basics - math

basics

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CONTENTS

This material is part of the **basics-books project**. It is also available as a .pdf document.

Contents.

- **Linear Algebra.** …, matrix factorization,…; basics of lots of numerical methods; **Vector and Tensor Algebra** provides the mathematical framework for manipulating vectors and tensors, the mathematical objects meant to represent **absolute quantities** - invariant under coordinate transformations - fundamental in geometry and physics.
- **Multivariable Calculus** provides the tools for working with continuous functions of many variables; **Differential Geometry** studies smooth curves, surfaces, volumes (and manifolds in general), within the framework of a Riemann structure, providing a metric to define distances, angles, curvatures and other geometric properties; **Vector and Tensor Calculus** studies vector and tensor fields defined on a manifold, along with their differentiation and integration, and thus being essential for a mature approach to geometry and physics;
- **Functional analysis** and **Complex Calculus** (complex analysis, transforms,…)
- ODEs
- PDEs
- Calculus of Variations: theoretical background; Lagrange multiplier method for constraints; sensitivty; gradient-based methods,…
- Optimization

Part I

Linear Algebra

CHAPTER

ONE

MATRICES

$\mathbf{A} \in \mathbb{K}^{m,n}$ with usually $\mathbb{K}^{m,n} = \mathbb{R}^{m,n}$ or $\mathbb{C}^{m,n}$

Hermitian matrix. The Hermitian matrix $\mathbf{A}^*$ of a matrix $\mathbf{A}$ is the transpose and complex conjugate matrix (if $\mathbb{K} = \mathbb{C}$),

$$[\mathbf{A}^*]_{ij} = A^*_{ji} \ ,$$

with the notation of a^* for the complex conjugate of a numerical quantity.

1.1 Subspaces

1.1.1 Range, Image

$$R(\mathbf{A}) = \{\mathbf{y} \in \mathbb{K}^m \mid \exists \mathbf{x} \in \mathbb{K}^m \ , \text{ s.t. } \mathbf{A}\mathbf{x} = \mathbf{y}\}$$

The range of a matrix $\mathbf{A}$ is the linear space built on the columns of $\mathbf{A}$, since the operation $\mathbf{A}\mathbf{x}$ represents nothing but a linear combination of the columns of matrix.

1.1.2 Null, Kernel

$$K(\mathbf{A}) = \{\mathbf{x} \in \mathbb{K}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0}\}$$

1.2 Theorem

1.2.1 Orthogonality of $R(\mathbf{A})$ and $K(\mathbf{A}^*)$

The following holds,

$$R(\mathbf{A}) \perp K(\mathbf{A}^*) \ ,$$

meaning that $\forall \mathbf{u} \in R(\mathbf{A})$ and $\forall \mathbf{v} \in K(\mathbf{A}^*)$, $\mathbf{u}^*\mathbf{v} = 0$.

Proof.

$$\mathbf{u} = \mathbf{A}\mathbf{x}$$
$$\mathbf{0} = \mathbf{A}^*\mathbf{v}$$

and thus, premultiplication by $\mathbf{x}^*$ of the second relation gives

$$0 = \mathbf{x}^*\mathbf{0} = \underbrace{\mathbf{x}^*\mathbf{A}^*}_{=(\mathbf{A}\mathbf{x})^*=\mathbf{u}^*} \mathbf{v} = \mathbf{u}^*\mathbf{v} \ .$$

This theorem becomes quite useful, e.g. for constrained linear systems and projections… (e.g. N-S, or other constrained linear systems…)

todo add links

CHAPTER

TWO

MATRIX FACTORIZATIONS

- **Singular Value Decomposition (SVD)**
- **Spectral decomposition** Eigenvalues, eigenvectors; Jordan canonical formula…
- **QR**
- **LU**
- **Schur**
- **Cholesky** Symmetric positive definite matrices have Choleski decomposition,

  $$\mathbf{M} = \mathbf{L}\mathbf{L}^* \ ,$$

  with $\mathbf{L}$ lower triangular matrix. And thus quite easy to "invert", for solving linear systems.

2.1 Singular Value Decomposition

Singular value decomposition of a matrix $\mathbf{A} \in \mathbb{C}^{m,n}$

$$\mathbf{A}_{(m,n)} = (\text{SVD}) = \mathbf{U}_{(m,m)}\mathbf{S}_{(m,n)}\mathbf{V}^*_{(n,n)}$$

with $\mathbf{U}^*\mathbf{U} = \mathbf{I}_{(m,m)}$, and $\mathbf{V}^*\mathbf{V} = \mathbf{I}_{(n,n)}$, $\mathbf{S} = \text{diag}\{\sigma_i\}$, $\sigma_i \geq 0$.

Exploiting the definition of matrix product, the SVD of matrix A can be written as

$$\mathbf{A} = \sum_{i=\min(m,n)} \sigma_i \mathbf{u}_i \mathbf{v}_i^* \ ,$$

see also economic decomposition below.

2.1.1 Properties

Relation with *range* and *kernel* the matrix.

$$\begin{aligned} \mathrm{R}(\mathbf{A}) &= \{\mathbf{u}_i | \sigma_i > 0\} \\ \mathrm{K}(\mathbf{A}) &= \{\mathbf{v}_i | \sigma_i = 0\} \\ \mathrm{R}(\mathbf{A}^*) &= \{\mathbf{v}_i | \sigma_i > 0\} \\ \mathrm{K}(\mathbf{A}^*) &= \{\mathbf{u}_i | \sigma_i = 0\} \end{aligned}$$

It's immediate to prove $R(\mathbf{A}) \perp K(\mathbf{A}^*)$.

Full or economic decomposition. In general the $\mathbf{S}$ of the full decompsition in not square.

$$\mathbf{A} = \text{(SVD)} = \mathbf{U}_{(m,m)}\mathbf{S}_{(m,n)}\mathbf{V}^*_{(n,n)} = \mathbf{U}^e_{(m,k)}\mathbf{S}^e_{(k,k)}\mathbf{V}^{e*}_{(k,n)} \ ,$$

with $k = \min(m, n)$.

2.1.2 Applications

Solution of under-determined linear systems

Norms and optimization

If L^2-norm is used for vector norms, see *Example 2.1.1*

$$\text{Find } \max_{||\mathbf{x}||=1} ||\mathbf{A}\mathbf{x}||^2$$

$$\mathbf{x}^*\mathbf{A}^*\mathbf{A}\mathbf{x} = \mathbf{x}^*\mathbf{V}\mathbf{S}^* \underbrace{\mathbf{U}^*\mathbf{U}}_{\mathbf{I}} \mathbf{S} \underbrace{\mathbf{V}^*\mathbf{x}}_{\mathbf{z}}$$

and $\mathbf{z} = \mathbf{V}^*\mathbf{x}$ is unitary as well (since $1 = \mathbf{x}^*\mathbf{x} = \mathbf{z}^*\mathbf{V}^*\mathbf{V}\mathbf{z} = \mathbf{z}^*\mathbf{z}$).

After defining $\mathbf{S}_2 := \mathbf{S}^*\mathbf{S}$, the problem thus becomes

$$\text{Find } \max_{||\mathbf{z}||=1} \mathbf{z}^*\mathbf{S}_2\mathbf{z}$$

Manipulating the objective function as $\sum_i \sigma_i^2 |z_i|^2$, the constraint optimization problem has global maximum σ_1^2 (sorted singular values from the largest to the smalles) when $\mathbf{z}_1 = (1, 0, 0, \dots, 0)^T$. Going back to the original variable, optimal condition

- is achieved for $\mathbf{x}_1 = \mathbf{v}_1$;
- has value $\max_{||\mathbf{x}||=1} ||\mathbf{A}\mathbf{x}|| = \sigma_1^2$
- and the response of the system is $\mathbf{y}_1 = \sigma_1 \mathbf{u}_1$ as

$$\mathbf{y}_1 := \mathbf{A}\mathbf{x}_1 = \mathbf{U}\mathbf{S}\mathbf{V}^*\mathbf{v}_1 = \sum_k \left(\sigma_k \mathbf{u}_k \mathbf{v}_k^*\right) \mathbf{v}_1 = \sigma_1 \mathbf{u}_1 \ .$$

Example 2.1.1 (Other norms - variations of the L^2-norm)

This kind of problem may results as the discrete counterpart of a continuous problem, as an example from *finite element methods*, where $\mathbf{x}$, $\mathbf{y}$ contain the coefficients of the basis functions. In this case, the discrete counterpart of the continous norm-measure of the continuous fields may involve a "mass matrix" (symmetric, definite positive - and thus with Choloeski factorization…),

$$\int_{\Omega_x} |x(\vec{r})|^2 d\vec{r} \simeq \mathbf{x}^*\mathbf{M}_x\mathbf{x}$$

$$\int_{\Omega_y} |y(\vec{r})|^2 d\vec{r} \simeq \mathbf{y}^*\mathbf{M}_y\mathbf{y}$$

Continuous and discrete optimization problems are

$$\text{Find } \max_{|x(\vec{r})|_{L^2(\Omega_x)}=1} |y|^2_{L^2(\Omega_y)}$$

$$\text{Find } \max_{\mathbf{x}^*\mathbf{M}_x\mathbf{x}=1} \mathbf{x}^*\mathbf{A}^*\mathbf{M}_y\mathbf{A}\mathbf{x}$$

with the relation $\mathbf{y} = \mathbf{A}\mathbf{x}$ between the discrete input and output.

This problem can be easily (and efficiently?) recast to the standard form of the problem, using *Cholesky decomposition* of matrix $\mathbf{M}_x = \mathbf{L}_x \mathbf{L}_x^*$, with the definition of the variable $\mathbf{z} = \mathbf{L}_x^* \mathbf{x}$

$$\text{Find } \max_{||\mathbf{z}||=1} \mathbf{z}^* \mathbf{L}_x^{-1} \mathbf{A}^* \mathbf{L}_y \mathbf{L}_y^* \mathbf{A} \mathbf{L}_x^{-*} \mathbf{z}$$

This problem can be efficiently solved with iterative algorithms to compute the SVD of the matrix $\widetilde{\mathbf{A}} := \mathbf{L}_y^* \mathbf{A} \mathbf{L}_x^{-*}$, that doesn't need the expensive full inversion of a matrix but only its action on a vector (instead of evaluating the inverse, a linear system - here triangular! Easier to solve - can be efficiently solved). Algorithms like **Arnoldi algorithm** evaluates the largest eigenvalues or singular values(if no options to set other goals) and the corresponding eigenvectors and singular vectors, alternating **power iterations** and **orthogonalization**. Here power iteration to evaluate the action of the matrix $\widetilde{\mathbf{A}} = \mathbf{L}_y^* \mathbf{A} \mathbf{L}_x^{-*}$ on a generic vector $\mathbf{z}$ is made of the following steps:

1. solution of the linear system $\mathbf{L}_x^* \mathbf{a} = \mathbf{z} \rightarrow \ \mathbf{a} = \dots$
2. matrix-vector multiplication $\mathbf{b} = \mathbf{A}\mathbf{a}$
3. matrix-vector multiplication $\mathbf{c} = \mathbf{L}_y^* \mathbf{b}$

Once the SVD is solved, with $\mathbf{z}_1 = \mathbf{v}_1$

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{L}_x^{-*} \mathbf{v}_1 \\ \mathbf{y}_1 &= \mathbf{A}\mathbf{x}_1 = \sigma_1 \mathbf{L}_y^{-*} \mathbf{u}_1 \end{aligned}$$

CHAPTER

THREE

LINEAR SYSTEMS

The linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

with $\mathbf{A} \in \mathbb{R}^{m,n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$ has solution if there exists (at least) a vector $\tilde{\mathbf{x}}$ whose product with $\mathbf{A}$ gives $\mathbf{b}$.

Condition for the existence of a solution. A solution exists if $\mathbf{b}$ belongs to the *range of* $\mathbf{A}$.

Uniqueness of a solution. If a solution $\tilde{\mathbf{x}}$ exists, it's unique if the *kernel of* $\mathbf{A}$ is empty, $K(\mathbf{A}) = \emptyset$. If the kernel is not empty,

$$\mathbf{b} = \mathbf{A}(\tilde{\mathbf{x}} + \mathbf{u}) = \mathbf{A}\tilde{\mathbf{x}} + \underbrace{\mathbf{A}\mathbf{u}}_{=\mathbf{0}} ,$$

for $\forall \mathbf{u} \in K(\mathbf{A})$, and thus an infinite number of solutions exists. Given a vector basis of the kernel $\mathbf{K}(\mathbf{A})$, where $\dim\left(K(\mathbf{A})\right) = n_K$, $\{\mathbf{u}_1, \dots, \mathbf{u}_{n_K}\}$, the general solution has n_K "degrees of arbitrarieness", since the general solution of the problem is

$$\tilde{\mathbf{x}} + \sum_{i=1}^{n_K} a_i \mathbf{u}_i = \tilde{\mathbf{x}} + \mathbf{U}\mathbf{a} .$$

todo treat under-, det-, over-determined lin sys

Part II

Multivariable Calculus

CHAPTER

FOUR

INTRODUCTION TO MULTI-VARIABLE CALCULUS

4.1 Function

4.2 Limit

4.3 Derivatives

4.4 Integrals

4.5 Theorems

4.5.1 Green's lemma

$$\begin{aligned}\int_S \frac{\partial F}{\partial y} dxdy &= -\oint_{\partial S} F dx \\ \int_S \frac{\partial G}{\partial x} dxdy &= \oint_{\partial S} G dy\end{aligned}$$

Proof for simple domains.

In a simple domain in x, so that the closed contour ∂S is delimited by the curves $y = Y_1(x)$, $y = Y_2(x) > Y_1(x)$, for $x \in [x_1, x_2]$,

$$\begin{aligned}\int_S \frac{\partial F}{\partial y} dxdy &= \int_{x=x_1}^{x_2} \int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial F}{\partial y} dy\, dx = \\ &= \int_{x=x_1}^{x_2} \left[F(x, Y_2(x)) - F(x, Y_1(x))\right] dx = \\ &= -\int_{x=x_1}^{x_2} F(x, Y_1(x)) - \int_{x=x_2}^{x_1} F(x, Y_2(x)) dx = \\ &= -\oint_{\partial S} F(x, y) dx\end{aligned}$$

In a simple domain in y, so that the closed contour ∂S is delimited by the curves $x = X_1(y)$, $x = X_2(y) > X_1(y)$ for $y \in [y_1, y_2]$,

$$\begin{aligned}
\int_S \frac{\partial G}{\partial x} dx dy = & \int_{y=y_1}^{y_2} \int_{x=X_1(y)}^{X_2(y)} \frac{\partial G}{\partial x} dx\, dy = \\
& = \int_{y=y_1}^{y_2} \left[G(X_2(y), y) - G(X_1(y), y) \right] dy = \\
& = \int_{y=y_1}^{y_2} G(X_1(y), y) dy + \int_{y=y_2}^{y_1} G(X_2(y), y) dy = \\
& = \oint_{\partial S} G(x, y) dy
\end{aligned}$$

Part III

Differential Geometry

CHAPTER

FIVE

INTRODUCTION TO DIFFERENTIAL GEOMETRY

5.1 Differential geometry in E^3

5.1.1 Curves

Parametric representation of curve in 3-dimensional (Euclidean) space E^3

$$\vec{r}(q)$$

Differential, $d\vec{r}$.

$$d\vec{r}(q) = \vec{r}\,'(q)\,dq\ .$$

Arc-length parameter, s. So that $ds = |d\vec{r}(s)|$ and thus

$$|d\vec{r}(s)| = |\vec{r}\,'(s)|\,|ds| \qquad \rightarrow \qquad |\vec{r}\,'(s)| = 1 \qquad \rightarrow \qquad \vec{r}\,'(s) = \hat{t}(s)\ .$$

Frenet basis. Using arc-length parameter, Frenet basis is naturally defined as the set $\{\hat{t}, \hat{n}, \hat{b}\}$:

- tangent unit vector, $\hat{t}(s) = \vec{r}\,'(s)$,
- normal unit vector, $\hat{r}''(s) = \hat{t}'(s) =: \kappa(s)\,\hat{n}(s)$, with $\kappa(s)$ local curvature
- binormal unit vector, $\hat{b}(s) = \hat{t}(s) \times \hat{n}(s)$

Using a general parameter, t, with some abuse of notation $\vec{r}(t) = \vec{r}(s(t))$ and indicating $\dot{()} = \frac{d}{dt}$,

- $\dot{\vec{r}} = \frac{ds}{dt}\frac{d\vec{r}}{ds} = \dot{s}\hat{t}$
- $\ddot{\vec{r}} = \dfrac{d}{dt}\dot{\vec{r}} = \dfrac{d}{dt}\left(\dot{s}\hat{t}\right) = \ddot{s}\hat{t} + \dfrac{ds}{dt}\dfrac{d}{ds}\hat{t} = \ddot{s}\hat{t} + \dot{s}^2\kappa\,\hat{n}$

Osculator circle. Circle with $R(s) = \frac{1}{\kappa(s)}$, in plane orthogonal to $\hat{b}(s)$, passing through $\vec{r}(s)$, and thus center in $\vec{r}_C(s) = \vec{r}(s) + \hat{n}R(s)$. Its parametric representation using its arc-length parameter p, with $\vec{r}(p=0) = \vec{r}(s)$ reads

$$\vec{r}(p) = \vec{r}_C(s) + R(s)\left[-\cos\left(\frac{p}{R(s)}\right)\hat{n}(s) + \sin\left(\frac{p}{R(s)}\right)\hat{t}(s)\right]\ .$$

Its first and second order derivatives w.r.t. the arc-length p evaluated in $p=0$, i.e. $\vec{r} = \vec{r}(s)$ read:

- first derivative in $p=0$,

  $$\hat{t}(p)\big|_{p=0} = \vec{r}\,'(p)\big|_{p=0} = \left[\sin\left(\frac{p}{R(s)}\right)\hat{n}(s) + \cos\left(\frac{p}{R(s)}\right)\hat{t}(s)\right]\Bigg|_{p=0} = \hat{t}(s)\ ,$$

  i.e. the osculator circle has the same tangent as the curve in the point.

- second derivative in $p=0$,

$$\kappa(p)\hat{n}(p)\big|_{p=0} = \vec{r}''(p)\big|_{p=0} = \frac{1}{R(s)}\left[\cos\left(\frac{p}{R(s)}\right)\hat{n}(s) - \sin\left(\frac{p}{R(s)}\right)\hat{t}(s)\right]\Bigg|_{p=0} = \frac{1}{R(s)}\hat{n}(s) = \kappa(s)\hat{n}(s)\ ,$$

i.e. the osculator circle has the same normal vector and curvature as the curve in the point.

5.1.2 Surfaces

$$\vec{r}(q^1, q^2)$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial q^1}\,dq^1 + \frac{\partial \vec{r}}{\partial q^2}\,dq^2 = \vec{b}_1\,dq^1 + \vec{b}_2\,dq^2$$

A third vector $\vec{b}_3 := \hat{n}$ can be defined so that $|\hat{n}| = 1$ and $\hat{n}\cdot\vec{b}_i = 0$, $i = 1:2$. For $i = 1:2$, $k = 1:2$

$$\frac{\partial \vec{b}_i}{\partial q^j} = \Gamma^k_{ij}\vec{b}_k = \Gamma^1_{ij}\vec{b}_1 + \Gamma^2_{ij}\vec{b}_2 + \Gamma^3_{ij}\vec{b}_3$$

so that

$$\Gamma^k_{ij} = \vec{b}^k \cdot \frac{\partial \vec{b}_i}{\partial q^j}$$

Normal vector.

$$\vec{n}(q^1, q^2) = \frac{\partial \vec{r}}{\partial q^1}(q^1, q^2) \times \frac{\partial \vec{r}}{\partial q^2}(q^1, q^2) = \vec{b}_1(q^1, q^2) \times \vec{b}_2(q^1, q^2)$$

Tangent plane.

$$(\vec{r} - \vec{r}(q^1, q^2)) \cdot \vec{n}(q^1, q^2) = 0$$

Length of elementary segment.

$$\begin{aligned}
|d\vec{r}|^2 &= d\vec{r}\cdot d\vec{r} = \\
&= \left(\vec{b}_1\,dq^1 + \vec{b}_2\,dq^2\right)\cdot\left(\vec{b}_1\,dq^1 + \vec{b}_2\,dq^2\right) = \\
&= g_{11}\,dq^1\,dq^1 + g_{12}\,dq^1\,dq^2 + g_{21}\,dq^2\,dq^1 + g_{22}\,dq^2\,dq^2 = g_{ij}\,dq^i\,dq^j
\end{aligned}$$

Second order approximation.

$$\begin{aligned}
\vec{r}(q^1 + dq^1, q^2 + dq^2) &= \vec{r}(q_1, q_2) + \frac{\partial \vec{r}}{\partial q^i}\,dq^i + \frac{\partial^2 \vec{r}}{\partial q^i \partial q^j}\,dq^i\,dq^j = \\
&= \vec{r}(q_1, q_2) + \vec{b}_i\,dq^i + \vec{b}_k\Gamma^k_{ij}\,dq^i\,dq^j + \hat{n}\,\Gamma^3_{ij}\,dq^i\,dq^j
\end{aligned}$$

so that

$$\begin{aligned}
\left[\vec{r}(q^1 + dq^1, q^2 + dq^2) - \vec{r}(q^1, q^2)\right]\cdot\hat{n} = \Gamma^3_{ij}\,dq^i\,dq^j &= \\
&= \hat{n}\cdot\frac{\partial^2 \vec{r}}{\partial q^i \partial q^j}\,dq^i\,dq^j = \\
&= \hat{n}\cdot\frac{\partial^2 \vec{r}}{\partial q^i \partial q^j}\,\vec{b}^i\cdot\vec{b}_k dq^k\,\vec{b}^j\cdot\vec{b}_l dq^l = \\
&= \underbrace{dq^k\vec{b}_k}_{d\vec{r}}\cdot\left[\hat{n}\cdot\frac{\partial^2 \vec{r}}{\partial q^i \partial q^j}\vec{b}^i\otimes\vec{b}^j\right]\cdot\underbrace{dq^l\vec{b}_l}_{d\vec{r}}
\end{aligned}$$

Curvature tensor.

Part IV

Vector and Tensor Algebra and Calculus

CHAPTER

SIX

TENSOR ALGEBRA

6.1 Basis

Definition 6.1.1 (Basis)

Definition 6.1.2 (Reciprocal basis)

In a inner product space, the reciprocal basis of a given basis $\{\vec{b}_a\}_{a=1:d}$ is the set of vectors $\{\vec{b}_b\}_{b=1:d}$, s.t.

$$\vec{b}^b \cdot \vec{b}_a = \delta^b_a \,.$$

6.2 Exterior algebra

$$\wedge$$

6.3 Exterior product

Generalization of the vector product

CHAPTER

SEVEN

TENSOR CALCULUS IN EUCLIDEAN SPACES

This section deals with tensor calculus in Euclidean space or on manifolds embedded in Euclidean spaces, focusing on d-dimensional spaces with $d \leq 3$, with *inner product*.

This section may rely on results of *differential geometry*.

7.1 Coordinates

A set of parameters $\{q^a\}_{a=1:d}$ to represent vector (or point) in space,

$$\vec{r}(q^a)$$

if $\vec{r} \in E^d$, $a = 1 : d$.

In E^3,

- **Coordinate lines**, 2-parameter family of lines, keeping 2 coordinates constant. As an example, coordinate lines with constant q^2, q^3

$$\vec{r}_1(q^1) = \vec{r}(q^1, \bar{q}^2, \bar{q}^3) \ .$$

- **Coordinate surfaces,** 1-parameter family of surfaces, keeping 1 coordinate constant. As an example, coordinate surfaces with constant q^1,

$$\vec{r}_{23}(q^2, q^3) = \vec{r}(\bar{q}^1, q^2, q^3) \ .$$

Definition 7.1.1 (Regular parametrization)

If $\frac{\partial \vec{r}}{\partial q^a} \neq 0$.

7.1.1 Natural basis

Definition 7.1.2 (Natural basis)

Vectors of natural basis

$$\vec{b}_a := \frac{\partial \vec{r}}{\partial q^a}$$

Definition 7.1.3 (Reciprocal basis (todo move to Tensor Algebra))

Given a basis $\{\vec{b}_a\}_a$, its reciprocal basis the set of vector $\{\vec{b}^b\}_b$ defined as

$$\vec{b}^b \cdot \vec{b}_a = \delta^b_a \ ,$$

being δ^b_a Kronecker delta.

Definition 7.1.4 (Christoffel symbols)

Christoffel symbols (of the 2^{nd} kind) are defined as the components of the partial derivatives of the vectors of a natural basis w.r.t. the coordinates referred to the natural basis itself,

$$\frac{\partial \vec{b}_a}{\partial q^b} = \Gamma^c_{ab} \vec{b}_c \tag{7.1}$$

Properties of Christoffel symbols

Exploiting the definition of reciprocal basis, Christoffel symbols can be written as

$$\Gamma^c_{ab} = \vec{b}^c \cdot \frac{\partial \vec{b}_a}{\partial q^b} \ .$$

Symmetry. Symmetry os the lower indices

$$\Gamma^c_{ab} = \Gamma^c_{ba} \ ,$$

readily follows Schwartz theorem about partial derivatives

$$\frac{\partial \vec{b}_a}{\partial q^b} = \frac{\partial}{\partial q^c}\frac{\partial \vec{r}}{\partial q^a} = \frac{\partial}{\partial q^a}\frac{\partial \vec{r}}{\partial q^b} = \frac{\partial \vec{b}_b}{\partial q^a}$$

7.2 Fields

Function of the points in space $F : E^d \to V^r$, being V^r a space of tensors of order r.

7.3 Differential operators

7.3.1 Directional derivative

$$F(\vec{r}) = F\left(\vec{r}\left(q^a\right)\right) = f(q^a)$$

$$f(q^a + \beta\Delta q^a) = F(\vec{r}(q^a + \beta\Delta q^a))$$

$$\vec{r}(q^a) + \alpha\vec{v} = \vec{r}(q^a + \beta\Delta q^a) \sim \vec{r}(q^a) + \frac{\partial \vec{r}}{\partial q^b}\beta\Delta q^b$$

$$\alpha\vec{v} \sim \beta\frac{\partial \vec{r}}{\partial q^b}(q^a)\,\Delta q^b = \beta\vec{b}_b(q^a)\Delta q^b \qquad \to \qquad \Delta q^b = \frac{\alpha}{\beta}\vec{b}^b(q^a)\cdot\vec{v}$$

The directional derivative for an arbitrary vector $\vec{v} \in V$

$$\frac{d}{d\alpha} F(\vec{r} + \alpha \vec{v}) \bigg|_{\alpha=0}$$

is evaluated as the limit for $\alpha \rightarrow 0$ of the incremental ratio

$$\begin{aligned}
\frac{F(\vec{r} + \alpha\vec{v}) - F(\vec{r})}{\alpha} & \sim \frac{f(q^a + \beta \Delta q^a) - f(q^a)}{\alpha} = \\
& \sim \frac{1}{\alpha} \frac{\partial f}{\partial q^b}(q^a) \beta \Delta q^b = \\
& \sim \vec{v} \cdot \vec{b}^b(q^a) \frac{\partial f}{\partial q^b}(q^a) = \\
& = \vec{v} \cdot \nabla F(\vec{r})
\end{aligned}$$

7.3.2 Gradient

The gradient is the differential operator is the first-order differential operator appearing in the definition of the directional derivative, $\nabla F(\vec{r})$. It takes a tensor field $F(\vec{r})$ of order r and gives a tensor field $\nabla F(\vec{r})$ of order $r + 1$. Given a set of coordinates $\{q^a\}_{a=1:d}$, the gradient can be written using the reciprocal basis of the natural basis as

$$\nabla F(\vec{r}) = \vec{b}^b(\vec{r}) \frac{\partial F}{\partial q^b}(\vec{r}) \tag{7.2}$$

Examples. …

Example 7.3.1 (Gradient of a scalar field - with general coordinates q^a)

Applying the definition (7.2) of gradient operator, it readily follows

$$\nabla F = \vec{b}^a \frac{\partial F}{\partial q^a}$$

Example 7.3.2 (Gradient of a vector field - with general coordinates q^a)

Applying the definition (7.2) of gradient operator, rule for the derivative of a product and the definition (7.1) of Christoffel symbols to write derivatives of base vectors,

$$\begin{aligned}
\nabla F & = \vec{b}^a \frac{\partial}{\partial q^a} \left(F^b \vec{b}_b \right) = \\
& = \vec{b}^a \left[\frac{\partial F^b}{\partial q^a} \vec{b}_b + F^b \frac{\partial \vec{b}_b}{\partial q^a} \right] = \\
& = \vec{b}^a \left[\frac{\partial F^b}{\partial q^a} \vec{b}_b + F^b \, \Gamma^c_{ab} \, \vec{b}_c \right] = \\
& = \vec{b}^a \otimes \vec{b}_b \left[\frac{\partial F^b}{\partial q^a} + \Gamma^b_{ac} \, F^c \right] .
\end{aligned}$$

Example 7.3.3 (Gradient of a 2^{nd}-order tensor field - with general coordinates q^a)

Applying the definition (7.2) of gradient operator, rule for the derivative of a product and the definition (7.1) of Christoffel symbols to write derivatives of base vectors,

$$
\begin{aligned}
\nabla F = \vec{b}^a \frac{\partial}{\partial q^a} \left(F^{bc} \vec{b}_b \otimes \vec{b}_c \right) = \\
= \vec{b}^a \left[\frac{\partial F^{bc}}{\partial q^a} \vec{b}_b \vec{b}_c + F^{bc} \frac{\partial \vec{b}_b}{\partial q^a} \vec{b}_c + F^{bc} \vec{b}_b \frac{\partial \vec{b}_c}{\partial q^a} \right] = \\
= \vec{b}^a \left[\frac{\partial F^{bc}}{\partial q^a} \vec{b}_b \vec{b}_c + F^{bc} \Gamma^d_{ab} \vec{b}_d \vec{b}_c + F^{bc} \Gamma^{ac}_d \vec{b}_b \vec{b}_d \right] = \\
= \vec{b}^a \otimes \vec{b}_b \otimes \vec{b}_c \left[\frac{\partial F^{bc}}{\partial q^a} + \Gamma^b_{ad} F^{dc} + \Gamma^c_{ad} F^{bd} \right] .
\end{aligned}
$$

7.3.3 Divergence

Divergence opearator is a first-order differential operator that can be defined as the contraction of the first two indices of the gradient,

$$
\nabla \cdot F = C_1^2 \left(\nabla F \right) .
$$

It takes a tensor field $F(\vec{r})$ of order $r \geq 1$ and gives a tensor field $\nabla \cdot F(\vec{r})$ of order $r - 1 \geq 0$.

Example 7.3.4 (Divergence of a vector field - with general coordiantes q^a)

Applying contraction to the gradient of a vector field, it readily follows,

$$
\begin{aligned}
\nabla \cdot \left(F^b \vec{b}_b \right) = C_1^2 \left(\nabla F \right) = \\
= C_1^2 \left(\vec{b}^a \otimes \vec{b}_b \left[\frac{\partial F^b}{\partial q^a} + \Gamma^b_{ac} F^c \right] \right) = \\
= \frac{\partial F^a}{\partial q^a} + \Gamma^a_{ac} F^c
\end{aligned}
$$

Example 7.3.5 (Divergence of a 2^{nd}-order tensor field - with general coordiantes q^a)

Applying contraction to the gradient of a vector field, it readily follows,

$$
\begin{aligned}
\nabla \cdot \left(F^{bc} \vec{b}_b \otimes \vec{b}_c \right) = C_1^2 \left(\nabla F \right) = \\
= C_1^2 \left(\vec{b}^a \otimes \vec{b}_b \otimes \vec{b}_c \left[\frac{\partial F^{bc}}{\partial q^a} + \Gamma^b_{ad} F^{dc} + \Gamma^c_{ad} F^{bd} \right] \right) = \\
= \vec{b}_c \left[\frac{\partial F^{ac}}{\partial q^a} + \Gamma^a_{ad} F^{dc} + \Gamma^c_{ad} F^{ad} \right]
\end{aligned}
$$

7.3.4 Laplacian

Laplacian operator is second-order differential operator that can be defined as the divergence of the gradient,

$$\Delta F = \nabla^2 F = \nabla \cdot \nabla F \; .$$

Example 7.3.6 (Laplacian of a scalar field - with general coordinates q^a)

$$\begin{aligned}
\nabla \cdot \nabla F = C_1^2 \left[\nabla \left(\nabla F \right) \right] = \\
&= C_1^2 \left[\nabla \left(\vec{b}^a \frac{\partial F}{\partial q^a} \right) \right] = \\
&= C_1^2 \left[\nabla \left(\vec{b}_b \, g^{ab} \, \frac{\partial F}{\partial q^a} \right) \right] = \\
&= C_1^2 \left[\vec{b}^c \frac{\partial}{\partial q^c} \left(\vec{b}_b \, g^{ab} \, \frac{\partial F}{\partial q^a} \right) \right] = \\
&= C_1^2 \left\{ \vec{b}^c \left[\vec{b}_b \, \frac{\partial}{\partial q^c} \left(g^{ab} \, \frac{\partial F}{\partial q^a} \right) + g^{ab} \frac{\partial F}{\partial q^a} \frac{\partial \vec{b}_b}{\partial q^c} \right] \right\} = \\
&= C_1^2 \left\{ \vec{b}^c \left[\vec{b}_b \, \frac{\partial}{\partial q^c} \left(g^{ab} \, \frac{\partial F}{\partial q^a} \right) + g^{ab} \frac{\partial F}{\partial q^a} \, \Gamma^d_{bc} \, \vec{b}_d \right] \right\} = \\
&= C_1^2 \left\{ \vec{b}^c \, \vec{b}_b \left[\frac{\partial}{\partial q^c} \left(g^{ab} \, \frac{\partial F}{\partial q^a} \right) + g^{ad} \, \Gamma^b_{cd} \, \frac{\partial F}{\partial q^a} \right] \right\} = \\
&= \frac{\partial}{\partial q^b} \left(g^{ab} \, \frac{\partial F}{\partial q^a} \right) + g^{ad} \, \Gamma^b_{bd} \, \frac{\partial F}{\partial q^a} \; .
\end{aligned}$$

Example 7.3.7 (Laplacian of a vector field - with general coordinates q^a)

$$\begin{aligned}
\nabla \cdot \nabla F = C_1^2 \left[\nabla \left(\nabla F \right) \right] = \\
&= C_1^2 \left\{ \nabla \left[\vec{b}^a \, \vec{b}_b \left(\frac{\partial F^b}{\partial q^a} + \Gamma^b_{ac} F^c \right) \right] \right\} = \\
&= C_1^2 \left\{ \nabla \left[\vec{b}_c \, \vec{b}_b \, g^{ac} \left(\frac{\partial F^b}{\partial q^a} + \Gamma^b_{ac} F^c \right) \right] \right\} = \\
&= C_1^2 \left\{ \nabla \cdot \left((\nabla F)^{cb} \, \vec{b}_c \, \vec{b}_b \right) \right\} = \\
&= C_1^2 \left\{ \vec{b}^a \, \vec{b}_c \, \vec{b}_b \, \left[\frac{\partial (\nabla F)^{cb}}{\partial q^a} + \Gamma^c_{ad} (\nabla F)^{db} + \Gamma^b_{ad} (\nabla F)^{cd} \right] \right\} = \\
&= \vec{b}_b \, \left[\frac{\partial (\nabla F)^{ab}}{\partial q^a} + \Gamma^a_{ad} (\nabla F)^{db} + \Gamma^b_{ad} (\nabla F)^{ad} \right] = \; .
\end{aligned}$$

7.3.5 Curl

7.4 Integrals in E^d, $d \leq 3$

7.4.1 Line integrals

Density

Integrals

$$\int_{\vec{r}\in\gamma} F(\vec{r})$$

represent the summation of contributions $F(\vec{r})$ over elementary segments of path γ, whose dimension is $|d\vec{r}|$, i.e. implicitly means

$$\int_{\vec{r}\in\gamma} F(\vec{r}) = \int_{\vec{r}\in\gamma} F(\vec{r})\,|d\vec{r}|\ .$$

Given a regular parametrization of the curve $\vec{r}(q^1)$ (with increasing q^1 so that $|dq^1| = dq^1$), and the differential $d\vec{r} = \vec{r}'(q^1)\,dq^1$, the integral can be written as an integral in the parameter q^1

$$\int_{q=q^1_a}^{q^1_b} F(\vec{r}(q^1))\,|\vec{r}'(q^1)|\,dq^1\ ,$$

with $\vec{r}(q^1_a)$, $\vec{r}(q^1_b)$ the extreme points of path γ.

Work

Integrals

$$\int_{\vec{r}\in\gamma} F(\vec{r})\cdot\hat{t}(\vec{r})$$

implicitly mean

$$\int_{\vec{r}\in\gamma} F(\vec{r})\cdot\hat{t}(\vec{r}) = \int_{\vec{r}\in\gamma} F(\vec{r})\cdot\hat{t}(\vec{r})|d\vec{r}| = \int_{\vec{r}\in\gamma} F(\vec{r})\cdot d\vec{r}\ ,$$

as $\hat{t} = \frac{d\vec{r}}{|d\vec{r}|}$. Given a regular parametrization of the curve $\vec{r}(q^1)$ (with increasing q^1 so that $|dq^1| = dq^1$), and the differential $d\vec{r} = \vec{r}'(q^1)\,dq^1$, the integral can be written as an integral in the parameter q^1

$$\int_{q^1=q^1_a}^{q^1_b} F(\vec{r}(q^1))\cdot\vec{r}'(q^1)\,dq^1$$

7.4.2 Surface integrals

Given two coordinates q^1, q^2 describing a surface, $\vec{r}(q^1,q^2)$ the elementary surface with unit normal reads

$$\hat{n}\,dS = d\vec{r}_1\times d\vec{r}_2 = \frac{\partial\vec{r}}{\partial q^1}\times\frac{\partial\vec{r}}{\partial q^2}\,dq^1\,dq^2\ ,$$

and the elementary surface thus reads

$$|dS| = |\hat{n}dS| = \left|\frac{\partial\vec{r}}{\partial q^1}\times\frac{\partial\vec{r}}{\partial q^2}\,dq^1\,dq^2\right|$$

Density

Integrals

$$\int_{\vec{r}\in S} F(\vec{r})$$

implicitly mean

$$\int_{\vec{r}\in S} F(\vec{r}) = \int_{\vec{r}\in S} F(\vec{r})|dS|\ .$$

Given regular parametrization of the surface, $\vec{r}(q^1,\ q^2),\ (q^1, q^2)\ \in\ Q^{12}$, the integral can be written as the multi-dimensional integral in coordinates $q^1,\ q^2$,

$$\int_{\vec{r}\in S} F(\vec{r}) = \int_{(q^1,q^2)\in Q^{12}} F(\vec{r}(q^1,q^2)) \left| \frac{\partial \vec{r}}{\partial q^1} \times \frac{\partial \vec{r}}{\partial q^2}\, dq^1\, dq^2 \right|$$

Flux

Integrals

$$\int_{\vec{r}\in S} \hat{n}(\vec{r}) \cdot F(\vec{r})$$

implicitly mean

$$\int_{\vec{r}\in S} \hat{n}(\vec{r}) \cdot F(\vec{r}) = \int_{\vec{r}\in S} \hat{n}(\vec{r}) \cdot F(\vec{r})|dS|$$

Given regular parametrization of the surface, $\vec{r}(q^1,\ q^2),\ (q^1, q^2)\ \in\ Q^{12}$, the integral can be written as the multi-dimensional integral in coordinates $q^1,\ q^2$,

$$\int_{\vec{r}\in S} \hat{n}(\vec{r}) \cdot F(\vec{r}) = \int_{(q^1,q^2)\in Q^{12}} \frac{\partial \vec{r}}{\partial q^1} \times \frac{\partial \vec{r}}{\partial q^2} \cdot F(\vec{r}(q^1,q^2))\, dq^1\, dq^2$$

7.4.3 Volume

$$dV = \frac{\partial \vec{r}}{\partial q^1} \cdot \frac{\partial \vec{r}}{\partial q^2} \times \frac{\partial \vec{r}}{\partial q^3}\, dq^1\, dq^2\, dq^3\ .$$

Density

Integrals

$$\int_{\vec{r}\in V} F(\vec{r})$$

implicitly mean

$$\int_{\vec{r}\in V} F(\vec{r}) = \int_{\vec{r}\in V} F(\vec{r})\, |dV|\ .$$

Given regular parametrization of the volume, $\vec{r}(q^1,\ q^2,\ q^3),\ (q^1, q^2, q^3) \in Q$, the integral can be written as the multi-dimensional integral in coordinates $q^1,\ q^2,\ q^3$,

$$\int_{\vec{r}\in V} F(\vec{r})|dV| = \int_{(q^1,q^2,q^3)\in Q} F(\vec{r}(q^1,q^2,q^3)) \left| \frac{\partial \vec{r}}{\partial q^1} \cdot \frac{\partial \vec{r}}{\partial q^2} \times \frac{\partial \vec{r}}{\partial q^3}\, dq^1\, dq^2\, dq^3 \right|\ .$$

7.4.4 Theorems

Gradient theorem

$$\int_V \nabla f = \oint_{\partial V} f \hat{n}$$

Divergence theorem

$$\int_V \nabla \cdot \vec{f} = \oint_{\partial V} \vec{f} \cdot \hat{n}$$

Curl theorem

$$\int_S \left[\nabla \times \vec{f}\right] \cdot \hat{n} = \oint_{\partial S} \vec{f} \cdot \hat{t}$$

7.5 Tensor Calculus in Euclidean Spaces - Cartesian coordinates in E^3

Using Cartesian coordinates $(q^1, q^2, q^3) = (r, \theta, z)$ and Cartesian base vectors (uniform in space, so that their derivatives are zero), a point in Euclidean vector space E^3 can be represented as

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}\,.$$

7.5.1 Natural basis, reciprocal basis, metric tensor, and Christoffel symbols

Cartesian coordinates in Euclidean spaces are a very special coordinate system, with reciprocal basis everywhere coinciding with natural basis, with uniform basis in space (zero second-order derivative of space w.r.t. coordinates, and thus zero first order derivative of base vectors, and thus identically zero Christoffel symbols), and components of the metric tensor equal to the identity matrix

$$\begin{cases} \vec{b}_1 = \vec{b}^1 = \hat{x} \\ \vec{b}_2 = \vec{b}^2 = \hat{y} \\ \vec{b}_3 = \vec{b}^3 = \hat{z} \end{cases}$$

$$[g_{ab}] = [g^{ab}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Gamma^c_{ab} = 0 \qquad , \qquad \forall a, b, c = 1:3\,.$$

7.5.2 Differential operators

Gradient

Example 7.5.1 (Gradient of a scalar field)

$$\nabla F = \hat{x}\,\partial_x F + \hat{y}\,\partial_y F + \hat{z}\,\partial_z F_z$$

Example 7.5.2 (Gradient of a vector field)

$$\begin{aligned}\nabla F &= \nabla (F_x \hat{x} + F_y \hat{y} + F_z \hat{z}) = \\ &= \hat{x} \otimes \hat{x}\,\partial_x F_x + \hat{x} \otimes \hat{y}\,\partial_x F_y + \hat{x} \otimes \hat{z}\,\partial_x F_z + \\ &+ \hat{y} \otimes \hat{x}\,\partial_y F_x + \hat{y} \otimes \hat{y}\,\partial_y F_y + \hat{y} \otimes \hat{z}\,\partial_y F_z + \\ &+ \hat{z} \otimes \hat{x}\,\partial_z F_x + \hat{z} \otimes \hat{y}\,\partial_z F_y + \hat{z} \otimes \hat{z}\,\partial_z F_z +\end{aligned}$$

Example 7.5.3 (Gradient of a 2^{nd}-order tensor field)

Directional derivative

Divergence

Example 7.5.4 (Divergence of a vector field)

$$\begin{aligned}\nabla \cdot F &= \nabla \cdot \left(F_x\,\hat{x} + F_y\,\hat{y} + F_z\,\hat{z} \right) = \\ &= \partial_x F_x + \partial_y F_y + \partial_z F_z \ .\end{aligned}$$

Example 7.5.5 (Divergence of a 2^{nd}-order tensor field)

$$\begin{aligned}\nabla \cdot F &= \nabla \cdot (F_{ab} \vec{e}_a \otimes \vec{e}_b) = \\ &= \vec{e}_c \frac{\partial F_{ab}}{\partial q^a} = \\ &= \hat{x} \left[\partial_x F_{xx} + \partial_y F_{yx} + \partial_z F_{zx} \right] + \\ &+ \hat{y} \left[\partial_x F_{xy} + \partial_y F_{yy} + \partial_z F_{zy} \right] + \\ &+ \hat{z} \left[\partial_x F_{xz} + \partial_y F_{yz} + \partial_z F_{zz} \right] \ .\end{aligned}$$

Laplacian

Example 7.5.6 (Laplacian of a scalar field)

$$\nabla^2 F = \partial_{xx} F + \partial_{yy} F + \partial_{zz} F$$

Example 7.5.7 (Laplacian of a vector field)

7.6 Tensor Calculus in Euclidean Spaces - cylindrical coordinates in E^3

7.6.1 Cylindrical coordiantes and cylindrical coordinates

Using cylindrical coordinates $(q^1, q^2, q^3) = (r, \theta, z)$ and cylindrical base vectors (uniform in space, so that their derivatives are zero), a point in Euclidean vector space E^3 can be represented as

$$\vec{r} = r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y} + z\,\hat{z}\ .$$

7.6.2 Natural basis, reciprocal basis, metric tensor, and Christoffel symbols

Natural basis

Natural basis reads

$$\begin{cases} \vec{b}_1 = \dfrac{\partial\vec{r}}{\partial q^1} = \dfrac{\partial\vec{r}}{\partial r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \\ \vec{b}_2 = \dfrac{\partial\vec{r}}{\partial q^2} = \dfrac{\partial\vec{r}}{\partial \theta} = -r\sin\theta\,\hat{x} + r\cos\theta\,\hat{y} \\ \vec{b}_3 = \dfrac{\partial\vec{r}}{\partial q^3} = \dfrac{\partial\vec{r}}{\partial z} = \hat{z} \end{cases}$$

Metric tensor

Covariant components of metric tensors,

$$g_{ab} = \vec{b}_a \cdot \vec{b}_b\ ,$$

can be collected in the diagonal matrix

$$[g_{ab}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ ,$$

while its contra-variant components can be collected in the inverse matrix (easy to compute, since $[g_{ab}]$ is diagonal),

$$[g^{ab}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\ .$$

Reciprocal basis

Reciprocal basis is readily evaluated using $\vec{b}^a = g^{ab}\,\vec{b}_b$,

$$\begin{cases}\vec{b}^1 = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \\ \vec{b}^2 = -\dfrac{1}{r}\sin\theta\,\hat{x} + \dfrac{1}{r}\cos\theta\,\hat{y} \\ \vec{b}^3 = \hat{z}\end{cases}$$

Physical basis

Since metric tensor is diagonal, the cylindrical coordinate system is orthogonal, and its natural and reciprocal basis are orthogonal. A unit orthogonal basis, usually named **physical basis** with unit vector with no physical dimension, is evalated by normalization process,

$$\begin{cases}\hat{r} = \hat{b}_1 = \dfrac{\vec{b}_1}{g_{11}} = \dfrac{\vec{b}^1}{g^{11}} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \\ \hat{\theta} = \hat{b}_2 = \dfrac{\vec{b}_2}{g_{22}} = \dfrac{\vec{b}^2}{g^{22}} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} \\ \hat{z} = \hat{b}_3 = \dfrac{\vec{b}_3}{g_{33}} = \dfrac{\vec{b}^3}{g^{33}} = \hat{z}\ .\end{cases}$$

Derivatives of natural basis and Christoffel symbols

Derivatives of the natural basis read

$$\begin{aligned}\frac{\partial\vec{b}_1}{\partial q^1} &= \vec{0} \\ \frac{\partial\vec{b}_2}{\partial q^2} &= -r\cos\theta\,\hat{x} - r\sin\theta\,\hat{y} = -q^1\,\vec{b}_1 \\ \frac{\partial\vec{b}_3}{\partial q^3} &= \vec{0} \\ \frac{\partial\vec{b}_2}{\partial q^1} = \frac{\partial\vec{b}_1}{\partial q^2} &= -\sin\theta\,\hat{x} + \cos\theta\hat{y} = \frac{1}{q^1}\,\vec{b}_2 \\ \frac{\partial\vec{b}_3}{\partial q^1} = \frac{\partial\vec{b}_1}{\partial q^3} &= \vec{0} \\ \frac{\partial\vec{b}_3}{\partial q^2} = \frac{\partial\vec{b}_2}{\partial q^3} &= \vec{0}\end{aligned}$$

so that non-zero Christoffel symbols of a cylindrical coordinate system are

$$\begin{aligned}\Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{q^1} \\ \Gamma^1_{22} &= -q^1\ .\end{aligned}$$

7.6.3 Differential operators

Gradient

Example 7.6.1 (Gradient of a scalar field)

$$\begin{aligned}\nabla F &= \vec{b}^a \frac{\partial F}{\partial q^a} = \\ &= \hat{b}_a\, g^{aa}\, \frac{\partial F}{\partial q^a} = \\ &= \hat{r}\, \frac{\partial F}{\partial r} + \hat{\theta}\, \frac{1}{r}\, \frac{\partial F}{\partial \theta} + \hat{z} \frac{\partial F}{\partial z}\ .\end{aligned}$$

Example 7.6.2 (Gradient of a vector field)

$$\begin{aligned}\nabla F = \vec{b}^a \otimes \vec{b}_b \left[\frac{\partial F^b}{\partial q^a} + \Gamma^b_{ac}\, F^c \right] = & & \\ = \cdots = & & \\ = \vec{b}^1 \otimes \vec{b}_1\, \partial_1 F^1 \qquad & + \vec{b}^1 \otimes \vec{b}_2\ [\partial_1 F^2 + \Gamma^2_{12} F^2] & + \vec{b}^1 \otimes \vec{b}_3\, \partial_1 F^3 \\ + \vec{b}^2 \otimes \vec{b}_1\ [\partial_2 F^1 + \Gamma^1_{22} F^2] & + \vec{b}^2 \otimes \vec{b}_2\ [\partial_2 F^2 + \Gamma^2_{21} F^1] & + \vec{b}^2 \otimes \vec{b}_3\, \partial_2 F^3 \\ + \vec{b}^3 \otimes \vec{b}_1\, \partial_3 F^1 \qquad & + \vec{b}^3 \otimes \vec{b}_2\, \partial_3 F^2 & + \vec{b}^3 \otimes \vec{b}_3\, \partial_3 F^3 \\ = \hat{r} \otimes \hat{r}\, \partial_r F_r \qquad & + \hat{r} \otimes \hat{\theta}\, \frac{1}{r}\, [\partial_r (r F_\theta) + F_\theta] & + \hat{r} \otimes \hat{z}\, \partial_r F_z \\ + \hat{\theta} \otimes \hat{r}\, \frac{1}{r}\, [\partial_\theta F_r - r F_\theta] & + \hat{\theta} \otimes \hat{\theta}\, \left[\partial_\theta \left(\frac{F_\theta}{r} \right) + \frac{F_r}{r} \right] & + \hat{\theta} \otimes \hat{z}\, \frac{1}{r} \partial_\theta F_z \\ + \hat{z} \otimes \hat{r}\, \partial_z F_x \qquad & + \hat{z} \otimes \hat{\theta}\, \frac{1}{r} \partial_\theta F_y & + \hat{z} \otimes \hat{z}\, \partial_z F_z\ .\end{aligned}$$

Example 7.6.3 (Gradient of a 2^{nd}-order tensor field)

Directional derivative

Divergence

Example 7.6.4 (Divergence of a vector field)

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial F^a}{\partial q^a} + \Gamma^a_{ac}\, F^c = \\ &= \frac{\partial F_r}{\partial r} + \frac{\partial}{\partial \theta} \left(\frac{F_\theta}{r} \right) + \frac{F_\theta}{r} + \frac{\partial F_z}{\partial z}\ .\end{aligned}$$

Example 7.6.5 (Divergence of a 2^{nd}-order tensor field)

Laplacian

Example 7.6.6 (Laplacian of a scalar field)

Example 7.6.7 (Laplacian of a vector field)

7.7 Tensor Calculus in Euclidean Spaces - Spehrical coordinates in E^3

Using spherical coordinates $(q^1, q^2, q^3) = (r, \phi, \theta)$ and spherical base vectors (uniform in space, so that their derivatives are zero), a point in Euclidean vector space E^3 can be represented as

$$\vec{r} = r\cos\theta\sin\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\phi\,\hat{z}\ .$$

7.7.1 Natural basis, reciprocal basis, metric tensor, and Christoffel symbols

7.7.2 Differential operators

Gradient

Example 7.7.1 (Gradient of a scalar field)

Example 7.7.2 (Gradient of a vector field)

Example 7.7.3 (Gradient of a 2^{nd}-order tensor field)

Directional derivative

Divergence

Example 7.7.4 (Divergence of a vector field)

Example 7.7.5 (Divergence of a 2^{nd}-order tensor field)

Laplacian

Example 7.7.6 (Laplacian of a scalar field)

Example 7.7.7 (Laplacian of a vector field)

CHAPTER

EIGHT

TIME DERIVATIVE OF INTEGRALS OVER MOVING DOMAINS

Some results about time derivatives over moving domains are collected here.

8.1 Volume density

Reynolds transport theorem. Given a volume $V(t)$ with boundary $\partial V(t)$, whose points $\vec{r} \in \partial V(t)$ have velocity $\vec{v}_b$,

$$\frac{d}{dt} \int_{V(t)} f = \int_{V(t)} \frac{\partial f}{\partial t} + \oint_{\partial V(t)} f \vec{v}_b \cdot \hat{n} \, .$$

“Proof”

8.2 Flux across a surface

$$\frac{d}{dt} \int_{S(t)} \vec{f} \cdot \hat{n} = \int_{S(t)} \frac{\partial \vec{f}}{\partial t} \cdot \hat{n} + \int_{S(t)} \nabla \cdot \vec{f} \; \vec{v}_b \cdot \hat{n} - \int_{\partial S(t)} \vec{v}_b \times \vec{f} \cdot \hat{t}$$

“Proof”

8.3 Work line integral along a line

$$\frac{d}{dt} \int_{\ell(t)} \vec{f} \cdot \hat{t} = \int_{\ell(t)} \frac{\partial \vec{f}}{\partial t} \cdot \hat{t} + \int_{\ell(t)} \nabla \times \vec{f} \cdot \vec{v}_b \times \hat{t} + \vec{f}_B \cdot \vec{v}_B - \vec{f}_A \cdot \vec{v}_A$$

“Proof”

Part V

Functional Analysis

CHAPTER NINE

INTRODUCTION TO FUNCTIONAL ANALYSIS

- Lebesgue integral
- L^p, H^p function spaces
- Banach and Hilbert spaces

CHAPTER

TEN

DIRAC'S DELTA

Dirac's delta $\delta(x)$ is a distribution, or generalized function, with the following properties

1. $$\int_D \delta(x-x_0)\,dx = 1 \quad \text{if } x_0 \in D \tag{10.1}$$

2. $$\int_D f(x)\delta(x-x_0)\,dx \quad \text{if } x_0 \in D \tag{10.2}$$

for $\forall f(x)$ "regular" **todo** *what does regular mean?*

10.1 Dirac's delta in terms of regular functions

10.1.1 Piece-wise constant

$$\delta(x) \sim r_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon} & x \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \\ 0 & \text{otherwise} \end{cases}$$

Properties - proof.

1. Unitariety

   $$\int_{x=-\infty}^{\infty} r_\varepsilon(x-x_0)\,dx = \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} \frac{1}{\varepsilon}\,dx = 1\ ,$$

   for $\forall\varepsilon$;

2. Shift property, using mean-value theorem of continuous functions

   $$\int_{x=-\infty}^{\infty} r_\varepsilon(x-x_0)f(x)\,dx = \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} \frac{1}{\varepsilon} f(x)\,dx = \frac{1}{\varepsilon}\varepsilon f(\xi)\ ,$$

   with $\xi \in [x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}]$, for the mean value theorem. As $\varepsilon \to 0$, $\xi \to x_0$, and thus

   $$\int_{x=-\infty}^{\infty} r_\varepsilon(x-x_0)f(x)\,dx \to f(x_0)$$

10.1.2 Piecewise-linear

$$\delta(x) \sim t_\varepsilon(x) = \begin{cases} \frac{2}{\varepsilon}\left(1 - \frac{2|x|}{\varepsilon}\right) & x \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \\ 0 & \text{otherwise} \end{cases}$$

Properties - proof

1. Unitariety

$$\int_{x=-\infty}^{\infty} t_\varepsilon(x - x_0)\, dx = \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} \frac{2}{\varepsilon}\left(1 - \frac{2|x|}{\varepsilon}\right)\, dx = \frac{1}{2}\varepsilon\frac{2}{\varepsilon} = 1\ ,$$

   for $\forall \varepsilon$;

2. Shift property, using mean-value integration scheme in $x \in [x_0 - \frac{\varepsilon}{2}, x_0]$, $x \in [x_0, x_0 + \frac{\varepsilon}{2}]$ (**todo** *why?*)

$$\begin{aligned}
\int_{x=-\infty}^{\infty} t_\varepsilon(x - x_0) f(x)\, dx &= \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} \frac{2}{\varepsilon}\left(1 - \frac{2|x-x_0|}{\varepsilon}\right) f(x)\, dx = \\
&= \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0} \frac{2}{\varepsilon}\left(1 - \frac{2|x-x_0|}{\varepsilon}\right) f(x)\, dx + \int_{x=x_0}^{x_0+\frac{\varepsilon}{2}} \frac{2}{\varepsilon}\left(1 - \frac{2|x-x_0|}{\varepsilon}\right) f(x)\, dx = \\
&= \frac{\varepsilon}{2}\frac{2}{\varepsilon}\left(1 - \frac{2}{\varepsilon}\frac{\varepsilon}{4}\right) f\left(x_0 - \frac{\varepsilon}{4}\right)\, dx + \frac{\varepsilon}{2}\frac{2}{\varepsilon}\left(1 - \frac{2}{\varepsilon}\frac{\varepsilon}{4}\right) f\left(x_0 + \frac{\varepsilon}{4}\right)\, dx = \\
&= \frac{1}{2} f\left(x_0 - \frac{\varepsilon}{4}\right) + \frac{1}{2} f\left(x_0 + \frac{\varepsilon}{4}\right)
\end{aligned}$$

   As $\varepsilon \to 0$

$$\int_{x=-\infty}^{\infty} t_\varepsilon(x - x_0) f(x)\, dx \to f(x_0)$$

10.1.3 Gaussian approximation

For $\alpha \to +\infty$,

$$\varphi_\alpha(x) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} \sim \delta(x)$$

Properties - proof

Fourier transform of $\varphi_\alpha(x)$ reads

$$\begin{aligned}
\mathscr{F}\{\varphi_\alpha(x)\}(k) &= \int_{x=-\infty}^{+\infty} \varphi_\alpha(x) e^{-ikx}\, dx = \\
&= \int_{x=-\infty}^{+\infty} \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} e^{-ikx}\, dx = \\
&= \sqrt{\frac{\alpha}{\pi}} \int_{x=-\infty}^{+\infty} e^{-\alpha\left(x + i\frac{k}{2\alpha}\right)^2}\, dx\, e^{-\frac{k^2}{4\alpha}} = \\
&= \sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{k^2}{4\alpha}} = e^{-\frac{k^2}{4\alpha}}\ ,
\end{aligned}$$

for $\alpha \to +\infty$,

$$\mathcal{F}\{\varphi_\alpha(x)\}(k) \to 1$$

Fourier transform of Dirac's delta is 1, as shown in (13.3), thus $\varphi_\alpha(x) \to \delta(x)$ for $\alpha \to +\infty$.

10.1.4 Fourier anti-transform

For $a \to +\infty$,

$$\delta(x) \sim \int_{y=-a}^{+a} e^{i2\pi yx}\, dy = \frac{1}{2\pi}\int_{k=-2\pi a}^{2\pi a} e^{ikx}\, dk\ , \tag{10.3}$$

or

$$\delta \sim 2\int_{y=0}^{a} \cos(2\pi yx)\, dy\ .$$

Proof of the equilvanece

$$\begin{aligned}\delta(x) &\sim \frac{1}{2\pi}\int_{k=-2\pi a}^{2\pi a} e^{ikx}\, dk = \frac{1}{2\pi}\left(\int_{k=-2\pi a}^{0} e^{ikx}\, dk + \int_{0}^{k=2\pi a} e^{ikx}\, dk\right) = \frac{1}{2\pi}\int_{k=0}^{2\pi a}\left(e^{ikx} + e^{ikx}\right)\, dx = \frac{1}{\pi}\int_{x=0}^{2\pi a}\cos(kx)\, dk \\ &= \int_{y=-a}^{+a} e^{i2\pi yx}\, dy = \cdots = \int_{y=0}^{a}\left(e^{i2\pi yx} + e^{i2\pi yx}\right) dy = 2\int_{y=0}^{a}\cos(2\pi yx)\, dy\ .\end{aligned}$$

10.1.5 sinc(x) approximation

For $a \to +\infty$

$$\delta(x) \sim \frac{\sin(2\pi xa)}{\pi x}$$

Proof

Directly follows from integral of the approximation (10.3)

$$\int_{y=-a}^{+a} e^{i2\pi yx}\, dy = \frac{1}{i2\pi x}\ e^{i2\pi yx}\Big|_{y=-a}^{+a} = \frac{1}{\pi x}\frac{e^{i2\pi ax} - e^{-i2\pi ax}}{2i} = \frac{\sin(2\pi xa)}{\pi x}$$

10.1.6 Fourier series

For $x \in [-\pi, \pi]$, and $N \to +\infty$, *Fourier series* of Dirac's delta (train with period 2π) reads

$$\delta(x) \sim \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx} = \frac{1}{2\pi}\frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}$$

or the T-periodic Dirac's delta train,

$$\delta(x) \sim \frac{1}{T}\sum_{n=-N}^{N} e^{in\frac{2\pi}{T}x}\ .$$

todo *Write the proof of the last expression, using the relation between complex exponentials and cosine and sine*

Proof

Coefficients of the Fourier series of Dirac's delta (train with period $T = 2\pi$) are evaluated using the expression (13.2)

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(t) e^{-in\frac{2\pi}{2\pi}t} = \frac{1}{2\pi} \ ,$$

and thus the complex Fourier series (13.1) of Dirac's delta reads

$$\delta(x) \sim \sum_{n=-\infty}^{+\infty} c_n e^{in\frac{2\pi}{T}x} = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{inx}$$

Obs. here, integration interval $[-\pi, \pi]$ to "avoid troubles" with Dirac's delta on the extreme points of the interval (it would give $1/2$ and $1/2$ contributions on both extremes…)

It's possible to write the T-periodic Dirac's delta train as

$$\delta(x) \sim \sum_{n=-\infty}^{+\infty} c_n e^{in\frac{2\pi}{T}x} = \frac{1}{T} \sum_{n=-\infty}^{+\infty} e^{in\frac{2\pi}{T}x}$$

Integral $I = \int_{-\infty}^{+\infty} e^{-\alpha x^2}\, dx$

$$\begin{aligned}
I^2 &= \int_{x=-\infty}^{+\infty} e^{-\alpha x^2}\, dx \int_{y=-\infty}^{+\infty} e^{-\alpha y^2}\, dy = \\
&= \int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{+\infty} e^{-\alpha(x^2+y^2)}\, dx\, dy = \\
&= \int_{\theta=0}^{2\pi} \int_{r=0}^{+\infty} e^{-\alpha r^2}\, r\, dr\, d\theta = \\
&= 2\pi \frac{1}{2\alpha} \int_{r=0}^{+\infty} e^{-\alpha r^2} d\left(\alpha r^2\right) = \\
&= \frac{\pi}{\alpha} \left[-e^{\alpha r^2}\right]\Big|_{r=0}^{+\infty} = \frac{\pi}{\alpha} \ .
\end{aligned}$$

Part VI

Complex Calculus

CHAPTER

ELEVEN

COMPLEX ANALYSIS

11.1 Complex functions, $f : \mathbb{C} \rightarrow \mathbb{C}$

A complex function f of complex variable $z = x + iy$, $f : \mathbb{C} \rightarrow \mathbb{C}$, can be written as

$$f(z) = \tilde{u}(z) + i\tilde{v}(z) = u(x,y) + iv(x,y) \ ,$$

as the sum of its real part $u(z)$ and i times its imaginary part $v(x,y)$. Here $x, y \in \mathbb{R}$, while $\tilde{u}(z), \tilde{v}(z) : \mathbb{C} \rightarrow \mathbb{R}$ and $u(x,y), v(x,y) : \mathbb{R}^2 \rightarrow \mathbb{R}$. With some abuse of notation, tilde won't be always explicitly written when arguments of real and imaginary parts of f functions won't be written.

11.1.1 Limit

$$\lim_{z \rightarrow z_0} f(z) = f(z_0) \qquad , \qquad \forall \varepsilon > 0 \ \exists \delta > 0 \ \text{ s.t. } \ |f(z) - f(z_0)| < \delta \ \forall z \text{ s.t. } |z - z_0| < \varepsilon, \ z \neq z_0 \ .$$

11.1.2 Derivative

Using the definition of *limit of complex functions*, the derivative of a function $f : \mathbb{C} \rightarrow \mathbb{C}$, if it exists, is the limit of incremental ratio,

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \ .$$

11.1.3 Line Integrals

Given a line $\gamma \in \mathbb{C}$, whose parametric form is $z(s)$, with regular parametrization with parameter $s \in [s_0, s_1]$,

$$\int_\gamma f(z)\, dz = \int_{s=s_0}^{s_1} f(z(s))\, z'(s)\, ds \ .$$

11.2 Holomorphic Functions - Analytic Functions

Definition 11.2.1

A holomorphic function is a function whose *derivative* exists.

Examples of analytic functions. todo...

11.2.1 Cauchy-Riemann conditions

For a holomorphic function $f(z) = u(x, y) + iv(x, y)$, Cauchy-Riemann conditions

$$\begin{cases} u_{/x} = v_{/y} \\ u_{/y} = -v_{/x} \end{cases}$$

hold. The evaluation of the derivative once with $\Delta z = \Delta x$ and once with $\Delta z = i\Delta y$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} =$$
$$= \begin{cases} \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x} = u_{/x} + iv_{/x} \\ \lim\limits_{\Delta y \to 0} \dfrac{f(x, y + \Delta y) - f(x, y)}{i\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y} = -iu_{/y} + v_{/y} \end{cases}$$

provides the proof.

11.2.2 Cauchy Theorem

For a holomorphic function f, $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$

$$\oint_\gamma f(z)\, dz = 0 \ ,$$

for $\forall \gamma \subset \Omega$. Proof follows from *Green's lemma*, and *Cauchy-Riemann conditions*

$$\begin{aligned} \oint_\gamma f(z) dz &= \oint_\gamma (u(x, y) + iv(x, y)) (dx + idy) = \\ &= \oint_\gamma (udx - vdy) + i \oint_\gamma (udy + vdx) = \\ &= -\int_S \bigg(\underbrace{u_{/y} + v_{/x}}_{=0} \bigg) \, dx\, dy + i \int_S \bigg(\underbrace{u_{/x} - v_{/y}}_{=0} \bigg) \, dx\, dy = 0 \ . \end{aligned}$$

11.3 Useful integrals

11.3.1 Independence of line integral for holomorphic functions

For a function $f(z)$ analytic in D, the line integral on paths $\ell_{ab,i}$ with the same extreme points a, b contained in D is independent on the path, but only depends on the extreme points a, b,

$$\int_{\ell_{ab,1}} f(z)\,dz = \int_{\ell_{ab,2}} f(z)\,dz$$

The proof readily follows, using *Cauchy theorem* applied to a function $f(z) : D \subseteq \mathbb{C} \to \mathbb{C}$, analytic in D, and splitting the closed path γ into two paths ℓ_1, ℓ_2 with the same extreme points, $\gamma = \ell_1 \cup (-\ell_2)$

$$0 = \oint_\gamma f(z)\,dz = \int_{\ell_1} f(z)\,dz + \int_{-\ell_2} f(z)\,dz = \int_{\ell_1} f(z)\,dz - \int_{\ell_2} f(z)\,dz\,.$$

11.3.2 Sum and difference of line integrals

11.3.3 Integral of z^n

Given a path γ embracing $z = 0$ only once in counter-clockwise direction, and $n \in \mathbb{Z}$

$$\oint_\gamma z^n\,dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

Since z^n is analytic everywhere (**todo** *prove it! Add a section with proofs for common functions*) except for $z = 0$, it's possible to evaluate the integral on a circle with center $z = 0$ and radius R. Using polar expression of the complex numbers on the circle, $z = Re^{i\theta}$, $\theta \in [0, 2\pi]$, R const, the differential becomes $dz = iRe^{i\theta}d\theta$ and the integral

$$\begin{aligned}
\oint_\gamma z^n\,dz &= \int_{\theta=0}^{2\pi} \left(Re^{i\theta}\right)^n iRe^{i\theta}d\theta = \\
&= i\int_{\theta=0}^{2\pi} R^{n+1}e^{i(n+1)\theta}d\theta = \\
&= \begin{cases} \text{if } n = -1 & : \quad i2\pi \\ \text{otherwise} & : \quad iR^{n+1}\dfrac{1}{i(n+1)}\, e^{i(n+1)\theta}\Big|_{\theta=0}^{2\pi} = \dfrac{R^{n+1}}{n+1}(1-1) = 0 \end{cases}
\end{aligned}$$

11.4 Meromorphic functions

Definition 11.4.1

A meromorphic function in a domain is a function holomorphic everywhere except for a (finite?) number of poles. **check**

11.4.1 Singularities

Definition 11.4.2 (Pole)

A pole of order n of a function $f(z)$ is a complex number a so that

$$f(z) = \frac{\phi(z)}{(z-a)^n} \ ,$$

with $\phi(z)$ holomorphic in $\phi(a) \neq 0$

Examples. …

Definition 11.4.3 (Branch)

Examples. $f(z) = z^{\frac{1}{2}}$

Definition 11.4.4 (Removable singularities)

Example. $f(z) = \frac{\sin z}{z}$

Other irregularities.

11.4.2 Laurent Series

Given a function $f(z)$, in a disk $D_{a,\varepsilon} : 0 < |z-a| < \varepsilon$, its Laurent series centered in a is the convergent (to $f(z)$, **todo** *which type of convergence?*) series

$$f(z) \sim \sum_{n=-\infty}^{+\infty} a_n (z-a)^n \ , \tag{11.1}$$

with

$$a_n = \frac{1}{2\pi i} \int_\gamma f(z)\,(z-a)^{-(n+1)}\,dz \tag{11.2}$$

and γ embracing $z = a$ once counter-clockwise. Proof follows immediately inserting the expressions of the coefficients a_n and using the *integral of z^n*. Evaluating the integral (11.2) of the coefficients of the Laurent series, using (11.1) to replace $f(z)$ with its series

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_\gamma \sum_{m=-\infty}^{+\infty} a_m (z-a)^m (z-a)^{-(n+1)} = \\ &= \frac{1}{2\pi i} \oint_\gamma \sum_{m=-\infty}^{+\infty} a_m (z-a)^{m-n-1}\,dz = \\ &= \frac{1}{2\pi i} \oint_\gamma a_n\, z^{-1}\,dz = \\ &= a_n \ . \end{aligned}$$

todo *Some freestyle with function and its convergent series…add some detail, and the meaning of convergence*

11.4.3 Cauchy formula

For an analytic function $f(z)$,

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a}\, dz$$

Proof readily follows using the *integral of z^n* on the Taylor series of $\frac{f(z)}{z-a}$ whose 0^{th} order term reads $f(a)$,

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(a) + \sum_{m=1}^{+\infty} f'(a)(z-a)^m}{z-a}\, dz = \frac{1}{2\pi i} \oint_\gamma \frac{f(a)}{z-a}\, dz = f(a)\frac{2\pi i}{2\pi i} = f(a)\ .$$

11.4.4 Residues

Definition 11.4.5 (Residue)

The residue of function f in a, $\text{Res}(f,a)$ is a complex number R so that $f(z) - \frac{R}{(z-a)}$ has analytic antiderivative in a disk $D_{a,\varepsilon}:\ 0 < |z-a| < \varepsilon$.

todo Explain this definition. Couldn't be possible to use $\text{Res}(f,a) = \frac{1}{2\pi i}\oint_\gamma f(z)\, dz = a_{-1}$ instead?

Properties.

- If $f(z)$ is analytic in $D_{a,\varepsilon}$ and has a pole of order n in $z=a$, its Laurent series has $a_m = 0$ for $m < n$ and reads

$$f(z) = \sum_{m=-n}^{+\infty} a_m (z-a)^m\ , \tag{11.3}$$

  with $a_{-n} \neq 0$. Since $f(z)$ has a pole of order n in $z=a$, it can be written as

$$f(z) = \frac{\phi(z)}{(z-a)^n}\ ,$$

  with $\phi(z)$ analytic in $D_{a,\varepsilon}$ and $\phi(a) \neq 0$. Since $\phi(z)$ is analytic, it has a Taylor series (or a Laurent series with non-negative powers),

$$\phi(z) \sim \sum_{m=0}^{+\infty} b_m (z-a)^m\ ,$$

  (**todo** *prove it! Extension of the real case. Add a link to the proof*) and thus

$$f(z) \sim \sum_{m=0}^{+\infty} b_m (z-a)^{m-n} = \sum_{m=-n}^{+\infty} b_{m+n} (z-a)^m = \sum_{m=-n}^{+\infty} a_m (z-a)^m\ ,$$

  with $a_m = b_{m+n}$.

- For simple closed path γ (embracing a only once counter-clokwise) in $D_{a,\varepsilon}$,

$$\oint_\gamma f(z)\, dz = 2\pi i a_{-1} = 2\pi i \text{Res}(f,a) \tag{11.4}$$

  The proof readily follows, using the *integral of z^n* and Laurent series (11.1) of $f(z)$,

$$\oint_\gamma f(z)\, dz = \oint_\gamma \sum_{m=-\infty}^{+\infty} a_m (z-a)^m\, dz = 2\pi i a_{-1}\ .$$

- For a pole a of order n, the following holds

$$a_{-1} = \frac{1}{(n+1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n \, f(z) \right]$$

The proof follows using Laurent series {eq}`eq:laurent:pole-n} for a function with pole of order n, and evaluating the $(n-1)^{th}$ order derivative

$$\begin{aligned}
\frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right] & = \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n \sum_{m=-n}^{+\infty} a_n (z-a)^m \right] = \\
& = \frac{d^{n-1}}{dz^{n-1}} \left[\sum_{m=-n}^{+\infty} a_n (z-a)^{m+n} \right] = \\
& = \frac{d^{n-1}}{dz^{n-1}} \left[\sum_{m=0}^{+\infty} a_{m-n} (z-a)^{m} \right] = \\
& = \frac{d^{n-2}}{dz^{n-2}} \left[\sum_{m=0}^{+\infty} m a_{m-n} (z-a)^{m-1} \right] = \\
& = \frac{d^{n-3}}{dz^{n-3}} \left[\sum_{m=0}^{+\infty} m(m-1) a_{m-n} (z-a)^{m-2} \right] = \\
& = \dots = \\
& = \left[\sum_{m=0}^{+\infty} m! \, a_{m-n} (z-a)^{m-n+1} \right]
\end{aligned}$$

and then letting $z \to a$, so that only the term with $m - n + 1 = 0$ survives

$$\lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n \sum_{m=-n}^{+\infty} a_n (z-a)^m \right] = (n-1)! \, a_{-1} \ .$$

11.4.5 Residue Theorem

Theorem 11.4.1 (Residue Theorem)

Given $f(z)$ with a finite number of poles $p_n \in D$, then

$$\int_\gamma f(z) \, dz = 2 \pi i \ \sum_n I(\gamma, p_n) \text{Res}(f, p_n) \ ,$$

being γ a path in D, and $I(\gamma, p_n)$ the winding index of the path γ around pole p_n (+1 for each counter-clockwise loop, -1 for each clockwise loop).

The proof readily follows extending the result for a single pole (11.4) to general number of poles and general paths γ embracing (with sign) each pole p_n $I(\gamma, p_n)$ times, with the same techinques shown in section *Sum and difference of line integrals*.

11.4.6 Evaluation of integrals

11.4.7 Inverse Laplace Transform

Given Laplace transform

$$F(s) := \mathcal{L}\{f(t)\}(s) := \int_{t=0^-}^{+\infty} f(t) e^{-st}\, dt \ ,$$

the inverse transform can be evaluated as

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) := \lim_{T\rightarrow+\infty} \frac{1}{2\pi i} \int_{s=a-iT}^{a+iT} e^{st} F(s)\, ds \ ,$$

with $a > \text{Re}\{p_n\}$ (**todo** *why?*) for each pole of the function $F(s)$, evaluated on the vertical line $s = a + iy$, $y \in [-T, T]$, $ds = i dy$,

$$\begin{aligned}
\lim_{T\rightarrow+\infty} \frac{1}{2\pi i} \int_{s=a-iT}^{a+iT} e^{st} F(s)\, ds &= \lim_{T\rightarrow+\infty} \frac{1}{2\pi i} \int_{s=a-iT}^{a+iT} e^{st} \int_{\tau=0^-}^{+\infty} f(\tau) e^{-s\tau}\, d\tau\, ds = \\
&= \lim_{T\rightarrow+\infty} \frac{1}{2\pi i} \int_{y=-T}^{T} e^{(a+iy)t} \int_{\tau=0^-}^{+\infty} f(\tau) e^{-(a+iy)\tau}\, d\tau\, i dy = \\
&= \lim_{T\rightarrow+\infty} \frac{1}{2\pi} \int_{y=-T}^{T} \int_{\tau=0^-}^{+\infty} e^{iy(t-\tau)} e^{a(t-\tau)} f(\tau)\, d\tau\, dy = \\
&= \dots \\
&= \int_{\tau=0^-}^{+\infty} \delta(t-\tau) e^{a(t-\tau)} f(\tau) d\tau = f(t) \ .
\end{aligned}$$

having used the transform of *Dirac's delta* $\delta(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} e^{-j\omega t}\, d\omega$.

todo *Ohter approach: if $a >$ Re$\{p_n\}$, the contour built with the vertical line with real part a and the arc of circumference on its...*

CHAPTER

TWELVE

LAPLACE TRANSFORM

$$\mathcal{L}\left\{f(t)\right\}(s) := \int_{t=0^-}^{+\infty} e^{-st} f(t)\, dt = F(s)\ .$$

12.1 Inverse transform

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \dots$$

12.2 Properties

Linearity.

$$\mathcal{L}\{af(t) + bg(t)\}(s) = aF(s) + bG(s)$$

***Dirac delta*.**

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{t=0^-}^{+\infty} \delta(t)\, e^{st}\, dt = 1$$

Time delay. If $f(t) = 0$ for $t < 0$ ("causality"), for $\tau > 0$,

$$\mathcal{L}\{f(t-\tau)\}(s) = e^{-s\tau} F(s)$$

Proof readily follows direct computation with change of variable $z = t - \tau$, $dt = dz$

$$\mathcal{L}\{f(t-\tau)\}(s) = \int_{t=0^-}^{+\infty} f(t-\tau) e^{-st}\, dt = \int_{z=-\tau}^{+\infty} f(z) e^{-sz}\, dz\, e^{-s\tau} = \int_{z=0}^{+\infty} f(z) e^{-sz}\, dz\, e^{-s\tau} = e^{-s\tau} F(s)\ .$$

"Frequency shift"

$$\mathcal{L}\{f(t)e^{at}\}(s) = F(s-a)$$

Direct computation gives

$$\mathcal{L}\{f(t)e^{at}\}(s) = \int_{t=0^-}^{+\infty} f(t) e^{at} e^{-st}\, dt = \int_{t=0^-}^{+\infty} f(t) e^{-(s-a)t}\, dt = F(s-a)$$

Derivative.

$$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0^-)\ .$$

Proof readily follows direct computation, with integration by parts

$$\mathcal{L}\{f'(t)\}(s) = \int_{t=0^-}^{+\infty} f'(t)e^{-st}\,dt = [f(t)e^{-st}]\,|_{t=0^-}^{+\infty} + s\int_{t=0^-}^{+\infty} f(t)e^{-st}\,dt = sF(s) - f(0^-)\ ,$$

provided that $\lim_{s\rightarrow+\infty} f(t)e^{-st} = 0$.

Integral.

$$\mathcal{L}\left\{\int_{\tau=0}^{t} f(\tau)\,d\tau\right\}(s) = \frac{1}{s}F(s)\ .$$

Proof readily follows direct computation, with integration by parts

$$\mathcal{L}\left\{\int_{\tau=0^-}^{t} f(\tau)\,d\tau\right\}(s) = \int_{t=0^-}^{+\infty}\int_{\tau=0^-}^{t} f(\tau)\,d\tau e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\int_{\tau=0^-}^{t} f(\tau)\,d\tau\right]_{t=0}^{+\infty} + \frac{1}{s}\int_{t=0}^{+\infty} f(t)e^{-st}\,dt = \frac{1}{s}F(s)\ ,$$

provided that $\int_{\tau=0^-}^{0} f(\tau)d\tau = 0$ and $\lim_{t\rightarrow+\infty}\frac{e^{-st}}{s}\int_{\tau=0^-}^{+\infty} f(\tau)\,d\tau = 0$.

Convolution.

$$\begin{aligned}\mathcal{L}\left\{f(t)*g(t)\right\} &= \int_{t=0^-}^{+\infty}\int_{\tau=-\infty}^{+\infty} f(t-\tau)g(\tau)\,d\tau\,e^{-st}\,dt = \qquad (1)\\ &= \int_{\tau=-\infty}^{+\infty}\int_{z=-\tau^-}^{+\infty} f(z)g(\tau)\,e^{-s(z+\tau)}\,d\tau\,dz = (2)\\ &= \int_{z=0^-}^{+\infty} f(z)\,e^{-sz}\,dz\int_{\tau=0^-}^{+\infty} g(\tau)e^{-s\tau} =\\ &= \mathcal{L}\{f(t)\}(s)\,\mathcal{L}\{g(t)\}(s)\ .\end{aligned} \tag{12.1}$$

having performed the change of coordinates $z = t-\tau$, $\tau = \tau$, with unitary Jacobian,

$$\frac{\partial(t,\tau)}{\partial(z,\tau)} = \partial_z t\partial_\tau\tau - \partial_z\tau\partial_z t = 1\cdot 1 - 1\cdot 0 = 1,$$

given the proper description of the domain of integration summarised in the extremes of integration in (1), and causality - i.e. all the functions $f(t)$ are identically zero for $t<0$ - in (2).

Initial value. If …

$$f(0^+) = \lim_{s\rightarrow+\infty} sF(s)$$

From direct computation,

$$\begin{aligned}\lim_{s\rightarrow+\infty} sF(s) &= \lim_{s\rightarrow+\infty} s\int_{t=0^-}^{+\infty} f(t)\,e^{-st}\,dt =\\ &= \lim_{s\rightarrow+\infty}\left\{\left[s\left(-\frac{e^{-st}}{s}\right)f(t)\right]\Big|_{t=0}^{+\infty} + \int_{t=0}^{+\infty} e^{-st}f'(t)\,dt\right\} =\\ &= \lim_{s\rightarrow+\infty}\left\{\left[-e^{-st}f(t)\right]\Big|_{t=0}^{+\infty} + \int_{t=0}^{+\infty} e^{-st}f'(t)\,dt\right\} =\\ &= f(0)\ ,\end{aligned}$$

provided that $\lim_{s\rightarrow+\infty}\lim_{t\rightarrow+\infty} e^{-st}f(t) = 0$ and $\lim_{s\rightarrow+\infty}\int_{t=0}^{+\infty} e^{-st}f'(t)\,dt = 0$.

Final value. If …

$$f(+\infty) = \lim_{s \to 0} sF(s)$$

From direct computation (**todo** *check and/or explain proof*),

$$\begin{aligned}
\lim_{s \to 0} sF(s) &= \lim_{s \to 0} s \int_{t=0^-}^{+\infty} f(t)\, e^{-st}\, dt = \\
&= \lim_{s \to 0} \left\{ \left[s \left(-\frac{e^{-st}}{s} \right) f(t) \right] \Big|_{t=0}^{+\infty} + \int_{t=0}^{+\infty} e^{-st} f'(t)\, dt \right\} = \\
&= \lim_{s \to 0} \left\{ \left[-e^{-st} f(t) \right] \Big|_{t=0}^{+\infty} + \int_{t=0}^{+\infty} e^{-st} f'(t)\, dt \right\} = \\
&= f(0) + f(+\infty) - f(0) = f(+\infty) \ ,
\end{aligned}$$

provided that $\lim_{s \to 0} \lim_{t \to +\infty} e^{-st} f(t) = 0$.

CHAPTER

THIRTEEN

FOURIER TRANSFORMS

Fourier transforms are linear transformations of functions usually relating a physical domain of time and/or space, with a domain of frequency and/or wave-vectors.

Fourier transforms can be useful in:

- highlighting the frequency content of functions
- solving problems: sometimes, it can be easier to transform a problem in frequency domain, solve it in frequency domain, and transform the solution back to the physical domain

Contents.

Fourier series. Fourier series is defined for finite-domain or periodic, time-continuous functions, or - more generally - continuous functions in the physical domain.

Fourier transform. Fourier transform is defined for infinite-domain non-periodic, time-continuous functions, or - more generally - continuous functions in the physical domain.

Relations between Fourier transforms and sampling. Fourier series, Fourier transform, discrete time Fourier transform and discrete Fourier transforms are presented, and their relations discussed. Fundamental results about **evenly-spaced sampling** seamlessly follows, as **Shannon-Nyquist theorem**, *Theorem 13.3.1*, shows.

Different Fourier transforms exist, depending if the original function is:

- time discrete/time continuous
- periodic/non-periodic

13.1 Fourier Series

For a T-periodic function,

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left[a_n \cos\left(n \frac{2\pi}{T} t \right) + b_n \sin\left(n \frac{2\pi}{T} t \right) \right] ,$$

todo Prove it with properties of integrals of sin and cos over $t \in [0, T]$; prove convergence to average value at jumps

The exponential form reads

$$g(t) \sim \sum_{n=-\infty}^{+\infty} c_n e^{in \frac{2\pi}{T} t} , \tag{13.1}$$

where

$$c_n = \frac{1}{T} \int_{t=0}^{T} f(t)\, e^{-in\frac{2\pi}{T}t} \,. \tag{13.2}$$

Proof

Exploiting the properties of integrals of complex exponentials with $k \in \mathbb{Z}$

$$\int_{t=0}^{T} e^{ik\frac{2\pi}{T}t}\, dt = \begin{cases} \frac{1}{ik\frac{2\pi}{T}} \left[e^{ik\frac{2\pi}{T}t} \right]\Big|_{t=0}^{T} = 0 & \text{if } k \neq 0 \\ T & \text{if } k = 0 \end{cases}$$

$$\int_{t=0}^{T} f(t) e^{-im\frac{2\pi}{T}t}\, dt \sim \int_{t=0}^{T} \sum_{n=-\infty}^{+\infty} c_n e^{in\frac{2\pi}{T}t} e^{-im\frac{2\pi}{T}t} \sim \sum_{n=-\infty}^{+\infty} c_n \sim \int_{t=0}^{T} e^{i(n-m)\frac{2\pi}{T}t} \sim T\, c_m \,.$$

13.2 Fourier Transform

Contents: *definition*; *properties*; *inverse transform*; *Plancherel's theorem*; *uncertainty relation*

13.2.1 Definition

…

$$\mathcal{F}\left\{ g(t) \right\}(f) := \int_{t=-\infty}^{+\infty} g(t)\, e^{-i2\pi f t}\, dt.$$

13.2.2 Properties

Linearity

***Dirac delta*.**

$$\mathcal{L}\left\{ \delta(t) \right\} = \int_{t=-\infty}^{+\infty} \delta(t)\, e^{-i2\pi f t}\, dt = 1 \tag{13.3}$$

Time delay.

Derivative.

Integral.

Initial value.

Final value.

13.2.3 Inverse Fourier Transform

Under the assumptions …**todo**, the inverse Fourier transform reads

$$\mathcal{F}^{-1}\left\{G(f)\right\}(t) := \int_{f=-\infty}^{+\infty} G(f)\, e^{i2\pi f t}\, df.$$

Proof using Dirac's delta expression.

$$\begin{aligned}\mathcal{F}^{-1}\left\{G(f)\right\}(t) := \int_{f=-\infty}^{+\infty} G(f)\, e^{i2\pi f t}\, df &= \int_{f=-\infty}^{+\infty}\int_{\tau=-\infty}^{+\infty} g(\tau)e^{-i2\pi f\tau}\, e^{i2\pi f t}\, df = \\ &= \int_{f=-\infty}^{+\infty}\int_{\tau=-\infty}^{+\infty} g(\tau)e^{-i2\pi f\tau}\, e^{i2\pi f t}\, df = \\ &= \int_{f=-\infty}^{+\infty}\int_{\tau=-\infty}^{+\infty} g(\tau)e^{i2\pi f(t-\tau)}\, df = \\ &= \int_{\tau=-\infty}^{+\infty} g(\tau)\delta(t-\tau)\, d\tau = g(t)\ .\end{aligned}$$

Proof using dominated convergence theorem and Fubini's lemma.

Proof. By the *dominated convergence theorem*, it follows that

$$\begin{aligned}\int_{\mathbb{R}} e^{i2\pi x\xi} F(\xi)\, d\xi &= \lim_{\varepsilon\rightarrow 0}\int_{\mathbb{R}} \underbrace{e^{-\pi\varepsilon^2\xi^2+i2\pi x\xi}}_{G(\xi;x,\varepsilon)} F(\xi)\, d\xi = \\ &= \lim_{\varepsilon\rightarrow 0}\int_{\mathbb{R}} g(y;x,\varepsilon) f(y)\, dy = \\ &= \lim_{\varepsilon\rightarrow 0}\int_{\mathbb{R}} \varphi_\varepsilon(x-y)\, f(y)\, dy = \\ &= \int_{\mathbb{R}} \delta(x-y)\, f(y)\, dy = f(x)\end{aligned}$$

Lemma 1. The Fourier transform of function $\varphi(t) := e^{-\pi|t|^2}$ reads

$$\begin{aligned}\mathcal{F}\{\varphi(t)\}(\omega) &= \int_{t=-\infty}^{+\infty} \varphi(t)e^{-i\omega t}\, dt = \\ &= \int_{t=-\infty}^{+\infty} e^{-\pi|t|^2}e^{-i\omega t}\, dt = \\ &= \int_{t=-\infty}^{+\infty} e^{-\pi\left(t^2+i\frac{\omega}{\pi}t-\frac{\omega^2}{4\pi^2}\right)}\, dt\, e^{-\frac{\omega^2}{4\pi^2}} = \\ &= \int_{t=-\infty}^{+\infty} e^{-\pi\left(t+i\frac{\omega}{2\pi}\right)^2}\, dt\, e^{-\frac{\omega^2}{4\pi}} = \\ &= e^{-\frac{\omega^2}{4\pi}}\ ,\end{aligned}$$

having evaluated *the integral* $\int_{-\infty}^{+\infty} e^{-\alpha x^2}$ with $\alpha = \pi$. **todo** *justify the result for complex exponential. Use Bromwich contour integrals*

Lemma 2. Fourier transform of $f(\alpha t)$, $\alpha > 0$

$$\mathcal{F}\{f(\alpha t)\}(\omega) = \int_{\mathbb{R}} f(\alpha t)e^{-j\omega t}\, dt = \int_{\tau\in\mathbb{R}} f(\tau)e^{-j\frac{\omega}{\alpha}\tau}\, d\tau\frac{1}{\alpha} = \frac{1}{\alpha}F\left(\frac{\omega}{\alpha}\right)$$

Lemma 3. $\frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right) \to \delta(x)$ for $\varepsilon \to 0$

$$\mathcal{F}\left\{\frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right)\right\}(\omega) = \frac{1}{\varepsilon}\varepsilon e^{-\frac{\omega^2}{4\pi\varepsilon^2}} = e^{-\frac{\omega^2}{4\pi\varepsilon^2}}$$

0. Fourier transform

$$G(f) = \int_{t=-\infty}^{\infty} e^{-i\omega t} g(t)\, dt$$

1.

$$g(t) = e^{i\alpha t}\psi(t)$$

$$\mathcal{F}\{g(t)\}(\omega) = \int_{t=-\infty}^{+\infty} g(t)e^{-i\omega t}\, dt = \int_{t=-\infty}^{+\infty} \psi(t)e^{i\alpha t}e^{-i\omega t}\, dt = \int_{t=-\infty}^{+\infty} \psi(t)e^{-i(\omega-\alpha)t}\, dt = \mathcal{F}\{\psi(t)\}(\omega-\alpha)\,.$$

2.

$$\psi(t) = \phi(\alpha t)$$

$$\mathcal{F}\{\psi(t)\} = \int_{t=-\infty}^{+\infty} \psi(t)e^{-i\omega t}\, dt = \int_{t=-\infty}^{+\infty} \phi(\alpha t)e^{-i\omega t}\, dt = \int_{\tau=-\infty}^{+\infty} \phi(\tau)e^{-i\frac{\omega}{\alpha}\tau}\, \frac{d\tau}{\alpha} = \frac{1}{\alpha}\mathcal{F}\{\phi(t)\}\left(\frac{\omega}{\alpha}\right)\,.$$

3. Fubini's theorem

4.

$$\varphi(t) := e^{-\pi t^2}$$

$$\mathcal{F}\{\varphi(t)\} = \int_{t=-\infty}^{+\infty} \varphi(t)e^{-i\omega t}\, dt = \int_{t=-\infty}^{+\infty} e^{-\pi t^2}e^{-i\omega t}\, dt$$

$$0 = \oint_\gamma e^{-\alpha|z|^2}\, dz = \int_{\dots} \dots$$

$$z = Re^{i\theta}, \quad dz = iRe^{i\theta}d\theta$$

$$\int_{C/4} e^{-\alpha|z|^2}\, dz = \int_{\theta=0}^{\frac{\pi}{2}} e^{-\alpha R^2} iRe^{i\theta}d\theta = iRe^{-\alpha R^2}\frac{e^{-i\theta}}{i}\Big|_{\theta=0}^{\frac{\pi}{2}}$$

$$\int_{t=0}^{+\infty} e^{-\pi t^2}e^{-i\omega t}\, dt = \int_{t=0}^{+\infty} e^{-\left(\pi t^2 + i\omega t - \frac{\omega^2}{4\pi}\right)}\, dt\, e^{-\frac{\omega^2}{4\pi}} =$$

$$= \int_{t=0}^{+\infty} e^{-\pi(t+i\frac{\omega}{2\pi})^2}\, dt\, e^{-\frac{\omega^2}{4\pi}}$$

5. $\varphi_\varepsilon(t) = \frac{1}{\varepsilon^n}\varphi\left(\frac{t}{\varepsilon}\right)$, $t \in \mathbb{R}^n$, is an approximation of Dirac's delta for $\varepsilon \to 0$, so that

$$\lim_{\varepsilon\to 0}\int_{t=-\infty}^{+\infty} \varphi_\varepsilon(t-\tau)f(t)\, dt = f(\tau)$$

$$\lim_{\varepsilon\to 0}\int_{t=-\infty}^{+\infty} \varphi_\varepsilon(t)\, dt = 1$$

As the Fourier transform $\mathcal{F}\{\varphi_\varepsilon(t)\}(\omega) \to 1$ for $\varepsilon \to 0$, then $\varphi_\varepsilon(t) \to \delta(t)$.

13.2.4 Plancherel's theorem

…assumptions…**todo**

$$\int_{f=-\infty}^{+\infty} |G(f)|^2 \, df = \int_{t=-\infty}^{+\infty} |g(t)|^2 \, dt \tag{13.4}$$

and

$$\int_{f=-\infty}^{+\infty} A^*(f) \, G(f) \, df = \int_{t=-\infty}^{+\infty} a^*(t) \, g(t) \, dt \; . \tag{13.5}$$

Proof of Plancherel's thm for the magnitude

$$\begin{aligned}
\int_{f=-\infty}^{+\infty} |G(f)|^2 \, df = \int_{f=-\infty}^{+\infty} G(f)^* G(f) \, df = & \\
= \int_{f=-\infty}^{+\infty} \left(\int_{t_1=-\infty}^{+\infty} g(t_1) e^{-i2\pi f t_1} dt_1 \right)^* \left(\int_{t_2=-\infty}^{+\infty} g(t_2) e^{-i2\pi f t_2} dt_2 \right) \, df = & \\
= \int_{t_1,t_2=-\infty}^{+\infty} g^*(t_1) \, g(t_2) \int_{f=-\infty}^{+\infty} e^{i2\pi f (t_1 - t_2)} \, df \, dt_1 \, dt_2 = & \qquad (1) \\
= \int_{t_1,t_2=-\infty}^{+\infty} g^*(t_1) \, g(t_2) \, \delta(t_1 - t_2) \, dt_1 \, dt_2 = & \qquad (2) \\
= \int_{t_1=-\infty}^{+\infty} g^*(t_1) \, g(t_1) \, dt_1 = & \qquad (3) \\
= \int_{t_1=-\infty}^{+\infty} |g(t_1)|^2 \, dt_1 \; . &
\end{aligned}$$

having used (1) the approximation (10.3) of Dirac's delta, and (2) property (10.2) of Dirac's delta, and (3) the expression of the absolute value of complex functions $g^*(t_1)g(t_1) = |g(t_1)|^2$.

Proof of Plancherel's thm for the product of functions

$$\begin{aligned}
\int_{f=-\infty}^{+\infty} A^*(f) \, G(f) \, df = \int_{f=-\infty}^{+\infty} G(f)^* G(f) \, df = & \\
= \int_{f=-\infty}^{+\infty} \left(\int_{t_1=-\infty}^{+\infty} a(t_1) e^{-i2\pi f t_1} dt_1 \right)^* \left(\int_{t_2=-\infty}^{+\infty} g(t_2) e^{-i2\pi f t_2} dt_2 \right) \, df = & \\
= \int_{t_1,t_2=-\infty}^{+\infty} a^*(t_1) \, g(t_2) \int_{f=-\infty}^{+\infty} e^{i2\pi f (t_1 - t_2)} \, df \, dt_1 \, dt_2 = & \qquad (1) \\
= \int_{t_1,t_2=-\infty}^{+\infty} a^*(t_1) \, g(t_2) \, \delta(t_1 - t_2) \, dt_1 \, dt_2 = & \qquad (2) \\
= \int_{t_1=-\infty}^{+\infty} a^*(t_1) \, g(t_1) \, dt_1 \; . &
\end{aligned}$$

having used (1) the approximation (10.3) of Dirac's delta, and (2) property (10.2) of Dirac's delta.

13.2.5 Uncertainty relation

An uncertainty relation holds linking standard deviations of a probability density function in time domain and a probability density function built with its Fourier transform. From this very same relation, Heisenberg uncertainty relation between position and momentum in Quantum Mechanics seamlessly follows.

Given a function $g(t)$ whose square of the absolute value is normalized to one, and thus it can be used as a probability density function in time domain,

$$\int_{t=-\infty}^{+\infty} |g(t)|^2 \, dt = \int_{t=-\infty}^{+\infty} g^*(t) \, g(t) \, dt = 1 \ .$$

for *Plancherel's theorem*, the square of the magnitude of Fourier transform $G(f)$ is unitary as well,

$$\int_{f=-\infty}^{+\infty} |G(f)|^2 \, df = \int_{f=-\infty}^{+\infty} G^*(f) \, G(f) \, df = 1 \ ,$$

and thus it can be interpreted as a probability density function in frequency domain. The following uncertainty relation holds

$$\sigma^2_{t,g}\sigma^2_{f,G} \geq \left(\frac{1}{2\pi}\frac{1}{2}\right)^2 \ ,$$

or in terms of pulsation $\omega = 2\pi \, f$,

$$\sigma^2_{t,g}\sigma^2_{\omega,G} \geq \left(\frac{1}{2}\right)^2 \ ,$$

Proof of the uncertainty relation

Assuming zero average $\bar{t} = 0$, $\bar{f} = 0$ (see below for proof without this assumption)

$$\begin{aligned}
\sigma^2_{t,g}\sigma^2_{f,G} &= \int_{t=-\infty}^{+\infty} |t|^2 \, g^*(t) \, g(t) \, dt \int_{f=-\infty}^{+\infty} |f|^2 \, G^*(f) \, G(f) \, df = \\
&= \int_{t=-\infty}^{+\infty} |t \, g(t)|^2 \, dt \int_{f=-\infty}^{+\infty} |f \, G(f)|^2 \, df = && (1) \\
&= \int_{t=-\infty}^{+\infty} |t \, g(t)|^2 \, dt \int_{t=-\infty}^{+\infty} \left|-\frac{i}{2\pi} \, \dot{g}(t)\right|^2 \, dt \geq && (2) \\
&= \left|\int_{t=-\infty}^{+\infty} -t \, g^*(t) \frac{i}{2\pi} \, \dot{g}(t) \, dt\right|^2 \geq && (3) \\
&= \left(\frac{1}{2\pi}\right)^2 \left(\frac{1}{2}\right)^2
\end{aligned}$$

having used in

(1)

$$\mathcal{F}\{\dot{g}(t)\}(f) = \int_{t=-\infty}^{+\infty} \dot{g}(t) e^{-i2\pi f t} \, dt = \cdots = i2\pi \, f \, G(f) \ ,$$

$$f \, G(f) = -i \, \frac{\mathcal{F}\{\dot{g}(t)\}}{2\pi}$$

and thus *Plancherel's theorem*

$$\begin{aligned}\int_{f=-\infty}^{+\infty} |f\,G(f)|^2\,df &= \int_{f=-\infty}^{+\infty} \left|-\frac{i}{2\pi}\,\mathcal{F}\{\dot g(t)\}\right|^2\,df = \\ &= \int_{t=-\infty}^{+\infty} \left|-\frac{i}{2\pi}\,\dot g(t)\right|^2\,dt\end{aligned}$$

in (2) Cauchy-Schwartz inequality,

(3)

$$\begin{aligned}a &:= \int_{t=-\infty}^{+\infty} t g^*(t)\dot g(t)\,dt = \\ &= \underbrace{[t\,g^*(t)\,g(t)]\,|_{-\infty}^{+\infty}}_{=0} - \int_{t=-\infty}^{+\infty}\frac{d}{dt}\,(tg^*(t))\,g(t)\,dt = \\ &= -\int_{t=-\infty}^{+\infty} g^*(t)\,g(t)\,dt - \int_{t=-\infty}^{+\infty} t\dot g^*(t)g(t)\,dt = \\ &= -1 - a^*\,.\end{aligned}$$

$$-1 = a + a^* = 2\,\mathrm{re}\{a\}\,,$$

and thus

$$|a|^2 \ge \mathrm{re}\{a\}^2 = \frac{1}{4}\,.$$

If $\overline{t} \neq 0$, or $\overline{f} \neq 0$,

$$\begin{aligned}\sigma^2_{t,g}\sigma^2_{f,G} &= \int_{t=-\infty}^{+\infty}\left|t-\overline{t}\right|^2\,g^*(t)\,g(t)\,dt\int_{f=-\infty}^{+\infty}\left|f-\overline{f}\right|^2\,G^*(f)\,G(f)\,df = \\ &= \int_{t=-\infty}^{+\infty}\left|(t-\overline{t})\,g(t)\right|^2\,dt\int_{f=-\infty}^{+\infty}\left|(f-\overline{f})\,G(f)\right|^2\,df = \qquad (1)\\ &= \dots\end{aligned}$$

13.3 Relations between Fourier transforms

Some freestyle in changing order of summations and integrals, and use of generalized functions here…check it!

Different Fourier transforms exist, depending if the original function is:

- time discrete/time continuous
- periodic/non-periodic

namely,

- FS, Fourier series: time continuous, periodic function (or finite domain, with a periodic extension)
- FT, Fourier transform: time continuous, non-periodic function
- DTFT, discrete-time Fourier transform: time didscrete, infinite-length sequence
- DFT, discrete Fourier transform: time discrete, finite-length sequence (and then with a periodic extension)

13.3.1 Fourier transform of integrable functions

$$F(\nu) := \mathcal{F}\left\{f(t)\right\}(\nu) := \int_{t=-\infty}^{+\infty} f(t) e^{-i2\pi\nu t}\,dt \ ,$$

13.3.2 Fourier transform of the sum of shifted integrable functions

The infinite sum of a shifted integrable function is defined as

$$\tilde{f}_T(t) = \sum_{n=-\infty}^{+\infty} f(t - nT) \ .$$

Its Fourier transform reads

$$\begin{aligned}
\mathcal{F}\left\{\tilde{f}_T(t)\right\}(\nu) &= \int_{t=-\infty}^{+\infty} \tilde{f}_T(t) e^{-i2\pi\nu t}\,dt = \\
&= \sum_{n=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} f(t-nT) e^{-i2\pi\nu t}\,dt = \qquad (1) \\
&= \sum_{n=-\infty}^{+\infty} F(\nu) e^{-i2\pi\nu nT} = \qquad (2) \\
&= F(\nu) \sum_{n=-\infty}^{+\infty} e^{-i2\pi\nu nT} = \qquad (3) \\
&= \Delta\nu\, F(\nu)\, \mathrm{III}_{\Delta\nu}(\nu) \ ,
\end{aligned}$$

having used properties of Fourier transform of shifted function in (1), and the properties of Dirac's comb in (3), having defined the frequency resolution

$$\Delta\nu := \frac{1}{T} \ .$$

This Fourier transform is proportional to the Fourier transform of the original function, sampled in frequency with elementary frequency $\Delta\nu$.

13.3.3 Fourier transform of the a function sampled with a Dirac comb - DTFT

Fourier transform of the original function sampled with $\Delta t\, \mathrm{III}_{\Delta t}(t)$ reads

$$\begin{aligned}
\mathcal{F}\left\{\Delta t\, f(t)\, \mathrm{III}_{\Delta t}(t)\right\} &= \Delta t \int_{t=-\infty}^{+\infty} f(t)\, \mathrm{III}_{\Delta t}(t) e^{-i2\pi\nu t}\,dt = \\
&\sim \Delta t \frac{1}{\Delta t} \int_{t=-\infty}^{+\infty} f(t) \sum_{n=-\infty}^{+\infty} e^{in\frac{2\pi}{\Delta t}t} e^{-i2\pi\nu t}\,dt = \\
&= \sum_{n=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} f(t)\, e^{-i2\pi(\nu - n\overline{\nu})t}\,dt = \\
&= \sum_{n=-\infty}^{+\infty} F\left(\nu - n\overline{\nu}\right) = \mathrm{DTFT}(f(t); \Delta t) \ ,
\end{aligned} \tag{13.6}$$

i.e. equals the periodic sum of the Fourier of the original function, with period

$$\overline{\nu} := \frac{1}{\Delta t} \ .$$

From this last sentence and from the *symmetry properties of Fourier transform*, **Nyquist-Shannon sampling theorem** follows seamlessly.

Theorem 13.3.1 (Nyquist-Shannon sampling theorem)

In order to **avoid aliasing** the sampling frequency must be twice the maxiumum[1] frequency in the signal,

$$\nu_s \ge 2\nu_{max} \ .$$

todo check alternative expressions if using the definition of train of impulses instead of the Fourier series of Dirac's comb.

$$\begin{aligned}
&= \Delta t \int_{t=-\infty}^{+\infty} f(t) \sum_{k=-\infty}^{+\infty} \delta(t-k\Delta t)\, e^{-i2\pi\nu t}\, dt = \\
&= \Delta t \sum_{k=-\infty}^{+\infty} f(k\Delta t) e^{-i2\pi\nu k\Delta t} = \text{DTFT}\left(f(t);\Delta t\right)
\end{aligned} \tag{13.7}$$

13.3.4 Fourier transform of the sum of shifted integral functions sampled with a Dirac comb

Fourier transform of the periodic sum

$$\Delta t\, \tilde{f}(t)\, \text{III}_{\Delta t}(t) = \Delta t \sum_{n=-\infty}^{+\infty} f(t-nT)\, \text{III}_{\Delta t}(t)$$

reads

$$\begin{aligned}
\mathcal{F}\left\{\Delta t\, \tilde{f}(t)\, \text{III}_{\Delta t}(t)\right\}(\nu) &= \Delta t \int_{t=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} f(t-nT) \sum_{k=-\infty}^{+\infty} \delta(t-k\Delta t)\, e^{-i2\pi\nu t}\, dt = \\
&= \Delta t \sum_{n=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} f(k\Delta t - nT)\, e^{-i2\pi\nu k\Delta t} =
\end{aligned}$$

and defining $k\Delta\tau_n := k\Delta t - nT$,

$$\begin{aligned}
&= \Delta t \sum_{n=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} f(k\Delta\tau_n) e^{-i2\pi\nu k\Delta\tau_n} e^{-i2\pi\nu nT} = \\
&= \underbrace{\Delta t \sum_{k=-\infty}^{+\infty} f(k\Delta\tau_n) e^{-i2\pi\nu k\Delta\tau_n}}_{=\text{DTFT}(f(t),\Delta t)} \underbrace{\sum_{n=-\infty}^{+\infty} e^{-i2\pi\nu nT}}_{=\Delta\nu\, \text{III}_{\Delta\nu}(\nu)} = \\
&= \text{DTFT}(f(t),\Delta t)\, \Delta\nu\, \text{III}_{\Delta\nu}(\nu) \ .
\end{aligned}$$

todo check! check the change of coordinates that makes DTFT appear

todo check! what follows or, using the relation between Δt and $T = N\Delta t$, $\Delta\nu = \frac{1}{T}$, and thus

$$\Delta t\, \Delta\nu = \Delta t\, \frac{1}{T} = \frac{1}{N} \ ,$$

it follows

$$= \frac{1}{N} \sum_{k=-\infty}^{+\infty} f(k\Delta\tau_n) e^{-i2\pi\nu k\Delta\tau_n}\, \text{III}_{\Delta\nu}(\nu) \ .$$

[1] Usually there's no such a frequency above which the signal is exactly zero, but usually there's a frequency above which the spectrum of the signal is approximately zero, i.e. below a threshold where it can be treated as zero, and introduce no aliasing.

13.3.5 Useful properties

Dirac's comb $\text{III}_T(t)$

Dirac comb $\text{III}_T(t)$ is defined as a train of Dirac's delta

$$\text{III}_T(t) = \sum_{m=-\infty}^{+\infty} \delta(t - mT) \ .$$

Coefficients (13.2) of the Fourier series (13.1) of a T-periodic train of Dirac delta for $t \in \left[-\frac{T}{2}, \frac{T}{2}\right]$, read

$$c_n = \frac{1}{T} \int_{t=0}^{T} \delta(t) \, e^{-in\frac{2\pi}{T}t} = \frac{1}{T} \ ,$$

and thus the Fourier series of Dirac comb $\text{III}_T(t)$ reads

$$\text{III}_T(t) = \sum_{m=-\infty}^{+\infty} \delta(t - mT) \sim \frac{1}{T} \sum_{n=-\infty}^{+\infty} e^{in\frac{2\pi}{T}t} \ .$$

Symmetry of Fourier transform

Part VII

Calculus of Variations

CHAPTER

FOURTEEN

INTRODUCTION TO CALCULUS OF VARIATIONS

Calculus of variation deals with variations - i.e. “small changes” - of functions and functionals.

The meaning of the term functional may vary on the subfield of interest. In the field of calculus of variation, a **functional** can be defined as a function of function, i.e. a function whose argument is another function.

Fields and applications

Fields and applications related to calculus of variations (give some examples below):

- gradient-based techniques like some methods in:
  - optimization, either free or constrained (via Lagrange multiplier methods)
  - sensitivity
- classical mechanics and physics in general:
  - analytical mechanics: Lagrangian formulation and Hamiltonian formulation of classical mechanics
- …

Examples

- Lagrange equations for general problem
- examples:
  - brachistochrone for minimum time,…
  - catenary, i.e. static solution of wire and cables with neglibile bending stiffness
  - isoperimetric inequality, i.e. circle is the plane closed curve with given perimeter enclosing the largest area
- sensitivity of results to parameters. Some interesting sensitivity, both in time and trasnformed domains
  - characteristics of a system:
    * equilibria
    * eigenvalues
    * …
- optimal control methods

14.1 Lagrange equations

Given the functional S, with arguments a function $q(t)$ and the independent variable t,

$$S[q(t), t] = \int_{t=t_0}^{t_1} L(\dot{q}(t),\, q(t),\, t)\, dt$$

its variation w.r.t. the function $q(t)$ reads

$$\delta S[q(t), t] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(S[q(t) + \varepsilon w(t),\, t] - S[q(t),\, t] \right)$$

where the function $w(t)$ is arbitrary, among those satisfying the constraint of the problems: as an example here, if the function $q(t)$ has prescribed values q^* for some values of the independent variable, t^*, the variation $w(t)$ of the function $q(t)$ is zero there, $w(t^*)$ so that the variated function $q(t)+\varepsilon w(t)$ satisfies the constraint as well, i.e. $q(t^*)+\varepsilon w(t^*) = q^*$.

Variation involves only small changes of function arguments, since these ones are the elements that can be effectively changed, while the independent variable is not.

Direct computation of the variation gives

$$\begin{aligned}
\delta S[q(t), t] & = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(S[q(t) + \varepsilon w(t),\, t] - S[q(t),\, t] \right) = \\
& = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\int_{t=t_0}^{t_1} L(\dot{q}(t) + \varepsilon \dot{w}(t),\, q(t) + \varepsilon w(t),\, t) - \int_{t=t_0}^{t_1} L(\dot{q}(t),\, q(t),\, t)\, dt \right) = \\
& = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t=t_0}^{t_1} \left(L(\dot{q}(t) + \varepsilon \dot{w}(t),\, q(t) + \varepsilon w(t),\, t) - L(\dot{q}(t),\, q(t),\, t) \right) dt = \\
& = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t=t_0}^{t_1} \left\{ L(\dot{q}(t),\, q(t),\, t) + \varepsilon \left[\frac{\partial L}{\partial \dot{q}} \dot{w}(t) + \frac{\partial L}{\partial q} w(t) \right] + o(\varepsilon) - L(\dot{q}(t),\, q(t),\, t) \right\} dt = \\
& = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t=t_0}^{t_1} \left\{ \varepsilon \left[\frac{\partial L}{\partial \dot{q}} \dot{w}(t) + \frac{\partial L}{\partial q} w(t) \right] + o(\varepsilon) \right\} dt = \\
& = \int_{t=t_0}^{t_1} \left\{ \frac{\partial L}{\partial \dot{q}} \dot{w}(t) + \frac{\partial L}{\partial q} w(t) \right\} dt = \\
& = \left[w(t) \frac{\partial L}{\partial \dot{q}} \right] \Bigg|_{t=t_0}^{t_1} + \int_{t=t_0}^{t_1} \left\{ -\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial q} \right\} w(t)\, dt\, .
\end{aligned}$$

The solution depends on the boundary conditions at the extreme points t_0, t_1. **If** the value of the function $q(t)$ is prescribed in t_0 and t_1, $q(t_0) = q_0$, $q(t_1) = q_1$, then its variation is zero, $w(t_0) = w(t_1) = 0$, for the reason that has been discussed above. The variation of the functional with prescribed boundary values of the argument function thus reads

$$\delta S[q(t), t] = \int_{t=t_0}^{t_1} \left\{ -\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial q} \right\} \delta q(t)\, dt\, ,$$

having called $w(t) =: \delta q(t)$ to stress that is the variation of function $q(t)$. This notation - it's just notation, it has no special properties - could be useful if the functional depends on several arguments.

Stationary conditions, $\delta S = 0$. Stationary condition of the functional S implies that $\delta S = 0$ for all the possible variations of the argument function, $\forall \delta q(t)$. This condition implies that the integrand is identically zero, i.e. **Lagrange equations**,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0\, ,$$

Higher-order derivatives

Method 1. If the Lagrangian function L depends on higher order derivatives,

$$L\left(q^{(n)}(t),\, q^{(n-1)}(t),\, \dots,\, q'(t),\, q(t),\, t\right)$$

it's possible to recast the problem defining the n-dimensional function, $\mathbf{q}(t)$,

$$\mathbf{q}(t) = \left(q^0(t), q^1(t), \dots, q^{n-1}(t)\right) := \left(q(t), q'(t), \dots, q^{(n-1)}(t)\right) \ .$$

With some abuse of notation in L, the functional S can be recasted as

$$\begin{aligned} S[q(t),t] & = \int_{t=t_0}^{t_1} L(q^{(n)}(t),\, \dots,\, q(t),\, t)\, dt = \\ & = \int_{t=t_0}^{t_1} L(\dot{\mathbf{q}}(t),\, \mathbf{q}(t),\, t)\, dt \ . \end{aligned}$$

todo *Add constraints on components of* $\mathbf{q}(t)$*?*

Repeating the computation, the variation of the functional reads

$$\delta S[\mathbf{q}(t),t] = \left[\delta \mathbf{q}^T(t) \frac{\partial L}{\partial \dot{\mathbf{q}}}\right]\Bigg|_{t=t_0}^{t_1} + \int_{t=t_0}^{t_1} \delta \mathbf{q}^T(t) \left\{ -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) + \frac{\partial L}{\partial \mathbf{q}} \right\} dt \ .$$

Method 2. …

14.1.1 Euler-Beltrami equation

If the Lagrangian function L is not an explicit function of the independent variable, $L(q'(x), q(x))$, Euler-Berltrami equation follows from the derivative of the Lagrangian,

$$\begin{aligned} \frac{dL}{dx} & = \frac{\partial L}{\partial q'} q''(x) + \frac{\partial L}{\partial q} q' = \\ & = \frac{\partial L}{\partial q'} q'' + \frac{d}{dx}\left(\frac{\partial L}{\partial q'}\right) q' = \\ & = \frac{d}{dx}\left(\frac{\partial L}{\partial q'} q'\right) \ , \end{aligned}$$

and thus

$$\frac{d}{dx}\left[L - q' \frac{\partial L}{\partial q'}\right] = 0 \qquad \rightarrow \qquad L - q' \frac{\partial L}{\partial q'} = C \quad \text{const.}$$

Note 1. While Lagrange equations are a set of N equations if the functional depends on N argument functions $q_k(t)$, $k = 1 : N$, Euler-Beltrami equation is an equation only. Indeed for multiple argument functions

$$\begin{aligned} \frac{dL}{dx} & = \frac{\partial L}{\partial q_k'} q_k''(x) + \frac{\partial L}{\partial q_k} q_k' = \\ & = \frac{\partial L}{\partial q_k'} q_k'' + \frac{d}{dx}\left(\frac{\partial L}{\partial q_k'}\right) q_k' = \frac{d}{dx}\left(\frac{\partial L}{\partial q_k'} q_k'\right) \ , \end{aligned}$$

where Einstein's summation notation of repeated index is used. Euler-Beltrami thus reads

$$L(q_l'(x), q_l(x)) - q_k'(x) \frac{\partial L}{\partial q_k'}(q_l'(x), q_l(x)) = C \ .$$

Note 2. If the Lagrangian function is an explicit function of the independent variable x, $L(q'(x), q(x), x)$, it's not hard to realize that the derivative of the Lagrangian function, along with the use of the Lagrange equation, gives

$$\frac{d}{dx}\left[L - q'\frac{\partial L}{\partial q'}\right] = \frac{\partial L}{\partial x} \ .$$

Example 14.1.1 (Euler-Beltrami with $L(q'(x), q(x), x)$, Hamiltonian, energy and E.Noether)

Euler-Beltrami equation shows that if $L(q'(x), q(x))$, thus $L - q'\partial_{q'}L$ is constant, (or an integral of motion in dynamics). In analytical mechanics (Lagrange mechanics, Hamiltonian mechanics), Lagrangian and Hamiltonian functions of a system read

$$\begin{aligned} L(\dot{q}_k(t), q_k(t), t) &= T(\dot{q}, q, t) + U(q, t) \\ H(p, q, t) &:= p_k \dot{q}_k - L = \dot{q}_k \frac{\partial L}{\partial \dot{q}^k} - L \ , \end{aligned}$$

having used the common definition of the generalized momenta $p_k := \frac{\partial L}{\partial \dot{q}^k}$. It should be immediate to realize that the Hamiltonian is just the quantity appearing in Euler-Beltrami equation (or in its "modified version" if $\partial_t L \neq 0$), and thus

$$\frac{dH}{dt} = \frac{\partial L}{\partial t} \ .$$

In mechanics, if $\partial_t L = 0$, the Hamiltonian is a constant of motion. In this case, it can be prove that the Hamiltonian is equal to the eneergy of the system.

Classical examples.

Example 14.1.2 (Brachistochrone)

Find the trajectory…

- Elementary length: $ds = v\,dt$
- Energy: $E(y) = \frac{1}{2}mv^2 - mgy + C$. Setting $E = 0$ at starting point, from rest, at $y_0 = 0$, it implies $C = 0$; thus $v = \sqrt{2gy}$
- $x(s)$, $y(s)$,
- $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2(x)}\,dx$

$$T = \int_{t_0}^{t_1} dt = \int_{s_0}^{s_1} \frac{ds}{v} = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2(x)}}{\sqrt{2gy(x)}}\,dx$$

The Lagrangian doesn't explicitly depend on x, thus Euler-Beltrami equation can be used. Partial derivative of the Lagrangian function w.r.t. q' reads

$$\frac{\partial L}{\partial y'} = ... = \frac{1}{\sqrt{2gy}\sqrt{1 + y'^2}} y' \ ,$$

and thus Euler-Beltrami equation reads

$$C = L - q'\frac{\partial L}{\partial y'} = \frac{1 + y'^2 - y'^2}{\sqrt{2gy}\sqrt{1 + y'^2}} = \frac{1}{\sqrt{2gy}\sqrt{1 + y'^2}}$$

Squaring $2gC^2 = \frac{1}{y(1+y'^2)}$, it's possible to write

$$y(x) = \frac{1}{2gC^2(1 + y'^2(x))} \ ,$$

Making the substitution $y(x)' = ...$

Example 14.1.3 (Catenary)

Example 14.1.4 (Isoperimetric problem)

Part VIII

Ordinary Differential Equations

CHAPTER

FIFTEEN

INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

CHAPTER

SIXTEEN

LINEAR TIME-INVARIANT SYSTEMS

A linear time invariant system is governed by a linear ODE with constant coefficients. These equations can be recast as a first order system of ODEs,

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \\ \mathbf{x}(0^-) = \mathbf{x}_0 \end{cases}$$

Exploiting the *properties of matrix exponential* the general expression of the state can be written as the sum of the free response to initial condition and the forced response.

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0 + \int_{\tau=0^-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau \\ \mathbf{y}(t) &= \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_{\tau=0^-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}(t) \end{aligned}$$

Proof in time domain

Multipying by $e^{-\mathbf{A}t}$,

$$\begin{aligned} e^{-\mathbf{A}t}(\dot{x}(t) - \mathbf{A}\mathbf{x}(t)) &= e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \\ \frac{d}{dt}\left(\mathbf{x}e^{-\mathbf{A}t}\right) &= e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \end{aligned}$$

and integrating from 0^- to a generic time value t,

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}_0 = \int_{\tau=0^-}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

The state $\mathbf{x}(t)$ can be written as the sum of the free response and a force response. The general expression of the state and the output as a function reads

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0 + \int_{\tau=0^-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau \\ \mathbf{y}(t) &= \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_{\tau=0^-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}(t) \end{aligned}$$

Laplace domain. The *Laplace transform* of the problem reads

$$\begin{cases} s\hat{\mathbf{x}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\hat{\mathbf{u}} + \mathbf{x}_0 \\ \hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}} + \mathbf{D}\hat{\mathbf{u}} \end{cases}$$

$$
\begin{aligned}
(s\mathbf{I}-\mathbf{A})\hat{\mathbf{x}} &= \mathbf{B}\hat{\mathbf{u}}+\mathbf{x}_0 \\
\hat{\mathbf{x}}(s) &= (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}_0+(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{u}}(s) \\
\hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}_0+\left[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right]\hat{\mathbf{u}}(s)
\end{aligned}
$$

Performing inverse Laplace transform allows to go back to time domain (just use Laplace inverse transform of a matrix exponential, and the formula (12.1) for Laplace transform of convolution).

16.1 Impulsive force

The effect of an impulsive force at time $t=0$ is equivalent to an instantaneous change in the initial state, from time 0^- before the impulse to time 0^+ after the impulse. Splitting the input $\mathbf{u}(t)$ as the sum of impulsive input and regular input,

$$
\begin{aligned}
\mathbf{u}(t) &= \mathbf{u}_r(t)+\mathbf{u}_\delta\delta(t) \\
\hat{\mathbf{u}}(s) &= \hat{\mathbf{u}}_r(s)+\mathbf{u}_\delta
\end{aligned}
$$

the solution in time and Laplace domain reads

$$
\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0+\int_{\tau=0^-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\left(\mathbf{u}_r(\tau)+\mathbf{u}_\delta\delta(\tau)\right)\,d\tau = \\
&= e^{\mathbf{A}t}\left(\mathbf{x}_0+\mathbf{B}\mathbf{u}_\delta\right)+\int_{\tau=0^-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}_r(\tau)\,d\tau \\
\mathbf{y}(t) &= \mathbf{C}e^{\mathbf{A}t}\left(\mathbf{x}_0+\mathbf{B}\mathbf{u}_\delta\right)+\int_{\tau=0^-}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}_r(\tau)\,d\tau+\mathbf{D}\mathbf{u}_r(t)+\mathbf{D}\mathbf{u}_\delta(t)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathbf{x}}(s) &= (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{u}}_r(s)+(s\mathbf{I}-\mathbf{A})^{-1}\left(\mathbf{x}_0+\mathbf{B}\mathbf{u}_\delta\right) \\
\hat{\mathbf{y}}(s) &= \left[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right]\hat{\mathbf{u}}_r(s)+\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\left(\mathbf{x}_0+\mathbf{B}\mathbf{u}_\delta\right)+\mathbf{D}\mathbf{u}_\delta
\end{aligned}
$$

16.2 Properties

Matrix exponential.

$$
e^{\mathbf{A}t}=\sum_{k=0}^{+\infty}\frac{\mathbf{A}^k t^k}{k!}\ .
$$

Assuming it's possible swap derivative operator and summation (when?), it's possible to write

$$
\frac{d}{dt}e^{\mathbf{A}t}=\frac{d}{dt}\sum_{k=0}^{+\infty}\frac{\mathbf{A}^k t^k}{k!}=\sum_{k=1}^{+\infty}kt^{k-1}\frac{\mathbf{A}^k t^{k-1}}{k!}=\mathbf{A}e^{\mathbf{A}t}\ .
$$

Laplace transform of exponential matrix.

$$
\begin{aligned}
\mathcal{L}\left\{e^{\mathbf{A}t}\right\}(s) &:= \int_{t=0^-}^{+\infty} e^{\mathbf{A}t}e^{-st}\,dt = \\
&= \int_{t=0^-}^{+\infty} e^{(-s\mathbf{I}+\mathbf{A})t}\,dt = \\
&= (-s\mathbf{I}+\mathbf{A})^{-1}\ e^{(-s\mathbf{I}+\mathbf{A})t}\Big|_{t=0^-}^{+\infty} = \\
&= (s\mathbf{I}-\mathbf{A})^{-1}\ ,
\end{aligned}
$$

for all the values of s for which $-s\mathbf{I} + \mathbf{A}$ is asymptotically stable, i.e. has all the eignevalues (thus, assuming that the matrix $\mathbf{A}$ can be diagonalizable. What happens if not? Exploit other matrix decompositions to draw conclusions) with negative real parts, and thus for all the values of $s > \max \text{re}\{s_k(\mathbf{A})\}$, as it's shown in *Example 16.2.1*

Example 16.2.1 (Asymptotic stability of a matrix A)

An $N \times N$ diagnonalizable matrix **A**,

$$\mathbf{A}\mathbf{v}_k = \mathbf{v}_k \, s_k \tag{16.1}$$

has all the eigenvalues with negative real part, re $\{s_k\} < 0$, $\forall k = 1 : N$.

The eigenvalues of a matrix $a\mathbf{I} + \mathbf{A}$ are $a + s_k$, while the eigenvectors are the same as those of the matrix **A**. This can be easily proved adding $a\mathbf{I}\mathbf{v}_k$ to both sides of equation (16.1),

$$(a\mathbf{I} + \mathbf{A})\, \mathbf{v}_k = \mathbf{v}_k(a + s_k) \ .$$

Transform of the convolution.

Part IX

Partial Differential Equations

CHAPTER

SEVENTEEN

INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations usually comes from balance equations in **continuum mechanics**. Integral equations are the most general form of these equations, and an equivalent differential problem only exists if the fields involved in the equations are regular enough, for their derivatives to exist - and to apply theorems requiring some regularity of the functions.

Classical numerical methods:

- **FVM**: directly solves the **integral problem**, solving integral balance equations for cells in which the domain is divided
- **FDM**: given the problem in **differential form**, FDM directly approximates space derivatives of the **strong formulation** of the problem
- **FEM**: given the problem in **differential form**, FEM projects the **weak formulation** of the problem on a finite-dimensional space
- **BEM**: *integro-differential equation*, *singularities*,…
- **Spectral methods**,…
- **SEM**,…

17.1 Examples

In Physics:

- Advection equation

$$\partial_t u + \vec{a} \cdot \nabla u = f$$

- Diffusion equation

$$\partial_t u - \nu \nabla^2 u = f$$

- Hyperbolic equation/system of equations

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{F}(\mathbf{u}) = \mathbf{f}$$

- Wave equation

$$\frac{1}{c^2} \partial_{tt} u - \nabla^2 u = f$$

17.2 Balance equations in physics

- Small-strain continuum mechanics

$$\rho \partial_{tt} \vec{s} = \rho_0 \vec{g} + \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{\varepsilon})$$

- Heat conduction
- Fluid dynamics
  - Navier-Stokes for compressible fluids (conservative or convective equations)

$$\{$$

  - Navier-Stokes for incompressible fluids (convective form,…)

$$\begin{cases} \rho \partial_t \vec{u} + \rho(\vec{u} \cdot \nabla)\vec{u} - \mu \nabla^2 \vec{u} + \nabla P = \rho \vec{g} \\ \nabla \cdot \vec{u} = 0 \end{cases}$$

todo

- Different forms of equations may be more or less convenient for different solution approaches
- Most of the physical laws comes from integral balance equation of the form

$$\frac{d}{dt} \int_{V_t} \rho \mathbf{u} = \int_{V_t} \rho \mathbf{r} + \oint_{\partial V_t} \hat{n} \cdot \mathbf{T}(\mathbf{u})$$

  whose local - differential - form (in case of differentiable functions) readily follows from the application of Reynolds' transport theorem and divergence theorem to transform time derivative and boundary terms

$$\begin{aligned} & \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \vec{u}) = \rho \mathbf{r} + \nabla \cdot \mathbf{T}(\mathbf{u}) \\ & \partial_t (\rho \mathbf{u}) + \nabla \cdot \mathbf{F}(\mathbf{u}) = \rho \mathbf{r} \ , \end{aligned}$$

  and the physical meaning of each term is evident and readily expalinable as flux or volume or surface sources.

- Further manipulation/simplification may cover the clear meaning of the terms of the differential equation. As an example, the conservative form of Navier-Stokes equations for incompressible fluids with constant and uniform density read

$$\begin{cases} \partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) = \rho \vec{g} + \nabla \cdot \mathbb{T} \\ \nabla \cdot \vec{u} = 0 \ , \end{cases}$$

  where the stress tensor for a Newtonian fluid reads

$$\begin{aligned} \mathbb{T} & = -p \mathbb{I} + 2 \mu \mathbb{D} + \lambda (\nabla \cdot \vec{u}) \mathbb{I} \\ & = -p \mathbb{I} + \mu (\nabla \vec{u} + \nabla^T \vec{u}) + \lambda (\nabla \cdot \vec{u}) \mathbb{I} \end{aligned}$$

  Using the incompressibility constraint $\nabla \cdot \vec{u}$, and treating the density ρ as a constant and uniform parameter, the convective form of the Navier-Stokes equations reads

$$\begin{cases} \rho \partial_t \vec{u} + \rho (\vec{u} \cdot \nabla) \vec{u} = \rho \vec{g} - \nabla P + 2 \mu \nabla \cdot \mathbb{D} \\ \nabla \cdot \vec{u} = 0 \ . \end{cases}$$

  The divergence of the viscous stress tensor becomes

$$2 \mu \nabla \cdot \mathbb{D} = \mu \nabla \cdot (\nabla \vec{u} + \nabla^T \vec{u}) = \mu \left(\nabla^2 \vec{u} + \nabla \underbrace{(\nabla \cdot \vec{u})}_{=0} \right) = \mu \nabla^2 \vec{u} \ ,$$

so that one of the most common form of incompressible Navier-Stokes equations follows

$$\begin{cases} \rho \partial_t \vec{u} + \rho(\vec{u} \cdot \nabla)\vec{u} - \mu \nabla^2 \vec{u} + \nabla P = \rho \vec{g} \\ \nabla \cdot \vec{u} = 0 \ . \end{cases}$$

It should be evident that in the latter form of Navier-Stokes equations no divergence explicitly appears, so that the right expression of surface source terms can't be found immediately. In momentum equation, surface source terms come from surface stress acting on the boundary of the domain, whose expression reads

$$\begin{aligned} \vec{t}_n & = \hat{n} \cdot \mathbb{T} = \\ & = \hat{n} \cdot (-P \mathbb{I} + 2 \mu \mathbb{D}) = \\ & = \hat{n} \cdot \left(-P \mathbb{I} + \mu \left(\nabla \vec{u} + \nabla^T \vec{u} \right) \right) = \\ & = -P \hat{n} + \hat{n} \cdot \left(\mu \left(\nabla \vec{u} + \nabla^T \vec{u} \right) \right) = \end{aligned}$$

As an example, in weak formulation of incompressible Navier-Stokes problem the **natural boundary condition** arising in the method depends on the expression of the strong formulation of the NS problem. If one needs to prescribe stress boundary conditions, it could be an idea to start from NS equations w/o extra simplifications.

CHAPTER

EIGHTEEN

ELLIPTIC EQUATIONS

18.1 Poisson equation

Given the volume density source $f(\vec{r})$ and the diffusivity $\nu(\vec{r})$, Poisson equation for the scalar field $\phi(\vec{r})$ reads

$$-\nabla \cdot (\nu \nabla \phi) = f \qquad \vec{r} \in V$$

with proper boundary conditions on ∂V. As an example, tipical boundary conditions are:

$$\begin{array}{lll} \phi(\vec{r}) = g(\vec{r}) & \vec{r} \in S_D & \text{esserntial - Dirichlet b.c.} \\ \nu \hat{n} \cdot \nabla \phi(\vec{r}) = h(\vec{r}) & \vec{r} \in S_N & \text{natural - Neumann b.c.} \\ a\phi(\vec{r}) + \nu \hat{n} \cdot \nabla \phi(\vec{r}) = b(\vec{r}) & \vec{r} \in S_R & \text{Robin b.c.} \end{array}$$

18.1.1 Weak formulation

For $\forall w \in$ ... (functional space, recall some results about existence and uniqueness of the solution, Lax-Milgram theorem,...)

$$\begin{aligned} 0 &= \int_V w \, \{\nabla \cdot (\nu \nabla \phi) + f\} = \\ &= \oint_{\partial V} w \hat{n} \cdot (\nu \nabla \phi) + \int_V \{-\nu \nabla \vec{w} \cdot \nabla \phi + w f\} = \end{aligned}$$

Splitting boundary contribution as the sum from single contributions from different regions, and applying boundary conditions, setting $w = 0$ for $\vec{r} \in S_D$ (see the ways to prescribe essential boundary conditions),

$$0 = \int_{S_D} \underbrace{w}_{=0} \hat{n} \cdot (\nu \nabla \phi) + \int_{S_N} w \underbrace{\hat{n} \cdot (\nu \nabla \phi)}_{=h} + \int_{S_R} w \underbrace{\hat{n} \cdot (\nu \nabla \phi)}_{=b - a\phi} + \int_V \{-\nu \nabla \vec{w} \cdot \nabla \phi + w f\} \ .$$

and rearranging the equation separating terms containing unknowns from known contributions,

$$\int_V \nu \nabla w \cdot \nabla \phi + \int_{S_R} w a \phi = \int_V w f + \int_{S_N} w h + \int_{S_R} w b \qquad \forall w \in \ldots \ ,$$

and $\phi = g$, for $\vec{r} \in S_D$.

Different ways to prescribe essential boundary conditions

Strong formulation.

Using Lagrance multiplier - weak formulation of essential boundary conditions. Adding a the essential boundary condition as a constraint with Lagrange multipliers in the weak formulation of the problem,

$$\cdots + \int_{S_D} w_D(\phi - g) \ ,$$

…

CHAPTER

NINETEEN

PARABOLIC EQUATIONS

19.1 Heat equation

Heat equation for a scalar field $\phi(\vec{r},t)$ can be interpreted as the unsteady equation of a *Poisson equation*,

$$\partial_t \phi - \nabla \cdot (\nu \nabla \phi) = f \qquad (\vec{r},t) \in V \times [0,T] \ ,$$

with proper boundary and initial conditions, $\phi(\vec{r},0) = \phi_0(\vec{r})$. Common boundary conditions are the same as the one discussed for Poisson problem.

19.1.1 Weak formulation

For $\forall w \in$ ... (functional space, recall some results about existence and uniqueness of the solution, Lax-Milgram theorem,…)

$$\begin{aligned} 0 & = \int_V w \left\{ -\partial_t \phi + \nabla \cdot (\nu \nabla \phi) + f \right\} = \\ & = \oint_{\partial V} w \hat{n} \cdot (\nu \nabla \phi) + \int_V \left\{ -\partial_t \phi - \nu \nabla \vec{w} \cdot \nabla \phi + w f \right\} = \end{aligned}$$

Splitting boundary contribution as the sum from single contributions from different regions, and applying boundary conditions, setting $w = 0$ for $\vec{r} \in S_D$ (see the ways to prescribe essential boundary conditions),

$$0 = \int_{S_D} \underbrace{w}_{=0} \hat{n} \cdot (\nu \nabla \phi) + \int_{S_N} w \underbrace{\hat{n} \cdot (\nu \nabla \phi)}_{=h} + \int_{S_R} w \underbrace{\hat{n} \cdot (\nu \nabla \phi)}_{=k - \phi} + \int_V \left\{ -\partial_t \phi - \nu \nabla \vec{w} \cdot \nabla \phi + w f \right\} \ .$$

and rearranging the equation separating terms containing unknowns from known contributions,

$$\int_V w \partial_t \phi + \int_V \nu \nabla w \cdot \nabla \phi + \int_{S_R} w \phi = \int_V w f + \int_{S_N} w h + \int_{S_R} w k \qquad \forall w \in \dots \ ,$$

and $\phi = g$, for $\vec{r} \in S_D$.

CHAPTER

TWENTY

HYPERBOLIC PROBLEMS

Hyperbolic problems often come from a small-amplitude linearization, or as the non-diffusion (or inviscid) limit of a more general problem.

As a result of these simplification, these problems may experience **shocks** (i.e. discontinuity in the solution, where the differential equations stop to hold, and integral equations and jump conditions are required). **todo** *classification of discontinuities on the massflow across the surface*

The very nature of these problem also suggest methods for the solution or the analysis of these equations, like **characteristic method**.

20.1 Scalar linear

20.1.1 1-dimensional

$$\partial_t u(x,t) + a \partial_x u(x,t) = f(x,t)$$

Caracteristic method. $U(t) = u(X(t),t)$, with the caracteristic curves $X(t)$ defined as those curves where the PDE becomes a ODE. Evaluating the time derivative of the function $u(X(t),t)$, the hyperbolic equation can be recast as

$$\frac{dU}{dt} + \left[a(X(t),t) - \frac{dX}{dt} \right] \partial_x u = f(X(t),t) \ .$$

The equation of characteristic lines is

$$\frac{dX}{dt} = a(X(t),t) \ ,$$

and the PDE on characteristic line becomes the ODE

$$\frac{dU}{dt}(X(t),t) = f(X(t),t) \ .$$

20.2 Scalar non-linear

20.3 System linear

20.3.1 1-dimensional

$$\mathbf{u}(x,t)$$

$$\partial_t \mathbf{u} + \mathbf{A} \partial_x \mathbf{u} = \mathbf{f}$$

Method of characteristics

Characteristics. $\mathbf{U}(t) = \mathbf{u}(X(t), t)$

$$\frac{d\mathbf{U}}{dt} - \frac{dX}{dt}\partial_x \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = \mathbf{f}$$

In order to get the equations of characteristic lines where PDE turns into ODEs, the eigenproblem

$$\mathbf{A}\partial_x \mathbf{u} = \frac{dX}{dt}\partial_x \mathbf{u} \ ,$$

holds. This problem has non trivial solution if $\frac{dX}{dt}$ and $\partial_x \mathbf{u}$ are pairs of eigenvalues and (right) eigenvectors of the array $\mathbf{A}$.

Diagonalization.

$$\mathbf{A} = \mathbf{R}\mathbf{\Lambda}\mathbf{L}$$

$$\mathbf{L}\left[\partial_t \mathbf{u} + \mathbf{R}\mathbf{\Lambda}\mathbf{L}\partial_x \mathbf{u}\right] = \mathbf{L}\mathbf{f}$$

Since $\mathbf{L} = \mathbf{R}^{-1}$, and defining the **characteristic variables** by $d\mathbf{q} = \mathbf{L} d\mathbf{u}$ - in linear problems matrix $\mathbf{A}$ is constant, and so its spectral decomoposition, and thus $\mathbf{q} = \mathbf{L}\mathbf{u}$ - , it's possible to recast the original problem in diagonal form

$$\partial_t \mathbf{q} + \mathbf{\Lambda}\partial_x \mathbf{q} = \mathbf{L}\mathbf{f}$$

$$\partial_t q_i + \Lambda_i \partial_x q_i = \sum_k L_{ik} f_k =: F_i \ .$$

Thus, on the i^{th} family of characteristic lines, $\dfrac{dX}{dt} = \lambda_i$, $Q_i(t) = q_i(x(t), t)$ evolves as

$$\frac{dQ_i}{dt} = F_i \ .$$

If $F_i = [\mathbf{L}\mathbf{f}]_i = 0$, the characteristic variable q_i is constant along the characteristic lines. Once the characteristic variables are determined, the conservative variables are evalauted as $\mathbf{u}(x,t) = \mathbf{R}\mathbf{q}(x,t)$.

Domain of influence and domain of dependence

Riemann problem

A Riemann problem is defined as the evolution of the initial state

$$\mathbf{u}(x, t_0) = \begin{cases} \mathbf{u}_a \ , & x < x_0 \\ \mathbf{u}_b \ , & x > x_0 \end{cases}$$

This problem is quite useful in quite a wide range of numerical methods for hyperbolic problems - Godunov schemes in Finite Volume Methods -, to evaluate the **boundary state** to be used numerical flux.

For linear problems, the matrix $\mathbf{A}$ is constant ad so it is its spectral decomposition, $\mathbf{A} = \mathbf{R}\mathbf{\Lambda}\mathbf{L}$, and the solution of a Riemann problem of an homogeneous linear hyperbolic system can be easily determined analytically with the method of characteristics,

Let's change the origin of space and time, so that the initial state is in $t = 0$, and the jump in the initial condiiton in $x = 0$. Each charactersitic variable $q_k(x,t)$ is constant on its family of characteristic lines, $x = X_k(t) = x_{0,k} + \lambda_k t$.

$$q_k(x,t) = q_k(x_{0,k} + \lambda_k t, t) = q_k(x_{0,k}, 0) = q_k(x - \lambda_k t, 0) = L_{ki} u_j(x - \lambda_k t, 0) \ .$$

Thus, the solution in conservative variables $\mathbf{u}(x,t)$ in x at time t reads

$$\begin{aligned}
&\mathbf{u}(x,t) = \mathbf{Rq}(x,t) \\
&u_i(x,t) = R_{ik} q_k(x,t) = R_{ik} q_k(x - \lambda_k t, 0) = R_{ik} L_{kj} u_j(x - \lambda_k t, 0)
\end{aligned}$$

In a Riemann problem for a N-dimensional linear system the solution shows $N+1$ homogeneous regions (at most, in general the same number as the number of the non-coincident eigenvalues $+1$), delimited by the characteristic lines with origin in the discontinuity. Sorting the eigenvalues in increasing order

$$\lambda_1 > \lambda_2 > \dots > \lambda_N \ ,$$

and defining the homogeneous regions

$$\begin{aligned}
S_0 &: \frac{x}{t} \in (-\infty, \lambda_1) \\
S_1 &: \frac{x}{t} \in (\lambda_1, \lambda_2) \\
&\dots \\
S_i &: \frac{x}{t} \in (\lambda_i, \lambda_{i+1}) \\
&\dots \\
S_{N-1} &: \frac{x}{t} \in (\lambda_{N-1}, \lambda_N) \\
S_N &: \frac{x}{t} \in (\lambda_N, +\infty)
\end{aligned}$$

the solution is in the S_i region is

$$u_i(x,t) = \sum_{\lambda_k > \frac{x}{t}} R_{ik} q_{a,k} + \sum_{\lambda_k < \frac{x}{t}} R_{ik} q_{b,k}$$

Example 20.3.1 (Linear(ized) P-system)

The linear(ized) P-system around a uniform reference state $\overline{\rho}$, $\overline{u}$ in convective form reads

$$\partial_t \begin{bmatrix} \rho \\ u \end{bmatrix} + \begin{bmatrix} \overline{u} & \overline{\rho} \\ \frac{a^2}{\overline{\rho}} & \overline{u} \end{bmatrix} \partial_x \begin{bmatrix} \rho \\ u \end{bmatrix} = \mathbf{0} \ .$$

Spectral decomposition.

$$0 = |-\lambda \mathbf{I} + \mathbf{A}| = (\overline{u} - \lambda)^2 - a^2$$

$$\begin{aligned}
\lambda_{12} = \overline{u} \mp a \qquad &, \qquad \mathbf{r}_{12} = \begin{bmatrix} \overline{\rho} \\ \mp a \end{bmatrix} \\
\mathbf{R} &= \begin{bmatrix} \overline{\rho} & \overline{\rho} \\ -a & a \end{bmatrix} \\
\mathbf{L} = \mathbf{R}^{-1} &= \frac{1}{2\overline{\rho} a} \begin{bmatrix} a & -\overline{\rho} \\ a & \overline{\rho} \end{bmatrix}
\end{aligned}$$

Reference state.

$$|u| \ : \ \begin{cases} = 0 & \text{at rest} \\ < a & \text{subsonic flow} \\ > a & \text{supersonic flow to the left/right} \end{cases}$$

Subsonic: the two families of characteristic lines have opposite direction; supersonic: the two families of characteristic lines have the same direction.

Example 20.3.2 (Linearized shallow water equations)

Example 20.3.3 (Linearized Euler equations (acoustics))

Example 20.3.4 (Wave equation)

A wave equation arises in many different fields of science. As an example, 1-dimensional wave equation descrives the axial dynamics of a truss

$$m\partial_{tt}u - EA\partial_{xx}u = f \ ,$$

that can be recast in the general expression of wave equation

$$\partial_{tt}u - c^2\partial_{xx}u = F$$

The 2^{nd} order differential operator appearing in 1-dimensional wave equation can be factored as the "product" of 2 1^{st} order differentail operators,

$$\left(\partial_{tt} - c^2\partial_{xx}\right)u = \left(\partial_t - c\partial_x\right)\left(\partial_t + c\partial_x\right)u$$

and thus a wave equation can be written as

$$\begin{cases} \partial_t u + c\partial_x u - w = 0 \\ \partial_t w - c\partial_x w = F \end{cases}$$

In the regime of small displacement, the velocity field is the partial time derivative of the dispalcement field, $v = \partial_t u$, and the axial force reads $N = EA\partial_x u$. Exploiting Schwartz's theorem about mixed partial derivatives to write $\partial_t N = EA\partial_x v$, it's possible to write the wave function as the following system of hyperbolic equations in the physical unknowns v, N

$$\begin{cases} \partial_t N - EA\partial_x v = 0 \\ \partial_t v - \frac{1}{m}\partial_x N = f \end{cases}$$

P-system and wave equation - reference state at rest, $\overline{u} = 0$.

$$\begin{cases} \partial_t \rho + \overline{\rho}\partial_x u = 0 \\ \partial_t u + \frac{a^2}{\overline{\rho}}\partial_x \rho = 0 \end{cases}$$

Taking time partial derivative of the first and space partial derivative of the second equation times $\overline{\rho}$, and evaluating their difference, a wave equation for rho appears

$$\partial_{tt}\rho - a^2\partial_{xx}\rho = 0 \ .$$

Analogously, taking space derivative of the first and time derivative of the second, a wave equation for the velocity field appears

$$\partial_{tt}u - a^2\partial_{xx}u = 0 \ .$$

20.4 System non-linear

20.4.1 1-dimensional space

$$
\begin{aligned}
&\mathbf{u}(x,t) \\
&\partial_t \mathbf{u} + \partial_x \mathbf{F}(\mathbf{u}) = \mathbf{f} && \text{(conservative form)} \\
&\partial_t \mathbf{u} + \partial_{\mathbf{u}} \mathbf{F}(\mathbf{u}) \partial_x \mathbf{u} = \mathbf{f} && \text{(convective form)}
\end{aligned}
$$

20.5 n-dimensional space

$$
\begin{aligned}
&\mathbf{u}(\vec{r},t) \\
&\partial_t \mathbf{u} + \nabla \cdot \mathbf{F}(\mathbf{u}) = \mathbf{f} && \text{(conservative form)} \\
&\partial_t \mathbf{u} + \nabla \mathbf{u} \cdot \partial_{\mathbf{u}} \mathbf{F}(\mathbf{u}) = \mathbf{f} && \text{(convective form)}
\end{aligned}
$$

Different descritpions of integral problem,

$$
\begin{aligned}
\frac{d}{dt}\int_V \mathbf{u} + \oint_{\partial V} \hat{n} \cdot \mathbf{F}(\mathbf{u}) &= \int_V \mathbf{f} && \text{(Eulerian)} \\
\frac{d}{dt}\int_{V_t} \mathbf{u} - \oint_{\partial V_t} \mathbf{u}\vec{u} \cdot \hat{n} + \oint_{\partial V_t} \hat{n} \cdot \mathbf{F}(\mathbf{u}) &= \int_{V_t} \mathbf{f} && \text{(Lagrangian)} \\
\frac{d}{dt}\int_{v_t} \mathbf{u} - \oint_{\partial v_t} \mathbf{u}\vec{u}_b \cdot \hat{n} + \oint_{\partial v_t} \hat{n} \cdot \mathbf{F}(\mathbf{u}) &= \int_{v_t} \mathbf{f} && \text{(arbitrary)}
\end{aligned}
$$

todo *in coordinates*

$$
\begin{aligned}
f_i &= \partial_t u_i + \partial_{x_k} F_{ki}(u_l) = \\
&= \partial_t u_i + \partial_{x_k} u_m \partial_{u_m} F_{ki}(u_l) =
\end{aligned}
$$

Example 20.5.1 (P-system in 1-dimensional domain)

$$
\begin{cases}
\partial_t \rho + u \partial_x \rho + \rho \partial_x u = 0 \\
\rho \partial_t u + \rho u \partial_x u + \partial_x P = 0
\end{cases}
$$

with $\partial_x P = a^2 \partial_x \rho$,

Convective form

$$
\partial_t \begin{bmatrix} \rho \\ u \end{bmatrix} + \begin{bmatrix} u & \rho \\ \frac{a^2}{\rho} & u \end{bmatrix} \partial_x \begin{bmatrix} \rho \\ u \end{bmatrix} = \underline{0} \, .
$$

Conservative form

$$
\partial_t \begin{bmatrix} \rho \\ \rho u \end{bmatrix} + \partial_x \begin{bmatrix} \rho u \\ \rho u^2 + \rho a^2 \end{bmatrix} = \underline{0} \, .
$$

Spectral decomposition of $\mathbf{A}(\mathbf{u})$ gives

$$
0 = \left| \begin{bmatrix} u - s & \rho \\ \frac{a^2}{\rho} & u - s \end{bmatrix} \right| = (u-s)^2 - a^2
$$

$$
\begin{aligned}
s_{1,2} &= u \mp a \\
\mathbf{R} &= \begin{bmatrix} \rho & \rho \\ a & -a \end{bmatrix} \\
\mathbf{L} &= \frac{1}{2\rho a}\begin{bmatrix} a & \rho \\ a & -\rho \end{bmatrix}
\end{aligned}
$$

Example 20.5.2 (Euler equations in 1-dimensional domain)

Conservative form

$$
\partial_t \begin{bmatrix} \rho \\ \rho u \\ \rho e^t \end{bmatrix} + \partial_x \begin{bmatrix} \rho u \\ \rho u^2 + P \\ \rho u h^t \end{bmatrix} = \underline{0} \, ,
$$

with $h^t = e^t + \frac{P}{\rho}$ and $e^t = e + \frac{u^2}{2}$, and the pressure field can be written as a function of the other thermodynamic variables. As an example, using conservative variables $(\rho, m, E^t) = (\rho, \rho u, \rho e^t) = \left(\rho, \rho u, \rho\left(e + \frac{u^2}{2}\right)\right)$

$$
P(\rho, e) = P\left(\rho, \frac{E^t}{\rho} - \frac{m^2}{2\rho^2}\right) = \Pi\left(\rho, m, E^t\right)
$$

so that

$$
\begin{aligned}
\partial_\rho \Pi &= \partial_\rho P\big|_e + \partial_e P\big|_\rho \left(-\frac{E^t}{\rho^2} + \frac{m^2}{\rho^3}\right) = \\
&= \partial_\rho P\big|_e + \partial_e P\big|_\rho \left(-\frac{e^t}{\rho} + \frac{u^2}{\rho}\right) \\
&= c^2 - \frac{P}{\rho^2}\partial_e P\big|_\rho + \partial_e P\big|_\rho \left(-\frac{e^t}{\rho} + \frac{u^2}{\rho}\right) \\
&= c^2 + \partial_e P\big|_\rho \left(-\frac{h^t}{\rho} + \frac{u^2}{\rho}\right) \\
\partial_m \Pi &= \partial_e P\big|_\rho \left(-\frac{m}{\rho^2}\right) \\
\partial_{E^t} \Pi &= \partial_e P\big|_\rho \left(\frac{1}{\rho}\right)
\end{aligned}
$$

The speed of sound reads

$$
\begin{aligned}
c^2 &= \partial_\rho P\big|_s = \\
&= \partial_\rho P\big|_e + \partial_e P\big|_\rho \partial_\rho e\big|_s = \\
&= \partial_\rho P\big|_e + \frac{P}{\rho^2}\partial_e P\big|_\rho \, ,
\end{aligned}
$$

Conservative form in conservative variables.

$$
\partial_t \begin{bmatrix} \rho \\ m \\ E^t \end{bmatrix} + \partial_x \begin{bmatrix} m \\ \frac{m^2}{\rho} + \Pi(\rho, m, E^t) \\ \frac{m}{\rho}\left(E^t + \Pi(\rho, m, E^t)\right) \end{bmatrix} = \underline{0} \, ,
$$

Convective form in conservative variables.

$$
\partial_t \begin{bmatrix} \rho \\ m \\ E^t \end{bmatrix} + \partial_x \begin{bmatrix} 0 & 1 & 0 \\ -\frac{m^2}{\rho^2} + \partial_\rho \Pi & \frac{2m}{\rho} + \partial_m \Pi & \partial_{E^t}\Pi \\ -\frac{m}{\rho^2}(E^t + \Pi) + \frac{m}{\rho}\partial_\rho \Pi & \frac{1}{\rho}(E^t + \Pi) + \frac{m}{\rho}\partial_m \Pi & \frac{m}{\rho}\left(1 + \partial_{E^t}\Pi\right) \end{bmatrix} \partial_x \begin{bmatrix} \rho \\ m \\ E^t \end{bmatrix} = \underline{0} \, ,
$$

Spectral decomposition of $\mathbf{A}(\mathbf{u})$

$$
\begin{aligned}
0 = & \left| \begin{bmatrix} -s & 1 & 0 \\ -u^2+\partial_\rho\Pi & 2u+\partial_m\Pi-s & \partial_{E^t}\Pi \\ -u\left(e^t+\frac{P}{\rho}\right)+u\partial_\rho\Pi & e^t+\frac{P}{\rho}+u\partial_m\Pi & u\left(1+\partial_{E^t}\Pi\right)-s \end{bmatrix} \right| = \\
& = -s\left[\left(2u+\partial_m\Pi-s\right)\left(u\left(1+\partial_{E^t}\Pi\right)-s\right)-\partial_{E^t}\Pi\left(h^t+u\partial_m\Pi\right)\right]+ \\
& \quad - uh^t\partial_{E^t}\Pi + u\partial_\rho\Pi\partial_{E^t}\Pi + \\
& \quad + \left(u^2-\partial_\rho\Pi\right)\left(u(1+\partial_{E^t}\Pi)-s\right) = \\
& = -s^3 + \\
& \quad + s^2\left(2u+\partial_m\Pi+u+u\partial_{E^t}\Pi\right)+ \\
& \quad + s\left(-2u^2-2u^2\partial_{E^t}\Pi-u\partial_m\Pi-u\partial_m\Pi\,\partial_{E^t}\Pi+\partial_{E^t}\Pi\,h^t+u\partial_{E^t}\Pi\,\partial_m\Pi-u^2+\partial_\rho\Pi\right)+ \\
& \quad + \left(-uh^t\partial_{E^t}\Pi+u\partial_\rho\Pi\,\partial_{E^t}\Pi+u^3+u^3\partial_{E^t}\Pi-u\partial_\rho\Pi-u\partial_\rho\Pi\,\partial_{E^t}\Pi\right)+ \\
& = -s^3 + \\
& \quad + s^2\left(3u+\partial_m\Pi+u\partial_{E^t}\Pi\right)+ \\
& \quad + s\left(-3u^2-2u^2\partial_{E^t}\Pi-u\partial_m\Pi+\partial_{E^t}\Pi\,h^t+\partial_\rho\Pi\right)+ \\
& \quad + \left(u^3-uh^t\partial_{E^t}\Pi+u^3\partial_{E^t}\Pi-u\partial_\rho\Pi\right) = \\
& = -(s-u)^3+(s-u)c^2 = \\
& = (s-u)\left[-(s-u)^2+c^2\right]
\end{aligned}
$$

being

$$
\begin{aligned}
\partial_m\Pi+u\partial_{E^t}\Pi &= \left(-\frac{u}{\rho}+\frac{u}{\rho}\right)\partial_\rho P\big|_e = 0 \\
-2u^2\partial_{E^t}\Pi-u\partial_m\Pi+\partial_{E^t}\Pi\,h^t+\partial_\rho\Pi &= -\frac{u^2}{\rho}\partial_\rho P\big|_e+\frac{1}{\rho}\partial_e P\big|_\rho h^t+\partial_\rho P\big|_e+\partial_e P\big|_\rho\left(-\frac{e^t}{\rho}+\frac{u^2}{\rho}\right) = \\
&= \partial_e P\big|_\rho\frac{P}{\rho^2}+\partial_\rho P\big|_e = \\
&= c^2 \\
-u\partial_\rho\Pi+u^3\partial_{E^t}\Pi-uh^t\partial_{E^t}\Pi &= u\left(-\partial_\rho P\big|_e-\partial_e P\big|_\rho\left(-\frac{e^t}{\rho}+\frac{u^2}{\rho}\right)+\frac{u^2}{\rho}\partial_e P\big|_\rho-\frac{h^t}{\rho}\partial_e P\big|_\rho\right) \\
&= u\left(-\partial_\rho P\big|_e-\frac{P}{\rho^2}\partial_e P\big|_\rho\right) = \\
&= -uc^2\,.
\end{aligned}
$$

Thus,

$$
s_{1,3} = u \mp c \quad , \quad s_2 = u
$$

$$
\mathbf{r}_{1,3} = \begin{bmatrix} 1 \\ u \mp c \\ \dots \end{bmatrix}\hat{\rho} \quad , \quad \mathbf{r}_2 = \begin{bmatrix} \dots \\ 0 \\ \dots \end{bmatrix}\hat{\rho}
$$

being

$$
\begin{aligned}
\hat{E}^t\partial_{E^t}\Pi &= \left[u^2-\partial_\rho\Pi+(-u\pm c-\partial_m\Pi)(u\mp c)\right]\hat{\rho} = \\
&= \left[u^2-\partial_\rho\Pi-u^2+\pm 2uc-c^2-\partial_m\Pi(u\mp c)\right]\hat{\rho} = \\
&= \left[-c^2-\partial_e P\big|_\rho\left(-\frac{h^t}{\rho}+\frac{u^2}{\rho}\right)\pm 2uc-c^2+\frac{u}{\rho}\partial_e P\big|_\rho(u\mp c)\right]\hat{\rho} = \\
&= \left[-2c^2+\partial_e P\big|_\rho\frac{h^t}{\rho}\pm 2uc\mp\frac{uc}{\rho}\partial_e P\big|_\rho\right]\hat{\rho} =
\end{aligned}
$$

$$\mathbf{R} = ...$$

$$\mathbf{L} = ...$$

Example 20.5.3 (Shallow water equation in 1-dimensional domain)

Let $b(x)$..., $h(x)$ the height of the free surface, $\eta(x) = h(x) - b(x)$ the depth.

Derivative of integrals with non-constant extremes

$$\partial_x \int_{z=0}^{\eta(x,t)} \rho u \, dz = \int_{z=0}^{\eta(x,t)} \partial_x(\rho u) \, dz + \rho u(x, \eta(x,t), t) \partial_x \eta(x,t) \ .$$

Continuity equation reads

$$\partial_t \rho + \partial_x(\rho u) + \partial_z(\rho w) = 0 \ ,$$

for fluids with constant and uniform density

$$\begin{aligned}
0 &= \int_{z=0}^{\eta(x,t)} \left(\partial_t \rho + \partial_x(\rho u) + \partial_z(\rho w) \right) \, dz = \\
&= \partial_x \int_{z=0}^{\eta(x,t)} (\rho u) \, dz - \rho u(x,\eta,t) \partial_x \eta + \rho w(x, \eta(x,t), t) = \\
&= \partial_x \int_{z=0}^{\eta(x,t)} (\rho u) \, dz + \rho \partial_t \eta = && \simeq \partial_x \left(\rho \eta u \right) + \partial_t \left(\rho \eta \right) \ .
\end{aligned}$$

having linked the velocity to the material derivative of the position, whose vertical component reads

$$w(x, \eta(x,t), t) = \frac{D\eta}{Dt} = \partial_t \eta(x,t) + u(x, \eta(x,t), t) \partial_x \eta \ .$$

Assuming hydrostatic pressure distribution, $p = P_a + \rho g z$ at depth z under the level of local free surface,

Momentum equation reads

$$0 = \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_z(\rho u w) + \partial_x P \ .$$

and integration in z-direction **todo** Explicitly treat the z term

$$\begin{aligned}
0 &= \partial_t(\rho \eta u) + \partial_x(\rho u^2 \eta) + \partial_x \int_{z=0}^{\eta(x)} (P_a + \rho g z) \, dz = \\
&= \partial_t(\rho \eta u) + \partial_x \left(\rho u^2 \eta + \frac{1}{2} \rho g \eta^2 \right) \ .
\end{aligned}$$

Conservative form of the equations.

$$\begin{cases}
\partial_t(\eta) + \partial_x m = 0 \\
\partial_t m + \partial_x \left(\frac{m^2}{\eta} + \frac{g\eta^2}{2} \right) = 0
\end{cases}$$

Convective form of the equations.

$$\partial_t \begin{bmatrix} \eta \\ m \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -\frac{m^2}{\eta^2} + g\eta & 2\frac{m}{\eta} \end{bmatrix} \partial_x \begin{bmatrix} \eta \\ m \end{bmatrix} = \underline{0}$$

Spectrum of matrix $\mathbf{A}(\mathbf{u})$.

$$0 = |\mathbf{A}(\mathbf{u}) - s^2\mathbf{I}| = -s\,(2u - s) + u^2 - g\eta = (s-u)^2 - g\eta \,.$$

Example 20.5.4 (P-system in n-dimensional domain)

- Conservative variables: $(\rho, \vec{m})$
- Physical variables: e.g. $(\rho, \vec{u})$, $(P, \vec{u})$,…

$$\begin{cases} \partial_t \rho + \nabla \cdot \vec{m} = 0 \\ \partial_t \vec{m} + \nabla \cdot \left[\frac{\vec{m} \otimes \vec{m}}{\rho} + \rho a^2 \mathbb{I} \right] = 0 \end{cases}$$

Example 20.5.5 (Euler system in n-dimensional domain)

- Conservative variables: $(\rho, \vec{m}, E^t)$
- Physical variables: e.g. $(\rho, \vec{u}, e)$,…

$$\begin{cases} \partial_t \rho + \nabla \cdot \vec{m} = 0 \\ \partial_t \vec{m} + \nabla \cdot \left[\frac{\vec{m} \otimes \vec{m}}{\rho} + \Pi\,\mathbb{I} \right] = \vec{0} \\ \partial_t E^t + \nabla \cdot \left[\frac{\vec{m}(E^t + \Pi)}{\rho} \right] = 0 \end{cases}$$

where Π represents the pressure field as a function of the conservative varaibles,

$$\Pi\,(\rho, \vec{m}, E^t) = P\,(\rho, e) = P\left(\rho, \frac{E^t}{\rho} - \frac{|\vec{m}|^2}{\rho^3}\right) \,,$$

and P the pressure field expressed by the **equation of state of the fluid** as a function of density and internal energy per unit mass as the pair of independent variables determining the thermodynamic state.

Example 20.5.6 (Shallow water equations in 2-dimensional domain)

$$\begin{cases} \partial_t (\rho\eta) + \nabla \cdot (\rho\eta\vec{u}) = 0 \\ \partial_t (\rho\eta\vec{u}) + \nabla \cdot \left(\rho\eta\vec{u}\vec{u} + \frac{1}{2}\rho g \eta^2 \mathbb{I}\right) = 0 \end{cases}$$

CHAPTER

TWENTYONE

NAVIER-CAUCHY EQUATIONS

Navier-Cauchy equations are the differential balance equation of the momentum of an elastic isotropic medium in the regime of small strain and displacement,

$$\rho_0 \partial_{tt} \vec{s} = \rho_0 \vec{g} + \nabla \cdot \boldsymbol{\sigma} \ .$$

Stress tensor for an isotropic medium reads

$$\begin{aligned}\boldsymbol{\sigma} &= 2\mu \boldsymbol{\varepsilon} + \lambda \, \mathrm{tr}\,(\boldsymbol{\varepsilon})\, \mathbb{I} = \\ &= \left(2\mu \boldsymbol{\varepsilon} - \frac{2}{3} \mu \, \mathrm{tr}(\boldsymbol{\varepsilon}) \mathbb{I} \right) + \left(\lambda + \frac{2}{3}\mu \right) \, \mathrm{tr}\,(\boldsymbol{\varepsilon})\, \mathbb{I} \ ,\end{aligned}$$

with the small strain tensor

$$\boldsymbol{\varepsilon} = \frac{1}{2} \left(\nabla \vec{s} + \nabla^T \vec{s} \right) \ .$$

Essential, natural and Robin boundary conditions read

$$\begin{aligned}&\vec{s} = \overline{\vec{s}} && \vec{r} \in S_D && \text{esserntial - Dirichlet b.c.} \\ &\hat{n} \cdot \boldsymbol{\sigma} = \overline{\vec{t}}_n && \vec{r} \in S_N && \text{natural - Neumann b.c.} \\ &a\vec{s} + \hat{n} \cdot \boldsymbol{\sigma} = \vec{b} && \vec{r} \in S_R && \text{Robin b.c.}\end{aligned}$$

21.1 Weak formulation

For $\forall \vec{w} \in \ldots$

$$\begin{aligned}0 &= -\int_V \rho \vec{w} \cdot \partial_{tt} \vec{s} + \int_V \rho_0 \vec{w} \cdot \vec{g} + \int_V \vec{w} \cdot \nabla \cdot \boldsymbol{\sigma} = \\ &= -\int_V \rho \vec{w} \cdot \partial_{tt} \vec{s} + \int_V \rho_0 \vec{w} \cdot \vec{g} + \int_{\partial V} \hat{n} \cdot \boldsymbol{\sigma} \cdot \vec{w} - \int_V \nabla \vec{w} : \boldsymbol{\sigma}\end{aligned}$$

The volume integral containing the stress tensor can be written either as

$$\begin{aligned}\int_V \nabla \vec{w} : \boldsymbol{\sigma} &= \int_V w_{i/j} \left[\mu \left(s_{i/j} + s_{j/i} \right) + \lambda s_{k/k} \delta_{ij} \right] = \\ &= \int_V \mu w_{i/j} \left(s_{i/j} + s_{j/i} \right) + \int_V \lambda w_{j/j} s_{k/k}\end{aligned}$$

or

$$\begin{aligned}\int_V \frac{1}{2} \left(\nabla \vec{w} + \nabla^T \vec{w} \right) : \boldsymbol{\sigma} &= \int_V \frac{1}{2} \left(w_{i/j} + w_{j/i} \right) \left[\mu \left(s_{i/j} + s_{j/i} \right) + \lambda s_{k/k} \delta_{ij} \right] = \\ &= \int_V \frac{\mu}{2} \left(w_{i/j} + w_{j/i} \right) \left(s_{i/j} + s_{j/i} \right) + \int_V \lambda w_{j/j} s_{k/k}\end{aligned}$$

The weak formulation of the Navier-Cauchy equations reads

$$\int_V \rho_0 \vec{w} \cdot \partial_{tt} \vec{s} + \int_V 2\mu \frac{\nabla \vec{w} + \nabla^T \vec{w}}{2} : \frac{\nabla \vec{s} + \nabla^T \vec{s}}{2} + \int_V \lambda \nabla \cdot \vec{w} \, \nabla \cdot \vec{s} + \int_{S_R} \vec{w} \cdot a \vec{s} = \int_V \rho_0 \vec{w} \cdot \vec{g} + \int_{S_N} \vec{w} \cdot \bar{\vec{t}}_n + \int_{S_R} \vec{w} \cdot \vec{b} \, ,$$

for $\forall \vec{w} \in$ ..., and with $\vec{s} = \bar{\vec{s}}$ for $\vec{r} \in S_D$.

CHAPTER

TWENTYTWO

NAVIER-STOKES EQUATIONS

Incompressible Navier-Stokes equations read

$$\begin{cases}\rho\partial_t\vec{u}+\rho\left(\vec{u}\cdot\nabla\right)\vec{u}-\mu\nabla^2\vec{u}+\nabla P=\rho\vec{g}\\ \nabla\cdot\vec{u}=0\;.\end{cases}$$

Mass balance equation is replaced by the incompressiblity kinematic constraint, $\nabla\cdot\vec{u}=0$: this constraint is not dynamic, as time derivative of density does not appear in the equation. With the incompressibility constraint, mass equation tells us that material particles keep their density constant,

$$0=\underbrace{\partial_t\rho+\vec{u}\cdot\nabla\rho}_{=\frac{D\bar{\rho}}{Dt}}+\rho\underbrace{\nabla\cdot\vec{u}}_{=0}=\frac{D\rho}{Dt}\;,$$

whose solution can be written using material coordinates $\vec{r}_0$ as $\rho(\vec{r}(\vec{r}_0,t),t)=\rho_0(\vec{r}_0,t)$.

22.1 Incompressibility constraint

Incompressibility constraint makes thermodynamic fade, while pressure field is replaced by/contains the contribution of a Lagrangian multiplier related to the incompressiblity constraint.

22.1.1 Wave-vector transformed space

Transforming the fields from physical space to the wave-vector space $\tilde{u}(\vec{\kappa},t)=\mathcal{F}\left\{\vec{u}(\vec{r},t)\right\}$, Navier-Stokes equations for incompressible fluids with uniform and constant density $\rho(\vec{r},t)=\rho$ becomes

$$\begin{cases}\rho\partial_t\tilde{u}+\mathcal{F}\left\{\left(\vec{u}\cdot\nabla\right)\vec{u}\right\}+\mu|\vec{\kappa}|^2\tilde{u}+i\vec{k}\widetilde{P}=\rho\tilde{g}\\ i\vec{\kappa}\cdot\tilde{u}=0\;.\end{cases}$$

Taking the divergence of the momentum balance equation, i.e. taking the scalar product with $i\vec{\kappa}$ in the transformed space, and using the incompressibility constraint to set $i\vec{\kappa}\cdot\tilde{u}=0$,

$$i\vec{\kappa}\cdot\mathcal{F}\left\{(\vec{u}\cdot\nabla)\vec{u}\right\}-|\vec{\kappa}|^2\widetilde{P}=i\vec{\kappa}\cdot\rho\tilde{g}\;,$$

so that the transformed pressure field becomes

$$\widetilde{P}=\frac{i\vec{\kappa}}{|\vec{\kappa}|^2}\cdot\mathcal{F}\left\{\left(\vec{u}\cdot\nabla\right)\vec{u}-\rho\vec{g}\right\}\;,$$

Replacing this expression in the transformed Navier-Stokes equations, the meaning of the pressure field as a Lagrange multiplier associated with incompressibility constraint becomes clear,

$$\rho\partial_t\tilde{u}+\mu|\vec{\kappa}|^2\tilde{u}=\left[1-\frac{\vec{\kappa}\vec{\kappa}}{|\vec{\kappa}|^2}\right]\cdot\mathcal{F}\left\{-(\vec{u}\cdot\nabla)\vec{u}+\rho\vec{g}\right\}$$

as the orthogonal projector $[1-\frac{\vec{\kappa}\vec{\kappa}}{|\vec{\kappa}|^2}]$ onto the space of divergence-free functions acts on the non-linear and forcing terms.

22.2 Weak formulation of the problem

$$\begin{aligned}
0 &= \int_V \vec{w} \cdot [\rho \partial_t \vec{u} + \rho(\vec{u}\cdot\nabla)\vec{u} - 2\mu\nabla\cdot\mathbb{D}(\vec{u}) + \nabla P - \rho\vec{g}] - \int_V v \nabla\cdot\vec{u} = \\
&= \int_V \vec{w} \cdot [\rho \partial_t \vec{u} + \rho(\vec{u}\cdot\nabla)\vec{u}] + \int_V 2\mu\nabla\vec{w} : \mathbb{D} - \int_V \nabla\cdot\vec{w}P - \int_V \vec{w}\cdot\rho\vec{g} - \int_V v\nabla\cdot\vec{u} - \int_{\partial V} \hat{n}\cdot(\mathbb{S} - P\mathbb{I})\cdot\vec{w}\,,
\end{aligned}$$

22.2.1 Weak formulation and incompressibility constraint

$$\vec{r}(\vec{r}_0, t) = \vec{r}(q(t), t)$$
$$\vec{u} = \frac{D\vec{r}}{Dt} = \dot{q}\frac{\partial\vec{r}}{\partial q} + \frac{\partial\vec{r}}{\partial t}$$

In the weak formulation, using $\vec{w} = \frac{\partial\vec{r}}{\partial q} = \frac{\partial\vec{u}}{\partial\dot{q}}$

$$0 = \int_V \vec{w}\cdot\rho\frac{D\vec{u}}{Dt} + \int_V 2\mu\nabla\vec{w} : \mathbb{D} - \int_V \nabla\cdot\vec{w}P - \int_V \rho\vec{w}\cdot\vec{g} - \int_V v\nabla\cdot\vec{u} - \int_{\partial V} \vec{t}_{\hat{n}}\cdot\vec{w}\,,$$

$$\begin{aligned}
\int_V \vec{w}\cdot\rho\frac{D\vec{u}}{Dt}\,dV &= \int_{V_0} \rho_0\frac{\partial\vec{u}}{\partial\dot{q}}\cdot\frac{D\vec{u}}{Dt} = \\
&= \int_{V_0} \rho_0\frac{D}{Dt}\left(\frac{\partial\vec{u}}{\partial\dot{q}}\cdot\vec{u}\right)dV_0 - \int_{V_0} \rho_0\frac{D}{Dt}\left(\frac{\partial\vec{r}}{\partial\dot{q}}\right)\cdot\vec{u}\,dV_0 = \\
&= \int_{V_0} \rho_0\frac{D}{Dt}\left(\frac{\partial}{\partial\dot{q}}\frac{|\vec{u}|^2}{2}\right)dV_0 - \int_{V_0} \rho_0\frac{\partial}{\partial q}\frac{|\vec{u}|^2}{2}\,dV_0 =
\end{aligned}$$

…

22.3 Non-linear term

Different ways to treat the non-linear term:

- Semi-linear approximation of the non-linear term

  $$(\vec{u}(\vec{r},t^n)\cdot\nabla)\vec{u}(\vec{r},t^n) \sim (\vec{u}^*(\vec{r},t^n)\cdot\nabla)\vec{u}(\vec{r},t^n)\,,$$

  with $\vec{u}^*(\vec{r},t^n)$ an approximation of $\vec{u}(\vec{r},t^n)$ involving values of the velocity field at previous time-steps, as an example

  $$\vec{u}^*(\vec{r},t^n) = \begin{cases} \vec{u}(\vec{r},t^{n-1}) & 1^{st}\text{-order} \\ 2\vec{u}(\vec{r},t^{n-1}) - \vec{u}(\vec{r},t^{n-2}) & 2^{nd}\text{-order} \end{cases}$$

CHAPTER

TWENTYTHREE

ARBITRARY LAGRANGIAN-EULERIAN DESCRIPTION

Reynold's transport theorem allows for the formulation of intergal equations, and grid-based methods like FVM, on moving grids and changing domains. Rules for derivatives of composite functions provide the relations between time derivatives in a Lagrangian, Eulerian, or arbitrary description,

$$\begin{aligned}
\left.\frac{\partial f}{\partial t}\right|_{\vec{r}_0} &= \left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + \vec{u} \cdot \nabla f \\
\left.\frac{\partial f}{\partial t}\right|_{\vec{r}_b} &= \left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + \vec{u}_b \cdot \nabla f
\end{aligned}$$

Equations governing the motion of the grid are usually required as well. E.g.:

- known and prescribed motion of the grid;
- boundary conditions only without changing grids (for small displacements)
- pseudo-elastic deformation (usually good for small strain and displacement;
- for large displacements of/or models with complex geometry, sliding and/or overlapping grids could an option for grid-based methods.

23.1 Integral problem

Application of Reynolds theorem to the balance equation of the quantity **u** for a material volume V_t

$$\frac{d}{dt}\int_{V_t} \rho \mathbf{u} = \int_{V_t} \rho \mathbf{f} + \oint_{\partial V_t} \hat{n} \cdot \mathbf{T} \, .$$

provides the expression of the balance equation for a geometrical volume v_t in arbitrary motion,

$$\frac{d}{dt}\int_{v_t} \rho \mathbf{u} + \oint_{\partial v_t} \rho \mathbf{u} \left(\vec{u} - \vec{u}_b\right) \cdot \hat{n} = \int_{v_t} \rho \mathbf{f} + \oint_{\partial v_t} \hat{n} \cdot \mathbf{T} \, .$$

Here, the integral forulation of the problem will be applied to each element of the grid in arbitrary motion, for domains with variable geometry.

23.2 Differential problem

Rules for derivatives of composite functions allows to write the differential w.r.t. the variables associated with the points of a moving grid. A balance equation in convective form can be written as

$$\begin{aligned}\rho \frac{D\mathbf{u}}{Dt} &= \rho \mathbf{f} + \nabla \cdot \mathbf{T} \\ \rho \left[\frac{\partial \mathbf{u}}{\partial t} + \vec{u} \cdot \nabla \mathbf{u} \right] &= \\ \rho \left[\left. \frac{\partial \mathbf{u}}{\partial t} \right|_{\vec{r}_b} + (\vec{u} - \vec{u}_b) \cdot \nabla \mathbf{u} \right] &= \end{aligned}$$

Part X

Numerical Methods for PDEs

CHAPTER TWENTYFOUR

INTRODUCTION TO NUMERICAL METHODS FOR PDES

Different numerical methods for PDEs rely on the discretization of different formulations of the continunuos problems. As an example,

- **FDM, Finite difference methods** rely on the approximation of derivatives of the **strong formulation** of the problem
- ***FEM, Finite element methods*** rely on a finite dimensional approximation of the **weak formulation** of the problem; usually the finite dimensional approximation can be interpreted as a projection of a infinite dimensional continuous problem onto a finite dimensional space, the space of the choosen finite elements
- ***FVM, Finite volume methods*** rely on an approximation of the **integral formulation** of the problem
- **BEM, Boundary elemement methods** rely on an approximation of a **boundary integral formulation** of the problem, when it's feasible and convenient
- …*spectral methods*, *spectral element methods*,…

Characteristics.

- grid: domain-grid-based, buondary-grid-based, grid-free methods
- range of interaction: short in physical space for FDV, FEM, FVM; long-range for boundary element methods, even though clustering techniques are available, like FMM; (usually) over the whole domain in space, short-range interaction in wave-number space for spectral methods;

Pros and cons. *More suited methods for each problems…; domain, order,…*

Let's take Poisson equations for a scalar function $u(\vec{r})$ to show all the possible approaches above. Here we start from the most general version of the equation, namely the integral form. As it's shown in Continuum Mechanics:Governing Equations, the most general form of balance equations is the integral form, while the differential form can be seamlessly derived only when the quantities involved are regular enough, to apply Stokes' theorem and for the derivatives appearing to exist.

The problem of interest can be interpreted as the problem of finding the temperature field $u(\vec{r})$ during steady heat conduction in the domain Ω, with distributed volume heat source $f(\vec{r})$. Fourier's law assume that condcution heat flux is proportional to the gradient of the temperature, $\vec{q} = -k\nabla u$. Temperature is prescribed, $u = g$ on the region of the boundary S_D, while the heat flux is prescribed $\hat{n} \cdot \vec{q} = h$ on the region of the boundary S_N.

The **integral formulation** of the problem for all the boundary reads

$$0 = -\oint_{\partial\Omega} \hat{n} \cdot \vec{q} + \int_{\Omega} f \, . \tag{24.1}$$

Example 24.1 (Finite volume methods)

Finite volume methods rely on a tassellation of the domain $\Omega = \cup_k \Omega_k$, to write and solve the integral balance equation (24.1) for all the elemantary domain Ω_k.

$$0 = -\oint_{\partial \Omega_k} \hat{n} \cdot \vec{q} + \int_{\Omega_k} f \ . \tag{24.2}$$

evaluating volume integrals with the internal variables of the cell Ω_k, and boundary flux with the variables of the cell Ω_k and the neighbouring cells $\Omega_i \in B_k$. Introducing the definition of numerical flux $F_{ik}(u_i, u_k)$ at the interface between a generic discrete version of the balance equation (24.2) for the k^{th} element becomes

$$0 = -\sum_{\Omega_i \in B_k} F_{ik}(u_i, u_k) + S_k(u_k) \ .$$

Different numerical methods differs in the evaluation of fluxes and sources. FVM is **conservative** as it evaluate flux at interfaces and then distribute it to neighoring cells, see *Property 24.2.1*. A rough elementwise-uniform variable with jump at interfaces and diffusive flux,

$$F_{ik} = -A_{ik}\, k \, \frac{u_i - u_k}{d_{ik}} \ ,$$

with d_{ik} equal to the distance of the centers of elements i and k, leads to the balance equation for the internal volume Ω_k,

$$0 = \sum_i A_{ik}\, k \, \frac{u_i - u_k}{d_{ik}} + V_k f_k \ .$$

Volumes at the boundary are influenced by fluxes at the boundaries and boundary conditions in general.

For regular cubic mesh, $A_{ik} = \Delta x$, $d_{ik} = \Delta x$, $V_k = \Delta x$, and thus the equations for internal elements become

$$0 = k \sum_{i \in \Omega_k} \frac{u_i - u_k}{\Delta x^2} + f_k \ ,$$

where the summation is exactly equal to the center-difference stencil for the Laplacian, used in finite difference method *Example 24.2*.

If fields are regular enough to apply Stokes' theorem, it's possible to derive differential problem from the integral formulation,

$$0 = -\oint_{\partial V} \hat{n} \cdot \vec{q} + \int_V f = -\int_V \nabla \cdot \vec{q} + \int_V f = \int_V (-\nabla \cdot \vec{q} + f)$$

Exploiting the arbitrariety of the volume V, the **strong form** of the differential problem is the Poisson equation with suitable boundary conditions,

$$\begin{cases} -\nabla \cdot (k \nabla u) = f(\vec{r}) \ , & \vec{r} \in V \\ u = g(\vec{r}) \ , & \vec{r} \in S_D \\ \hat{n} \cdot \nabla u = h(\vec{r}) \ , & \vec{r} \in S_N \end{cases}$$

only holds in regions of the domain where the functions involved are continuous and differentiable, i.e. everything written in the problem is not meaningful, i.e. at least it exists.

Example 24.2 (Finite difference methods)

Finite different method approximates the strong form of the problem, building a stencil to evaluate an approximation of the Laplacian. On a cubic regular grid ($\Delta x = \Delta y = \Delta z$), the Laplacian van be evaluated with a 7-point stencil

$$\begin{aligned}\nabla^2 u = \partial_{xx} u + \partial_{yy} u + \partial_{zz} u = \\ &= \frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{\Delta x^2} + \frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{\Delta y^2} + \frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}}{\Delta z^2} = \\ &= \frac{1}{\Delta x^2}\left[-6u_{i,j,k} + u_{i+1,j,k} + u_{i-1,j,k} + u_{i,j+1,k} + u_{i,j-1,k} + u_{i,j,k+1} + u_{i,j,k-1}\right] = \\ &= \frac{1}{\Delta x^2}\sum_{i \in B_k}(u_i - u_k)\end{aligned}$$

and it's equal to the same expression already provided for the rough finite volume method with regular grid, in *Example 24.1*

Starting from strong formulation of the differential problem, the **weak formulation** of the problem is derived:

- multiplying the strong problem by an arbitrary test function $w(\vec{r})$ compatible with the essential constraints
- integrating over the whole domain

$$\begin{aligned}0 &= \int_V w(\vec{r}) \cdot [-\nabla \cdot (k\nabla u) - f(\vec{r})] = \\ &= \int_V \{k\nabla w(\vec{r}) \cdot \nabla u(\vec{r}) - w(\vec{r}) f(\vec{r})\} - \oint_{\partial V} w(\vec{r})\, \hat{n} \cdot \nabla u \qquad \forall w(\vec{r}) \ .\end{aligned}$$

Using a test function $w(\vec{r})$ equal to zero on the boundary where essential boundary conditions are prescribed $w|_{S_D} = 0$, and introducing the natural boundary conditions of the Neumann boundary S_N, $\hat{n} \cdot \nabla u|_{S_N} = h$,

$$\int_V k\nabla w \cdot \nabla u = \int_V wf + \int_{S_N} wh \ , \quad \forall w$$

compatible with essential boundary conditions.

Example 24.3 (Finite element methods)

Finite element methods build an N-dimensional systems:

- approximating the solution as a linear combination of N base functions, $u(\vec{r}) = \sum_j \phi_j(\vec{r}) u_j$
- testing the equation over N independent test functions $\psi_i(\vec{r})$

i.e. the i^{th} equation becomes

$$\int_V k\nabla\psi_i(\vec{r}) \cdot \nabla\phi_j(\vec{r})\, u_j = \int_V \psi_i(\vec{r}) f + \int_{S_N} \psi_i(\vec{r}) h \ ,$$

or briefly

$$K_{ij} u_j = f_i \qquad , \qquad \mathbf{K}\mathbf{u} = \mathbf{f} \ .$$

A common choice uses the same functions both as test and base functions, $\psi_i(\vec{r}) = \phi_i(\vec{r})$.

The differential problem in strong form can be recast as a boundary element problem exploiting the properties of Green's functions $G(\vec{r}; \vec{r}_0)$. The expression of *Green's function $G(\vec{r}, \vec{r}_0)$* is known for some problems of interest like Poisson equation, Helmholtz equation or wave equation. Exploiting the properties of Green's function and integration by parts, the integral buondary problem is derived as follow

$$\begin{aligned}
E(\vec{r}_0)u(\vec{r}_0) &= \int_{\vec{r}\in V} u(\vec{r})\delta(\vec{r}-\vec{r}_0)\,dV = \\
&= -\int_{\vec{r}\in V} u(\vec{r})\nabla^2 G(\vec{r};\vec{r}_0)\,dV = \\
&= -\oint_{\vec{r}\in\partial V} u(\vec{r})\hat{n}(\vec{r})\cdot\nabla G(\vec{r};\vec{r}_0)\,dS + \oint_{\vec{r}\in\partial V} G(\vec{r};\vec{r}_0)\hat{n}(\vec{r})\cdot\nabla u(\vec{r})\,dS - \int_{\vec{r}\in V} G(\vec{r};\vec{r}_0)\nabla^2 u(\vec{r})\,dV = \\
&= -\oint_{\vec{r}\in\partial V} u(\vec{r})\hat{n}(\vec{r})\cdot\nabla G(\vec{r};\vec{r}_0)\,dS + \oint_{\vec{r}\in\partial V} G(\vec{r};\vec{r}_0)\hat{n}(\vec{r})\cdot\nabla u(\vec{r})\,dS + \int_{\vec{r}\in V} G(\vec{r};\vec{r}_0)f(\vec{r})\,dV\ .
\end{aligned}$$

It must be paid attention that the integrals may be singular and may produce discontintuities in the fields across the surface ∂V (*so what's the value of a field in presence of a discontinuity?…virtual singularities, regularization close to the singularities…one way to correctly interpret them is via Cauchy principal value… evaluating some integrals in a check-point just inside or outside the domain may make the value of the function* $E(\vec{r}_0)$ *change, but the integral involving* $\hat{n}\cdot\nabla G$ *- related to the solid angle seen by the check-point of the surface - changes accordingly and keep the sum of these two terms constant=.*

The value of the unknown function or the flux $\hat{n}\cdot\nabla u$ are known on the Dirichlet and Neumann regions of the boundary respectively, and so the integro-differential problem becomes

$$E(\vec{r}_0)u(\vec{r}_0) + \int_{S_N} u\hat{n}\cdot\nabla G\,dS - \int_{S_D} G\,\hat{n}\cdot\nabla u\,dS = \int_{S_D} g\hat{n}\cdot\nabla G\,dS - \int_{S_N} Gh\,dS + \int_{\vec{r}\in V} Gf\,dV$$

The unknown function is approximated on the boundary of the domain ∂V as an N-dimensional approximation, as an example

$$u(\vec{r}) = \sum_j \phi_j(\vec{r})u_j \qquad , \qquad \vec{r}\in\partial V\ ,$$

and the integro-differential equation is evaluated in N different points $\vec{r}_{0,i}$ in order to get a N, N linear equation in the amplitudes u_j of the base functions,

$$[\mathbf{E} + \mathbf{D}_N + \mathbf{S}_D]\,\mathbf{u} = \mathbf{D}_D\mathbf{g} + \mathbf{S}_N\mathbf{h} + \mathbf{S}_V\mathbf{f}\ .$$

24.1 Finite Element Method

24.2 Finite Volume Method

Property 24.2.1

Evaluate flux on interfaces between cells, and distribute between neighboring cells.

24.3 Boundary Element Method

Part XI

Boundary Methods for PDEs

CHAPTER

TWENTYFIVE

GREEN'S FUNCTION METHOD

25.1 Poisson equation

General Poisson's problem

$$\begin{cases} -\nabla^2 \mathbf{u}(\mathbf{r},t) = \mathbf{f}(\mathbf{r},t) \\ + \text{b.c.} \end{cases}$$

with common boundary conditions

$$\begin{cases} \mathbf{u} = \mathbf{g} & \text{on } S_D \\ \hat{\mathbf{n}} \cdot \nabla \mathbf{u} = \mathbf{h} & \text{on } S_N \end{cases}$$

over Dirichlet and Neumann regions of the boundary.

Poisson's problem for Green's function, in infinite domain

$$-\nabla^2_{\mathbf{r}} G(\mathbf{r};\mathbf{r}_0) = \delta(\mathbf{r}-\mathbf{r}_0)$$

Green's function method

$$\begin{aligned} E(\mathbf{r}_0,t)u_i(\mathbf{r}_0,t) &= \int_{\mathbf{r}\in\Omega} u_i(\mathbf{r},t)\delta(\mathbf{r}-\mathbf{r}_0) = \\ &= -\int_{\mathbf{r}\in\Omega} u_i(\mathbf{r},t)\nabla^2_{\mathbf{r}} G(\mathbf{r}-\mathbf{r}_0) = \\ &= -\int_{\mathbf{r}\in\Omega} \nabla_{\mathbf{r}} \cdot (u_i \nabla_{\mathbf{r}} G - G\nabla_{\mathbf{r}} u_i) - \int_{\mathbf{r}\in\Omega} G\nabla^2 u_i = \\ &= -\oint_{\mathbf{r}\in\partial\Omega} \hat{\mathbf{n}} \cdot (u_i \nabla_{\mathbf{r}} G - G\nabla_{\mathbf{r}} u_i) + \int_{\mathbf{r}\in\Omega} G(\mathbf{r}-\mathbf{r}_0) f_i(\mathbf{r},t). \end{aligned}$$

An integro-differential boundary problem can be written using boundary conditions. As an example, using Dirichlet and Neumann boundary conditions, the integro-differential problem reads

$$\begin{aligned} & E(\mathbf{r}_0,t)\mathbf{u}(\mathbf{r}_0,t) + \int_{\mathbf{r}\in S_N} \mathbf{u}(\mathbf{r},t)\,\hat{\mathbf{n}} \cdot \nabla_{\mathbf{r}} G(\mathbf{r}-\mathbf{r}_0) - \int_{\mathbf{r}\in S_D} G(\mathbf{r}-\mathbf{r}_0)\,\hat{\mathbf{n}} \cdot \nabla_{\mathbf{r}} \mathbf{u}(\mathbf{r},t) = \\ & = -\int_{\mathbf{r}\in S_D} \mathbf{g}(\mathbf{r},t)\,\hat{\mathbf{n}} \cdot \nabla_{\mathbf{r}} G(\mathbf{r}-\mathbf{r}_0) + \int_{\mathbf{r}\in S_N} G(\mathbf{r}-\mathbf{r}_0)\,\mathbf{h}(\mathbf{r},t) + \int_{\mathbf{r}\in\Omega} G(\mathbf{r}-\mathbf{r}_0)\,\mathbf{f}(\mathbf{r},t). \end{aligned}$$

Green's function of the Poisson-Laplace equation reads

$$G(\mathbf{r};\mathbf{r}_0) = \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}_0|} \, .$$

Green's function of the Laplace equation

$$-\nabla^2 G = 0 \qquad \text{for } \mathbf{r} \neq \mathbf{r}_0$$

Solutions with spherical symmetry,

$$0 = \nabla^2 G = \frac{1}{r^2}\left(r^2 G'\right)' \quad \rightarrow \quad G'(r) = \frac{A}{r^2} \quad \rightarrow \quad G(r) = -\frac{A}{r} + B$$

Choosing $B = 0$ s.t. $G(r) \rightarrow 0$ as $r \rightarrow \infty$, and integrating over a sphere centered in $r = 0$ to get $A = -\frac{1}{4\pi}$,

$$1 = \int_V \delta(r) = -\int_V \nabla^2 G = -\oint_{\partial V} \hat{\mathbf{n}} \cdot \nabla G = -\oint_{\partial V} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \frac{A}{r^2} = -4\pi\, A$$

25.2 Helmholtz equation

todo from Fourier to Laplace trasnform in the first lines of this section

A Helmholtz's equation can be thouought as the time Fourier transform of a wave equation,

$$\begin{cases} \dfrac{1}{c^2}\partial_{tt}\mathbf{u}(\mathbf{r},t) - \nabla^2 \mathbf{u}(\mathbf{r},t) = \mathbf{f}(\mathbf{r},t) \\ + \text{b.c.} \\ + \text{i.c.}\ , \end{cases}$$

Fourier transform in time of field $\mathbf{u}(\mathbf{r},t)$ reads

$$\tilde{\mathbf{u}}(\mathbf{r},\omega) = \mathcal{F}\{\mathbf{u}(\mathbf{r},t)\} = \int_{t=-\infty}^{+\infty} \mathbf{u}(\mathbf{r},t) e^{-i\omega t}\, d\omega$$

and, if $\mathbf{u}(\mathbf{r},t)$ is compact in time, Fourier transform of its time partial derivatives read

$$\begin{aligned} \mathcal{F}\{\dot{\mathbf{u}}(\mathbf{r},t)\} &= \int_{t=-\infty}^{+\infty} \dot{\mathbf{u}}(\mathbf{r},t) e^{-i\omega t}\, d\omega = \\ &= \mathbf{u}(\mathbf{r},t) e^{-i\omega t}\big|_{t=-\infty}^{+\infty} + i\omega \int_{t=-\infty}^{+\infty} \mathbf{u}(\mathbf{r},t) e^{-i\omega t}\, d\omega = \\ &= i\omega \mathcal{F}\{\mathbf{u}(\mathbf{r},t)\} \end{aligned}$$

$$\mathcal{F}\{\partial_t^n \mathbf{u}(\mathbf{r},t)\} = (i\omega)^n \tilde{\mathbf{u}}\ .$$

The differential problem in the transformed domain thus reads

$$-\frac{\omega^2}{c^2}\tilde{\mathbf{u}} - \nabla^2 \tilde{\mathbf{u}} = \tilde{\mathbf{f}}$$

Green's function of Helmholtz'e equation reads

$$G(\mathbf{r},s) = \alpha^+ \frac{e^{\frac{s|\mathbf{r}-\mathbf{r}_0|}{c}}}{|\mathbf{r}-\mathbf{r}_0|} + \alpha^- \frac{e^{-\frac{s|\mathbf{r}-\mathbf{r}_0|}{c}}}{|\mathbf{r}-\mathbf{r}_0|}$$

with $\alpha^+ + \alpha^- = \frac{1}{4\pi}$.

Being the Laplace transform,

$$\mathcal{L}\{f(t)\} = \int_{t=0^-}^{+\infty} f(t) e^{-st} dt\ ,$$

the Laplace transform of a causal function with time delay $\tau \ge 0$ reads

$$\mathcal{L}\{f(t-\tau)\} = \int_{t=0^-}^{+\infty} f(t-\tau) e^{-st} dt = \int_{z=-\tau}^{+\infty} f(z) e^{-s(z+\tau)}\, dz = e^{-s\tau} \int_{z=0}^{+\infty} f(z) e^{-sz}\, dz = e^{-s\tau}\, \mathcal{L}\{f(t)\}$$

having used causality $f(t) = 0$ for $t < 0$. Laplace transform of Dirac's delta $\delta(t)$ reads

$$\mathcal{L}\{\delta(t)\} = \int_{t=0^-}^{+\infty} \delta(t)\, dt = 1 \ ,$$

so that $e^{-s\tau} = e^{-s\tau}\, 1 = \mathcal{L}\{\delta(t-\tau)\}$.

Thus, Green's function for the wave equation reads

$$G(\mathbf{r}, t; \mathbf{r}_0, t_0) = \alpha^+ \frac{\delta\left(t - t_0 + \frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right)}{|\mathbf{r}-\mathbf{r}_0|} + \alpha^- \frac{\delta\left(t - t_0 - \frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right)}{|\mathbf{r}-\mathbf{r}_0|}$$

If $t \ge t_0$, and $G(\mathbf{r}, t; \mathbf{r}_0, t_0)$ connects the past t_0 with the future t, the first term is not causal, and thus $\alpha^+ = 0$ and

$$G(\mathbf{r}, t; \mathbf{r}_0, t_0) = \frac{1}{4\pi} \frac{\delta\left(t - t_0 - \frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right)}{|\mathbf{r}-\mathbf{r}_0|} \ .$$

Green's function of Helmholtz's equation

$$\frac{s^2}{c^2} G - \nabla^2 G = \delta(r)$$

$$G(r) = \frac{\alpha e^{kr} + \beta e^{-kr}}{r}$$

Proof:

- Gradient

$$\nabla G(r) = \hat{\mathbf{r}} \partial_r G = \hat{\mathbf{r}} \frac{\alpha(kr-1)e^{kr} + \beta(-kr-1)e^{-kr}}{r^2}$$

- Laplacian

$$\begin{aligned}
\nabla^2 G(r) &= \frac{1}{r^2} \left(r^2 G'(r)\right)' = \\
&= \frac{1}{r^2} \left(\alpha(kr-1)e^{kr} + \beta(-kr-1)e^{-kr}\right)' = \\
&= \frac{1}{r^2} \left(\alpha k e^{kr} + \alpha k^2 r e^{kr} - \alpha k e^{kr} - \beta k e^{-kr} + \beta k^2 r e^{-kr} + \beta k e^{-kr}\right) = \\
&= \frac{1}{r} \left(\alpha e^{kr} + \beta e^{-kr}\right) k^2 = k^2 G(r) \ .
\end{aligned}$$

  and thus $k^2 G(r) - \nabla^2 G = 0$, for $r \ne 0$;

- Unity

$$1 = \int_V \delta(r) = \int_V \left(k^2 G - \nabla^2 G\right) = \int_V k^2 G - \oint_{\partial V} \hat{\mathbf{n}} \cdot \nabla G$$

  the second term is the sum of two contributions of the form

$$\oint_{\partial V} \hat{\mathbf{n}} \cdot \nabla G^\pm = \oint_{\partial V} \frac{\alpha^\pm(\pm kr - 1)e^{\pm kr}}{r^2} = 4\pi \alpha^\pm (\pm kr - 1) e^{\pm kr}$$

the first term is the sum of two contributions of the form

$$\begin{aligned}
k^2 \int_V G(r) = k^2 \int_V \frac{\alpha^\pm e^{\pm kr}}{r} = \\
&= k^2 \alpha^\pm \int_{R=0}^{r} \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \frac{e^{\pm kR}}{R} R^2 \sin\phi \, dR \, d\phi \, d\theta = \\
&= k^2 \alpha^\pm \, 4\pi \int_{R=0}^{r} R \, e^{\pm kR} \, dR \, .
\end{aligned}$$

the last integral can be evaluated with integration by parts

$$\begin{aligned}
\int_{R=0}^{r} R \, e^{\pm kR} \, dR &= \left[\frac{1}{\pm k} e^{\pm kR} R \right] \Big|_{R=0}^{r} \mp \frac{1}{k} \int_{R=0}^{r} e^{\pm kR} \, dR = \\
&= \frac{1}{\pm k} e^{\pm kr} r - \frac{1}{k^2} e^{\pm kR} + \frac{1}{k^2} =
\end{aligned}$$

Thus summing everything together,

$$\begin{aligned}
1 &= \alpha^+ \left[4\pi k^2 \left(\frac{r}{k} e^{kr} - \frac{1}{k^2} e^{kr} + \frac{1}{k^2} \right) - 4\pi \left(kr - 1 \right) e^{kr} \right] + \alpha^- \left[... \right] = \\
&= 4\pi \left(\alpha^+ + \alpha^- \right) \ .
\end{aligned}$$

25.3 Wave equation

Wave equation general problem

$$\begin{cases}
\frac{1}{c^2} \partial_{tt} \mathbf{u}(\mathbf{r},t) - \nabla^2 \mathbf{u}(\mathbf{r},t) = \mathbf{f}(\mathbf{r},t) \\
+ \text{b.c.} \\
+ \text{i.c.}
\end{cases}$$

Green's problem of the wave equation

$$\frac{1}{c^2} \partial_{tt} G(\mathbf{r}, t; \mathbf{r}_0, t_0) - \nabla^2_{\mathbf{r}} G(\mathbf{r}, t; \mathbf{r}_0, t_0) = \delta(\mathbf{r} - \mathbf{r}_0) \delta(t - t_0)$$

Integration by parts

$$\begin{aligned}
E(\mathbf{r}_\alpha, t_\alpha) \mathbf{u}(\mathbf{r}_\alpha, t_\alpha) &= \int_{t \in T} \int_{\mathbf{r} \in V} \delta(t - t_\alpha) \delta(\mathbf{r} - \mathbf{r}_\alpha) \mathbf{u}(\mathbf{r}, t) = \\
&= \int_{t \in T} \int_{\mathbf{r} \in V} \left\{ \frac{1}{c^2} \partial_{tt} G - \nabla^2_{\mathbf{r}} G \right\} \mathbf{u} = \\
&= \int_{t \in T} \int_{\mathbf{r} \in V} \left\{ \frac{1}{c^2} \left[\partial_t \left(\mathbf{u} \partial_t G - G \partial_t \mathbf{u} \right) + G \partial_{tt} \mathbf{u} \right] - \nabla_{\mathbf{r}} \cdot \left(\nabla_{\mathbf{r}} G \, \mathbf{u} - G \nabla_{\mathbf{r}} \mathbf{u} \right) - G \, \nabla^2_{\mathbf{r}} \mathbf{u} \right\} = \\
&= \int_{\mathbf{r} \in V} \frac{1}{c^2} \left[\mathbf{u}(\mathbf{r}, t) \partial_t G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) - G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) \partial_t \mathbf{u}(\mathbf{r}, t) \right] \Big|_{t_0}^{t_1} + \\
&\quad + \int_{t \in T} \oint_{\mathbf{r} \in \partial V} \left\{ -\hat{\mathbf{n}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{r}} G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) \, \mathbf{u}(\mathbf{r}, t) + G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) \, \hat{\mathbf{n}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{r}} \mathbf{u}(\mathbf{r}, t) \right\} + \\
&\quad + \int_{t \in T} \int_{\mathbf{r} \in V} G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) \underbrace{\left\{ \frac{1}{c^2} \partial_{tt} \mathbf{u}(\mathbf{r}, t) - \nabla^2_{\mathbf{r}} \mathbf{u}(\mathbf{r}, t) \right\}}_{= \mathbf{f}(\mathbf{r}, t)}
\end{aligned}$$

$$\int_{t \in T} \int_{\mathbf{r} \in V} \frac{1}{4\pi} \frac{\delta \left(t - t_\alpha + \frac{|\mathbf{r} - \mathbf{r}_\alpha|}{c} \right)}{|\mathbf{r} - \mathbf{r}_\alpha|} \mathbf{f}(\mathbf{r}, t) = \int_{\mathbf{r} \in V \cap B_{|\mathbf{r} - \mathbf{r}_\alpha| \le c(t_\alpha - t)}} \frac{1}{4\pi |\mathbf{r} - \mathbf{r}_\alpha|} \mathbf{f} \left(\mathbf{r}, t_\alpha - \frac{|\mathbf{r} - \mathbf{r}_\alpha|}{c} \right)$$

PROOF INDEX

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thm-shannon-nyquist