continuum mechanics

basics

Dec 05, 2025

CONTENTS

This material is part of the **basics-books project**. It is also available as a .pdf document.

work-in-progress! Feel free to reach out on Github, leave a comment, a suggestion, a request or contribute

General approach and equations in continuum mechanics are first presented, and then specialized to the most common models of solids - mainly elastic solids - and fluids - mainly Newtonian fluids.

Introduction to Continuum Mechanics

Kinematics of continuum media. Lagrangian, Eulerian and arbitrary descriptions of the motion of continuous media is presented, and kinematic quantities are introduced.

Balance equations of physical quantities. Balance equations of physical quantities are introduced here for continuous media, both in integral and differential forms - in regular domains with "smooth" distribution of physical properties. Reynolds theorem and derivatives of composite functions are exploited to provide Lagrangian, Eulerian and arbitrary descriptions - and their relationship - both for integral and differential equations respectively.

First Lavoisier principle for mass conservation, Newton principles and equations of motion for momentum and angular momentu balance equation, and first principle of thermodynamics or balance equation of total energy are written for closed systems - and derived for arbitrary systems.

The need for constitutive equations and state equations is discussed. Properties of stress tensors and heat conduction flux are described.

Then, balance equations for other physical quantities are derived, e.g. for kinetic energy, internal energy, and entropy. Balance equation of entropy and second principle of thermodynamics prescribe some constraints on stress tensor and heat conduction flux.

Solid Mechanics

Fluid Mechanics

Part I

Continuum Mechanics

CHAPTER ONE

KINEMATICS

Let

- $\vec{r}$ the physical space coordinates
- $\vec{r}_0$ the material coordinates, labels associated to material points of the continuum
- $\vec{r}_b$ arbitrary coordinates, labels associated to arbitrary points - e.g. geometric points

1.1 Material points in physical space

Position. The position in physical space of material points labeled with material coordinates $\vec{r}_0$ can be written as a function

$$\vec{r}(\vec{r}_0, t) \ , \tag{1.1}$$

providing the position in physical space of a material point, as a function of its label $\vec{r}_0$ and time t.

Velocity. The velocity of each material point is the time-derivative of function (1.1) at constant $\vec{r}_0$ (since one is interested here in the velocity of material points),

$$\vec{u} = \left.\frac{\partial \vec{r}}{\partial t}\right|_{\vec{r}_0} =: \frac{D\vec{r}}{Dt} \ , \tag{1.2}$$

having introduced the definition of **material derivative**, $\dfrac{D}{Dt} := \left.\dfrac{\partial}{\partial t}\right|_{\vec{r}_0}$.

Independent variables

In formula (1.2), independent variables are not explicitly written. If $\vec{r}\,(\vec{r}_0, t)$, the velocity field $\vec{u}$ can be readily written as functions of the same independent variables,

$$\vec{u}_0\,(\vec{r}_0, t) = \left.\frac{\partial \vec{r}}{\partial t}\right|_{\vec{r}_0} (\vec{r}_0, t) \ ,$$

and it provides the velocity field as a function of the material coordinates, namely the **Lagrangian description**, following material points in their evolution in space.

Eulerian description of the problem requires physical properties to be written as functions of physical coordinates, $\vec{r}$, t. If the inverse transformation of (1.1) exists, it's possible to write $\vec{r}_0\,(\vec{r}, t)$, and the velocity field as

$$\vec{u}\,(\vec{r}, t) = \vec{u}_0\,(\vec{r}_0\,(\vec{r}, t)\,, t) \ \ ,$$

or, for invertible transformations,

$$\vec{u}_0\,(\vec{r}_0, t) = \vec{u}\,(\vec{r}\,(\vec{r}_0, t)\,, t) \ \ ,$$

having used indices to mathematcally discern functions of different independent variables, even if they represent the same phsyical quantity. In many situations, this inverse transformation between the position in physical space and the material coordinates is not well-defined, often for fluid systems or solid mechanics with (very) large deformations: in these cases, it's always (?) possible to update the reference configuration at some closer time instant in order to find a well-defined inverse transformation, if needed.

Acceleration. Acceleration of a material point labeled with material coordinate $\vec{r}_0$ is the second order derivative of the physical position (1.1) w.r.t. time t keeping $\vec{r}_0$ constant, or the first order derivative of the velocity (1.2),

$$\vec{a} = \left.\frac{\partial \vec{u}}{\partial t}\right|_{\vec{r}_0} = \left.\frac{\partial \vec{r}}{\partial t}\right|_{\vec{r}_0} \cdot \left.\frac{\partial \vec{u}}{\partial \vec{r}}\right|_t + \left.\frac{\partial \vec{u}}{\partial t}\right|_{\vec{r}} = \vec{u} \cdot \nabla \vec{u} + \frac{\partial \vec{u}}{\partial t} \ ,$$

having written the partial derivative in time at constant physical coordianate $\vec{r}$ as $\left.\frac{\partial}{\partial t}\right|_{\vec{r}} = \frac{\partial}{\partial t}$, and the gradient w.r.t. the physical coordinate as $\nabla_{\vec{r}} = \nabla$.

1.2 Arbitrary points in physical space

Following the same process as the one used for *material points*, the position, the velocity and the acceleration of a set of arbitrary points labeled with $\vec{r}_b$ coordinates read

$$\begin{aligned}
& \vec{r}\left(\vec{r}_b, t\right) \\
\vec{u}_b &= \left.\frac{\partial \vec{r}}{\partial t}\right|_{\vec{r}_b} \\
\vec{a}_b &= \left.\frac{\partial \vec{u}_b}{\partial t}\right|_{\vec{r}_b} = \frac{\partial \vec{u}_b}{\partial t} + \vec{u}_b \cdot \nabla \vec{u}_b
\end{aligned}$$

1.3 Time derivatives of a function from different descriptions

Coordinate transformations implies the rules to compute the relations between time derivatives of a field f keeping physical, material or arbitrary coordinates constant, namely

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_{\vec{r}_0} f(\vec{r}(\vec{r}_0, t), t) &= \left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + \left.\frac{\partial \vec{r}}{\partial t}\right|_{\vec{r}_0} \cdot \left.\frac{\partial f}{\partial \vec{r}}\right|_t = \left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + \vec{u} \cdot \nabla f \\
\left.\frac{\partial}{\partial t}\right|_{\vec{r}_b} f(\vec{r}(\vec{r}_b, t), t) &= \left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + \left.\frac{\partial \vec{r}}{\partial t}\right|_{\vec{r}_b} \cdot \left.\frac{\partial f}{\partial \vec{r}}\right|_t = \left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + \vec{u}_b \cdot \nabla f
\end{aligned}$$

and thus

$$\frac{Df}{Dt} = \left.\frac{\partial}{\partial t}\right|_{\vec{r}_0} f = \left.\frac{\partial}{\partial t}\right|_{\vec{r}} f + \vec{u} \cdot \nabla f = \left.\frac{\partial}{\partial t}\right|_{\vec{r}_b} f + (\vec{u} - \vec{u}_b) \cdot \nabla f \ . \qquad (1.3)$$

1.4 Kinematics of two points

$$\vec{r}_2(t) - \vec{r}_1(t) = \vec{r}\left(\vec{r}_{0,2}, t\right) - \vec{r}\left(\vec{r}_{0,1}, t\right)$$
$$\vec{v}_2(t) - \vec{v}_1(t) = \frac{d}{dt}\vec{r}_2(t) - \frac{d}{dt}\vec{r}_1(t)$$

strain velocity tensor

$$\mathbb{D} = \frac{1}{2}\left[\nabla\vec{u} + \nabla^T\vec{u}\right] \tag{1.4}$$

1.5 Kinematics in reference space

Let $\vec{r}(\vec{r}_0, t)$, it's differential - keeping t constant - reads

$$d\vec{r} = d\vec{r}_0 \cdot \nabla_0\vec{r} = d\vec{r}_0 \cdot \mathbb{F}$$

or using a Cartesian base in the reference space

$$d\vec{r} = \hat{e}_i^0 dx_k^0 \frac{\partial x_i}{\partial x_k^0} = \hat{e}_i^0\, dx_k^0\, F_{ki} = \hat{e}_i^0\, dx_i\ .$$

$$|d\vec{r}|^2 = d\vec{r}\cdot d\vec{r} = d\vec{r}_0 \cdot \mathbb{F}\cdot\mathbb{F}^T \cdot d\vec{r}_0$$
$$|d\vec{r}|^2 - |d\vec{r}_0|^2 = d\vec{r}_0 \cdot \left[\mathbb{F}\cdot\mathbb{F}^T - \mathbb{I}\right]\cdot d\vec{r}_0$$

1.5.1 Strain

Green-Lagrange tensor

Green-Lagrange strain tensor is defined as

$$\epsilon := \frac{1}{2}\left[\mathbb{F}\cdot\mathbb{F}^T - \mathbb{I}\right] \tag{1.5}$$

or in Cartesian coordinates in the reference space

$$\epsilon_{ij} = \frac{1}{2}\left[F_{ik}F_{jk} - \delta_{ij}\right] = \frac{1}{2}\left[\frac{\partial x_i}{\partial x_k^0}\frac{\partial x_j}{\partial x_k^0} - \delta_{ij}\right]\ .$$

Its (material) time derivative reads (**todo** *pay attention to vector basis in reference and physical space. Can they be compared/confused?*)

$$\begin{aligned}
\frac{D\epsilon_{ij}}{Dt} &= \frac{1}{2}\left[\frac{D}{Dt}\frac{\partial x_k}{\partial x_i^0}\frac{\partial x_k}{\partial x_j^0} + \frac{\partial x_k}{\partial x_i^0}\frac{D}{Dt}\frac{\partial x_k}{\partial x_j^0}\right] = \\
&= \frac{1}{2}\left[\frac{\partial v_k}{\partial x_i^0}\frac{\partial x_k}{\partial x_j^0} + \frac{\partial x_k}{\partial x_i^0}\frac{\partial v_k}{\partial x_j^0}\right] = \\
&= \frac{1}{2}\left[\frac{\partial x_l}{\partial x_i^0}\frac{\partial v_k}{\partial x_l}\frac{\partial x_k}{\partial x_j^0} + \frac{\partial x_k}{\partial x_i^0}\frac{\partial x_l}{\partial x_j^0}\frac{\partial v_k}{\partial x^l}\right] \\
&= \frac{\partial x_l}{\partial x_i^0}\left[\frac{1}{2}\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x^k}\right)\right]\frac{\partial x_k}{\partial x_j^0} = \\
&= F_{il}\, D_{lk}\, F_{jk}\ .
\end{aligned}$$

$$
\begin{aligned}
\frac{D\epsilon}{Dt} &= \frac{1}{2}\left[\frac{D\mathbb{F}}{Dt}\cdot\mathbb{F}^T + \mathbb{F}\cdot\frac{D\mathbb{F}^T}{Dt}\right] = \\
&= \frac{1}{2}\left[\,\nabla_0\vec{v}\cdot\mathbb{F}^T + \mathbb{F}\cdot\nabla_0^T\vec{v}\,\right] = \\
&= \frac{1}{2}\left[\,\mathbb{F}\cdot\nabla\vec{v}\cdot\mathbb{F}^T + \mathbb{F}\cdot\nabla^T\vec{v}\cdot\mathbb{F}^T\,\right] = \\
&= \mathbb{F}\cdot\left[\frac{1}{2}\left(\,\nabla\vec{v} + \nabla^T\vec{v}\,\right)\right]\cdot\mathbb{F}^T = \\
&= \mathbb{F}\cdot\mathbb{D}\cdot\mathbb{F}^T\,.
\end{aligned}
\tag{1.6}
$$

CHAPTER TWO

GOVERNING EQUATIONS

The following process is detailed in the following sections

Integral balance equations for primary physical quantities. First, integral balance equations for closed systems are written as a manifestation of principles of classical mechanics for closed systems, namely mass conservation, second principle of mehcanics, and first principle of thermodynamics. Starting from integral balance equations for closed systems (material systems, Lagrange description), Reynolds transport theorem is used to derive integral balance equations for open systems, either stationary in space (control volume, Eulerian description) or with arbitrary motion (arbitrary description).

Differential balance equations for primary physical quantities. Starting from integral balance equations, under the assumption of sufficient regularity of the physical quantities, divergence theorem and arbitrariety of the domain is used to derive differential (local) balance equations of primary physical quantities.

Differential balance equations for derived physical quantities. Starting from differential equations of primary physical quantities, differential balance equations are derived for other physical quantities, as an example kinetic energy, internal energy and entropy.

Integral balance equations for derived physical quantities. Starting from differential balance equations, and exploiting divergence theorem (in the "opposite direction" w.r.t. what has been done before, to get differential from integral equations), integral balance equations are derived for derived quantities.

2.1 Integral Balance Equations of aaa physical quantities

Classical physics relies on a small set of principles, usually formulated for closed systems.

- classical physics and chemistry rely on Lavoisier principle, or mass conservation in closed systems
- classical (Newton) mechanics is built on 3 principles:
  - 1^{st} principle, or principle of inertia, dealing with the invariance of classical physics w.r.t. Galileian transformations
  - 2^{nd} principle, or balance of momentum
  - 3^{rd} principle, or action/reaction principle
- classical thermodynamics:
  - 1^{st} principle, or balance of total energy
  - 2^{nd} principle, describing irreversibility or natural tendencies in physical processes - positive dissipation of mechanical (macroscopic) energy and heat transfer "from hot to cold bodies" - in terms of entropy
  - 3^{rd} principle, relating energy, entropy and thermodynamic temperature as positive physical quantity (it sets an absolute zero of the thermodynamic temperature, in the thermodynamic scale of temperature - Kelvin K)
- classical electromagnetism:

- Electric charge conservation
- Maxwell's equations, relating electromagnetic field with charges and currents
- Lorentz's force, acting on charges in an electromagnetic field

Here, electromagnetic processes are not investigated. Dynamical equations for angular momentum and kinetic energy derived in classical mechanics are discussed later: integral balance equation of angular momentum relates changes of angular momentum of the system with external moments acting on it; differential balance equation of angular momentum reduces to the an identity - and thus it adds no information - for **non-polar media**; kinetic energy integral balance relates changes of kinetic energy of the system with the total mechanical power acting on the system, and it can be substracted from total energy to get internal energy of the system.

2.1.1 Principles of classical mechanics for closed systems - Lagrangian description

Mass balance equation: Lavoisier principle.

$$\frac{d}{dt}\int_{V_t} \rho = 0 \ .$$

Momentum balance equation: 2^{nd} **principle of Newton mechanics.**

$$\frac{d}{dt}\int_{V_t} \rho\vec{u} = \int_{V_t} \rho\vec{g} + \oint_{\partial V_t} \vec{t}_{\hat{n}} \ .$$

Total energy balance equation: 1^{st} **principle of thermodynamics.**

$$\frac{d}{dt}\int_{V_t} \rho e^t = \int_{V_t} \rho\vec{g}\cdot\vec{u} + \oint_{\partial V_t} \vec{t}_{\hat{n}}\cdot\vec{u} - \oint_{\partial V_t} \vec{q}\cdot\hat{n} + \int_{V_t} \rho r \ .$$

2.1.2 Integral balance equations for arbitrary domains - arbitrary description

Using Reynolds transport theorem, time derivative over the material volume V_t can be written in terms of the time derivative over volume v_t in arbitrary motion and a flux contribution across its boundary.

Mass balance equation.

$$\frac{d}{dt}\int_{v_t} \rho + \oint_{\partial v_t} \rho(\vec{u}-\vec{u}_b)\cdot\vec{\hat{n}} = 0 \ .$$

Momentum balance equation.

$$\frac{d}{dt}\int_{v_t} \rho\vec{u} + \oint_{\partial v_t} \rho\vec{u}(\vec{u}-\vec{u}_b)\cdot\vec{\hat{n}} = \int_{v_t} \rho\vec{g} + \oint_{\partial v_t} \vec{t}_{\vec{\hat{n}}} \ .$$

Total energy balance equation.

$$\frac{d}{dt}\int_{v_t} \rho e^t + \oint_{\partial v_t} \rho e^t(\vec{u}-\vec{u}_b)\cdot\vec{\hat{n}} = \int_{v_t} \rho\vec{g}\cdot\vec{u} + \oint_{\partial v_t} \vec{t}_{\vec{\hat{n}}}\cdot\vec{u} - \oint_{\partial v_t} \vec{q}\cdot\vec{\hat{n}} + \int_{v_t} \rho r \ .$$

How to correctly apply Reynolds's transport theorem in continuum mechanics

Apply Reynold's transport both to material volume V_t and arbitrary volume v_t

$$\begin{aligned}
\frac{d}{dt}\int_{V_t} f &= \int_{V_t} \frac{\partial f}{\partial t} + \oint_{\partial V_t} f\,\vec{v}\cdot\hat{n} \\
\frac{d}{dt}\int_{v_t} f &= \int_{v_t} \frac{\partial f}{\partial t} + \oint_{\partial v_t} f\,\vec{v}_b\cdot\hat{n}
\end{aligned}$$

and compare these two expressions, after setting $v_t \equiv V_t$, i.e. considering the material volume at time t conciding with the arbitrary volume at time t (in general, at any time t there's a different material volume V_t coinciding with the arbitrary volume v_t - i.e. a different set of material particles in the arbitrary volume - but this is not a problem at all in the manipulation),

$$\frac{d}{dt}\int_{V_t\equiv v_t} f = \frac{d}{dt}\int_{v_t} f + \oint_{\partial V_t\equiv\partial v_t} f(\vec{v}-\vec{v}_b)\cdot\hat{n}\,.$$

2.1.3 Integral balance equations for control volumes - Eulerian description

Eulerian description of integral balance equations in continuum mechanics relies on stationary control volume, V. Integral balance equations are readily derived from *balance equations for arbitrary volumes* setting the velocity of the boundary of the domain equal to zero, i.e. $\vec{v}_b = \vec{0}$, and the Eulerian controlo volume equal to the "instantanteously coinciding material volume", $V \equiv V_t$.

Mass balance equation

$$\frac{d}{dt}\int_V \rho + \oint_{\partial V} \rho\vec{u}\cdot\hat{\vec{n}} = 0\,.$$

Momentum balance equation

$$\frac{d}{dt}\int_V \rho\vec{u} + \oint_{\partial V} \rho\vec{u}\vec{u}\cdot\hat{\vec{n}} = \int_V \rho\vec{g} + \oint_{\partial V} \vec{t}_{\vec{n}}\,.$$

Total energy balance equation.

$$\frac{d}{dt}\int_V \rho e^t + \oint_{\partial V} \rho e^t\vec{u}\cdot\hat{\vec{n}} = \int_V \rho\vec{g}\cdot\vec{u} + \oint_{\partial V} \vec{t}_{\vec{n}}\cdot\vec{u} - \oint_{\partial V} \vec{q}\cdot\hat{\vec{n}} + \int_V \rho r\,.$$

2.2 Differential Balance Equations of aaa physical quantities

2.2.1 Balance equation in physical space

In this section, differential form of balance equations is derived using time t and physical coordinate $\vec{r}$ as independent variables of fields representing physical quantities $f(\vec{r},t)$.

Conservative form - Eulerian description in physical space.

$$\begin{aligned}
&\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \\
&\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla\cdot(\rho\vec{v}\otimes\vec{v}) = \rho\vec{g} + \nabla\cdot\mathbb{T} \\
&\frac{\partial}{\partial t}(\rho e^t) + \nabla\cdot(\rho e^t\vec{v}) = \rho\vec{g}\cdot\vec{v} + \nabla\cdot(\mathbb{T}\cdot\vec{v}) - \nabla\cdot\vec{q} + \rho r
\end{aligned} \tag{2.1}$$

Convective form - Lagrangian description in physical space. Using vector calculus identities to evaluate partial derivatives of products, mass equation and relation (1.3) to write partial derivative w.r.t. material derivative,

$$\begin{aligned}
&\frac{D\rho}{Dt} = -\rho\nabla\cdot\vec{v} \\
&\rho\frac{D\vec{v}}{Dt} = \rho\vec{g} + \nabla\cdot\mathbb{T} \\
&\rho\frac{De^t}{Dt} = \rho\vec{g}\cdot\vec{v} + \nabla\cdot(\mathbb{T}\cdot\vec{v}) - \nabla\cdot\vec{q} + \rho r
\end{aligned}$$

Proof

todo

Arbitrary description in physical space. Using relation (1.3) to write material derivatives w.r.t. time derivative at constant $\vec{r}_b$

$$\begin{aligned}
&\left.\frac{\partial\rho}{\partial t}\right|_{\vec{r}_b} + (\vec{v}-\vec{v}_b)\cdot\nabla\rho = -\rho\nabla\cdot\vec{v} \\
&\rho\left.\frac{\partial\vec{v}}{\partial t}\right|_{\vec{r}_b} + \rho\,(\vec{v}-\vec{v}_b)\cdot\nabla\vec{v} = \rho\vec{g} + \nabla\cdot\mathbb{T} \\
&\rho\left.\frac{\partial e^t}{\partial t}\right|_{\vec{r}_b} + \rho\,(\vec{v}-\vec{v}_b)\cdot\nabla e^t = \rho\vec{g}\cdot\vec{v} + \nabla\cdot(\mathbb{T}\cdot\vec{v}) - \nabla\cdot\vec{q} + \rho r
\end{aligned}$$

2.2.2 Balance equations in reference space

In this section, differential form of balance equations is derived using time t and material coordinate $\vec{r}_0$ as independent variables of fields representing physical quantities $f_0(\vec{r}_0,t) = f(\vec{r}(\vec{r}_0,t),t)$, exploiting the change of variables $\vec{r}(\vec{r}_0,t)$ and its inverse transformation - assumed to exist (with the same consideration done in the kinematics sections: while it's likely that a global invertible transformation w.r.t. the original reference configuration doesn't exist, limiting the time interval and space domain a "piecewise" invertible transformation w.r.t. intermetdiate states exists).

2.3 Differential Balance Equations of ddd physical quantities

Balance equations of kinetic energy, internal energy and entropy

$$k = \frac{|\vec{v}|^2}{2} \quad , \qquad e = e^t - k \quad , \qquad s = \dots$$

Convective form - Lagrangian description in physical space. Kinetic energy equation is derived multiplying the momentum equation by the velocity field; internal energy equation is derived subtracting kinetic energy equation from the total energy equation; entropy equation strongly depends on the constitutive equation of the material, as it's shown for elastic solids and Newtonian fluids

$$\begin{aligned}
&\rho\frac{Dk}{Dt} = \rho\vec{g}\cdot\vec{v} + \vec{v}\cdot\nabla\cdot\mathbb{T} \\
&\rho\frac{De}{Dt} = \mathbb{T}:\nabla\vec{v} - \nabla\cdot\vec{q} + \rho r \\
&\dots
\end{aligned}$$

Conservative form - Eulerian description in physical space.

$$\begin{aligned}&\frac{\partial}{\partial t}(\rho k) + \nabla\cdot(\rho k \vec{v}) = \rho \vec{g}\cdot\vec{v} + \vec{v}\cdot\nabla\cdot\mathbb{T}\\&\frac{\partial}{\partial t}(\rho e) + \nabla\cdot(\rho e \vec{v}) = \mathbb{T}:\nabla\vec{v} - \nabla\cdot\vec{q} + \rho r\\&\dots\end{aligned}$$

Arbitrary description in physical space.

$$\begin{aligned}&\rho \left.\frac{\partial k}{\partial t}\right|_{\vec{r}_b} + \rho\,(\vec{v}-\vec{v}_b)\cdot\nabla k = \rho\vec{g}\cdot\vec{v} + \vec{v}\cdot\nabla\cdot\mathbb{T}\\&\rho \left.\frac{\partial e}{\partial t}\right|_{\vec{r}_b} + \rho\,(\vec{v}-\vec{v}_b)\cdot\nabla e = \mathbb{T}:\nabla\vec{v} - \nabla\cdot\vec{q} + \rho r\\&\dots\end{aligned}$$

2.4 Integral Balance Equations of ddd physical quantities

2.5 Jump conditions

Jump conditions comes from integral balances for an anrbitrary domain. These conditions hold both across discontinuities - where fields are not regular enough for differential equations to hold - and in regular domains.

2.6 Integral Balance Equations in reference space

2.6.1 Mass

Integal balance for a material volume V_t reads

$$\begin{aligned}0 &= \frac{d}{dt}\int_{V_t}\rho(\vec{r},t)\,dV =\\&= \frac{d}{dt}\int_{V_0}\rho(\vec{r}(\vec{r}_0,t),t)\,J(\vec{r}_0,t)\,dV_0 =\\&= \frac{d}{dt}\int_{V_0}\rho_0(\vec{r}_0,t)\,J(\vec{r}_0,t)\,dV_0 =\\&= \int_{V_0}\frac{D}{Dt}\left(\rho_0(\vec{r}_0,t)\,J(\vec{r}_0,t)\right)\,dV_0\,.\end{aligned}$$

Since the domain V_0 is arbitrary, with some abuse of notation to indicate the density field as ρ, hidihg the dependence on the independet fvariables $\rho_0(\vec{r}_0,t)$, the differential balance in reference space follows

$$\frac{D}{Dt}(\rho\,J) = 0$$

or

$$\rho\,J = \rho^0\,,$$

i.e. the product $\rho\,J$ equals the initial density field ρ^0, assuming that the determinant of the transformation is $J^0 = 1$, in the reference configuration.

2.6.2 Momentum

Integral balance for a material volume V_t reads

$$\begin{aligned}
\frac{d}{dt}\int_{V_t} \rho\vec{v}\,dV &= \int_{V_t} \rho\vec{g} + \oint_{\partial V_t} \hat{n}\cdot\mathbb{T}\,dS \\
\frac{d}{dt}\int_{V_0} \rho\,J\,\vec{v}\,dV_0 &= \int_{V_0} \rho\,J\,\vec{g} + \oint_{\partial V_t} \hat{n}_0\cdot\left(J\mathbb{F}^{-T}\cdot\mathbb{T}\right)\,dS_0 \\
\frac{d}{dt}\int_{V_0} \rho^0\,\vec{v}\,dV_0 &= \int_{V_0} \rho^0\,\vec{g} + \oint_{\partial V_0} \hat{n}_0\cdot\Sigma_n\,dS_0 \\
\int_{V_0} \rho^0\,\frac{D}{Dt}\vec{v}\,dV_0 &= \int_{V_0} \rho^0\,\vec{g} + \oint_{V_0} \nabla_0\cdot\Sigma_n\,dS_0
\end{aligned}$$

Since the domain V_0 is arbitrary, the differential balance in reference space follows

$$\rho^0\frac{D\vec{v}}{Dt} = \rho^0\,\vec{g} + \nabla_0\cdot\Sigma_n$$

Nanson's formula

$$d\vec{r} = d\vec{r}_0\cdot\frac{\partial\vec{r}}{\partial\vec{r}_0} = d\vec{r}_0\cdot\nabla_0\vec{r} = d\vec{r}_0\cdot\mathbb{F}$$

$$\begin{aligned}
dV &= J\,dV_0 \\
d\vec{r}\cdot\hat{n}dS &= J\,d\vec{r}_0\cdot\hat{n}_0\,dS_0 \\
d\vec{r}_0\cdot\mathbb{F}\cdot\hat{n}\,dS &= J\,d\vec{r}_0\cdot\hat{n}_0\,dS_0
\end{aligned}$$

must be true for all $\vec{r}_0$ arbitrary, so that

$$\mathbb{F}\cdot\hat{n}\,dS = J\,\hat{n}_0\,dS_0$$

and

$$\begin{aligned}
\hat{n}\,dS &= J\,\mathbb{F}^{-1}\cdot\hat{n}_0\,dS_0 = \\
&= J\,\hat{n}_0\cdot\mathbb{F}^{-T}\,dS_0
\end{aligned}$$

Stress tensors

Cauchy stress tensor.

Piola-Kirchhoff I - transpose of normal stress tensors.

Piola-Kirchhoff II

Example 2.6.1 (Relation between description in physical and reference space)

$$\begin{aligned}
\rho^0\frac{D\vec{v}}{Dt} &= \rho^0\,\vec{g} + \nabla_0\cdot\Sigma_n \\
J\,\rho\,\frac{D\vec{v}}{Dt} &= J\,\rho\,\vec{g} + \nabla_0\cdot\Sigma_n \\
\rho\,\frac{D\vec{v}}{Dt} &= \rho\,\vec{g} + \frac{1}{J}\nabla_0\cdot\Sigma_n
\end{aligned}$$

thus,

$$\nabla \cdot \mathbb{T} = \frac{1}{J}\nabla_0 \cdot \Sigma_n = $$
$$= \frac{1}{J}\nabla_0 \cdot \left(J\,\mathbb{F}^{-T} \cdot \mathbb{T}\right)$$

todo Prove it with derivation!

2.6.3 Kinetic energy

$$0 = \vec{v} \cdot \left\{\rho^0 \frac{D\vec{v}}{Dt} - \rho^0 \vec{g} - \nabla_0 \cdot \Sigma_n\right\} =$$
$$= \rho^0 \frac{D}{Dt}\frac{|\vec{v}|^2}{2} - \rho^0 \vec{v}\cdot\vec{g} - \vec{v}\cdot\nabla_0\cdot\Sigma_n =$$
$$= \rho^0 \frac{D}{Dt}\frac{|\vec{v}|^2}{2} - \rho^0 \vec{v}\cdot\vec{g} - \nabla_0\cdot\left(\vec{v}\cdot\Sigma_n\right) + \nabla_0\vec{v} : \Sigma_n$$
$$v_i \partial_k^0 \Sigma_{ki} = \partial_k^0 \left(v_i \Sigma_{ki}\right) - \partial_k^0 v_i \Sigma_{ki}$$
$$dV = J dV_0$$
$$dr_i n_i\, dS = J dr_k^0 n_k^0\, dS_0$$
$$dr_k^0 \frac{\partial r_i}{\partial r_k^0} n_i\, dS = dr_k^0 J n_k^0\, dS_0$$
$$\frac{\partial r_i}{\partial r_k^0} n_i\, dS = J n_k^0\, dS_0$$
$$\underbrace{\frac{\partial r_k^0}{\partial r_j}\frac{\partial r_i}{\partial r_k^0}}_{=\delta_{ij}} n_i\, dS = J \frac{\partial r_k^0}{\partial r_j} n_k^0\, dS_0$$
$$n_j\, dS = J \frac{\partial r_k^0}{\partial r_j} n_k^0\, dS_0$$
$$\mathbb{F} = \hat{e}_k^0 \hat{e}_i^0 \frac{\partial r_i}{\partial r_k^0}$$
$$\mathbb{F}^{-1} = \hat{e}_j^0 \hat{e}_k^0 \frac{\partial r_k^0}{\partial r_j}$$
$$\mathbb{F}^{-1}\cdot\mathbb{F} = \left(\hat{e}_j^0 \hat{e}_k^0 \frac{\partial r_k^0}{\partial r_j}\right)\cdot\left(\hat{e}_l^0 \hat{e}_i^0 \frac{\partial r_i}{\partial r_l^0}\right) = \hat{e}_j^0 \hat{e}_i^0 \frac{\partial r^i}{\partial r_j} = \hat{e}_j^0 \hat{e}_i^0 \delta_{ij}$$
$$\mathbb{F}\cdot\mathbb{F}^{-1} = \left(\hat{e}_l^0 \hat{e}_i^0 \frac{\partial r_i}{\partial r_l^0}\right)\cdot\left(\hat{e}_j^0 \hat{e}_k^0 \frac{\partial r_k^0}{\partial r_j}\right) = \hat{e}_l^0 \hat{e}_k^0 \frac{\partial r^k}{\partial r_l} = \hat{e}_l^0 \hat{e}_k^0 \delta_{lk}$$
$$\frac{\partial r_k^0}{\partial r_i}\frac{\partial r_i}{\partial r_l^0} = \delta_{kl}$$
$$\Sigma := \Sigma_n \cdot \mathbb{F}^{-1} = J\mathbb{F}^{-T}\cdot\mathbb{T}\cdot\mathbb{F}^{-1}$$
$$\Sigma_n = \Sigma\cdot\mathbb{F}$$
$$\Sigma_{ik} = \Sigma_{n,ij}\left(\mathbb{F}^{-1}\right)_{jk} = \Sigma_{n,ij}\frac{\partial r_k^0}{\partial r_j}$$

$$\Sigma_{n,ij} = \Sigma_{ik} \frac{\partial x_j}{\partial x^0_k}$$

$$\begin{aligned}
\nabla_0 \vec{v} : \Sigma_n = & \frac{D}{Dt}\mathbb{F} : \Sigma_n = \\
= & \frac{\partial v_j}{\partial x^0_i} \Sigma_{n,ij} = \\
= & \frac{\partial v_j}{\partial x^0_i} \Sigma_{ik} \frac{\partial x_j}{\partial x^0_k} = \\
= & \Sigma_{ik} \frac{1}{2} \left(\frac{\partial v_j}{\partial x^0_i} \frac{\partial x_j}{\partial x^0_k} + \frac{\partial v_j}{\partial x^0_k} \frac{\partial x_j}{\partial x^0_i} \right)
\end{aligned}$$

if Σ is symmetric, $\Sigma_{ik} = \Sigma_{ki}$, or with tensor notation

$$\begin{aligned}
\nabla_0 \vec{v} : \Sigma_n = & \frac{D}{Dt}\mathbb{F} : \Sigma_n = \\
= & \nabla_0 \vec{v} : (\Sigma \cdot \mathbb{F}) = \\
= & \Sigma : \frac{1}{2} \left(\frac{D\mathbb{F}}{Dt} \cdot \mathbb{F}^T + \mathbb{F} \cdot \frac{D\mathbb{F}^T}{Dt} \right) = \\
= & \Sigma : \frac{D}{Dt}\mathbb{E} \, ,
\end{aligned}$$

having recognized the time derivative (1.6) of the *Green-Lagrange tensor* (1.5).

Integral of the volume stress in the reference space can be recast as the volume in the physical space

$$\int_{V_0} \nabla_0 \vec{v} : \Sigma_n \, dV_0 = \int_{V_0} \Sigma : \frac{D\mathbb{E}}{Dt} \, dV_0$$

$$\Sigma_{n,ik} = J \, \frac{\partial x^0_i}{\partial x_k} \, T_{jk}$$

$$\begin{aligned}
\int_{V_0} \nabla_0 \vec{v} : \Sigma_n \, dV_0 = & \int_{V_0} \frac{\partial v_k}{\partial x^0_i} \Sigma_{n,ik} \, dV_0 = \\
= & \int_{V_0} \underbrace{\frac{\partial v_k}{\partial x^0_i} \left(\frac{\partial x^0_i}{\partial x_j}} \, T_{kj} \underbrace{J \right) dV_0}_{dV} = \\
= & \int_V \frac{\partial v_k}{\partial x^j} \, T_{kj} \, dV = \\
= & \int_V \frac{1}{2} \left(\frac{\partial v_k}{\partial x^j} + \frac{\partial v_j}{\partial x^k} \right) \, T_{kj} \, dV = \\
= & \int_V D_{jk} \, T_{kj} \, dV = \\
= & \int_V \mathbb{D} : \mathbb{T} \, dV \, .
\end{aligned}$$

Variational principles

Using an arbitrary test function $\vec{w}(\vec{r}_0)$,

$$0 = \vec{w} \cdot \left\{ \rho^0 \frac{D\vec{v}}{Dt} - \rho^0 \vec{g} - \nabla_0 \cdot \Sigma_n \right\}$$

and using rule of product

$$w_i \frac{\partial \Sigma_{n,ji}}{\partial x^0_j} = \frac{\partial}{\partial x^0_j} \left(w_i \Sigma_{n,ji} \right) - \frac{\partial w_i}{\partial x^0_j} \Sigma_{n,ji} =$$

and the second term can be transformed using the relation bewteen normal stress and second Piola-Kirchhoff tensor

$$\frac{\partial w_i}{\partial x^0_j} \Sigma_{n,ji} = \frac{\partial w_i}{\partial x^0_j} \Sigma_{jk} \frac{\partial x_i}{\partial x^0_k} = \Sigma_{jk} \, \frac{1}{2} \left[\frac{\partial x_i}{\partial x^0_k} \frac{\partial w_i}{\partial x^0_j} + \frac{\partial x_i}{\partial x^0_j} \frac{\partial w_i}{\partial x^0_k} \right] = \Sigma_{jk} \, W_{ij}(\vec{w}) \ ,$$

having defined the tensor

$$\mathbb{W}(\vec{w}) := \frac{1}{2} \left[\nabla_0 \vec{w} \cdot \mathbb{F}^T + \mathbb{F} \cdot \nabla^T_0 \vec{w} \right] \ ,$$

with the evident analogy with the time derivative of Green-Lagrange strain tensor, namely

$$\epsilon = \mathbb{W}(\vec{v}) \ ,$$

being $\vec{v}$ the velocity field. Integrating on the domain V_0 and using divergence theorem, the problem is written in its weak form

$$\int_{V_0} \left\{ \rho^0 \vec{w} \cdot \frac{D\vec{v}}{Dt} + \mathbb{W}(\vec{w}) : \Sigma \right\} = \int_{V_0} \rho^0 \vec{w} \cdot \vec{g} + \oint_{\partial V_0} \hat{n}_0 \cdot \Sigma_n \cdot \vec{w} \ ,$$

with the proper boundary conditions and the corresponding conditions on the test function $\vec{w}$. As an example, if the boundary is composed of two different regions, $S_{D,0} \cup S_{N,0} = \partial V_0$, $S_D \cap S_N = \emptyset$ where either position (called S_D from Dirichlet boundary) and stress (called S_N from Neumann boundary) are prescribed

$$\begin{aligned} \vec{r} &= \vec{r}_b \quad , \quad \vec{w} = \vec{0} && \text{(on } S_{D,0} \text{ Dirichlet - essential - boundary)} \\ \hat{n}_0 \cdot \Sigma_n &= \vec{t}_{0,\hat{n}_0} \quad , && \text{(on } S_{N,0} \text{ Neumann - natural - boundary)} \end{aligned}$$

the weak form of the equation reads

$$\int_{V_0} \left\{ \rho^0 \vec{w} \cdot \frac{D\vec{v}}{Dt} + \mathbb{W}(\vec{w}) : \Sigma \right\} = \int_{V_0} \rho^0 \vec{w} \cdot \vec{g} + \int_{S_{n,0}} \hat{n}_0 \cdot \vec{t}_{\hat{n}_0}$$

2.6.4 Total energy

Using Nanson's relation $\hat{n} \, dS = \hat{n}_0 \cdot \left(J \mathbb{F}^{-T} \right) \, dS_0$,

$$\begin{aligned} \frac{d}{dt} \int_{V_t} \rho e^t \, dV &= \int_{V_t} \rho \vec{g} \cdot \vec{v} \, dV + \oint_{\partial V_t} \vec{t}_{\hat{n}} \cdot \vec{v} \, dS - \oint_{\partial V_t} \hat{n} \cdot \vec{q} + \int_{V_t} \rho \, r \, dV = \\ &= \int_{V_t} \rho \vec{g} \cdot \vec{v} \, dV + \oint_{\partial V_t} \hat{n} \cdot \mathbb{T} \cdot \vec{v} \, dS - \oint_{\partial V_t} \hat{n} \cdot \vec{q} \, dS + \int_{V_t} \rho \, r \, dV \\ &= \int_{V_0} \rho^0 \, \vec{g} \cdot \vec{v} \, dV_0 + \oint_{\partial V_0} \hat{n}_0 \cdot \left(J \mathbb{F}^{-T} \cdot \mathbb{T} \right) \cdot \vec{v} \, dS_0 - \oint_{\partial V_0} \hat{n}_0 \cdot \left(J \mathbb{F}^{-T} \cdot \vec{q} \right) \, dS_0 + \int_{V_0} \rho^0 \, r \, dV_0 \ . \end{aligned}$$

and the differential form reads

$$\rho^0 \frac{De^t}{Dt} = \rho^0 \vec{g} \cdot \vec{v} + \nabla_0 \cdot (J \mathbb{F}^{-T} \cdot \mathbb{T} \cdot \vec{v}) - \nabla_0 \cdot (J \mathbb{F}^{-T} \cdot \vec{q}) + \rho^0 \, r \, .$$

or

$$\rho^0 \frac{De^t}{Dt} = \rho^0 \vec{g} \cdot \vec{v} + \nabla_0 \cdot (\Sigma_n \cdot \vec{v}) - \nabla_0 \cdot \vec{q}_0 + \rho^0 \, r \, .$$

and dividing by J and using the relation (**see below**) $\nabla_0 \cdot (J \mathbb{F}^{-T} \cdot \vec{a}) = J \, \nabla \cdot \vec{a}$,

$$\rho \frac{De^t}{Dt} = \rho \vec{g} \cdot \vec{v} + \nabla \cdot (\mathbb{T} \cdot \vec{v}) - \nabla \cdot \vec{q} + \rho r \, .$$

Comparison with equation in physical space (dividing by J) suggests the identity

$$\frac{1}{J} \nabla_0 \cdot (J \mathbb{F}^{-T} \cdot \vec{a}) = \nabla \cdot \vec{a} \, ,$$

and thus

$$\nabla_0 \cdot (J \mathbb{F}^{-T}) = \vec{0} \, ,$$

since

$$\nabla_0 \cdot (J \mathbb{F}^{-T} \cdot \vec{a}) = \nabla_0 \cdot (J \mathbb{F}^{-T}) \cdot \vec{a} + J \mathbb{F}^{-T} : \nabla_0 \vec{a} = \nabla_0 \cdot (J \mathbb{F}^{-T}) \cdot \vec{a} + J \nabla \cdot \vec{a}$$

Proof.

$$J = \left| \frac{\partial r_k}{\partial r^0_i} \right| = \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r^0_1} \frac{\partial r_{i_2}}{\partial r^0_2} \frac{\partial r_{i_3}}{\partial r^0_3}$$

$$\left(\mathbb{F}^{-T} \right)_{ij} = \frac{\partial r^0_i}{\partial r_j}$$

$$\begin{aligned}
\left\{ \nabla_0 \cdot (J \, \mathbb{F}^{-T}) \right\}_j = & \frac{\partial}{\partial r^0_i} \left(\varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r^0_1} \frac{\partial r_{i_2}}{\partial r^0_2} \frac{\partial r_{i_3}}{\partial r^0_3} \frac{\partial r^0_i}{\partial r_j} \right) = \\
& = \varepsilon_{i_1, i_2, i_3} \frac{\partial}{\partial r^0_i} \left(\frac{\partial r_{i_1}}{\partial r^0_1} \right) \frac{\partial r_{i_2}}{\partial r^0_2} \frac{\partial r_{i_3}}{\partial r^0_3} \frac{\partial r^0_i}{\partial r_j} + \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r^0_1} \frac{\partial}{\partial r^0_i} \left(\frac{\partial r_{i_2}}{\partial r^0_2} \right) \frac{\partial r_{i_3}}{\partial r^0_3} \frac{\partial r^0_i}{\partial r_j} + \\
& + \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r^0_1} \frac{\partial r_{i_2}}{\partial r^0_2} \frac{\partial}{\partial r^0_i} \left(\frac{\partial r_{i_3}}{\partial r^0_3} \right) \frac{\partial r^0_i}{\partial r_j} + \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r^0_1} \frac{\partial r_{i_2}}{\partial r^0_2} \frac{\partial r_{i_3}}{\partial r^0_3} \frac{\partial}{\partial r^0_i} \left(\frac{\partial r^0_i}{\partial r_j} \right) = \\
& = \varepsilon_{i_1, i_2, i_3} \underbrace{\frac{\partial r^0_i}{\partial r_j} \frac{\partial}{\partial r^0_i}}_{= \frac{\partial}{\partial r_j}} \left(\frac{\partial r_{i_1}}{\partial r^0_1} \right) \frac{\partial r_{i_2}}{\partial r^0_2} \frac{\partial r_{i_3}}{\partial r^0_3} + \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r^0_1} \underbrace{\frac{\partial r^0_i}{\partial r_j} \frac{\partial}{\partial r^0_i}}_{= \frac{\partial}{\partial r_j}} \left(\frac{\partial r_{i_2}}{\partial r^0_2} \right) \frac{\partial r_{i_3}}{\partial r^0_3} + \\
& + \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r^0_1} \frac{\partial r_{i_2}}{\partial r^0_2} \underbrace{\frac{\partial r^0_i}{\partial r_j} \frac{\partial}{\partial r^0_i}}_{= \frac{\partial}{\partial r_j}} \left(\frac{\partial r_{i_3}}{\partial r^0_3} \right) + \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r^0_1} \frac{\partial r_{i_2}}{\partial r^0_2} \frac{\partial r_{i_3}}{\partial r^0_3} \frac{\partial}{\partial r_j} \underbrace{\left(\frac{\partial r^0_i}{\partial r^0_i} \right)}_{=3} = \\
& = 0 \, ,
\end{aligned}$$

since

$$\frac{\partial}{\partial r_j} \left(\frac{\partial r_{i_k}}{\partial r^0_k} \right) = \frac{\partial}{\partial r^0_k} \left(\frac{\partial r_{i_k}}{\partial r^j} \right) = \frac{\partial}{\partial r^0_k} \delta_{i_k j} = 0 \, .$$

Thus

$$\frac{1}{J} \frac{\partial}{\partial r^0_i} \left(J \frac{\partial r^0_i}{\partial r_j} a_j \right) = \frac{1}{J} \frac{\partial}{\partial r^0_i} \left(J \frac{\partial r^0_i}{\partial r_j} \right) a_j + \frac{1}{J} \, J \, \frac{\partial r^0_i}{\partial r_j} \frac{\partial a_j}{\partial r^0_i} = \frac{\partial a_j}{\partial r_j} = \nabla \cdot \vec{a} \, .$$

2.6.5 Internal energy

$$\frac{d}{dt}\int_{V_t}\rho e\,dV = \int_{V_t}\mathbb{T}:\nabla\vec{v}\,dV - \oint_{\partial V_t}\hat{n}\cdot\vec{q}\,dS + \int_{V_t}\rho\,r\,dV =$$
$$\frac{d}{dt}\int_{V_0}\rho^0 e\,dV_0 = \int_{V_0}J\,\mathbb{T}:\nabla\vec{v}\,dV_0 - \oint_{\partial V_0}\hat{n}_0\cdot\left(J\mathbb{F}^{-T}\cdot\vec{q}\right)\,dS_0 + \int_{V_0}\rho^0\,r\,dV_0 =$$

and the differential form reads

$$\rho^0\frac{De}{Dt} = \Sigma_n : \nabla_0\vec{v} - \nabla_0\cdot\vec{q}^0 + \rho^0 r\,.$$

todo *pay attention at the definition - choose one and keep using it! - of the product* $\mathbb{A}:\mathbb{B}$, in components

$$\mathbb{A}:\mathbb{B} = A_{ij}B_{ij} \qquad \text{or} \qquad = A_{ij}B_{ji}$$

CHAPTER

THREE

EQUAZIONI DI STATO ED EQUAZIONI COSTITUTIVE

CHAPTER

FOUR

EQUAZIONI DI BILANCIO DI ALTRE GRANDEZZE FISICHE

Partendo dai bilanci di massa, quantità di moto e di energia totale, si possono ricare le le equazioni di bilancio di altre grandezze fisiche come l'*energia cinetica*, l'*energia interna*, l'*entropia*.

4.1 Bilanci in forma differenziale, convettiva

Energia cinetica. L'energia cinetica (macroscopica) per unità di massa è $k = \frac{1}{2}|\mathbf{u}|^2 = \frac{1}{2}\mathbf{u}\cdot\mathbf{u}$. L'equazione di bilancio dell'energia cinetica viene derivata moltiplicando scalarmente l'equazione della quantità di moto

$$\rho \frac{D\mathbf{u}}{Dt} = \rho\mathbf{g} + \nabla\cdot\mathbb{T} \ ,$$

per il campo di velocità **u**,

$$\rho \frac{Dk}{Dt} = \rho\mathbf{g}\cdot\mathbf{u} + \nabla\cdot\mathbb{T}\cdot\mathbf{u} \ ,$$

avendo usato $\mathbf{u}\cdot d\mathbf{u} = d\left(\dfrac{\mathbf{u}\cdot\mathbf{u}}{2}\right) = dk$.

Energia interna. L'energia interna per unità di massa è la differenza tra l'energia totale e l'energia cinetica, $e = e^{tot} - k$. L'equazione di bilancio dell'energia interna viene ottenuta come differenza dell'equazione dell'energia totale

$$\rho \frac{De^{tot}}{Dt} = \rho\mathbf{g}\cdot\mathbf{u} + \nabla\cdot(\mathbb{T}\cdot\mathbf{u}) - \nabla\cdot\mathbf{q} + \rho r \ ,$$

e quella dell'energia cinetica, per ottenere

$$\rho \frac{De}{Dt} = \mathbb{T} : \nabla\mathbf{u} - \nabla\cdot\mathbf{q} + \rho r \ .$$

Entropia.

- **Entropia nei fluidi.** Se l'entropia può essere scritta come funzione dell'energia interna e della densità, e il primo principio della termodinamica viene scritto come

  $$de = \frac{P}{\rho^2}\, d\rho + T\, ds \ ,$$

  e il tensore degli sforzi può essere rappresentato come somma degli sforzi di pressione e degli sforzi viscosi **todo** *riferimento alle leggi costitutive*,

  $$\mathbb{T} = -P\mathbb{I} + \mathbb{S} = -P\mathbb{I} + 2\mu\mathbb{D} + \lambda(\nabla\cdot\mathbf{u})\mathbb{I} \ ,$$

si può ricavare l'equazione di governo dell'entropia usando il differenziale $ds = \dfrac{1}{T}\,de - \dfrac{P}{T\rho^2}\,d\rho$

$$\begin{aligned}
\rho \frac{Ds}{Dt} & = \frac{\rho}{T}\left(\frac{De}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt}\right) = \\
& = \frac{1}{T}\left(\rho\frac{De}{Dt} - \frac{P}{\rho}\frac{D\rho}{Dt}\right) = \\
& = \frac{1}{T}\left(\mathbb{T} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q} + \rho r - \frac{P}{\rho}\left(\rho \nabla \cdot \mathbf{u}\right)\right) = \\
& = \frac{1}{T}\left(-P\nabla \cdot \mathbf{u} + \mathbb{S} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q} + \rho r + P\,\nabla \cdot \mathbf{u}\right) = \\
& = \frac{1}{T}\left(\mathbb{S} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q} + \rho r\right) = \\
& = \frac{1}{T}\left(2\mu|\mathbb{D}|^2 + \lambda|(\nabla \cdot \mathbf{u})|^2 - \nabla \cdot \mathbf{q} + \rho r\right) = \\
& = \frac{2\mu|\mathbb{D}|^2 + \lambda|(\nabla \cdot \mathbf{u})|^2}{T} - \frac{\mathbf{q} \cdot \nabla T}{T^2} - \nabla \cdot \left(\frac{\mathbf{q}}{T}\right) + \frac{\rho r}{T}\ ,
\end{aligned}$$

avendo usato la degola del prodotto $\nabla \cdot \left(\dfrac{\mathbf{q}}{T}\right) = \dfrac{\nabla \cdot \mathbf{q}}{T} - \dfrac{\mathbf{q} \cdot \nabla T}{T^2}$.

Gli ultimi due termini sono legati alla **sorgenti di entropia** nel sistema, dovute alla sorgente di calore nel sistema e al flusso di calore tramite la frontiera del sistema.

I primi due termini possono essere ricondotti alla **dissipazione viscosa** e dovuta alla **conduzione termica** all'interno del volume: entrambi devono essere non-negativi per il secondo principio della termodinamica **todo**. Il primo termine è positivo se i coefficienti di viscosità del modello di fluido newtoniano sono non-negativi

$$\mu, \lambda \ge 0$$

. Il secondo termine impone che il flusso di calore avvenga in direzione opposta al gradiente di temperatura locale, e quindi la proiezione su di esso sia negativa (traducendo il concetto che il calore trasferisce energia da un corpo caldo a uno freddo),

$$-\mathbf{q} \cdot \nabla T \ge 0\ ,$$

come è facile da verificare per il modello di Fourier per la conduzione in mezzi isotropi, $\mathbf{q} = -k\nabla T$, $-\mathbf{q} \cdot \nabla T = k|\nabla T|^2 \ge 0$ se

$$k \ge 0\ .$$

Nel caso di modello lineare per la conduzione in mezzi non isotrpi, il flusso di conduzione può essere descritto usando un tensore del secondo ordine $\mathbb{K}$, $\mathbf{q} = -\mathbb{K} \cdot \nabla T$ (**todo** simmetria?) e la condizione diventa

$$0 \le -\nabla T \cdot \mathbf{q} = \nabla T \cdot \mathbb{K} \cdot \nabla T\ ,$$

che impone che il tensore di conduzione sia (semi-)definito positivo, a causa dell'arbitrarietà del vettore ∇T.

Se questi due termini sono non-negativi, il bilancio di entropia può essere riscritto come la disuguaglianza

$$\begin{aligned}
\rho \frac{Ds}{Dt} & = \underbrace{\frac{2\mu|\mathbb{D}|^2 + \lambda|(\nabla \cdot \mathbf{u})|^2}{T} - \frac{\mathbf{q} \cdot \nabla T}{T^2}}_{\ge 0} - \nabla \cdot \left(\frac{\mathbf{q}}{T}\right) + \frac{\rho r}{T} = \\
& \ge -\nabla \cdot \left(\frac{\mathbf{q}}{T}\right) + \frac{\rho r}{T}\ ,
\end{aligned}$$

o nella forma integrale per un volume materiale

$$\frac{d}{dt}\int_{V_t} \rho s \ge -\oint_{\partial V_t} \hat{\mathbf{n}} \cdot \frac{\mathbf{q}}{T} + \int_{V_t} \rho \frac{r}{T}\ ,$$

che richiama alla mente la disuguaglianza di Clausius **todo** *aggiungere riferimento*

$$dS \geq \frac{\delta Q^e}{T} \ .$$

La differenza di segno deriva dalla definizione di dQ^e come flusso di calore dall'ambiente verso il sistema e del vettore flusso di calore $\mathbf{q}$ come flusso di calore "uscente dal sistema" **todo**

Part II

Solid Mechanics

CHAPTER

FIVE

INTRODUCTION TO SOLID MECHANICS

Small displacements (and small strains)

Structural models

- **3-dimensional isotropic elastic medium.** Introduction to equilibrium and congruence.
- **Beam models.**
  - de Saint Venant
  - thin-walled
  - Timoshenko
  - Bernoulli
  - aeronautical

Statics

Weak form of equilibrium and congruence and energy theorems

- 3-dimensional structures
- Beam structures (with increasing assumptions)

CHAPTER

SIX

SMALL DISPLACEMENT - STATICS

6.1 \texttt{todo}

todo list

- "Kinematics": labile, isostatic, hyperstatics
- Stress tensor: Cauchy relation, proof of symmetry (under non-polar assumption)
- Small strain tensor: definition and compatibility conditions
- …
- **Labile - Undetermined.**
- **Isostatic - "determined".**
- **Hyperstatic - "overdetermined".**

6.2 Linear isotropic elastic medium

6.2.1 Constitutive equation

An isotropic elastic medium has no prefereed orientation. The most general linear relation between stress tensor $\boldsymbol{\sigma}$ and strain tensor $\boldsymbol{\varepsilon}$, and temperature difference ΔT w.r.t. a reference temperature, $\Delta T := T - T_0$,

$$\boldsymbol{\sigma} = \mathbf{D} : \boldsymbol{\varepsilon} - \boldsymbol{\beta} \Delta T \ ,$$

for isotropic media involves the rank-2 and rank-4 isotropic tensors, see rank-2-iso, and rank-4-iso. Since stress tensor $\boldsymbol{\sigma}$ and strain tensor $\boldsymbol{\varepsilon}$ are symmetric the constitutive law for isotropic elastic media reads

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \operatorname{tr}(\boldsymbol{\varepsilon})\mathbb{I} - \beta \Delta T \mathbb{I} \ , \tag{6.1}$$

being μ, λ the Lamé coefficients.

Decomposition of tensor in hydrostatic (proportional to $\mathbb{I}$) and deviatoric (traceless) parts reads

$$\boldsymbol{\sigma} = \left(2\mu\boldsymbol{\varepsilon} - \frac{2}{3}\mu \operatorname{tr}(\varepsilon)\mathbb{I} \right) + \left(\lambda + \frac{2}{3}\mu \right) \operatorname{tr}(\varepsilon)\mathbb{I} - \beta\Delta T\mathbb{I} \tag{6.2}$$

The expression of strain tensor $\boldsymbol{\varepsilon}$ as a function of stress tensor and temperature difference

$$\boldsymbol{\varepsilon} =$$

can be easily evaluated, using the relation between the traces of the tensors

Rank-2 and rank-4 isotropic tensors

An isotropic tensor has the same components after rotations of the vectors of the basis.

Rank-2 tensor. A rank-2 tensor $\mathbf{A}$ can be expressed in an original basis $\{\mathbf{b}_i\}_i$ and a rotated basis $\{\tilde{\mathbf{b}}_j\}_j$, related by the transformation law

$$\tilde{\mathbf{b}}_j = \left(\mathbf{b}^i \cdot \tilde{\mathbf{b}}_j\right) \mathbf{b}_i = R^i_j \mathbf{b}_i \ .$$

A rank-2 tensor can be written as

$$\mathbf{A} = A^{ij} \mathbf{b}_i \mathbf{b}_j = \widetilde{A}^{kl} \tilde{\mathbf{b}}_k \tilde{\mathbf{b}}_l = \underbrace{\widetilde{A}^{kl} R^i_k R^j_l}_{=A^{ij}} \mathbf{b}_i \mathbf{b}_j \ .$$

An isotropic tensor has the same components in both the original and the rotated bases, namely $\widetilde{A}^{ij} = A^{ij} = \widetilde{A}^{kl} R^i_k R^j_l$. An arbitrary infinitesimal rotation can be represented as $R^i_k = \delta_{ik} + \varepsilon^i{}_{mk} \theta^m$. The condition of isotropic tensors for an infinitesimal rotation reads

$$\begin{aligned}
\widetilde{A}^{ij} & = \widetilde{A}^{kl} (\delta_{ik} + \varepsilon^i{}_{mk} \theta^m)(\delta_{jl} + \varepsilon^j{}_{nl} \theta^n) = \\
& = \widetilde{A}^{ij} + \widetilde{A}^{kj} \varepsilon_{imk} \theta_m + \widetilde{A}^{il} \varepsilon_{jnl} \theta^n + \widetilde{A}^{kl} \begin{vmatrix} \delta_{ij} & \delta_{in} & \delta_{il} \\ \delta_{mj} & \delta_{mn} & \delta_{ml} \\ \delta_{kj} & \delta_{kn} & \delta_{kl} \end{vmatrix} \theta_m \theta_n = \\
& = \widetilde{A}^{ij} + \left(\widetilde{A}^{kj} \varepsilon_{imk} + \widetilde{A}^{ik} \varepsilon_{jmk} \right) \theta^m + \widetilde{A}^{kl} \theta_m \theta_n \left(\delta_{ij} \delta_{mn} \delta_{kl} + \dots \right) = \\
& = \dots + \theta_m \theta_m \widetilde{A}^{kk} \delta_{ij} + \theta_i \theta_l \widetilde{A}^{jl} + \theta_j \theta_k \widetilde{A}^{ki} - \theta_l \theta_k \widetilde{A}^{kl} \delta_{ij} - \theta_j \theta_i \widetilde{A}^{ll} - \theta_m \theta_m \widetilde{A}^{ji} =
\end{aligned}$$

Zero-order condition is identically satisfied. First-order conditions

$$\begin{aligned}
i,j,m = 1,1,1: & \quad 0 = 0 \\
i,j,m = 1,1,2: & \quad 0 = \widetilde{A}^{31} + \widetilde{A}^{13} \\
i,j,m = 1,2,1: & \quad 0 = 0 - \widetilde{A}^{13} \\
i,j,m = 1,2,2: & \quad 0 = \widetilde{A}^{31} + 0 \\
i,j,m = 1,2,3: & \quad 0 = -\widetilde{A}^{22} + \widetilde{A}^{11} \\
& \dots
\end{aligned}$$

imply

$$\begin{aligned}
& \widetilde{A}^{11} = \widetilde{A}^{22} = \widetilde{A}^{33} =: a \\
& \widetilde{A}^{ij} = 0 \quad \text{if } i \neq j
\end{aligned}$$

Second order conditions…(do they need to hold, in this first-order approximation of small rotations?)

Rank-4 tensor. For an rank-4 tensor the isotropic condition under a first order approximation of a rotation reads

$$A^{ijkl} = A^{mnpq} \left(\delta_{im} + \varepsilon_{ium} \theta_u \right) \left(\delta_{jn} + \varepsilon_{jvn} \theta_v \right) \left(\delta_{kp} + \varepsilon_{kwp} \theta_w \right) \left(\delta_{lq} + \varepsilon_{lxq} \theta_x \right) \ ,$$

Zero-order condition is identically satisfied. First order conditions read

$$\begin{aligned}
0 & = \varepsilon_{ium} A^{mjkl} \theta_u + \varepsilon_{jvn} A^{inkl} \theta_v + \varepsilon_{kwp} A^{ijpl} \theta_w + \varepsilon_{lxq} A^{ijkq} \theta_x = \\
& = \left(V \varepsilon_{ium} A^{mjkl} + \varepsilon_{jun} A^{inkl} + \varepsilon_{kup} A^{ijpl} + \varepsilon_{luq} A^{ijkq} \right) \theta_u =
\end{aligned}$$

Some combinations:

- 4 equal indices, $i = j = k = l$. Identity, if the index u has the same value. If the index u has different value, there's some info if m is different from i and u

$$0 = \varepsilon_{ium} \left(A^{miii} + A^{imii} + A^{iimi} + A^{iiim} \right) \theta_u \quad \text{no sum}$$

- 3 equal indices, $i = j = k \neq l$.

$$\begin{aligned} 0 &= \left(\varepsilon_{ium} A^{mjkl} + \varepsilon_{jun} A^{inkl} + \varepsilon_{kup} A^{ijpl} + \varepsilon_{luq} A^{ijkq}\right) \theta_u = \\ &= \left(\varepsilon_{ium} \left(A^{miil} + A^{imil} + A^{iiml}\right) + \varepsilon_{lum} A^{iiim}\right) \theta_u \end{aligned}$$

For $u = i$, the only non-trivial equation is for $m \neq i, l$:

$$0 = \varepsilon_{lim} A^{iiim}$$

Check and Uncomment

- 2 equal indices, $i = j \neq k, l$. Here two sub-cases may occur depending if 1) $k = l$ or 2) not. In case 1), need for evaluating the cases 1.1) $u = i$ (or equivalently $u = k$) or 1.2) $u \neq i, k$; in case 2), need for evaluating 2.1) $u = i$, or 2.2) $u = k$ (or equivalently, $u = l$).
  - Case 1.1)

$$\begin{aligned} 0 &= \left(\varepsilon_{ium} A^{mjkl} + \varepsilon_{jun} A^{inkl} + \varepsilon_{kup} A^{ijpl} + \varepsilon_{luq} A^{ijkq}\right) \theta_u = \\ &= \left(\varepsilon_{ium} (A^{mill} + A^{imll}) + \varepsilon_{lum} A^{iiml} + \varepsilon_{lum} A^{iilm}\right) \theta_u = \\ &= \left(\varepsilon_{lim} (A^{iiml} + A^{iilm})\right) \theta_i \end{aligned}$$

    so switching the last two indices gives $A^{iiml} = A^{iilm}$.
  - Case 1.2)

$$\begin{aligned} 0 &= \left(\varepsilon_{ium} A^{mjkl} + \varepsilon_{jun} A^{inkl} + \varepsilon_{kup} A^{ijpl} + \varepsilon_{luq} A^{ijkq}\right) \theta_u = \\ &= \left(\varepsilon_{ium} (A^{mill} + A^{imll}) + \varepsilon_{lum} A^{iiml} + \varepsilon_{lum} A^{iilm}\right) \theta_u = \\ &= \left(\varepsilon_{iul} (A^{lill} + A^{illl}) + \varepsilon_{lui} A^{iiil} + \varepsilon_{lui} A^{iili}\right) \theta_u = \\ &= 0 \end{aligned}$$

  - Case 2.1)

$$\begin{aligned} 0 &= \left(\varepsilon_{ium} A^{mjkl} + \varepsilon_{jun} A^{inkl} + \varepsilon_{kup} A^{ijpl} + \varepsilon_{luq} A^{ijkq}\right) \theta_u = \\ &= \left(0 + 0 + \varepsilon_{kip} A^{iipl} + \varepsilon_{lip} A^{iikp}\right) \theta_i = \\ &= \left(\varepsilon_{kil} A^{iill} + \varepsilon_{lik} A^{iikk}\right) \theta_i = \\ &= \varepsilon_{kil} \left(A^{iill} - A^{iikk}\right) \theta_i \ . \end{aligned}$$

    so $A^{iill} = A^{iikk}$.
  - Case 2.2) $i \neq k, l$, $k \neq l$. If $u = k$

$$\begin{aligned} 0 &= \left(\varepsilon_{ium} A^{mjkl} + \varepsilon_{jun} A^{inkl} + \varepsilon_{kup} A^{ijpl} + \varepsilon_{luq} A^{ijkq}\right) \theta_u = \\ &= \left(\varepsilon_{ikm} A^{mikl} + \varepsilon_{ikm} A^{imkl} + \varepsilon_{kkm} A^{iiml} + \varepsilon_{lkm} A^{iikm}\right) \theta_k = \\ &= \left(\varepsilon_{ikl} A^{likl} + \varepsilon_{ikl} A^{ilkl} + 0 + \varepsilon_{lki} \underbrace{A^{iiki}}_{=0}\right) \theta_k = \\ &= \varepsilon_{ikl} \left(A^{likl} + A^{ilkl}\right) \theta_k = \end{aligned}$$

    so switching the first two indices gives $A^{likl} = A^{ilkl}$.

Putting everything together…, $A^{iiij} = 0$, $A^{iijk} = 0$. So only the components with pairs of equal indices may be non-zero. The conditions not used yet are

$$\begin{aligned} 0 &= \\ 0 &= A^{iill} - A^{iikk} \end{aligned}$$

Thermodynamic constraints on parameters of the constitutive law

todo Discuss constraints on the sign of the parameters of the constitutive law, due to thermodynamics

Different expression of constitutive laws, and sets of parameters.

Remembering that $\mathrm{tr}(\mathbb{I}) = 3$ in the 3-dimensional space, evaluating the trace of the relation (6.1) provides the relation between the traces of strain and stress tensors and the temeprature difference

$$\mathrm{tr}(\boldsymbol{\sigma}) = (2\mu + 3\lambda)\,\mathrm{tr}(\boldsymbol{\varepsilon}) - 3\beta\Delta T \ ,$$

and thus

$$\mathrm{tr}(\boldsymbol{\varepsilon}) = \frac{1}{2\mu + 3\lambda}\mathrm{tr}(\boldsymbol{\sigma}) + \frac{3\beta}{2\mu + 3\lambda}\Delta T \ .$$

Using the relation between traces, it's possible to find $\boldsymbol{\varepsilon}\,(\boldsymbol{\sigma}, \Delta T)$,

$$\begin{aligned}
\boldsymbol{\varepsilon} &= \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu}\mathrm{tr}(\boldsymbol{\varepsilon})\mathbb{I} + \frac{\beta}{2\mu}\Delta T\mathbb{I} = \\
&= \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu}\left[\frac{1}{2\mu + 3\lambda}\mathrm{tr}(\boldsymbol{\sigma}) + \frac{3\beta}{2\mu + 3\lambda}\Delta T\right]\mathbb{I} + \frac{\beta}{2\mu}\Delta T\mathbb{I} = \\
&= \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\mathrm{tr}(\boldsymbol{\sigma})\mathbb{I} + \frac{1}{2\mu}\left[1 - \frac{3\lambda}{2\mu + 3\lambda}\right]\beta\Delta T\mathbb{I} = \\
&= \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\mathrm{tr}(\boldsymbol{\sigma})\mathbb{I} + \frac{1}{2\mu + 3\lambda}\beta\Delta T\mathbb{I} \ .
\end{aligned}$$

Modulo elastico. Il modulo elastico E è definito dalla relazione

$$\begin{aligned}
\frac{1}{E} &= \frac{1}{2\mu} - \frac{\lambda}{2\mu(2\mu + 3\lambda)} = \\
&= \frac{1}{2\mu}\frac{2(\mu + \lambda)}{2\mu + 3\lambda} \ ,
\end{aligned}$$

e quindi

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda}$$

Modulo di Poisson. Il modulo di Poisson ν è definito dalla relazione

$$\frac{\nu}{E} = \frac{\lambda}{2\mu(2\mu + 3\lambda)} \ ,$$

e quindi

$$\nu = \frac{\lambda}{2(\mu + \lambda)}$$

Thermodynamics

Helmholtz's free energy is the thermodynamic potential F defined w.r.t. temperature T and generalized displacements $\mathbf{X}_i$ as independet variables. Here **unit-volume** relation (Explain why! Link to general continuum mechanics, and equations in reference coordinates). Using difference of temperature $\Delta T = T - T_0$ w.r.t. a reference temperature T_0 instead of temperature T, differential of Helmhtoltz's free energy per unit volume for a linear elastic medium reads

$$d\mathcal{F}(\varepsilon_{ij}, \Delta T) = -\mathcal{S} d\Delta T + \sigma_{ij} d\varepsilon_{ij} \ ,$$

so that

$$\sigma_{ij} = \left(\frac{\partial \mathcal{F}}{\partial \varepsilon_{ij}}\right)_{\mathcal{S}} \qquad , \qquad \mathcal{S} = -\left(\frac{\partial \mathcal{F}}{\partial T}\right)_{\varepsilon_{ij}} .$$

Using the constitutive law (6.1) for a linear isotropic elastic medium, here written in Cartesian coordinates as

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} - \beta\Delta T\delta_{ij} ,$$

integration w.r.t. ε_{ij} gives

$$\begin{aligned}\mathcal{F}(\varepsilon_{ij}, \Delta T) &= \frac{1}{2}\left(2\mu\varepsilon_{ij}\varepsilon_{ij} + \lambda\varepsilon_{kk}\varepsilon_{ll}\right) - \beta\Delta T\varepsilon_{ll} + \mathcal{F}(0, \Delta T) = \\ &= \frac{1}{2}\left(2\mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + \lambda\left(\mathrm{tr}(\boldsymbol{\varepsilon})\right)^2\right) - \beta\Delta T\mathrm{tr}(\boldsymbol{\varepsilon}) + \mathcal{F}(0, \Delta T)\end{aligned}$$

being $\mathcal{F}(0, \Delta T) = F(\Delta T)$ a function of ΔT appearing from integration in ε_{ij}. If you don't trust this, try do evaluate the partial derivative w.r.t. the components of the strain tensor of the last relation.

Entropy. Entropy is the parital derivative w.r.t. T of Helmholtz's free energy and, assuming constant parameters, its expression reads

$$\mathcal{S}(\varepsilon_{ij}, \Delta T) = -\left(\frac{\partial \mathcal{F}}{\partial T}\right)_{\varepsilon_{ij}} = \beta\varepsilon_{ll} - F'(\Delta T) .$$

Heat coefficients. Heat coefficient at constant strain per unit-volume reads

$$C_{\varepsilon_{ij}} := T\left(\frac{\partial \mathcal{S}}{\partial T}\right)_{\varepsilon_{ij}} = -(T_0 + \Delta T)F''(\Delta T) .$$

Assuming that heat coefficient $C_{\varepsilon_{ij}}$ is independent from T, integration in ΔT gives

$$C_{\varepsilon_{ij}} \ln\left(1 + \frac{\Delta T}{T_0}\right) = -F'(\Delta T) + F'(0) ,$$

and thus an expression for $F'(\Delta T)$

$$F'(\Delta T) = -C_{\varepsilon_{ij}} \ln\left(1 + \frac{\Delta T}{T_0}\right) + F'(0) ,$$

to be inserted in the expression of entropy

$$\mathcal{S}(\varepsilon_{ij}, \Delta T) = \beta\varepsilon_{ll} + C_{\varepsilon_{ij}} \ln\left(1 + \frac{\Delta T}{T_0}\right) - F'(0) ,$$

and $-F'(0) = \mathcal{S}_0$ can be recognized as the entropy per unit volume in the reference condition with $\varepsilon_{ij} = 0$, $\Delta T = 0$. The differential of the last expression reads

$$d\mathcal{S} = \beta d\varepsilon_{ll} + \frac{C_{\varepsilon_{ij}}}{T_0}\frac{1}{1 + \frac{\Delta T}{T_0}}\, dT = \beta d\varepsilon_{ll} + \frac{C_{\varepsilon_{ij}}}{T}\, dT$$

or

$$dT = \frac{T}{C_{\varepsilon_{ij}}} d\mathcal{S} - \frac{\beta T}{C_{\varepsilon_{ij}}}\, d\varepsilon_{ll}$$

Furhter integration in ΔT of

$$F'(\Delta T) = -C_{\varepsilon_{ij}} \ln\left(1 + \frac{\Delta T}{T_0}\right) - \mathcal{S}_0 ,$$

gives and expression of funtion $F(\Delta T)$,

$$F(\Delta T) - F(0) = -C_{\varepsilon_{ij}} \left[(T_0 + \Delta T) \ln \left(1 + \frac{\Delta T}{T_0} \right) - \Delta T \right] - \mathcal{S}_0 \Delta T \ ,$$

that can be used int he expression of Helmholtz free energy, as an example, as show later.

Internal energy. The relation between the internal energy and Hemlholtz free energy $\mathcal{F} = \mathcal{E} - T\mathcal{S}$ allows to find the expression of the internal energy per unit volume of an elastic linear isotropic media,

$$\begin{aligned}
\mathcal{E} &= \mathcal{F} + T\mathcal{S} = \\
&= \mu \varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2} \left(\varepsilon_{kk} \right)^2 - \beta \Delta T \varepsilon_{kk} + F(0) - C_{\varepsilon_{ij}} \left[(T_0 + \Delta T) \ln \left(1 + \frac{\Delta T}{T_0} \right) - \Delta T \right] \\
&\quad - \mathcal{S}_0 \Delta T + (T_0 + \Delta T) \left(\beta \varepsilon_{ll} + C_{\varepsilon_{ij}} \ln \left(1 + \frac{\Delta T}{T_0} \right) + \mathcal{S}_0 \right) = \\
&= \mu \varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2} \left(\varepsilon_{kk} \right)^2 + \beta T_0 \varepsilon_{kk} + C_{\varepsilon_{ij}} \Delta T + F(0) + T_0 \mathcal{S}_0 \ ,
\end{aligned}$$

so that the reference internal energy and reference Helmholtz free energy can be recognized as

$$\mathcal{F}_0 = F(0) \qquad , \qquad \mathcal{E}_0 = \mathcal{F}_0 + T_0 \mathcal{S}_0 \ ,$$

In order to write the differential of the internal energy w.r.t. its natural independent variables $\mathcal{E}(\varepsilon_{ij}, \mathcal{S})$, the temperature difference must be written as a function of strain and entropy. Using relation **todo**, and performing derivatives of the composite funciton $\mathcal{E}(\varepsilon_{ij}, \Delta T(\varepsilon_{ij}, \mathcal{S}))$

$$\begin{aligned}
d\mathcal{E} &= \left(\frac{\partial \mathcal{E}}{\partial \varepsilon_{ij}} \right)_{\Delta T} d\varepsilon_{ij} + \left(\frac{\partial \mathcal{E}}{\partial \Delta T} \right)_{\varepsilon_{ij}} \left[\left(\frac{\partial \Delta T}{\partial \varepsilon_{ij}} \right)_{\mathcal{S}} d\varepsilon_{ij} + \left(\frac{\partial \Delta T}{\partial \mathcal{S}} \right)_{\varepsilon_{ij}} d\mathcal{S} \right] = \\
&= \left(2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij} + \beta T_0 \delta_{ij} \right) d\varepsilon_{ij} + C_{\varepsilon_{ij}} \left[-\frac{\beta T}{C_{\varepsilon_{ij}}} \delta_{ij} \, d\varepsilon_{ij} + \frac{T}{C_{\varepsilon_{ij}}} \, d\mathcal{S} \right] = \\
&= \left(2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij} - \beta \Delta T \delta_{ij} \right) d\varepsilon_{ij} + T \, d\mathcal{S} \ ,
\end{aligned}$$

i.e. we found what we should have expected, i.e.

$$d\mathcal{E} = \sigma_{ij} \, d\varepsilon_{ij} + T \, d\mathcal{S} \ ,$$

and thus

$$\begin{aligned}
\sigma_{ij} &= \left(\frac{\partial \mathcal{F}}{\partial \varepsilon_{ij}} \right)_T = \left(\frac{\partial \mathcal{E}}{\partial \varepsilon_{ij}} \right)_{\mathcal{S}} \\
T &= \left(\frac{\partial \mathcal{E}}{\partial \mathcal{S}} \right)_{\varepsilon_{ij}} \\
\mathcal{S} &= -\left(\frac{\partial \mathcal{F}}{\partial T} \right)_{\varepsilon_{ij}}
\end{aligned}$$

Isothermal and isentropic elastic coefficients

Assuming small enough $\Delta T = T - T_0$ so that linear approximation of the relation between entropy, temperature and strain holds,

$$\Delta \mathcal{S} = \beta \, \varepsilon_{ll} + \frac{C_{\varepsilon_{ij}}}{T_0} \Delta T \ ,$$

it's possible to write the stress tensor as a function of strain and entropy

$$
\begin{aligned}
\sigma_{ij} &= 2\mu\varepsilon_{ij} + \lambda\varepsilon_{ll}\delta_{ij} - \beta\Delta T\delta_{ij} = \\
&= 2\mu\varepsilon_{ij} + \lambda\varepsilon_{ll}\delta_{ij} - \beta\left[\frac{T_0}{C_{\varepsilon_{ij}}}\Delta\mathcal{S} - \frac{T_0}{C_{\varepsilon_{ij}}}\beta\,\varepsilon_{ll}\right]\delta_{ij} = \\
&= 2\mu\varepsilon_{ij} + \left(\lambda + \frac{\beta^2 T_0}{C_{\varepsilon_{ij}}}\right)\varepsilon_{ll}\delta_{ij} - \beta\frac{T_0}{C_{\varepsilon_{ij}}}\Delta\mathcal{S} = \\
&= 2\mu_s\varepsilon_{ij} + \lambda_s\varepsilon_{ll}\delta_{ij} - \beta\frac{T_0}{C_{\varepsilon_{ij}}}\Delta\mathcal{S}\ .
\end{aligned}
$$

having defined Lamé coefficients in isentropic conditions as functions of the cofficients in isothermal conditions,

$$
\begin{aligned}
\lambda_s &= \lambda + \frac{\beta^2 T_0}{C_{\varepsilon_{ij}}} \\
\mu_s &= \mu
\end{aligned}
$$

Elastic modulus and Poisson ratio. Starting from the relations

$$
\begin{aligned}
E &= \frac{\mu(2\mu+3\lambda)}{\mu+\lambda} \\
\frac{E}{\nu} &= \frac{2\mu(2\mu+3\lambda)}{\lambda}
\end{aligned}
$$

$$GE + \lambda E = 2G^2 + 3G\lambda$$

$$\lambda = \frac{G(E-2G)}{3G-E}$$

$$
\begin{aligned}
\nu &= \frac{E\lambda}{2G(2G+3\lambda)} = \\
&= \frac{\lambda}{2G(2G+3\lambda)}\frac{G(2G+3\lambda)}{G+\lambda} = \\
&= \frac{\lambda}{2(G+\lambda)} = \\
&= \frac{G(E-2G)}{3G-E}\frac{1}{2\left(G+\frac{G(E-2G)}{3G-E}\right)} = \\
&= \frac{G}{2G}\frac{E-2G}{3G-E}\frac{3G-E}{3G-E+E-2G} = \\
&= \frac{1}{2}\frac{E-2G}{G}\ .
\end{aligned}
$$

so that

$$\nu = \frac{E-2G}{2G} \qquad , \qquad G = \frac{E}{2(1+\nu)}$$

Heat capacity, thermal expansion coefficients, compressibility coefficients

Thermal expansion coefficient reads

$$\alpha_x := \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_x$$

with $\dfrac{dV}{V} = d\mathrm{tr}(\boldsymbol{\varepsilon})$ for small displacement and strain regime. At constant strain, $d\sigma_{ij} = 0$,

$$\alpha_\sigma = \left(\frac{\partial \varepsilon_{ll}}{\partial T}\right)_\sigma$$

Some algebra

$$\begin{aligned}
&= \frac{3\beta}{2\mu + 3\lambda} = \\
&= 3\beta\frac{2G\nu}{E\lambda} = \\
&= 3\beta\frac{\nu}{(1+\nu)\lambda} = \\
&= 3\beta\frac{\nu}{(1+\nu)}\frac{3G-E}{G(E-2G)} = \\
&= 3\beta\frac{\nu}{(1+\nu)}\frac{3-2(1+\nu)}{G\left(2(1+\nu)-2\right)} = \\
&= 3\frac{\beta}{2G}\frac{\nu}{(1+\nu)}\frac{1-2\nu}{\nu} = \\
&= 3\frac{\beta}{E}(1-2\nu)\ .
\end{aligned}$$

so that

$$\beta = K\alpha_\sigma = \left(\frac{2}{3}\mu + \lambda\right)\,\alpha_\sigma = \frac{E}{3(1-2\nu)}\,\alpha_\sigma$$

Heat capacity. The heat capacity at constant strain $C_{\varepsilon_{ij}}$ has been assumed to be constant (or just independent from temperature). Entropy can be written as functions of strain and temperature or - exploiting the relation between traces of tensors - stress and temperature

$$\begin{aligned}
\mathcal{S} - \mathcal{S}_0 &= \beta\varepsilon_{ll} + C_{\varepsilon_{ij}}\ln\left(1 + \frac{\Delta T}{T_0}\right) \\
&= \beta\left(\frac{1}{2\mu + 3\lambda}\sigma_{ll} + \alpha_\sigma\Delta T\right) + C_{\varepsilon_{ij}}\ln\left(1 + \frac{\Delta T}{T_0}\right) \\
&= \beta\left(\frac{1}{3\,K}\sigma_{ll} + \alpha_\sigma\Delta T\right) + C_{\varepsilon_{ij}}\ln\left(1 + \frac{\Delta T}{T_0}\right)
\end{aligned}$$

and thus, constaint strain heat capacity per unit volume reads

$$C_{\varepsilon_{ij}} := T\left(\frac{\partial \mathcal{S}}{\partial T}\right)_\varepsilon = C_{\varepsilon_{ij}}\ ,$$

as defined above, whilte constant stress heat capacity per unit volume reads

$$\begin{aligned}
C_{\sigma_{ij}} &:= T\left(\frac{\partial \mathcal{S}}{\partial T}\right)_\sigma = \\
&= T\left(\beta\alpha_\sigma + C_{\varepsilon_{ij}}\frac{1}{T_0}\frac{1}{1 + \frac{\Delta T}{T_0}}\right) \\
&= TK\alpha_\sigma^2 + C_{\varepsilon_{ij}}
\end{aligned}$$

Compressibility coefficients. **todo** *check: Below, T or T_0? It should be a minor change, since we're assuming small ΔT so that linearized constitutive equation holds?*

$$K = \frac{2}{3}\mu + \lambda$$

$$K_s = \frac{2}{3}\mu_s + \lambda_s = \frac{2}{3}\mu + \lambda + \frac{\beta^2 T}{C_{\varepsilon_{ij}}} = K + \frac{\beta^2 T}{C_{\varepsilon_{ij}}} = K + \frac{K^2 \alpha_\sigma^2 T}{C_\varepsilon}$$

$$K_s = K\left(1 + \frac{\alpha^2 K T}{C_\varepsilon}\right)$$

$$\rightarrow \quad \frac{1}{K_s} = \frac{1}{K}\frac{C_\varepsilon}{C_\sigma}$$

$$\rightarrow \quad \frac{1}{K_s} = \frac{1}{K}\frac{C_\sigma - \alpha_\sigma^2 K T}{C_\sigma}$$

$$\rightarrow \quad \frac{1}{K_s} = \frac{1}{K} - \frac{\alpha_\sigma^2 T}{C_\sigma}$$

Elastic modulus and Poisson ratio.

$$E_s = \frac{E}{1 - E\,\dfrac{\alpha_\sigma^2 T}{9C_\sigma}} \qquad , \qquad \nu_s = \frac{\nu + E\,\dfrac{\alpha_\sigma^2 T}{9C_\sigma}}{1 - E\,\dfrac{\alpha_\sigma^2 T}{9C_\sigma}}$$

Some algebra

$$\begin{aligned}
K &= \lambda + \frac{2}{3}G = \\
&= \frac{GE - 2G^2}{3G - E} + \frac{2}{3}G = \\
&= G\frac{3E - 6G + 6G - 2E}{3(3G - E)} = \frac{EG}{3(3G - E)}
\end{aligned}$$

$$E = \frac{9GK}{3K + G} = \frac{1}{\frac{1}{3G} + \frac{1}{9K}}$$

$$\begin{aligned}
E_s &= \frac{1}{\dfrac{1}{3G_s} + \dfrac{1}{9K_s}} = \\
&= \frac{1}{\dfrac{1}{3G} + \dfrac{1}{9K} - \dfrac{\alpha_\sigma^2 T}{9C_\sigma}} = \\
&= \frac{\left(\dfrac{1}{3G} + \dfrac{1}{9K}\right)^{-1}}{\left(\dfrac{1}{3G} + \dfrac{1}{9K} - \dfrac{\alpha_\sigma^2 T}{9C_\sigma}\right)\left(\dfrac{1}{3G} + \dfrac{1}{9K}\right)^{-1}} = \\
&= \frac{E}{1 - E\,\dfrac{\alpha_\sigma^2 T}{9C_\sigma}} =
\end{aligned}$$

Fix admonition reference

Isotropic tensor

An isotropic tensor is a tensor whose components do not change after a rotation of the vector basis. **todo** *Examples,...*

Rank-2 isotropic tensor

The most general expression of a rank-2 isotropic tensor is proportional to the rank-2 identity tensor, and can be written in a Cartesina basis using the Kroeneker delta,

$$a\mathbf{I} = a\delta_{ij}\hat{e}_i \otimes \hat{e}_j \ .$$

todo *Proof*

Rank-4 isotropic tensor

The most general expression of a rank-4 isotropic tensor can be written using a Cartesian basis as

$$\mathbf{D} = D_{ijkl}\hat{e}_i \otimes \hat{e}_j \otimes \hat{e}_k \otimes \hat{e}_l \ ,$$

where

$$D_{ijkl} = a\delta_{ij}\delta_{kl} + b\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk} \ ,$$

i.e. depends on three possible combinations of rank-2 identity tensor, with 3 scalar parameters, a, b, c. In isotropic relation between symmetric tensors, $A_{ij} = A_{ji}$, $B_{kl} = B_{lk}$ only two parameters are enough since

$$\begin{aligned} A_{ij} & = D_{ijkl}B_{kl} = \\ & = \left(a\delta_{ij}\delta_{kl} + b\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk}\right)B_{kl} = \\ & = a\delta_{ij}B_{ll} + bB_{ij} + cB_{ji} = \qquad (1) \\ & = a\delta_{ij}B_{ll} + (b+c)B_{ij} \ , \end{aligned}$$

having used (1) the symmetry of tensor **B**, $B_{ji} = B_{ij}$.

References.

- Fluid Mechanics, R. Fitzpatrick, University of Texas, Austin
- P.G. Hodge, *On Isotropic Cartesian Tensors*, 1961, The American Mathematical Monthly

---

6.3 Beam models

6.4 Beam structures

CHAPTER

SEVEN

SMALL DISPLACEMENT - STATICS - WEAK FORMULATION AND "ENERGY" THEOREMS

This section presents different weak formulations in the elastic problem, for both equilibrium conditions and congruence conditions, for different models/structural elements.

Models. In this section, weak formulations for different models of elastic structures are explicitly derived:

- *Three-dimensional solid*
- *General beam model*
- *Simplified beam models*

Approaches. The results shown here can be classified into two different approaches, depending on the independent variables of the methods:

- **displacement approach**: 1) weak form of the equilibrium equations, 2) principle of virtual work, and 3) stationariety of the total potential energy
- **force approach**: 1) weak form of the congruence conditionss, 2) principle of complementary virtual work, and 3) stationariety of the total complementary potential energy.

A *force approach* can be efficiently used to *evaluate hyperstatics* and *point displacements/rotations*, common questions in simple problems/exercises of the introductory courses in structural mechanics, as a result of low-dimensional linear systems that can be solved with a pen-and-paper approach. *Displacement methods* usually require some assumptions about the displacement field, and provides a common general methods for solving the *whole structure*. As an example, *finite element method* can be formulated as a discrete counterpart (usually a projection over a finite-dimensional basis) of the the weak form of equilibrium equations: *FEM* usually results in a *"higher"-dimensional* linear system[1] (if compared with those of a force approach), that makes from little to no sense at all to solve *by-hand*, but can be easily built and solved with a computer.

Examples, problems and exercises. Some *problems about beam structures* can be conveniently approached with the contents of this section.

Reading guidance

There is no mandatory or recommended order for addressing these paragraphs. The treatment of the **elastic solid** requires some familiarity with *tensors*. Beam models can be treated as simplified models of the 3-dimensional solid, under assumptions on stress (or strain) fields. The formulation of the **general elastic beam** introduces a vector-based framework applicable to any elastic beam: it may appear obscure or cryptic at first, but it offers a level of generality that greatly simplifies the *numerical implementation* of arbitrary beam models. The **structurally decoupled elastic beam** is a very

[1] Finite element methods for linear structural problems usually provide a linear problem, $\mathbf{K}\mathbf{u} = \mathbf{f}$, with some useful properties that can be exploited while building and solving the problem: the stiffness matrix $\mathbf{K}$ is usually **sparse** (this properties allows to *save memory*, using sparse matrix format to avoid storing in an inefficient way lots of information — lots of zero; efficients *algorithms* exist for matrix operations with sparse matrices) and **symmetric** (many *algorithms* work well with symmetric matrices).

specific case of this general treatment, yet it enables the construction of *simple models* suitable for solving structural problems by hand, using only paper, pencil, and manageable amounts of algebra. Simple models become even simpler for *slender beams* with Bernoulli's kinematic assumption - i.e. if displacement due to axial and shear actions are negligible if compared with bending. This approach helps internalize the core principles by allowing practice on simpler problems without unnecessary algebraic complexity — after all, computers exist for the heavy calculations.

7.1 3-dimensional solid

Starting from strong form of equilibrium equations, inner compatibility and congruence with essential boundary conditions, it's possible to:

- derive a weak formulation of the problem
- derive energy theorems, with a proper choice of the test function involved in the weak formulation.

Summary

- Strong formulation of the problem
- Weak formulation
  - Existence and uniqueness of the solution
  - Principle of virtual work and complementary virtual work
  - Principle of stationariety of the total potential energy Π and total complementary potential energy Π^*
  - Classical theorems: Maxwell-Betti ($F^1 s^2 = F^2 s^1$), Menabrea-Castigliano ($s_i = \partial_{F_i} \Pi$, $F_i = \partial_{s_i} \Pi$)

7.1.1 Strong formulation of the problem

Indefinite equilibrium and natural boundary conditions on S_N.

$$\begin{cases} \nabla \cdot \sigma + \bar{\mathbf{f}} = \mathbf{0} & \text{in } V \\ \hat{\mathbf{n}} \cdot \sigma = \bar{\mathbf{t}}_{\mathbf{n}} & \text{on } S_N \end{cases}$$

Internal congruence and compatibility with essential constraints on S_D.

$$\begin{cases} \varepsilon = \nabla^S \mathbf{s} = \dfrac{1}{2} \left(\nabla \mathbf{s} + \nabla^T \mathbf{s} \right) & \text{in } V \\ \mathbf{s} = \bar{\mathbf{s}} & \text{on } S_D \end{cases}$$

Other boundary conditions - e.g. Robin. Beside essential boundary conditions (prescribing the displacement) and natural boundary condtions (prescribing the stress vector), other boundary conditions may exist. As an example, Robin boundary conditions are defined as a boundary condition prescribing a linear combination of displacement and stress, and may represent *flexible constraints*. The most general affine relation between dispacement and stress vector reads

$$\mathbf{a} \cdot \mathbf{s} = \mathbf{b} \cdot \sigma \cdot \hat{\mathbf{n}} + \mathbf{c} \qquad \text{on } S_R \ ,$$

having exploited here the symmetry (**todo**) of the stress tensor σ. If α is invertible, the latter relation may be written in the form

$$\mathbf{s} = \mathbf{b} \cdot \sigma \cdot \hat{\mathbf{n}} + \mathbf{c} \qquad \text{on } S_R \ .$$

Linear elastic constitutive equation. Constitutive equation of linear elastic media in the regime of small displacement reads

$$\varepsilon = \mathbf{D} : \sigma + \alpha\,\Delta T \ ,$$

being the latter the contribution of thermal strains. The "inverse" relation reads

$$\sigma = \mathbf{C} : \varepsilon - \beta\,\Delta T \ .$$

If temperature field is prescribed and known it can be treated as a forcing, otherwise its an unkown physical quantity and the internal energy (or temperature) balance equation needs to be solved along with the mechanical equilibrium equation.

7.1.2 Weak formulations of the problem

Weak formulation of equilibrium conditions

For every[1] function $\mathbf{w}$

$$\begin{aligned}
0 &= \int_V \mathbf{w}\cdot\left\{\nabla\cdot\sigma + \overline{f}\right\} = \\
&= \int_V w_j \left\{\sigma_{ij/i} + \overline{f}_j\right\} = \\
&= \oint_{\partial V} n_i \sigma_{ij} w_j - \int_V w_{j/i}\sigma_{ij} + \int_V w_j \overline{f}_j = \\
&= -\int_V w_{j/i}\sigma_{ij} + \int_V w_j \overline{f}_j + \int_{S_N} w_j \bar{t}_n + \int_{\partial V/S_N} w_j t_j = \\
&= -\int_V \frac{1}{2}\left(w_{j/i} + w_{i/j}\right)\sigma_{ij} + \int_V w_j \overline{f}_j + \int_{S_N} w_j \bar{t}_n + \int_{\partial V/S_N} w_j t_j =
\end{aligned}$$

having exploited symmetry of stress tensor σ.

Weak formulation of congruence conditions

For every 2^{nd} order tensor function Ω

$$\begin{aligned}
0 &= \int_V \Omega : \left\{\varepsilon - \frac{1}{2}\left(\nabla\mathbf{s} + \nabla^T\mathbf{s}\right)\right\} = \\
&= \int_V \Omega_{ij}\left\{\varepsilon_{ij} - \frac{1}{2}\left(s_{i/j} + s_{j/i}\right)\right\} = \\
&= \int_V \frac{1}{2}\left\{\Omega_{ij/j} s_i + \Omega_{ij/i} s_j\right\} + \int_V \Omega_{ij}\varepsilon_{ij} - \oint_{\partial V} \frac{1}{2}\left\{n_j \Omega_{ij} s_i + n_i \Omega_{ij} s_j\right\} \ .
\end{aligned}$$

If Ω is chosen to be symmetric,

$$\begin{aligned}
0 &= \int_V \Omega_{ij/j} s_i + \int_V \Omega_{ij}\varepsilon_{ij} - \oint_{\partial V} n_i \Omega_{ij} s_j = \\
&= \int_V \Omega_{ij/j} s_i + \int_V \Omega_{ij}\varepsilon_{ij} - \int_{S_D} n_i \Omega_{ij} \overline{s}_j - \int_{\partial V/S_D} n_i \Omega_{ij} s_j \ .
\end{aligned}$$

[1] For every test function that is **regular enough**, meaning that everything that is written in the equatiions exist.

Existence and uniqueness of the solution

Theorem 7.1.1 (Existence and uniqueness of the solution of the small-strain, small-displacement elastic problem)

Under the assumptions …, there exists a unique solution of the elastic problem that is at the same time congruent and equilibrated.

In structural mechanics, it's quite common to deal with congruent displacement and strain fields, and equilibrated stress fields. Stress and strain fields are related via the constitutive law of the medium. A congruent displacement and strain field may produce non-equilibrated stress fields; an equilibrated stress field may produce non-congruent displacement field. Under the assumptions… it's possible to prove that the lienar elastic problem has a unique solution corresponding to the congruent strain and displacement and equilibrated stress fields, i.e. the unique set of fields simultaneously satisfying equilibrium equations and constraints.

The proof of existence of a solution requires some functional analysis tools (not developed here, yet). The proof of uniqueness and the necessary conditions can be discussed with a proof by contradiction, assuming that two solutions of the same problem exists and evaluating the norm of the difference of the solution on the whole domain, quantifying the difference between these solutions, and eventually assessing that this norm is identically zero under certain assumptions.

The proof follows a similar procedure as the proof of uniqueness of the solution of elliptic problems.

Proof.

Equilibrium equation for statics can be written as a function of strain and displacement, exploiting the constitutive equation $\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - \beta_{ij}\Delta T$,

$$\begin{aligned}
& 0 = \sigma_{ij/i} + f_j = \left(C_{ijkl}\varepsilon_{kl} - \beta_{ij}\Delta T\right)_{/i} + f_j && \text{in } V \\
& g_i = s_i && \text{on } S_D \\
& t_j = n_i\sigma_{ij} = n_i(C_{ijkl}\varepsilon_{kl} - \beta_{ij}\Delta T) && \text{on } S_N
\end{aligned}$$

Let's assume two strain fields exists s.t. their solution of the elastic problem, and let's define $\delta\varepsilon = \varepsilon_2(\mathbf{r}) - \varepsilon_1(\mathbf{r})$. This difference satisfies the homogeneous problem

$$\begin{aligned}
& 0 = C_{ijkl}\delta\varepsilon_{kl} && \text{in } V \\
& 0 = \delta s_i && \text{on } S_D \\
& 0 = n_i\, C_{ijkl}\delta\varepsilon_{kl} && \text{on } S_N \ ,
\end{aligned}$$

given a prescribed temperature field ΔT.

…

$$\begin{aligned}
\int_V \delta\varepsilon_{ij} C_{ijkl}\delta\varepsilon_{kl} & = \int_V \frac{1}{2}\left(\delta s_{i/j} + \delta s_{j/i}\right) C_{ijkl}\delta\varepsilon_{kl} = \\
& = \int_V \delta s_{j/i} C_{ijkl}\delta\varepsilon_{kl} = \\
& = \int_V \left\{ \left(\delta s_j C_{ijkl}\delta\varepsilon_{kl}\right)_{/i} - \delta s_j\left(C_{ijkl}\delta\varepsilon_{kl}\right)_{/i} \right\} = \\
& = \oint_{\partial V} n_i\,\delta s_j C_{ijkl}\delta\varepsilon_{kl} - \int_V \delta s_j\left(C_{ijkl}\delta\varepsilon_{kl}\right)_{/i} = \\
& = \int_{S_D} n_i \underbrace{\delta s_j}_{\delta s_j|_{S_D}=0} C_{ijkl}\delta\varepsilon_{kl} + \int_{S_N} n_i\,\delta s_j \underbrace{C_{ijkl}\delta\varepsilon_{kl}}_{C_{ijkl}\delta\varepsilon|_{S_N}=0} - \int_V \delta s_j \underbrace{\left(C_{ijkl}\delta\varepsilon_{kl}\right)_{/i}}_{=0} = \\
& = 0\ .
\end{aligned}$$

As $\delta\varepsilon : \mathbf{C} : \delta\varepsilon \ge 0$ for every symmetric 2^{nd}-order tensor (**todo** this is related to constraints on deformation energy…), the integral condition implies $\delta\varepsilon : \mathbf{C} : \delta\varepsilon = 0$ (**todo** check it). As $\mathbf{C}$ … (**todo** which property required?), this further implies $\delta\varepsilon = \mathbf{0}$, and thus the two solution may differ at most by a rigid motion,

$$s_2(\mathbf{r}) = s_1(\mathbf{r}) + \mathbf{a} + \theta_\times \mathbf{r} \ .$$

If the problem has boundary conditions preventing rigid motion, i.e. no rigid degree of freedom, there's no arbitrariety in the solution as the boundary conditions set $\mathbf{a} = \mathbf{0}$, and $\theta = \mathbf{0}$, and thus

$$\mathbf{s}_1(\mathbf{r}) = \mathbf{s}_2(\mathbf{r}) \ .$$

Comments and todo.

- **todo** Comment condition $\varepsilon : \mathbf{C} : \varepsilon = \varepsilon_{ij} C_{ijkl} \varepsilon \ge 0$, with internal energy balance equation
- Show explicitly the case of a linear isotropic medium, governed by the constitutive law $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{ll}\delta_{ij}$, i.e.

  $$C_{ijkl} = \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk} + \lambda\delta_{ij}\delta_{kl} \ ,$$

  i.e. the most general expression of an isotropic rank-4 tensor relating two isotropic and symmetric rank-2 tensors, like the stress and small strain tensors[2]. In this case the double dot product of the stress and strain tensors reads

  $$\varepsilon : \sigma = \varepsilon_{ij}\sigma_{ij} = \varepsilon_{ij}\left(2\mu\varepsilon_{ij} + \lambda\varepsilon_{ll}\delta_{ij}\right) = 2\mu\varepsilon_{ij}\varepsilon_{ij} + \lambda\varepsilon_{ll}\varepsilon_{kk}\delta_{ij} = 2\mu|\varepsilon|^2 + \lambda \mathrm{tr}\left(\varepsilon\right)^2 \ge 0 \ .$$

- **todo** Any thermodynamic condition prescribing the sign of the coefficients μ, λ? Anything like minimum energy or maximum entropy principle at thermodynamic equilibrium?

Principle of virtual work

Starting from the *weak form of equilibrium conditions*, and choosing $\mathbf{w}$ to be the variation of a **congruent displacement field** $\tilde{s}$ with internal congruence in V and compatibility with *given* essential constraints on S_D, i.e. that satisfies the conditions

$$\begin{aligned} &\tilde{\varepsilon} := \frac{1}{2}\left(\nabla\tilde{\mathbf{s}} + \nabla^T\tilde{\mathbf{s}}\right) && \text{in } V \\ &\tilde{\mathbf{s}} = \bar{\mathbf{s}} && \text{on } S_D \ , \end{aligned}$$

with **no other conditions** on $\tilde{\sigma}$ in V and $\tilde{\mathbf{t}}_\mathbf{n}$ on S_N. From the definition $\mathbf{w} = \delta\tilde{\mathbf{s}}$, it follows

$$\begin{aligned} &\delta\tilde{\varepsilon} = \frac{1}{2}\left(\nabla\delta\tilde{\mathbf{s}} + \nabla^T\delta\tilde{\mathbf{s}}\right) && \text{in } V \\ &\delta\tilde{\mathbf{s}} = \mathbf{0} && \text{on } S_D \ , \end{aligned}$$

and

$$0 = -\int_V \delta\tilde{\varepsilon}_{ij}\sigma_{ij} + \int_V \delta\tilde{s}_j \overline{f}_j + \int_{S_N} \delta\tilde{s}_j \bar{t}_j + \int_{S_R} \delta\tilde{s}_j t_j \ .$$

Principle of complementary virtual work

Starting from the *weak form of congruence conditions*, and choosing Ω to be the variation of an **equilibrated stress field** $\tilde{\sigma}$ due to *given* external loads $\tilde{\mathbf{f}}$ in V and $\tilde{\mathbf{t}}_\mathbf{n}$ on S_N, i.e. satisfying the conditions

$$\begin{cases} \tilde{\sigma}_{ij/i} + \tilde{f}_j = 0 & \text{in } V \\ n_i\tilde{\sigma}_{ij} = \tilde{t}_j & \text{in } S_N \end{cases}$$

[2] Otherwise the first and second scalar factors are different, $C_{ijkl} = \mu\delta_{ik}\delta_{jl} + \eta\delta_{il}\delta_{jk} + \lambda\delta_{ij}\delta_{kl}$.

with **no other conditions** on ε and $\mathbf{s}$ in V and S_D. From the definition $\Omega_{ij} = \delta\tilde{\sigma}_{ij}$, it follows

$$\begin{cases} \delta\tilde{\sigma}_{ij/i} = 0 & \text{in } V \\ n_i\,\delta\tilde{\sigma}_{ij} = 0 & \text{in } S_N \end{cases}$$

and

$$\begin{aligned} 0 &= \int_V \underbrace{\delta\tilde{\sigma}_{ij/i}}_{=0} s_j + \int_V \delta\tilde{\sigma}_{ij}\varepsilon_{ij} - \int_{S_D} n_i\delta\tilde{\sigma}_{ij}\bar{s}_j - \int_{S_N} \underbrace{n_i\delta\tilde{\sigma}_{ij}}_{=0} s_j - \int_{S_R} n_i\delta\tilde{\sigma}_{ij}s_j = \\ &= \int_V \delta\tilde{\sigma}_{ij}\varepsilon_{ij} - \int_{S_D} n_i\delta\tilde{\sigma}_{ij}\bar{s}_j - \int_{S_R} n_i\delta\tilde{\sigma}_{ij}s_j \,. \end{aligned}$$

Principle of stationariety of total potential energy

Choosing the (unique) solution of the elastic problem as the compatible field used in the *principle of virtual work*, $\tilde{\mathbf{s}} = \mathbf{s}$, $\tilde{\varepsilon} = \varepsilon$, it follows that

$$0 = -\int_V \delta\varepsilon : \sigma + \int_V \delta\mathbf{s}\cdot\bar{\mathbf{f}} + \int_{S_N} \delta\mathbf{s}\cdot\bar{\mathbf{t}} + \int_{S_R} \delta\mathbf{s}\cdot\mathbf{t} =$$

Different expressions of variation of the “internal energy”

todo check which kind of thermodynamic potential it really is.

$$\begin{aligned} \int_V \delta\varepsilon : \sigma &= \int_V \delta\varepsilon : (\mathbf{C} : \varepsilon - \beta\Delta T) = \\ &= \delta \int_V \left\{ \frac{1}{2}\varepsilon : \mathbf{C} : \varepsilon - \varepsilon : \beta\Delta T \right\} + \int_V \varepsilon : \beta\delta\Delta T \end{aligned}$$

$$\begin{aligned} \int_V \delta\sigma : \varepsilon &= \int_V \delta\sigma : (\mathbf{D} : \sigma + \alpha\Delta T) = \\ &= \delta \int_V \left\{ \frac{1}{2}\sigma : \mathbf{D} : \sigma + \sigma : \alpha\Delta T \right\} - \int_V \sigma : \alpha\delta\Delta T \end{aligned}$$

If the stress vector on Robin boundary reads $\mathbf{t} = -\mathbf{K}\cdot\mathbf{s} + \bar{\mathbf{h}}$, it follows

$$\begin{aligned} 0 &= -\int_V \delta\varepsilon : \sigma + \int_V \delta\mathbf{s}\cdot\bar{\mathbf{f}} + \int_{S_N} \delta\mathbf{s}\cdot\bar{\mathbf{t}} + \int_{S_R} \delta\mathbf{s}\cdot\left(-\mathbf{K}\cdot\mathbf{s} + \bar{\mathbf{h}}\right) = \\ &= \delta \underbrace{\left\{ -\int_V \frac{1}{2}\varepsilon : \mathbf{C} : \varepsilon + \int_V \varepsilon : \beta\Delta T + \int_V \mathbf{s}\cdot\bar{\mathbf{f}} + \int_{S_N} \mathbf{s}\cdot\bar{\mathbf{t}} - \int_{S_R} \frac{1}{2}\mathbf{s}\cdot\mathbf{K}\cdot\mathbf{s} + \int_{S_R} \mathbf{s}\cdot\bar{\mathbf{h}} \right\}}_{=:\Pi(\varepsilon,\mathbf{s})} + \int_V \varepsilon : \beta\,\delta\Delta T \,, \end{aligned}$$

and if ΔT is prescribed, it follows $\delta\Delta T = 0$, and

$$0 = \delta\Pi(\varepsilon, \mathbf{s}) \,.$$

Theorem 7.1.2 (Principle of stationariety of total potential energy)

Among all the equilibrated solutions, the congruent solution (and thus the unique solution of the elastic problem) is the one that makes the total potential energy stationary.

Principle of stationariety of total complementary potential energy

Choosing the (unique) solution of the elastic problem as the **equilibrated stress field** in the *principle of complementary virtual work*, $\tilde{\sigma} = \sigma$ with given loads $\tilde{\mathbf{f}}$, if the displacement of the Robin boundary reads $\mathbf{s} = -\mathbf{S} \cdot \mathbf{t_n} + \bar{\mathbf{r}}$,

$$\begin{aligned}
0 &= \int_V \delta\sigma : \varepsilon - \int_{S_D} \delta\mathbf{t_n} \cdot \bar{\mathbf{s}} - \int_{S_R} \delta\mathbf{t_n} \cdot \mathbf{s} = \\
&= \delta \underbrace{\left\{ \int_V \frac{1}{2}\sigma : \mathbf{D} : \sigma + \int_V \sigma : \alpha\Delta T - \int_{S_D} \mathbf{t_n} \cdot \bar{\mathbf{s}} + \int_{S_R} \frac{1}{2}\mathbf{t}_n \cdot \mathbf{S} \cdot \mathbf{t_n} - \int_{S_R} \mathbf{t_n} \cdot \bar{\mathbf{r}} \right\}}_{\Pi^*(\sigma, \mathbf{t_n})} - \int_V \sigma : \alpha\,\delta\Delta T
\end{aligned}$$

If ΔT is prescribed, it follows $\delta\,\Delta T = 0$, and

$$\delta\Pi^*(\sigma, \mathbf{t_n}) = 0 \; .$$

Theorem 7.1.3 (Principle of stationariety of total complementary potential energy)

Among all the congruent solutions, the equilibrated solution (and thus the unique solution of the elastic problem) is the one that makes the total complementary potential energy stationary.

7.1.3 Classical theorems

Uncomment (and move to beam structures?)

Maxwell-Betti

Menabrea-Castigliano

7.2 General elastic beam structures

In this section, theorems for elastic structures are specialized for beam structures.

todo Beam model used here…

7.2.1 Strong formulation of the problem

Indefinite equilibrium. External distributed loads are equilibrated by internal actions, resultant of stress field on beam sections.

$$\begin{aligned}
\mathbf{0} &= \mathbf{F}' + \mathbf{f} \\
\mathbf{0} &= \mathbf{M}' + \hat{\mathbf{z}} \times \mathbf{F} + \mathbf{m}
\end{aligned}$$

Kinematics. Displacement

$$\mathbf{s}(x, y, z) = \mathbf{s}_P(z) - \mathbf{r}_P(x, y) \times \theta(z) + \mathbf{w}(x, y, z)$$

Strain

$$
\begin{aligned}
\varepsilon_{zz} &= s'_{Pz} - x\theta'_y + y\theta'_x + w_{z/z} \\
\varepsilon_{xx} &= w_{x/x} \\
\varepsilon_{yy} &= w_{y/y} \\
2\varepsilon_{zx} &= s'_{Px} - \theta_y - y\theta'_z + w_{x/z} + w_{z/x} \\
2\varepsilon_{zy} &= s'_{Py} + \theta_x - x\theta'_z + w_{y/z} + w_{z/y} \\
2\varepsilon_{xy} &= w_{x/y} + w_{y/x}
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_z &:= \begin{bmatrix}\gamma_{zx}\\ \gamma_{zy}\\ \varepsilon_z\end{bmatrix} = \mathbf{s}'_P - \mathbf{r}_P \times \theta' + \hat{\mathbf{z}} \times \theta + \mathbf{v}_1(w_{i/j}) \\
\varepsilon_2 &:= \begin{bmatrix}\gamma_{xy}\\ \varepsilon_{xx}\\ \varepsilon_{yy}\end{bmatrix} = \mathbf{v}_2(w_{i/j})
\end{aligned}
$$

Constitutive equations. Under the assumptions … (kinematic assumptions, decoupling,…),

$$
\begin{aligned}
\mathbf{F}(z) &:= \int_{A(z)} \hat{\mathbf{n}} \cdot \sigma = \int_{A(z)} \sigma_z \\
\mathbf{M}(z) &:= \int_{A(z)} \mathbf{r}_P \times \hat{\mathbf{n}} \cdot \sigma = \int_{A(z)} \mathbf{r}_P \times \sigma_z =
\end{aligned}
$$

$$
\begin{aligned}
\sigma_z := \begin{bmatrix}\sigma_{zx}\\ \sigma_{zy}\\ \sigma_{zz}\end{bmatrix} &= \mathbf{C}_1 \cdot \varepsilon_z + \mathbf{C}_2 \cdot \varepsilon_2 - \mathbf{b}\Delta T = \\
&= \mathbf{C}_1 \cdot \begin{bmatrix}\gamma_{zx}\\ \gamma_{zy}\\ \varepsilon_{zz}\end{bmatrix} + \mathbf{C}_2 \cdot \begin{bmatrix}\gamma_{xy}\\ \varepsilon_{xx}\\ \varepsilon_{yy}\end{bmatrix} - \mathbf{b}\Delta T = \\
&= \mathbf{C}_1 \cdot \left(\mathbf{s}'_P - \mathbf{r}_P \times \theta' + \hat{\mathbf{z}} \times \theta + \mathbf{v}_1(w_{i/j})\right) + \mathbf{D}_2 \cdot \mathbf{v}_2(w_{i/j}) - \mathbf{b}\Delta T
\end{aligned}
$$

Neglecting (*or evaluating and finding that their contribution is zero*) the contribution of ε_2,

$$
\begin{aligned}
\mathbf{F} &= \int_A \{\mathbf{C}_1\varepsilon_z - \mathbf{b}\Delta T\} = \\
&= \int_A \mathbf{C}_1(\mathbf{s}'_P + \hat{\mathbf{z}} \times \theta) - \int_A \mathbf{C}_1\mathbf{r}_\times\theta' - \int_A \mathbf{b}\Delta T = \\
&= \mathbf{K}_{fs}\,(\mathbf{s}'_P + \hat{\mathbf{z}} \times \theta) + \mathbf{K}_{f\theta}\theta' - \mathbf{b}_f\Delta T \\
\mathbf{M} &= \int_A \mathbf{r}_\times \{\mathbf{C}_1\varepsilon_z - \mathbf{b}\Delta T\} = \\
&= \int_A \mathbf{r}_\times\mathbf{C}_1(\mathbf{s}'_P + \hat{\mathbf{z}} \times \theta) - \int_A \mathbf{r}_\times\mathbf{C}_1\mathbf{r}_\times\theta' - \int_A \mathbf{r}_\times\mathbf{b}\Delta T = \\
&= \mathbf{K}_{ms}\,(\mathbf{s}'_P + \hat{\mathbf{z}} \times \theta) + \mathbf{K}_{m\theta}\theta' - \mathbf{b}_m\Delta T\ ,
\end{aligned}
$$

with $\mathbf{K}_{ms} = \mathbf{K}_{f\theta}$, as $\mathbf{C}_1 = \mathbf{C}_1^T$ and $\mathbf{r}_\times = -\mathbf{r}_\times^T$.

$$
\begin{bmatrix}\mathbf{F}\\ \mathbf{M}\end{bmatrix} = \begin{bmatrix}\mathbf{K}_{11} & \mathbf{K}_{12}\\ \mathbf{K}_{21} & \mathbf{K}_{22}\end{bmatrix} \begin{bmatrix}\mathbf{s}' + \hat{\mathbf{z}} \times \theta\\ \theta'\end{bmatrix} - \begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{bmatrix}\Delta T
$$

$$
\begin{bmatrix}\mathbf{s}' + \hat{\mathbf{z}} \times \theta\\ \theta'\end{bmatrix} = \begin{bmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ \mathbf{S}_{21} & \mathbf{S}_{22}\end{bmatrix} \begin{bmatrix}\mathbf{F}\\ \mathbf{M}\end{bmatrix} + \begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{bmatrix}\Delta T\ ,
$$

with

$$\mathbf{I} = \mathbf{K}\mathbf{S} = \mathbf{S}\mathbf{K} \quad , \quad \mathbf{a} = \mathbf{S}\mathbf{b}$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11}\mathbf{K}_{11} + \mathbf{S}_{12}\mathbf{K}_{21} & \mathbf{S}_{11}\mathbf{K}_{12} + \mathbf{S}_{12}\mathbf{K}_{22} \\ \mathbf{S}_{21}\mathbf{K}_{11} + \mathbf{S}_{22}\mathbf{K}_{21} & \mathbf{S}_{21}\mathbf{K}_{12} + \mathbf{S}_{22}\mathbf{K}_{22} \end{bmatrix}$$

Thus, stress as a function of internal actions reads

$$\begin{aligned} \sigma_z &= \mathbf{C}_1 \begin{bmatrix} \mathbf{I} & -\mathbf{r}_\times \end{bmatrix} \begin{bmatrix} (\mathbf{s}'_P + \hat{\mathbf{z}} \times \theta) \\ \theta' \end{bmatrix} - \mathbf{b}\Delta T = \\ &= \mathbf{C}_1 \begin{bmatrix} \mathbf{I} & -\mathbf{r}_\times \end{bmatrix} \left(\begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix} + \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} \Delta T \right) - \mathbf{b}\Delta T \;. \end{aligned}$$

Evaluation of the term $\int_V \tilde{\sigma}_z^T \varepsilon_z$, with $\widetilde{\Delta T} = 0$

$$\begin{aligned} \int_V \tilde{\sigma}_z^T \varepsilon_z &= \int_\ell \begin{bmatrix} \tilde{\mathbf{F}}^T & \widetilde{\mathbf{M}}^T \end{bmatrix} \mathbf{S} \int_A \begin{bmatrix} \mathbf{I} \\ \mathbf{r}_\times \end{bmatrix} \mathbf{C}_1 \begin{bmatrix} \mathbf{I} & -\mathbf{r}_\times \end{bmatrix} dA \left(\mathbf{S} \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix} + \mathbf{a}\Delta T \right) d\ell = \\ &= \int_\ell \begin{bmatrix} \tilde{\mathbf{F}}^T & \widetilde{\mathbf{M}}^T \end{bmatrix} \underbrace{\mathbf{S}\mathbf{K}}_{=\mathbf{I}} \left(\mathbf{S} \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix} + \mathbf{a}\Delta T \right) d\ell = \\ &= \int_\ell \begin{bmatrix} \tilde{\mathbf{F}}^T & \widetilde{\mathbf{M}}^T \end{bmatrix} \left(\mathbf{S} \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix} + \mathbf{a}\Delta T \right) d\ell = \end{aligned}$$

For an elastic isotropic medium with structural decoupling,

$$\mathbf{S} = \begin{bmatrix} \frac{\chi_x}{GA} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{\chi_y}{GA} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{EA} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{EJ_x} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \frac{1}{EJ_y} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{GJ_z} \end{bmatrix} ,$$

s.t.

$$\begin{aligned} s'_x - \theta_y &= \frac{\chi_x}{GA} F_x \\ s'_y + \theta_x &= \frac{\chi_y}{GA} F_y \\ s'_z &= \frac{1}{EA} F_z \\ \theta'_x &= \frac{1}{EJ_x} M_x \\ \theta'_y &= \frac{1}{EJ_y} M_y \\ \theta'_z &= \frac{1}{GJ_z} M_z \end{aligned}$$

Null contribution of warping and transverse strain to internal force and moment

No contribution of warping to internal actions

$$\int_A \mathbf{D}_1 \mathbf{v}_1 = \dots = \mathbf{0} \ .$$

Bernoulli beam. If Bernoulli kinematic assumption

$$(\mathbb{I} - \hat{\mathbf{z}} \otimes \hat{\mathbf{z}}) \cdot (\mathbf{s}'_P + \hat{\mathbf{z}} \times \theta) = \mathbf{0} \ ,$$

holds, …

7.2.2 Weak formulations of the problem

Weak formulation of equilibrium conditions

$\forall \mathbf{u}(z), \ \mathbf{v}(z),$

$$\begin{aligned}
0 &= \int_\ell \{ \mathbf{u} \cdot (\mathbf{F}' + \mathbf{f}) + \mathbf{v} \cdot (\mathbf{M}' + \hat{\mathbf{z}} \times \mathbf{F} + \mathbf{m}) \} = \\
&= - \int_\ell \{ (\mathbf{u}' + \hat{\mathbf{z}} \times \mathbf{v}) \cdot \mathbf{F} + \mathbf{v}' \cdot \mathbf{M} \} + \int_\ell \{ \mathbf{u} \cdot \mathbf{f} + \mathbf{v} \cdot \mathbf{m} \} + [\mathbf{u} \cdot \mathbf{F} + \mathbf{v} \cdot \mathbf{M}]\big|_{\partial\ell} \ ,
\end{aligned}$$

having exploited the properties of the mixed vector product.

If $\mathbf{u}$, $\mathbf{v}$ respectively represent a displacement and a rotation field $\mathbf{u} = \tilde{\mathbf{s}}$, $\mathbf{v} = \tilde{\theta}$,

$$0 = - \int_\ell \left\{ (\tilde{\mathbf{s}}' + \hat{\mathbf{z}} \times \tilde{\theta}) \cdot \mathbf{F} + \tilde{\theta}' \cdot \mathbf{M} \right\} + \int_\ell \left\{ \tilde{\mathbf{s}} \cdot \mathbf{f} + \tilde{\theta} \cdot \mathbf{m} \right\} + \left[\tilde{\mathbf{s}} \cdot \mathbf{F} + \tilde{\theta} \cdot \mathbf{M} \right]\Big|_{\partial\ell}$$

If the constitutive equation is introduced to use the kinematic variables as the independent variables the **symmetry** of this formulation of the elastic problem naturally arises

$$\begin{aligned}
0 = &- \int_\ell (\tilde{\mathbf{s}}' + \hat{\mathbf{z}} \times \tilde{\theta}) \cdot \left[\mathbf{K}_{fs} \left(\mathbf{s}'_P + \hat{\mathbf{z}} \times \theta \right) + \mathbf{K}_{f\theta} \theta' - \mathbf{b}_f \Delta T \right] + \\
&- \int_\ell \tilde{\theta}' \cdot \left[\mathbf{K}_{ms} \left(\mathbf{s}'_P + \hat{\mathbf{z}} \times \theta \right) + \mathbf{K}_{m\theta} \theta' - \mathbf{b}_m \Delta T \right] + \\
&+ \int_\ell \left\{ \tilde{\mathbf{s}} \cdot \mathbf{f} + \tilde{\theta} \cdot \mathbf{m} \right\} + \left[\tilde{\mathbf{s}} \cdot \mathbf{F} + \tilde{\theta} \cdot \mathbf{M} \right]\Big|_{\partial\ell} \ .
\end{aligned}$$

Weak formulation of congruence conditions

$\forall \Sigma_z,$

$$\begin{aligned}
0 &= \int_V \Sigma_z \cdot \left(\varepsilon_z - \mathbf{s}'_P + \mathbf{r}_P \times \theta' - \hat{\mathbf{z}} \times \theta - \mathbf{v}_1(w_{i/j}) \right) = \\
&= \int_V \Sigma_z \cdot \varepsilon_z + \int_V \left\{ \Sigma'_z \cdot \mathbf{s}_P + (\mathbf{r}_P \times \Sigma'_z + \hat{\mathbf{z}} \times \Sigma_z) \cdot \theta \right\} - \int_A \left[\Sigma_z \cdot \mathbf{s}_P + (\mathbf{r}_P \times \Sigma_z) \cdot \theta \right]\big|_{\partial\ell}
\end{aligned}$$

If the test function is an equilibrated stress field $\Sigma_z = \tilde{\sigma}_z$,

$$\int_A \Sigma_z =: \tilde{\mathbf{F}}$$

$$\int_A \mathbf{r}_P \times \Sigma_z =: \widetilde{\mathbf{M}}$$

with

$$\mathbf{0} = \tilde{\mathbf{F}}' + \tilde{\mathbf{f}}$$
$$\mathbf{0} = \widetilde{\mathbf{M}}' + \hat{\mathbf{z}} \times \tilde{\mathbf{F}} + \widetilde{\mathbf{m}}$$

thus the weak form of congruence conditions becomes

$$\begin{aligned} 0 &= \int_V \tilde{\sigma}_z \cdot \varepsilon_z + \int_\ell \left\{ \tilde{\mathbf{F}}' \cdot \mathbf{s}_P + \left(\widetilde{\mathbf{M}}' + \hat{\mathbf{z}} \times \tilde{\mathbf{F}} \right) \cdot \theta \right\} - \left[\tilde{\mathbf{F}} \cdot \mathbf{s}_P + \widetilde{\mathbf{M}} \cdot \theta \right]_{\partial \ell} = \\ &= \int_V \tilde{\sigma}_z \cdot \varepsilon_z - \int_\ell \left\{ \tilde{\mathbf{f}} \cdot \mathbf{s}_P + \widetilde{\mathbf{m}} \cdot \theta \right\} - \left[\tilde{\mathbf{F}} \cdot \mathbf{s}_P + \widetilde{\mathbf{M}} \cdot \theta \right]_{\partial \ell} = \\ &= \dots \end{aligned}$$

todo *discuss useful choices of boundary conditions (along with no distributed loads to set the second integral identically zero), e.g. for the evaluation of hyperstatics*

todo *evaluate the first integral in terms of internal actions. See above*

$$\int_V \tilde{\sigma}_z \cdot \varepsilon_z = \int_\ell \begin{bmatrix} \tilde{\mathbf{F}} \\ \widetilde{\mathbf{M}} \end{bmatrix}^T \left(\mathbf{S} \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix} + \mathbf{a} \Delta T \right)$$

Principle of virtual work

…test functions: variations…the problem can be recast as the principle of stationariety of total potential energy

If test function $\mathbf{u}$, $\mathbf{v}$ in the weak form of the equilibrium conditions respectively represent the variation of a congruent displacement and a rotation field $\mathbf{u} = \delta\tilde{\mathbf{s}}$, $\mathbf{v} = \delta\tilde{\theta}$, with $\delta\tilde{\mathbf{s}}|_{S_D} = \mathbf{0}$ and $\delta\tilde{\theta}|_{S_D} = \mathbf{0}$ (**these relations may hold just for the constrained components**)

$$0 = -\int_\ell \left\{ (\delta\tilde{\mathbf{s}}' + \hat{\mathbf{z}} \times \delta\tilde{\theta}) \cdot \mathbf{F} + \tilde{\theta}' \cdot \mathbf{M} \right\} + \int_\ell \left\{ \delta\tilde{\mathbf{s}} \cdot \mathbf{f} + \delta\tilde{\theta} \cdot \mathbf{m} \right\} + \left[\delta\tilde{\mathbf{s}} \cdot \mathbf{F} + \delta\tilde{\theta} \cdot \mathbf{M} \right] \Big|_{\partial\ell / S_D}$$

It looks like that this relation doesn't add too much info beyond the weak form of the problem! Is that true? **Spaces of test and basis functions and the prescription of essential boundary conditions need some words!**

If the constitutive equation is introduced to use the kinematic variables as the independent variables the symmetry of this formulation of the elastic problem naturally arises, as already shown in the weak form of the equilibrium equations.

Principle of complementary virtual work

If the test function of the weak form of the congruence conditions is chosen to be the variation of an equilibrated stress field, $\Sigma_z = \delta\sigma_z$, s.t.

$$\mathbf{0} = \delta\tilde{\mathbf{F}}'$$
$$\mathbf{0} = \delta\widetilde{\mathbf{M}}' + \hat{\mathbf{z}} \times \delta\tilde{\mathbf{F}}$$

and $\delta\tilde{\mathbf{F}}\Big|_{S_D} = \mathbf{0}$, $\delta\widetilde{\mathbf{M}}\Big|_{S_D} = \mathbf{0}$, where essential boundary conditions are prescribed (some constraints prescribe only some components),

$$\begin{aligned} 0 &= \int_V \delta\tilde{\sigma}_z \cdot \varepsilon_z - \left[\delta\widetilde{\mathbf{F}} \cdot \mathbf{s}_P + \delta\widetilde{\mathbf{M}} \cdot \theta \right]_{\partial\ell / S_N} = \\ &= \int_V \begin{bmatrix} \delta\tilde{\mathbf{F}} \\ \delta\widetilde{\mathbf{M}} \end{bmatrix}^T \left(\mathbf{S} \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix} + \mathbf{a}\Delta T \right) - \left[\delta\widetilde{\mathbf{F}} \cdot \mathbf{s}_P + \delta\widetilde{\mathbf{M}} \cdot \theta \right]_{\partial\ell / S_N} = \end{aligned}$$

Principle of stationariety of total potential energy

Principle of virtual work can be recast as a principle of stationariety of total potential energy, being $\tilde{\mathbf{s}} = \mathbf{s}$, $\tilde{\theta} = \theta$,

$$\begin{aligned}
0 = & - \int_\ell (\delta \mathbf{s}' + \hat{\mathbf{z}} \times \delta\theta) \cdot \left[\mathbf{K}_{fs} \left(\mathbf{s}'_P + \hat{\mathbf{z}} \times \theta \right) + \mathbf{K}_{f\theta} \theta' - \mathbf{b}_f \Delta T \right] + \\
& - \int_\ell \delta\theta' \cdot \left[\mathbf{K}_{ms} \left(\mathbf{s}'_P + \hat{\mathbf{z}} \times \theta \right) + \mathbf{K}_{m\theta} \theta' - \mathbf{b}_m \Delta T \right] + \\
& + \int_\ell \{ \delta \mathbf{s} \cdot \mathbf{f} + \delta\theta \cdot \mathbf{m} \} + \left[\delta \mathbf{s} \cdot \mathbf{F} + \delta\theta \cdot \mathbf{M} \right] \Big|_{\partial\ell / \partial S_D} = \\
= & \delta \left\{ -\frac{1}{2} \int_\ell \begin{bmatrix} \mathbf{s}'_P + \hat{\mathbf{z}} \times \theta \\ \theta' \end{bmatrix} \cdot \begin{bmatrix} \mathbf{K}_{fs} & \mathbf{K}_{f\theta} \\ \mathbf{K}_{ms} & \mathbf{K}_{m\theta} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{s}'_P + \hat{\mathbf{z}} \times \theta \\ \theta' \end{bmatrix} d\ell - \frac{1}{2} \begin{bmatrix} \mathbf{s} \\ \theta \end{bmatrix} \cdot \mathbf{K}_R \cdot \begin{bmatrix} \mathbf{s}_P \\ \theta \end{bmatrix} \Bigg|_{S_R} + \right. \\
& \left. + \int_\ell \begin{bmatrix} \mathbf{s}_P \\ \theta \end{bmatrix} \cdot \begin{bmatrix} \mathbf{f} \\ \mathbf{m} \end{bmatrix} d\ell + \begin{bmatrix} \mathbf{s}_P \\ \theta \end{bmatrix} \cdot \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix} \Bigg|_{S_N} \right\} ,
\end{aligned}$$

with some notation abuse, and prescribed temperature ΔT, s.t. $\delta \Delta T = 0$, and loads.

Principle of stationariety of total complementary potential energy

If test functions of the congruence conditions are internal actions of an equilibrated and congruent solution (and thus the only solution of the elastic problem, if well-posed), and thus $\delta\widetilde{\mathbf{F}} = \delta\mathbf{F}$ and $\delta\widetilde{\mathbf{M}} = \delta\mathbf{M}$, the principle of complementary virtual work can be recast as the principle of stationariety of the total complementary potential energy, for given $\mathbf{s}_P$, θ on S_N,

$$\begin{aligned}
0 = & \delta \left\{ \int_\ell \left(\frac{1}{2} \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix}^T \mathbf{S} \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix} + \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix}^T \mathbf{a} \Delta T \right) d\ell - \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix}^T \begin{bmatrix} \mathbf{s}_P \\ \theta \end{bmatrix} \Bigg|_{S_N} + \frac{1}{2} \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix}^T \mathbf{S}_R \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix} \Bigg|_{S_R} \right\} = \\
= & \delta \Pi^* ,
\end{aligned}$$

if the b.c. on Robin boundaries reads…

7.3 Isotropic elastic beam structures

In this section, theorems for elastic structures are specialized for beam structures.

todo Beam model used here…

7.3.1 Strong formulation of the problem

Indefinite equilibrium. External distributed loads are equilibrated by internal actions, resultant of stress field on beam sections.

$$\begin{aligned}
0 &= F'_z(z) + n(z) && \text{(axial loads)} \\
0 &= F'_x(z) + f_x(z) && \text{(shear loads)} \\
0 &= F'_y(z) + f_y(z) && \\
0 &= M'_x(z) - F_y(z) + m_x(z) && \text{(bending)} \\
0 &= M'_y(z) + F_x(z) + m_y(z) && \\
0 &= M'_z(z) + m_z(z) && \text{(torsion)}
\end{aligned}$$

Kinematics.

Constitutive equations. Under the assumptions … (kinematic assumptions, decoupling,…),

$$\begin{aligned}
F_z(z) &= EAs'_z(z)\\
F_x(z) &= \chi_x^{-1}GA(s'_x(z) - \theta_y(z))\\
F_y(z) &= \chi_y^{-1}GA(s'_y(z) + \theta_x(z))\\
M_x(z) &= EJ_x\theta'_x(z)\\
M_y(z) &= EJ_y\theta'_y(z)\\
M_z(z) &= GJ_y\theta'_z(z)
\end{aligned}$$

With thermal strains due to temperature distribution $T(z) = T_0(z) + \Delta_x T(z)\frac{x}{h_x} + \Delta_y T(z)\frac{y}{h_y}$, these constitutive equations hold for the mechanical part only, while the most general constitutive equations read

$$\begin{aligned}
u'_z &= u_z^{mech\,\prime} + u_z^{th\,\prime} = \frac{T_z}{EA} + \alpha\Delta T\\
\theta'_x &= \theta_x^{mech\,\prime} + \theta_x^{th\,\prime} = \frac{M_x}{EJ_x} + \alpha\frac{\Delta_y T}{h_y}\\
\theta'_y &= \theta_y^{mech\,\prime} + \theta_y^{th\,\prime} = \frac{M_y}{EJ_y} - \alpha\frac{\Delta_x T}{h_x}\\
&\ldots
\end{aligned}$$

Bernoulli beam. If Bernoulli kinematic assumption

$$\begin{aligned}
\theta_x(z) &= -u'_y(z)\\
\theta_y(z) &= u'_x(z)
\end{aligned}$$

holds, bending equilibrium equations read

$$\begin{aligned}
0 &= f_x + T'_x = f_x - M''_y = f_x - (EJ_y\theta'_y)'' = f_x - (EJ_y u''_x)''\\
0 &= f_y + T'_y = f_y + M''_x = f_y + (EJ_x\theta'_x)'' = f_y - (EJ_x u''_y)''
\end{aligned}$$

7.3.2 Weak formulations of the problem

Weak formulation of equilibrium conditions

Under the assumption of negligible contribution of shear stress and deformation, and Bernoulli kinematic assumption, the weak form of equilibrium equation is derived as follows: axial and bending indefinite equilibrium equations for each beam (or structural element in general) are multiplied by arbitrary test functions, these products are integrated over the beam length. Then, natural boundary conditions are applied.

Timoshenko beam

The weak form of the equilibrium equations for a beam structure modelled with *Timoshenko beams* reads

$$\begin{aligned}
0 = \int_{\text{beams}} &\big\{u_x(F'_x + f_x) + u_y(F'_y + f_y) + u_z(F'_z + f_z) +\\
&+ \phi_x(M'_x - F_y + m_x) + \phi_y(M'_y + F_x + m_y) + \phi_z(M'_z + m_z)\big\} =\\
= -\int_{\text{beams}} &\big\{F_x(u'_x - \phi_y) + F_y(u'_y + \phi_x) + F_z u'_z + M_x\phi'_x + M_y\phi'_y + M_z\phi'_z\big\} +\\
+ \int_{\text{beams}} &\big\{f_x u_x + f_y u_y + f_z u_z + m_x\phi_x + m_y\phi_y + m_z\phi_z\big\} +\\
+ &\big[F_x u_x + F_y u_y + F_z u_z + M_x\phi_x + M_y\phi_y + M_z\phi_z\big]\big|_{\partial\,\text{str}}
\end{aligned}$$

having used integration by parts. This relation holds **for every** u_i, ϕ_i.

Bernoulli beam

$$
\begin{aligned}
0 = \int_{\text{beams}} & \left\{ u_x(-M_y'' + f_x - m_y') + u_y(M_x'' + f_y + m_x') + u_z(F_z' + f_z) + \phi_z(M_z' + m_z) \right\} = \\
= -\int_{\text{beams}} & \left\{ F_z u_z' + M_z \phi_z' + M_y u_x'' - M_x u_y'' \right\} + \\
+ \int_{\text{beams}} & \left\{ f_x u_x + f_y u_y + f_z u_z - m_x u_y' + m_y u_x' + m_z \phi_z \right\} + \\
+ & \left[F_z u_z + M_z \phi_z + u_x \left(-M_y' - m_y \right) + u_y \left(M_x' + m_x \right) + u_x' M_y - u_y' M_x \right] \Big|_{\partial \text{str}} = \\
= -\int_{\text{beams}} & \left\{ F_z u_z' + M_z \phi_z' + M_y u_x'' - M_x u_y'' \right\} + \\
+ \int_{\text{beams}} & \left\{ f_x u_x + f_y u_y + f_z u_z - m_x u_y' + m_y u_x' + m_z \phi_z \right\} + \\
+ & \left[F_z u_z + M_z \phi_z + u_x F_x + u_y F_y + u_x' M_y - u_y' M_x \right] \Big|_{\partial \text{str}}
\end{aligned}
$$

having used moment balance equations $M_x' + m_x = F_y$ and $M_y' + m_y = -F_x$.

Comparision with Timoshenko beam. The formulation for a Bernoulli beam structure looks the same as the formulation for a Timoshenko beam structure, under the assumption of negligible shear volume contributions, and/or that *test functions* are not completely independent but they satisfy Bernoulli kinematic assumption, $u_x' = \phi_y$ and $u_y' = -\phi_x$.

Example 7.3.1 (Finite element method. Example: 1-element clamped Bernoulli beam)

Approximation 1. Choosing only one test function $u_{y,2}(z) = z^2$ (that satisfies the essential boundary conditions, ($u_{y,2}(0) = 0$, $u_{y,2}'(0) = 0$), and assuming that the y-displacement s_y is proportional to that test function (same test and base function, to get a symmetric problem - even when multi-dimensional),

$$s_y(z) = a_2 z^2$$

and the bending moment reads

$$M_x(z) = -EJ_x s_y''(z) = -2EJ_x a_2 \ ,$$

the weak formulation of equilibrium equation reads (assuming Bernoulli beam, no distributed loads, and lumped force F_y at the free end as the only load)

$$
\begin{aligned}
0 &= -\int_{z=0}^{b} (2EJ_x a_2)(2)\, dz + F_y b^2 = \\
&= -4EJ_x b\, a_2 + F_y b^2 \ ,
\end{aligned}
$$

so that the only coefficient of the approximation is $a_2 = \frac{b}{4EJ} F_y$. Thus the approximate solution of the problem is $s_y(z) = \frac{F_y b}{4EJ_x} z^2$, and the displacement of the free end is $w(b) = \frac{Fb^3}{4EJ_x}$.

Comment. The displacement of the approximate solution is smaller than the displacement of the exact solution $w_B = \frac{Fb^3}{3EJ_x}$, as the approximated solution can't represent the exact solution, $s_y(z) = \frac{Fb^3}{EJ} \left(-\frac{1}{6} \left(\frac{z}{b}\right)^3 + \frac{1}{2} \left(\frac{z}{b}\right)^2 \right)$, since it doesn't contain any z^3 term, and this introduce an *extra numerical additional constraint* that makes the approximate model stiffer than the continuous model.

Approximation 2.

$$s_y(z) = a_2 z^2 + a_3 z^3 = \begin{bmatrix} z^2 & z^3 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}$$

$$s''_y(z) = 2a_2 + 6a_3 z = \begin{bmatrix} 2 & 6z \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}$$

$$\begin{aligned}
\mathbf{0} &= -\int_{z=0}^{b} \begin{bmatrix} 2 \\ 6z \end{bmatrix} EJ_x \begin{bmatrix} 2 & 6z \end{bmatrix} \, dz \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b^2 \\ b^3 \end{bmatrix} F_y = \\
&= -EJ_x \begin{bmatrix} 4b & 6b^2 \\ 6b^2 & 12b^3 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b^2 \\ b^3 \end{bmatrix} F_y \ ,
\end{aligned}$$

and the solution of the system provides the value of the coefficients of the linear combination in the approximate solution

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \frac{1}{12b^4} \begin{bmatrix} 12b^3 & -6b^2 \\ -6b^2 & 4b \end{bmatrix} \begin{bmatrix} b^2 \\ b^3 \end{bmatrix} \frac{F_y}{EJ_x} = \begin{bmatrix} \frac{b}{2} \\ -\frac{1}{6} \end{bmatrix} \frac{F_y}{EJ_x} \ ,$$

This approximation allows to get the exact solution of the continuous problem,

$$s_y(z) = \frac{F_y b}{2EJ_x} z^2 - \frac{F_y}{6EJ_x} z^3 \ .$$

Weak formulation of congruence conditions

Timoshenko beam

Neglecting transverse strain and warping contributions (their contribution is zero when multiplied by constant and linear functions and integrated), strain field in a Timoshenko beam reads

$$\begin{aligned}
\varepsilon_{zz} &= s'_{Pz} + y\theta'_x - x\theta'_y \\
\varepsilon_{xx} &= 0 \\
\varepsilon_{yy} &= 0 \\
2\varepsilon_{zx} &= s'_{Px} - \theta_y - y\theta'_z \\
2\varepsilon_{zy} &= s'_{Py} + \theta_x + x\theta'_z \\
2\varepsilon_{xy} &= 0
\end{aligned}$$

A weak form of congruence conditions is obtained multiplying by test functions…(**todo** discuss the choice of test functions)

$$\begin{aligned}
0 = &-\int_V \left\{ (\Sigma_z + y\Phi_x - x\Phi_y)(-\varepsilon_{zz} + s'_{Pz} + y\theta'_x - x\theta'_y) + (\Sigma_x - y\Phi_z)(-2\varepsilon_{zx} + s'_{Px} - \theta_y - y\theta'_z) + (\Sigma_y - x\Phi_z)(-2\varepsilon_{zy} + s'_{Py} + \theta \right. \\
= &\int_V \left\{ (\Sigma_z + y\Phi_x - x\Phi_y)\varepsilon_{zz} + (2\Sigma_x)\varepsilon_{zx} + (2\Sigma_y)\varepsilon_{zy} \right\} + \\
&+ \int_V \left\{ (\Sigma_z + y\Phi_x - x\Phi_y)'(s_{Pz} + y\theta_x - x\theta_y) + (\Sigma'_x - y\Phi'_z)(s_{Px} - y\theta_z) + (\Sigma_x - y\Phi_z)\theta_y + (\Sigma'_y - x\Phi'_z)(s_{Py} - x\theta_z) - (\Sigma_y - \right. \\
&- \left[\int_A \left\{ (\Sigma_z + y\Phi_x - x\Phi_y)(s_{Pz} + y\theta_x - x\theta_y) + (\Sigma_x - y\Phi_z)(s_{Px} - y\theta_z) + (\Sigma_y - z\Phi_z)(s_{Py} - x\theta_z) \right\} \right]_{\partial l}
\end{aligned}$$

For simplicity, considering first a structurally decoupled system, so that static moments are zero and the inertia tensor is diagonal,

$$\begin{aligned}
0 = &\int_V \left\{ (\Sigma_z + y\Phi_x - x\Phi_y)\varepsilon_{zz} + 2(\Sigma_x - y\Phi_z)\varepsilon_{zx} + 2(\Sigma_y - x\Phi_z)\varepsilon_{zy} \right\} + \\
&+ \int_\ell \left\{ \Sigma'_x A s_{Px} + \Sigma'_y A s_{Py} + \Sigma'_z A s_{Pz} + \left(\Phi'_x J_x - \Sigma_y A \right) \theta_x + (\Phi'_y J_y + \Sigma_x A)\theta_y + \Phi'_z J_z \theta_z \right\} + \\
&- \left[\Sigma_z A s_{Pz} + \Phi_x J_x \theta_x + \Phi_y J_y \theta_y + \Sigma_x s_{Px} + \Sigma_y s_{Py} + \Phi_z \theta_z \right]_{\partial l}
\end{aligned}$$

If the test functions are related to equilibrated internal actions,

$$\begin{aligned}
0 &= \widetilde{F}_z'(z) + \tilde{f}_z(z) && \text{(axial loads)} \\
0 &= \widetilde{F}_x'(z) + \tilde{f}_x(z) && \text{(shear loads)} \\
0 &= \widetilde{F}_y'(z) + \tilde{f}_y(z) && \\
0 &= \widetilde{M}_x'(z) - \widetilde{F}_y(z) + \widetilde{m}_x(z) && \text{(bending)} \\
0 &= \widetilde{M}_y'(z) + \widetilde{F}_x(z) + \widetilde{m}_y(z) && \\
0 &= \widetilde{M}_z'(z) + \widetilde{m}_z(z) && \text{(torsion)}
\end{aligned}$$

and defined as $\widetilde{F}_i = \Sigma_i A$, $\widetilde{M}_i = \Phi_i J_i$, the weak form of the problem becomes

$$\begin{aligned}
0 = &\int_V \left\{ \left(\frac{\widetilde{F}_z}{A} + y\frac{\widetilde{M}_x}{J_x} - x\frac{\widetilde{M}_y}{J_y} \right) \varepsilon_{zz} + 2\left(\frac{\widetilde{F}_x}{A} - y\frac{\widetilde{M}_z}{J_z} \right) \varepsilon_{zx} + 2\left(\frac{\widetilde{F}_y}{A} - x\frac{\widetilde{M}_z}{J_z} \right) \varepsilon_{zy} \right\} + \\
&- \int_\ell \left\{ \tilde{f}_x s_{Px} + \tilde{f}_y s_{Py} + \tilde{f}_z s_{Pz} + \widetilde{m}_x \theta_x - \widetilde{m}_y \theta_y - \widetilde{m}_z \theta_z \right\} + \\
&- \left[\widetilde{F}_x s_{Px} + \widetilde{F}_y s_{Py} + \widetilde{F}_z s_{Pz} + \widetilde{M}_x \theta_x + \widetilde{M}_y \theta_y + \widetilde{M}_z \theta_z \right]_{\partial l} .
\end{aligned}$$

Principle of virtual work

Starting from the *weak form of equilibrium conditions*, and choosing the test functions $u_i = \delta s_i$, $\phi_i = \delta\theta_i$ to be variations of congruent displacement fields,

$$\begin{aligned}
\delta s_x &= 0 && \text{where } x\text{-transverse displacement is prescribed} \\
\delta s_y &= 0 && \text{where } y\text{-transverse displacement is prescribed} \\
\delta s_z &= 0 && \text{where } z\text{-axial displacement is prescribed} \\
\delta \theta_x &= 0 && \text{where } x\text{-rotation is prescribed} \\
\delta \theta_y &= 0 && \text{where } y\text{-rotation is prescribed} \\
\delta \theta_z &= 0 && \text{where } z\text{-rotation is prescribed}
\end{aligned}$$

Principle of complementary virtual work

Timoshenko beam

If the test functions are related to equilibrated internal actions,

$$\begin{aligned}
0 &= \delta\widetilde{F}_z'(z) && \text{(axial loads)} \\
0 &= \delta\widetilde{F}_x'(z) && \text{(shear loads)} \\
0 &= \delta\widetilde{F}_y'(z) && \\
0 &= \delta\widetilde{M}_x'(z) - \delta\widetilde{F}_y(z) && \text{(bending)} \\
0 &= \delta\widetilde{M}_y'(z) + \delta\widetilde{F}_x(z) && \\
0 &= \delta\widetilde{M}_z'(z) && \text{(torsion)}
\end{aligned}$$

and defined as $\widetilde{F}_i = \Sigma_i A$, $\widetilde{M}_i = \Phi_i J_i$, the weak form of the problem becomes

$$\begin{aligned}
0 = &\int_V \left\{ \left(\frac{\delta\widetilde{F}_z}{A} + y\frac{\delta\widetilde{M}_x}{J_x} - x\frac{\delta\widetilde{M}_y}{J_y} \right) \varepsilon_{zz} + 2\left(\frac{\delta\widetilde{F}_x}{A} - y\frac{\delta\widetilde{M}_z}{J_z} \right) \varepsilon_{zx} + 2\left(\frac{\delta\widetilde{F}_y}{A} - x\frac{\delta\widetilde{M}_z}{J_z} \right) \varepsilon_{zy} \right\} + \\
&- \left[\delta\widetilde{F}_x s_{Px} + \delta\widetilde{F}_y s_{Py} + \delta\widetilde{F}_z s_{Pz} + \delta\widetilde{M}_x \theta_x + \delta\widetilde{M}_y \theta_y + \delta\widetilde{M}_z \theta_z \right]_{\partial l / S_D} .
\end{aligned}$$

Introducing the constitutive law for an isotropic elastic beam with structural decoupling, it's possible to write strain as a function of the internal actions

$$\begin{aligned}
\varepsilon_{zz} &= \frac{F_z}{EA} + y\frac{M_x}{J_x} - x\frac{M_y}{J_y} \\
2\varepsilon_{zx} &= \chi_x \frac{F_x}{GA} \\
2\varepsilon_{zy} &= \chi_y \frac{F_y}{GA} \ ,
\end{aligned}$$

s.t. the PCVW (remember structural decoupling) becomes

$$\begin{aligned}
0 = \int_\ell \left\{ \frac{\delta\widetilde{F}_z F_z}{EA} + \frac{\delta\widetilde{F}_x F_x}{\chi_x^{-1}GA} + \frac{\delta\widetilde{F}_y F_y}{\chi_y^{-1}GA} + \frac{\delta\widetilde{M}_x M_x}{EJ_x} + \frac{\delta\widetilde{M}_y M_y}{EJ_y} + \frac{\delta\widetilde{M}_z M_z}{GJ_z} \right\} + \\
- \left[\delta\widetilde{F}_x s_{Px} + \delta\widetilde{F}_y s_{Py} + \delta\widetilde{F}_z s_{Pz} + \delta\widetilde{M}_x \theta_x + \delta\widetilde{M}_y \theta_y + \delta\widetilde{M}_z \theta_z \right]_{\partial l / S_D} \ .
\end{aligned}$$

Principle of stationariety of total potential energy

Principle of stationariety of total complementary potential energy

---

Example 7.3.2 (Clamp-cart beam)

A beam of length b is clamped in A and constrained with a cart in B preventing transverse displacement. A transverse distributed uniform load q is applied. Thermal deformation is induced by a linear distribution of ΔT across all the sections of the beam. Assuming slender beam model and considering only the bending deformation, solve the (hyperstatic) structure: determine the internal bending moment $M(z)$ and the displacement $w(z)$.

Let's solve the problem with different approaches, after calling X the cart reaction and choosing it as the independent unknown hyperstatics. The reaction in A are $V_A = -X + qb$, $H_A = 0$, $M_A = -\frac{qb^2}{2} + Xb$ (clockwise), and the internal bending moment from $A: \ z = 0$ to $B: \ z = b$ (counter-clockwise)

$$\begin{aligned}
M(z) &= M_A + V_A z - \frac{qz^2}{2} = \\
&= -\frac{qb^2}{2} + Xb + (-X + qb)\, z - \frac{qz^2}{2} = \\
&= -\frac{1}{2} q\,(b - z)^2 + X(b - z) \ .
\end{aligned}$$

Method 1 - Elastic line. The axial equilibrium is trivial with axial internal action $N(z) = 0$, and displacement $u(z) = 0$. Focusing on bending equilibrium and transverse displacement

$$\theta'(z) = \theta^{mech\,\prime}(z) + \theta^{th\,\prime}(z) = \frac{M(z)}{EJ} + \frac{\alpha\Delta T}{h} \ ,$$

with the approximation $\theta(z) \simeq w'(z)$, the problem is governed by the differential problem

$$w''(z) = \frac{1}{EJ}\left[-\frac{1}{2} q\,(b - z)^2 + X(b - z) \right] + \frac{\alpha\Delta T}{h} \ ,$$

supplied with the boundary conditions $w(z = 0) = 0$, $w'(z = 0) = 0$, $w(z = b) = 0$. The solution reads

$$\begin{aligned}
w'(z) &= \frac{q}{EJ}\left[-\frac{b^2 z}{2} + \frac{bz^2}{2} - \frac{z^3}{6} \right] + \frac{X}{EJ}\left[\left(bz - \frac{z^2}{2} \right) \right] + \frac{\alpha\Delta T}{h} z + A \\
w(z) &= \frac{q}{EJ}\left[-\frac{b^2 z^2}{4} + \frac{bz^3}{6} - \frac{z^4}{24} \right] + \frac{X}{EJ}\left[\left(b\frac{z^2}{2} - \frac{z^3}{6} \right) \right] + \frac{\alpha\Delta T}{2h} z^2 + Az + B \ .
\end{aligned}$$

Boundary conditions in $z=0$ forces $A=0$, $B=0$. The boundary condition in $B: \ z=b$ is used to evaluate the hyperstatics X

$$0 = w(b) = -\frac{qb^4}{8EJ} + \frac{Xb^3}{3EJ} + \frac{\alpha\Delta T\, b^2}{2h} \ ,$$

so that:

- the hyperstatics is

$$X = \frac{3}{8}qb - \frac{3}{2}\frac{\alpha\Delta T\, EJ}{hb} \ ,$$

- the internal bending moment is

$$M(z) = -\frac{qz^2}{2} + \left(\frac{5}{8}qb + \frac{3}{2}\frac{\alpha\Delta T\, EJ}{h}\right) z + \left(-\frac{1}{8}qb^2 - \frac{3}{2}\alpha\Delta T\, EJb\right)$$

- the transverse displacement is

$$w(z) = -\frac{qz^4}{24\,EJ} + \left(\frac{5}{8}qb + \frac{3}{2}\frac{\alpha\Delta T\, EJ}{h}\right)\frac{z^3}{6} + \left(-\frac{1}{8}qb^2 - \frac{3}{2}\alpha\Delta T\, EJb\right)\frac{z^2}{2} \ .$$

Proof, $w(b)=0$**.** $-\frac{1}{24}+\frac{5}{48}-\frac{1}{16}=\frac{-2+5-3}{48}=0$.

Method 2 - Force method. An equilibrated solution with unit external loads at the hyperstatics, $\widetilde{X}=1$, has intenral bending moment $\widetilde{M}(z)=X(b-z)$. The PCVW with this equilibrated solution as the test function in the weak formulation reads

$$\begin{aligned}
0 &= \int_{z=0}^{b} \widetilde{M}(z)\theta'(z)\, dz - \widetilde{X}\,\underbrace{w_B}_{=0} = \\
&= \int_{z=0}^{b} \widetilde{M}(z)\left(\frac{M(z)}{EJ} + \frac{\alpha\Delta T}{h}\right) dz = \\
&= \int_{z=0}^{b} (b-z)\left[\frac{1}{EJ}\left(-\frac{1}{2}q(b-z)^2 + X(b-z)\right) + \frac{\alpha\Delta T}{h}\right] dz = \\
&= -\frac{1}{8}\frac{qb^4}{EJ} + \frac{1}{3}\frac{Xb^3}{EJ} + \frac{\alpha\Delta T\, b^2}{2h} \ ,
\end{aligned}$$

and thus the same expression of the hyperstatics is found, $X=\frac{3}{8}qb-\frac{3}{2}\frac{\alpha\Delta T\, EJ}{hb}$.

Method 3 - Stationariety of total complementary potential energy. Total complementary potential energy reads

$$\Pi^* = \int_{z=0}^{b} \frac{1}{2}\frac{M^2(z;X)}{EJ}\, dz + \int_{z=0}^{b} M(z;X)\frac{\alpha\Delta T}{h}\, dz \ ,$$

and its derivative[1] w.r.t. X (the only independent variable; equilibrium and constitutive law are already used to get the latter expression),

$$0 = \frac{\partial\Pi^*}{\partial X} = \int_{z=0}^{b} \frac{M(z;X)}{EJ}\frac{\partial M}{\partial X}(z;X)\, dz + \int_{z=0}^{b} \frac{\partial M}{\partial X}(z;X)\frac{\alpha\Delta T}{h}\, dz \ .$$

Using the expression of teh internal bending moment $M(z;X)=-\frac{1}{2}q(b-z)^2+X(b-z)$, the same expression as the one provided by **method 2 - force method** is found. Thus, doing algebra properly, this method gives the same results as the other two methods.

---

[1] The variation of bending moment becomes $\delta M(z;X)=\frac{\partial M}{\partial X}\delta X$, and thus $\delta\Pi^*=\frac{\partial\Pi^*}{\partial X}\delta X$. If the total complementary potential energy functional needs to be stationary, $\delta\Pi^*$ for every possible δX, it follows $\frac{\partial\Pi^*}{\partial X}=0$.

CHAPTER

EIGHT

WAVES IN LINEAR ELASTIC HOMOGENEOUS ISOTROPIC MEDIA

8.1 Navier-Cauchy equation: displacement formulation of the momentum equation

Momentum balance equation in differential form for continuous media in the small-displacement regime

$$\rho_0 \, \partial_{tt} \mathbf{s} = \rho_0 \, \mathbf{g} + \nabla \cdot \sigma \ .$$

- Introducing the constitutive equation for linear elastic homogeneous isotropic media,

$$\sigma = 2\mu \, \varepsilon + \lambda \, \mathrm{tr}\,(\varepsilon) \, \mathbf{I} \ ,$$

- using the definition of the strain tensor

$$\varepsilon = \frac{1}{2} \left[\nabla \mathbf{s} + \nabla^T \mathbf{s} \right] \ ,$$

- and under the assumption of no volume force $\mathbf{g} = \mathbf{0}$,

the momentum equation becomes

$$\rho_0 \partial_{tt} \mathbf{s} = \mu \nabla^2 \mathbf{s} + (\mu + \lambda) \, \nabla \nabla \cdot \mathbf{s} \ ,$$

8.2 Helmholtz decomposition and sum of waves equation for p and s waves

Displacement field can be written using Helmholtz decomposition as the sum of a potential $\mathbf{s}_p = \nabla \phi$ (s.t. $\nabla \times \nabla \phi = \mathbf{0}$) and a divergence-free $\mathbf{s}_s = \nabla \times \mathbf{a}$ (s.t. $\nabla \cdot \nabla \times \mathbf{a} = \mathbf{0}$) part,

$$\mathbf{s} = \mathbf{s}_p + \mathbf{s}_s = \nabla \phi + \nabla \times \mathbf{a} \ .$$

Introducing the last expression in the momentum equation, using vector identity

$$\nabla^2 \mathbf{v} = \nabla \nabla \cdot \mathbf{v} - \nabla \times \nabla \times \mathbf{v} \ ,$$

the equation can be written as

$$\begin{aligned} \mathbf{0} & = \rho_0 \partial_{tt} \nabla \phi - (2\mu + \lambda) \nabla^2 \nabla \phi + \rho_0 \partial_{tt} \nabla \times \mathbf{a} + \mu \nabla^2 \nabla \times \mathbf{a} = \\ & = \rho_0 \partial_{tt} \mathbf{s}_p - (2\mu + \lambda) \nabla^2 \mathbf{s}_p + \rho_0 \partial_{tt} \mathbf{s}_s - \mu \nabla^2 \mathbf{s}_s \ , \end{aligned}$$

i.e. as the "sum of two wave equations" for the potential part $\mathbf{s}_p$ and the divergence-free part $\mathbf{s}_s$ of the displacement. Speed of propagation of p- and s-displacement read

$$c_p = \sqrt{\frac{2\mu + \lambda}{\rho_0}} \qquad , \qquad c_s = \sqrt{\frac{\mu}{\rho_0}} \ .$$

8.3 Fourier decomposition: p is longitudinal, s is transverse

Using **Fourier decomposition** of fields as sum of harmonic plane waves,

$$\mathbf{s}(\mathbf{r},t) = \sum_{\mathbf{k},\omega} \mathbf{s}_{\mathbf{k},\omega} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \ ,$$

it's immediate to prove that the potential part can be associated to a longitudinal perturbation (i.e. with displacement in the same direction of the wave vector $\mathbf{k}$, representing the direction of propagation of the perturbation, while the divergence-free part can be associated to a transverse perturnation (i.e. with displacement orthogonal to the wave vector $\mathbf{k}$). Helmholtz's decomposition of the field in Fourier domain reads

$$\begin{aligned}
\mathbf{s}(\mathbf{r},t) & = \sum_{\mathbf{k},\omega} \mathbf{s}_{\mathbf{k},\omega} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = \\
& = \sum_{\mathbf{k},\omega} \left(\mathbf{s}^p_{\mathbf{k},\omega} + \mathbf{s}^s_{\mathbf{k},\omega} \right) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = \\
& = \sum_{\mathbf{k},\omega} \left(i\,\mathbf{k}\phi_{\mathbf{k},\omega} + i\,\mathbf{k}\times\mathbf{a}_{\mathbf{k},\omega} \right) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = \ .
\end{aligned}$$

For each individual harmonic contribution, the potential part is thus proportional, i.e. aligned, to wave vector $\mathbf{k}$,

$$\mathbf{s}^p_{\mathbf{k},\omega} = i\,\mathbf{k}\phi_{\mathbf{k},\omega} \ ,$$

while the divergence-free is orthogonal, and thus transverse, w.r.t. the direction of wave propagation,

$$\mathbf{k}\cdot\mathbf{s}^s_{\mathbf{k},\omega} = i\,\mathbf{k}\cdot\left(\mathbf{k}\times\mathbf{a}_{\mathbf{k},\omega}\right) = 0 \ .$$

CHAPTER

NINE

FINITE ELEMENT METHODS FOR BEAM STRUCTURES

Strong form

Momentum and angular momentum balance equation for a straight beam

$$\mathbf{M}\begin{bmatrix}\ddot{\mathbf{s}}\\ \ddot{\theta}\end{bmatrix} - \begin{bmatrix}\mathbf{F}\\ \mathbf{M}\end{bmatrix}' - \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\times\mathbf{F}\end{bmatrix} = \begin{bmatrix}\mathbf{f}\\ \mathbf{m}\end{bmatrix} ,$$

can be written as a function of kinematic variables,

$$\mathbf{M}\ddot{\mathbf{s}} + \left(\mathbf{K}_1\mathbf{s}' + \mathbf{K}_2\mathbf{s}\right)' + \mathbf{H}_1\mathbf{s}' + \mathbf{H}_2\mathbf{s} = \mathbf{f} ,$$

where the local displacement w.r.t. the beam coordinates $\mathbf{s}(z,t)$ are function of the axial coordinate z and time t.

Weak form

$$\begin{aligned}
0 = & \sum_{b\in\text{beams}} \int_{\ell_b} \begin{bmatrix}\mathbf{w}_b\\ \varphi_b\end{bmatrix}^T \left\{\mathbf{M}_b\begin{bmatrix}\ddot{\mathbf{s}}_b\\ \ddot{\theta}\end{bmatrix} - \begin{bmatrix}\mathbf{F}\\ \mathbf{M}\end{bmatrix}' - \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\times\mathbf{F}\end{bmatrix} - \begin{bmatrix}\mathbf{f}_b\\ \mathbf{m}_b\end{bmatrix}\right\} = \\
= & \sum_{b\in\text{beams}} \int_{\ell_b} \left\{\mathbf{w}_b^T\left(\mathbf{M}_{11}\ddot{\mathbf{s}} + \mathbf{M}_{12}\ddot{\theta}\right) + \mathbf{w}_b'^{\,T}\mathbf{F} - \mathbf{w}_b^T\mathbf{f}_b\right\} + \\
& + \sum_{b\in\text{beams}} \left\{\varphi_b^T\left(\mathbf{M}_{21}\ddot{\mathbf{s}} + \mathbf{M}_{22}\ddot{\theta}\right) + \varphi_b'^{\,T}\mathbf{M} - \varphi_b^T\hat{\mathbf{z}}\times\mathbf{F} - \varphi_b^T\mathbf{m}_b\right\} + \\
& - \left[\mathbf{w}_b^T\mathbf{F} + \varphi_b^T\mathbf{M}\right]\big|_{\partial b}
\end{aligned}$$

Using the constitutive law

$$\begin{aligned}
\begin{bmatrix}\mathbf{F}_b\\ \mathbf{M}_b\end{bmatrix} & = \begin{bmatrix}\mathbf{K}_{11} & \mathbf{K}_{12}\\ \mathbf{K}_{21} & \mathbf{K}_{22}\end{bmatrix}\left(\begin{bmatrix}\mathbf{s}_b\\ \theta_b\end{bmatrix}' + \begin{bmatrix}\hat{\mathbf{z}}\times\theta_b\\ \mathbf{0}\end{bmatrix}\right) - \begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{bmatrix}\Delta T = \\
& = \mathbf{K}\left(\begin{bmatrix}\mathbf{s}_b\\ \theta_b\end{bmatrix}' + \begin{bmatrix}\mathbf{0} & \hat{\mathbf{z}}_\times\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{s}_b\\ \theta_b\end{bmatrix}\right) - \begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{bmatrix}\Delta T ,
\end{aligned}$$

the problem can be written with kinetic variables (displacements and rotations) as the independent unknowns. The volume contribution of the internal actions due to displacements in beam b becomes

$$\begin{aligned}
&\int_{\ell_b} \left\{ \begin{bmatrix} \mathbf{w}_b'^{\,T} & \varphi_b'^{\,T} \end{bmatrix} - \begin{bmatrix} \mathbf{w}_b^T & \varphi_b^T \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{z}}_\times & \mathbf{0} \end{bmatrix} \right\} \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix}^{mech} = \\
&= \int_{\ell_b} \left\{ \begin{bmatrix} \mathbf{w}_b'^{\,T} & \varphi_b'^{\,T} \end{bmatrix} + \begin{bmatrix} \mathbf{w}_b^T & \varphi_b^T \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{z}}_\times^T & \mathbf{0} \end{bmatrix} \right\} \left\{ \mathbf{K} \left(\begin{bmatrix} \mathbf{s}_b \\ \theta_b \end{bmatrix}' + \begin{bmatrix} \mathbf{0} & \hat{\mathbf{z}}_\times \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} \mathbf{s}_b \\ \theta_b \end{bmatrix} \right\} = \\
&= \int_{\ell_b} \left\{ \begin{bmatrix} \mathbf{w} \\ \varphi \end{bmatrix}_b^{\prime\,T} \mathbf{K} \begin{bmatrix} \mathbf{s} \\ \theta \end{bmatrix}_b' + \begin{bmatrix} \mathbf{w} \\ \varphi \end{bmatrix}_b^T \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{z}}_\times^T & \mathbf{0} \end{bmatrix} \mathbf{K} \begin{bmatrix} \mathbf{s} \\ \theta \end{bmatrix}_b' + \begin{bmatrix} \mathbf{w} \\ \varphi \end{bmatrix}_b^{\prime\,T} \mathbf{K} \begin{bmatrix} \mathbf{0} & \hat{\mathbf{z}}_\times \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{s} \\ \theta \end{bmatrix}_b + \begin{bmatrix} \mathbf{w} \\ \varphi \end{bmatrix}_b^T \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{z}}_\times^T & \mathbf{0} \end{bmatrix} \mathbf{K} \begin{bmatrix} \mathbf{0} & \hat{\mathbf{z}}_\times \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{s} \\ \theta \end{bmatrix}_b \right\} = \\
&= \int_{\ell_b} \left\{ \mathbf{v}_b'^{\,T} \mathbf{K} \mathbf{u}_b' + \mathbf{v}_b^T \mathbf{K}_1 \mathbf{u}_b' + \mathbf{v}_b'^{\,T} \mathbf{K}_1^T \mathbf{u}_b + \mathbf{v}_b^T \mathbf{K}_2 \mathbf{u}_b \right\} \ .
\end{aligned}$$

having exploited $-\hat{\mathbf{z}}_\times = \hat{\mathbf{z}}_\times^T$, and defined $\mathbf{u} = \begin{bmatrix} \mathbf{s} \\ \theta \end{bmatrix}$. The PVW thus becomes

$$0 = \sum_b \left\{ \int_{\ell_b} \left\{ \mathbf{v}_b^T \mathbf{M} \ddot{\mathbf{u}}_b + \mathbf{v}_b'^{\,T} \mathbf{K} \mathbf{u}_b' + \mathbf{v}_b^T \mathbf{K}_1 \mathbf{u}_b' + \mathbf{v}_b'^{\,T} \mathbf{K}_1^T \mathbf{u}_b + \mathbf{v}_b^T \mathbf{K}_2 \mathbf{u}_b - \left(\mathbf{v}_b'^{\,T} \mathbf{b} + \mathbf{v}_b^T \tilde{\mathbf{b}} \right) \Delta T - \mathbf{v}_b^T \tilde{\mathbf{f}}_b \right\} - \left[\mathbf{v}_b^T \tilde{\mathbf{F}}_b \right] \Big|_{\partial b} \right\} \ .$$

Finite element method

Using the same functions as test and base functions, i.e. testing for every $\mathbf{n}(z)$, with an expansion of the variable $\mathbf{u}_b(z,t) = \mathbf{N}(z) \mathbf{u}_b^{loc}(t)$, with a minor abuse of notation,

$$\begin{aligned}
\mathbf{0} &= \sum_b \left\{ \int_{\ell_b} \mathbf{N}_b^T \mathbf{M} \mathbf{N}_b \, \ddot{\mathbf{u}}_b + \int_{\ell_b} \left\{ \mathbf{N}_b'^{\,T} \mathbf{K} \mathbf{N}_b + \mathbf{N}_b^T \mathbf{K}_1 \mathbf{N}_b' + \mathbf{N}_b'^{\,T} \mathbf{K}_1 \mathbf{N}_b + \mathbf{N}_b^T \mathbf{K}_2 \mathbf{N}_b \right\} \mathbf{u}_b - \int_{\ell_b} \left\{ \mathbf{N}_b' \mathbf{b} + \mathbf{N}_b \tilde{\mathbf{b}} \right\} \Delta T - \int_{\ell_b} \mathbf{N}_b^T \tilde{\mathbf{f}} - \left[\mathbf{N}_b^T \tilde{\mathbf{F}}_b \right] \Big|_{\partial} \right. \\
&= \sum_b \left\{ \mathbf{M}_b^{loc} \ddot{\mathbf{u}}_b^{loc} + \mathbf{K}_b^{loc} \mathbf{u}_b^{loc} - \mathbf{B}_b^{loc} \Delta \mathbf{T} - \mathbf{q}_b^{loc} + \dots \right\}
\end{aligned}$$

Transofrming local *nodal variables* into global nodal variables, (and updating the projection as well, $\mathbf{v}_b^{loc} = \mathbf{T} \mathbf{v}_b$),

$$\mathbf{u}_b^{loc} = \mathbf{T}_b \mathbf{u}_b \ ,$$

the problem becomes

$$\mathbf{0} = \sum_b \left\{ \mathbf{M}_b \ddot{\mathbf{u}}_b + \mathbf{K}_b \mathbf{u}_b - \mathbf{B}_b \Delta \mathbf{T} - \mathbf{q}_b \right\}$$

Assembly of the finite element linear system.

$$\mathbf{M} \ddot{\mathbf{u}} + \mathbf{K} \mathbf{u} = \mathbf{f} + \mathbf{B} \Delta T \ .$$

- It can be formally represented by matrix multiplication
- It's performed assembling matrices in **sparse format**

CHAPTER

TEN

BEAMS

- *de Saint Venant*
- *Thin-walled*
- *Timoshenko*
- *Bernoulli*
- *Aeronautical*

10.1 de Saint Venant beam

10.1.1 Assumptions

10.1.2 Internal actions

Axial

Shear

…

The axial equilibrium of an infinitesimal section of the beam between z and $z + dz$ and $y = y^*$, under linear axial stress, $\sigma_z = \sigma_{z/y}\, y = \frac{M_x}{J_x}\, y$, reads for beams with constant section (**todo** is this assumption really required?)

$$\begin{aligned} 0 &= -\int_{x \in b(y*)} \int_{dz} \tau_{zy}\, dz\, dx + \int_{A^*(z+dz)} \sigma_z\, dz\, dy - \int_{A^*(z)} \sigma_z\, dz\, dy = \\ &\simeq -dz \int_{x \in b(y^*)} \tau_{zy} + M_x(z+dz) \int_{A^*(z+dz)} \frac{y}{J_x} - M_x(z) \int_{A^*(z)} \frac{y}{J_x} = \\ &\simeq dz \left[- \int_{x \in b(y*)} \tau_{zy} + M_x'(z) \frac{S^*(z)}{J_x} \right] , \end{aligned}$$

with $S^*(z) = \int_{A^*(z)} y$. Expliting the rotational equilibrium, $0 = M_x'(z) - T(z)$, and the definition of the average shear stress $\overline{\tau}_{zy} = \frac{1}{b(y^*)} \int_{x \in b(y^*)} \tau_{zy}$, it follows that

$$\overline{\tau}_{zy}(z, y) = \frac{S^*(y)}{b^*(y) J_x} T_y \ .$$

…

Shear stiffness. With $\gamma_{zy} = 2\varepsilon_{zy} = \partial_y s_z + \partial_z s_y = \frac{\tau_{zy}}{G}$, and an equilibrated shaer load $\widetilde{T}(z) = \widetilde{T}(z+dz) = 1$, so that $\widetilde{M}(z+dz) = \widetilde{M}(z) + \widetilde{T}(z)dz$ with $\widetilde{M}(z) = 0$, and $\tilde{\tau} = \frac{S^*}{b^* J}\widetilde{T}$, and $\tilde{\sigma} = \frac{\widetilde{M}}{J} y$, it follows

$$\begin{aligned}
0 &= \int_V \tilde{\sigma}_{ij}\varepsilon_{ij} - \int_{S_D} n_i \tilde{\sigma}_{ij} s_j = \dots \\
&= \int_V 2\tilde{\tau}(z)\frac{\tau(z)}{2G} + \widetilde{T}_y(z)s_y(z) - \widetilde{T}_y(z+dz)s_y(z+dz) - \widetilde{M}_x(z+dz)\theta_x(z+dz) = \\
&= \int_\ell \int_A \frac{S^*}{b^* J}\widetilde{T}\frac{1}{G}\frac{S^*}{b^* J} T \, dA \, d\ell - \widetilde{T}_y(z)s'_y(z)\, dz - \widetilde{T}(z)dz \; (\theta(z) + \theta'(z)dz) \simeq \\
&= dz \left[\frac{1}{GA} \underbrace{A\int_A \frac{S^{*2}}{b^{*2}J^2}}_{\chi} T(z) - \left(s'_y(z) + \theta_x(z)\right) \right]
\end{aligned}$$

and thus

$$s'_y(z) + \theta_x(z) = \frac{\chi_y}{GA} T_y(z) \; ,$$

having introduced the definition of the **shear factor** χ into the shear stiffness $\frac{GA}{\chi}$.

Example 10.1.1 (Shear factor of a rectangular section)

As the static moment $S^*(y)$ of a rectangular section with base a and height b reads

$$S^*(y) = a\left(\frac{b}{2} - y\right) \cdot \frac{1}{2}\left(\frac{b}{2} + y\right) = \frac{1}{2}\left(\frac{b^2}{4} - y^2\right) a \; .$$

the shear factor χ of a rectangular section is

$$\begin{aligned}
\chi &= ab \int_{x=-a/2}^{a/2} \int_{y=-b/2}^{b/2} \frac{\frac{1}{4}\left(\frac{b^4}{16} - \frac{b^2 y^2}{2} + y^4\right)}{b^2\left(\frac{1}{12}ab^3\right)^2 a^2} = \\
&= \frac{36\, ab}{b^8} a \left(\frac{1}{16} - \frac{1}{6}\frac{1}{4} + \frac{1}{5}\frac{1}{16}\right) b^5 = \\
&= \frac{ab}{b^8} a \left(\frac{9}{4} - \frac{3}{2} + \frac{9}{20}\right) b^5 = \\
&= \frac{6}{5}\frac{a^2}{b^2} \; .
\end{aligned}$$

Bending

Torsion

10.2 Thin-walled beam

Shear and torsion approximation exists for thin-walled beams. Different approximations apply to open and closed section beams

…

10.2.1 Open section

…

10.2.2 Closed section

…

10.2.3 Multiple-loop section

…

10.3 Timoshenko beam

10.3.1 Kinematic assumptions

Let z coordinate the axial coordinate of a beam, and x, y a pair of cartesian coordinates to represent the points on a beam section.

Displacement. Displacement of the points of a beam can written as

$$\begin{aligned} \mathbf{s}(x,y,z,t) &= \mathbf{s}_P(z,t) + \theta(z,t)\times\mathbf{r}_P(x,y) + \mathbf{s}^{\nu+w}(x,y,z,t) = \\ &= \mathbf{s}_P(z,t) + \hat{\mathbf{x}}\left(-y\theta_z\right) + \hat{\mathbf{y}}\left(x\theta_z\right) + \hat{\mathbf{z}}\left(+y\theta_x - x\theta_y\right) + \mathbf{s}^{\nu+w}\ , \end{aligned}$$

where the first two contributions represent a rigid motion of the section identified by the value z of the axial coordinate, and $\mathbf{s}^{\nu+w}$ the contribution of strain (due to non-zero Poisson ration) and warping of the section. Here the vector $\mathbf{r}_P$(x,y) lies in the same section of referencce point P, i.e. $\mathbf{r}_P = (x-x_P)\hat{\mathbf{x}} + (y-y_P)\hat{\mathbf{y}}$, so that the motion of points on section $A(z)$ only depends on the displacement of $P(z)$ and the rotation of the section $A(z)$.

Strain.

$$\begin{aligned} \varepsilon_{zz} &= s'_{Pz} + y\theta'_x - x\theta'_y + s^{\nu+w}_{Pz/z} \\ \varepsilon_{xx} &= s^{\nu+w}_{Px/x} \\ \varepsilon_{yy} &= s^{\nu+w}_{Py/y} \\ 2\varepsilon_{zx} &= s'_{Px} - \theta_y - y\theta'_z + s^{\nu+w}_{x/z} + s^{\nu+w}_{z/x} \\ 2\varepsilon_{zy} &= s'_{Py} + \theta_x + x\theta'_z + s^{\nu+w}_{y/z} + s^{\nu+w}_{z/y} \\ 2\varepsilon_{xy} &= s^{\nu+w}_{y/z} + s^{\nu+w}_{z/y}\ . \end{aligned}$$

Stress. …todo… usually stiffness matrix is defined providing axial, bending, shear and torsion stiffness, and cross-coupling terms. Here, using a simplified (or modified, so that no contribution of ε_{xx}, ε_{yy} exists) version of the constitutive law for **elastic isotropic media**,

$$\begin{aligned} \sigma_{zz} &= E\varepsilon_{zz} \\ \tau_{zx} &= 2G\varepsilon_{zx} \\ \tau_{zy} &= 2G\varepsilon_{zy} \end{aligned}$$

10.3.2 Internal actions

$$
\begin{aligned}
\mathbf{F} &= \int_A \hat{\mathbf{n}} \cdot \sigma = \int_A \hat{\mathbf{x}}\tau_{zx} + \hat{\mathbf{y}}\tau_{zy} + \hat{\mathbf{z}}\sigma_{zz} \\
\mathbf{M} &= \int_A \mathbf{r} \times (\hat{\mathbf{n}} \cdot \sigma) = \int_A \hat{\mathbf{x}}\,(y\sigma_{zz}) + \hat{\mathbf{y}}\,(-x\sigma_{zz}) + \hat{\mathbf{z}}\,(x\tau_{zy} - y\tau_{zx})
\end{aligned}
$$

Internal actions as function of displacement - elastic isotropic media. Neglecting warping and strain due to non-zero Poisson ratio,

$$
\begin{aligned}
\mathbf{F} &= \int_A \hat{\mathbf{x}}\tau_{zx} + \hat{\mathbf{y}}\tau_{zy} + \hat{\mathbf{z}}\sigma_{zz} = \\
&= \int_A \hat{\mathbf{x}} G(s'_{Px} - \theta_y - y\theta'_z) + \hat{\mathbf{y}} G(s'_{Py} + \theta_x + x\theta'_z) + \hat{\mathbf{z}} E(s'_{Pz} + y\theta'_x - x\theta'_y) = \\
&= \hat{\mathbf{x}}\left(\chi_x GA(s'_{Px} - \theta_y) - GS_x\theta'_z\right) + \hat{\mathbf{y}}\left(\chi_y GA(s'_{Py} + \theta_x) + GS_y\theta'_z\right) + \hat{\mathbf{z}}\left(EAs'_{Pz} + ES_x\theta'_x - ES_y\theta'_y\right) \ , \\
\mathbf{M} &= \int_A \hat{\mathbf{x}}\,(y\sigma_{zz}) + \hat{\mathbf{y}}\,(-x\sigma_{zz}) + \hat{\mathbf{z}}\,(x\tau_{zy} - y\tau_{zx}) = \\
&= \int_A \hat{\mathbf{x}} yE\left(s'_{Pz} + y\theta'_x - x\theta'_y\right) - \hat{\mathbf{y}} xE\left(s'_{Pz} + y\theta'_x - x\theta'_y\right) + \hat{\mathbf{z}} G\left(x(s'_{Py} + \theta_x + x\theta'_z) - y(s'_{Px} - \theta_y - y\theta'_z)\right) = \\
&= \hat{\mathbf{x}}\left(ES_x s'_{Pz} + EJ_x\theta'_x - EJ_{xy}\theta'_y\right) + \hat{\mathbf{y}}\left(-ES_y s'_{Pz} - EJ_{xy}\theta'_x + EJ_y\theta'_y\right) + \hat{\mathbf{z}}\left(GS_y(s'_{Py} + \theta_x) - GS_x(s'_{Px} - \theta_y) + GJ_z\theta'_z\right)
\end{aligned}
$$

or introducing matrix notation,

$$
\begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix} =
\begin{bmatrix}
\chi_x^{-1}GA & & & & & -GS_x \\
 & \chi_y^{-1}GA & & & & GS_y \\
 & & EA & ES_x & -ES_y & \\
 & & ES_x & EJ_x & -EJ_{xy} & \\
 & & -ES_y & -EJ_{xy} & EJ_y & \\
-GS_x & GS_y & & & & GJ_z
\end{bmatrix}
\begin{bmatrix} s'_{Px} - \theta_y \\ s'_{Py} + \theta_x \\ s'_{Pz} \\ \theta'_x \\ \theta'_y \\ \theta'_z \end{bmatrix}
$$

Structural decoupling. $S_i = 0$, $J_{xy} = 0$

$$
\begin{aligned}
\mathbf{F} &= \hat{\mathbf{x}}\chi_x GA(s'_{Px} - \theta_y) + \hat{\mathbf{y}}\chi_y GA(s'_{Py} + \theta_x) + \hat{\mathbf{z}} EAs'_{Pz}, \\
\mathbf{M} &= \hat{\mathbf{x}} EJ_x\theta'_x + \hat{\mathbf{y}} EJ_y\theta'_y + \hat{\mathbf{z}} GJ_z\theta'_z
\end{aligned}
$$

10.3.3 Balance equations

Balance equations for a beam can be obtained integrating indefinite balance equations for a 3-dimensional solid on the sections $A(z)$ of the beam, with some further assumption on non-rigid contributions to displacement.

Momentum equation +

$$
\rho_0\ddot{\mathbf{s}} = \nabla \cdot \sigma + \mathbf{f}
$$

gives

$$
\begin{aligned}
\mathbf{0} &= -\int_{\Delta V} \rho_0\ddot{\mathbf{s}} + \int_{\Delta V} \nabla \cdot \sigma + \int_{\Delta V} \rho_0\mathbf{g} = \\
&= -\Delta z \int_A \rho_0\left(\ddot{\mathbf{s}}_P - \mathbf{r}_P \times \ddot{\theta}\right) + \int_{A(z)} \hat{\mathbf{n}} \cdot \sigma + \int_{A(z+\Delta z)} \hat{\mathbf{n}} \cdot \sigma + \int_{\Delta A_{lat}} \hat{\mathbf{n}} \cdot \sigma + \Delta z \int_A \rho_0\mathbf{g} = \\
&= \Delta z\left[-m\ddot{\mathbf{s}}_P - \mathbf{S}_P \cdot \ddot{\theta} + \mathbf{F}' + \mathbf{f}\right]
\end{aligned}
$$

Angular momentum equation

$$\mathbf{r}_P \times \rho_0 \ddot{\mathbf{s}} = \mathbf{r}_P \times (\nabla \cdot \sigma + \mathbf{f})$$

gives

$$\begin{aligned}
\mathbf{0} &= -\int_{\Delta V} \rho_0 \mathbf{r}_P \times \ddot{\mathbf{s}} + \int_{\Delta V} \mathbf{r}_P \times \nabla \cdot \sigma + \int_{\Delta V} \rho_0 \mathbf{r}_P \times \mathbf{g} = \\
&= -\Delta z \int_A \rho_0 \mathbf{r}_P \times \left(\ddot{\mathbf{s}}_P - \mathbf{r}_P \times \ddot{\theta} \right) + \ldots \\
&= \Delta z \left[-\mathbf{S}_P^T \cdot \ddot{\mathbf{s}}_P - \mathbf{I}_P \cdot \ddot{\theta} + \mathbf{M}' + \hat{\mathbf{z}} \times \mathbf{F} + \mathbf{m} \right] \ .
\end{aligned}$$

Contribution of stress to moment

$$\begin{aligned}
\int_V \mathbf{r} \times \nabla \cdot \sigma &= \hat{\mathbf{e}}_i \int_V \varepsilon_{ijk} r_j \sigma_{lk/l} = \\
&= \hat{\mathbf{e}}_i \int_V \varepsilon_{ijk} \left[\left(r_j \sigma_{lk} \right)_{/l} - r_{j/l} \sigma_{lk} \right] = \\
&= \hat{\mathbf{e}}_i \int_{\partial V} \varepsilon_{ijk} r_j n_l \sigma_{lk} - \hat{\mathbf{e}}_i \int_V \varepsilon_{ijk} r_{j/l} \sigma_{lk} = \\
&= \int_{\partial V} \mathbf{r}_P \times (\hat{\mathbf{n}} \cdot \sigma) - \hat{\mathbf{e}}_i \int_V \varepsilon_{ijk} \delta^{(2)}_{jl} \sigma_{lk} =
\end{aligned}$$

For an elementary beam element Δz, the first contribution contains internal moments on two sections at z and $z + \Delta z$ and the contribution of the lateral surface, that can be summed with the volume contribution to get load from linear density loads,

$$\int_{\partial \Delta V} \mathbf{r}_P \times (\hat{\mathbf{n}} \cdot \sigma) = \Delta z \, \mathbf{M}'(z) + \Delta z \int_{\partial A} \mathbf{r}_P \times (\hat{\mathbf{n}} \cdot \sigma)$$

The second contribution becomes (as $\delta^{(2)}_{xx} = \delta^{(2)}_{yy} = 1$, but $\delta^{(2)}_{zz} = 0$),

$$\begin{aligned}
-\hat{\mathbf{e}}_i \int_V \varepsilon_{ijk} \delta^{(2)}_{jl} \sigma_{lk} &= -\int_{\Delta V} \left\{ \hat{\mathbf{z}}(\sigma_{xy} - \sigma_{yx}) + \hat{\mathbf{x}}(\sigma_{yz}) + \hat{\mathbf{y}}(-\sigma_{xz}) \right\} = \\
&= \Delta z \left[-\hat{\mathbf{x}} T_y + \hat{\mathbf{y}} T_x \right] = \\
&= \Delta z \, \hat{\mathbf{z}} \times \mathbf{F} \ .
\end{aligned}$$

In components, for an inertially decoupled set of Cartesian coordinates,

$$\begin{aligned}
0 &= -m \ddot{s}_{Px} + F_x' + f_x \\
0 &= -m \ddot{s}_{Py} + F_y' + f_y \\
0 &= -m \ddot{s}_{Pz} + F_z' + f_z \\
0 &= -I_x \ddot{\theta}_x + M_x' - T_y + m_x \\
0 &= -I_y \ddot{\theta}_y + M_y' + T_x + m_y \\
0 &= -I_z \ddot{\theta}_z + M_z' + m_z
\end{aligned}$$

Using matrix formalism, momentum and angular momentum equations for an isotropic elastic beam read

$$\mathbf{0} = -\begin{bmatrix} m & & & & & -S_y \\ & m & & & & S_x \\ & & m & S_y & -S_x & \\ & & S_y & I_x & I_{xy} & I_{xz} \\ & & -S_x & I_{xy} & I_y & I_{yz} \\ -S_y & S_x & & I_{xz} & I_{yz} & I_z \end{bmatrix} \begin{bmatrix} \ddot{s}_{Px} \\ \ddot{s}_{Py} \\ \ddot{s}_{Pz} \\ \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z \end{bmatrix} + \left(\begin{bmatrix} \chi_x^{-1} GA & & & & & -GS_x \\ & \chi_y^{-1} GA & & & & GS_y \\ & & EA & ES_x & -ES_y & \\ & & ES_x & EJ_x & -EJ_{xy} & \\ & & -ES_y & -EJ_{xy} & EJ_y & \\ -GS_x & GS_y & & & & GJ_z \end{bmatrix} \begin{bmatrix} s_{Px}' - \theta_y \\ s_{Py}' + \theta_x \\ s_{Pz}' \\ \theta_x' \\ \theta_y' \\ \theta_z' \end{bmatrix} \right)$$

Structural and inertial simoultaneously decoupled isotropic elastic beam.

$$\begin{aligned}
0 &= -m\ddot{s}_{Px} + \left(\chi_x^{-1} GA (s'_{Px} - \theta_y)\right)' + f_x \\
0 &= -m\ddot{s}_{Py} + \left(\chi_y^{-1} GA (s'_{Py} + \theta_x)\right)' + f_y \\
0 &= -m\ddot{s}_{Pz} + (EAs'_{Pz})' + f_z \\
0 &= -I_x\ddot{\theta}_x + (EJ_x\theta'_x)' - \chi_y^{-1} GA (s'_{Py} + \theta_x) + m_x \\
0 &= -I_y\ddot{\theta}_y + \left(EJ_y\theta'_y\right)' + \chi_x^{-1} GA (s'_{Px} - \theta_y) + m_y \\
0 &= -I_z\ddot{\theta}_z + (GJ_z\theta'_z)' + m_z
\end{aligned}$$

10.4 Bernoulli beam

10.5 Aeronautical beam

10.6 Slender beams

Bernoulli kinematic assumption.

$$\begin{aligned}
0 &= s'_{Px} - \theta_y && \rightarrow && \theta_y = s'_{Px} \\
0 &= s'_{Py} + \theta_x && \rightarrow && \theta_x = -s'_{Py}
\end{aligned}$$

As an example, the constitutive law for bending in a structurally decoupled elastic beam becomes

$$\begin{aligned}
M_x &= EJ_x\theta_x'^{\,mech} = -EJ_x s_{Py}''^{\,mech} \\
M_y &= EJ_y\theta_y'^{\,mech} = EJ_y s_{Px}''^{\,mech}
\end{aligned}$$

Example 10.6.1 (Clamped beam)

Internal equilibrium equations read

$$\begin{aligned}
0 &= F'_z + f_z && \text{(axial)} \\
0 &= F'_y + f_y && \text{(shear)} \\
0 &= M'_x - T_y + m_x && \text{(bending)}
\end{aligned}$$

with axial force $F_z = EAs'_{Pz}$, shear force $F_y = \chi^{-1} GA(s'_{Py} + \theta_x)$, and bending moment $M_x = EJ\theta'_x$. The beam has length b, it's clamped in $A : z = 0$, s.t. essential boundary conditions in A read

$$s_{Pz}(z=0) = 0 \quad , \quad s_{Py}(z=0) = 0 \quad , \quad \theta_x(z=0) = 0 \, ,$$

and loaded lumped force and moment in $B : z = b$, s.t. natural boundary conditions in B read

$$F_z(z=b) = Z \quad , \quad F_y(z=b) = Y \quad , \quad M_x(z=b) = M$$

and no distributed actions $f_y = f_z = 0$, $m_z = 0$, s.t. equilibrium equation becomes

$$\begin{aligned}
0 &= F'_z && \text{(axial)} \\
0 &= F'_y && \text{(shear)} \\
0 &= M'_x - F_y && \text{(bending)}
\end{aligned}$$

From the equilibrium equations and the boundary conditions in $z = b$, it immediately follows the distribution of the internal actions along the beam

$$F_z(z) = Z \quad , \quad F_y(z) = Y \quad , \quad M_x(z) = M + Y(z-b) \; .$$

Using the constitutive laws and the essential boundary conditions in $z = 0$, it immediately follows the displacement field along the beam,

$$\begin{aligned}
s_{Pz}(z) &= \frac{Z}{EA} z \\
s_{Py}(z) &= -\frac{1}{EJ}\left(Y\frac{z^3}{6} + (M - Yb)\frac{z^2}{2} \right) + \frac{Y}{\chi^{-1}GA} z \\
\theta_x(z) &= \frac{1}{EJ}\left(Y\frac{z^2}{2} + (M - Yb) z \right)
\end{aligned}$$

The displacement of the extreme point B thus reads

$$\begin{aligned}
s_{Bz} = s_{Pz}(z=b) \quad &= \frac{Zb}{EA} \\
s_{By} = s_{Py}(z=b) \quad &= -\frac{Mb^2}{2EJ} + \frac{Yb^3}{3EJ} + \frac{Yb}{\chi^{-1}GA} \\
\theta_{Bx} = \theta_x(z=b) \quad &= \frac{Mb}{EJ} - \frac{Yb^2}{2EJ}
\end{aligned}$$

Let's discuss the order of magnitude of the two terms in s_{By} due to Y.

Transverse displacement: bending and shear contributions. For an elastic medium, the shear modulus G can be written as a function of the elastic modulus E and the Poisson ration ν,

$$G = \frac{E}{2(1+\nu)} \; .$$

While the value of the Poisson ratio i limited to $-1 \le \nu \le 0.5$, it's usually in the range $[0, 0.5]$. If Poisson ratio belongs to the latter range, the ratio $\frac{G}{E}$ belongs to the range $\left[\frac{1}{3}, \frac{1}{2}\right]$, and thus G has the same order of magnitude as E.

The properties of a square section are

$$A = a^2 \quad , \quad J = \frac{1}{12}a^4 \quad , \quad \chi = \frac{6}{5} \; .$$

Thus the ratio of the two contributions to transverse displacement has order of magnitude

$$\frac{\frac{Yb^3}{3EJ}}{\frac{Yb}{\chi^{-1}GA}} = \frac{G}{E}\chi\frac{b^2A}{J} = \frac{G}{E}\frac{6}{5}12\frac{b^2}{a^2} = \frac{24}{5} \div \frac{36}{5}\frac{b^2}{a^2} = 4.8 \div 7.2 \left(\frac{b}{a}\right)^2 \; .$$

It immediately follows that the displacement of B due to shear deformation for a beam with square section with side a and length $b = 10a$ is $480 \div 720$ times larger than the contribution due to bending (and the following transverse displacement of the axis of the beam associated with the rotation of tits sections).

Transverse and axial displacement. The comparison of the transverse displacement and the axial displacement gives the ratio

$$\frac{\frac{Yb^3}{3EJ}}{\frac{Zb}{EA}} = \frac{Y}{Z}\frac{b^2A}{J} = \frac{Y}{Z}12\left(\frac{b}{a}\right)^2 \; ,$$

and if the components of the force have similar magnitude $Y \sim Z$, the order of magnitude of the ratio becomes $12\left(\frac{b}{a}\right)^2$, so that axial displacement of slender beams $\frac{b}{a} \gg 1$ becomes negligible if compared with transverse displacement.

10.7 Problems

CHAPTER

ELEVEN

MODAL METHODS FOR STRUCTURAL PROBLEMS

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \ .$$

11.1 No free rigid motion

If a structure has no free rigid motion, the stiffness matrix of mechanical systems is symmetric **definite positive**.

Spectral decomposition of the problem.

$$\left[s_i^2 \mathbf{M} + \mathbf{K} \right] \hat{\mathbf{u}}_i = \mathbf{0} \ ,$$

or in index and matrix form

$$\begin{aligned} 0 &= s_i^2 M_{jk} U_{ki} + K_{jk} U_{ki} = \\ &= \mathbf{M}\mathbf{U}\mathbf{S}^2 + \mathbf{K}\mathbf{U} \ , \end{aligned}$$

with the diagonal matrix **S** collecting the eigenvalues,

$$\mathbf{S} = \text{diag}\left\{ s_i \right\} \ .$$

Properties. For eigenvectors with different eigenvalues,

$$\hat{\mathbf{u}}_j^* \mathbf{M} \hat{\mathbf{u}}_i = 0 \qquad , \qquad \hat{\mathbf{u}}_j^* \mathbf{K} \hat{\mathbf{u}}_i = 0 \ .$$

Nodal and modal unknowns. The nodal vector can be written as a combination of modes, being **q** the vector of modal amplitudes,

$$\mathbf{u} = \mathbf{U}\mathbf{q} = \left[\hat{\mathbf{u}}_1 | \dots | \hat{\mathbf{u}}_N \right] \mathbf{q} \ .$$

Laplace domain. In Laplace domain

$$\begin{aligned} \left[s^2 \mathbf{U}^* \mathbf{M} \mathbf{U} + \mathbf{U}^* \mathbf{K} \mathbf{U} \right] \mathbf{q}(s) &= \mathbf{U}^* \mathbf{f}(s) \\ \text{diag}\left[s^2 m_i + k_i \right] \mathbf{q}(s) &= \mathbf{U}^* \mathbf{f}(s) \ . \end{aligned}$$

Modal damping Adding modal damping, with simultaneous diagonalization with mass and stiffness matrices,

$$\begin{aligned} \text{diag}\left[s^2 m_i + s c_i + k_i \right] \mathbf{q}(s) &= \mathbf{U}^* \mathbf{f}(s) \\ \mathbf{q}(s) &= \text{diag}\left[\frac{1}{m_i (s^2 + 2\xi_i \omega_i s + \omega_i^2)} \right] \mathbf{U}^* \mathbf{f}(s) \ . \end{aligned}$$

The original equation becomes

$$\left[s^2 \mathbf{M} + s \mathbf{C} + \mathbf{K} \right] \mathbf{u} = \mathbf{f} \ ,$$

with

$$\begin{aligned}\mathbf{M} &= \mathbf{U}\mathrm{diag}\left\{m_i\right\}\mathbf{U}^*\\ \mathbf{C} &= \mathbf{U}\mathrm{diag}\left\{c_i\right\}\mathbf{U}^*\\ \mathbf{K} &= \mathbf{U}\mathrm{diag}\left\{k_i\right\}\mathbf{U}^*\end{aligned}$$

and the eigenproblem reads

$$\begin{aligned}\mathbf{0} &= \left[s_i^2\mathbf{M} + s_i\mathbf{C} + \mathbf{K}\right]\hat{\mathbf{u}}_i\\ \mathbf{0} &= \mathbf{M}\mathbf{U}\mathbf{S}^2 + \mathbf{C}\mathbf{U}\mathbf{S} + \mathbf{K}\mathbf{U}\end{aligned}$$

Nodal vector. Nodal vector thus reads

$$\mathbf{u}(s) = \mathbf{U}\mathbf{q}(s) = \mathbf{U}\mathrm{diag}\left[\frac{1}{m_i(s^2 + 2\xi_i\omega_i s + \omega_i^2)}\right]\mathbf{U}^*\mathbf{f}(s)$$

and the internal forces usually derived from a manipulation of the term $\mathbf{K}\mathbf{u}(s)$,

$$\mathbf{K}\mathbf{u}(s) = \mathbf{K}\mathbf{U}\mathrm{diag}\left[\frac{1}{m_i(s^2 + 2\xi_i\omega_i s + \omega_i^2)}\right]\mathbf{U}^*\mathbf{f}(s)\ .$$

11.1.1 Dimension reduction

Modal unknowns can usually partitioned in slow (dynamical, resolved) and fast modes (with natural frequencies well above the frequency content of the forcing, and the dynamics of the system; can be treated as static modes),

$$\mathbf{q} = \begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_f\end{bmatrix}\ ,$$

and the sum of their contributions give the nodal unknown,

$$\mathbf{u} = \mathbf{U}\mathbf{q} = \begin{bmatrix}\mathbf{U}_s & \mathbf{U}_f\end{bmatrix}\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_f\end{bmatrix} = \mathbf{U}_s\mathbf{q}_s + \mathbf{U}_f\mathbf{q}_f\ .$$

Truncation and direct recovery of loads

$$\mathbf{u} = \mathbf{U}_s\mathrm{diag}\left[\frac{1}{m_s(s^2 + 2\xi_s\omega_s s + \omega_s^2)}\right]\mathbf{U}_s^*\mathbf{f} + \mathbf{U}_f\mathrm{diag}\left[\frac{1}{m_f(s^2 + 2\xi_f\omega_f s + \omega_f^2)}\right]\mathbf{U}_f^*\mathbf{f}$$

Mode acceleration and static recovery of fast modes

Static approximation of fast modes gives

$$\begin{aligned}\mathbf{u} &= \mathbf{u}_s + \mathbf{u}_f = \\ &= \mathbf{U}_s\mathrm{diag}\left[\frac{1}{m_s(s^2 + 2\xi_s\omega_s s + \omega_s^2)}\right]\mathbf{U}_s^*\mathbf{f} + \mathbf{U}_f\mathrm{diag}\left[\frac{1}{m_f(s^2 + 2\xi_f\omega_f s + \omega_f^2)}\right]\mathbf{U}_f^*\mathbf{f} \simeq \\ &\simeq \mathbf{u}_s + \mathbf{u}_{f,static} = \\ &= \mathbf{U}_s\mathrm{diag}\left[\frac{1}{m_s(s^2 + 2\xi_s\omega_s s + \omega_s^2)}\right]\mathbf{U}_s^*\mathbf{f} + \mathbf{U}_f\mathrm{diag}\left[\frac{1}{m_f\omega_f^2}\right]\mathbf{U}_f^*\mathbf{f}\end{aligned}$$

and adding and subtracting the static response of the slow modes, with the assumption that stiffness matrix $\mathbf{K}$ is invertible (i.e. no rigid motion exists; *rigid motion* will be treated in another section later)

$$\begin{aligned}
\mathbf{u} &\simeq \mathbf{u}_s - \mathbf{u}_{s,static} + \mathbf{u}_{s,static} + \mathbf{u}_{f,static} = \\
&= \underbrace{\mathbf{U}_s \text{diag}\left[\frac{1}{m_s(s^2+2\xi_s\omega_s s+\omega_s^2)}\right]\mathbf{U}_s^*\mathbf{f}}_{\mathbf{U}_s\mathbf{q}_s} - \mathbf{U}_s\text{diag}\left[\frac{1}{m_s\omega_s^2}\right]\mathbf{U}_s^*\mathbf{f} + \\
&+ \underbrace{\mathbf{U}_s\text{diag}\left[\frac{1}{m_s\omega_s^2}\right]\mathbf{U}_s^*\mathbf{f} + \mathbf{U}_f\text{diag}\left[\frac{1}{m_f\omega_f^2}\right]\mathbf{U}_f^*\mathbf{f}}_{=\mathbf{U}\text{diag}\left[\frac{1}{k_i}\right]\mathbf{U}^*\mathbf{f}=\mathbf{K}^{-1}\mathbf{f}} = \\
&= \mathbf{U}_s\text{diag}\left[\frac{-s^2-2\xi_s\omega_s s}{m_s\omega_s^2(s^2+2\xi_s\omega_s s+\omega_s^2)}\right]\mathbf{U}_s^*\mathbf{f} + \mathbf{K}^{-1}\mathbf{f} \ .
\end{aligned}$$

The internal stress/actions are represente here as

$$\begin{aligned}
\mathbf{K}\mathbf{u} &= \mathbf{K}\left(\mathbf{U}_s\text{diag}\left[\frac{-s^2-2\xi_s\omega_s s}{m_s\omega_s^2(s^2+2\xi_s\omega_s s+\omega_s^2)}\right]\mathbf{U}_s^*\mathbf{f} + \mathbf{K}^{-1}\mathbf{f}\right) \\
&= \mathbf{M}\mathbf{U}_s\text{diag}\left[\frac{-s^2-2\xi_s\omega_s s}{m_s(s^2+2\xi_s\omega_s s+\omega_s^2)}\right]\mathbf{U}_s^*\mathbf{f} + \mathbf{f}
\end{aligned}$$

State space system.

$$\begin{bmatrix}\mathbf{u}\\ \mathbf{K}\mathbf{u}\\ \dot{\mathbf{u}}_s\\ \ddot{\mathbf{u}}_s\end{bmatrix} = \begin{bmatrix}\mathbf{U}_s & \mathbf{0}\\ \mathbf{K}\mathbf{U}_s & \mathbf{0}\\ \mathbf{0} & \mathbf{U}_s\\ -\text{diag}\{\omega_s^2\} & -\text{diag}\{2\xi_s\omega_s\}\end{bmatrix}\begin{bmatrix}\mathbf{q}_s\\ \dot{\mathbf{q}}_s\end{bmatrix} + \begin{bmatrix}\mathbf{K}^{-1} - \mathbf{U}_s\text{diag}\{(m_s\omega_s^2)^{-1}\}\mathbf{U}_s^*\\ \mathbf{I} - \mathbf{M}\mathbf{U}_s\text{diag}\{m_s^{-1}\}\mathbf{U}_s^*\\ \mathbf{0}\\ \text{diag}\{m_s^{-1}\}\mathbf{U}_s^T\end{bmatrix}\mathbf{f} \ .$$

having used the identity (proved below, under negligible damping assumption) $\mathbf{K}\mathbf{U}_s = \mathbf{M}\mathbf{U}_s\text{diag}\{\omega_i^2\}$.

Convergence. For $s = j\omega$, and $\omega \gg \omega_0$ the leading term in the displacement goes as $-\frac{1}{m_i\omega_i^2}$, and the stress with \frac{}{}; for $\omega \ll \omega_0$ the leading term goes as $-2\frac{\xi_i\omega}{m_i\omega_i^3}$ if the damping is not negligible, as $-\frac{\omega^2}{m_i\omega_i^4}$ if the damping is negligible.

todo Choose one expression to comment (and motivate the choice)

$\mathbf{K}^{-1}$

The diagonal stiffness matrix using the modal basis (the basis needs to be complete, otherwise this is just a projection onto a lower-dimensional space and the equality of the two expressions of $\mathbf{u}$ doesn't hold) reads

$$\text{diag}\{k_i\} = \mathbf{U}_i^*\mathbf{K}\mathbf{U}_i \ .$$

The static solution of the problem $\mathbf{K}\mathbf{u} = \mathbf{f}$, may be recast in modal basis as $\text{diag}\{k_i\}\mathbf{q}_i = \mathbf{U}_i^*\mathbf{f}$, having introduced the amplitudes of the modes $\mathbf{q}_i$, defined by the change of coordinates $\mathbf{u} = \mathbf{U}_i\mathbf{q}_i$. Now, under the assumption of invertible stiffness matrix, the solution reads

$$\mathbf{q}_i = \text{diag}\left\{\frac{1}{k_i}\right\}\mathbf{U}_i^*\mathbf{f} \ .$$

From the comparison of two expressions of the displacement $\mathbf{u}$

$$\begin{aligned}
\mathbf{u} &= \mathbf{K}^{-1}\mathbf{f} = \\
&= \mathbf{U}_i\mathbf{q}_i = \mathbf{U}_i\text{diag}\left\{\frac{1}{k_i}\right\}\mathbf{U}_i^*\mathbf{f} \ ,
\end{aligned}$$

from the arbitrariety of $\mathbf{f}$ (is this condition enough?), it follows that

$$\mathbf{K}^{-1} = \mathbf{U}_i \text{diag}\left\{\frac{1}{k_i}\right\}\mathbf{U}_i^* \ .$$

Some proofs/identities of the modal basis

As the modal problem reads $\mathbf{M}\mathbf{U}_i \text{diag}\{s_i^2\} + \mathbf{C}\mathbf{U}_i\text{diag}\{s_i\} + \mathbf{K}\mathbf{U}_i = \mathbf{0}$, it immediately follows that

$$\mathbf{K}\mathbf{U}_i = -\mathbf{M}\mathbf{U}_i \text{diag}\{s_i^2\} - \mathbf{C}\mathbf{U}_i\text{diag}\{s_i\} \ .$$

Let the system be under-critically dumped so that eigenvalues can be written as $s_i = \sigma_i + j\hat{\omega}_i = \omega_i \left(-\xi_i \mp j\sqrt{1-\xi_i^2}\right)$. Then, the latter expression can be recast as

$$\mathbf{0} = \mathbf{M}\mathbf{U}_i \text{diag}\left\{\omega_i^2\left(2\xi_i^2 - 1 \pm j\,2\xi_i\sqrt{1-\xi_i^2}\right)\right\} + \mathbf{C}\mathbf{U}_i\text{diag}\left\{-\xi_i\omega_i \mp j\omega_i\sqrt{1-\xi^2}\right\} + \mathbf{K}\mathbf{U}_i \ ,$$

and its real and imaginary parts (is $\mathbf{U}_i$ real, even with non-zero damping?) read

$$\begin{aligned}
\text{real:} \ \ & \mathbf{0} = \mathbf{M}\mathbf{U}_i \text{diag}\left\{\omega_i^2\left(2\xi_i^2 - 1\right)\right\} - \mathbf{C}\mathbf{U}_i\text{diag}\left\{\xi_i\omega_i\right\} + \mathbf{K}\mathbf{U}_i \\
\text{imag:} \ \ & \mathbf{0} = \pm\mathbf{M}\mathbf{U}_i \text{diag}\left\{2\xi_i\omega_i^2\right\} \mp \mathbf{C}\mathbf{U}_i\text{diag}\left\{\omega_i\right\}
\end{aligned}$$

and thus

$$\begin{aligned}
\mathbf{C}\mathbf{U}_i &= \mathbf{M}\mathbf{U}_i\text{diag}\left\{2\xi_i\omega_i\right\} \\
\mathbf{K}\mathbf{U}_i &= \mathbf{M}\mathbf{U}_i\text{diag}\left\{\omega_i^2\right\}
\end{aligned}$$

11.2 With free rigid motion

Free rigid degrees of freedom are associated with vectors of the kernel of the stiffness matrix. Rigid motion is associated with no structural stiffness and damping, i.e. no elastic and linear damping actions occur in a structure performing a rigid motion. Then, damping and stiffness matrices are singular.

Mathematical consequences of rigid modes

Deformable modes have non-zero eigenvalues s_d. Thus, the eigenvalue problem reads

$$\mathbf{0} = \mathbf{M}\mathbf{U}_d\mathbf{S}_d^2 + \mathbf{C}\mathbf{U}_d\mathbf{S}_d + \mathbf{K}\mathbf{U}_d \ ,$$

with $\mathbf{S}_d = \text{diag}\left\{s_d\right\}$. Rigid modes have zero eigenbalues $s_r = 0$, and thus it follows

$$\begin{aligned}
\mathbf{0} &= \mathbf{K}\mathbf{U}_r \\
\mathbf{0} &= \mathbf{C}\mathbf{U}_r \ .
\end{aligned}$$

If damping and stiffenss matrices are symmetric, projecting the eigenvalue problem for deformable modes onto the subspace of rigid modes it immediately follows that

$$\mathbf{0} = \mathbf{U}_r^*\mathbf{M}\mathbf{U}_d \ .$$

It's possible to divide the modes of the system in two sets: rigid modes, r, and deformable modes, d,

$$\mathbf{u} = \mathbf{U}\mathbf{q} = \begin{bmatrix}\mathbf{U}_r & \mathbf{U}_d\end{bmatrix}\begin{bmatrix}\mathbf{q}_r \\ \mathbf{q}_d\end{bmatrix} \ .$$

Projection of the equations of motion on the modes gives

$$\mathbf{0} = \begin{bmatrix} \mathbf{U}_r^* \\ \mathbf{U}_d^* \end{bmatrix} \left(\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} - \mathbf{f} \right) \ ,$$

and exploiting orthogonality of the deformable modes $\mathbf{U}_d^*\mathbf{M}\mathbf{U}_d = \text{diag}\left\{m_d\right\}$, and the mathematical consequences of the definition of rigid modes, namely $\mathbf{K}\mathbf{U}_r = \mathbf{0}$, $\mathbf{C}\mathbf{U}_r = \mathbf{0}$, and $\mathbf{U}_r^*\mathbf{M}\mathbf{U}_d = \mathbf{0}$ the dynamical equations of rigid and deformable d.o.f.s read

$$\begin{bmatrix} \mathbf{U}_r^*\mathbf{M}\mathbf{U}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_d^*\mathbf{M}\mathbf{U}_d \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{q}}_d \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_d^*\mathbf{C}\mathbf{U}_d \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{q}}_d \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_d^*\mathbf{K}\mathbf{U}_d \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{q}}_d \end{bmatrix} + = \begin{bmatrix} \mathbf{U}_r^*\mathbf{f} \\ \mathbf{U}_d^*\mathbf{f} \end{bmatrix}$$

or

$$\begin{cases} \mathbf{U}_r^*\mathbf{f} = \mathbf{M}_r\ddot{\mathbf{q}}_r \\ \mathbf{U}_d^*\mathbf{f} = \text{diag}\left\{m_d\right\}\ddot{\mathbf{q}}_d + \text{diag}\left\{c_d\right\}\dot{\mathbf{q}}_d + \text{diag}\left\{k_d\right\}\mathbf{q}_d \end{cases}$$

…

CHAPTER

TWELVE

STRUCTURAL DAMPING

As a first approximation, a large number of structures can be treated as undamped structures that, if constrained so that there's no rigid motion allowed, can be represented by the second order system

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \ ,$$

being mass and stiffness matrices $\mathbf{M}$, $\mathbf{K}$ that are positive definite, and symmetric if derived as an example from a Lagrangian formulation of the problem.

todo Add reference to Lagrange mechanics and its properties in the classical mechanics bbooks.

This kind of systems are conveniently described using **modal basis**, as modes (or free/natural modes of vibrations) are orthogonal w.r.t. both mass and stiffness matrix.

todo Add reference; add comment: diagonal, or diagonalizable with coincident eigenvectors.

Free response using modal basis

12.1 Small damping

Structural small damping can be treated as a first order perturbation of the undamped system,

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \ ,$$

or in Laplace domain

$$\left[s^2 \mathbf{M} + s\mathbf{C} + \mathbf{K} \right] \mathbf{u} = \mathbf{f} \ .$$

Here two assumptions are made and justified later:

- matrix $\mathbf{C}$ is (semi)positive symmetric
- matrix $\mathbf{C}$ becomes diagonal in the modal basis, i.e. modal basis simoultaneously diagonalize mass, damping and stiffness matrices

If these two assumption holds, using the modal base collected in matrix $\mathbf{U}$,

$$\mathbf{u} = \mathbf{U}\mathbf{q} \ ,$$

the diagonalization reads

$$\begin{aligned}\mathbf{U}^T\mathbf{f} = \mathbf{U}^T \left\{ \mathbf{M}\mathbf{U}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{U}\dot{\mathbf{q}} + \mathbf{K}\mathbf{U}\mathbf{q} \right\} = \\ = \text{diag}\{m_i\}\ddot{\mathbf{q}} + \text{diag}\{c_i\}\dot{\mathbf{q}} + \text{diag}\{k_i\}\mathbf{q} = \\ = \text{diag}\{m_i \ddot{q}_i + c_i \dot{q}_i + k_i q_i\} \ .\end{aligned}$$

being $m_i := \mathbf{u}_i^T \mathbf{M} \mathbf{u}_i$, $c_i := \mathbf{u}_i^T \mathbf{C} \mathbf{u}_i$, $k_i := \mathbf{u}_i^T \mathbf{K} \mathbf{u}_i$, the modal mass, damping and stiffness.

(Semi)definite positive damping matrix

Starting from the equations of motion

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \ ,$$

the kinetic energy and the mechanical energy balance is derived (**todo** add references) with scalar multiplication by $\dot{\mathbf{u}}$. For constant matrices,

$$\begin{aligned}
\dot{\mathbf{u}}^T\mathbf{M}\ddot{\mathbf{u}} + \dot{\mathbf{u}}^T\mathbf{C}\dot{\mathbf{u}} + \dot{\mathbf{u}}^T\mathbf{K}\mathbf{u} &= \dot{\mathbf{u}}^T\mathbf{f} \\
\frac{d}{dt}\left[\frac{1}{2}\dot{\mathbf{u}}^T\mathbf{M}\dot{\mathbf{u}} + \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u}\right] &= \dot{\mathbf{u}}^T\mathbf{f} - \dot{\mathbf{u}}^T\mathbf{C}\dot{\mathbf{u}} \\
\frac{d}{dt}\left(K + V\right) &= \dot{\mathbf{u}}^T\mathbf{f} - \underbrace{\dot{\mathbf{u}}^T\mathbf{C}\dot{\mathbf{u}}}_{D\geq 0} \ ,
\end{aligned}$$

having recognized $D = \dot{\mathbf{u}}^T\mathbf{C}\dot{\mathbf{u}} \geq 0$ as the dissipation from damping, that can't make the mechanical energy of the system $K + V$ increase. This condition implies that $\mathbf{C}$ is (semi)definite positive.

Diagonal damping in modal basis

Let's write here the perturbed free damped system in Laplace domain using modal basis,

$$\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]\mathbf{u} = \mathbf{0} \ ,$$

and evaluate the derivative of this relation w.r.t. a parameter p associated to the damping, and not influencing mass or stiffness properties, $\mathbf{M}_{/p} = \mathbf{0}$, $\mathbf{K}_{/p} = \mathbf{0}$,

$$\begin{aligned}
\mathbf{0} &= \left\{\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]\mathbf{u}\right\}_{/p} = \\
&= \left[(2s\mathbf{M} + \mathbf{C})s_{/p} + s\mathbf{C}_{/p}\right]\mathbf{u} + \left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]\mathbf{u}_{/p} \ .
\end{aligned}$$

Let's investigate the effect of small damping on the eigensolution $(s_i, \mathbf{u}_i)$. Exploiting the symmetry of the matrices of the system (following from the assumed simultaneous diagonalization of the damping matrix $\mathbf{C} = \mathbf{U}\text{diag}\{c_i\}\mathbf{U}^*$), and evaluating the dot product of the latter relation for the i-th eigensolution with the eigenvector $\mathbf{u}_i$,

$$\begin{aligned}
0 &= \mathbf{u}_i^T\left\{\left[(2s_i\mathbf{M} + \mathbf{C})s_{i/p} + s_i\mathbf{C}_{/p}\right]\mathbf{u}_i + \left[s_i^2\mathbf{M} + s_i\mathbf{C} + \mathbf{K}\right]\mathbf{u}_{i/p}\right\} = \\
&= \mathbf{u}_i^T\left[(2s_i\mathbf{M} + \mathbf{C})s_{i/p} + s_i\mathbf{C}_{/p}\right]\mathbf{u}_i + \mathbf{u}_{i/p}^T\underbrace{\left[s_i^2\mathbf{M} + s_i\mathbf{C} + \mathbf{K}\right]\mathbf{u}_i}_{=\mathbf{0}} =
\end{aligned}$$

it follows that the derivative of the i^{th} eigenvalue w.r.t. the parameter p reads

$$s_{i/p} = -\frac{s_i\mathbf{u}_i^T\mathbf{C}_{/p}\mathbf{u}_i}{\mathbf{u}_i^T(2s_i\mathbf{M} + \mathbf{C})\mathbf{u}_i} \ .$$

This derivative evaluated for the reference undamped condition $\mathbf{C} = \mathbf{0}$ becomes

$$s_{i/p} = -\frac{1}{2}\frac{\mathbf{u}_i^T\mathbf{C}_{/p}\mathbf{u}_i}{\mathbf{u}_i^T\mathbf{M}\mathbf{u}_i} = -\frac{1}{2}\frac{\mathbf{u}_i^T\mathbf{C}_{/p}\mathbf{u}_i}{m_i} \ ,$$

having recognized the modal mass $m_i := \mathbf{u}_i^T\mathbf{M}\mathbf{u}_i$ associated to the i^{th} mode. Now, let's evaluate the derivative of the eigenvalue s_i w.r.t. the components of the damping matrix $\mathbf{C}$, i.e. $\mathbf{C}_{/C_{jk}}$ that is a matrix full of zero, except for the component (j, k) equal to one,

$$s_{i/C_{jk}} = -\frac{1}{2}\frac{\mathbf{u}_i^T\mathbf{C}_{/C_{jk}}\mathbf{u}_i}{m_i} = -\frac{1}{2}\frac{u_j^{(i)}u_k^{(i)}}{m_i} \ ,$$

and the first order polynomial expansion of s_i in coefficients C_{jk} reads

$$\begin{aligned}
s_i & = s_{i,0} + s_{i/C_{jk}} C_{jk} = \\
& = s_{i,0} - \frac{1}{2} \frac{u_j^{(i)} C_{jk} u_k^{(i)}}{m_i} = \\
& = s_{i,0} - \frac{1}{2} \frac{\mathbf{u}_i^T \mathbf{C} \mathbf{u}_i}{m_i} \ .
\end{aligned}$$

From this expression, it's possible to deduce that the i^{th} eigenvalue of the slightly damped system differs from the i^{th} eigenvalue of the undamped system $s_{i,0} = \mp j\omega_i$ of a real non-positive (as $\mathbf{C} \geq 0$ for dissipative damping actions) term $\Delta s_i = -\frac{1}{2}\frac{\mathbf{u}_i^T \mathbf{C} \mathbf{u}_i}{m_i} \in \mathbb{R}$, $\Delta s_i \leq 0$, depending only on the damping matrix and the i^{th} mode. This term shifts the eigenvalue s_i to the left in the complex plane, and thus makes it asymptotically stable.

As the variation Δs_i only depends on the i^{th} eigenvector, and not on other eigenvectors, the assumption of siultaneously diagonalizable damping matrix is consistent with the results from this assumption.

12.2 Sensitivity of eigenvalues and eigenvectors

Link to Sensitivity of spectral decomposition, for first order equations. The sensitivity of the i^{th} eigenvalue to a general parameter reads

$$s_{i/p} = -\frac{s_i \mathbf{u}_i^* \mathbf{C}_{/p} \mathbf{u}_i}{\mathbf{u}_i^* (2 s_i \mathbf{M} + \mathbf{C}) \mathbf{u}_i} \ .$$

The sensitivity of this eigenvalue to damping $\mathbf{C}$ of the undamped system as reference condition $\mathbf{C} = \mathbf{0}$ reads

$$s_{i/\mathbf{C}} = -\frac{1}{2} \frac{\mathbf{u}_i \otimes \mathbf{u}_i}{m_i} \ .$$

The sensitivity of the eigenvector $\mathbf{u}_{i/p}$ can be evaluated as the solution of a linear system derived from the derivation of the eigenvalue problem w.r.t. the parameter p

$$\begin{aligned}
\mathbf{0} & = \frac{d}{dp} \left\{ \left(s^2 \mathbf{M} + s \mathbf{C} + \mathbf{K} \right) \mathbf{u} \right\} = \\
& = \left(s^2 \mathbf{M} + s \mathbf{C} + \mathbf{K} \right) \mathbf{u}_{/p} + \left(2 s s_{/p} \mathbf{M} + s_{/p} \mathbf{C} + s \mathbf{C}_{/p} \right) \mathbf{u} \ ,
\end{aligned}$$

having assumed here $\mathbf{M}_{/p} = \mathbf{0}$ and $\mathbf{K}_{/p} = \mathbf{0}$. The linear system thus becomes

$$\left(s^2 \mathbf{M} + s \mathbf{C} + \mathbf{K} \right) \mathbf{u}_{/p} = \underbrace{-\left(2 s s_{/p} \mathbf{M} + s_{/p} \mathbf{C} + s \mathbf{C}_{/p} \right) \mathbf{u}}_{=\mathbf{b}} \ .$$

- This linear system is singular, as s is an eigenvalue of the system, and

  $$\left(s^2 \mathbf{M} + s \mathbf{C} + \mathbf{K} \right) \mathbf{x} = \mathbf{0} \ ,$$

  for every $\mathbf{x}$ that is a linear combination of the eigenvectors with eigenvalue s. Thus, if the linear system has a solution, it has infinite solution. The linear system has a solution if the RHS $\mathbf{b}$ is in the range of the matrix $\mathbf{A}(s) := s^2 \mathbf{M} + s \mathbf{C} + \mathbf{K}$.

- If $\mathbf{b} \in \mathrm{R}(\mathbf{A})$ then it's orthogonal to $\mathrm{K}(\mathbf{A}^*)$, and viceversa. Kernel of $\mathbf{A}^* := s^{*2} \mathbf{M}^* + s^* \mathbf{C}^* + \mathbf{K}^*$ is spanned by the left eigenvectors $\mathbf{v}$ associated with the eigenvalue s. Direct evaluation of the product $\mathbf{v}^* \mathbf{b}$ proves that $\mathbf{b} \perp \mathrm{K}(\mathbf{A}^*)$ and thus $\mathbf{b} \in \mathrm{R}(\mathbf{A})$, and thus a solution exists,

  $$\begin{aligned}
  \mathbf{v}^* \mathbf{b} & = \mathbf{v}^* \left(-2 s s_{/p} \mathbf{M} - s_{/p} \mathbf{C} - s \mathbf{C}_{/p} \right) \mathbf{u} = \\
  & = \mathbf{v}^* (-2 s \mathbf{M} - \mathbf{C}) \mathbf{u} s_{/p} - s \mathbf{v}^* \mathbf{C}_{/p} \mathbf{u} = \\
  & = \mathbf{v}^* (-2 s \mathbf{M} - \mathbf{C}) \mathbf{u} \frac{-s \mathbf{v}^* \mathbf{C}_{/p} \mathbf{u}}{\mathbf{v}^* (2 s \mathbf{M} + \mathbf{C}) \mathbf{u}} - s \mathbf{v}^* \mathbf{C}_{/p} \mathbf{u} = 0 \ .
  \end{aligned}$$

- Since a solution exists, an infinite number of solution exists. Given a solution $\tilde{\mathbf{u}}_{/p}$, adding a linear combination of the eigenvectors with eigenvalue s produces a solution as well,

$$\tilde{\mathbf{u}}_{/p} + \mathbf{U}\beta \ .$$

- In order to remove the arbitariness, it's possible to introduce some conditions, like the orthogonality condition $\mathbf{V}^* \mathbf{M} \mathbf{u}_{/p}) \mathbf{0}$
- Writing the solution as a linear combination of eigenvectors $\mathbf{U}$ of the linear system (are they a basis?) and explicitly discern the eigenvectors with eigenvalue s_i fro the other ones,

$$\mathbf{u}_{/p} = \mathbf{U}_{\notin i}\alpha + \mathbf{U}_i\beta \ ,$$

the linear system and the orthogonality condition $\mathbf{V}_i^* \mathbf{M} \mathbf{U}_{\notin i} = \mathbf{0}$ give the decoupled linear system

$$\begin{cases} \mathbf{V}^*_{\notin i} \left(s_i^2 \mathbf{M} + s_i \mathbf{C} + \mathbf{K} \right) \mathbf{U}_{\notin i}\alpha = \mathbf{V}^*_{\notin i}\mathbf{b} \\ \mathbf{V}_i \mathbf{M} \mathbf{U}_i \beta = \mathbf{0} \ . \end{cases}$$

Under the assumption that mass, damping and stiffness matrices are simultaneously diagnoalized, it immediately follows that this linear system is diagonal,

$$\begin{cases} \text{diag}\left\{ s_i^2 m_{j\notin i} + s_i c_{j \notin i} + k_{j\notin i} \right\} \alpha = \mathbf{V}^*_{\notin i}\mathbf{b} \\ \text{diag}\left\{ m_i \right\} \beta = \mathbf{0} \ . \end{cases}$$

If mass normalization is chosen, $\mathbf{V}^*\mathbf{M}\mathbf{U} = \mathbf{I}$, i.e. $m_j = 1$, $k_j = \omega_j^2 = |s_j|^2$, and $c_i = 2\xi_j\omega_j = -2\,\text{re}\{s_j\}$, being $s_j = \omega_j \left(-\xi_j \mp i \sqrt{1 - \xi_j^2} \right)$. It's immediate to prove that

$$s^2 - 2s\,\text{re}\{s_j\} + |s_j|^2 = (s - s_j^*)(s - s_j) \ ,$$

as

$$(s - s_j^*)(s - s^j) = s^2 - s(s_j + s_j^*) + s_j^* s_j = s^2 - 2\,\text{re}\{s_j\} + |s_j|^2 \ .$$

The solution of the linear system thus reads

$$\begin{cases} \alpha = \text{diag}\left\{ \dfrac{1}{s_i^2 + 2\xi_j\omega_j s_i + \omega_j^2} \right\} \mathbf{V}^*_{\notin i}\mathbf{b} \\ \beta = \mathbf{0} \ , \end{cases}$$

s.t. the unique solution of the augmented problem reads

$$\begin{aligned} \mathbf{u}_{i/p} & = \mathbf{U}_{\notin i}\text{diag}\left\{ \frac{1}{s_i^2 + 2\xi_j\omega_j s_i + \omega_j^2} \right\} \mathbf{V}^*_{\notin i}\mathbf{b}_i = \\ & = -\mathbf{U}_{\notin i}\text{diag}\left\{ \frac{1}{s_i^2 + 2\xi_j\omega_j s_i + \omega_j^2} \right\} \mathbf{V}^*_{\notin i} \left(2 s_i s_{i/p} \mathbf{M} + s_{i/p}\mathbf{C} + s_i \mathbf{C}_{/p} \right) \mathbf{u}_i = \\ & = -s_i \mathbf{U}_{\notin i}\text{diag}\left\{ \frac{1}{s_i^2 + 2\xi_j\omega_j s_i + \omega_j^2} \right\} \mathbf{V}^*_{\notin i}\mathbf{C}_{/p}\mathbf{u}_i = \\ & = \dots \end{aligned}$$

Algebra with components

Let the matrix $\mathbf{U}_{\notin i} = \left[\dots | \mathbf{u}_j | \dots \right]$, and $U^{\notin i}_{ab} = u^{(b)}_a$, $V^{\notin i}_{ab} = v^{(b)}_a$,

$$\begin{aligned}
u^{(i)}_{a/p} & = -s_i U^{\notin i}_{ab} \frac{\delta_{be}}{s_i^2 + 2\xi_b \omega_b s_i + \omega_b^2} V^{\notin i,\,*}_{ce} C_{cd/p} u^{(i)}_d = \\
& = -u^{(b)}_a \frac{s_i}{s_i^2 + 2\xi_b \omega_b s_i + \omega_b^2} v^{(b)\,*}_c C_{cd/p} u^{(i)}_d = \\
& = -\sum_{b \neq i} \frac{s_i}{s_i^2 + 2\xi_b \omega_b s_i + \omega_b^2} \mathbf{v}^*_b \mathbf{C}_{/p} \mathbf{u}_i \, \mathbf{u}_b
\end{aligned}$$

Parameter $p = C_{rs}$

If $p = C_{rs}$, then $C_{cd/p} = C_{cd/C_{rs}} = \delta_{cr} \delta_{ds}$, and

$$\begin{aligned}
u^{(i)}_{a/C_{rs}} & = -s_i U^{\notin i}_{ab} \frac{\delta_{be}}{s_i^2 + 2\xi_b \omega_b s_i + \omega_b^2} V^{\notin i,\,*}_{ce} C_{cd/C_{rs}} u^{(i)}_d = \\
& = -\sum_{b \neq i} u^{(b)}_a \frac{s_i}{s_i^2 + 2\xi_b \omega_b s_i + \omega_b^2} v^{(b)\,*}_c \delta_{cr} \delta_{ds} u^{(i)}_d = \\
& = -\sum_{b \neq i} u^{(b)}_a \frac{s_i}{s_i^2 + 2\xi_b \omega_b s_i + \omega_b^2} v^{(b)\,*}_r u^{(i)}_s =
\end{aligned}$$

is a *third-order tensor*, as it's the derivative of a vector quantity w.r.t. a second-order tensor $\mathbf{C}$[1], and its index representation has 3 non-dummy indices, namely a, r, s.

The first-order approximation of the eigenvector reads

$$\begin{aligned}
u^{(i)}_a & \simeq u^{(i)}_{a,0} + \sum_{r,s} C_{rs} u^{(i)}_{a/C_{rs}} = \\
& = u^{(i)}_{a,0} - \sum_{r,s} C_{rs} \sum_{b \neq i} u^{(b)}_a \frac{s_i}{s_i^2 + 2\xi_b \omega_b s_i + \omega_b^2} v^{(b)\,*}_r u^{(i)}_s = \\
& = u^{(i)}_{a,0} - \sum_{b \neq i} u^{(b)}_a \frac{s_i}{s_i^2 + 2\xi_b \omega_b s_i + \omega_b^2} \sum_{r,s} v^{(b)\,*}_r C_{rs} u^{(i)}_s \ ,
\end{aligned}$$

or using vector formalism

$$\mathbf{u}_i = \mathbf{u}_{i,0} - \sum_{b \neq i} \frac{s_i}{s_i^2 + 2\xi_b \omega_b s_i + \omega_b^2} \mathbf{v}^*_{b,0} \mathbf{C} \mathbf{u}_{i,0} \, \mathbf{u}_{i,0} \ .$$

[1] Is this really a tensor? Should we recall the definition of tensor here? What's the right name of the extension of a matrix with 3 indices?

Part III

Fluid Mechanics

CHAPTER

THIRTEEN

INTRODUCTION TO FLUID MECHANICS

- Statics and definition of fluids, as medium that has no shear stress at rest.
- Kinematics
- Dynamics
- Models:
  - Incompressible flows
    - Governing equations, theorems and regimes of motion
      - Inviscid
      - Irrotational
  - Compressible flows
    - Inviscid
    - …

CHAPTER

FOURTEEN

STATICS

The behavior of continuous medium in static conditions can be used to define a fluid.

Definition 14.1 (Fluid)

A fluid can be defined as a continuous medium with no shear stress in static conditions. Thus, the stress tensor of an *isotropic fluid* under static conditions reads

$$\mathbb{T}^s = -p \mathbb{I} ,$$

where p is *pressure*. (**todo** mechanical? Thermodynamical?)

CHAPTER

FIFTEEN

CONSTITUTIVE EQUATIONS OF FLUID MECHANICS

15.1 Newtonian Fluids

A Newtonian fluid is the model of a fluid as a continuous medium whose stress tensor can be written as the sum of the hydrostatic pressure stress tensor $-p\mathbb{I}$ - the only contribution holding in *statics* - and a viscous stress tensor $\mathbb{S}$

$$\mathbb{T} = -p\mathbb{I} + \mathbb{S} \ ,$$

and the viscous stress tensor is isotropic and **linear** in the first-order spatial derivatives of the velocity field,

$$\mathbb{S} = 2\mu\mathbb{D} + \lambda(\nabla \cdot \vec{u})\mathbb{I} \ , \tag{15.1}$$

being μ, λ the viscosity coefficients, and $\mathbb{D}$ the strain velocity tensor (1.4). Thus, the definition

Definition 15.1.1 (Newtonian fluid)

A Newtonian fluid is a continuous medium whose stress tensor reads

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D} + \lambda(\nabla \cdot \vec{u})\mathbb{I} \ . \tag{15.2}$$

Note: The expression (15.1) of the viscosity stress tensor is the most general expression of a 2-nd order symmetric isotropic tensor proportional to 1-st order derivatives of a vector field.

CHAPTER

SIXTEEN

GOVERNING EQUATIONS OF FLUID MECHANICS

16.1 Newtonian Fluid

The differential conservative form of the governing equations of a *Newtonian fluid* directly follows from the expression (2.1) of *governing equations of a continuum medium in differential form*,

$$\begin{cases}\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \\ \dfrac{\partial}{\partial t}(\rho \vec{v}) + \nabla \cdot (\rho \vec{v} \otimes \vec{v}) = \rho \vec{g} + \nabla \cdot \mathbb{T} \\ \dfrac{\partial}{\partial t}(\rho e^t) + \nabla \cdot (\rho e^t \vec{v}) = \rho \vec{g} \cdot \vec{v} + \nabla \cdot (\mathbb{T} \cdot \vec{v}) - \nabla \cdot \vec{q} + \rho r\end{cases}$$

using the expression (15.2) of the stress tensor of a Newtonian fluid,

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D} + \lambda(\nabla \cdot \vec{v})\mathbb{I} \ ,$$

a constitutive equation for conduction heat flux $\vec{q}$, as an example **Fourier's law**

$$\vec{q} = -k\nabla T \ ,$$

and the required state equations characterizing the behavior of the medium linking thermodynamic variables (assumption of **local thermodynamic equilibrium todo** *discuss this principle*), and required to get a well-defined mathematical problem, with the same number of equations and unknowns.

Example 16.1.1 (Equations of state)

As an example, the required equations of states need to provide and expression of thermodynamic quantities as a function of the dynamical physical quantities. This could be quite a common choice in numerical methods using conservative form of the equations. Namely, defining momentum and total energy per unit volume

$$\vec{m} := \rho \vec{v} \quad , \quad E^t := \rho e^t \ ,$$

equations of state should provide the expression of pressure p, temperature T, viscosity coefficients μ, ν and thermal conductivity k as functions of "dynamic quantities" ρ, $\vec{m}$, E^t,

$$\begin{aligned} &p(\rho, \vec{m}, E^t) \\ &T(\rho, \vec{m}, E^t) \\ &\mu(\rho, \vec{m}, E^t) \\ &\lambda(\rho, \vec{m}, E^t) \\ &k(\rho, \vec{m}, E^t) \end{aligned}$$

Usually, in thermodynamics pressure and temperature can be written as functions of other two thermodynamic variables, as an example density ρ and internal energy (per unit mass)

$$e = e^t - \frac{|\vec{v}|^2}{2} = \frac{E^t}{\rho} - \frac{1}{2}\frac{|\vec{m}|^2}{\rho^2}$$

so that - avoiding here notation abuses and using two different symbols for functions with different independent variables representing the same physical quantity -,

$$\begin{aligned}
\Pi\,(\rho, e) &= \Pi\left(\rho, \frac{E^t}{\rho} - \frac{1}{2}\frac{|\vec{m}|^2}{\rho^2}\right) = p(\rho, \vec{m}, E^t) \\
\Theta\,(\rho, e) &= \Theta\left(\rho, \frac{E^t}{\rho} - \frac{1}{2}\frac{|\vec{m}|^2}{\rho^2}\right) = T(\rho, \vec{m}, E^t)
\end{aligned}$$

16.2 Derived quantities

Balance equations of kinetic energy and internal energy readily follows from balance equations of continuum media in covective form.

Kinetic energy.

Internal energy.

$$\begin{aligned}
\rho\frac{De}{Dt} &= \mathbb{T} : \nabla\vec{v} - \nabla\cdot\vec{q} + \rho r = \\
&= (-p\mathbb{I} + 2\mu\mathbb{D} + \lambda(\nabla\cdot\vec{v})\mathbb{I}) : \nabla\vec{v} - \nabla\cdot\vec{q} + \rho r = \\
&= -p\nabla\cdot\vec{v} + 2\mu\mathbb{D} : \mathbb{D} + \lambda(\nabla\cdot\vec{v})^2 - \nabla\cdot\vec{q} + \rho r
\end{aligned}$$

Details

todo

Entropy equation. The first principle of thermodynamics for non-reactive fluid, with no electric charge and other processes, provides the expression of the differential of entropy as a function of internal energy and density, $s(e, \rho)$

$$de = T\,ds + \frac{P}{\rho^2}d\rho \qquad , \qquad ds = \frac{1}{T}\,de - \frac{P}{T\rho^2}\,d\rho\ ,$$

and thus the balance equation for entropy directly follows from the evaluation of the material derivative of entropy field, exploiting balance equations of mass and internal energy

$$\begin{aligned}
\rho\frac{Ds}{Dt} &= \frac{1}{T}\left[\rho\frac{De}{Dt} - \frac{P}{\rho}\frac{D\rho}{Dt}\right] = \\
&= \frac{1}{T}\left[-p\nabla\cdot\vec{v} + 2\mu\mathbb{D} : \mathbb{D} + \lambda(\nabla\cdot\vec{v})^2 - \nabla\cdot\vec{q} + \rho r - \frac{P}{\rho}\left(-\rho\nabla\cdot\vec{v}\right)\right] = \\
&= \frac{1}{T}\left[2\mu\mathbb{D} : \mathbb{D} + \lambda(\nabla\cdot\vec{v})^2 - \nabla\cdot\vec{q} + \rho r\right] = \\
&= \frac{2\mu\mathbb{D} : \mathbb{D} + \lambda(\nabla\cdot\vec{v})^2}{T} - \frac{\vec{q}\cdot\nabla T}{T^2} - \nabla\cdot\left(\frac{\vec{q}}{T}\right) + \frac{\rho r}{T} = \\
&= \frac{2\mu\mathbb{D} : \mathbb{D} + \lambda(\nabla\cdot\vec{v})^2}{T} + \frac{k|\nabla T|^2}{T^2} - \nabla\cdot\left(\frac{\vec{q}}{T}\right) + \frac{\rho r}{T}\ .
\end{aligned}$$

Details

$$\nabla \cdot \left(\frac{\vec{q}}{T} \right) = \frac{\nabla \cdot \vec{q}}{T} - \frac{\vec{q} \cdot \nabla T}{T^2}$$

Second principle of thermodyanamics and continuum mechanics. Second principle of thermodynamics implies some constraints on the behavior of continuous media, and thus on the contitutive equations. Namely, Clausis statement of the second principle reads

$$dS \ge \frac{\delta Q}{T} \ ,$$

i.e. the variation of entropy is greater or equal to the ratio of the heat flux added to the system and the temperature of the system itself. This can be written for a simple homogeneous system, or for a composite systems where physical quantities are not homogeneous in space **todo** ref

Integral form of balance equation of entropy of a system reads

$$\frac{dS}{dt} = \frac{d}{dt}\int_{V_t} \rho s = \int_{V_t} \left\{ \frac{2\mu \mathbb{D} : \mathbb{D} + \lambda (\nabla \cdot \vec{v})^2}{T} + \frac{k|\nabla T|^2}{T^2} \right\} \underbrace{- \oint_{\partial V_t} \frac{\vec{q}}{T} \cdot \hat{n} + \int_{V_t} \frac{\rho r}{T}}_{\text{"} \frac{\delta Q}{T} \text{"}} \ ,$$

and Clausius statement of the second principle implies

$$0 \le \frac{d}{dt}\int_{V_t} \rho s - \left(- \oint_{\partial V_t} \frac{\vec{q}}{T} \cdot \hat{n} + \int_{V_t} \frac{\rho r}{T} \right) = \int_{V_t} \left\{ \frac{2\mu \mathbb{D} : \mathbb{D} + \lambda (\nabla \cdot \vec{v})^2}{T} + \frac{k|\nabla T|^2}{T^2} \right\} \ ,$$

and, since this must hold for any volume V_t and state of the system - namley every velocity and temperature field - and thermodynamic temperature is positive, it follows that

$$\mu \ge 0 \quad , \quad \lambda \ge 0 \quad , \quad k \ge 0$$

Example 16.2.1 (Sign of physical quantity)

todo pay attention, that temperature, viscosity ccoefficients and thermal conductivity have physical dimensions…explain the meaning of positive physical quantities (scalar, w.r.t. a unit of measurement)…

CHAPTER

SEVENTEEN

NON-DIMENSIONAL EQUATIONS OF FLUID MECHANICS

If $\rho(P, s)$,

$$d\rho = \left(\frac{\partial \rho}{\partial P}\right)_s dP + \left(\frac{\partial \rho}{\partial s}\right)_\rho ds$$

$$\begin{cases} \dfrac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0 \\ \rho \dfrac{D\vec{v}}{Dt} = \rho\vec{g} + \nabla \cdot (-p\mathbb{I} + 2\mu\mathbb{D} + \lambda(\nabla \cdot \vec{v})\mathbb{I}) \\ \rho \dfrac{De^t}{Dt} = \rho\vec{g} \cdot \vec{v} + \nabla \cdot (\mathbb{T} \cdot \vec{v}) - \nabla \cdot \vec{q} + \rho r \end{cases}$$

CHAPTER

EIGHTEEN

INCOMPRESSIBLE FLUID MECHANICS

Chapter of a introductory course in incompressible fluid mechanics:

- statics
- kinematics
- governing equations
- non-dimensional equations
- vorticity dynamics
- low-Re exact solutions
- high-Re flows, incompressible inviscid irrotational flows:
  - vorticity dynamics and Bernoulli theorems
  - aeronautical applications
- boundary layer
- instability and turbulence

18.1 Navier-Stokes Equations

The kinematic constraints (link to *Non-dimensional Equations of Fluid Mechanics*?)

$$\nabla \cdot \vec{v} = 0$$

replaces mass balance in the governing equation and implies $\frac{D\rho}{Dt} = 0$, i.e. all the material particles have constant density in time.

If …

$$\begin{cases} \rho \left[\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla)\vec{u} \right] - \mu \nabla^2 \vec{u} + \nabla P = \rho \vec{g} \\ \nabla \cdot \vec{u} = 0 \end{cases} \tag{18.1}$$

with the proper initial conditions, boundary conditions and - if required - *compatibility conditions*.

Compatibility condition

A compatibility condition is needed if the velocity field is prescribed on the whole boundary ∂V of the domain V,

$$\vec{u}\Big|_{\partial V} = \vec{b}_n \ .$$

The compatibility condition reads

$$\oint_{\partial V} \vec{b} \cdot \hat{n} = 0 \ ,$$

to ensure that the conudary conditions are consistent with the incompressibility constraint, as it is readily proved using divergence theorem on the velocity field in V,

$$0 \equiv \int_V \underbrace{\nabla \cdot \vec{u}}_{=0} = \oint_{\partial V} \hat{v} \cdot \hat{n} = \oint_{\partial V} \vec{b} \cdot \hat{n} \ .$$

18.2 Vorticity

A dynamical equation for vorticity $\vec{\omega} := \nabla \times \vec{u}$ reailty follows taking the curl of Navier-Stokes equations (18.1)

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{u} + \nu \Delta \vec{\omega} \ , \tag{18.2}$$

i.e. vorticity can be stretched-tilted by the term $(\vec{\omega} \cdot \nabla)\vec{u}$, or diffused by the term $\nu \Delta \vec{\omega}$.

…

18.3 Bernoulli theorems

For an incompressible fluid, the advective term $(\vec{u} \cdot \nabla) \cdot \vec{u}$ can be recasted as

$$(\vec{u} \cdot \nabla) \cdot \vec{u} = \vec{\omega} \times \vec{u} + \nabla \frac{|\vec{u}|^2}{2} \ ,$$

so that the momentum equation in Navier-Stokes equations (18.1) for fluids with uniform density ρ reads

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u} + \nabla \frac{|\vec{u}|^2}{2} \right] - \mu \Delta \vec{u} + \nabla P = \rho \vec{g} \ . \tag{18.3}$$

Starting from the form (18.3), different forms of Bernoulli theorems are readilty derived with the proper assumptions.

Theorem 18.3.1 (Bernoulli theorem along path and vortex lines in steady flows)

In a steady incompressible inviscid flow with conservative volume forces, $\vec{g} = -\nabla \chi$, the Bernoulli polynomial is constant along path (everywhere tangent to the velocity field, $\hat{t}(\vec{r}) \parallel \vec{u}(\vec{r})$) and vortex lines (everywhere tangent to the vorticity field, $\hat{t}(\vec{r}) \parallel \vec{\omega}(\vec{r})$), i.e. the directional derivative of the Bernoulli polynomial in the direction of the velocity or the vorticity field is identically zero,

$$\hat{t} \cdot \nabla \left(\frac{|\vec{u}|^2}{2} + \frac{P}{\rho} + \chi \right) = 0 \ .$$

The proof readily follows taking the scalar product with a unit-norm vector $\hat{t}$ parallel to the local velocity or vorticity, and noting that $\hat{t} \cdot \vec{u} \times \vec{\omega}$ is zero if either $\hat{t} \parallel \vec{v}$ or $\hat{t} \parallel \vec{\omega}$.

Theorem 18.3.2 (Bernoulli theorem in irrotational inviscid steady flows)

In a steady incompressible inviscid irrotational flow with conservative volume forces, $\vec{g} = -\nabla\chi$, the Bernoulli polynomial is uniform in the whole domain, since its gradient is identically zero

$$\nabla \left(\frac{|\vec{u}|^2}{2} + \frac{P}{\rho} + \chi \right) = 0 \qquad \rightarrow \qquad \frac{|\vec{u}|^2}{2} + \frac{P}{\rho} + \chi = 0 \ .$$

Theorem 18.3.3 (Bernoulli theorem in irrotational inviscid flows)

In an incompressible inviscid irrotational flow with conservative volume forces, $\vec{g} = -\nabla\chi$, the Bernoulli polynomial is uniform in the connected irrotational regions of the domain - but not constant in time in general - , since its gradient is identically zero

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{|\vec{u}|^2}{2} + \frac{P}{\rho} + \chi \right) = 0 \qquad \rightarrow \qquad \frac{\partial \phi}{\partial t} + \frac{|\vec{u}|^2}{2} + \frac{P}{\rho} + \chi = C(t) \ .$$

being ϕ the velocity potential used to write the irrotational velocity field as the gradient of a scalar function $\vec{u} = \nabla\phi$.

Note: The assumption of inviscid flow is not directly required if irrotationality holds. Anyway the inviscid flow assumption may be required to make irrotationality condition holds. Lookinig at the vorticity equation (18.2) the assumption of negligible viscosity prevents diffusion of vorticity from rotational regions to irrotational regions.

Note: A barotropic fluid is defined as a fluid where the pressure is a function of density only, $P(\rho)$. For this kind of flows it's possible to find a function Π so that

$$d\Pi = \frac{dP}{\rho} \ .$$

The results of this section derived for a uniform density flow hold for a barotropic fluid as well, replacing $\frac{P}{\rho}$ with Π.

CHAPTER

NINETEEN

COMPRESSIBLE FLUID MECHANICS

19.1 Compressible Inviscid Fluid Mechanics

19.1.1 Shocks

19.1.2 Quasi-1d flows

If no shock occurs in the flow, Euler equations in differential form governs the dynamics of the flow

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0 \\ \partial_t \left(\rho \vec{u} \right) + \nabla \cdot (\rho \vec{u} \otimes \vec{u} + P \mathbb{I}) = 0 \\ \partial_t \left(\rho e^t \right) + \nabla \cdot (\rho \vec{u} h^t) = 0 \end{cases}$$

Quasi-1d model for steady flows is a simple model that provides good-enough results for flows delimited by streamlines that varies gently in the streamwise direction (or by solid walls, in the limit of the high-Reynolds flow without separations, where viscous effects are confined to a thin region - boundary layer - close to the walls).

This model is derived integrating over the sections of the stream tube, so that the physical quantities are functions of the streamwise coordinate x only. Integration over an elementary volume between sections at x and $x + dx$ of the mass and momentum equations, in steady conditions, gives

$$\begin{cases} d(\rho u A) = 0 \\ d(\rho u^2 A) + d(PA) + \displaystyle\int_{S_{lat}(x,dx)} P n_x = 0 \ , \end{cases}$$

where the last equation comes from the contribution of the lateral surfaces, that has non zero contribution in the streamwise component of momentum equation if sections is not constant, and thus the unit normal vector $\hat{n}$ on the lateral surface has non-zero x-component n_x. This contribution on the elementary lateral surface (where pressure is assumed to be uniform), can be evaluated summing and subracting the contributions on the $A(x)$ and $A(x + dx)$ surface,

$$\underbrace{\int_{S_{lat}(x,dx)} P n_x + P(A + dA) - PA}_{=P \oint_{\partial V} n_x = 0} - P(A + dA) + PA = -P dA$$

Thus the equations become

$$\begin{cases} 0 = d(\rho u A) = 0 \\ 0 = d(\rho u^2 A) + A dP = \rho u A \, du + A \, dP \ , \end{cases}$$

having used the mass equation to simplify the first term in the momentum equation, since

$$d(\rho u^2 A) = \underbrace{d(\rho u A)}_{=0} u + \rho u A \, du \ .$$

Now, from momentum equation a relation from changes in velocity and pressure holds

$$\rho u \, du = -dP \ .$$

If the flow is isentropic (i.e. negligible viscous effects, no heat conduction, no shocks), $s = \overline{s}$, the definition of the speed of sound

$$c^2(\rho, s) = \left(\frac{\partial P}{\partial \rho} \right)_s \ ,$$

can be used to write a relation between changes in pressure and density $dP = c^2 d\rho$. Using this formula and mass equation $d(\rho u A) = 0$, it's possible to write relations between physical quantities and the variation of the section of the stream tube. Mass equation can be recast as

$$\begin{aligned}
0 &= \frac{dA}{A} + \frac{d\rho}{\rho} + \frac{du}{u} = \\
&= \frac{dA}{A} + \frac{1}{c^2}\frac{dP}{\rho} + \frac{du}{u} = \\
&= \frac{dA}{A} + \frac{1}{c^2}\frac{dP}{\rho} - \frac{dP}{\rho u^2}
\end{aligned}$$

and thus, introducing Mach number $M := \dfrac{u}{c}$,

$$\frac{dA}{A} = (1 - M^2)\frac{dP}{\rho u^2} = -(1 - M^2)\frac{du}{u} \ .$$

From this equation, it's immediate to realize that:

- for subsonic flows $M < 1$, if section of the stream tube increases, $dA > 0$, thus velocity descreases $du < 0$, and pressure increases $dP > 0$, and viceversa
- for supersonic flows $M > 1$, if section of the stream tube increases, $dA > 0$, thus velocity increases $du > 0$, and pressure decreases $dP < 0$

PROOF INDEX