}$, can be written as

$$f(z) = \tilde{u}(z) + i\tilde{v}(z) = u(x, y) + iv(x, y) \ ,$$

as the sum of its real part $u(z)$ and i times its imaginary part $v(x, y)$. Here $x, y \in \mathbb{R}$, while $\tilde{u}(z), \tilde{v}(z) : \mathbb{C} \rightarrow \mathbb{R}$ and $u(x, y), v(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$. With some abuse of notation, tilde won't be always explicitly written when arguments of real and imaginary parts of f functions won't be written.

7.1.1 Limit

$$\lim_{z \rightarrow z_0} f(z) = f(z_0) \qquad , \qquad \forall \varepsilon > 0 \ \exists \delta > 0 \ \text{ s.t. } \ |f(z) - f(z_0)| < \delta \ \forall z \text{ s.t. } |z - z_0| < \varepsilon, \ z \neq z_0 \ .$$

7.1.2 Derivative

Using the definition of *limit of complex functions*, the derivative of a function $f : \mathbb{C} \rightarrow \mathbb{C}$, if it exists, is the limit of incremental ratio,

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \ .$$

7.1.3 Line Integrals

Given a line $\gamma \in \mathbb{C}$, whose parametric form is $z(s)$, with regular parametrization with parameter $s \in [s_0, s_1]$,

$$\int_\gamma f(z) \, dz = \int_{s=s_0}^{s_1} f(z(s)) \, z'(s) \, ds \ .$$

7.2 Holomorphic Functions - Analytic Functions

Definition 6

A holomorphic function is a function whose *derivative* exists.

Examples of analytic functions. todo…

7.2.1 Cauchy-Riemann conditions

For a holomorphic function $f(z) = u(x, y) + iv(x, y)$, Cauchy-Riemann conditions

$$\begin{cases} u_{/x} = v_{/y} \\ u_{/y} = -v_{/x} \end{cases}$$

hold. The evaluation of the derivative once with $\Delta z = \Delta x$ and once with $\Delta z = i\Delta y$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} =$$
$$= \begin{cases} \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x} = u_{/x} + iv_{/x} \\ \lim\limits_{\Delta y \to 0} \dfrac{f(x, y + \Delta y) - f(x, y)}{i\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y} = -iu_{/y} + v_{/y} \end{cases}$$

provides the proof.

7.2.2 Cauchy Theorem

For a holomorphic function f, $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$

$$\oint_\gamma f(z)\, dz = 0 \,,$$

for $\forall \gamma \subset \Omega$. Proof follows from *Green's lemma*, and *Cauchy-Riemann conditions*

$$\begin{aligned} \oint_\gamma f(z) dz &= \oint_\gamma \left(u(x, y) + iv(x, y)\right)(dx + idy) = \\ &= \oint_\gamma (udx - vdy) + i \oint_\gamma (udy + vdx) = \\ &= -\int_S \Bigg(\underbrace{u_{/y} + v_{/x}}_{=0} \Bigg)\, dx\, dy + i \int_S \Bigg(\underbrace{u_{/x} - v_{/y}}_{=0} \Bigg)\, dx\, dy = 0 \,. \end{aligned}$$

7.3 Useful integrals

7.3.1 Independence of line integral for holomorphic functions

For a function $f(z)$ analytic in D, the line integral on paths $\ell_{ab,i}$ with the same extreme points a, b contained in D is independent on the path, but only depends on the extreme points a, b,

$$\int_{\ell_{ab,1}} f(z)\,dz = \int_{\ell_{ab,2}} f(z)\,dz$$

The proof readily follows, using *Cauchy theorem* applied to a function $f(z) : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$, analytic in D, and splitting the closed path γ into two paths ℓ_1, ℓ_2 with the same extreme points, $\gamma = \ell_1 \cup (-\ell_2)$

$$0 = \oint_\gamma f(z)\,dz = \int_{\ell_1} f(z)\,dz + \int_{-\ell_2} f(z)\,dz = \int_{\ell_1} f(z)\,dz - \int_{\ell_2} f(z)\,dz\,.$$

7.3.2 Sum and difference of line integrals

7.3.3 Integral of z^n

Given a path γ embracing $z = 0$ only once in counter-clockwise direction, and $n \in \mathbb{Z}$

$$\oint_\gamma z^n\,dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

Since z^n is analytic everywhere (**todo** *prove it! Add a section with proofs for common functions*) except for $z = 0$, it's possible to evaluate the integral on a circle with center $z = 0$ and radius R. Using polar expression of the complex numbers on the circle, $z = Re^{i\theta}$, $\theta \in [0, 2\pi]$, R const, the differential becomes $dz = iRe^{i\theta} d\theta$ and the integral

$$\begin{aligned} \oint_\gamma z^n\,dz &= \int_{\theta=0}^{2\pi} \left(Re^{i\theta}\right)^n iRe^{i\theta} d\theta = \\ &= i \int_{\theta=0}^{2\pi} R^{n+1} e^{i(n+1)\theta} d\theta = \\ &= \begin{cases} \text{if } n = -1 & : \quad i2\pi \\ \text{otherwise} & : \quad iR^{n+1} \dfrac{1}{i(n+1)}\, e^{i(n+1)\theta}\Big|_{\theta=0}^{2\pi} = \dfrac{R^{n+1}}{n+1}(1-1) = 0 \end{cases} \end{aligned}$$

7.4 Meromorphic functions

Definition 7

A meromorphic function in a domain is a function holomorphic everywhere except for a (finite?) number of poles. **check**

7.4.1 Singularities

Definition 8 (Pole)

A pole of order n of a function $f(z)$ is a complex number a so that

$$f(z) = \frac{\phi(z)}{(z-a)^n} \ ,$$

with $\phi(z)$ holomorphic in $\phi(a) \neq 0$

Examples. ...

Definition 9 (Branch)

Examples. $f(z) = z^{\frac{1}{2}}$

Definition 10 (Removable singularities)

Example. $f(z) = \frac{\sin z}{z}$

Other irregularities.

7.4.2 Laurent Series

Given a function $f(z)$, in a disk $D_{a,\varepsilon} : 0 < |z-a| < \varepsilon$, its Laurent series centered in a is the convergent (to $f(z)$, **todo** *which type of convergence?*) series

$$f(z) \sim \sum_{n=-\infty}^{+\infty} a_n (z-a)^n \ , \tag{7.1}$$

with

$$a_n = \frac{1}{2\pi i} \int_\gamma f(z)\, (z-a)^{-(n+1)}\, dz \tag{7.2}$$

and γ embracing $z = a$ once counter-clockwise. Proof follows immediately inserting the expressions of the coefficients a_n and using the *integral of z^n*. Evaluating the integral (7.2) of the coefficients of the Laurent series, using (7.1) to replace $f(z)$ with its series

$$\begin{aligned}
a_n &= \frac{1}{2\pi i} \oint_\gamma \sum_{m=-\infty}^{+\infty} a_m (z-a)^m (z-a)^{-(n+1)} = \\
&= \frac{1}{2\pi i} \oint_\gamma \sum_{m=-\infty}^{+\infty} a_m (z-a)^{m-n-1}\, dz = \\
&= \frac{1}{2\pi i} \oint_\gamma a_n\, z^{-1}\, dz = \\
&= a_n \ .
\end{aligned}$$

todo *Some freestyle with function and its convergent series...add some detail, and the meaning of convergence*

7.4.3 Cauchy formula

For an analytic function $f(z)$,

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a}\, dz$$

Proof readily follows using the *integral of z^n* on the Taylor series of $\frac{f(z)}{z-a}$ whose 0^{th} order term reads $f(a)$,

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(a) + \sum_{m=1}^{+\infty} f'(a)(z-a)^m}{z-a}\, dz = \frac{1}{2\pi i} \oint_\gamma \frac{f(a)}{z-a}\, dz = f(a)\frac{2\pi i}{2\pi i} = f(a)\ .$$

7.4.4 Residues

Definition 11 (Residue)

The residue of function f in a, $\text{Res}(f, a)$ is a complex number R so that $f(z) - \frac{R}{(z-a)}$ has analytic antiderivative in a disk $D_{a,\varepsilon}:\ 0 < |z-a| < \varepsilon$.

todo Explain this definition. Couldn't be possible to use $\text{Res}(f, a) = \frac{1}{2\pi i} \oint_\gamma f(z)\, dz = a_{-1}$ instead?

Properties.

- If $f(z)$ is analytic in $D_{a,\varepsilon}$ and has a pole of order n in $z = a$, its Laurent series has $a_m = 0$ for $m < n$ and reads

$$f(z) = \sum_{m=-n}^{+\infty} a_m (z-a)^m\ , \tag{7.3}$$

  with $a_{-n} \neq 0$. Since $f(z)$ has a pole of order n in $z = a$, it can be written as

$$f(z) = \frac{\phi(z)}{(z-a)^n}\ ,$$

  with $\phi(z)$ analytic in $D_{a,\varepsilon}$ and $\phi(a) \neq 0$. Since $\phi(z)$ is analytic, it has a Taylor series (or a Laurent series with non-negative powers),

$$\phi(z) \sim \sum_{m=0}^{+\infty} b_m (z-a)^m\ ,$$

  (**todo** *prove it! Extension of the real case. Add a link to the proof*) and thus

$$f(z) \sim \sum_{m=0}^{+\infty} b_m (z-a)^{m-n} = \sum_{m=-n}^{+\infty} b_{m+n} (z-a)^m = \sum_{m=-n}^{+\infty} a_m (z-a)^m\ ,$$

  with $a_m = b_{m+n}$.

- For simple closed path γ (embracing a only once counter-clokwise) in $D_{a,\varepsilon}$,

$$\oint_\gamma f(z)\, dz = 2\pi i a_{-1} = 2\pi i \text{Res}(f, a) \tag{7.4}$$

  The proof readily follows, using the *integral of z^n* and Laurent series (7.1) of $f(z)$,

$$\oint_\gamma f(z)\, dz = \oint_\gamma \sum_{m=-\infty}^{+\infty} a_m (z-a)^m\, dz = 2\pi i a_{-1}\ .$$

- For a pole a of order n, the following holds

$$a_{-1} = \frac{1}{(n+1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]$$

The proof follows using Laurent series {eq}`eq:laurent:pole-n} for a function with pole of order n, and evaluating the $(n-1)^{th}$ order derivative

$$\begin{aligned}
\frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right] & = \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n \sum_{m=-n}^{+\infty} a_n (z-a)^m \right] = \\
& = \frac{d^{n-1}}{dz^{n-1}} \left[\sum_{m=-n}^{+\infty} a_n (z-a)^{m+n} \right] = \\
& = \frac{d^{n-1}}{dz^{n-1}} \left[\sum_{m=0}^{+\infty} a_{m-n} (z-a)^{m} \right] = \\
& = \frac{d^{n-2}}{dz^{n-2}} \left[\sum_{m=0}^{+\infty} m a_{m-n} (z-a)^{m-1} \right] = \\
& = \frac{d^{n-3}}{dz^{n-3}} \left[\sum_{m=0}^{+\infty} m(m-1) a_{m-n} (z-a)^{m-2} \right] = \\
& = \dots = \\
& = \left[\sum_{m=0}^{+\infty} m! \, a_{m-n} (z-a)^{m-n+1} \right]
\end{aligned}$$

and then letting $z \to a$, so that only the term with $m - n + 1 = 0$ survives

$$\lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n \sum_{m=-n}^{+\infty} a_n (z-a)^m \right] = (n-1)! \, a_{-1} \ .$$

7.4.5 Residue Theorem

Theorem 1 (Residue Theorem)

Given $f(z)$ with a finite number of poles $p_n \in D$, then

$$\int_\gamma f(z) \, dz = 2 \pi i \sum_n I(\gamma, p_n) \text{Res}(f, p_n) \ ,$$

being γ a path in D, and $I(\gamma, p_n)$ the winding index of the path γ around pole p_n (+1 for each counter-clockwise loop, -1 for each clockwise loop).

The proof readily follows extending the result for a single pole (7.4) to general number of poles and general paths γ embracing (with sign) each pole p_n $I(\gamma, p_n)$ times, with the same techinques shown in section *Sum and difference of line integrals*.

7.4.6 Evaluation of integrals

7.4.7 Inverse Laplace Transform

Given Laplace transform

$$F(s) := \mathcal{L}\{f(t)\}(s) := \int_{t=0^-}^{+\infty} f(t) e^{-st} \, dt \ ,$$

the inverse transform can be evaluated as

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) := \lim_{T \to +\infty} \frac{1}{2\pi i} \int_{s=a-iT}^{a+iT} e^{st} F(s) \, ds \ ,$$

with $a > \text{Re}\{p_n\}$ (**todo** *why?*) for each pole of the function $F(s)$, evaluated on the vertical line $s = a + iy$, $y \in [-T, T]$, $ds = i dy$,

$$\begin{aligned}
\lim_{T \to +\infty} \frac{1}{2\pi i} \int_{s=a-iT}^{a+iT} e^{st} F(s) \, ds &= \lim_{T \to +\infty} \frac{1}{2\pi i} \int_{s=a-iT}^{a+iT} e^{st} \int_{\tau=0^-}^{+\infty} f(\tau) e^{-s\tau} \, d\tau \, ds = \\
&= \lim_{T \to +\infty} \frac{1}{2\pi i} \int_{y=-T}^{T} e^{(a+iy)t} \int_{\tau=0^-}^{+\infty} f(\tau) e^{-(a+iy)\tau} \, d\tau \, i dy = \\
&= \lim_{T \to +\infty} \frac{1}{2\pi} \int_{y=-T}^{T} \int_{\tau=0^-}^{+\infty} e^{iy(t-\tau)} e^{a(t-\tau)} f(\tau) \, d\tau \, dy = \\
&= \dots \\
&= \int_{\tau=0^-}^{+\infty} \delta(t-\tau) e^{a(t-\tau)} f(\tau) d\tau = f(t) \ .
\end{aligned}$$

having used the transform of *Dirac's delta* $\delta(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} e^{-j\omega t} \, d\omega$.

todo *Ohter approach: if $a >$ Re$\{p_n\}$, the contour built with the vertical line with real part a and the arc of circumference on its…*

CHAPTER

EIGHT

LAPLACE TRANSFORM

$$\mathcal{L}\left\{f(t)\right\}(s) := \int_{t=0^-}^{+\infty} e^{-st} f(t)\, dt = F(s)\ .$$

8.1 Inverse transform

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = ...$$

8.2 Properties

Linearity.

$$\mathcal{L}\{af(t) + bg(t)\}(s) = aF(s) + bG(s)$$

***Dirac delta*.**

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{t=0^-}^{+\infty} \delta(t)\, e^{st}\, dt = 1$$

Time delay. If $f(t) = 0$ for $t < 0$ ("causality"), for $\tau > 0$,

$$\mathcal{L}\{f(t-\tau)\}(s) = e^{-s\tau} F(s)$$

Proof readily follows direct computation with change of variable $z = t - \tau$, $dt = dz$

$$\mathcal{L}\{f(t-\tau)\}(s) = \int_{t=0^-}^{+\infty} f(t-\tau) e^{-st}\, dt = \int_{z=-\tau}^{+\infty} f(z) e^{-sz}\, dz\, e^{-s\tau} = \int_{z=0}^{+\infty} f(z) e^{-sz}\, dz\, e^{-s\tau} = e^{-s\tau} F(s)\ .$$

"Frequency shift"

$$\mathcal{L}\{f(t)e^{at}\}(s) = F(s-a)$$

Direct computation gives

$$\mathcal{L}\{f(t)e^{at}\}(s) = \int_{t=0^-}^{+\infty} f(t) e^{at} e^{-st}\, dt = \int_{t=0^-}^{+\infty} f(t) e^{-(s-a)t}\, dt = F(s-a)$$

Derivative.

$$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0^-)\ .$$

Proof readily follows direct computation, with integration by parts

$$\mathcal{L}\{f'(t)\}(s) = \int_{t=0^-}^{+\infty} f'(t)e^{-st}\,dt = \left[f(t)e^{-st}\right]\Big|_{t=0^-}^{+\infty} + s\int_{t=0^-}^{+\infty} f(t)e^{-st}\,dt = sF(s) - f(0^-)\ ,$$

provided that $\lim_{s\to+\infty} f(t)e^{-st} = 0$.

Integral.

$$\mathcal{L}\left\{\int_{\tau=0}^{t} f(\tau)\,d\tau\right\}(s) = \frac{1}{s}F(s)\ .$$

Proof readily follows direct computation, with integration by parts

$$\mathcal{L}\left\{\int_{\tau=0^-}^{t} f(\tau)\,d\tau\right\}(s) = \int_{t=0^-}^{+\infty}\int_{\tau=0^-}^{t} f(\tau)\,d\tau e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\int_{\tau=0^-}^{t} f(\tau)\,d\tau\right]_{t=0}^{+\infty} + \frac{1}{s}\int_{t=0}^{+\infty} f(t)e^{-st}\,dt = \frac{1}{s}F(s)\ ,$$

provided that $\int_{\tau=0^-}^{0} f(\tau)d\tau = 0$ and $\lim_{t\to+\infty}\frac{e^{-st}}{s}\int_{\tau=0^-}^{+\infty} f(\tau)\,d\tau = 0$.

Initial value. If …

$$f(0^+) = \lim_{s\to+\infty} sF(s)$$

From direct computation,

$$\begin{aligned}
\lim_{s\to+\infty} sF(s) &= \lim_{s\to+\infty} s\int_{t=0^-}^{+\infty} f(t)\,e^{-st}\,dt = \\
&= \lim_{s\to+\infty}\left\{\left[s\left(-\frac{e^{-st}}{s}\right)f(t)\right]\Bigg|_{t=0}^{+\infty} + \int_{t=0}^{+\infty} e^{-st}f'(t)\,dt\right\} = \\
&= \lim_{s\to+\infty}\left\{\left[-e^{-st}f(t)\right]\Bigg|_{t=0}^{+\infty} + \int_{t=0}^{+\infty} e^{-st}f'(t)\,dt\right\} = \\
&= f(0)\ ,
\end{aligned}$$

provided that $\lim_{s\to+\infty}\lim_{t\to+\infty} e^{-st}f(t) = 0$ and $\lim_{s\to+\infty}\int_{t=0}^{+\infty} e^{-st}f'(t)\,dt = 0$.

Final value. If …

$$f(+\infty) = \lim_{s\to 0} sF(s)$$

From direct computation (**todo** *check and/or explain proof*),

$$\begin{aligned}
\lim_{s\to 0} sF(s) &= \lim_{s\to 0} s\int_{t=0^-}^{+\infty} f(t)\,e^{-st}\,dt = \\
&= \lim_{s\to 0}\left\{\left[s\left(-\frac{e^{-st}}{s}\right)f(t)\right]\Bigg|_{t=0}^{+\infty} + \int_{t=0}^{+\infty} e^{-st}f'(t)\,dt\right\} = \\
&= \lim_{s\to 0}\left\{\left[-e^{-st}f(t)\right]\Bigg|_{t=0}^{+\infty} + \int_{t=0}^{+\infty} e^{-st}f'(t)\,dt\right\} = \\
&= f(0) + f(+\infty) - f(0) = f(+\infty)\ ,
\end{aligned}$$

provided that $\lim_{s\to 0}\lim_{t\to+\infty} e^{-st}f(t) = 0$.

CHAPTER

NINE

FOURIER TRANSFORMS

- Fourier series: continuous time, periodic function in time
- Fourier transform: continuous time, non-periodic function in time
- Discrete Fourier transform (DFT):
- Discrete time Fourier transform (DTFT):

9.1 Fourier Series

For a T-periodic function,

$$g(t)$$

9.2 Fourier Transform

$$\mathcal{F}\left\{g(t)\right\}(f) := \int_{t=-\infty}^{+\infty} g(t)\, e^{-i2\pi ft}\, dt.$$

9.2.1 Properties

Linearity.

***Dirac delta*.**

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{t=-\infty}^{+\infty} \delta(t)\, e^{-i2\pi ft}\, dt = 1$$

Time delay.

Derivative.

Integral.

Initial value.

Final value.

9.2.2 Inverse Fourier Transform

$$\mathcal{F}^{-1}\left\{G(f)\right\}(t) := \int_{f=-\infty}^{+\infty} G(f)\, e^{i2\pi f t}\, df .$$

Proof using Dirac's delta expression.

$$\begin{aligned}
\mathcal{F}^{-1}\left\{G(f)\right\}(t) := \int_{f=-\infty}^{+\infty} G(f)\, e^{i2\pi f t}\, df &= \int_{f=-\infty}^{+\infty} \int_{\tau=-\infty}^{+\infty} g(\tau) e^{-i2\pi f\tau}\, e^{i2\pi f t}\, df = \\
&= \int_{f=-\infty}^{+\infty} \int_{\tau=-\infty}^{+\infty} g(\tau) e^{-i2\pi f\tau}\, e^{i2\pi f t}\, df = \\
&= \int_{f=-\infty}^{+\infty} \int_{\tau=-\infty}^{+\infty} g(\tau) e^{i2\pi f(t-\tau)}\, df = \\
&= \int_{\tau=-\infty}^{+\infty} g(\tau)\delta(t-\tau)\, d\tau = g(t) \, .
\end{aligned}$$

Proof. By the *dominated convergence theorem*, it follows that

$$\begin{aligned}
\int_{\mathbb{R}} e^{i2\pi x\xi} F(\xi)\, d\xi &= \lim_{\varepsilon\rightarrow 0} \int_{\mathbb{R}} \underbrace{e^{-\pi\varepsilon^2\xi^2 + i2\pi x\xi}}_{G(\xi;x,\varepsilon)} F(\xi)\, d\xi = \\
&= \lim_{\varepsilon\rightarrow 0} \int_{\mathbb{R}} g(y;x,\varepsilon) f(y)\, dy = \\
&= \lim_{\varepsilon\rightarrow 0} \int_{\mathbb{R}} \varphi_\varepsilon(x-y)\, f(y)\, dy = \\
&= \int_{\mathbb{R}} \delta(x-y)\, f(y)\, dy = f(x)
\end{aligned}$$

Lemma 1. The Fourier transform of function $\varphi(t) := e^{-\pi|t|^2}$ reads

$$\begin{aligned}
\mathcal{F}\{\varphi(t)\}(\omega) &= \int_{t=-\infty}^{+\infty} \varphi(t) e^{-i\omega t}\, dt = \\
&= \int_{t=-\infty}^{+\infty} e^{-\pi|t|^2} e^{-i\omega t}\, dt = \\
&= \int_{t=-\infty}^{+\infty} e^{-\pi\left(t^2 + i\frac{\omega}{\pi}t - \frac{\omega^2}{4\pi^2}\right)}\, dt\, e^{-\frac{\omega^2}{4\pi^2}} = \\
&= \int_{t=-\infty}^{+\infty} e^{-\pi\left(t + i\frac{\omega}{2\pi}\right)^2}\, dt\, e^{-\frac{\omega^2}{4\pi}} = \\
&= e^{-\frac{\omega^2}{4\pi}} \, ,
\end{aligned}$$

having evaluated *the integral* $\int_{-\infty}^{+\infty} e^{-\alpha x^2}$ with $\alpha = \pi$. **todo** *justify the result for complex exponential. Use Bromwich contour integrals*

Lemma 2. Fourier transform of $f(\alpha t)$, $\alpha > 0$

$$\mathcal{F}\{f(\alpha t)\}(\omega) = \int_{\mathbb{R}} f(\alpha t) e^{-j\omega t}\, dt = \int_{\tau\in\mathbb{R}} f(\tau) e^{-j\frac{\omega}{\alpha}\tau}\, d\tau \frac{1}{\alpha} = \frac{1}{\alpha} F\left(\frac{\omega}{\alpha}\right)$$

Lemma 3. $\frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right) \rightarrow \delta(x)$ for $\varepsilon \rightarrow 0$

$$\mathcal{F}\left\{\frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right)\right\}(\omega) = \frac{1}{\varepsilon}\varepsilon e^{-\frac{\omega^2}{4\pi\varepsilon^2}} = e^{-\frac{\omega^2}{4\pi\varepsilon^2}}$$

0. Fourier transform

$$G(f) = \int_{t=-\infty}^{\infty} e^{-i\omega t} g(t)\, dt$$

1.

$$g(t) = e^{i\alpha t}\psi(t)$$

$$\mathcal{F}\{g(t)\}(\omega) = \int_{t=-\infty}^{+\infty} g(t)e^{-i\omega t}\, dt = \int_{t=-\infty}^{+\infty} \psi(t)e^{i\alpha t}e^{-i\omega t}\, dt = \int_{t=-\infty}^{+\infty} \psi(t)e^{-i(\omega-\alpha)t}\, dt = \mathcal{F}\{\psi(t)\}(\omega-\alpha)\ .$$

2.

$$\psi(t) = \phi(\alpha t)$$

$$\mathcal{F}\{\psi(t)\} = \int_{t=-\infty}^{+\infty} \psi(t)e^{-i\omega t}\, dt = \int_{t=-\infty}^{+\infty} \phi(\alpha t)e^{-i\omega t}\, dt = \int_{\tau=-\infty}^{+\infty} \phi(\tau)e^{-i\frac{\omega}{\alpha}\tau}\, \frac{d\tau}{\alpha} = \frac{1}{\alpha}\mathcal{F}\{\phi(t)\}\left(\frac{\omega}{\alpha}\right)\ .$$

3. Fubini's theorem

4.

$$\varphi(t) := e^{-\pi t^2}$$

$$\mathcal{F}\{\varphi(t)\} = \int_{t=-\infty}^{+\infty} \varphi(t)e^{-i\omega t}\, dt = \int_{t=-\infty}^{+\infty} e^{-\pi t^2}e^{-i\omega t}\, dt$$

$$0 = \oint_{\gamma} e^{-\alpha|z|^2}\, dz = \int_{\dots} \dots$$

$$z = Re^{i\theta}, \quad dz = iRe^{i\theta}d\theta$$

$$\int_{C/4} e^{-\alpha|z|^2}\, dz = \int_{\theta=0}^{\frac{\pi}{2}} e^{-\alpha R^2} iRe^{i\theta}d\theta = iRe^{-\alpha R^2}\frac{e^{-i\theta}}{i}|_{\theta=0}^{\frac{\pi}{2}}$$

$$\int_{t=0}^{+\infty} e^{-\pi t^2}e^{-i\omega t}\, dt = \int_{t=0}^{+\infty} e^{-\left(\pi t^2 + i\omega t - \frac{\omega^2}{4\pi}\right)}\, dt\, e^{-\frac{\omega^2}{4\pi}} =$$

$$= \int_{t=0}^{+\infty} e^{-\pi\left(t+i\frac{\omega}{2\pi}\right)^2}\, dt\, e^{-\frac{\omega^2}{4\pi}}$$

5. $\varphi_\varepsilon(t) = \frac{1}{\varepsilon^n}\varphi\left(\frac{t}{\varepsilon}\right)$, $t \in \mathbb{R}^n$, is an approximation of Dirac's delta for $\varepsilon \to 0$, so that

$$\lim_{\varepsilon\to 0}\int_{t=-\infty}^{+\infty} \varphi_\varepsilon(t-\tau)f(t)\, dt = f(\tau)$$

$$\lim_{\varepsilon\to 0}\int_{t=-\infty}^{+\infty} \varphi_\varepsilon(t)\, dt = 1$$

As the Fourier transform $\mathcal{F}\left\{\varphi_\varepsilon(t)\right\}(\omega) \to 1$ for $\varepsilon \to 0$, then $\varphi_\varepsilon(t) \to \delta(t)$.

Part VI

Calculus of Variations

CHAPTER

TEN

INTRODUCTION TO CALCULUS OF VARIATIONS

Given the functional S,

$$S[q(t),t] = \int_{t=t_0}^{t_1} L(\dot{q}(t),\, q(t),\, t)\, dt$$

its variation reads

$$\delta S[q(t),t] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(S[q(t) + \varepsilon w(t),\, t] - S[q(t),\, t] \right)$$

$$\begin{aligned}
\delta S[q(t),t] & = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(S[q(t) + \varepsilon w(t),\, t] - S[q(t),\, t] \right) = \\
& = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t=t_0}^{t_1} \left(L(\dot{q}(t) + \varepsilon w(t),\, q(t) + \varepsilon w(t),\, t) - L(\dot{q}(t),\, q(t),\, t) \right) dt = \\
& = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t=t_0}^{t_1} \left\{ L(\dot{q}(t),\, q(t),\, t) + \varepsilon \left[\frac{\partial L}{\partial \dot{q}} \dot{w}(t) + \frac{\partial L}{\partial q} w(t) \right] + o(\varepsilon) - L(\dot{q}(t),\, q(t),\, t) \right\} dt = \\
& = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t=t_0}^{t_1} \left\{ \varepsilon \left[\frac{\partial L}{\partial \dot{q}} \dot{w}(t) + \frac{\partial L}{\partial q} w(t) \right] + o(\varepsilon) \right\} dt = \\
& = \int_{t=t_0}^{t_1} \left\{ \frac{\partial L}{\partial \dot{q}} \dot{w}(t) + \frac{\partial L}{\partial q} w(t) \right\} dt = \\
& = \left[w(t) \frac{\partial L}{\partial \dot{q}} \right] \Bigg|_{t=t_0}^{t_1} + \int_{t=t_0}^{t_1} \left\{ -\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial q} \right\} w(t)\, dt\ .
\end{aligned}$$

If $q(t_0)$, $q(t_1)$ are prescribed, then $w(t_0) = w(t_1) = 0$ and thus

$$\delta S[q(t),t] = \int_{t=t_0}^{t_1} \left\{ -\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial q} \right\} \delta q(t)\, dt\ ,$$

having called $w(t) = \delta q(t)$ to stress that is the variation of function $q(t)$.

Stationary conditions, $\delta S = 0$. Stationary condition for $\forall \delta q(t)$ implies Lagrange equation,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0\ .$$

10.1 Higher-order derivatives

Method 1. If the Lagrangian function L depends on higher order derivatives,

$$L\left(q^{(n)}(t),\, q^{(n-1)}(t),\, \dots,\, q'(t),\, q(t),\, t\right)$$

it's possible to recast the problem defining the n-dimensional function, $\mathbf{q}(t)$,

$$\mathbf{q}(t) = \left(q^0(t), q^1(t), \dots, q^{n-1}(t)\right) := \left(q(t), q'(t), \dots, q^{(n-1)}(t)\right) \ .$$

With some abuse of notation in L, the functional S can be recast as

$$\begin{aligned} S[q(t),t] & = \int_{t=t_0}^{t_1} L(q^{(n)}(t),\, \dots,\, q(t),\, t)\, dt = \\ & = \int_{t=t_0}^{t_1} L(\dot{\mathbf{q}}(t),\, \mathbf{q}(t),\, t)\, dt \ . \end{aligned}$$

todo *Add constraints on components of* $\mathbf{q}(t)$

Repeating the computation, the variation of the functional reads

$$\delta S[\mathbf{q}(t),t] = \left[\delta \mathbf{q}^T(t) \frac{\partial L}{\partial \dot{\mathbf{q}}}\right]\Bigg|_{t=t_0}^{t_1} + \int_{t=t_0}^{t_1} \delta \mathbf{q}^T(t) \left\{ -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) + \frac{\partial L}{\partial \mathbf{q}} \right\} dt \ .$$

Method 2.

PROOF INDEX