basics - math

basics

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CONTENTS

Argomenti.

- **Calcolo**
  - Calcolo multivariabile e calcolo vettoriale in spazi euclidei 2D e 3D
  - Algebra lineare e multilineare su spazi con prodotto interno
  - Calcolo lineare e multilineare su spazi con prodotto interno
- **Geometria**
  - Geometria differenziale
- **Calcolo delle variazioni** (qui e/o nella scuola superiore?)
- **Calcolo complesso**
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  - **Teoria delle trasformate**: Fourier, Laplace

Part I

Multivariable Calculus

CHAPTER

ONE

INTRODUCTION TO MULTI-VARIABLE CALCULUS

1.1 Function

1.2 Limit

1.3 Derivatives

1.4 Integrals

1.5 Theorems

1.5.1 Green's lemma

$$\begin{aligned}
\int_S \frac{\partial F}{\partial y} dx dy &= -\oint_{\partial S} F dx \\
\int_S \frac{\partial G}{\partial x} dx dy &= \oint_{\partial S} G dy
\end{aligned}$$

Proof for simple domains.

In a simple domain in x, so that the closed contour ∂S is delimited by the curves $y = Y_1(x)$, $y = Y_2(x) > Y_1(x)$, for $x \in [x_1, x_2]$,

$$\begin{aligned}
\int_S \frac{\partial F}{\partial y} dx dy &= \int_{x=x_1}^{x_2} \int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial F}{\partial y} dy\, dx = \\
&= \int_{x=x_1}^{x_2} \left[F(x, Y_2(x)) - F(x, Y_1(x)) \right] dx = \\
&= -\int_{x=x_1}^{x_2} F(x, Y_1(x)) - \int_{x=x_2}^{x_1} F(x, Y_2(x)) dx = \\
&= -\oint_{\partial S} F(x, y) dx
\end{aligned}$$

In a simple domain in y, so that the closed contour ∂S is delimited by the curves $x = X_1(y)$, $x = X_2(y) > X_1(y)$ for $y \in [y_1, y_2]$,

$$\begin{aligned}
\int_S \frac{\partial G}{\partial x} dx dy &= \int_{y=y_1}^{y_2} \int_{x=X_1(y)}^{X_2(y)} \frac{\partial G}{\partial x} dx\, dy = \\
&= \int_{y=y_1}^{y_2} \left[G(X_2(y), y) - G(X_1(y), y) \right] dy = \\
&= \int_{y=y_1}^{y_2} G(X_1(y), y) dy + \int_{y=y_2}^{y_1} G(X_2(y), y) dy = \\
&= \oint_{\partial S} G(x, y) dy
\end{aligned}$$

Part II

Differential Geometry

CHAPTER

TWO

INTRODUCTION TO DIFFERENTIAL GEOMETRY

2.1 Differential geometry in E^3

2.1.1 Curves

Parametric representation of curve in 3-dimensional (Euclidean) space E^3

$$\vec{r}(q)$$

Differential, $d\vec{r}$.

$$d\vec{r}(q) = \vec{r}\,'(q)\,dq \ .$$

Arc-length parameter, s. So that $ds = |d\vec{r}(s)|$ and thus

$$|d\vec{r}(s)| = |\vec{r}\,'(s)|\,|ds| \qquad \rightarrow \qquad |\vec{r}\,'(s)| = 1 \qquad \rightarrow \qquad \vec{r}\,'(s) = \hat{t}(s) \ .$$

Frenet basis. Using arc-length parameter, Frenet basis is naturally defined as the set $\{\hat{t}, \hat{n}, \hat{b}\}$:

- tangent unit vector, $\hat{t}(s) = \vec{r}\,'(s)$,
- normal unit vector, $\hat{r}''(s) = \hat{t}'(s) =: \kappa(s)\,\hat{n}(s)$, with $\kappa(s)$ local curvature
- binormal unit vector, $\hat{b}(s) = \hat{t}(s) \times \hat{n}(s)$

Using a general parameter, t, with some abuse of notation $\vec{r}(t) = \vec{r}(s(t))$ and indicating $\dot{()} = \frac{d}{dt}$,

- $\dot{\vec{r}} = \frac{ds}{dt}\frac{d\vec{r}}{ds} = \dot{s}\hat{t}$
- $\ddot{\vec{r}} = \dfrac{d}{dt}\dot{\vec{r}} = \dfrac{d}{dt}\left(\dot{s}\hat{t}\right) = \ddot{s}\hat{t} + \dfrac{ds}{dt}\dfrac{d}{ds}\hat{t} = \ddot{s}\hat{t} + \dot{s}^2\kappa\,\hat{n}$

Osculator circle. Circle with $R(s) = \frac{1}{\kappa(s)}$, in plane orthogonal to $\hat{b}(s)$, passing through $\vec{r}(s)$, and thus center in $\vec{r}_C(s) = \vec{r}(s) + \hat{n}R(s)$. Its parametric representation using its arc-length parameter p, with $\vec{r}(p=0) = \vec{r}(s)$ reads

$$\vec{r}(p) = \vec{r}_C(s) + R(s)\left[-\cos\left(\frac{p}{R(s)}\right)\hat{n}(s) + \sin\left(\frac{p}{R(s)}\right)\hat{t}(s)\right] \ .$$

Its first and second order derivatives w.r.t. the arc-length p evaluated in $p=0$, i.e. $\vec{r} = \vec{r}(s)$ read:

- first derivative in $p=0$,

  $$\hat{t}(p)\big|_{p=0} = \vec{r}\,'(p)\big|_{p=0} = \left[\sin\left(\frac{p}{R(s)}\right)\hat{n}(s) + \cos\left(\frac{p}{R(s)}\right)\hat{t}(s)\right]\Bigg|_{p=0} = \hat{t}(s) \ ,$$

  i.e. the osculator circle has the same tangent as the curve in the point.

- second derivative in $p=0$,

$$\kappa(p)\hat{n}(p)\big|_{p=0} = \vec{r}''(p)\big|_{p=0} = \frac{1}{R(s)}\left[\cos\left(\frac{p}{R(s)}\right)\hat{n}(s) - \sin\left(\frac{p}{R(s)}\right)\hat{t}(s)\right]\Bigg|_{p=0} = \frac{1}{R(s)}\hat{n}(s) = \kappa(s)\hat{n}(s)\ ,$$

i.e. the osculator circle has the same normal vector and curvature as the curve in the point.

2.1.2 Surfaces

$$\vec{r}(q^1, q^2)$$
$$d\vec{r} = \frac{\partial \vec{r}}{\partial q^1}\,dq^1 + \frac{\partial \vec{r}}{\partial q^2}\,dq^2 = \vec{b}_1\,dq^1 + \vec{b}_2\,dq^2$$

A third vector $\vec{b}_3 := \hat{n}$ can be defined so that $|\hat{n}| = 1$ and $\hat{n}\cdot\vec{b}_i = 0$, $i = 1:2$. For $i = 1:2$, $k = 1:2$

$$\frac{\partial \vec{b}_i}{\partial q^j} = \Gamma^k_{ij}\vec{b}_k = \Gamma^1_{ij}\vec{b}_1 + \Gamma^2_{ij}\vec{b}_2 + \Gamma^3_{ij}\vec{b}_3$$

so that

$$\Gamma^k_{ij} = \vec{b}^k \cdot \frac{\partial \vec{b}_i}{\partial q^j}$$

Normal vector.

$$\vec{n}(q^1, q^2) = \frac{\partial \vec{r}}{\partial q^1}(q^1, q^2) \times \frac{\partial \vec{r}}{\partial q^2}(q^1, q^2) = \vec{b}_1(q^1, q^2) \times \vec{b}_2(q^1, q^2)$$

Tangent plane.

$$(\vec{r} - \vec{r}(q^1, q^2)) \cdot \vec{n}(q^1, q^2) = 0$$

Length of elementary segment.

$$\begin{aligned} |d\vec{r}|^2 &= d\vec{r}\cdot d\vec{r} = \\ &= \left(\vec{b}_1\,dq^1 + \vec{b}_2\,dq^2\right)\cdot\left(\vec{b}_1\,dq^1 + \vec{b}_2\,dq^2\right) = \\ &= g_{11}\,dq^1\,dq^1 + g_{12}\,dq^1\,dq^2 + g_{21}\,dq^2\,dq^1 + g_{22}\,dq^2\,dq^2 = g_{ij}\,dq^i\,dq^j \end{aligned}$$

Second order approximation.

$$\begin{aligned} \vec{r}(q^1 + dq^1, q^2 + dq^2) &= \vec{r}(q_1, q_2) + \frac{\partial \vec{r}}{\partial q^i}\,dq^i + \frac{\partial^2 \vec{r}}{\partial q^i \partial q^j}\,dq^i\,dq^j = \\ &= \vec{r}(q_1, q_2) + \vec{b}_i\,dq^i + \vec{b}_k\Gamma^k_{ij}\,dq^i\,dq^j + \hat{n}\,\Gamma^3_{ij}\,dq^i\,dq^j \end{aligned}$$

so that

$$\begin{aligned} \left[\vec{r}(q^1 + dq^1, q^2 + dq^2) - \vec{r}(q^1, q^2)\right]\cdot\hat{n} = \Gamma^3_{ij}\,dq^i\,dq^j &= \\ &= \hat{n}\cdot\frac{\partial^2 \vec{r}}{\partial q^i \partial q^j}\,dq^i\,dq^j = \\ &= \hat{n}\cdot\frac{\partial^2 \vec{r}}{\partial q^i \partial q^j}\,\vec{b}^i\cdot\vec{b}_k dq^k\,\vec{b}^j\cdot\vec{b}_l dq^l = \\ &= \underbrace{dq^k\vec{b}_k}_{d\vec{r}}\cdot\left[\hat{n}\cdot\frac{\partial^2 \vec{r}}{\partial q^i \partial q^j}\vec{b}^i\otimes\vec{b}^j\right]\cdot\underbrace{dq^l\vec{b}_l}_{d\vec{r}} \end{aligned}$$

Curvature tensor.

Part III

Vector and Tensor Algebra and Calculus

CHAPTER

THREE

TENSOR ALGEBRA

3.1 Basis

Definition 1 (Basis)

Definition 2 (Reciprocal basis)

In a inner product space, the reciprocal basis of a given basis $\{\vec{b}_a\}_{a=1:d}$ is the set of vectors $\{\vec{b}_b\}_{b=1:d}$, s.t.

$$\vec{b}^b \cdot \vec{b}_a = \delta_a^b \,.$$

3.2 Exterior algebra

$$\wedge$$

3.3 Exterior product

Generalization of the vector product

CHAPTER

FOUR

TENSOR CALCULUS IN EUCLIDEAN SPACES

This section deals with tensor calculus in Euclidean space or on manifolds embedded in Euclidean spaces, focusing on d-dimensional spaces with $d \leq 3$, with *inner product*.

This section may rely on results of *differential geometry*.

4.1 Coordinates

A set of parameters $\{q^a\}_{a=1:d}$ to represent vector (or point) in space,

$$\vec{r}(q^a)$$

if $\vec{r} \in E^d$, $a = 1 : d$.

In E^3,

- **Coordinate lines**, 2-parameter family of lines, keeping 2 coordinates constant. As an example, coordinate lines with constant q^2, q^3

$$\vec{r}_1(q^1) = \vec{r}(q^1, \bar{q}^2, \bar{q}^3) \ .$$

- **Coordinate surfaces,** 1-parameter family of surfaces, keeping 1 coordinate constant. As an example, coordinate surfaces with constant q^1,

$$\vec{r}_{23}(q^2, q^3) = \vec{r}(\bar{q}^1, q^2, q^3) \ .$$

Definition 3 (Regular parametrization)

If $\frac{\partial \vec{r}}{\partial q^a} \neq 0$.

4.1.1 Natural basis

Definition 4 (Natural basis)

Vectors of natural basis

$$\vec{b}_a := \frac{\partial \vec{r}}{\partial q^a}$$

Definition 5 (Reciprocal basis (todo move to Tensor Algebra))

Given a basis $\{\vec{b}_a\}_a$, its reciprocal basis the set of vector $\{\vec{b}^b\}_b$ defined as

$$\vec{b}^b \cdot \vec{b}_a = \delta^b_a \ ,$$

being δ^b_a Kronecker delta.

Definition 6 (Christoffel symbols)

Christoffel symbols (of the 2^{nd} kind) are defined as the components of the partial derivatives of the vectors of a natural basis w.r.t. the coordinates referred to the natural basis itself,

$$\frac{\partial \vec{b}_a}{\partial q^b} = \Gamma^c_{ab} \, \vec{b}_c \tag{4.1}$$

Properties of Christoffel symbols

Exploiting the definition of reciprocal basis, Christoffel symbols can be written as

$$\Gamma^c_{ab} = \vec{b}^c \cdot \frac{\partial \vec{b}_a}{\partial q^b} \ .$$

Symmetry. Symmetry os the lower indices

$$\Gamma^c_{ab} = \Gamma^c_{ba} \ ,$$

readily follows Schwartz theorem about partial derivatives

$$\frac{\partial \vec{b}_a}{\partial q^b} = \frac{\partial}{\partial q^c} \frac{\partial \vec{r}}{\partial q^a} = \frac{\partial}{\partial q^a} \frac{\partial \vec{r}}{\partial q^b} = \frac{\partial \vec{b}_b}{\partial q^a}$$

4.2 Fields

Function of the points in space $F : E^d \to V^r$, being V^r a space of tensors of order r.

4.3 Differential operators

4.3.1 Directional derivative

$$F(\vec{r}) = F\left(\vec{r}\left(q^a\right)\right) = f(q^a)$$

$$f(q^a + \beta \Delta q^a) = F(\vec{r}(q^a + \beta \Delta q^a))$$

$$\vec{r}(q^a) + \alpha \vec{v} = \vec{r}(q^a + \beta \Delta q^a) \sim \vec{r}(q^a) + \frac{\partial \vec{r}}{\partial q^b} \beta \Delta q^b$$

$$\alpha \vec{v} \sim \beta \frac{\partial \vec{r}}{\partial q^b} (q^a) \, \Delta q^b = \beta \vec{b}_b(q^a) \Delta q^b \qquad \to \qquad \Delta q^b = \frac{\alpha}{\beta} \vec{b}^b(q^a) \cdot \vec{v}$$

The directional derivative for an arbitrary vector $\vec{v} \in V$

$$\frac{d}{d\alpha} F(\vec{r} + \alpha \vec{v})\bigg|_{\alpha=0}$$

is evaluated as the limit for $\alpha \to 0$ of the incremental ratio

$$\begin{aligned}
\frac{F(\vec{r} + \alpha\vec{v}) - F(\vec{r})}{\alpha} \sim \frac{f(q^a + \beta\Delta q^a) - f(q^a)}{\alpha} = \\
\sim \frac{1}{\alpha}\frac{\partial f}{\partial q^b}(q^a)\beta\Delta q^b = \\
\sim \vec{v} \cdot \vec{b}^b(q^a)\frac{\partial f}{\partial q^b}(q^a) = \\
= \vec{v} \cdot \nabla F(\vec{r})
\end{aligned}$$

4.3.2 Gradient

The gradient is the differential operator is the first-order differential operator appearing in the definition of the directional derivative, $\nabla F(\vec{r})$. It takes a tensor field $F(\vec{r})$ of order r and gives a tensor field $\nabla F(\vec{r})$ of order $r+1$. Given a set of coordinates $\{q^a\}_{a=1:d}$, the gradient can be written using the reciprocal basis of the natural basis as

$$\nabla F(\vec{r}) = \vec{b}^b(\vec{r})\frac{\partial F}{\partial q^b}(\vec{r}) \tag{4.2}$$

Examples. ...

Example 1 (Gradient of a scalar field - with general coordinates q^a)

Applying the definition (4.2) of gradient operator, it readily follows

$$\nabla F = \vec{b}^a \frac{\partial F}{\partial q^a}$$

Example 2 (Gradient of a vector field - with general coordinates q^a)

Applying the definition (4.2) of gradient operator, rule for the derivative of a product and the definition (4.1) of Christoffel symbols to write derivatives of base vectors,

$$\begin{aligned}
\nabla F &= \vec{b}^a \frac{\partial}{\partial q^a}\left(F^b \vec{b}_b\right) = \\
&= \vec{b}^a \left[\frac{\partial F^b}{\partial q^a}\vec{b}_b + F^b \frac{\partial \vec{b}_b}{\partial q^a}\right] = \\
&= \vec{b}^a \left[\frac{\partial F^b}{\partial q^a}\vec{b}_b + F^b\, \Gamma^c_{ab}\, \vec{b}_c\right] = \\
&= \vec{b}^a \otimes \vec{b}_b \left[\frac{\partial F^b}{\partial q^a} + \Gamma^b_{ac}\, F^c\right] .
\end{aligned}$$

Example 3 (Gradient of a 2^{nd}-order tensor field - with general coordinates q^a)

Applying the definition (4.2) of gradient operator, rule for the derivative of a product and the definition (4.1) of Christoffel symbols to write derivatives of base vectors,

$$\begin{aligned}
\nabla F &= \vec{b}^a \frac{\partial}{\partial q^a} \left(F^{bc} \vec{b}_b \otimes \vec{b}_c \right) = \\
&= \vec{b}^a \left[\frac{\partial F^{bc}}{\partial q^a} \vec{b}_b \, \vec{b}_c + F^{bc} \frac{\partial \vec{b}_b}{\partial q^a} \, \vec{b}_c + F^{bc} \vec{b}_b \frac{\partial \vec{b}_c}{\partial q^a} \right] = \\
&= \vec{b}^a \left[\frac{\partial F^{bc}}{\partial q^a} \vec{b}_b \, \vec{b}_c + F^{bc} \Gamma^d_{ab} \, \vec{b}_d \, \vec{b}_c + F^{bc} \, \Gamma^{ac}_{d} \, \vec{b}_b \, \vec{b}_d \right] = \\
&= \vec{b}^a \otimes \vec{b}_b \otimes \vec{b}_c \left[\frac{\partial F^{bc}}{\partial q^a} + \Gamma^b_{ad} \, F^{dc} + \Gamma^c_{ad} \, F^{bd} \right] \ .
\end{aligned}$$

4.3.3 Divergence

Divergence opearator is a first-order differential operator that can be defined as the contraction of the first two indices of the gradient,

$$\nabla \cdot F = C_1^2 \left(\nabla F \right) \ .$$

It takes a tensor field $F(\vec{r})$ of order $r \geq 1$ and gives a tensor field $\nabla \cdot F(\vec{r})$ of order $r - 1 \geq 0$.

Example 4 (Divergence of a vector field - with general coordiantes q^a)

Applying contraction to the gradient of a vector field, it readily follows,

$$\begin{aligned}
\nabla \cdot \left(F^b \vec{b}_b \right) &= C_1^2 \left(\nabla F \right) = \\
&= C_1^2 \left(\vec{b}^a \otimes \vec{b}_b \left[\frac{\partial F^b}{\partial q^a} + \Gamma^b_{ac} \, F^c \right] \right) = \\
&= \frac{\partial F^a}{\partial q^a} + \Gamma^a_{ac} \, F^c
\end{aligned}$$

Example 5 (Divergence of a 2^{nd}-order tensor field - with general coordiantes q^a)

Applying contraction to the gradient of a vector field, it readily follows,

$$\begin{aligned}
\nabla \cdot \left(F^{bc} \vec{b}_b \otimes \vec{b}_c \right) &= C_1^2 \left(\nabla F \right) = \\
&= C_1^2 \left(\vec{b}^a \otimes \vec{b}_b \otimes \vec{b}_c \left[\frac{\partial F^{bc}}{\partial q^a} + \Gamma^b_{ad} \, F^{dc} + \Gamma^c_{ad} \, F^{bd} \right] \right) = \\
&= \vec{b}_c \left[\frac{\partial F^{ac}}{\partial q^a} + \Gamma^a_{ad} \, F^{dc} + \Gamma^c_{ad} \, F^{ad} \right]
\end{aligned}$$

4.3.4 Laplacian

Laplacian operator is second-order differential operator that can be defined as the divergence of the gradient,

$$\Delta F = \nabla^2 F = \nabla \cdot \nabla F \; .$$

Example 6 (Laplacian of a scalar field - with general coordinates q^a)

$$\begin{aligned}
\nabla \cdot \nabla F &= C_1^2 \left[\nabla \left(\nabla F \right) \right] = \\
&= C_1^2 \left[\nabla \left(\vec{b}^a \frac{\partial F}{\partial q^a} \right) \right] = \\
&= C_1^2 \left[\nabla \left(\vec{b}_b \, g^{ab} \, \frac{\partial F}{\partial q^a} \right) \right] = \\
&= C_1^2 \left[\vec{b}^c \frac{\partial}{\partial q^c} \left(\vec{b}_b \, g^{ab} \, \frac{\partial F}{\partial q^a} \right) \right] = \\
&= C_1^2 \left\{ \vec{b}^c \left[\vec{b}_b \, \frac{\partial}{\partial q^c} \left(g^{ab} \, \frac{\partial F}{\partial q^a} \right) + g^{ab} \frac{\partial F}{\partial q^a} \frac{\partial \vec{b}_b}{\partial q^c} \right] \right\} = \\
&= C_1^2 \left\{ \vec{b}^c \left[\vec{b}_b \, \frac{\partial}{\partial q^c} \left(g^{ab} \, \frac{\partial F}{\partial q^a} \right) + g^{ab} \frac{\partial F}{\partial q^a} \, \Gamma^d_{bc} \, \vec{b}_d \right] \right\} = \\
&= C_1^2 \left\{ \vec{b}^c \, \vec{b}_b \left[\frac{\partial}{\partial q^c} \left(g^{ab} \, \frac{\partial F}{\partial q^a} \right) + g^{ad} \, \Gamma^b_{cd} \, \frac{\partial F}{\partial q^a} \right] \right\} = \\
&= \frac{\partial}{\partial q^b} \left(g^{ab} \, \frac{\partial F}{\partial q^a} \right) + g^{ad} \, \Gamma^b_{bd} \, \frac{\partial F}{\partial q^a} \; .
\end{aligned}$$

Example 7 (Laplacian of a vector field - with general coordinates q^a)

$$\begin{aligned}
\nabla \cdot \nabla F &= C_1^2 \left[\nabla \left(\nabla F \right) \right] = \\
&= C_1^2 \left\{ \nabla \left[\vec{b}^a \, \vec{b}_b \left(\frac{\partial F^b}{\partial q^a} + \Gamma^b_{ac} F^c \right) \right] \right\} = \\
&= C_1^2 \left\{ \nabla \left[\vec{b}_c \, \vec{b}_b \, g^{ac} \left(\frac{\partial F^b}{\partial q^a} + \Gamma^b_{ac} F^c \right) \right] \right\} = \\
&= C_1^2 \left\{ \nabla \cdot \left((\nabla F)^{cb} \, \vec{b}_c \, \vec{b}_b \right) \right\} = \\
&= C_1^2 \left\{ \vec{b}^a \, \vec{b}_c \, \vec{b}_b \; \left[\frac{\partial (\nabla F)^{cb}}{\partial q^a} + \Gamma^c_{ad} (\nabla F)^{db} + \Gamma^b_{ad} (\nabla F)^{cd} \right] \right\} = \\
&= \vec{b}_b \; \left[\frac{\partial (\nabla F)^{ab}}{\partial q^a} + \Gamma^a_{ad} (\nabla F)^{db} + \Gamma^b_{ad} (\nabla F)^{ad} \right] = \; .
\end{aligned}$$

4.3.5 Curl

4.4 Integrals in E^d, $d \le 3$

4.4.1 Line integrals

Density

Integrals

$$\int_{\vec{r}\in\gamma} F(\vec{r})$$

represent the summation of contributions $F(\vec{r})$ over elementary segments of path γ, whose dimension is $|d\vec{r}|$, i.e. implicitly means

$$\int_{\vec{r}\in\gamma} F(\vec{r}) = \int_{\vec{r}\in\gamma} F(\vec{r})\,|d\vec{r}|\ .$$

Given a regular parametrization of the curve $\vec{r}(q^1)$ (with increasing q^1 so that $|dq^1| = dq^1$), and the differential $d\vec{r} = \vec{r}'(q^1)\,dq^1$, the integral can be written as an integral in the parameter q^1

$$\int_{q=q^1_a}^{q^1_b} F(\vec{r}(q^1))\,|\vec{r}'(q^1)|\,dq^1\ ,$$

with $\vec{r}(q^1_a)$, $\vec{r}(q^1_b)$ the extreme points of path γ.

Work

Integrals

$$\int_{\vec{r}\in\gamma} F(\vec{r})\cdot\hat{t}(\vec{r})$$

implicitly mean

$$\int_{\vec{r}\in\gamma} F(\vec{r})\cdot\hat{t}(\vec{r}) = \int_{\vec{r}\in\gamma} F(\vec{r})\cdot\hat{t}(\vec{r})|d\vec{r}| = \int_{\vec{r}\in\gamma} F(\vec{r})\cdot d\vec{r}\ ,$$

as $\hat{t} = \frac{d\vec{r}}{|d\vec{r}|}$. Given a regular parametrization of the curve $\vec{r}(q^1)$ (with increasing q^1 so that $|dq^1| = dq^1$), and the differential $d\vec{r} = \vec{r}'(q^1)\,dq^1$, the integral can be written as an integral in the parameter q^1

$$\int_{q^1=q^1_a}^{q^1_b} F(\vec{r}(q^1))\cdot\vec{r}'(q^1)\,dq^1$$

4.4.2 Surface integrals

Given two coordinates q^1, q^2 describing a surface, $\vec{r}(q^1, q^2)$ the elementary surface with unit normal reads

$$\hat{n}\,dS = d\vec{r}_1 \times d\vec{r}_2 = \frac{\partial\vec{r}}{\partial q^1}\times\frac{\partial\vec{r}}{\partial q^2}\,dq^1\,dq^2\ ,$$

and the elementary surface thus reads

$$|dS| = |\hat{n}dS| = \left|\frac{\partial\vec{r}}{\partial q^1}\times\frac{\partial\vec{r}}{\partial q^2}\,dq^1\,dq^2\right|$$

Density

Integrals

$$\int_{\vec{r}\in S} F(\vec{r})$$

implicitly mean

$$\int_{\vec{r}\in S} F(\vec{r}) = \int_{\vec{r}\in S} F(\vec{r})|dS| \ .$$

Given regular parametrization of the surface, $\vec{r}(q^1,\, q^2),\ (q^1, q^2) \in Q^{12}$, the integral can be written as the multi-dimensional integral in coordinates $q^1,\ q^2$,

$$\int_{\vec{r}\in S} F(\vec{r}) = \int_{(q^1,q^2)\in Q^{12}} F(\vec{r}(q^1,q^2)) \left| \frac{\partial\vec{r}}{\partial q^1} \times \frac{\partial\vec{r}}{\partial q^2}\, dq^1\, dq^2 \right|$$

Flux

Integrals

$$\int_{\vec{r}\in S} \hat{n}(\vec{r}) \cdot F(\vec{r})$$

implicitly mean

$$\int_{\vec{r}\in S} \hat{n}(\vec{r}) \cdot F(\vec{r}) = \int_{\vec{r}\in S} \hat{n}(\vec{r}) \cdot F(\vec{r})|dS|$$

Given regular parametrization of the surface, $\vec{r}(q^1,\, q^2),\ (q^1, q^2) \in Q^{12}$, the integral can be written as the multi-dimensional integral in coordinates $q^1,\ q^2$,

$$\int_{\vec{r}\in S} \hat{n}(\vec{r}) \cdot F(\vec{r}) = \int_{(q^1,q^2)\in Q^{12}} \frac{\partial\vec{r}}{\partial q^1} \times \frac{\partial\vec{r}}{\partial q^2} \cdot F(\vec{r}(q^1,q^2))\, dq^1\, dq^2$$

4.4.3 Volume

$$dV = \frac{\partial\vec{r}}{\partial q^1} \cdot \frac{\partial\vec{r}}{\partial q^2} \times \frac{\partial\vec{r}}{\partial q^3}\, dq^1\, dq^2\, dq^3 \ .$$

Density

Integrals

$$\int_{\vec{r}\in V} F(\vec{r})$$

implicitly mean

$$\int_{\vec{r}\in V} F(\vec{r}) = \int_{\vec{r}\in V} F(\vec{r})\, |dV| \ .$$

Given regular parametrization of the volume, $\vec{r}(q^1,\, q^2,\, q^3),\ (q^1, q^2, q^3) \in Q$, the integral can be written as the multi-dimensional integral in coordinates $q^1,\ q^2,\ q^3$,

$$\int_{\vec{r}\in V} F(\vec{r})|dV| = \int_{(q^1,q^2,q^3)\in Q} F(\vec{r}(q^1,q^2,q^3)) \left| \frac{\partial\vec{r}}{\partial q^1} \cdot \frac{\partial\vec{r}}{\partial q^2} \times \frac{\partial\vec{r}}{\partial q^3}\, dq^1\, dq^2\, dq^3 \right| \ .$$

4.4.4 Theorems

Gradient theorem

$$\int_V \nabla f = \oint_{\partial V} f \hat{n}$$

Divergence theorem

$$\int_V \nabla \cdot \vec{f} = \oint_{\partial V} \vec{f} \cdot \hat{n}$$

Curl theorem

$$\int_S \left[\nabla \times \vec{f} \right] \cdot \hat{n} = \oint_{\partial S} \vec{f} \cdot \hat{t}$$

4.5 Tensor Calculus in Euclidean Spaces - Cartesian coordinates in E^3

Using Cartesian coordinates $(q^1, q^2, q^3) = (r, \theta, z)$ and Cartesian base vectors (uniform in space, so that their derivatives are zero), a point in Euclidean vector space E^3 can be represented as

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}\,.$$

4.5.1 Natural basis, reciprocal basis, metric tensor, and Christoffel symbols

Cartesian coordinates in Euclidean spaces are a very special coordinate system, with reciprocal basis everywhere coinciding with natural basis, with uniform basis in space (zero second-order derivative of space w.r.t. coordinates, and thus zero first order derivative of base vectors, and thus identically zero Christoffel symbols), and components of the metric tensor equal to the identity matrix

$$\begin{cases} \vec{b}_1 = \vec{b}^1 = \hat{x} \\ \vec{b}_2 = \vec{b}^2 = \hat{y} \\ \vec{b}_3 = \vec{b}^3 = \hat{z} \end{cases}$$

$$[g_{ab}] = [g^{ab}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Gamma^c_{ab} = 0 \qquad , \qquad \forall a, b, c = 1:3\,.$$

4.5.2 Differential operators

Gradient

Example 8 (Gradient of a scalar field)

$$\nabla F = \hat{x}\,\partial_x F + \hat{y}\,\partial_y F + \hat{z}\,\partial_z F_z$$

Example 9 (Gradient of a vector field)

$$\begin{aligned}\nabla F = &\nabla(F_x\hat{x} + F_y\hat{y} + F_z\hat{z}) = \\ &= \hat{x}\otimes\hat{x}\,\partial_x F_x + \hat{x}\otimes\hat{y}\,\partial_x F_y + \hat{x}\otimes\hat{z}\,\partial_x F_z + \\ &+ \hat{y}\otimes\hat{x}\,\partial_y F_x + \hat{y}\otimes\hat{y}\,\partial_y F_y + \hat{y}\otimes\hat{z}\,\partial_y F_z + \\ &+ \hat{z}\otimes\hat{x}\,\partial_z F_x + \hat{z}\otimes\hat{y}\,\partial_z F_y + \hat{z}\otimes\hat{z}\,\partial_z F_z +\end{aligned}$$

Example 10 (Gradient of a 2^{nd}-order tensor field)

Directional derivative

Divergence

Example 11 (Divergence of a vector field)

$$\begin{aligned}\nabla\cdot F &= \nabla\cdot\left(F_x\,\hat{x} + F_y\,\hat{y} + F_z\,\hat{z}\right) = \\ &= \partial_x F_x + \partial_y F_y + \partial_z F_z\ .\end{aligned}$$

Example 12 (Divergence of a 2^{nd}-order tensor field)

$$\begin{aligned}\nabla\cdot F = &\nabla\cdot(F_{ab}\vec{e}_a\otimes\vec{e}_b) = \\ &= \vec{e}_c\frac{\partial F_{ab}}{\partial q^a} = \\ &= \hat{x}\left[\partial_x F_{xx} + \partial_y F_{yx} + \partial_z F_{zx}\right] + \\ &+ \hat{y}\left[\partial_x F_{xy} + \partial_y F_{yy} + \partial_z F_{zy}\right] + \\ &+ \hat{z}\left[\partial_x F_{xz} + \partial_y F_{yz} + \partial_z F_{zz}\right]\ .\end{aligned}$$

Laplacian

Example 13 (Laplacian of a scalar field)

$$\nabla^2 F = \partial_{xx} F + \partial_{yy} F + \partial_{zz} F$$

Example 14 (Laplacian of a vector field)

4.6 Tensor Calculus in Euclidean Spaces - cylindrical coordinates in E^3

4.6.1 Cylindrical coordiantes and cylindrical coordinates

Using cylindrical coordinates $(q^1, q^2, q^3) = (r, \theta, z)$ and cylindrical base vectors (uniform in space, so that their derivatives are zero), a point in Euclidean vector space E^3 can be represented as

$$\vec{r} = r \cos\theta \, \hat{x} + r \sin\theta \, \hat{y} + z \, \hat{z} \, .$$

4.6.2 Natural basis, reciprocal basis, metric tensor, and Christoffel symbols

Natural basis

Natural basis reads

$$\begin{cases} \vec{b}_1 = \dfrac{\partial \vec{r}}{\partial q^1} = \dfrac{\partial \vec{r}}{\partial r} = \cos\theta \, \hat{x} + \sin\theta \, \hat{y} \\ \vec{b}_2 = \dfrac{\partial \vec{r}}{\partial q^2} = \dfrac{\partial \vec{r}}{\partial \theta} = -r \sin\theta \, \hat{x} + r \cos\theta \, \hat{y} \\ \vec{b}_3 = \dfrac{\partial \vec{r}}{\partial q^3} = \dfrac{\partial \vec{r}}{\partial z} = \hat{z} \end{cases}$$

Metric tensor

Covariant components of metric tensors,

$$g_{ab} = \vec{b}_a \cdot \vec{b}_b \, ,$$

can be collected in the diagonal matrix

$$[g_{ab}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \, ,$$

while its contra-variant components can be collected in the inverse matrix (easy to compute, since $[g_{ab}]$ is diagonal),

$$[g^{ab}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \, .$$

Reciprocal basis

Reciprocal basis is readily evaluated using $\vec{b}^a = g^{ab}\,\vec{b}_b$,

$$\begin{cases}\vec{b}^1 = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \\ \vec{b}^2 = -\dfrac{1}{r}\sin\theta\,\hat{x} + \dfrac{1}{r}\cos\theta\,\hat{y} \\ \vec{b}^3 = \hat{z}\end{cases}$$

Physical basis

Since metric tensor is diagonal, the cylindrical coordinate system is orthogonal, and its natural and reciprocal basis are orthogonal. A unit orthogonal basis, usually named **physical basis** with unit vector with no physical dimension, is evalated by normalization process,

$$\begin{cases}\hat{r} = \hat{b}_1 = \dfrac{\vec{b}_1}{g_{11}} = \dfrac{\vec{b}^1}{g^{11}} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \\ \hat{\theta} = \hat{b}_2 = \dfrac{\vec{b}_2}{g_{22}} = \dfrac{\vec{b}^2}{g^{22}} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} \\ \hat{z} = \hat{b}_3 = \dfrac{\vec{b}_3}{g_{33}} = \dfrac{\vec{b}^3}{g^{33}} = \hat{z}\,.\end{cases}$$

Derivatives of natural basis and Christoffel symbols

Derivatives of the natural basis read

$$\begin{aligned}\frac{\partial\vec{b}_1}{\partial q^1} &= \vec{0} \\ \frac{\partial\vec{b}_2}{\partial q^2} &= -r\cos\theta\,\hat{x} - r\sin\theta\,\hat{y} = -q^1\,\vec{b}_1 \\ \frac{\partial\vec{b}_3}{\partial q^3} &= \vec{0} \\ \frac{\partial\vec{b}_2}{\partial q^1} = \frac{\partial\vec{b}_1}{\partial q^2} &= -\sin\theta\,\hat{x} + \cos\theta\hat{y} = \frac{1}{q^1}\,\vec{b}_2 \\ \frac{\partial\vec{b}_3}{\partial q^1} = \frac{\partial\vec{b}_1}{\partial q^3} &= \vec{0} \\ \frac{\partial\vec{b}_3}{\partial q^2} = \frac{\partial\vec{b}_2}{\partial q^3} &= \vec{0}\end{aligned}$$

so that non-zero Christoffel symbols of a cylindrical coordinate system are

$$\begin{aligned}\Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{q^1} \\ \Gamma^1_{22} &= -q^1\,.\end{aligned}$$

4.6.3 Differential operators

Gradient

Example 15 (Gradient of a scalar field)

$$
\begin{aligned}
\nabla F &= \vec{b}^a \frac{\partial F}{\partial q^a} = \\
&= \hat{b}_a \, g^{aa} \, \frac{\partial F}{\partial q^a} = \\
&= \hat{r} \, \frac{\partial F}{\partial r} + \hat{\theta} \, \frac{1}{r} \, \frac{\partial F}{\partial \theta} + \hat{z} \frac{\partial F}{\partial z} \, .
\end{aligned}
$$

Example 16 (Gradient of a vector field)

$$
\begin{aligned}
\nabla F = \vec{b}^a \otimes \vec{b}_b \left[\frac{\partial F^b}{\partial q^a} + \Gamma^b_{ac} \, F^c \right] = \\
= \cdots = \\
&= \vec{b}^1 \otimes \vec{b}_1 \, \partial_1 F^1 && + \vec{b}^1 \otimes \vec{b}_2 \, \left[\partial_1 F^2 + \Gamma^2_{12} F^2 \right] && + \vec{b}^1 \otimes \vec{b}_3 \, \partial_1 F^3 \\
&+ \vec{b}^2 \otimes \vec{b}_1 \, \left[\partial_2 F^1 + \Gamma^1_{22} F^2 \right] && + \vec{b}^2 \otimes \vec{b}_2 \, \left[\partial_2 F^2 + \Gamma^2_{21} F^1 \right] && + \vec{b}^2 \otimes \vec{b}_3 \, \partial_2 F^3 \\
&+ \vec{b}^3 \otimes \vec{b}_1 \, \partial_3 F^1 && + \vec{b}^3 \otimes \vec{b}_2 \, \partial_3 F^2 && + \vec{b}^3 \otimes \vec{b}_3 \, \partial_3 F^3 \\
&= \hat{r} \otimes \hat{r} \, \partial_r F_r && + \hat{r} \otimes \hat{\theta} \, \frac{1}{r} \left[\partial_r (r F_\theta) + F_\theta \right] && + \hat{r} \otimes \hat{z} \, \partial_r F_z \\
&+ \hat{\theta} \otimes \hat{r} \, \frac{1}{r} \left[\partial_\theta F_r - r F_\theta \right] && + \hat{\theta} \otimes \hat{\theta} \, \left[\partial_\theta \left(\frac{F_\theta}{r} \right) + \frac{F_r}{r} \right] && + \hat{\theta} \otimes \hat{z} \, \frac{1}{r} \partial_\theta F_z \\
&+ \hat{z} \otimes \hat{r} \, \partial_z F_x && + \hat{z} \otimes \hat{\theta} \, \frac{1}{r} \partial_\theta F_y && + \hat{z} \otimes \hat{z} \, \partial_z F_z \, .
\end{aligned}
$$

Example 17 (Gradient of a 2^{nd}-order tensor field)

Directional derivative

Divergence

Example 18 (Divergence of a vector field)

$$
\begin{aligned}
\nabla \cdot \vec{F} &= \frac{\partial F^a}{\partial q^a} + \Gamma^a_{ac} \, F^c = \\
&= \frac{\partial F_r}{\partial r} + \frac{\partial}{\partial \theta} \left(\frac{F_\theta}{r} \right) + \frac{F_\theta}{r} + \frac{\partial F_z}{\partial z} \, .
\end{aligned}
$$

Example 19 (Divergence of a 2^{nd}-order tensor field)

Laplacian

Example 20 (Laplacian of a scalar field)

Example 21 (Laplacian of a vector field)

4.7 Tensor Calculus in Euclidean Spaces - Spehrical coordinates in E^3

Using spherical coordinates $(q^1, q^2, q^3) = (r, \phi, \theta)$ and spherical base vectors (uniform in space, so that their derivatives are zero), a point in Euclidean vector space E^3 can be represented as

$$\vec{r} = r\cos\theta\sin\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\phi\,\hat{z}\ .$$

4.7.1 Natural basis, reciprocal basis, metric tensor, and Christoffel symbols

4.7.2 Differential operators

Gradient

Example 22 (Gradient of a scalar field)

Example 23 (Gradient of a vector field)

Example 24 (Gradient of a 2^{nd}-order tensor field)

Directional derivative

Divergence

Example 25 (Divergence of a vector field)

Example 26 (Divergence of a 2^{nd}-order tensor field)

Laplacian

Example 27 (Laplacian of a scalar field)

Example 28 (Laplacian of a vector field)

CHAPTER

FIVE

TIME DERIVATIVE OF INTEGRALS OVER MOVING DOMAINS

Some results about time derivatives over moving domains are collected here.

5.1 Volume density

Reynolds transport theorem. Given a volume $V(t)$ with boundary $\partial V(t)$, whose points $\vec{r} \in \partial V(t)$ have velocity $\vec{v}_b$,

$$\frac{d}{dt} \int_{V(t)} f = \int_{V(t)} \frac{\partial f}{\partial t} + \oint_{\partial V(t)} f \vec{v}_b \cdot \hat{n} \, .$$

“Proof”

5.2 Flux across a surface

$$\frac{d}{dt} \int_{S(t)} \vec{f} \cdot \hat{n} = \int_{S(t)} \frac{\partial \vec{f}}{\partial t} \cdot \hat{n} + \int_{S(t)} \nabla \cdot \vec{f} \, \vec{v}_b \cdot \hat{n} - \int_{\partial S(t)} \vec{v}_b \times \vec{f} \cdot \hat{t}$$

“Proof”

5.3 Work line integral along a line

$$\frac{d}{dt} \int_{\ell(t)} \vec{f} \cdot \hat{t} = \int_{\ell(t)} \frac{\partial \vec{f}}{\partial t} \cdot \hat{t} + \int_{\ell(t)} \nabla \times \vec{f} \cdot \vec{v}_b \times \hat{t} + \vec{f}_B \cdot \vec{v}_B - \vec{f}_A \cdot \vec{v}_A$$

“Proof”

Part IV

Functional Analysis

CHAPTER

SIX

INTRODUCTION TO FUNCTIONAL ANALYSIS

- Lebesgue integral
- L^p, H^p function spaces
- Banach and Hilbert spaces

CHAPTER

SEVEN

DISTRIBUTIONS (OR GENERALIZED FUNCTIONS)

…

7.1 Dirac's delta

Dirac's delta $\delta(x)$ is a distribution, or generalized function, with the following properties

1. $$\int_D \delta(x-x_0)\,dx = 1 \quad \text{if } x_0 \in D$$

2. $$\int_D f(x)\delta(x-x_0)\,dx \quad \text{if } x_0 \in D$$

for $\forall f(x)$ "regular" **todo** *what does regular mean?*

7.1.1 Dirac's delta in terms of regular functions

Approximations …

$$\delta(x) \sim r_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon} & x \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \\ 0 & \text{otherwise} \end{cases}$$

as

1. Unitariety

   $$\int_{x=-\infty}^{\infty} r_\varepsilon(x-x_0)\,dx = \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} \frac{1}{\varepsilon}\,dx = 1 \ ,$$

   for $\forall \varepsilon$;

2. Shift property, using mean-value theorem of continuous functions

   $$\int_{x=-\infty}^{\infty} r_\varepsilon(x-x_0)f(x)\,dx = \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} \frac{1}{\varepsilon}f(x)\,dx = \frac{1}{\varepsilon}\varepsilon f(\xi) \ ,$$

   with $\xi \in [x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}]$, for the mean value theorem. As $\varepsilon \to 0$, $\xi \to x_0$, and thus

   $$\int_{x=-\infty}^{\infty} r_\varepsilon(x-x_0)f(x)\,dx \to f(x_0)$$

$$\delta(x) \sim t_\varepsilon(x) = \begin{cases} \frac{2}{\varepsilon}\left(1 - \frac{2|x|}{\varepsilon}\right) & x \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \\ 0 & \text{otherwise} \end{cases}$$

as

1. Unitariety

$$\int_{x=-\infty}^{\infty} t_\varepsilon(x - x_0)\, dx = \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} \frac{2}{\varepsilon}\left(1 - \frac{2|x|}{\varepsilon}\right)\, dx = \frac{1}{2}\varepsilon\frac{2}{\varepsilon} = 1\ ,$$

for $\forall \varepsilon$;

2. Shift property, using mean-value integration scheme in $x \in [x_0 - \frac{\varepsilon}{2}, x_0]$, $x \in [x_0, x_0 + \frac{\varepsilon}{2}]$ (**todo** *why?*)

$$\begin{aligned}\int_{x=-\infty}^{\infty} t_\varepsilon(x - x_0) f(x)\, dx &= \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} \frac{2}{\varepsilon}\left(1 - \frac{2|x-x_0|}{\varepsilon}\right) f(x)\, dx = \\ &= \int_{x=x_0-\frac{\varepsilon}{2}}^{x_0} \frac{2}{\varepsilon}\left(1 - \frac{2|x-x_0|}{\varepsilon}\right) f(x)\, dx + \int_{x=x_0}^{x_0+\frac{\varepsilon}{2}} \frac{2}{\varepsilon}\left(1 - \frac{2|x-x_0|}{\varepsilon}\right) f(x)\, dx = \\ &= \frac{\varepsilon}{2}\frac{2}{\varepsilon}\left(1 - \frac{2}{\varepsilon}\frac{\varepsilon}{4}\right) f\left(x_0 - \frac{\varepsilon}{4}\right)\, dx + \frac{\varepsilon}{2}\frac{2}{\varepsilon}\left(1 - \frac{2}{\varepsilon}\frac{\varepsilon}{4}\right) f\left(x_0 + \frac{\varepsilon}{4}\right)\, dx = \\ &= \frac{1}{2} f\left(x_0 - \frac{\varepsilon}{4}\right) + \frac{1}{2} f\left(x_0 + \frac{\varepsilon}{4}\right)\end{aligned}$$

As $\varepsilon \to 0$

$$\int_{x=-\infty}^{\infty} t_\varepsilon(x - x_0) f(x)\, dx \to f(x_0)$$

Approximation 1. For $\alpha \to +\infty$,

$$\varphi_\alpha(x) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} \sim \delta(x)$$

Fourier transform of $\varphi_\alpha(x)$ reads

$$\begin{aligned}\mathcal{F}\{\varphi_\alpha(x)\}(k) &= \int_{x=-\infty}^{+\infty} \varphi_\alpha(x) e^{-ikx}\, dx = \\ &= \int_{x=-\infty}^{+\infty} \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} e^{-ikx}\, dx = \\ &= \sqrt{\frac{\alpha}{\pi}} \int_{x=-\infty}^{+\infty} e^{-\alpha\left(x + i\frac{k}{2\alpha}\right)^2}\, dx\, e^{-\frac{k^2}{4\alpha}} = \\ &= \sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{\pi}{\alpha}} e^{-\frac{k^2}{4\alpha}} = e^{-\frac{k^2}{4\alpha}}\ ,\end{aligned}$$

for $\alpha \to +\infty$,

$$\mathcal{F}\{\varphi_\alpha(x)\}(k) \to 1$$

and thus $\varphi_\alpha(x) \to \delta(x)$ for $\alpha \to +\infty$.

Approximation 2. For $a \to +\infty$

$$\frac{1}{2\pi} \int_{k=-2\pi a}^{2\pi a} e^{ikx}\, dk = \int_{y=-a}^{+a} e^{i2\pi yx}\, dy \sim \delta(x)$$

or

$$\delta(x) \sim \frac{1}{2\pi}\int_{k=-2\pi a}^{2\pi a} e^{ikx}\,dk = \frac{1}{2\pi}\left(\int_{k=-2\pi a}^{0} e^{ikx}\,dk + \int_{0}^{k=2\pi a} e^{ikx}\,dk\right) = \frac{1}{2\pi}\int_{k=0}^{2\pi a}\left(e^{ikx}+e^{ikx}\right)\,dx = \frac{1}{\pi}\int_{x=0}^{2\pi a}\cos(kx)\,dk$$
$$= \int_{y=-a}^{+a} e^{i2\pi yx}\,dy = \dots = \int_{y=0}^{a}(e^{i2\pi yx}+e^{i2\pi yx})\,dy = 2\int_{y=0}^{a}\cos(2\pi yx)\,dy\ .$$

Approximation 3. For $a \to +\infty$

$$\frac{\sin(2\pi xa)}{\pi x} \sim \delta(x)$$

Directly follows from integral of approximation 2,

$$\int_{y=-a}^{+a} e^{i2\pi yx}\,dy = \frac{1}{i2\pi x}\ e^{i2\pi yx}\Big|_{y=-a}^{+a} = \frac{1}{\pi x}\frac{e^{i2\pi ax}-e^{-i2\pi ax}}{2i} = \frac{\sin(2\pi xa)}{\pi x}$$

Approximation 4. For $x \in [-\pi, \pi]$, and $N \to +\infty$

$$\frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx} = \frac{1}{2\pi}\frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)} \sim \delta(x)$$

Integral $I = \int_{-\infty}^{+\infty} e^{-\alpha x^2}\,dx$

$$\begin{aligned}
I^2 &= \int_{x=-\infty}^{+\infty} e^{-\alpha x^2}\,dx \int_{y=-\infty}^{+\infty} e^{-\alpha y^2}\,dy = \\
&= \int_{x=-\infty}^{+\infty}\int_{y=-\infty}^{+\infty} e^{-\alpha(x^2+y^2)}\,dx\,dy = \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{+\infty} e^{-\alpha r^2}\,r\,dr\,d\theta = \\
&= 2\pi\frac{1}{2\alpha}\int_{r=0}^{+\infty} e^{-\alpha r^2}\,d\left(\alpha r^2\right) = \\
&= \frac{\pi}{\alpha}\left[-e^{\alpha r^2}\right]\Big|_{r=0}^{+\infty} = \frac{\pi}{\alpha}\ .
\end{aligned}$$

Part V

Complex Calculus

CHAPTER

EIGHT

COMPLEX ANALYSIS

8.1 Complex functions, $f : \mathbb{C} \rightarrow \mathbb{C}$

A complex function f of complex variable $z = x + iy$, $f : \mathbb{C} \rightarrow \mathbb{C}$, can be written as

$$f(z) = \tilde{u}(z) + i\tilde{v}(z) = u(x, y) + iv(x, y) \ ,$$

as the sum of its real part $u(z)$ and i times its imaginary part $v(x, y)$. Here $x, y \in \mathbb{R}$, while $\tilde{u}(z), \tilde{v}(z) : \mathbb{C} \rightarrow \mathbb{R}$ and $u(x, y), v(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$. With some abuse of notation, tilde won't be always explicitly written when arguments of real and imaginary parts of f functions won't be written.

8.1.1 Limit

$$\lim_{z \rightarrow z_0} f(z) = f(z_0) \qquad , \qquad \forall \varepsilon > 0 \ \exists \delta > 0 \ \text{ s.t. } \ |f(z) - f(z_0)| < \delta \ \forall z \text{ s.t. } |z - z_0| < \varepsilon, \ z \neq z_0 \ .$$

8.1.2 Derivative

Using the definition of *limit of complex functions*, the derivative of a function $f : \mathbb{C} \rightarrow \mathbb{C}$, if it exists, is the limit of incremental ratio,

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \ .$$

8.1.3 Line Integrals

Given a line $\gamma \in \mathbb{C}$, whose parametric form is $z(s)$, with regular parametrization with parameter $s \in [s_0, s_1]$,

$$\int_\gamma f(z) \, dz = \int_{s=s_0}^{s_1} f(z(s)) \, z'(s) \, ds \ .$$

8.2 Holomorphic Functions - Analytic Functions

Definition 7

A holomorphic function is a function whose *derivative* exists.

Examples of analytic functions. todo…

8.2.1 Cauchy-Riemann conditions

For a holomorphic function $f(z) = u(x, y) + iv(x, y)$, Cauchy-Riemann conditions

$$\begin{cases} u_{/x} = v_{/y} \\ u_{/y} = -v_{/x} \end{cases}$$

hold. The evaluation of the derivative once with $\Delta z = \Delta x$ and once with $\Delta z = i\Delta y$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} =$$
$$= \begin{cases} \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x} = u_{/x} + iv_{/x} \\ \lim\limits_{\Delta y \to 0} \dfrac{f(x, y + \Delta y) - f(x, y)}{i\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y} = -iu_{/y} + v_{/y} \end{cases}$$

provides the proof.

8.2.2 Cauchy Theorem

For a holomorphic function f, $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$

$$\oint_\gamma f(z)\, dz = 0 \,,$$

for $\forall \gamma \subset \Omega$. Proof follows from *Green's lemma*, and *Cauchy-Riemann conditions*

$$\begin{aligned} \oint_\gamma f(z) dz &= \oint_\gamma (u(x, y) + iv(x, y)) (dx + idy) = \\ &= \oint_\gamma (udx - vdy) + i \oint_\gamma (udy + vdx) = \\ &= -\int_S \Bigg(\underbrace{u_{/y} + v_{/x}}_{=0} \Bigg)\, dx\, dy + i \int_S \Bigg(\underbrace{u_{/x} - v_{/y}}_{=0} \Bigg)\, dx\, dy = 0 \,. \end{aligned}$$

8.3 Useful integrals

8.3.1 Independence of line integral for holomorphic functions

For a function $f(z)$ analytic in D, the line integral on paths $\ell_{ab,i}$ with the same extreme points a, b contained in D is independent on the path, but only depends on the extreme points a, b,

$$\int_{\ell_{ab,1}} f(z)\, dz = \int_{\ell_{ab,2}} f(z)\, dz$$

The proof readily follows, using *Cauchy theorem* applied to a function $f(z) : D \subseteq \mathbb{C} \to \mathbb{C}$, analytic in D, and splitting the closed path γ into two paths ℓ_1, ℓ_2 with the same extreme points, $\gamma = \ell_1 \cup (-\ell_2)$

$$0 = \oint_\gamma f(z)\, dz = \int_{\ell_1} f(z)\, dz + \int_{-\ell_2} f(z)\, dz = \int_{\ell_1} f(z)\, dz - \int_{\ell_2} f(z)\, dz\, .$$

8.3.2 Sum and difference of line integrals

8.3.3 Integral of z^n

Given a path γ embracing $z = 0$ only once in counter-clockwise direction, and $n \in \mathbb{Z}$

$$\oint_\gamma z^n\, dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

Since z^n is analytic everywhere (**todo** *prove it! Add a section with proofs for common functions*) except for $z = 0$, it's possible to evaluate the integral on a circle with center $z = 0$ and radius R. Using polar expression of the complex numbers on the circle, $z = Re^{i\theta}$, $\theta \in [0, 2\pi]$, R const, the differential becomes $dz = iRe^{i\theta} d\theta$ and the integral

$$\begin{aligned}
\oint_\gamma z^n\, dz &= \int_{\theta=0}^{2\pi} \left(Re^{i\theta}\right)^n iRe^{i\theta} d\theta = \\
&= i \int_{\theta=0}^{2\pi} R^{n+1} e^{i(n+1)\theta} d\theta = \\
&= \begin{cases} \text{if } n = -1 & : \quad i2\pi \\ \text{otherwise} & : \quad iR^{n+1} \dfrac{1}{i(n+1)}\, e^{i(n+1)\theta}\Big|_{\theta=0}^{2\pi} = \dfrac{R^{n+1}}{n+1}(1-1) = 0 \end{cases}
\end{aligned}$$

8.4 Meromorphic functions

Definition 8

A meromorphic function in a domain is a function holomorphic everywhere except for a (finite?) number of poles. **check**

8.4.1 Singularities

Definition 9 (Pole)

A pole of order n of a function $f(z)$ is a complex number a so that

$$f(z) = \frac{\phi(z)}{(z-a)^n} \ ,$$

with $\phi(z)$ holomorphic in $\phi(a) \neq 0$

Examples. ...

Definition 10 (Branch)

Examples. $f(z) = z^{\frac{1}{2}}$

Definition 11 (Removable singularities)

Example. $f(z) = \frac{\sin z}{z}$

Other irregularities.

8.4.2 Laurent Series

Given a function $f(z)$, in a disk $D_{a,\varepsilon} : 0 < |z-a| < \varepsilon$, its Laurent series centered in a is the convergent (to $f(z)$, **todo** *which type of convergence?*) series

$$f(z) \sim \sum_{n=-\infty}^{+\infty} a_n (z-a)^n \ , \tag{8.1}$$

with

$$a_n = \frac{1}{2\pi i} \int_\gamma f(z)\,(z-a)^{-(n+1)}\,dz \tag{8.2}$$

and γ embracing $z = a$ once counter-clockwise. Proof follows immediately inserting the expressions of the coefficients a_n and using the *integral of z^n*. Evaluating the integral (8.2) of the coefficients of the Laurent series, using (8.1) to replace $f(z)$ with its series

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_\gamma \sum_{m=-\infty}^{+\infty} a_m (z-a)^m (z-a)^{-(n+1)} = \\ &= \frac{1}{2\pi i} \oint_\gamma \sum_{m=-\infty}^{+\infty} a_m (z-a)^{m-n-1}\,dz = \\ &= \frac{1}{2\pi i} \oint_\gamma a_n\, z^{-1}\,dz = \\ &= a_n \ . \end{aligned}$$

todo *Some freestyle with function and its convergent series...add some detail, and the meaning of convergence*

8.4.3 Cauchy formula

For an analytic function $f(z)$,

$$f(a) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}\,dz$$

Proof readily follows using the *integral of z^n* on the Taylor series of $\frac{f(z)}{z-a}$ whose 0^{th} order term reads $f(a)$,

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(a)+\sum_{m=1}^{+\infty} f'(a)(z-a)^m}{z-a}\,dz = \frac{1}{2\pi i}\oint_\gamma \frac{f(a)}{z-a}\,dz = f(a)\frac{2\pi i}{2\pi i} = f(a)\ .$$

8.4.4 Residues

Definition 12 (Residue)

The residue of function f in a, $\text{Res}(f,a)$ is a complex number R so that $f(z) - \frac{R}{(z-a)}$ has analytic antiderivative in a disk $D_{a,\varepsilon}:\ 0<|z-a|<\varepsilon$.

todo Explain this definition. Couldn't be possible to use $\text{Res}(f,a) = \frac{1}{2\pi i}\oint_\gamma f(z)\,dz = a_{-1}$ instead?

Properties.

- If $f(z)$ is analytic in $D_{a,\varepsilon}$ and has a pole of order n in $z=a$, its Laurent series has $a_m = 0$ for $m<n$ and reads

  $$f(z) = \sum_{m=-n}^{+\infty} a_m (z-a)^m\ , \tag{8.3}$$

  with $a_{-n}\neq 0$. Since $f(z)$ has a pole of order n in $z=a$, it can be written as

  $$f(z) = \frac{\phi(z)}{(z-a)^n}\ ,$$

  with $\phi(z)$ analytic in $D_{a,\varepsilon}$ and $\phi(a)\neq 0$. Since $\phi(z)$ is analytic, it has a Taylor series (or a Laurent series with non-negative powers),

  $$\phi(z) \sim \sum_{m=0}^{+\infty} b_m (z-a)^m\ ,$$

  (**todo** *prove it! Extension of the real case. Add a link to the proof*) and thus

  $$f(z) \sim \sum_{m=0}^{+\infty} b_m (z-a)^{m-n} = \sum_{m=-n}^{+\infty} b_{m+n}(z-a)^m = \sum_{m=-n}^{+\infty} a_m (z-a)^m\ ,$$

  with $a_m = b_{m+n}$.

- For simple closed path γ (embracing a only once counter-clokwise) in $D_{a,\varepsilon}$,

  $$\oint_\gamma f(z)\,dz = 2\pi i a_{-1} = 2\pi i\text{Res}(f,a) \tag{8.4}$$

  The proof readily follows, using the *integral of z^n* and Laurent series (8.1) of $f(z)$,

  $$\oint_\gamma f(z)\,dz = \oint_\gamma \sum_{m=-\infty}^{+\infty} a_m (z-a)^m\,dz = 2\pi i a_{-1}\ .$$

- For a pole a of order n, the following holds

$$a_{-1} = \frac{1}{(n+1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]$$

The proof follows using Laurent series {eq}`eq:laurent:pole-n} for a function with pole of order n, and evaluating the $(n-1)^{th}$ order derivative

$$\begin{aligned}
\frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right] & = \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n \sum_{m=-n}^{+\infty} a_n (z-a)^m \right] = \\
& = \frac{d^{n-1}}{dz^{n-1}} \left[\sum_{m=-n}^{+\infty} a_n (z-a)^{m+n} \right] = \\
& = \frac{d^{n-1}}{dz^{n-1}} \left[\sum_{m=0}^{+\infty} a_{m-n} (z-a)^{m} \right] = \\
& = \frac{d^{n-2}}{dz^{n-2}} \left[\sum_{m=0}^{+\infty} m a_{m-n} (z-a)^{m-1} \right] = \\
& = \frac{d^{n-3}}{dz^{n-3}} \left[\sum_{m=0}^{+\infty} m(m-1) a_{m-n} (z-a)^{m-2} \right] = \\
& = \dots = \\
& = \left[\sum_{m=0}^{+\infty} m! \, a_{m-n} (z-a)^{m-n+1} \right]
\end{aligned}$$

and then letting $z \to a$, so that only the term with $m - n + 1 = 0$ survives

$$\lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n \sum_{m=-n}^{+\infty} a_n (z-a)^m \right] = (n-1)! \, a_{-1} \ .$$

8.4.5 Residue Theorem

Theorem 1 (Residue Theorem)

Given $f(z)$ with a finite number of poles $p_n \in D$, then

$$\int_\gamma f(z) \, dz = 2 \pi i \sum_n I(\gamma, p_n) \text{Res}(f, p_n) \ ,$$

being γ a path in D, and $I(\gamma, p_n)$ the winding index of the path γ around pole p_n (+1 for each counter-clockwise loop, -1 for each clockwise loop).

The proof readily follows extending the result for a single pole (8.4) to general number of poles and general paths γ embracing (with sign) each pole p_n $I(\gamma, p_n)$ times, with the same techinques shown in section *Sum and difference of line integrals*.

8.4.6 Evaluation of integrals

8.4.7 Inverse Laplace Transform

Given Laplace transform

$$F(s) := \mathcal{L}\{f(t)\}(s) := \int_{t=0^-}^{+\infty} f(t) e^{-st}\, dt \ ,$$

the inverse transform can be evaluated as

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) := \lim_{T\rightarrow+\infty} \frac{1}{2\pi i} \int_{s=a-iT}^{a+iT} e^{st} F(s)\, ds \ ,$$

with $a > \text{Re}\{p_n\}$ (**todo** *why?*) for each pole of the function $F(s)$, evaluated on the vertical line $s = a + iy$, $y \in [-T, T]$, $ds = i dy$,

$$\begin{aligned}
\lim_{T\rightarrow+\infty} \frac{1}{2\pi i} \int_{s=a-iT}^{a+iT} e^{st} F(s)\, ds &= \lim_{T\rightarrow+\infty} \frac{1}{2\pi i} \int_{s=a-iT}^{a+iT} e^{st} \int_{\tau=0^-}^{+\infty} f(\tau) e^{-s\tau}\, d\tau\, ds = \\
&= \lim_{T\rightarrow+\infty} \frac{1}{2\pi i} \int_{y=-T}^{T} e^{(a+iy)t} \int_{\tau=0^-}^{+\infty} f(\tau) e^{-(a+iy)\tau}\, d\tau\, i dy = \\
&= \lim_{T\rightarrow+\infty} \frac{1}{2\pi} \int_{y=-T}^{T} \int_{\tau=0^-}^{+\infty} e^{iy(t-\tau)} e^{a(t-\tau)} f(\tau)\, d\tau\, dy = \\
&= \dots \\
&= \int_{\tau=0^-}^{+\infty} \delta(t-\tau) e^{a(t-\tau)} f(\tau) d\tau = f(t) \ .
\end{aligned}$$

having used the transform of *Dirac's delta* $\delta(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} e^{-j\omega t}\, d\omega$.

todo *Ohter approach: if* $a >$ *Re*$\{p_n\}$*, the contour built with the vertical line with real part* a *and the arc of circumference on its…*

CHAPTER

NINE

LAPLACE TRANSFORM

$$\mathcal{L}\left\{f(t)\right\}(s) := \int_{t=0^-}^{+\infty} e^{-st} f(t)\, dt = F(s)\ .$$

9.1 Inverse transform

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \dots$$

9.2 Properties

Linearity.

$$\mathcal{L}\{af(t) + bg(t)\}(s) = aF(s) + bG(s)$$

***Dirac delta*.**

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{t=0^-}^{+\infty} \delta(t)\, e^{st}\, dt = 1$$

Time delay. If $f(t) = 0$ for $t < 0$ ("causality"), for $\tau > 0$,

$$\mathcal{L}\{f(t-\tau)\}(s) = e^{-s\tau}F(s)$$

Proof readily follows direct computation with change of variable $z = t - \tau$, $dt = dz$

$$\mathcal{L}\{f(t-\tau)\}(s) = \int_{t=0^-}^{+\infty} f(t-\tau)e^{-st}\, dt = \int_{z=-\tau}^{+\infty} f(z)e^{-sz}\, dz\, e^{-s\tau} = \int_{z=0}^{+\infty} f(z)e^{-sz}\, dz\, e^{-s\tau} = e^{-s\tau}F(s)\ .$$

"Frequency shift"

$$\mathcal{L}\{f(t)e^{at}\}(s) = F(s-a)$$

Direct computation gives

$$\mathcal{L}\{f(t)e^{at}\}(s) = \int_{t=0^-}^{+\infty} f(t)e^{at}e^{-st}\, dt = \int_{t=0^-}^{+\infty} f(t)e^{-(s-a)t}\, dt = F(s-a)$$

Derivative.

$$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0^-)\ .$$

Proof readily follows direct computation, with integration by parts

$$\mathcal{L}\{f'(t)\}(s) = \int_{t=0^-}^{+\infty} f'(t)e^{-st}\,dt = \left[f(t)e^{-st}\right]\Big|_{t=0^-}^{+\infty} + s\int_{t=0^-}^{+\infty} f(t)e^{-st}\,dt = sF(s) - f(0^-)\;,$$

provided that $\lim_{s\rightarrow+\infty} f(t)e^{-st} = 0$.

Integral.

$$\mathcal{L}\left\{\int_{\tau=0}^{t} f(\tau)\,d\tau\right\}(s) = \frac{1}{s}F(s)\;.$$

Proof readily follows direct computation, with integration by parts

$$\mathcal{L}\left\{\int_{\tau=0^-}^{t} f(\tau)\,d\tau\right\}(s) = \int_{t=0^-}^{+\infty}\int_{\tau=0^-}^{t} f(\tau)\,d\tau e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\int_{\tau=0^-}^{t} f(\tau)\,d\tau\right]_{t=0}^{+\infty} + \frac{1}{s}\int_{t=0}^{+\infty} f(t)e^{-st}\,dt = \frac{1}{s}F(s)\;,$$

provided that $\int_{\tau=0^-}^{0} f(\tau)d\tau = 0$ and $\lim_{t\rightarrow+\infty}\frac{e^{-st}}{s}\int_{\tau=0^-}^{+\infty} f(\tau)\,d\tau = 0$.

Initial value. If …

$$f(0^+) = \lim_{s\rightarrow+\infty} sF(s)$$

From direct computation,

$$\begin{aligned}
\lim_{s\rightarrow+\infty} sF(s) &= \lim_{s\rightarrow+\infty} s\int_{t=0^-}^{+\infty} f(t)\,e^{-st}\,dt = \\
&= \lim_{s\rightarrow+\infty}\left\{\left[s\left(-\frac{e^{-st}}{s}\right)f(t)\right]\Bigg|_{t=0}^{+\infty} + \int_{t=0}^{+\infty} e^{-st}f'(t)\,dt\right\} = \\
&= \lim_{s\rightarrow+\infty}\left\{\left[-e^{-st}f(t)\right]\Big|_{t=0}^{+\infty} + \int_{t=0}^{+\infty} e^{-st}f'(t)\,dt\right\} = \\
&= f(0)\;,
\end{aligned}$$

provided that $\lim_{s\rightarrow+\infty}\lim_{t\rightarrow+\infty} e^{-st}f(t) = 0$ and $\lim_{s\rightarrow+\infty}\int_{t=0}^{+\infty} e^{-st}f'(t)\,dt = 0$.

Final value. If …

$$f(+\infty) = \lim_{s\rightarrow0} sF(s)$$

From direct computation (**todo** *check and/or explain proof*),

$$\begin{aligned}
\lim_{s\rightarrow0} sF(s) &= \lim_{s\rightarrow0} s\int_{t=0^-}^{+\infty} f(t)\,e^{-st}\,dt = \\
&= \lim_{s\rightarrow0}\left\{\left[s\left(-\frac{e^{-st}}{s}\right)f(t)\right]\Bigg|_{t=0}^{+\infty} + \int_{t=0}^{+\infty} e^{-st}f'(t)\,dt\right\} = \\
&= \lim_{s\rightarrow0}\left\{\left[-e^{-st}f(t)\right]\Big|_{t=0}^{+\infty} + \int_{t=0}^{+\infty} e^{-st}f'(t)\,dt\right\} = \\
&= f(0) + f(+\infty) - f(0) = f(+\infty)\;,
\end{aligned}$$

provided that $\lim_{s\rightarrow0}\lim_{t\rightarrow+\infty} e^{-st}f(t) = 0$.

CHAPTER

TEN

FOURIER TRANSFORMS

- Fourier series: continuous time, periodic function in time
- Fourier transform: continuous time, non-periodic function in time
- Discrete Fourier transform (DFT):
- Discrete time Fourier transform (DTFT):

10.1 Fourier Series

For a T-periodic function,

$$g(t)$$

10.2 Fourier Transform

$$\mathcal{F}\left\{g(t)\right\}(f) := \int_{t=-\infty}^{+\infty} g(t)\, e^{-i2\pi ft}\, dt.$$

10.2.1 Properties

Linearity.

***Dirac delta*.**

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{t=-\infty}^{+\infty} \delta(t)\, e^{-i2\pi ft}\, dt = 1$$

Time delay.

Derivative.

Integral.

Initial value.

Final value.

10.2.2 Inverse Fourier Transform

$$\mathcal{F}^{-1}\left\{G(f)\right\}(t) := \int_{f=-\infty}^{+\infty} G(f)\, e^{i2\pi f t}\, df.$$

Proof using Dirac's delta expression.

$$\begin{aligned}
\mathcal{F}^{-1}\left\{G(f)\right\}(t) := \int_{f=-\infty}^{+\infty} G(f)\, e^{i2\pi f t}\, df &= \int_{f=-\infty}^{+\infty}\int_{\tau=-\infty}^{+\infty} g(\tau) e^{-i2\pi f\tau}\, e^{i2\pi f t}\, df = \\
&= \int_{f=-\infty}^{+\infty}\int_{\tau=-\infty}^{+\infty} g(\tau) e^{-i2\pi f\tau}\, e^{i2\pi f t}\, df = \\
&= \int_{f=-\infty}^{+\infty}\int_{\tau=-\infty}^{+\infty} g(\tau) e^{i2\pi f(t-\tau)}\, df = \\
&= \int_{\tau=-\infty}^{+\infty} g(\tau)\delta(t-\tau)\, d\tau = g(t)\ .
\end{aligned}$$

Proof. By the *dominated convergence theorem*, it follows that

$$\begin{aligned}
\int_{\mathbb{R}} e^{i2\pi x\xi} F(\xi)\, d\xi &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \underbrace{e^{-\pi\varepsilon^2\xi^2 + i2\pi x\xi}}_{G(\xi;x,\varepsilon)} F(\xi)\, d\xi = \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}} g(y;x,\varepsilon) f(y)\, dy = \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \varphi_\varepsilon(x-y)\, f(y)\, dy = \\
&= \int_{\mathbb{R}} \delta(x-y)\, f(y)\, dy = f(x)
\end{aligned}$$

Lemma 1. The Fourier transform of function $\varphi(t) := e^{-\pi|t|^2}$ reads

$$\begin{aligned}
\mathcal{F}\{\varphi(t)\}(\omega) &= \int_{t=-\infty}^{+\infty} \varphi(t) e^{-i\omega t}\, dt = \\
&= \int_{t=-\infty}^{+\infty} e^{-\pi|t|^2} e^{-i\omega t}\, dt = \\
&= \int_{t=-\infty}^{+\infty} e^{-\pi\left(t^2 + i\frac{\omega}{\pi}t - \frac{\omega^2}{4\pi^2}\right)}\, dt\, e^{-\frac{\omega^2}{4\pi^2}} = \\
&= \int_{t=-\infty}^{+\infty} e^{-\pi\left(t + i\frac{\omega}{2\pi}\right)^2}\, dt\, e^{-\frac{\omega^2}{4\pi}} = \\
&= e^{-\frac{\omega^2}{4\pi}}\ ,
\end{aligned}$$

having evaluated *the integral* $\int_{-\infty}^{+\infty} e^{-\alpha x^2}$ with $\alpha = \pi$. **todo** *justify the result for complex exponential. Use Bromwich contour integrals*

Lemma 2. Fourier transform of $f(\alpha t)$, $\alpha > 0$

$$\mathcal{F}\{f(\alpha t)\}(\omega) = \int_{\mathbb{R}} f(\alpha t) e^{-j\omega t}\, dt = \int_{\tau\in\mathbb{R}} f(\tau) e^{-j\frac{\omega}{\alpha}\tau}\, d\tau \frac{1}{\alpha} = \frac{1}{\alpha} F\left(\frac{\omega}{\alpha}\right)$$

Lemma 3. $\frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right) \to \delta(x)$ for $\varepsilon \to 0$

$$\mathcal{F}\left\{\frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right)\right\}(\omega) = \frac{1}{\varepsilon}\varepsilon e^{-\frac{\omega^2}{4\pi\varepsilon^2}} = e^{-\frac{\omega^2}{4\pi\varepsilon^2}}$$

0. Fourier transform

$$G(f) = \int_{t=-\infty}^{\infty} e^{-i\omega t} g(t)\, dt$$

1.

$$g(t) = e^{i\alpha t}\psi(t)$$

$$\mathcal{F}\{g(t)\}(\omega) = \int_{t=-\infty}^{+\infty} g(t)e^{-i\omega t}\, dt = \int_{t=-\infty}^{+\infty} \psi(t)e^{i\alpha t}e^{-i\omega t}\, dt = \int_{t=-\infty}^{+\infty} \psi(t)e^{-i(\omega-\alpha)t}\, dt = \mathcal{F}\{\psi(t)\}(\omega-\alpha)\,.$$

2.

$$\psi(t) = \phi(\alpha t)$$

$$\mathcal{F}\{\psi(t)\} = \int_{t=-\infty}^{+\infty} \psi(t)e^{-i\omega t}\, dt = \int_{t=-\infty}^{+\infty} \phi(\alpha t)e^{-i\omega t}\, dt = \int_{\tau=-\infty}^{+\infty} \phi(\tau)e^{-i\frac{\omega}{\alpha}\tau}\, \frac{d\tau}{\alpha} = \frac{1}{\alpha}\mathcal{F}\{\phi(t)\}\left(\frac{\omega}{\alpha}\right)\,.$$

3. Fubini's theorem

4.

$$\varphi(t) := e^{-\pi t^2}$$

$$\mathcal{F}\{\varphi(t)\} = \int_{t=-\infty}^{+\infty} \varphi(t)e^{-i\omega t}\, dt = \int_{t=-\infty}^{+\infty} e^{-\pi t^2}e^{-i\omega t}\, dt$$

$$0 = \oint_{\gamma} e^{-\alpha|z|^2}\, dz = \int_{\dots} \dots$$

$$z = Re^{i\theta}, \quad dz = iRe^{i\theta}d\theta$$

$$\int_{C/4} e^{-\alpha|z|^2}\, dz = \int_{\theta=0}^{\frac{\pi}{2}} e^{-\alpha R^2} iRe^{i\theta}d\theta = iRe^{-\alpha R^2}\frac{e^{-i\theta}}{i}\Big|_{\theta=0}^{\frac{\pi}{2}}$$

$$\int_{t=0}^{+\infty} e^{-\pi t^2}e^{-i\omega t}\, dt = \int_{t=0}^{+\infty} e^{-\left(\pi t^2 + i\omega t - \frac{\omega^2}{4\pi}\right)}\, dt\, e^{-\frac{\omega^2}{4\pi}} =$$

$$= \int_{t=0}^{+\infty} e^{-\pi\left(t+i\frac{\omega}{2\pi}\right)^2}\, dt\, e^{-\frac{\omega^2}{4\pi}}$$

5. $\varphi_\varepsilon(t) = \frac{1}{\varepsilon^n}\varphi\left(\frac{t}{\varepsilon}\right)$, $t \in \mathbb{R}^n$, is an approximation of Dirac's delta for $\varepsilon \to 0$, so that

$$\lim_{\varepsilon\to 0}\int_{t=-\infty}^{+\infty} \varphi_\varepsilon(t-\tau)f(t)\, dt = f(\tau)$$

$$\lim_{\varepsilon\to 0}\int_{t=-\infty}^{+\infty} \varphi_\varepsilon(t)\, dt = 1$$

As the Fourier transform $\mathcal{F}\left\{\varphi_\varepsilon(t)\right\}(\omega) \to 1$ for $\varepsilon \to 0$, then $\varphi_\varepsilon(t) \to \delta(t)$.

Part VI

Calculus of Variations

CHAPTER

ELEVEN

INTRODUCTION TO CALCULUS OF VARIATIONS

Given the functional S,

$$S[q(t),t] = \int_{t=t_0}^{t_1} L(\dot{q}(t),\, q(t),\, t)\, dt$$

its variation reads

$$\delta S[q(t),t] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(S[q(t) + \varepsilon w(t),\, t] - S[q(t),\, t] \right)$$

$$\begin{aligned}
\delta S[q(t),t] &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(S[q(t) + \varepsilon w(t),\, t] - S[q(t),\, t] \right) = \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t=t_0}^{t_1} \left(L(\dot{q}(t) + \varepsilon w(t),\, q(t) + \varepsilon w(t),\, t) - L(\dot{q}(t),\, q(t),\, t) \right) \, dt = \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t=t_0}^{t_1} \left\{ L(\dot{q}(t),\, q(t),\, t) + \varepsilon \left[\frac{\partial L}{\partial \dot{q}} \dot{w}(t) + \frac{\partial L}{\partial q} w(t) \right] + o(\varepsilon) - L(\dot{q}(t),\, q(t),\, t) \right\} \, dt = \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t=t_0}^{t_1} \left\{ \varepsilon \left[\frac{\partial L}{\partial \dot{q}} \dot{w}(t) + \frac{\partial L}{\partial q} w(t) \right] + o(\varepsilon) \right\} \, dt = \\
&= \int_{t=t_0}^{t_1} \left\{ \frac{\partial L}{\partial \dot{q}} \dot{w}(t) + \frac{\partial L}{\partial q} w(t) \right\} \, dt = \\
&= \left[w(t) \frac{\partial L}{\partial \dot{q}} \right] \Bigg|_{t=t_0}^{t_1} + \int_{t=t_0}^{t_1} \left\{ -\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial q} \right\} w(t) \, dt \; .
\end{aligned}$$

If $q(t_0)$, $q(t_1)$ are prescribed, then $w(t_0) = w(t_1) = 0$ and thus

$$\delta S[q(t),t] = \int_{t=t_0}^{t_1} \left\{ -\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial q} \right\} \delta q(t) \, dt \; ,$$

having called $w(t) = \delta q(t)$ to stress that is the variation of function $q(t)$.

Stationary conditions, $\delta S = 0$. Stationary condition for $\forall \delta q(t)$ implies Lagrange equation,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \; .$$

11.1 Higher-order derivatives

Method 1. If the Lagrangian function L depends on higher order derivatives,

$$L\left(q^{(n)}(t),\, q^{(n-1)}(t),\, \dots,\, q'(t),\, q(t),\, t\right)$$

it's possible to recast the problem defining the n-dimensional function, $\mathbf{q}(t)$,

$$\mathbf{q}(t) = \left(q^0(t), q^1(t), \dots, q^{n-1}(t)\right) := \left(q(t), q'(t), \dots, q^{(n-1)}(t)\right) \ .$$

With some abuse of notation in L, the functional S can be recast as

$$\begin{aligned} S[q(t), t] & = \int_{t=t_0}^{t_1} L(q^{(n)}(t),\, \dots,\, q(t),\, t)\, dt = \\ & = \int_{t=t_0}^{t_1} L(\dot{\mathbf{q}}(t),\, \mathbf{q}(t),\, t)\, dt \ . \end{aligned}$$

todo *Add constraints on components of* $\mathbf{q}(t)$

Repeating the computation, the variation of the functional reads

$$\delta S[\mathbf{q}(t), t] = \left[\delta \mathbf{q}^T(t) \frac{\partial L}{\partial \dot{\mathbf{q}}}\right]\Bigg|_{t=t_0}^{t_1} + \int_{t=t_0}^{t_1} \delta \mathbf{q}^T(t) \left\{ -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) + \frac{\partial L}{\partial \mathbf{q}} \right\} dt \ .$$

Method 2.

Part VII

Ordinary Differential Equations

CHAPTER

TWELVE

INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

Part VIII

Partial Differential Equations

CHAPTER

THIRTEEN

INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations usually comes from balance equations in **continuum mechanics**. Integral equations are the most general form of these equations, and an equivalent differential problem only exists if the fields involved in the equations are regular enough, for their derivatives to exist - and to apply theorems requiring some regularity of the functions.

Classical numerical methods:

- **FVM**: directly solves the **integral problem**, solving integral balance equations for cells in which the domain is divided
- **FDM**: given the problem in **differential form**, FDM directly approximates space derivatives of the **strong formulation** of the problem
- **FEM**: given the problem in **differential form**, FEM projects the **weak formulation** of the problem on a finite-dimensional space
- **BEM**: *integro-differential equation*, *singularities*,...
- **Spectral methods**,...
- **SEM**,...

13.1 Examples

In Physics:

- Advection equation

$$\partial_t u + \vec{a} \cdot \nabla u = f$$

- Diffusion equation

$$\partial_t u - \nu \nabla^2 u = f$$

- Hyperbolic equation/system of equations

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{F}(\mathbf{u}) = \mathbf{f}$$

- Wave equation

$$\frac{1}{c^2} \partial_{tt} u - \nabla^2 u = f$$

13.2 Balance equations in physics

- Small-strain continuum mechanics

$$\rho \partial_{tt} \vec{s} = \rho_0 \vec{g} + \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{\varepsilon})$$

- Heat conduction
- Fluid dynamics
  - Navier-Stokes for compressible fluids (conservative or convective equations)

$$\{$$

  - Navier-Stokes for incompressible fluids (convective form,…)

$$\begin{cases} \rho \partial_t \vec{u} + \rho(\vec{u} \cdot \nabla)\vec{u} - \mu \nabla^2 \vec{u} + \nabla P = \rho \vec{g} \\ \nabla \cdot \vec{u} = 0 \end{cases}$$

todo

- Different forms of equations may be more or less convenient for different solution approaches
- Most of the physical laws comes from integral balance equation of the form

$$\frac{d}{dt} \int_{V_t} \rho \mathbf{u} = \int_{V_t} \rho \mathbf{r} + \oint_{\partial V_t} \hat{n} \cdot \mathbf{T}(\mathbf{u})$$

  whose local - differential - form (in case of differentiable functions) readily follows from the application of Reynolds' transport theorem and divergence theorem to transform time derivative and boundary terms

$$\begin{aligned} &\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \vec{u}) = \rho \mathbf{r} + \nabla \cdot \mathbf{T}(\mathbf{u}) \\ &\partial_t (\rho \mathbf{u}) + \nabla \cdot \mathbf{F}(\mathbf{u}) = \rho \mathbf{r} \ , \end{aligned}$$

  and the physical meaning of each term is evident and readily expalinable as flux or volume or surface sources.

- Further manipulation/simplification may cover the clear meaning of the terms of the differential equation. As an example, the conservative form of Navier-Stokes equations for incompressible fluids with constant and uniform density read

$$\begin{cases} \partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) = \rho \vec{g} + \nabla \cdot \mathbb{T} \\ \nabla \cdot \vec{u} = 0 \ , \end{cases}$$

  where the stress tensor for a Newtonian fluid reads

$$\begin{aligned} \mathbb{T} &= -p \mathbb{I} + 2 \mu \mathbb{D} + \lambda (\nabla \cdot \vec{u}) \mathbb{I} \\ &= -p \mathbb{I} + \mu (\nabla \vec{u} + \nabla^T \vec{u}) + \lambda (\nabla \cdot \vec{u}) \mathbb{I} \end{aligned}$$

  Using the incompressibility constraint $\nabla \cdot \vec{u}$, and treating the density ρ as a constant and uniform parameter, the convective form of the Navier-Stokes equations reads

$$\begin{cases} \rho \partial_t \vec{u} + \rho (\vec{u} \cdot \nabla) \vec{u} = \rho \vec{g} - \nabla P + 2 \mu \nabla \cdot \mathbb{D} \\ \nabla \cdot \vec{u} = 0 \ . \end{cases}$$

  The divergence of the viscous stress tensor becomes

$$2 \mu \nabla \cdot \mathbb{D} = \mu \nabla \cdot (\nabla \vec{u} + \nabla^T \vec{u}) = \mu \left(\nabla^2 \vec{u} + \nabla \underbrace{(\nabla \cdot \vec{u})}_{=0} \right) = \mu \nabla^2 \vec{u} \ ,$$

so that one of the most common form of incompressible Navier-Stokes equations follows

$$\begin{cases}\rho\partial_t\vec{u}+\rho(\vec{u}\cdot\nabla)\vec{u}-\mu\nabla^2\vec{u}+\nabla P=\rho\vec{g}\\ \nabla\cdot\vec{u}=0\ .\end{cases}$$

It should be evident that in the latter form of Navier-Stokes equations no divergence explicitly appears, so that the right expression of surface source terms can't be found immediately. In momentum equation, surface source terms come from surface stress acting on the boundary of the domain, whose expression reads

$$\begin{aligned}\vec{t}_n &= \hat{n}\cdot\mathbb{T} = \\ &= \hat{n}\cdot(-P\mathbb{I}+2\mu\mathbb{D}) = \\ &= \hat{n}\cdot\left(-P\mathbb{I}+\mu\left(\nabla\vec{u}+\nabla^T\vec{u}\right)\right) = \\ &= -P\hat{n}+\hat{n}\cdot\left(\mu\left(\nabla\vec{u}+\nabla^T\vec{u}\right)\right) =\end{aligned}$$

As an example, in weak formulation of incompressible Navier-Stokes problem the **natural boundary condition** arising in the method depends on the expression of the strong formulation of the NS problem. If one needs to prescribe stress boundary conditions, it could be an idea to start from NS equations w/o extra simplifications.

CHAPTER

FOURTEEN

ELLIPTIC EQUATIONS

14.1 Poisson equation

Given the volume density source $f(\vec{r})$ and the diffusivity $\nu(\vec{r})$, Poisson equation for the scalar field $\phi(\vec{r})$ reads

$$-\nabla \cdot (\nu \nabla \phi) = f \qquad \vec{r} \in V$$

with proper boundary conditions on ∂V. As an example, tipical boundary conditions are:

$$\begin{array}{lll} \phi(\vec{r}) = g(\vec{r}) & \vec{r} \in S_D & \text{esserntial - Dirichlet b.c.} \\ \nu \hat{n} \cdot \nabla \phi(\vec{r}) = h(\vec{r}) & \vec{r} \in S_N & \text{natural - Neumann b.c.} \\ a\phi(\vec{r}) + \nu \hat{n} \cdot \nabla \phi(\vec{r}) = b(\vec{r}) & \vec{r} \in S_R & \text{Robin b.c.} \end{array}$$

14.1.1 Weak formulation

For $\forall w \in ...$ (functional space, recall some results about existence and uniqueness of the solution, Lax-Milgram theorem,…)

$$\begin{aligned} 0 &= \int_V w \,\{\nabla \cdot (\nu \nabla \phi) + f\} = \\ &= \oint_{\partial V} w \hat{n} \cdot (\nu \nabla \phi) + \int_V \{-\nu \nabla \vec{w} \cdot \nabla \phi + wf\} = \end{aligned}$$

Splitting boundary contribution as the sum from single contributions from different regions, and applying boundary conditions, setting $w = 0$ for $\vec{r} \in S_D$ (see the ways to prescribe essential boundary conditions),

$$0 = \int_{S_D} \underbrace{w}_{=0} \hat{n} \cdot (\nu \nabla \phi) + \int_{S_N} w \underbrace{\hat{n} \cdot (\nu \nabla \phi)}_{=h} + \int_{S_R} w \underbrace{\hat{n} \cdot (\nu \nabla \phi)}_{=b - a\phi} + \int_V \{-\nu \nabla \vec{w} \cdot \nabla \phi + wf\} \ .$$

and rearranging the equation separating terms containing unknowns from known contributions,

$$\int_V \nu \nabla w \cdot \nabla \phi + \int_{S_R} w a \phi = \int_V w f + \int_{S_N} w h + \int_{S_R} w b \qquad \forall w \in ... \ ,$$

and $\phi = g$, for $\vec{r} \in S_D$.

Different ways to prescribe essential boundary conditions

Strong formulation.

Using Lagrance multiplier - weak formulation of essential boundary conditions. Adding a the essential boundary condition as a constraint with Lagrange multipliers in the weak formulation of the problem,

$$\cdots + \int_{S_D} w_D(\phi - g) \ ,$$

…

CHAPTER

FIFTEEN

PARABOLIC EQUATIONS

15.1 Heat equation

Heat equation for a scalar field $\phi(\vec{r}, t)$ can be interpreted as the unsteady equation of a *Poisson equation*,

$$\partial_t \phi - \nabla \cdot (\nu \nabla \phi) = f \qquad (\vec{r}, t) \in V \times [0, T] \ ,$$

with proper boundary and initial conditions, $\phi(\vec{r}, 0) = \phi_0(\vec{r})$. Common boundary conditions are the same as the one discussed for Poisson problem.

15.1.1 Weak formulation

For $\forall w \in$ ... (functional space, recall some results about existence and uniqueness of the solution, Lax-Milgram theorem,…)

$$\begin{aligned} 0 & = \int_V w \left\{ -\partial_t \phi + \nabla \cdot (\nu \nabla \phi) + f \right\} = \\ & = \oint_{\partial V} w \hat{n} \cdot (\nu \nabla \phi) + \int_V \left\{ -\partial_t \phi - \nu \nabla \vec{w} \cdot \nabla \phi + w f \right\} = \end{aligned}$$

Splitting boundary contribution as the sum from single contributions from different regions, and applying boundary conditions, setting $w = 0$ for $\vec{r} \in S_D$ (see the ways to prescribe essential boundary conditions),

$$0 = \int_{S_D} \underbrace{w}_{=0} \hat{n} \cdot (\nu \nabla \phi) + \int_{S_N} w \underbrace{\hat{n} \cdot (\nu \nabla \phi)}_{=h} + \int_{S_R} w \underbrace{\hat{n} \cdot (\nu \nabla \phi)}_{=k - \phi} + \int_V \left\{ -\partial_t \phi - \nu \nabla \vec{w} \cdot \nabla \phi + w f \right\} \ .$$

and rearranging the equation separating terms containing unknowns from known contributions,

$$\int_V w \partial_t \phi + \int_V \nu \nabla w \cdot \nabla \phi + \int_{S_R} w \phi = \int_V w f + \int_{S_N} w h + \int_{S_R} w k \qquad \forall w \in \ldots \ ,$$

and $\phi = g$, for $\vec{r} \in S_D$.

CHAPTER

SIXTEEN

HYPERBOLIC PROBLEMS

Hyperbolic problems often come from a small-amplitude linearization, or as the non-diffusion (or inviscid) limit of a more general problem.

As a result of these simplification, these problems may experience **shocks** (i.e. discontinuity in the solution, where the differential equations stop to hold, and integral equations and jump conditions are required). **todo** *classification of discontinuities on the massflow across the surface*

The very nature of these problem also suggest methods for the solution or the analysis of these equations, like **characteristic method**.

16.1 Scalar linear

16.1.1 1-dimensional

$$\partial_t u(x,t) + a \partial_x u(x,t) = f(x,t)$$

Caracteristic method. $U(t) = u(X(t),t)$, with the caracteristic curves $X(t)$ defined as those curves where the PDE becomes a ODE. Evaluating the time derivative of the function $u(X(t),t)$, the hyperbolic equation can be recast as

$$\frac{dU}{dt} + \left[a(X(t),t) - \frac{dX}{dt} \right] \partial_x u = f(X(t),t) \ .$$

The equation of characteristic lines is

$$\frac{dX}{dt} = a(X(t),t) \ ,$$

and the PDE on characteristic line becomes the ODE

$$\frac{dU}{dt}(X(t),t) = f(X(t),t) \ .$$

16.2 Scalar non-linear

16.3 System linear

16.3.1 1-dimensional

$$\mathbf{u}(x,t)$$

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = \mathbf{f}$$

Characteristics. $\mathbf{U}(t) = \mathbf{u}(X(t), t)$

$$\frac{d\mathbf{U}}{dt} - \frac{dX}{dt}\partial_x \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = \mathbf{f}$$

In order to get the equations of characteristic lines where PDE turns into ODEs, the eigenproblem

$$\mathbf{A}\partial_x \mathbf{u} = \frac{dX}{dt}\partial_x \mathbf{u} \ ,$$

holds. This problem has non trivial solution if $\frac{dX}{dt}$ and $\partial_x \mathbf{u}$ are pairs of eigenvalues and (right) eigenvectors of the array $\mathbf{A}$.

Diagonalization.

$$\mathbf{A} = \mathbf{R}\mathbf{\Lambda}\mathbf{L}$$

$$\mathbf{L}\left[\partial_t \mathbf{u} + \mathbf{R}\mathbf{\Lambda}\mathbf{L}\partial_x \mathbf{u}\right] = \mathbf{L}\mathbf{f}$$

Since $\mathbf{L} = \mathbf{R}^{-1}$, and defining $d\mathbf{q} = \mathbf{L}d\mathbf{u}$, it's possible to recast the original problem in diagonal form

$$\partial_t \mathbf{q} + \mathbf{\Lambda}\partial_x \mathbf{q} = \mathbf{L}\mathbf{f}$$

$$\partial_t q_i + \Lambda_i \partial_x q_i = \sum_k R_{ik}\, f_k =: F_i \ .$$

Thus, on the i^{th} family of characteristic lines, $\dfrac{dX}{dt} = \lambda_i$,

$$\frac{dQ_i}{dt} = F_i$$

16.4 System non-linear

16.4.1 1-dimensional space

$$\mathbf{u}(x,t)$$

$$\begin{aligned} &\partial_t \mathbf{u} + \partial_x \mathbf{F}(\mathbf{u}) = \mathbf{f} \qquad && \text{(conservative form)} \\ &\partial_t \mathbf{u} + \partial_{\mathbf{u}} \mathbf{F}(\mathbf{u})\partial_x \mathbf{u} = \mathbf{f} && \text{(convective form)} \end{aligned}$$

16.5 n-dimensional space

$$\mathbf{u}(\vec{r},t)$$

$$\begin{aligned} &\partial_t \mathbf{u} + \nabla\cdot\mathbf{F}(\mathbf{u}) = \mathbf{f} \qquad && \text{(conservative form)} \\ &\partial_t \mathbf{u} + \nabla\mathbf{u}\cdot\partial_{\mathbf{u}} \mathbf{F}(\mathbf{u}) = \mathbf{f} && \text{(convective form)} \end{aligned}$$

Different descritpions of integral problem,

$$\begin{aligned} \frac{d}{dt}\int_V \mathbf{u} + \oint_{\partial V} \hat{n}\cdot\mathbf{F}(\mathbf{u}) &= \int_V \mathbf{f} \qquad && \text{(Eulerian)} \\ \frac{d}{dt}\int_{V_t} \mathbf{u} - \oint_{\partial V_t} \mathbf{u}\vec{u}\cdot\hat{n} + \oint_{\partial V_t} \hat{n}\cdot\mathbf{F}(\mathbf{u}) &= \int_{V_t} \mathbf{f} && \text{(Lagrangian)} \\ \frac{d}{dt}\int_{v_t} \mathbf{u} - \oint_{\partial v_t} \mathbf{u}\vec{u}_b\cdot\hat{n} + \oint_{\partial v_t} \hat{n}\cdot\mathbf{F}(\mathbf{u}) &= \int_{v_t} \mathbf{f} && \text{(arbitrary)} \end{aligned}$$

todo *in coordinates*

$$\begin{aligned} f_i &= \partial_t u_i + \partial_{x_k} F_{ki}(u_l) = \\ &= \partial_t u_i + \partial_{x_k} u_m \partial_{u_m} F_{ki}(u_l) = \end{aligned}$$

Example 29 (P-system in 1-dimensional domain)

$$\begin{cases} \partial_t \rho + u \partial_x \rho + \rho \partial_x u = 0 \\ \rho \partial_t u + \rho u \partial_x u + \partial_x P = 0 \end{cases}$$

with $\partial_x P = a^2 \partial_x \rho$,

Convective form

$$\partial_t \begin{bmatrix} \rho \\ u \end{bmatrix} + \begin{bmatrix} u & \rho \\ \frac{a^2}{\rho} & u \end{bmatrix} \partial_x \begin{bmatrix} \rho \\ u \end{bmatrix} = \underline{0} \, .$$

Conservative form

$$\partial_t \begin{bmatrix} \rho \\ \rho u \end{bmatrix} + \partial_x \begin{bmatrix} \rho u \\ \rho u^2 + \rho a^2 \end{bmatrix} = \underline{0} \, .$$

Spectral decomposition of $\mathbf{A}(\mathbf{u})$ gives

$$\begin{aligned} 0 = \left| \begin{bmatrix} u-s & \rho \\ \frac{a^2}{\rho} & u-s \end{bmatrix} \right| &= (u-s)^2 - a^2 \\ s_{1,2} &= u \mp a \\ \mathbf{R} &= \begin{bmatrix} \rho & \rho \\ a & -a \end{bmatrix} \\ \mathbf{L} &= \frac{1}{2\rho a} \begin{bmatrix} a & \rho \\ a & -\rho \end{bmatrix} \end{aligned}$$

Example 30 (Euler equations in 1-dimensional domain)

Conservative form

$$\partial_t \begin{bmatrix} \rho \\ \rho u \\ \rho e^t \end{bmatrix} + \partial_x \begin{bmatrix} \rho u \\ \rho u^2 + P \\ \rho u h^t \end{bmatrix} = \underline{0} \, ,$$

with $h^t = e^t + \frac{P}{\rho}$ and $e^t = e + \frac{u^2}{2}$, and the pressure field can be written as a function of the other thermodynamic variables. As an example, using conservative variables $(\rho, m, E^t) = (\rho, \rho u, \rho e^t) = \left(\rho, \rho u, \rho \left(e + \frac{u^2}{2}\right)\right)$

$$P(\rho, e) = P\left(\rho, \frac{E^t}{\rho} - \frac{m^2}{2\rho^2}\right) = \Pi\left(\rho, m, E^t\right)$$

so that

$$\begin{aligned} \partial_\rho \Pi &= \partial_\rho P\big|_e + \partial_e P\big|_\rho \left(-\frac{E^t}{\rho^2} + \frac{m^2}{\rho^3}\right) \\ \partial_m \Pi &= \partial_e P\big|_\rho \left(-2\frac{m}{\rho^2}\right) \\ \partial_{E^t} \Pi &= \partial_e P\big|_\rho \left(\frac{1}{\rho}\right) \end{aligned}$$

$$dP = \left(\frac{\partial P}{\partial \rho} \right)_e d\rho + \left(\frac{\partial P}{\partial e} \right)_\rho de$$

while starting from $P(s, \rho)$

$$\begin{aligned} dP &= \left(\frac{\partial P}{\partial \rho} \right)_s d\rho + \left(\frac{\partial P}{\partial s} \right)_\rho ds = \\ &= c^2(\rho, s) d\rho + \left(\frac{\partial P}{\partial s} \right)_\rho ds \,. \end{aligned}$$

$$P(\rho, s(\rho, e))$$

$$dP = \partial_\rho P\big|_s d\rho + \partial_s P\big|_\rho \left(\partial_\rho s\big|_e d\rho + \partial_e s\big|_\rho de \right)$$

and using $ds = \frac{de}{T} - \frac{P}{\rho^2 T} d\rho$,

$$P(\rho, s) = P(\rho, e(\rho, s))$$

$$c^2 = \partial_\rho P\big|_s = \partial_\rho P\big|_e + \partial_e P\big|_\rho \partial_\rho e\big|_s$$

Conservative form in conservative variables.

$$\partial_t \begin{bmatrix} \rho \\ m \\ E^t \end{bmatrix} + \partial_x \begin{bmatrix} m \\ \frac{m^2}{\rho} + \Pi(\rho, m, E^t) \\ \frac{m}{\rho} \left(E^t + \Pi(\rho, m, E^t) \right) \end{bmatrix} = \underline{0} \,,$$

Convective form in conservative variables.

$$\partial_t \begin{bmatrix} \rho \\ m \\ E^t \end{bmatrix} + \partial_x \begin{bmatrix} 0 & 1 & 0 \\ -\frac{m^2}{\rho^2} + \partial_\rho \Pi & \frac{2m}{\rho} + \partial_m \Pi & \partial_{E^t} \Pi \\ -\frac{m}{\rho^2}(E^t + \Pi) + \frac{m}{\rho} \partial_\rho \Pi & \frac{1}{\rho}(E^t + \Pi) + \frac{m}{\rho} \partial_m \Pi & \frac{m}{\rho} \left(1 + \partial_{E^t} \Pi \right) \end{bmatrix} \partial_x \begin{bmatrix} \rho \\ m \\ E^t \end{bmatrix} = \underline{0} \,,$$

Spectral decomposition of $\mathbf{A}(\mathbf{u})$

$$\begin{aligned} 0 &= \left| \begin{bmatrix} -s & 1 & 0 \\ -u^2 + \partial_\rho \Pi & 2u + \partial_m \Pi - s & \partial_{E^t} \Pi \\ -u \left(e^t + \frac{P}{\rho} \right) + u \partial_\rho \Pi & e^t + \frac{P}{\rho} + u \partial_m \Pi & u \left(1 + \partial_{E^t} \Pi \right) - s \end{bmatrix} \right| = \\ &= -s \left[\left(2u + \partial_m \Pi - s \right) \left(u \left(1 + \partial_{E^t} \Pi \right) - s \right) - \partial_{E^t} \Pi \left(h^t + u \partial_m \Pi \right) \right] + \\ &\quad - u h^t \partial_{E^t} \Pi + u \partial_\rho \Pi \partial_{E^t} \Pi + \\ &\quad + \left(u^2 - \partial_\rho \Pi \right) \left(u (1 + \partial_{E^t} \Pi) - s \right) \end{aligned}$$

CHAPTER

SEVENTEEN

NAVIER-CAUCHY EQUATIONS

Navier-Cauchy equations are the differential balance equation of the momentum of an elastic isotropic medium in the regime of small strain and displacement,

$$\rho_0 \partial_{tt} \vec{s} = \rho_0 \vec{g} + \nabla \cdot \boldsymbol{\sigma} \ .$$

Stress tensor for an isotropic medium reads

$$\begin{aligned}\boldsymbol{\sigma} &= 2\mu\boldsymbol{\varepsilon} + \lambda \operatorname{tr}(\boldsymbol{\varepsilon})\,\mathbb{I} = \\ &= \left(2\mu\boldsymbol{\varepsilon} - \frac{2}{3}\mu \operatorname{tr}(\boldsymbol{\varepsilon})\mathbb{I}\right) + \left(\lambda + \frac{2}{3}\mu\right) \operatorname{tr}(\boldsymbol{\varepsilon})\,\mathbb{I} \ ,\end{aligned}$$

with the small strain tensor

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\vec{s} + \nabla^T\vec{s}\right) \ .$$

Essential, natural and Robin boundary conditions read

$$\begin{aligned} &\vec{s} = \overline{\vec{s}} && \vec{r} \in S_D && \text{esserntial - Dirichlet b.c.} \\ &\hat{n}\cdot\boldsymbol{\sigma} = \overline{\vec{t}}_n && \vec{r} \in S_N && \text{natural - Neumann b.c.} \\ &a\vec{s} + \hat{n}\cdot\boldsymbol{\sigma} = \vec{b} && \vec{r} \in S_R && \text{Robin b.c.} \end{aligned}$$

17.1 Weak formulation

For $\forall \vec{w} \in$ ...

$$\begin{aligned} 0 &= -\int_V \rho\vec{w}\cdot\partial_{tt}\vec{s} + \int_V \rho_0\vec{w}\cdot\vec{g} + \int_V \vec{w}\cdot\nabla\cdot\boldsymbol{\sigma} = \\ &= -\int_V \rho\vec{w}\cdot\partial_{tt}\vec{s} + \int_V \rho_0\vec{w}\cdot\vec{g} + \int_{\partial V} \hat{n}\cdot\boldsymbol{\sigma}\cdot\vec{w} - \int_V \nabla\vec{w} : \boldsymbol{\sigma} \end{aligned}$$

The volume integral containing the stress tensor can be written either as

$$\begin{aligned} \int_V \nabla\vec{w} : \boldsymbol{\sigma} &= \int_V w_{i/j}\left[\mu\left(s_{i/j} + s_{j/i}\right) + \lambda s_{k/k}\delta_{ij}\right] = \\ &= \int_V \mu w_{i/j}\left(s_{i/j} + s_{j/i}\right) + \int_V \lambda w_{j/j} s_{k/k} \end{aligned}$$

or

$$\begin{aligned} \int_V \frac{1}{2}\left(\nabla\vec{w} + \nabla^T\vec{w}\right) : \boldsymbol{\sigma} &= \int_V \frac{1}{2}\left(w_{i/j} + w_{j/i}\right)\left[\mu\left(s_{i/j} + s_{j/i}\right) + \lambda s_{k/k}\delta_{ij}\right] = \\ &= \int_V \frac{\mu}{2}\left(w_{i/j} + w_{j/i}\right)\left(s_{i/j} + s_{j/i}\right) + \int_V \lambda w_{j/j} s_{k/k} \end{aligned}$$

The weak formulation of the Navier-Cauchy equations reads

$$\int_V \rho_0 \vec{w} \cdot \partial_{tt} \vec{s} + \int_V 2\mu \frac{\nabla \vec{w} + \nabla^T \vec{w}}{2} : \frac{\nabla \vec{s} + \nabla^T \vec{s}}{2} + \int_V \lambda \nabla \cdot \vec{w} \, \nabla \cdot \vec{s} + \int_{S_R} \vec{w} \cdot a \vec{s} = \int_V \rho_0 \vec{w} \cdot \vec{g} + \int_{S_N} \vec{w} \cdot \bar{\vec{t}}_n + \int_{S_R} \vec{w} \cdot \vec{b} \,,$$

for $\forall \vec{w} \in$ ..., and with $\vec{s} = \bar{\vec{s}}$ for $\vec{r} \in S_D$.

CHAPTER

EIGHTEEN

NAVIER-STOKES EQUATIONS

Incompressible Navier-Stokes equations read

$$\begin{cases}\rho\partial_t\vec{u} + \rho\left(\vec{u}\cdot\nabla\right)\vec{u} - \mu\nabla^2\vec{u} + \nabla P = \rho\vec{g} \\ \nabla\cdot\vec{u} = 0\ .\end{cases}$$

Mass balance equation is replaced by the incompressiblity kinematic constraint, $\nabla\cdot\vec{u} = 0$: this constraint is not dynamic, as time derivative of density does not appear in the equation. With the incompressibility constraint, mass equation tells us that material particles keep their density constant,

$$0 = \underbrace{\partial_t\rho + \vec{u}\cdot\nabla\rho}_{=\frac{D\bar{\rho}}{Dt}} + \rho\underbrace{\nabla\cdot\vec{u}}_{=0} = \frac{D\rho}{Dt}\ ,$$

whose solution can be written using material coordinates $\vec{r}_0$ as $\rho(\vec{r}(\vec{r}_0,t),t) = \rho_0(\vec{r}_0,t)$.

18.1 Incompressibility constraint

Incompressibility constraint makes thermodynamic fade, while pressure field is replaced by/contains the contribution of a Lagrangian multiplier related to the incompressiblity constraint.

18.1.1 Wave-vector transformed space

Transforming the fields from physical space to the wave-vector space $\tilde{u}(\vec{\kappa},t) = \mathscr{F}\left\{\vec{u}(\vec{r},t)\right\}$, Navier-Stokes equations for incompressible fluids with uniform and constant density $\rho(\vec{r},t) = \rho$ becomes

$$\begin{cases}\rho\partial_t\tilde{u} + \mathscr{F}\left\{\left(\vec{u}\cdot\nabla\right)\vec{u}\right\} + \mu|\vec{\kappa}|^2\tilde{u} + i\vec{k}\widetilde{P} = \rho\tilde{g} \\ i\vec{\kappa}\cdot\tilde{u} = 0\ .\end{cases}$$

Taking the divergence of the momentum balance equation, i.e. taking the scalar product with $i\vec{\kappa}$ in the transformed space, and using the incompressibility constraint to set $i\vec{\kappa}\cdot\tilde{u} = 0$,

$$i\vec{\kappa}\cdot\mathscr{F}\left\{(\vec{u}\cdot\nabla)\vec{u}\right\} - |\vec{\kappa}|^2\widetilde{P} = i\vec{\kappa}\cdot\rho\tilde{g}\ ,$$

so that the transformed pressure field becomes

$$\widetilde{P} = \frac{i\vec{\kappa}}{|\vec{\kappa}|^2}\cdot\mathscr{F}\left\{\left(\vec{u}\cdot\nabla\right)\vec{u} - \rho\vec{g}\right\}\ ,$$

Replacing this expression in the transformed Navier-Stokes equations, the meaning of the pressure field as a Lagrange multiplier associated with incompressibility constraint becomes clear,

$$\rho\partial_t\tilde{u} + \mu|\vec{\kappa}|^2\tilde{u} = \left[1 - \frac{\vec{\kappa}\vec{\kappa}}{|\vec{\kappa}|^2}\right]\cdot\mathscr{F}\left\{-(\vec{u}\cdot\nabla)\vec{u} + \rho\vec{g}\right\}$$

as the orthogonal projector $[1 - \frac{\vec{\kappa}\vec{\kappa}}{|\vec{\kappa}|^2}]$ onto the space of divergence-free functions acts on the non-linear and forcing terms.

18.2 Weak formulation of the problem

$$\begin{aligned}0 &= \int_V \vec{w}\cdot\left[\rho\partial_t\vec{u} + \rho(\vec{u}\cdot\nabla)\vec{u} - 2\mu\nabla\cdot\mathbb{D}(\vec{u}) + \nabla P - \rho\vec{g}\right] - \int_V v\nabla\cdot\vec{u} = \\ &= \int_V \vec{w}\cdot\left[\rho\partial_t\vec{u} + \rho(\vec{u}\cdot\nabla)\vec{u}\right] + \int_V 2\mu\nabla\vec{w} : \mathbb{D} - \int_V \nabla\cdot\vec{w}P - \int_V \vec{w}\cdot\rho\vec{g} - \int_V v\nabla\cdot\vec{u} - \int_{\partial V}\hat{n}\cdot(\mathbb{S} - P\mathbb{I})\cdot\vec{w}\,,\end{aligned}$$

18.2.1 Weak formulation and incompressibility constraint

$$\begin{aligned}\vec{r}(\vec{r}_0, t) &= \vec{r}(q(t), t) \\ \vec{u} &= \frac{D\vec{r}}{Dt} = \dot{q}\frac{\partial\vec{r}}{\partial q} + \frac{\partial\vec{r}}{\partial t}\end{aligned}$$

In the weak formulation, using $\vec{w} = \frac{\partial\vec{r}}{\partial q} = \frac{\partial\vec{u}}{\partial\dot{q}}$

$$0 = \int_V \vec{w}\cdot\rho\frac{D\vec{u}}{Dt} + \int_V 2\mu\nabla\vec{w} : \mathbb{D} - \int_V \nabla\cdot\vec{w}P - \int_V \rho\vec{w}\cdot\vec{g} - \int_V v\nabla\cdot\vec{u} - \int_{\partial V}\vec{t}_{\hat{n}}\cdot\vec{w}\,,$$

$$\begin{aligned}\int_V \vec{w}\cdot\rho\frac{D\vec{u}}{Dt}\,dV &= \int_{V_0}\rho_0\frac{\partial\vec{u}}{\partial\dot{q}}\cdot\frac{D\vec{u}}{Dt} = \\ &= \int_{V_0}\rho_0\frac{D}{Dt}\left(\frac{\partial\vec{u}}{\partial\dot{q}}\cdot\vec{u}\right)dV_0 - \int_{V_0}\rho_0\frac{D}{Dt}\left(\frac{\partial\vec{r}}{\partial\dot{q}}\right)\cdot\vec{u}\,dV_0 = \\ &= \int_{V_0}\rho_0\frac{D}{Dt}\left(\frac{\partial}{\partial\dot{q}}\frac{|\vec{u}|^2}{2}\right)dV_0 - \int_{V_0}\rho_0\frac{\partial}{\partial q}\frac{|\vec{u}|^2}{2}\,dV_0 =\end{aligned}$$

…

18.3 Non-linear term

Different ways to treat the non-linear term:

- Semi-linear approximation of the non-linear term

$$(\vec{u}(\vec{r}, t^n)\cdot\nabla)\vec{u}(\vec{r}, t^n) \sim (\vec{u}^*(\vec{r}, t^n)\cdot\nabla)\vec{u}(\vec{r}, t^n)\,,$$

  with $\vec{u}^*(\vec{r}, t^n)$ an approximation of $\vec{u}(\vec{r}, t^n)$ involving values of the velocity field at previous time-steps, as an example

$$\vec{u}^*(\vec{r}, t^n) = \begin{cases}\vec{u}(\vec{r}, t^{n-1}) & 1^{st}\text{-order} \\ 2\vec{u}(\vec{r}, t^{n-1}) - \vec{u}(\vec{r}, t^{n-2}) & 2^{nd}\text{-order}\end{cases}$$

CHAPTER

NINETEEN

ARBITRARY LAGRANGIAN-EULERIAN DESCRIPTION

Reynold's transport theorem allows for the formulation of intergal equations, and grid-based methods like FVM, on moving grids and changing domains. Rules for derivatives of composite functions provide the relations between time derivatives in a Lagrangian, Eulerian, or arbitrary description,

$$\begin{aligned}
\left.\frac{\partial f}{\partial t}\right|_{\vec{r}_0} &= \left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + \vec{u}\cdot\nabla f \\
\left.\frac{\partial f}{\partial t}\right|_{\vec{r}_b} &= \left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + \vec{u}_b\cdot\nabla f
\end{aligned}$$

Equations governing the motion of the grid are usually required as well. E.g.:

- known and prescribed motion of the grid;
- boundary conditions only without changing grids (for small displacements)
- pseudo-elastic deformation (usually good for small strain and displacement;
- for large displacements of/or models with complex geometry, sliding and/or overlapping grids could an option for grid-based methods.

19.1 Integral problem

Application of Reynolds theorem to the balance equation of the quantity **u** for a material volume V_t

$$\frac{d}{dt}\int_{V_t}\rho\mathbf{u} = \int_{V_t}\rho\mathbf{f} + \oint_{\partial V_t}\hat{n}\cdot\mathbf{T}\,.$$

provides the expression of the balance equation for a geometrical volume v_t in arbitrary motion,

$$\frac{d}{dt}\int_{v_t}\rho\mathbf{u} + \oint_{\partial v_t}\rho\mathbf{u}\,(\vec{u}-\vec{u}_b)\cdot\hat{n} = \int_{v_t}\rho\mathbf{f} + \oint_{\partial v_t}\hat{n}\cdot\mathbf{T}\,.$$

Here, the integral forulation of the problem will be applied to each element of the grid in arbitrary motion, for domains with variable geometry.

19.2 Differential problem

Rules for derivatives of composite functions allows to write the differential w.r.t. the variables associated with the points of a moving grid. A balance equation in convective form can be written as

$$\begin{aligned}\rho \frac{D\mathbf{u}}{Dt} &= \rho \mathbf{f} + \nabla \cdot \mathbf{T} \\ \rho \left[\frac{\partial \mathbf{u}}{\partial t} + \vec{u} \cdot \nabla \mathbf{u} \right] &= \\ \rho \left[\left. \frac{\partial \mathbf{u}}{\partial t} \right|_{\vec{r}_b} + (\vec{u} - \vec{u}_b) \cdot \nabla \mathbf{u} \right] &= \end{aligned}$$

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