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# **continuum mechanics**

**basics**

**Feb 27, 2025**



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This material is part of the [basics-books project](#). It is also available as a .pdf document.

General approach and equations in continuum mechanics are first presented, and then specialized to the most common models of solids - mainly elastic solids - and fluids - mainly Newtonian fluids.

## Introduction to Continuum Mechanics

**Kinematics of continuum media.** Lagrangian, Eulerian and arbitrary descriptions of the motion of continuous media is presented, and kinematic quantities are introduced.

**Balance equations of physical quantities.** Balance equations of physical quantities are introduced here for continuous media, both in integral and differential forms - in regular domains with “smooth” distribution of physical properties. Reynolds theorem and derivatives of composite functions are exploited to provide Lagrangian, Eulerian and arbitrary descriptions - and their relationship - both for integral and differential equations respectively.

First Lavoisier principle for mass conservation, Newton principles and equations of motion for momentum and angular momentum balance equation, and first principle of thermodynamics or balance equation of total energy are written for closed systems - and derived for arbitrary systems.

The need for constitutive equations and state equations is discussed. Properties of stress tensors and heat conduction flux are described.

Then, balance equations for other physical quantities are derived, e.g. for kinetic energy, internal energy, and entropy. Balance equation of entropy and second principle of thermodynamics prescribe some constraints on stress tensor and heat conduction flux.

## Solid Mechanics

## Fluid Mechanics



**Part I**

**Continuum Mechanics**





## KINEMATICS

Let

- $\vec{r}$  the physical space coordinates
- $\vec{r}_0$  the material coordinates, labels associated to material points of the continuum
- $\vec{r}_b$  arbitrary coordinates, labels associated to arbitrary points - e.g. geometric points

### 1.1 Material points in physical space

**Position.** The position in physical space of material points labeled with material coordinates  $\vec{r}_0$  can be written as a function

$$\vec{r}(\vec{r}_0, t), \quad (1.1)$$

providing the position in physical space of a material point, as a function of its label  $\vec{r}_0$  and time  $t$ .

**Velocity.** The velocity of each material point is the time-derivative of function (1.1) at constant  $\vec{r}_0$  (since one is interested here in the velocity of material points),

$$\vec{u} = \left. \frac{\partial \vec{r}}{\partial t} \right|_{\vec{r}_0} =: \frac{D\vec{r}}{Dt}, \quad (1.2)$$

having introduced the definition of **material derivative**,  $\frac{D}{Dt} := \left. \frac{\partial}{\partial t} \right|_{\vec{r}_0}$ .

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#### Independent variables

In formula (1.2), independent variables are not explicitly written. If  $\vec{r}(\vec{r}_0, t)$ , the velocity field  $\vec{u}$  can be readily written as functions of the same independent variables,

$$\vec{u}_0(\vec{r}_0, t) = \left. \frac{\partial \vec{r}}{\partial t} \right|_{\vec{r}_0}(\vec{r}_0, t),$$

and it provides the velocity field as a function of the material coordinates, namely the **Lagrangian description**, following material points in their evolution in space.

Eulerian description of the problem requires physical properties to be written as functions of physical coordinates,  $\vec{r}$ ,  $t$ . If the inverse transformation of (1.1) exists, it's possible to write  $\vec{r}_0(\vec{r}, t)$ , and the velocity field as

$$\vec{u}(\vec{r}, t) = \vec{u}_0(\vec{r}_0(\vec{r}, t), t),$$

or, for invertible transformations,

$$\vec{u}_0(\vec{r}_0, t) = \vec{u}(\vec{r}(\vec{r}_0, t), t),$$

having used indices to mathematically discern functions of different independent variables, even if they represent the same physical quantity. In many situations, this inverse transformation between the position in physical space and the material coordinates is not well-defined, often for fluid systems or solid mechanics with (very) large deformations: in these cases, it's always (?) possible to update the reference configuration at some closer time instant in order to find a well-defined inverse transformation, if needed.

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**Acceleration.** Acceleration of a material point labeled with material coordinate  $\vec{r}_0$  is the second order derivative of the physical position (1.1) w.r.t. time  $t$  keeping  $\vec{r}_0$  constant, or the first order derivative of the velocity (1.2),

$$\vec{a} = \frac{\partial \vec{u}}{\partial t} \Big|_{\vec{r}_0} = \frac{\partial \vec{r}}{\partial t} \Big|_{\vec{r}_0} \cdot \frac{\partial \vec{u}}{\partial \vec{r}} \Big|_t + \frac{\partial \vec{u}}{\partial t} \Big|_{\vec{r}} = \vec{u} \cdot \nabla \vec{u} + \frac{\partial \vec{u}}{\partial t} \Big|_{\vec{r}},$$

having written the partial derivative in time at constant physical coordinate  $\vec{r}$  as  $\frac{\partial}{\partial t} \Big|_{\vec{r}} = \frac{\partial}{\partial t}$ , and the gradient w.r.t. the physical coordinate as  $\nabla_{\vec{r}} = \nabla$ .

## 1.2 Arbitrary points in physical space

Following the same process as the one used for *material points*, the position, the velocity and the acceleration of a set of arbitrary points labeled with  $\vec{r}_b$  coordinates read

$$\begin{aligned} \vec{r}(\vec{r}_b, t) \\ \vec{u}_b &= \frac{\partial \vec{r}}{\partial t} \Big|_{\vec{r}_b} \\ \vec{a}_b &= \frac{\partial \vec{u}_b}{\partial t} \Big|_{\vec{r}_b} = \frac{\partial \vec{u}_b}{\partial t} + \vec{u}_b \cdot \nabla \vec{u}_b \end{aligned}$$

## 1.3 Time derivatives of a function from different descriptions

Coordinate transformations implies the rules to compute the relations between time derivatives of a field  $f$  keeping physical, material or arbitrary coordinates constant, namely

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{\vec{r}_0} f(\vec{r}(\vec{r}_0, t), t) &= \frac{\partial f}{\partial t} \Big|_{\vec{r}} + \frac{\partial \vec{r}}{\partial t} \Big|_{\vec{r}_0} \cdot \frac{\partial f}{\partial \vec{r}} \Big|_t = \frac{\partial f}{\partial t} \Big|_{\vec{r}} + \vec{u} \cdot \nabla f \\ \frac{\partial}{\partial t} \Big|_{\vec{r}_b} f(\vec{r}(\vec{r}_b, t), t) &= \frac{\partial f}{\partial t} \Big|_{\vec{r}} + \frac{\partial \vec{r}}{\partial t} \Big|_{\vec{r}_b} \cdot \frac{\partial f}{\partial \vec{r}} \Big|_t = \frac{\partial f}{\partial t} \Big|_{\vec{r}} + \vec{u}_b \cdot \nabla f \end{aligned}$$

and thus

$$\frac{Df}{Dt} = \frac{\partial}{\partial t} \Big|_{\vec{r}_0} f = \frac{\partial}{\partial t} \Big|_{\vec{r}} f + \vec{u} \cdot \nabla f = \frac{\partial}{\partial t} \Big|_{\vec{r}_b} f + (\vec{u} - \vec{u}_b) \cdot \nabla f. \quad (1.3)$$

## 1.4 Kinematics of two points

$$\vec{r}_2(t) - \vec{r}_1(t) = \vec{r}(\vec{r}_{0,2}, t) - \vec{r}(\vec{r}_{0,1}, t)$$

$$\vec{v}_2(t) - \vec{v}_1(t) = \frac{d}{dt}\vec{r}_2(t) - \frac{d}{dt}\vec{r}_1(t)$$

strain velocity tensor

$$\mathbb{D} = \frac{1}{2} [\nabla \vec{u} + \nabla^T \vec{u}] \quad (1.4)$$

## 1.5 Kinematics in reference space

Let  $\vec{r}(\vec{r}_0, t)$ , it's differential - keeping  $t$  constant - reads

$$d\vec{r} = d\vec{r}_0 \cdot \nabla_0 \vec{r} = d\vec{r}_0 \cdot \mathbb{F}$$

or using a Cartesian base in the reference space

$$d\vec{r} = \hat{e}_i^0 dx_k^0 \frac{\partial x_i}{\partial x_k^0} = \hat{e}_i^0 dx_k^0 F_{ki} = \hat{e}_i^0 dx_i.$$

$$|d\vec{r}|^2 = d\vec{r} \cdot d\vec{r} = d\vec{r}_0 \cdot \mathbb{F} \cdot \mathbb{F}^T \cdot d\vec{r}_0$$

$$|d\vec{r}|^2 - |d\vec{r}_0|^2 = d\vec{r}_0 \cdot [\mathbb{F} \cdot \mathbb{F}^T - \mathbb{I}] \cdot d\vec{r}_0$$

### 1.5.1 Strain

#### Green-Lagrange tensor

Green-Lagrange strain tensor is defined as

$$\epsilon := \frac{1}{2} [\mathbb{F} \cdot \mathbb{F}^T - \mathbb{I}] \quad (1.5)$$

or in Cartesian coordinates in the reference space

$$\epsilon_{ij} = \frac{1}{2} [F_{ik} F_{jk} - \delta_{ij}] = \frac{1}{2} \left[ \frac{\partial x_i}{\partial x_k^0} \frac{\partial x_j}{\partial x_k^0} - \delta_{ij} \right].$$

Its (material) time derivative reads (**todo** pay attention to vector basis in reference and physical space. Can they be compared/confused?)

$$\begin{aligned} \frac{D\epsilon_{ij}}{Dt} &= \frac{1}{2} \left[ \frac{D}{Dt} \frac{\partial x_k}{\partial x_i^0} \frac{\partial x_k}{\partial x_j^0} + \frac{\partial x_k}{\partial x_i^0} \frac{D}{Dt} \frac{\partial x_k}{\partial x_j^0} \right] = \\ &= \frac{1}{2} \left[ \frac{\partial v_k}{\partial x_i^0} \frac{\partial x_k}{\partial x_j^0} + \frac{\partial x_k}{\partial x_i^0} \frac{\partial v_k}{\partial x_j^0} \right] = \\ &= \frac{1}{2} \left[ \frac{\partial x_l}{\partial x_i^0} \frac{\partial v_k}{\partial x_l} \frac{\partial x_k}{\partial x_j^0} + \frac{\partial x_k}{\partial x_i^0} \frac{\partial x_l}{\partial x_j^0} \frac{\partial v_k}{\partial x_l} \right] \\ &= \frac{\partial x_l}{\partial x_i^0} \left[ \frac{1}{2} \left( \frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} \right) \right] \frac{\partial x_k}{\partial x_j^0} = \\ &= F_{il} D_{lk} F_{jk}. \end{aligned}$$

$$\begin{aligned}
 \frac{D\epsilon}{Dt} &= \frac{1}{2} \left[ \frac{D\mathbb{F}}{Dt} \cdot \mathbb{F}^T + \mathbb{F} \cdot \frac{D\mathbb{F}^T}{Dt} \right] = \\
 &= \frac{1}{2} \left[ \nabla_0 \vec{v} \cdot \mathbb{F}^T + \mathbb{F} \cdot \nabla_0^T \vec{v} \right] = \\
 &= \frac{1}{2} \left[ \mathbb{F} \cdot \nabla \vec{v} \cdot \mathbb{F}^T + \mathbb{F} \cdot \nabla^T \vec{v} \cdot \mathbb{F}^T \right] = \\
 &= \mathbb{F} \cdot \left[ \frac{1}{2} (\nabla \vec{v} + \nabla^T \vec{v}) \right] \cdot \mathbb{F}^T = \\
 &= \mathbb{F} \cdot \mathbb{D} \cdot \mathbb{F}^T .
 \end{aligned} \tag{1.6}$$

## GOVERNING EQUATIONS

The following process is detailed in the following sections

**Integral balance equations for primary physical quantities.** First, integral balance equations for closed systems are written as a manifestation of principles of classical mechanics for closed systems, namely mass conservation, second principle of mechanics, and first principle of thermodynamics. Starting from integral balance equations for closed systems (material systems, Lagrange description), Reynolds transport theorem is used to derive integral balance equations for open systems, either stationary in space (control volume, Eulerian description) or with arbitrary motion (arbitrary description).

**Differential balance equations for primary physical quantities.** Starting from integral balance equations, under the assumption of sufficient regularity of the physical quantities, divergence theorem and arbitrariness of the domain is used to derive differential (local) balance equations of primary physical quantities.

**Differential balance equations for derived physical quantities.** Starting from differential equations of primary physical quantities, differential balance equations are derived for other physical quantities, as an example kinetic energy, internal energy and entropy.

**Integral balance equations for derived physical quantities.** Starting from differential balance equations, and exploiting divergence theorem (in the “opposite direction” w.r.t. what has been done before, to get differential from integral equations), integral balance equations are derived for derived quantities.

### 2.1 Integral Balance Equations of primary physical quantities

Classical physics relies on a small set of principles, usually formulated for closed systems.

- classical physics and chemistry rely on Lavoisier principle, or mass conservation in closed systems
- classical (Newton) mechanics is built on 3 principles:
  - 1<sup>st</sup> principle, or principle of inertia, dealing with the invariance of classical physics w.r.t. Galileian transformations
  - 2<sup>nd</sup> principle, or balance of momentum
  - 3<sup>rd</sup> principle, or action/reaction principle
- classical thermodynamics:
  - 1<sup>st</sup> principle, or balance of total energy
  - 2<sup>nd</sup> principle, describing irreversibility or natural tendencies in physical processes - positive dissipation of mechanical (macroscopic) energy and heat transfer “from hot to cold bodies” - in terms of entropy
  - 3<sup>rd</sup> principle, relating energy, entropy and thermodynamic temperature as positive physical quantity (it sets an absolute zero of the thermodynamic temperature, in the thermodynamic scale of temperature - Kelvin  $K$ )
- classical electromagnetism:

- Electric charge conservation
- Maxwell's equations, relating electromagnetic field with charges and currents
- Lorentz's force, acting on charges in an electromagnetic field

Here, electromagnetic processes are not investigated. Dynamical equations for angular momentum and kinetic energy derived in classical mechanics are discussed later: integral balance equation of angular momentum relates changes of angular momentum of the system with external moments acting on it; differential balance equation of angular momentum reduces to the an identity - and thus it adds no information - for **non-polar media**; kinetic energy integral balance relates changes of kinetic energy of the system with the total mechanical power acting on the system, and it can be subtracted from total energy to get internal energy of the system.

### 2.1.1 Principles of classical mechanics for closed systems - Lagrangian description

**Mass balance equation: Lavoisier principle.**

$$\frac{d}{dt} \int_{V_t} \rho = 0 .$$

**Momentum balance equation: 2<sup>nd</sup> principle of Newton mechanics.**

$$\frac{d}{dt} \int_{V_t} \rho \vec{u} = \int_{V_t} \rho \vec{g} + \oint_{\partial V_t} \vec{t}_{\vec{n}} .$$

**Total energy balance equation: 1<sup>st</sup> principle of thermodynamics.**

$$\frac{d}{dt} \int_{V_t} \rho e^t = \int_{V_t} \rho \vec{g} \cdot \vec{u} + \oint_{\partial V_t} \vec{t}_{\vec{n}} \cdot \vec{u} - \oint_{\partial V_t} \vec{q} \cdot \vec{n} + \int_{V_t} \rho r .$$

### 2.1.2 Integral balance equations for arbitrary domains - arbitrary description

Using [Reynolds transport theorem](#), time derivative over the material volume  $V_t$  can be written in terms of the time derivative over volume  $v_t$  in arbitrary motion and a flux contribution across its boundary.

**Mass balance equation.**

$$\frac{d}{dt} \int_{v_t} \rho + \oint_{\partial v_t} \rho (\vec{u} - \vec{u}_b) \cdot \vec{n} = 0 .$$

**Momentum balance equation.**

$$\frac{d}{dt} \int_{v_t} \rho \vec{u} + \oint_{\partial v_t} \rho \vec{u} (\vec{u} - \vec{u}_b) \cdot \vec{n} = \int_{v_t} \rho \vec{g} + \oint_{\partial v_t} \vec{t}_{\vec{n}} .$$

**Total energy balance equation.**

$$\frac{d}{dt} \int_{v_t} \rho e^t + \oint_{\partial v_t} \rho e^t (\vec{u} - \vec{u}_b) \cdot \vec{n} = \int_{v_t} \rho \vec{g} \cdot \vec{u} + \oint_{\partial v_t} \vec{t}_{\vec{n}} \cdot \vec{u} - \oint_{\partial v_t} \vec{q} \cdot \vec{n} + \int_{v_t} \rho r .$$

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**How to correctly apply Reynolds's transport theorem in continuum mechanics**

Apply Reynold's transport both to material volume  $V_t$  and arbitrary volume  $v_t$

$$\begin{aligned}\frac{d}{dt} \int_{V_t} f &= \int_{V_t} \frac{\partial f}{\partial t} + \oint_{\partial V_t} f \vec{v} \cdot \hat{n} \\ \frac{d}{dt} \int_{v_t} f &= \int_{v_t} \frac{\partial f}{\partial t} + \oint_{\partial v_t} f \vec{v}_b \cdot \hat{n}\end{aligned}$$

and compare these two expressions, after setting  $v_t \equiv V_t$ , i.e. considering the material volume at time  $t$  coinciding with the arbitrary volume at time  $t$  (in general, at any time  $t$  there's a different material volume  $V_t$  coinciding with the arbitrary volume  $v_t$  - i.e. a different set of material particles in the arbitrary volume - but this is not a problem at all in the manipulation),

$$\frac{d}{dt} \int_{V_t \equiv v_t} f = \frac{d}{dt} \int_{v_t} f + \oint_{\partial V_t \equiv \partial v_t} f(\vec{v} - \vec{v}_b) \cdot \hat{n}.$$

### 2.1.3 Integral balance equations for control volumes - Eulerian description

Eulerian description of integral balance equations in continuum mechanics relies on stationary control volume,  $V$ . Integral balance equations are readily derived from *balance equations for arbitrary volumes* setting the velocity of the boundary of the domain equal to zero, i.e.  $\vec{v}_b = \vec{0}$ , and the Eulerian control volume equal to the "instantaneously coinciding material volume",  $V \equiv V_t$ .

**Mass balance equation**

$$\frac{d}{dt} \int_V \rho + \oint_{\partial V} \rho \vec{u} \cdot \vec{\hat{n}} = 0.$$

**Momentum balance equation**

$$\frac{d}{dt} \int_V \rho \vec{u} + \oint_{\partial V} \rho \vec{u} \vec{u} \cdot \vec{\hat{n}} = \int_V \rho \vec{g} + \oint_{\partial V} \vec{t}_{\vec{n}}.$$

**Total energy balance equation.**

$$\frac{d}{dt} \int_V \rho e^t + \oint_{\partial V} \rho e^t \vec{u} \cdot \vec{\hat{n}} = \int_V \rho \vec{g} \cdot \vec{u} + \oint_{\partial V} \vec{t}_{\vec{n}} \cdot \vec{u} - \oint_{\partial V} \vec{q} \cdot \vec{\hat{n}} + \int_V \rho r.$$

## 2.2 Differential Balance Equations of aaa physical quantities

### 2.2.1 Balance equation in physical space

In this section, differential form of balance equations is derived using time  $t$  and physical coordinate  $\vec{r}$  as independent variables of fields representing physical quantities  $f(\vec{r}, t)$ .

**Conservative form - Eulerian description in physical space.**

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) &= 0 \\ \frac{\partial}{\partial t} (\rho \vec{v}) + \nabla \cdot (\rho \vec{v} \otimes \vec{v}) &= \rho \vec{g} + \nabla \cdot \mathbb{T} \\ \frac{\partial}{\partial t} (\rho e^t) + \nabla \cdot (\rho e^t \vec{v}) &= \rho \vec{g} \cdot \vec{v} + \nabla \cdot (\mathbb{T} \cdot \vec{v}) - \nabla \cdot \vec{q} + \rho r\end{aligned}\tag{2.1}$$

**Convective form - Lagrangian description in physical space.** Using vector calculus identities to evaluate partial derivatives of products, mass equation and relation (1.3) to write partial derivative w.r.t. material derivative,

$$\begin{aligned}\frac{D\rho}{Dt} &= -\rho \nabla \cdot \vec{v} \\ \rho \frac{D\vec{v}}{Dt} &= \rho \vec{g} + \nabla \cdot \mathbb{T} \\ \rho \frac{De^t}{Dt} &= \rho \vec{g} \cdot \vec{v} + \nabla \cdot (\mathbb{T} \cdot \vec{v}) - \nabla \cdot \vec{q} + \rho r\end{aligned}$$

## Proof

todo

**Arbitrary description in physical space.** Using relation (1.3) to write material derivatives w.r.t. time derivative at constant  $\vec{r}_b$

$$\begin{aligned}\left. \frac{\partial \rho}{\partial t} \right|_{\vec{r}_b} + (\vec{v} - \vec{v}_b) \cdot \nabla \rho &= -\rho \nabla \cdot \vec{v} \\ \rho \left. \frac{\partial \vec{v}}{\partial t} \right|_{\vec{r}_b} + \rho (\vec{v} - \vec{v}_b) \cdot \nabla \vec{v} &= \rho \vec{g} + \nabla \cdot \mathbb{T} \\ \rho \left. \frac{\partial e^t}{\partial t} \right|_{\vec{r}_b} + \rho (\vec{v} - \vec{v}_b) \cdot \nabla e^t &= \rho \vec{g} \cdot \vec{v} + \nabla \cdot (\mathbb{T} \cdot \vec{v}) - \nabla \cdot \vec{q} + \rho r\end{aligned}$$

## 2.2.2 Balance equations in reference space

In this section, differential form of balance equations is derived using time  $t$  and material coordinate  $\vec{r}_0$  as independent variables of fields representing physical quantities  $f_0(\vec{r}_0, t) = f(\vec{r}(\vec{r}_0, t), t)$ , exploiting the change of variables  $\vec{r}(\vec{r}_0, t)$  and its inverse transformation - assumed to exist (with the same consideration done in the kinematics sections: while it's likely that a global invertible transformation w.r.t. the original reference configuration doesn't exist, limiting the time interval and space domain a "piecewise" invertible transformation w.r.t. intermediate states exists).

## 2.3 Differential Balance Equations of ddd physical quantities

Balance equations of kinetic energy, internal energy and entropy

$$k = \frac{|\vec{v}|^2}{2}, \quad e = e^t - k, \quad s = \dots$$

**Convective form - Lagrangian description in physical space.** Kinetic energy equation is derived multiplying the momentum equation by the velocity field; internal energy equation is derived subtracting kinetic energy equation from the total energy equation; entropy equation strongly depends on the constitutive equation of the material, as it's shown for elastic solids and Newtonian fluids

$$\begin{aligned}\rho \frac{Dk}{Dt} &= \rho \vec{g} \cdot \vec{v} + \vec{v} \cdot \nabla \cdot \mathbb{T} \\ \rho \frac{De}{Dt} &= \mathbb{T} : \nabla \vec{v} - \nabla \cdot \vec{q} + \rho r \\ &\dots\end{aligned}$$



**Conservative form - Eulerian description in physical space.**

$$\begin{aligned}\frac{\partial}{\partial t}(\rho k) + \nabla \cdot (\rho k \vec{v}) &= \rho \vec{g} \cdot \vec{v} + \vec{v} \cdot \nabla \cdot \mathbb{T} \\ \frac{\partial}{\partial t}(\rho e) + \nabla \cdot (\rho e \vec{v}) &= \mathbb{T} : \nabla \vec{v} - \nabla \cdot \vec{q} + \rho r \\ &\dots\end{aligned}$$

**Arbitrary description in physical space.**

$$\begin{aligned}\rho \left. \frac{\partial k}{\partial t} \right|_{\vec{r}_b} + \rho (\vec{v} - \vec{v}_b) \cdot \nabla k &= \rho \vec{g} \cdot \vec{v} + \vec{v} \cdot \nabla \cdot \mathbb{T} \\ \rho \left. \frac{\partial e}{\partial t} \right|_{\vec{r}_b} + \rho (\vec{v} - \vec{v}_b) \cdot \nabla e &= \mathbb{T} : \nabla \vec{v} - \nabla \cdot \vec{q} + \rho r \\ &\dots\end{aligned}$$

## 2.4 Integral Balance Equations of ddd physical quantities

## 2.5 Jump conditions

Jump conditions comes from integral balances for an arbitrary domain. These conditions hold both across discontinuities - where fields are not regular enough for differential equations to hold - and in regular domains.

## 2.6 Integral Balance Equations in reference space

### 2.6.1 Mass

Integral balance for a material volume  $V_t$  reads

$$\begin{aligned}0 &= \frac{d}{dt} \int_{V_t} \rho(\vec{r}, t) dV = \\ &= \frac{d}{dt} \int_{V_0} \rho(\vec{r}(\vec{r}_0, t), t) J(\vec{r}_0, t) dV_0 = \\ &= \frac{d}{dt} \int_{V_0} \rho_0(\vec{r}_0, t) J(\vec{r}_0, t) dV_0 = \\ &= \int_{V_0} \frac{D}{Dt} (\rho_0(\vec{r}_0, t) J(\vec{r}_0, t)) dV_0 .\end{aligned}$$

Since the domain  $V_0$  is arbitrary, with some abuse of notation to indicate the density field as  $\rho$ , hiding the dependence on the independent variables  $\rho_0(\vec{r}_0, t)$ , the differential balance in reference space follows

$$\frac{D}{Dt} (\rho J) = 0$$

or

$$\rho J = \rho^0 ,$$

i.e. the product  $\rho J$  equals the initial density field  $\rho^0$ , assuming that the determinant of the transformation is  $J^0 = 1$ , in the reference configuration.

## 2.6.2 Momentum

Integral balance for a material volume  $V_t$  reads

$$\begin{aligned}\frac{d}{dt} \int_{V_t} \rho \vec{v} dV &= \int_{V_t} \rho \vec{g} + \oint_{\partial V_t} \hat{n} \cdot \mathbb{T} dS \\ \frac{d}{dt} \int_{V_0} \rho J \vec{v} dV_0 &= \int_{V_0} \rho J \vec{g} + \oint_{\partial V_t} \hat{n}_0 \cdot (J \mathbb{F}^{-T} \cdot \mathbb{T}) dS_0 \\ \frac{d}{dt} \int_{V_0} \rho^0 \vec{v} dV_0 &= \int_{V_0} \rho^0 \vec{g} + \oint_{\partial V_0} \hat{n}_0 \cdot \Sigma_n dS_0 \\ \int_{V_0} \rho^0 \frac{D}{Dt} \vec{v} dV_0 &= \int_{V_0} \rho^0 \vec{g} + \oint_{V_0} \nabla_0 \cdot \Sigma_n dS_0\end{aligned}$$

Since the domain  $V_0$  is arbitrary, the differential balance in reference space follows

$$\rho^0 \frac{D\vec{v}}{Dt} = \rho^0 \vec{g} + \nabla_0 \cdot \Sigma_n$$

### Nanson's formula

$$\begin{aligned}d\vec{r} &= d\vec{r}_0 \cdot \frac{\partial \vec{r}}{\partial \vec{r}_0} = d\vec{r}_0 \cdot \nabla_0 \vec{r} = d\vec{r}_0 \cdot \mathbb{F} \\ dV &= J dV_0 \\ d\vec{r} \cdot \hat{n} dS &= J d\vec{r}_0 \cdot \hat{n}_0 dS_0 \\ d\vec{r}_0 \cdot \mathbb{F} \cdot \hat{n} dS &= J d\vec{r}_0 \cdot \hat{n}_0 dS_0\end{aligned}$$

must be true for all  $\vec{r}_0$  arbitrary, so that

$$\mathbb{F} \cdot \hat{n} dS = J \hat{n}_0 dS_0$$

and

$$\begin{aligned}\hat{n} dS &= J \mathbb{F}^{-1} \cdot \hat{n}_0 dS_0 = \\ &= J \hat{n}_0 \cdot \mathbb{F}^{-T} dS_0\end{aligned}$$

### Stress tensors

**Cauchy stress tensor.**

**Piola-Kirchhoff I - transpose of normal stress tensors.**

**Piola-Kirchhoff II**

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**Example 2.6.1 (Relation between description in physical and reference space)**

$$\begin{aligned}\rho^0 \frac{D\vec{v}}{Dt} &= \rho^0 \vec{g} + \nabla_0 \cdot \Sigma_n \\ J \rho \frac{D\vec{v}}{Dt} &= J \rho \vec{g} + \nabla_0 \cdot \Sigma_n \\ \rho \frac{D\vec{v}}{Dt} &= \rho \vec{g} + \frac{1}{J} \nabla_0 \cdot \Sigma_n\end{aligned}$$

thus,

$$\begin{aligned}\nabla \cdot \mathbb{T} &= \frac{1}{J} \nabla_0 \cdot \Sigma_n = \\ &= \frac{1}{J} \nabla_0 \cdot (J \mathbb{F}^{-T} \cdot \mathbb{T})\end{aligned}$$

**todo** Prove it with derivation!

### 2.6.3 Kinetic energy

$$\begin{aligned}0 &= \vec{v} \cdot \left\{ \rho^0 \frac{D\vec{v}}{Dt} - \rho^0 \vec{g} - \nabla_0 \cdot \Sigma_n \right\} = \\ &= \rho^0 \frac{D}{Dt} \frac{|\vec{v}|^2}{2} - \rho^0 \vec{v} \cdot \vec{g} - \vec{v} \cdot \nabla_0 \cdot \Sigma_n = \\ &= \rho^0 \frac{D}{Dt} \frac{|\vec{v}|^2}{2} - \rho^0 \vec{v} \cdot \vec{g} - \nabla_0 \cdot (\vec{v} \cdot \Sigma_n) + \nabla_0 \vec{v} : \Sigma_n \\ &\quad v_i \partial_k^0 \Sigma_{ki} = \partial_k^0 (v_i \Sigma_{ki}) - \partial_k^0 v_i \Sigma_{ki} \\ &\quad dV = J dV_0 \\ &\quad dr_i n_i dS = J dr_k^0 n_k^0 dS_0 \\ &\quad dr_k^0 \frac{\partial r_i}{\partial r_k^0} n_i dS = dr_k^0 J n_k^0 dS_0 \\ &\quad \frac{\partial r_i}{\partial r_k^0} n_i dS = J n_k^0 dS_0 \\ &\quad \underbrace{\frac{\partial r_k^0}{\partial r_j} \frac{\partial r_i}{\partial r_k^0} n_i dS}_{=\delta_{ij}} = J \frac{\partial r_k^0}{\partial r_j} n_k^0 dS_0 \\ &\quad n_j dS = J \frac{\partial r_k^0}{\partial r_j} n_k^0 dS_0 \\ &\quad \mathbb{F} = \hat{e}_k^0 \hat{e}_i^0 \frac{\partial r_i}{\partial r_k^0} \\ &\quad \mathbb{F}^{-1} = \hat{e}_j^0 \hat{e}_k^0 \frac{\partial r_k^0}{\partial r_j} \\ &\quad \mathbb{F}^{-1} \cdot \mathbb{F} = \left( \hat{e}_j^0 \hat{e}_k^0 \frac{\partial r_k^0}{\partial r_j} \right) \cdot \left( \hat{e}_l^0 \hat{e}_i^0 \frac{\partial r_i}{\partial r_l^0} \right) = \hat{e}_j^0 \hat{e}_i^0 \frac{\partial r_i}{\partial r_j} = \hat{e}_j^0 \hat{e}_i^0 \delta_{ij} \\ &\quad \mathbb{F} \cdot \mathbb{F}^{-1} = \left( \hat{e}_l^0 \hat{e}_i^0 \frac{\partial r_i}{\partial r_l^0} \right) \cdot \left( \hat{e}_j^0 \hat{e}_k^0 \frac{\partial r_k^0}{\partial r_j} \right) = \hat{e}_l^0 \hat{e}_k^0 \frac{\partial r_k^0}{\partial r_l} = \hat{e}_l^0 \hat{e}_k^0 \delta_{lk} \\ &\quad \frac{\partial r_k^0}{\partial r_i} \frac{\partial r_i}{\partial r_l^0} = \delta_{kl} \\ &\quad \Sigma := \Sigma_n \cdot \mathbb{F}^{-1} = J \mathbb{F}^{-T} \cdot \mathbb{T} \cdot \mathbb{F}^{-1} \\ &\quad \Sigma_n = \Sigma \cdot \mathbb{F} \\ &\quad \Sigma_{ik} = \Sigma_{n,ij} (\mathbb{F}^{-1})_{jk} = \Sigma_{n,ij} \frac{\partial r_k^0}{\partial r_j}\end{aligned}$$

$$\begin{aligned}
 \Sigma_{n,ij} &= \Sigma_{ik} \frac{\partial x_j}{\partial x_k^0} \\
 \nabla_0 \vec{v} : \Sigma_n &= \frac{D}{Dt} \mathbb{F} : \Sigma_n = \\
 &= \frac{\partial v_j}{\partial x_i^0} \Sigma_{n,ij} = \\
 &= \frac{\partial v_j}{\partial x_i^0} \Sigma_{ik} \frac{\partial x_j}{\partial x_k^0} = \\
 &= \Sigma_{ik} \frac{1}{2} \left( \frac{\partial v_j}{\partial x_i^0} \frac{\partial x_j}{\partial x_k^0} + \frac{\partial v_j}{\partial x_k^0} \frac{\partial x_j}{\partial x_i^0} \right)
 \end{aligned}$$

if  $\Sigma$  is symmetric,  $\Sigma_{ik} = \Sigma_{ki}$ , or with tensor notation

$$\begin{aligned}
 \nabla_0 \vec{v} : \Sigma_n &= \frac{D}{Dt} \mathbb{F} : \Sigma_n = \\
 &= \nabla_0 \vec{v} : (\Sigma \cdot \mathbb{F}) = \\
 &= \Sigma : \frac{1}{2} \left( \frac{D\mathbb{F}}{Dt} \cdot \mathbb{F}^T + \mathbb{F} \cdot \frac{D\mathbb{F}^T}{Dt} \right) = \\
 &= \Sigma : \frac{D}{Dt} \mathbb{E} ,
 \end{aligned}$$

having recognized the time derivative (1.6) of the *Green-Lagrange tensor* (1.5).

Integral of the volume stress in the reference space can be recast as the volume in the physical space

$$\begin{aligned}
 \int_{V_0} \nabla_0 \vec{v} : \Sigma_n dV_0 &= \int_{V_0} \Sigma : \frac{D\mathbb{E}}{Dt} dV_0 \\
 \Sigma_{n,ik} &= J \frac{\partial x_i^0}{\partial x_k} T_{jk} \\
 \int_{V_0} \nabla_0 \vec{v} : \Sigma_n dV_0 &= \int_{V_0} \frac{\partial v_k}{\partial x_i^0} \Sigma_{n,ik} dV_0 = \\
 &= \int_{V_0} \underbrace{\frac{\partial v_k}{\partial x_i^0} \left( \frac{\partial x_i^0}{\partial x_j} T_{kj} J \right)}_{dV} dV_0 = \\
 &= \int_V \frac{\partial v_k}{\partial x^j} T_{kj} dV = \\
 &= \int_V \frac{1}{2} \left( \frac{\partial v_k}{\partial x^j} + \frac{\partial v_j}{\partial x^k} \right) T_{kj} dV = \\
 &= \int_V D_{jk} T_{kj} dV = \\
 &= \int_V \mathbb{D} : \mathbb{T} dV .
 \end{aligned}$$

## Variational principles

Using an arbitrary test function  $\vec{w}(\vec{r}_0)$ ,

$$0 = \vec{w} \cdot \left\{ \rho^0 \frac{D\vec{v}}{Dt} - \rho^0 \vec{g} - \nabla_0 \cdot \Sigma_n \right\}$$

and using rule of product

$$w_i \frac{\partial \Sigma_{n,ji}}{\partial x_j^0} = \frac{\partial}{\partial x_j^0} (w_i \Sigma_{n,ji}) - \frac{\partial w_i}{\partial x_j^0} \Sigma_{n,ji} =$$

and the second term can be transformed using the relation between normal stress and second Piola-Kirchhoff tensor

$$\frac{\partial w_i}{\partial x_j^0} \Sigma_{n,ji} = \frac{\partial w_i}{\partial x_j^0} \Sigma_{jk} \frac{\partial x_i}{\partial x_k^0} = \Sigma_{jk} \frac{1}{2} \left[ \frac{\partial x_i}{\partial x_k^0} \frac{\partial w_i}{\partial x_j^0} + \frac{\partial x_i}{\partial x_j^0} \frac{\partial w_i}{\partial x_k^0} \right] = \Sigma_{jk} W_{ij}(\vec{w}),$$

having defined the tensor

$$\mathbb{W}(\vec{w}) := \frac{1}{2} [\nabla_0 \vec{w} \cdot \mathbb{F}^T + \mathbb{F} \cdot \nabla_0^T \vec{w}],$$

with the evident analogy with the time derivative of Green-Lagrange strain tensor, namely

$$\epsilon = \mathbb{W}(\vec{v}),$$

being  $\vec{v}$  the velocity field. Integrating on the domain  $V_0$  and using divergence theorem, the problem is written in its weak form

$$\int_{V_0} \left\{ \rho^0 \vec{w} \cdot \frac{D\vec{v}}{Dt} + \mathbb{W}(\vec{w}) : \Sigma \right\} = \int_{V_0} \rho^0 \vec{w} \cdot \vec{g} + \oint_{\partial V_0} \hat{n}_0 \cdot \Sigma_n \cdot \vec{w},$$

with the proper boundary conditions and the corresponding conditions on the test function  $\vec{w}$ . As an example, if the boundary is composed of two different regions,  $S_{D,0} \cup S_{N,0} = \partial V_0$ ,  $S_D \cap S_N = \emptyset$  where either position (called  $S_D$  from Dirichlet boundary) and stress (called  $S_N$  from Neumann boundary) are prescribed

$$\begin{aligned} \vec{r} &= \vec{r}_b, & \vec{w} &= \vec{0} & (\text{on } S_{D,0} \text{ Dirichlet - essential - boundary}) \\ \hat{n}_0 \cdot \Sigma_n &= \vec{t}_{0,\hat{n}_0}, & & & (\text{on } S_{N,0} \text{ Neumann - natural - boundary}) \end{aligned}$$

the weak form of the equation reads

$$\int_{V_0} \left\{ \rho^0 \vec{w} \cdot \frac{D\vec{v}}{Dt} + \mathbb{W}(\vec{w}) : \Sigma \right\} = \int_{V_0} \rho^0 \vec{w} \cdot \vec{g} + \int_{S_{n,0}} \hat{n}_0 \cdot \vec{t}_{\hat{n}_0}$$

### 2.6.4 Total energy

Using Nanson's relation  $\hat{n} dS = \hat{n}_0 \cdot (J\mathbb{F}^{-T}) dS_0$ ,

$$\begin{aligned} \frac{d}{dt} \int_{V_t} \rho e^t dV &= \int_{V_t} \rho \vec{g} \cdot \vec{v} dV + \oint_{\partial V_t} \vec{t}_{\hat{n}} \cdot \vec{v} dS - \oint_{\partial V_t} \hat{n} \cdot \vec{q} + \int_{V_t} \rho r dV = \\ &= \int_{V_t} \rho \vec{g} \cdot \vec{v} dV + \oint_{\partial V_t} \hat{n} \cdot \mathbb{T} \cdot \vec{v} dS - \oint_{\partial V_t} \hat{n} \cdot \vec{q} dS + \int_{V_t} \rho r dV \\ &= \int_{V_0} \rho^0 \vec{g} \cdot \vec{v} dV_0 + \oint_{\partial V_0} \hat{n}_0 \cdot (J\mathbb{F}^{-T} \cdot \mathbb{T}) \cdot \vec{v} dS_0 - \oint_{\partial V_0} \hat{n}_0 \cdot (J\mathbb{F}^{-T} \cdot \vec{q}) dS_0 + \int_{V_0} \rho^0 r dV_0. \end{aligned}$$

and the differential form reads

$$\rho^0 \frac{De^t}{Dt} = \rho^0 \vec{g} \cdot \vec{v} + \nabla_0 \cdot (J\mathbb{F}^{-T} \cdot \mathbb{T} \cdot \vec{v}) - \nabla_0 \cdot (J\mathbb{F}^{-T} \cdot \vec{q}) + \rho^0 r.$$

or

$$\rho^0 \frac{De^t}{Dt} = \rho^0 \vec{g} \cdot \vec{v} + \nabla_0 \cdot (\Sigma_n \cdot \vec{v}) - \nabla_0 \cdot \vec{q}_0 + \rho^0 r.$$

and dividing by  $J$  and using the relation (**see below**)  $\nabla_0 \cdot (J\mathbb{F}^{-T} \cdot \vec{a}) = J \nabla \cdot \vec{a}$ ,

$$\rho \frac{De^t}{Dt} = \rho \vec{g} \cdot \vec{v} + \nabla \cdot (\mathbb{T} \cdot \vec{v}) - \nabla \cdot \vec{q} + \rho r.$$

Comparison with equation in physical space (dividing by  $J$ ) suggests the identity

$$\frac{1}{J} \nabla_0 \cdot (J\mathbb{F}^{-T} \cdot \vec{a}) = \nabla \cdot \vec{a},$$

and thus

$$\nabla_0 \cdot (J\mathbb{F}^{-T}) = \vec{0},$$

since

$$\nabla_0 \cdot (J\mathbb{F}^{-T} \cdot \vec{a}) = \nabla_0 \cdot (J\mathbb{F}^{-T}) \cdot \vec{a} + J\mathbb{F}^{-T} : \nabla_0 \vec{a} = \nabla_0 \cdot (J\mathbb{F}^{-T}) \cdot \vec{a} + J \nabla \cdot \vec{a}$$

**Proof.**

$$\begin{aligned} J &= \left| \frac{\partial r_k}{\partial r_i^0} \right| = \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r_1^0} \frac{\partial r_{i_2}}{\partial r_2^0} \frac{\partial r_{i_3}}{\partial r_3^0} \\ (\mathbb{F}^{-T})_{ij} &= \frac{\partial r_i^0}{\partial r_j} \\ \{\nabla_0 \cdot (J\mathbb{F}^{-T})\}_j &= \frac{\partial}{\partial r_i^0} \left( \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r_1^0} \frac{\partial r_{i_2}}{\partial r_2^0} \frac{\partial r_{i_3}}{\partial r_3^0} \frac{\partial r_i^0}{\partial r_j} \right) = \\ &= \varepsilon_{i_1, i_2, i_3} \frac{\partial}{\partial r_i^0} \left( \frac{\partial r_{i_1}}{\partial r_1^0} \right) \frac{\partial r_{i_2}}{\partial r_2^0} \frac{\partial r_{i_3}}{\partial r_3^0} \frac{\partial r_i^0}{\partial r_j} + \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r_1^0} \frac{\partial}{\partial r_i^0} \left( \frac{\partial r_{i_2}}{\partial r_2^0} \right) \frac{\partial r_{i_3}}{\partial r_3^0} \frac{\partial r_i^0}{\partial r_j} + \\ &+ \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r_1^0} \frac{\partial r_{i_2}}{\partial r_2^0} \frac{\partial}{\partial r_i^0} \left( \frac{\partial r_{i_3}}{\partial r_3^0} \right) \frac{\partial r_i^0}{\partial r_j} + \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r_1^0} \frac{\partial r_{i_2}}{\partial r_2^0} \frac{\partial r_{i_3}}{\partial r_3^0} \frac{\partial}{\partial r_i^0} \left( \frac{\partial r_i^0}{\partial r_j} \right) = \\ &= \varepsilon_{i_1, i_2, i_3} \underbrace{\frac{\partial r_i^0}{\partial r_j} \frac{\partial}{\partial r_i^0} \left( \frac{\partial r_{i_1}}{\partial r_1^0} \right)}_{=\frac{\partial}{\partial r_j}} \frac{\partial r_{i_2}}{\partial r_2^0} \frac{\partial r_{i_3}}{\partial r_3^0} + \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r_1^0} \underbrace{\frac{\partial r_i^0}{\partial r_j} \frac{\partial}{\partial r_i^0} \left( \frac{\partial r_{i_2}}{\partial r_2^0} \right)}_{=\frac{\partial}{\partial r_j}} \frac{\partial r_{i_3}}{\partial r_3^0} + \\ &+ \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r_1^0} \frac{\partial r_{i_2}}{\partial r_2^0} \underbrace{\frac{\partial r_i^0}{\partial r_j} \frac{\partial}{\partial r_i^0} \left( \frac{\partial r_{i_3}}{\partial r_3^0} \right)}_{=\frac{\partial}{\partial r_j}} + \varepsilon_{i_1, i_2, i_3} \frac{\partial r_{i_1}}{\partial r_1^0} \frac{\partial r_{i_2}}{\partial r_2^0} \frac{\partial r_{i_3}}{\partial r_3^0} \underbrace{\frac{\partial}{\partial r_i^0} \left( \frac{\partial r_i^0}{\partial r_j} \right)}_{=3} = \\ &= 0, \end{aligned}$$

since

$$\frac{\partial}{\partial r_j} \left( \frac{\partial r_{i_k}}{\partial r_k^0} \right) = \frac{\partial}{\partial r_k^0} \left( \frac{\partial r_{i_k}}{\partial r_j} \right) = \frac{\partial}{\partial r_k^0} \delta_{i_k j} = 0.$$

Thus

$$\frac{1}{J} \frac{\partial}{\partial r_i^0} \left( J \frac{\partial r_i^0}{\partial r_j} a_j \right) = \frac{1}{J} \frac{\partial}{\partial r_i^0} \left( J \frac{\partial r_i^0}{\partial r_j} \right) a_j + \frac{1}{J} J \frac{\partial r_i^0}{\partial r_j} \frac{\partial a_j}{\partial r_i^0} = \frac{\partial a_j}{\partial r_j} = \nabla \cdot \vec{a}.$$

### 2.6.5 Internal energy

$$\begin{aligned}\frac{d}{dt} \int_{V_t} \rho e \, dV &= \int_{V_t} \mathbb{T} : \nabla \vec{v} \, dV - \oint_{\partial V_t} \hat{n} \cdot \vec{q} \, dS + \int_{V_t} \rho r \, dV = \\ \frac{d}{dt} \int_{V_0} \rho^0 e \, dV_0 &= \int_{V_0} J \mathbb{T} : \nabla \vec{v} \, dV_0 - \oint_{\partial V_0} \hat{n}_0 \cdot (J \mathbb{F}^{-T} \cdot \vec{q}) \, dS_0 + \int_{V_0} \rho^0 r \, dV_0 =\end{aligned}$$

and the differential form reads

$$\rho^0 \frac{De}{Dt} = \Sigma_n : \nabla_0 \vec{v} - \nabla_0 \cdot \vec{q}^0 + \rho^0 r .$$

**todo** pay attention at the definition - choose one and keep using it! - of the product  $\mathbb{A} : \mathbb{B}$ , in components

$$\mathbb{A} : \mathbb{B} = A_{ij} B_{ij} \quad \text{or} \quad = A_{ij} B_{ji}$$





**EQUAZIONI DI STATO ED EQUAZIONI COSTITUTIVE**



## EQUAZIONI DI BILANCIO DI ALTRE GRANDEZZE FISICHE

Partendo dai bilanci di massa, quantità di moto e di energia totale, si possono ricare le equazioni di bilancio di altre grandezze fisiche come l'*energia cinetica*, l'*energia interna*, l'*entropia*.

### 4.1 Bilanci in forma differenziale, convettiva

**Energia cinetica.** L'energia cinetica (macroscopica) per unità di massa è  $k = \frac{1}{2}|\mathbf{u}|^2 = \frac{1}{2}\mathbf{u} \cdot \mathbf{u}$ . L'equazione di bilancio dell'energia cinetica viene derivata moltiplicando scalarmente l'equazione della quantità di moto

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} + \nabla \cdot \mathbb{T} ,$$

per il campo di velocità  $\mathbf{u}$ ,

$$\rho \frac{Dk}{Dt} = \rho \mathbf{g} \cdot \mathbf{u} + \nabla \cdot \mathbb{T} \cdot \mathbf{u} ,$$

avendo usato  $\mathbf{u} \cdot d\mathbf{u} = d\left(\frac{\mathbf{u} \cdot \mathbf{u}}{2}\right) = dk$ .

**Energia interna.** L'energia interna per unità di massa è la differenza tra l'energia totale e l'energia cinetica,  $e = e^{tot} - k$ . L'equazione di bilancio dell'energia interna viene ottenuta come differenza dell'equazione dell'energia totale

$$\rho \frac{De^{tot}}{Dt} = \rho \mathbf{g} \cdot \mathbf{u} + \nabla \cdot (\mathbb{T} \cdot \mathbf{u}) - \nabla \cdot \mathbf{q} + \rho r ,$$

e quella dell'energia cinetica, per ottenere

$$\rho \frac{De}{Dt} = \mathbb{T} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q} + \rho r .$$

**Entropia.**

- **Entropia nei fluidi.** Se l'entropia può essere scritta come funzione dell'energia interna e della densità, e il primo principio della termodinamica viene scritto come

$$de = \frac{P}{\rho^2} d\rho + T ds ,$$

e il tensore degli sforzi può essere rappresentato come somma degli sforzi di pressione e degli sforzi viscosi **tutto** riferimento alle leggi costitutive,

$$\mathbb{T} = -P\mathbb{I} + \mathbb{S} = -P\mathbb{I} + 2\mu\mathbb{D} + \lambda(\nabla \cdot \mathbf{u})\mathbb{I} ,$$

si può ricavare l'equazione di governo dell'entropia usando il differenziale  $ds = \frac{1}{T} de - \frac{P}{T\rho^2} d\rho$

$$\begin{aligned}
 \rho \frac{Ds}{Dt} &= \frac{\rho}{T} \left( \frac{De}{Dt} - \frac{P}{\rho^2} \frac{D\rho}{Dt} \right) = \\
 &= \frac{1}{T} \left( \rho \frac{De}{Dt} - \frac{P}{\rho} \frac{D\rho}{Dt} \right) = \\
 &= \frac{1}{T} \left( \mathbb{T} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q} + \rho r - \frac{P}{\rho} (\rho \nabla \cdot \mathbf{u}) \right) = \\
 &= \frac{1}{T} (-P \nabla \cdot \mathbf{u} + \mathbb{S} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q} + \rho r + P \nabla \cdot \mathbf{u}) = \\
 &= \frac{1}{T} (\mathbb{S} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q} + \rho r) = \\
 &= \frac{1}{T} (2\mu |\mathbb{D}|^2 + \lambda |(\nabla \cdot \mathbf{u})|^2 - \nabla \cdot \mathbf{q} + \rho r) = \\
 &= \frac{2\mu |\mathbb{D}|^2 + \lambda |(\nabla \cdot \mathbf{u})|^2}{T} - \frac{\mathbf{q} \cdot \nabla T}{T^2} - \nabla \cdot \left( \frac{\mathbf{q}}{T} \right) + \frac{\rho r}{T},
 \end{aligned}$$

avendo usato la degola del prodotto  $\nabla \cdot \left( \frac{\mathbf{q}}{T} \right) = \frac{\nabla \cdot \mathbf{q}}{T} - \frac{\mathbf{q} \cdot \nabla T}{T^2}$ .

Gli ultimi due termini sono legati alla **sorgenti di entropia** nel sistema, dovute alla sorgente di calore nel sistema e al flusso di calore tramite la frontiera del sistema.

I primi due termini possono essere ricondotti alla **dissipazione viscosa** e dovuta alla **conduzione termica** all'interno del volume: entrambi devono essere non-negativi per il secondo principio della termodinamica **todo**. Il primo termine è positivo se i coefficienti di viscosità del modello di fluido newtoniano sono non-negativi

$$\mu, \lambda \geq 0$$

. Il secondo termine impone che il flusso di calore avvenga in direzione opposta al gradiente di temperatura locale, e quindi la proiezione su di esso sia negativa (traducendo il concetto che il calore trasferisce energia da un corpo caldo a uno freddo),

$$-\mathbf{q} \cdot \nabla T \geq 0,$$

come è facile da verificare per il modello di Fourier per la conduzione in mezzi isotropi,  $\mathbf{q} = -k \nabla T$ ,  $-\mathbf{q} \cdot \nabla T = k |\nabla T|^2 \geq 0$  se

$$k \geq 0.$$

Nel caso di modello lineare per la conduzione in mezzi non isotrpi, il flusso di conduzione può essere descritto usando un tensore del secondo ordine  $\mathbb{K}$ ,  $\mathbf{q} = -\mathbb{K} \cdot \nabla T$  (**todo** simmetria?) e la condizione diventa

$$0 \leq -\nabla T \cdot \mathbf{q} = \nabla T \cdot \mathbb{K} \cdot \nabla T,$$

che impone che il tensore di conduzione sia (semi-)definito positivo, a causa dell'arbitrarietà del vettore  $\nabla T$ .

Se questi due termini sono non-negativi, il bilancio di entropia può essere riscritto come la disuguaglianza

$$\begin{aligned}
 \rho \frac{Ds}{Dt} &= \underbrace{\frac{2\mu |\mathbb{D}|^2 + \lambda |(\nabla \cdot \mathbf{u})|^2}{T}}_{\geq 0} - \frac{\mathbf{q} \cdot \nabla T}{T^2} - \nabla \cdot \left( \frac{\mathbf{q}}{T} \right) + \frac{\rho r}{T} = \\
 &\geq -\nabla \cdot \left( \frac{\mathbf{q}}{T} \right) + \frac{\rho r}{T},
 \end{aligned}$$

o nella forma integrale per un volume materiale

$$\frac{d}{dt} \int_{V_t} \rho s \geq - \oint_{\partial V_t} \hat{\mathbf{n}} \cdot \frac{\mathbf{q}}{T} + \int_{V_t} \rho \frac{r}{T},$$

che richiama alla mente la disuguaglianza di Clausius **todo** *aggiungere riferimento*

$$dS \geq \frac{\delta Q^e}{T} .$$

La differenza di segno deriva dalla definizione di  $dQ^e$  come flusso di calore dall'ambiente verso il sistema e del vettore flusso di calore  $\mathbf{q}$  come flusso di calore "uscente dal sistema" **todo**



# **Part II**

## **Solid Mechanics**





## INTRODUCTION TO SOLID MECHANICS



## SMALL DISPLACEMENT - STATICS

- Labile - Undetermined.
- Isostatic - “determined”.
- Hyperstatic - “overdetermined”.

### 6.1 Isotropic elastic medium

#### 6.1.1 Constitutive equation

An isotropic elastic medium has no preferred orientation. The most general relation between stress tensor  $\sigma$  and strain tensor  $\varepsilon$ , and temperature difference  $\Delta T$  w.r.t. a reference temperature,  $\Delta T := T - T_0$ ,

$$\sigma = \mathbf{D} : \varepsilon - \beta \Delta T \mathbf{I} ,$$

for isotropic media involves the rank-2 and rank-4 isotropic tensors, see rank-2-iso, and rank-4-iso. Since stress tensor  $\sigma$  and strain tensor  $\varepsilon$  are symmetric the constitutive law for isotropic elastic media

---

#### Isotropic tensor

An isotropic tensor is a tensor whose components do not change after a rotation of the vector basis. **todo** Examples,...

---



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#### Rank-2 isotropic tensor

The most general expression of a rank-2 isotropic tensor is proportional to the rank-2 identity tensor, and can be written in a Cartesian basis using the Kronecker delta,

$$a \mathbf{I} = a \delta_{ij} \hat{e}_i \otimes \hat{e}_j .$$

**todo** Proof

---



---

#### Rank-4 isotropic tensor

The most general expression of a rank-4 isotropic tensor can be written using a Cartesian basis as

$$\mathbf{D} = D_{ijkl} \hat{e}_i \otimes \hat{e}_j \otimes \hat{e}_k \otimes \hat{e}_l ,$$

where

$$D_{ijkl} = a \delta_{ij} \delta_{kl} + b \delta_{ik} \delta_{jl} + c \delta_{il} \delta_{jk} ,$$

i.e. depends on three possible combinations of rank-2 identity tensor, with 3 scalar parameters,  $a$ ,  $b$ ,  $c$ . In isotropic relation between symmetric tensors,  $A_{ij} = A_{ji}$ ,  $B_{kl} = B_{lk}$  only two parameters are enough since

$$\begin{aligned}
 A_{ij} &= D_{ijkl} B_{kl} = \\
 &= (a\delta_{ij}\delta_{kl} + b\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk}) B_{kl} = \\
 &= a\delta_{ij}B_{ll} + bB_{ij} + cB_{ji} = \\
 &= a\delta_{ij}B_{ll} + (b+c)B_{ij},
 \end{aligned} \tag{1}$$

having used (1) the symmetry of tensor  $\mathbf{B}$ ,  $B_{ji} = B_{ij}$ .

#### References.

- [Fluid Mechanics](#), R. Fitzpatrick, University of Texas, Austin
  - P.G. Hodge, *On Isotropic Cartesian Tensors*, 1961, The American Mathematical Monthly
- 

## 6.2 Beam models

## 6.3 Beam structures

**Part III**

**Fluid Mechanics**



## INTRODUCTION TO FLUID MECHANICS

- Statics and definition of fluids, as medium that has no shear stress at rest.
- Kinematics
- Dynamics
- Models:
  - Incompressible flows
    - \* Governing equations, theorems and regimes of motion
      - Inviscid
      - Irrotational
  - Compressible flows
    - \* Inviscid
    - \* ...





## STATICS

The behavior of continuous medium in static conditions can be used to define a fluid.

---

### Definition 8.1 (Fluid)

A fluid can be defined as a continuous medium with no shear stress in static conditions. Thus, the stress tensor of an *isotropic fluid* under static conditions reads

$$\mathbb{T}^s = -p\mathbb{I} ,$$

where  $p$  is *pressure*. (**todo** mechanical? Thermodynamical?)

---



## CONSTITUTIVE EQUATIONS OF FLUID MECHANICS

### 9.1 Newtonian Fluids

A Newtonian fluid is the model of a fluid as a continuous medium whose stress tensor can be written as the sum of the hydrostatic pressure stress tensor  $-p\mathbb{I}$  - the only contribution holding in *statics* - and a viscous stress tensor  $\mathbb{S}$

$$\mathbb{T} = -p\mathbb{I} + \mathbb{S} ,$$

and the viscous stress tensor is isotropic and **linear** in the first-order spatial derivatives of the velocity field,

$$\mathbb{S} = 2\mu\mathbb{D} + \lambda(\nabla \cdot \vec{u})\mathbb{I} , \quad (9.1)$$

being  $\mu, \lambda$  the viscosity coefficients, and  $\mathbb{D}$  the strain velocity tensor (1.4). Thus, the definition

---

**Definition 9.1.1 (Newtonian fluid)**

A Newtonian fluid is a continuous medium whose stress tensor reads

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D} + \lambda(\nabla \cdot \vec{u})\mathbb{I} . \quad (9.2)$$

---

---

**Note:** The expression (9.1) of the viscosity stress tensor is the most general expression of a 2-nd order symmetric isotropic tensor proportional to 1-st order derivatives of a vector field.

---



## GOVERNING EQUATIONS OF FLUID MECHANICS

### 10.1 Newtonian Fluid

The differential conservative form of the governing equations of a *Newtonian fluid* directly follows from the expression (2.1) of *governing equations of a continuum medium in differential form*,

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \\ \frac{\partial}{\partial t} (\rho \vec{v}) + \nabla \cdot (\rho \vec{v} \otimes \vec{v}) = \rho \vec{g} + \nabla \cdot \mathbb{T} \\ \frac{\partial}{\partial t} (\rho e^t) + \nabla \cdot (\rho e^t \vec{v}) = \rho \vec{g} \cdot \vec{v} + \nabla \cdot (\mathbb{T} \cdot \vec{v}) - \nabla \cdot \vec{q} + \rho r \end{cases}$$

using the expression (9.2) of the stress tensor of a Newtonian fluid,

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D} + \lambda(\nabla \cdot \vec{v})\mathbb{I},$$

a constitutive equation for conduction heat flux  $\vec{q}$ , as an example **Fourier's law**

$$\vec{q} = -k\nabla T,$$

and the required state equations characterizing the behavior of the medium linking thermodynamic variables (assumption of **local thermodynamic equilibrium** *todo discuss this principle*), and required to get a well-defined mathematical problem, with the same number of equations and unknowns.

---

#### Example 10.1.1 (Equations of state)

As an example, the required equations of states need to provide an expression of thermodynamic quantities as a function of the dynamical physical quantities. This could be quite a common choice in numerical methods using conservative form of the equations. Namely, defining momentum and total energy per unit volume

$$\vec{m} := \rho \vec{v} \quad , \quad E^t := \rho e^t,$$

equations of state should provide the expression of pressure  $p$ , temperature  $T$ , viscosity coefficients  $\mu$ ,  $\nu$  and thermal conductivity  $k$  as functions of “dynamic quantities”  $\rho$ ,  $\vec{m}$ ,  $E^t$ ,

$$\begin{aligned} p(\rho, \vec{m}, E^t) \\ T(\rho, \vec{m}, E^t) \\ \mu(\rho, \vec{m}, E^t) \\ \lambda(\rho, \vec{m}, E^t) \\ k(\rho, \vec{m}, E^t) \end{aligned}$$

Usually, in thermodynamics pressure and temperature can be written as functions of other two thermodynamic variables, as an example density  $\rho$  and internal energy (per unit mass)

$$e = e^t - \frac{|\vec{v}|^2}{2} = \frac{E^t}{\rho} - \frac{1}{2} \frac{|\vec{m}|^2}{\rho^2}$$

so that - avoiding here notation abuses and using two different symbols for functions with different independent variables representing the same physical quantity -,

$$\begin{aligned}\Pi(\rho, e) &= \Pi\left(\rho, \frac{E^t}{\rho} - \frac{1}{2} \frac{|\vec{m}|^2}{\rho^2}\right) = p(\rho, \vec{m}, E^t) \\ \Theta(\rho, e) &= \Theta\left(\rho, \frac{E^t}{\rho} - \frac{1}{2} \frac{|\vec{m}|^2}{\rho^2}\right) = T(\rho, \vec{m}, E^t)\end{aligned}$$


---

## 10.2 Derived quantities

Balance equations of kinetic energy and internal energy readily follows from balance equations of continuum media in convective form.

**Kinetic energy.**

**Internal energy.**

$$\begin{aligned}\rho \frac{De}{Dt} &= \mathbb{T} : \nabla \vec{v} - \nabla \cdot \vec{q} + \rho r = \\ &= (-p\mathbb{I} + 2\mu\mathbb{D} + \lambda(\nabla \cdot \vec{v})\mathbb{I}) : \nabla \vec{v} - \nabla \cdot \vec{q} + \rho r = \\ &= -p\nabla \cdot \vec{v} + 2\mu\mathbb{D} : \mathbb{D} + \lambda(\nabla \cdot \vec{v})^2 - \nabla \cdot \vec{q} + \rho r\end{aligned}$$

### Details

**todo**

**Entropy equation.** The first principle of thermodynamics for non-reactive fluid, with no electric charge and other processes, provides the expression of the differential of entropy as a function of internal energy and density,  $s(e, \rho)$

$$de = T ds + \frac{P}{\rho^2} d\rho \quad , \quad ds = \frac{1}{T} de - \frac{P}{T\rho^2} d\rho \quad ,$$

and thus the balance equation for entropy directly follows from the evaluation of the material derivative of entropy field, exploiting balance equations of mass and internal energy

$$\begin{aligned}\rho \frac{Ds}{Dt} &= \frac{1}{T} \left[ \rho \frac{De}{Dt} - \frac{P}{\rho} \frac{D\rho}{Dt} \right] = \\ &= \frac{1}{T} \left[ -p\nabla \cdot \vec{v} + 2\mu\mathbb{D} : \mathbb{D} + \lambda(\nabla \cdot \vec{v})^2 - \nabla \cdot \vec{q} + \rho r - \frac{P}{\rho} (-\rho\nabla \cdot \vec{v}) \right] = \\ &= \frac{1}{T} [2\mu\mathbb{D} : \mathbb{D} + \lambda(\nabla \cdot \vec{v})^2 - \nabla \cdot \vec{q} + \rho r] = \\ &= \frac{2\mu\mathbb{D} : \mathbb{D} + \lambda(\nabla \cdot \vec{v})^2}{T} - \frac{\vec{q} \cdot \nabla T}{T^2} - \nabla \cdot \left( \frac{\vec{q}}{T} \right) + \frac{\rho r}{T} = \\ &= \frac{2\mu\mathbb{D} : \mathbb{D} + \lambda(\nabla \cdot \vec{v})^2}{T} + \frac{k|\nabla T|^2}{T^2} - \nabla \cdot \left( \frac{\vec{q}}{T} \right) + \frac{\rho r}{T} .\end{aligned}$$

## Details

$$\nabla \cdot \left( \frac{\vec{q}}{T} \right) = \frac{\nabla \cdot \vec{q}}{T} - \frac{\vec{q} \cdot \nabla T}{T^2}$$

**Second principle of thermodynamics and continuum mechanics.** Second principle of thermodynamics implies some constraints on the behavior of continuous media, and thus on the constitutive equations. Namely, Clausius statement of the second principle reads

$$dS \geq \frac{\delta Q}{T} ,$$

i.e. the variation of entropy is greater or equal to the ratio of the heat flux added to the system and the temperature of the system itself. This can be written for a simple homogeneous system, or for a composite systems where physical quantities are not homogeneous in space **todo** ref

Integral form of balance equation of entropy of a system reads

$$\frac{dS}{dt} = \frac{d}{dt} \int_{V_t} \rho s = \int_{V_t} \left\{ \frac{2\mu \mathbb{D} : \mathbb{D} + \lambda (\nabla \cdot \vec{v})^2}{T} + \frac{k |\nabla T|^2}{T^2} \right\} - \underbrace{\oint_{\partial V_t} \frac{\vec{q}}{T} \cdot \hat{n} + \int_{V_t} \frac{\rho r}{T}}_{* \frac{\delta Q}{T} *},$$

and Clausius statement of the second principle implies

$$0 \leq \frac{d}{dt} \int_{V_t} \rho s - \left( - \oint_{\partial V_t} \frac{\vec{q}}{T} \cdot \hat{n} + \int_{V_t} \frac{\rho r}{T} \right) = \int_{V_t} \left\{ \frac{2\mu \mathbb{D} : \mathbb{D} + \lambda (\nabla \cdot \vec{v})^2}{T} + \frac{k |\nabla T|^2}{T^2} \right\} ,$$

and, since this must hold for any volume  $V_t$  and state of the system - namely every velocity and temperature field - and thermodynamic temperature is positive, it follows that

$$\mu \geq 0 \quad , \quad \lambda \geq 0 \quad , \quad k \geq 0$$

### Example 10.2.1 (Sign of physical quantity)

**todo** pay attention, that temperature, viscosity coefficients and thermal conductivity have physical dimensions...explain the meaning of positive physical quantities (scalar, w.r.t. a unit of measurement)...





## NON-DIMENSIONAL EQUATIONS OF FLUID MECHANICS

If  $\rho(P, s)$ ,

$$d\rho = \left( \frac{\partial \rho}{\partial P} \right)_s dP + \left( \frac{\partial \rho}{\partial s} \right)_P ds$$
$$\begin{cases} \frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0 \\ \rho \frac{D\vec{v}}{Dt} = \rho \vec{g} + \nabla \cdot (-p\mathbb{I} + 2\mu\mathbb{D} + \lambda(\nabla \cdot \vec{v})\mathbb{I}) \\ \rho \frac{De^t}{Dt} = \rho \vec{g} \cdot \vec{v} + \nabla \cdot (\mathbb{T} \cdot \vec{v}) - \nabla \cdot \vec{q} + \rho r \end{cases}$$



## INCOMPRESSIBLE FLUID MECHANICS

Chapter of a introductory course in incompressible fluid mechanics:

- statics
- kinematics
- governing equations
- non-dimensional equations
- vorticity dynamics
- low- $Re$  exact solutions
- high- $Re$  flows, incompressible inviscid irrotational flows:
  - vorticity dynamics and Bernoulli theorems
  - aeronautical applications
- boundary layer
- instability and turbulence

### 12.1 Navier-Stokes Equations

The kinematic constraints (link to *Non-dimensional Equations of Fluid Mechanics?*)

$$\nabla \cdot \vec{v} = 0$$

replaces mass balance in the governing equation and implies  $\frac{D\rho}{Dt} = 0$ , i.e. all the material particles have constant density in time.

If ...

$$\begin{cases} \rho \left[ \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right] - \mu \nabla^2 \vec{u} + \nabla P = \rho \vec{g} \\ \nabla \cdot \vec{u} = 0 \end{cases} \quad (12.1)$$

with the proper initial conditions, boundary conditions and - if required - *compatibility conditions*.

## Compatibility condition

A compatibility condition is needed if the velocity field is prescribed on the whole boundary  $\partial V$  of the domain  $V$ ,

$$\vec{u} \Big|_{\partial V} = \vec{b}_n .$$

The compatibility condition reads

$$\oint_{\partial V} \vec{b} \cdot \hat{n} = 0 ,$$

to ensure that the boundary conditions are consistent with the incompressibility constraint, as it is readily proved using divergence theorem on the velocity field in  $V$ ,

$$0 \equiv \int_V \underbrace{\nabla \cdot \vec{u}}_{=0} = \oint_{\partial V} \vec{u} \cdot \hat{n} = \oint_{\partial V} \vec{b} \cdot \hat{n} .$$

## 12.2 Vorticity

A dynamical equation for vorticity  $\vec{\omega} := \nabla \times \vec{u}$  readily follows taking the curl of Navier-Stokes equations (12.1)

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla) \vec{u} + \nu \Delta \vec{\omega} , \quad (12.2)$$

i.e. vorticity can be stretched-tilted by the term  $(\vec{\omega} \cdot \nabla) \vec{u}$ , or diffused by the term  $\nu \Delta \vec{\omega}$ .

...

## 12.3 Bernoulli theorems

For an incompressible fluid, the advective term  $(\vec{u} \cdot \nabla) \cdot \vec{u}$  can be recasted as

$$(\vec{u} \cdot \nabla) \cdot \vec{u} = \vec{\omega} \times \vec{u} + \nabla \frac{|\vec{u}|^2}{2} ,$$

so that the momentum equation in Navier-Stokes equations (12.1) for fluids with uniform density  $\rho$  reads

$$\rho \left[ \frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u} + \nabla \frac{|\vec{u}|^2}{2} \right] - \mu \Delta \vec{u} + \nabla P = \rho \vec{g} . \quad (12.3)$$

Starting from the form (12.3), different forms of Bernoulli theorems are readily derived with the proper assumptions.

---

### Theorem 12.3.1 (Bernoulli theorem along path and vortex lines in steady flows)

In a steady incompressible inviscid flow with conservative volume forces,  $\vec{g} = -\nabla \chi$ , the Bernoulli polynomial is constant along path (everywhere tangent to the velocity field,  $\hat{t}(\vec{r}) \parallel \vec{u}(\vec{r})$ ) and vortex lines (everywhere tangent to the vorticity field,  $\hat{t}(\vec{r}) \parallel \vec{\omega}(\vec{r})$ ), i.e. the directional derivative of the Bernoulli polynomial in the direction of the velocity or the vorticity field is identically zero,

$$\hat{t} \cdot \nabla \left( \frac{|\vec{u}|^2}{2} + \frac{P}{\rho} + \chi \right) = 0 .$$


---

The proof readily follows taking the scalar product with a unit-norm vector  $\hat{t}$  parallel to the local velocity or vorticity, and noting that  $\hat{t} \cdot \vec{u} \times \vec{\omega}$  is zero if either  $\hat{t} \parallel \vec{v}$  or  $\hat{t} \parallel \vec{\omega}$ .

---

**Theorem 12.3.2 (Bernoulli theorem in irrotational inviscid steady flows)**

In a steady incompressible inviscid irrotational flow with conservative volume forces,  $\vec{g} = -\nabla\chi$ , the Bernoulli polynomial is uniform in the whole domain, since its gradient is identically zero

$$\nabla \left( \frac{|\vec{u}|^2}{2} + \frac{P}{\rho} + \chi \right) = 0 \quad \rightarrow \quad \frac{|\vec{u}|^2}{2} + \frac{P}{\rho} + \chi = 0 .$$


---

---

**Theorem 12.3.3 (Bernoulli theorem in irrotational inviscid flows)**

In an incompressible inviscid irrotational flow with conservative volume forces,  $\vec{g} = -\nabla\chi$ , the Bernoulli polynomial is uniform in the connected irrotational regions of the domain - but not constant in time in general - , since its gradient is identically zero

$$\nabla \left( \frac{\partial\phi}{\partial t} + \frac{|\vec{u}|^2}{2} + \frac{P}{\rho} + \chi \right) = 0 \quad \rightarrow \quad \frac{\partial\phi}{\partial t} + \frac{|\vec{u}|^2}{2} + \frac{P}{\rho} + \chi = C(t) .$$

being  $\phi$  the velocity potential used to write the irrotational velocity field as the gradient of a scalar function  $\vec{u} = \nabla\phi$ .

---

**Note:** The assumption of inviscid flow is not directly required if irrotationality holds. Anyway the inviscid flow assumption may be required to make irrotationality condition holds. Looking at the vorticity equation (12.2) the assumption of negligible viscosity prevents diffusion of vorticity from rotational regions to irrotational regions.

---

**Note:** A barotropic fluid is defined as a fluid where the pressure is a function of density only,  $P(\rho)$ . For this kind of flows it's possible to find a function  $\Pi$  so that

$$d\Pi = \frac{dP}{\rho} .$$

The results of this section derived for a uniform density flow hold for a barotropic fluid as well, replacing  $\frac{P}{\rho}$  with  $\Pi$ .

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## COMPRESSIBLE FLUID MECHANICS

### 13.1 Compressible Inviscid Fluid Mechanics

#### 13.1.1 Shocks





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