Classical Electromagnetism and Principles of Electrical Engineering

basics

Apr 29, 2025

CONTENTS

This material is part of the **basics-books project**. It is also available as a .pdf document.

Classical electromagnetism

Brief history of electromagnetism. **todo**

Principles of classical electromagnetism. Principles of electromagnetism (charge conservation, Lorentz's force and Maxwell's equations) are first introduced for *electromagentic phenomena in free-space*, in both differential and integral form. Then, equations governing *electromagnetism in matter* are discussed: free charge and current are distinguished from bound charte and current, resulting from polarization and the magnetization of matter as a response to external fields are described, and introduced into the **constitutive equations** characterizing the behavior of matter. Integral form of governing equations is provided for both control volumes and arbitrary domains in motion w.r.t. the observer, and this description is used to introduce the *low-speed relativity* of physical quantities involved in electromagnetism.

Electromagentic potentials and wave equations. *Electromagnetic potentials* are introduced, along with gauge conditions. *Wave equations* for physical quantities in electromagnetism are then introduced. *Plane waves* are discussed along with interface phenomena like rifraction and reflecion.

Force, Moments on charges, Momentum and Energy of the electromagnetic field.

Regimes.

Einstein's special relativity and electromagnetism.

Electric Engineering

Electric circuits.

Electromagentic systems.

Electromagneto-mechanics systems.

Part I

Electromagnetism

CHAPTER

ONE

BRIEF HISTORY OF ELECTROMAGNETISM

CHAPTER

TWO

PRINCIPLES OF CLASSICAL ELECTROMAGNETISM

2.1 Principles of Classical Electromagnetism in Free Space

The progress in the study of electromagnetic phenomena during the 19th century allowed James Clerk Maxwell to formulate what are now known as *Maxwell's equations*, which can be considered the first consistent formulation of the principles of classical electromagnetism, together with the charge conservation law and the expression for the Lorentz force on an electric charge immersed in an electromagnetic field.

Principles are introduced here for total charges and the electric and the magnetic field, in the form that is known as **equations of electromagnetism in vacuum**. Equations of electromagnetism in matter (1) separate the contribution of free and bound charges and currents, and (2) introduce **polarization** and **magnetization** of matter in constitutive equations representing the macroscopic response of the media as a result of local microscopic charge distribution induced by "external" fields.

Here principles of electromagnetism are first shown in their *differential form*: (1) continuity equation of electric charge describes the conservation of electric charge, (2) Maxwell's equations govern the generation of the electromagnteic field by electric charge and currents, while (3) the differential form of Lorentz's force gives the expression of the force per unit volume acting on a distribution of electric charge and currents immersed in an electromagnetic field. Then, the more general *integral form*[1] is presented for *control volume*, at rest w.r.t. the observer - an inertial one? - and then derived form *arbitrary domains*, using the rules for time derivatives of integrals over moving domains, and this description is used to have a first discussion about relativity in electromagnetism.

2.1.1 Principles in Differential Form

The principles in differential form can be derived from the more general *integral form*, provided the fields satisfy the necessary minimal regularity conditions, which can be qualitatively stated as "all operations must make sense."

Conservation of Electric Charge. Differential form of conservation of electric charge is described by a continuity equation for the electric charge density $\rho(\vec{r}, t)$, with electric current $\vec{j} = \rho\vec{v}$ as the flux - where $\vec{v}(\vec{r}, t)$ is the average velocity of the charges in the point $\vec{r}$ of space at time t

$$\partial_t \rho + \nabla \cdot \vec{j} = 0 \ .$$

Maxwell's Equations. Maxwell's equations give the relations between the electric charge and current densities, with the

[1] As in continuum mechanics, integral equations are the most general form of the equations that governs the global behavior of a system and requires no assumption of regularity of the physical quantities involved. Under the assumptions of regularity, differential equations can be derived from integral equations using theorems of calculus involving differential operators of the fields: differential equations provide local balances. If the fields are piece-wise regular in different regions of the domain, it's possible to derive and use differenial equations in each sub-domain, and link them through jump conditions.

electromagnetic field $\vec{e}(\vec{r},t)$, $\vec{b}(\vec{r},t)$,

$$\begin{cases} \nabla \cdot \vec{e} = \frac{\rho}{\varepsilon_0} \\ \nabla \times \vec{e} + \partial_t \vec{b} = \vec{0} \\ \nabla \cdot \vec{b} = 0 \\ \nabla \times \vec{b} - \varepsilon_0 \mu_0 \, \partial_t \vec{e} = \mu_0 \vec{j} \, , \end{cases}$$

with the **permittivity of free space** - or the **dielectric constant of free space** - ε_0 and the **permeability of the free space**, μ_0

$$\begin{aligned} \varepsilon_0 &= 8.85 \cdot 10^{-12} \, \mathrm{F\,m^{-1}} \\ \mu_0 &= 4\pi \cdot 10^{-7} \, \mathrm{N\,A^{-2}} \end{aligned}$$

Lorentz Force. The force per unit volume acting on the electric charges at point $\vec{r}$ and time t is governed by differential form of Lorent'z force

$$\begin{aligned} \vec{f}(\vec{r},t) &= \rho(\vec{r},t)\,\vec{e}(\vec{r},t) + \vec{j}(\vec{r},t) \times \vec{b}(\vec{r},t) = \\ &= \rho(\vec{r},t)\left[\vec{e}(\vec{r}) + \vec{v}(\vec{r},t) \times \vec{b}(\vec{r},t)\right] = \\ &= \rho^*(\vec{r},t)\,\vec{e}^*(\vec{r},t) \end{aligned}$$

having defined $\rho^*(\vec{r}), t\, \vec{e}^*(\vec{r},t)$ as the current desntiy and the electric field **as seen by the moving charge**

2.1.2 Principles in Integral Form: Electromagnetic Equations and Galilean Relativity

Integral Form on Control Volumes

The integral form of the principles of electromagnetism for fixed volumes V and surfaces S in space is obtained by integrating the differential equations over the domains and using the divergence theorem to obtain flux terms, and Stokes' theorem to obtain circulation terms.

Continuity of Electric Charge.

$$\frac{d}{dt}\int_V \rho + \oint_{\partial V} \vec{j} \cdot \hat{n} = 0$$

$$\frac{d}{dt} Q_V + \Phi_{\partial V}\left(\vec{j}\right) = 0$$

Gauss's Law for the Field $\vec{e}(\vec{r},t)$.

$$\oint_{\partial V} \vec{e} \cdot \hat{n} = \int_V \frac{\rho}{\varepsilon_0}$$

$$\Phi_{\partial V}(\vec{e}) = \frac{Q_V}{\varepsilon_0}$$

Gauss's Law for the Field $\vec{b}(\vec{r},t)$.

$$\oint_{\partial V} \vec{b} \cdot \hat{n} = 0$$

$$\Phi_{\partial V}\left(\vec{b}\right) = 0$$

Faraday–Neumann–Lenz Law for Electromagnetic Induction.

$$\oint_{\partial S} \vec{e} \cdot \hat{t} + \frac{d}{dt} \int_S \vec{b} \cdot \hat{n} = 0$$

$$\Gamma_S \left(\vec{e} \right) + \frac{d}{dt} \Phi_S \left(\vec{b} \right) = 0$$

Ampère–Maxwell Law.

$$\oint_{\partial S} \vec{b} \cdot \hat{t} - \frac{d}{dt} \int_S \varepsilon_0 \mu_0 \, \vec{e} \cdot \hat{n} = \int_S \mu_0 \, \vec{j} \cdot \hat{n}$$

$$\Gamma_{\partial S} \left(\vec{b} \right) - \frac{1}{c_0^2} \frac{d}{dt} \Phi_S \left(\vec{e} \right) = \mu_0 \Phi_S \left(\vec{j} \right) \ ,$$

having introduced the speed of velocity in free space, $c_0 = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$.

Maxwell's equations and continuity equation of electric charge are overdetermined

Introducing (1) the time derivative of Gauss law of the electric field $\vec{e}(\vec{r}, t)$ and (2) the Ampére-Maxwell law in the continuity equation of the electric charge

$$\begin{aligned}
0 &= \dot{Q}_V + \Phi_{\partial V}(\vec{j}) = && (1) \\
&= \varepsilon_0 \dot{\Phi}_{\partial V}(\vec{e}) + \Phi_{\partial V}(\vec{j}) = \\
&= \frac{1}{\mu_0} \left[\mu_0 \varepsilon_0 \dot{\Phi}_{\partial V}(\vec{e}) + \mu_0 \Phi_{\partial V}(\vec{j}) \right] = && (2) \\
&= \frac{1}{\mu_0} \Gamma_{\partial \partial V}(\vec{b}) = 0 \ ,
\end{aligned}$$

an identity appears as the contour ∂S of a closed surface $S = \partial V$ has zero dimension. Thus, these equations are not linearly independent and the system is over-determined.

Integral Form on Arbitrary Volumes

Due to their importance in fundamental applications such as electric motors, and to avoid confusion or leaps in logic when dealing with electromagnetic induction, it is crucial to provide the correct expression of the electromagnetic principles when moving volumes are involved in space. Not only is the form of these principles shown, but also the correct procedure to derive them starting from the fixed-control-volume version. This is done using rules for time derivative for fundamental integrals over moving domains, such as the integral of a density function over a volume, the flux of a vector field through a surface, or the circulation along a curve.

These three derivative rules are listed here and proved in the material about Mathematics:Vector and Tensor Algebra and Calculus:Time derivatives of integrals over moving domains

$$\begin{aligned}
\frac{d}{dt} \int_{v_t} f &= \int_{v_t} \frac{\partial f}{\partial t} + \oint_{\partial v_t} f \, \vec{v}_b \cdot \hat{n} && \text{(density)} \\
\frac{d}{dt} \int_{s_t} \vec{f} \cdot \hat{n} &= \int_{s_t} \frac{\partial \vec{f}}{\partial t} \cdot \hat{n} + \int_{s_t} \nabla \cdot \vec{f} \, \vec{v}_b \cdot \hat{n} - \oint_{\partial s_t} \vec{v}_b \times \vec{f} \cdot \hat{t} && \text{(flux)} \\
\frac{d}{dt} \int_{\ell_t} \vec{f} \cdot \hat{t} &= \int_{\ell_t} \frac{\partial \vec{f}}{\partial t} \cdot \hat{t} + \int_{\ell_t} \nabla \times \vec{f} \cdot \vec{v}_b \times \hat{t} + \vec{f}_B \cdot \vec{v}_B - \vec{f}_A \cdot \vec{v}_A && \text{(circulation)}
\end{aligned}$$

Continuity of Electric Charge.

$$\begin{aligned}
0 &= \frac{d}{dt}\int_V \rho + \oint_{\partial V} \vec{j}\cdot\hat{n} = \\
&= \frac{d}{dt}\int_{v_t} \rho - \oint_{\partial v_t} \rho \vec{v}_b \cdot \hat{n} + \oint_{\partial v_t} \vec{j}\cdot\hat{n} \\
& \frac{d}{dt}\int_{v_t} \rho + \oint_{\partial v_t} \underbrace{\rho(\vec{v}-\vec{v}_b)}_{\vec{j}^*} \cdot \hat{n}
\end{aligned}$$

Gauss's Law for the Field $\vec{d}(\vec{r},t)$.

$$\oint_{\partial v_t} \vec{d}\cdot\hat{n} = \int_{v_t} \rho$$

Gauss's Law for the Field $\vec{b}(\vec{r},t)$.

$$\oint_{\partial v_t} \vec{b}\cdot\hat{n} = 0$$

Faraday–Neumann–Lenz Law for Electromagnetic Induction.

$$\begin{aligned}
\vec{0} &= \oint_{\partial S} \vec{e}\cdot\hat{t} + \frac{d}{dt}\int_S \vec{b}\cdot\hat{n} = \\
&= \oint_{\partial s_t} \vec{e}\cdot\hat{t} + \frac{d}{dt}\int_{s_t} \vec{b}\cdot\hat{n} - \int_{s_t} \underbrace{\nabla\cdot\vec{b}}_{=0}\, \vec{v}_b\cdot\hat{n} + \oint_{s_t} \vec{v}_b \times \vec{b}\cdot\hat{t} = \\
& \oint_{\partial s_t} \vec{e}^*\cdot\hat{t} + \frac{d}{dt}\int_{s_t} \vec{b}\cdot\hat{n}\,,
\end{aligned}$$

with the definition $\vec{e}^* := \vec{e} + \vec{v}_b \cdot \vec{b}$, already used in the expression of the Lorentz force law.

Ampère–Maxwell Law.

$$\begin{aligned}
\vec{0} &= \oint_{\partial s_t} \vec{h}\cdot\hat{t} - \frac{d}{dt}\int_{s_t} \vec{d}\cdot\hat{n} - \int_{s_t} \vec{j}\cdot\hat{n} = \\
&= \oint_{\partial s_t} \vec{h}\cdot\hat{t} - \frac{d}{dt}\int_{s_t} \vec{d}\cdot\hat{n} + \int_{s_t} \underbrace{\nabla\cdot\vec{d}}_{=\rho}\, \vec{v}_b\cdot\hat{n} - \oint_{s_t} \vec{v}_b \times \vec{d}\cdot\hat{t} - \int_{s_t} \vec{j}\cdot\hat{n} = \\
& \oint_{\partial s_t} \vec{h}^*\cdot\hat{t} - \frac{d}{dt}\int_{s_t} \vec{b}\cdot\hat{n} = \int_{s_t} \vec{j}^*\cdot\hat{n}\,,
\end{aligned}$$

having defined $\vec{h}^* := \vec{h} - \vec{v}_b \times \vec{d}$, and using the previously introduced definition $\vec{j}^* := \vec{j} - \rho\vec{v}_b$.

Adding the definitions:

$$\begin{aligned}
\rho^* &= \rho \\
\vec{d}^* &= \vec{d} \\
\vec{b}^* &= \vec{b}
\end{aligned}$$

one obtains equations having the same form as those written for stationary domains in space, but which can be applied to moving domains. The definitions:

$$\begin{aligned}
\rho^* &= \rho \quad &,\quad \vec{j}^* &= \vec{j} - \rho\vec{v}_b \\
\vec{d}^* &= \vec{d} \quad &,\quad \vec{e}^* &= \vec{e} + \vec{v}_b \times \vec{b} \\
\vec{b}^* &= \vec{b} \quad &,\quad \vec{h}^* &= \vec{h} - \vec{v}_b \times \vec{d}
\end{aligned}$$

are nothing more than the transformation of the fields for two observers in relative motion, and correspond to the low-speed limit of Lorentz transformations from special relativity for velocities $|\vec{v}_b| \ll c$: in this procedure, the transformations for low relative speeds are obtained, as no transformation of spatial and temporal dimensions has been considered, unlike Einstein's theory of relativity.

todo Reference Galilean and Lorentz transformations for relativity in electromagnetism.

2.2 Electromagnetism in Matter

Electromagnetism in matter requires the description of the behavior of the matter involved in the process. In general, a medium immersed in an electromagnetic field may respond with local charge distributions, resulting in ***polarization*** and ***magnetization***. Total electric $\vec{e}(\vec{r},t)$ and magnetic field $\vec{b}(\vec{r},t)$ can be written as the sum of contributions of free charges ρ_f and currents $\vec{j}_f$ and bound charges ρ_b and currents $\vec{j}_b$.

Bound charge density represents local separation of charges of molecules of dielectric media immersed in electric field, that can be represented as a volume distribution of charge dipole,

$$\rho = \rho_f + \rho_b = \rho_f + \rho_P \ .$$

Bound current density represents two effects: the variation of polarization charge and the orientation of Amperian currents - "non random" currents in the molecules of the medium, producing net contribution to the magnetic field, and can be represented as a volume distribution of elementary loop currents.

$$\vec{j} = \vec{j}_f + \vec{j}_b = \vec{j}_f + \vec{j}_P + \vec{j}_M \ .$$

As it will shown below, the bound current can be written as the divergence of the polarization field $\vec{p}$, representing the volume density fo the dipole distribution, and the magnetization current as the curl of the magnetization field $\vec{m}$,

$$\rho_p = -\nabla\cdot\vec{p} \quad , \quad \vec{j}_M = \nabla\times\vec{m} \ .$$

2.2.1 Equations of electromagnetism in matter

Introducing the splitting of free and bound charge and current into the equations of the electromagnetism, namely electric charge continuity and Maxwell's equations,

$$\partial_t\rho + \nabla\cdot\vec{j} = 0 \qquad , \qquad \begin{cases} \nabla\cdot\vec{e} = \frac{\rho}{\varepsilon_0} \\ \nabla\times\vec{e} + \partial_t\vec{b} = \vec{0} \\ \nabla\cdot\vec{b} = 0 \\ \nabla\times\vec{b} - \mu_0\varepsilon_0\partial_t\vec{e} = \mu_0\vec{j} \end{cases}$$

and more precisely

- into Gauss' law for the electric field

$$0 = \nabla\cdot\vec{e} - \frac{\rho}{\varepsilon_0} = \nabla\cdot\vec{e} - \frac{\rho_f - \nabla\cdot\vec{p}}{\varepsilon_0}$$

$$\rightarrow \qquad \nabla\cdot\vec{d} = \rho_f \ ,$$

  with $\vec{d} = \varepsilon_0\vec{e} + \vec{p}$ defined as the **displacement field**.

- into continuity equation

$$\begin{aligned} 0 &= \partial_t \rho + \nabla \cdot \vec{j} = \\ &= \partial_t \rho_f + \nabla \cdot \vec{j}_f + \partial_t \rho_b + \nabla \cdot (\vec{j}_P + \nabla \times \vec{j}_M) = \end{aligned}$$

since $\nabla \cdot \nabla \times \vec{m} \equiv 0$, and keeping separated the contributions of free and bound charges,

$$\begin{aligned} &\partial_t \rho_f + \nabla \cdot \vec{j}_f = 0 \\ &\partial_t \rho_P + \nabla \cdot \vec{j}_P = 0 \end{aligned}$$

$$\rightarrow \qquad \vec{j}_P = \partial_t \vec{p} \, .$$

- and into Ampére-Maxwell's law

$$\begin{aligned} \vec{0} &= \nabla \times \vec{b} - \mu_0 \varepsilon_0 \partial_t \vec{e} - \mu_0 \vec{j} = \\ &= \nabla \times \vec{b} - \mu_0 \, \partial_t \left(\vec{d} - \vec{p} \right) - \mu_0 \vec{j}_f - \mu_0 \partial_t \vec{p} - \mu_0 \nabla \times \vec{m} \\ &= \nabla \times \left(\vec{b} - \mu_0 \vec{m} \right) - \; mu_0 \, \partial_t \vec{d} - \mu_0 \vec{j}_f \end{aligned}$$

$$\rightarrow \qquad \nabla \times \vec{h} - \partial_t \vec{d} = \vec{j}_f \, ,$$

where $\vec{h} := \vec{b} - \mu_0 \vec{m}$, the **magnetic field strength**.

2.2.2 Examples

- conductors
- ferromagnetic and weak magnetism (para-, dia-, anti-)

2.2.3 Governing equations in differential form

Differential form of Maxwell's equations

$$\begin{cases} \nabla \cdot \vec{d} = \rho_f \\ \nabla \times \vec{e} + \partial_t \vec{b} = \vec{0} \\ \nabla \cdot \vec{b} = 0 \\ \nabla \times \vec{h} - \partial_t \vec{d} = \vec{j}_f \end{cases}$$

todo *continuity equation for charge*

2.2.4 Governing equation in integral form

Integral form of Maxwell's equations

$$\begin{cases} \displaystyle\oint_{\partial V} \vec{d} \cdot \hat{\vec{n}} = \int_V \rho_f \\ \displaystyle\oint_{\partial S} \vec{e} \cdot \hat{\vec{t}} + \frac{d}{dt} \int_S \vec{b} \cdot \hat{\vec{n}} = 0 \\ \displaystyle\oint_{\partial V} \vec{b} \cdot \hat{\vec{n}} = 0 \\ \displaystyle\oint_{\partial S} \vec{h} \cdot \hat{\vec{t}} - \frac{d}{dt} \int_S \vec{d} \cdot \hat{\vec{n}} = \int_S \vec{j}_f \cdot \hat{\vec{n}} \end{cases}$$

- control volume
- arbitrary domain
- low-speed relativity

2.2.5 Jump Conditions

Letting V and S "collapse on a discontinuity"…

$$\begin{cases} [d_n] = \sigma_f \\ [e_t] = 0 \\ [b_n] = 0 \\ [h_t] = \iota_f \ , \end{cases} \tag{2.1}$$

being σ_f and ι_f surface charge and current density, with physical dimension $\frac{\text{charge}}{\text{surface}}$, and $\frac{\text{current}}{\text{surface}}$ respectively. These contributions can be thought of as Dirac delta contributions in volume density, namely

$$\rho(\vec{r},t) = \rho_0(\vec{r},t) + \sigma(\vec{r}_s,t)\delta_1(\vec{r}-\vec{r}_s) \ ,$$

being $\rho(\vec{r},t)$ the regular part of the volume density in all the points of the domain $\vec{r} \in V$, $\sigma(\vec{r}_s,t)$ the surface density on 2-dimensional surfaces $\vec{r}_s \in S$, $\delta_1()$ the Dirac's delta with physical dimension $\frac{1}{\text{length}}$.

If there's no free surface charge and currents, jump conditions for linear media become

$$\begin{cases} [d_n] = 0 \\ [e_t] = 0 \\ [b_n] = 0 \\ [h_t] = 0 \ , \end{cases} \qquad \rightarrow \qquad \begin{cases} d_{n,1} = d_{n,2} & \rightarrow \quad \varepsilon_1 e_{n,1} = \varepsilon_2 e_{n,2} \\ e_{t,1} = e_{t,2} & \\ b_{n,1} = b_{n,2} & \\ h_{t,1} = h_{t,2} & \rightarrow \quad \frac{1}{\mu_1} b_{t,1} = \frac{1}{\mu_2} b_{t,2} \end{cases} \tag{2.2}$$

2.2.6 Polarization

Single Electric Dipole

A discrete electric dipole is formed by two equal and opposite electric charges q, $-q$, at points P_+, $P_- = P_+\vec{l}$, in the limit $q \rightarrow +\infty$, $|\vec{l}| \rightarrow 0$ with $q|\vec{l}|$ finite.

The electric field (stationary **todo** *check what happens in the non-stationary case. Perhaps after deriving the general solution to the problem, as a solution to the wave equations in terms of EM potentials*) generated at the point in space $\vec{r}$ by an electric dipole at the point $\vec{r}_0$ is calculated as the limit of the electric field generated by two equal and opposite charges $q^{\mp}$ at the points $\vec{r}_0 \mp \frac{\vec{l}}{2}$,

$$\vec{e}(\vec{r}) = -\frac{q}{4\pi\varepsilon_0} \frac{\vec{r} - \left(\vec{r}_0 - \frac{\vec{l}}{2}\right)}{\left|\vec{r} - \left(\vec{r}_0 - \frac{\vec{l}}{2}\right)\right|^3} + \frac{q}{4\pi\varepsilon_0} \frac{\vec{r} - \left(\vec{r}_0 + \frac{\vec{l}}{2}\right)}{\left|\vec{r} - \left(\vec{r}_0 + \frac{\vec{l}}{2}\right)\right|^3} \ .$$

Using the formula for the derivative of the terms

$$\partial_{\ell_k} \frac{x_i \pm \frac{\ell_i}{2}}{\left|\vec{x} \pm \frac{\vec{l}}{2}\right|^3} = \frac{1}{2}\left[\pm\frac{\delta_{ik}}{r^3} - 3r^{-4}\left(\pm\frac{x_k \pm \frac{\ell_k}{2}}{r}\right)\right]$$

$$\partial_{\ell_k} \frac{x_i \pm \frac{\ell_i}{2}}{\left|\vec{x} \pm \frac{\vec{l}}{2}\right|^3}\Bigg|_{\vec{l}=\vec{0}} = \mp\frac{1}{2}\left[-\frac{\delta_{ik}}{|\vec{x}|^3} + 3\left(\frac{x_k}{r^5}\right)\right] = \mp\frac{1}{2}\partial_{r_{0k}}\frac{r_i - r_{0i}}{|\vec{r}-\vec{r}_0|^3} = \mp\frac{1}{2}\nabla_{\vec{r}_0}\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}$$

we derive the first-order approximation in $\vec{l}$ of the two terms

$$\frac{\vec{r} - \left(\vec{r}_0 \mp \frac{\vec{l}}{2}\right)}{\left|\vec{r} - \left(\vec{r}_0 \mp \frac{\vec{l}}{2}\right)\right|^3} = \frac{\vec{r}-\vec{r}_0}{\left|\vec{r}-\vec{r}_0\right|^3} \pm \vec{l}\cdot\frac{1}{2}\nabla_{\vec{r}_0}\left(\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}\right) + o(|\vec{l}|)$$

and, defining the dipole intensity $\vec{P}_0 := q\vec{l}$ and taking the quantities to the desired limit, that of the electric field

$$\begin{aligned}
\vec{e}(\vec{r}) &= -\frac{1}{4\pi\varepsilon_0}\,\vec{P}_0\cdot\nabla_{\vec{r}_0}\left(\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}\right) = \\
&= -\frac{1}{4\pi\varepsilon_0}\left[\frac{(\vec{r}-\vec{r}_0)(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^5}\cdot\vec{P}_0 - \frac{\vec{P}_0}{|\vec{r}-\vec{r}_0|^3}\right] = \\
&= -\frac{1}{4\pi\varepsilon_0}\left[\frac{(\vec{r}-\vec{r}_0)\otimes(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^5} - \frac{\mathbb{I}}{|\vec{r}-\vec{r}_0|^3}\right]\cdot\vec{P}_0\,.
\end{aligned}$$

todo In the general case, it would be necessary to pay attention to the order of the factors in the product between vectors and tensors, but in this case, the symmetry of the second-order tensor (or of the operations) can be exploited.

Continuous Distribution of Dipoles

A distribution of dipoles with volume density $\vec{p}(\vec{r_0})$, which produces the elementary dipole $\Delta\vec{P}(\vec{r}_0) = \vec{p}(\vec{r}_0)dV_0$ in the volume dV_0, produces the electric field

$$\vec{e}(\vec{r}) = \int_{\vec{r}_0\in V_0}\frac{1}{4\pi\varepsilon_0}\vec{p}(\vec{r}_0)\cdot\nabla_{\vec{r}_0}\left(\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}\right)\ ,$$

whose expression can be rewritten using the rules of integration by parts

$$\begin{aligned}
\vec{e}(\vec{r}) &= \int_{\vec{r}_0\in V_0}\frac{1}{4\pi\varepsilon_0}\vec{p}(\vec{r}_0)\cdot\nabla_{\vec{r}_0}\left(\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}\right) = \\
&= \oint_{\vec{r}_0\in\partial V_0}\frac{1}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}\underbrace{\hat{n}(\vec{r}_0)\cdot\vec{p}(\vec{r}_0)}_{=:\sigma_P(\vec{r}_0)} + \int_{\vec{r}_0\in V_0}\frac{1}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}\underbrace{\left(-\nabla_{\vec{r}_0}\cdot\vec{p}(\vec{r}_0)\right)}_{=:\rho_P(\vec{r}_0)}\ ,
\end{aligned}$$

having defined the surface polarization charge density σ_P and the volume polarization charge density ρ_P as the intensities of the distributed sources of the electric field, in analogy with the expression of Coulomb's law.

Reformulation of Maxwell's Equations and Charge Continuity

Gauss's equation determines the volume flux density of the electric field $\vec{e}$,

$$\nabla\cdot\vec{e} = \frac{\rho}{\varepsilon_0}\,.$$

By decomposing the charge density as the sum of **free charges** ρ_f and **polarization charges** $\rho_P := -\nabla\cdot\vec{p}$, we can rework Gauss's equation,

$$\begin{aligned}
&\nabla\cdot\vec{e} = \frac{\rho_f+\rho_P}{\varepsilon_0} \\
&\nabla\cdot(\varepsilon_0\vec{e}+\vec{p}) = \rho_f \\
\\
&\nabla\cdot\vec{d} = \rho_f\ ,
\end{aligned}$$

having introduced the **displacement field**, $\vec{d} := \varepsilon_0 \vec{e} + \vec{p}$.

The decomposition of the electric current as the sum $\vec{j} = \vec{j}_f + \vec{j}_P$ of the free current $\vec{j}_f$ and the polarization current $\vec{j}_P$, allows us to rework the continuity equation of electric charge

$$\begin{aligned}
0 &= \partial_t \rho + \nabla \cdot \vec{j} = \\
&= \partial_t (\rho_f + \rho_P) + \nabla \cdot \left(\vec{j}_f + \vec{j}_P \right) = \\
&= \partial_t \rho_f + \nabla \cdot \vec{j}_f + \partial_t \rho_P + \nabla \cdot \vec{j}_P \ ,
\end{aligned}$$

and write the continuity equations for the two charge distributions (of different nature, it is assumed that both must satisfy charge continuity independently, if free charges remain free and polarization charges remain polarization charges),

$$\begin{aligned}
& \partial_t \rho_f + \nabla \cdot \vec{j}_f = 0 \\
& \partial_t \rho_P + \nabla \cdot \vec{j}_P = 0 \qquad \rightarrow \qquad 0 = \nabla \cdot (-\partial_t \vec{p} + \vec{j}_P) \qquad \rightarrow \qquad \vec{j}_P = \partial_t \vec{p}
\end{aligned}$$

todo *justify absence of constant field*

2.2.7 Magnetization

Single Magnetic Moment (Limit of an Elementary Loop)

Using the Biot-Savart law, specialized for a conductor carrying current $i(\vec{r}_0)$

$$\begin{aligned}
d\vec{b}(\vec{r}) &= -\frac{\mu}{4\pi} \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3} \times \vec{j}(\vec{r}_0) dV_0 = \\
&= -\frac{\mu}{4\pi} i(\vec{r}_0) \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3} \times \hat{\vec{t}}(\vec{r}_0) d\ell_0 \ ,
\end{aligned}$$

we can calculate the magnetic field generated by a loop with path $\ell_0 = \partial S_0$ using the PSCE

$$\begin{aligned}
\vec{b}(\vec{r}) &= \oint_{\ell_0} d\vec{b}(\vec{r}_0) = \\
&= -\frac{\mu}{4\pi} i_0 \oint_{\vec{r}_0 \in \ell_0} \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3} \times \hat{\vec{t}}(\vec{r}_0) = \\
&= \frac{\mu}{4\pi} i_0 \int_{\vec{r}_0 \in S_0} \hat{\vec{n}}(\vec{r}_0) \cdot \nabla_{\vec{r}_0} \left(\frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3} \right)
\end{aligned}$$

The field generated by an elementary loop of surface S_0 with normal $\hat{\vec{n}}_0$, using the mean value theorem, is

$$\vec{b}(\vec{r}) = \frac{\mu}{4\pi} i_0 S_0 \hat{\vec{n}}_0 \cdot \nabla_{\vec{r}_0} \left(\frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3} \right) + o(S_0)$$

and as $i_0 \rightarrow \infty$, $S_0 \rightarrow 0$ such that $\vec{M}_0 := i_0 S_0 \hat{\vec{n}}_0$

$$\begin{aligned}
\vec{b}(\vec{r}) &= \frac{\mu}{4\pi} \vec{M}_0 \cdot \nabla_{\vec{r}_0} \left(\frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3} \right) \\
&= -\frac{\mu_0}{4\pi} \left[\frac{(\vec{r} - \vec{r}_0)(\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^5} \cdot \vec{M}_0 - \frac{\vec{M}_0}{|\vec{r} - \vec{r}_0|^3} \right] = \\
&= -\frac{\mu_0}{4\pi} \left[\frac{(\vec{r} - \vec{r}_0) \otimes (\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^5} - \frac{\mathbb{I}}{|\vec{r} - \vec{r}_0|^3} \right] \cdot \vec{M}_0 \ .
\end{aligned}$$

todo Analogy with the electric field produced by a distribution of dipoles.

Details

$$\oint_{\partial S} A\, t_i = \int_S \varepsilon_{ijk}\, n_j\, \partial_k A \qquad , \qquad \oint_{\partial S} A\, \hat{\vec{t}} = \int_S \hat{\vec{n}} \times \nabla A$$

$$\begin{aligned}
\oint_{\vec{r}_0 \in \ell_0} \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \times \hat{\vec{t}}(\vec{r}_0) d\ell_0 = & \oint_{\vec{r}_0 \in \ell_0} \varepsilon_{ijk} \frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} t_k = \\
= & \int_{\vec{r}_0 \in S_0} \varepsilon_{krs} n_r \partial^0_s \left(\varepsilon_{ijk} \frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} \right) = \\
= & \int_{\vec{r}_0 \in S_0} \left(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr} \right) n_r \partial^0_s \left(\frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} \right) = \\
= & \int_{\vec{r}_0 \in S_0} \left\{ n_i \underbrace{\partial^0_j \left(\frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} \right)}_{=0} - n_j \partial^0_i \left(\frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} \right) \right\} = \\
= & - \int_{\vec{r}_0 \in S_0} n_j \partial^0_i \left(\frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} \right) .
\end{aligned}$$

Continuous Distribution of Magnetic Moment

To calculate the magnetic field generated by a volume distribution of magnetic moment, we can proceed in analogy with what was done to calculate the electric field generated by a distribution of dipoles

$$\begin{aligned}
\vec{b}(\vec{r}) = & \int_{\vec{r}_0 \in V_0} \frac{\mu_0}{4\pi} \vec{m}(\vec{r}_0) \cdot \nabla_{\vec{r}_0} \left(\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \right) = \\
= & \oint_{\vec{r}_0 \in \partial V_0} \frac{\mu_0}{4\pi} \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \hat{\vec{n}}(\vec{r}_0) \cdot \vec{m}(\vec{r}_0) + \int_{\vec{r}_0 \in V_0} \frac{\mu_0}{4\pi} \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \left(-\nabla_{\vec{r}_0} \cdot \vec{m}(\vec{r}_0) \right) ,
\end{aligned}$$

but without obtaining an analogy with the expression of the Biot-Savart law, which involves the cross product between the term $\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}$ and a current density $\vec{j}(\vec{r}_0)$.

Details

We can rewrite

$$\begin{aligned}
&\oint_{\vec{r}_0 \in \partial V_0} \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \times \left(\hat{\vec{n}}(\vec{r}_0) \times \vec{m}(\vec{r}_0)\right) \\
&\qquad = \oint_{\vec{r}_0 \in \partial V_0} \varepsilon_{ijk} \frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} \varepsilon_{krs} n_r m_s = \\
&\qquad = \int_{\vec{r}_0 \in V_0} \left(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}\right) \partial^0_r \left(\frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} m_s \right) = \\
&\qquad = \int_{\vec{r}_0 \in V_0} \left\{ \partial^0_i \left(\frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} m_j \right) - \partial^0_j \left(\frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} m_i \right) \right\} = \\
&\qquad = \int_{\vec{r}_0 \in V_0} \left\{ \partial^0_i \frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} m_j + \frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} \partial^0_i m_j - \frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} \partial^0_j m_i - \underbrace{\partial^0_j \frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3}}_{=0} m_i \right\} = \\
&\qquad = \int_{\vec{r}_0 \in V_0} \left\{ \partial^0_i \frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} m_j + \varepsilon_{ijk}\varepsilon_{krs} \frac{r_j - r_{0,j}}{|\vec{r}-\vec{r}_0|^3} \partial^0_r m_s \right\} = \\
&\qquad = \int_{\vec{r}_0 \in V_0} \left\{ \nabla_{\vec{r}_0} \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \cdot \vec{m}(\vec{r}_0) + \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \times \left(\nabla_{\vec{r}_0} \times \vec{m}(\vec{r}_0) \right) \right\} =
\end{aligned}$$

using vector calculus identities,

$$\begin{aligned}
\vec{a} \times (\vec{b} \times \vec{c}) &= \varepsilon_{ijk} a_j \varepsilon_{krs} b_r c_s = \\
&= (\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}) a_j b_r c_s = \\
&= a_j b_i c_j - c_i b_j a_j = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})
\end{aligned}$$

$$\begin{aligned}
a_j \partial_i m_j - a_j \partial_j m_i &= (\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}) a_j \partial_r m_s = \\
&= \varepsilon_{ijk}\varepsilon_{krs} a_j \partial_r m_s = \\
&= \vec{a} \times (\nabla \times \vec{m})
\end{aligned}$$

The magnetic field generated by a distribution of magnetic moment can therefore be rewritten as

$$\begin{aligned}
\vec{b}(\vec{r}) &= \int_{\vec{r}_0 \in V_0} \frac{\mu_0}{4\pi} \vec{m}(\vec{r}_0) \cdot \nabla_{\vec{r}_0} \left(\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \right) = \\
&= -\frac{\mu_0}{4\pi} \oint_{\vec{r}_0 \in \partial V_0} \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \times \underbrace{\left(-\hat{\vec{n}}(\vec{r}_0) \times \vec{m}(\vec{r}_0)\right)}_{\vec{j}^s_M} - \frac{\mu_0}{4\pi} \int_{\vec{r}_0 \in V_0} \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \times \underbrace{\left(\nabla_{\vec{r}_0} \times \vec{m}(\vec{r}_0)\right)}_{\vec{j}_M} ,
\end{aligned}$$

having defined the surface magnetization current density $\vec{j}^s_M$ and the volume magnetization current density $\vec{j}_M$ as the intensities of the distributed singularities, in analogy with the expression of the Biot-Savart law.

Reformulation of Maxwell's Equations and Charge Continuity

The Ampère-Maxwell law can be rewritten

$$\begin{aligned}
&\nabla \times \vec{b} - \mu_0 \varepsilon_0 \partial_t \vec{e} = \mu_0 \vec{j} \\
&\nabla \times \vec{b} - \mu_0 \partial_t \left(\vec{d} - \vec{p}\right) = \mu_0 \left(\vec{j}_f + \vec{j}_P + \vec{j}_M\right) \\
&\nabla \times \underbrace{\left(\vec{b} - \mu_0 \vec{m}\right)}_{=:\mu_0 \vec{h}} - \mu_0 \partial_t \vec{d} + \mu_0 \underbrace{\left(\partial_t \vec{p} - \vec{j}_P\right)}_{=\vec{0}} = \mu_0 \vec{j}_f \\
\\
&\nabla \times \vec{h} - \partial_t \vec{d} = \vec{j}_f
\end{aligned}$$

From the continuity equation of electric current,

$$\partial_t \rho + \nabla \cdot \vec{j} = 0\ ,$$

we derive the continuity equation for magnetization charges

$$\begin{aligned}
0 &= \partial_t \rho_M + \nabla \cdot \vec{j}_M = \\
&= \partial_t \rho_M + \underbrace{\nabla \cdot \nabla \times \vec{m}}_{\equiv \vec{0}}\ .
\end{aligned}$$

2.3 Galileian relativity in electromagnetism

CHAPTER

THREE

ELECTROMAGNETIC WAVES

3.1 Electromagnetic Potentials

It is possible to demonstrate that the system of Maxwell's equations and the charge continuity equation is overdetermined. Specifically, it can be shown that, given the distribution of charge and current density—considered as the generating causes of the electric field—and the constitutive laws of the material, four unknowns are sufficient to define the six unknowns (three components for two vector fields) of the problem. Therefore, the problem can be formulated in terms of a scalar potential φ and a vector potential $\vec{a}$, along with a gauge condition that eliminates the remaining two arbitrary factors (irrelevant for the calculation of physical fields).

3.1.1 Vector Potential and Scalar Potential

Starting from Maxwell's equations, the potentials of the electromagnetic field can be defined. Using Gauss's law for the magnetic field, the vector potential $\vec{a}(\vec{r},t)$ can be introduced,

$$0 = \nabla \cdot \vec{b} \qquad \rightarrow \qquad \vec{b} = \nabla \times \vec{a} \ ,$$

since the divergence of a curl is identically zero. Introducing this relationship into the Faraday-Neumann-Lenz equation, assuming sufficient regularity of the fields to allow the inversion of the order of derivatives,

$$0 = \nabla \times \vec{e} + \partial_t \vec{b} = \nabla \times \vec{e} + \partial_t \nabla \times \vec{a} = \nabla \times (\vec{e} + \partial_t \vec{a}) \qquad \rightarrow \qquad \vec{e} + \partial_t \vec{a} = -\nabla \varphi \ ,$$

since the curl of a gradient is identically zero. The "physical" quantities of the electric field $\vec{e}(\vec{r},t)$ and the magnetic field $\vec{b}(\vec{r},t)$ can therefore be written using the electromagnetic potentials as

$$\begin{aligned} \vec{e} &= -\nabla\varphi - \partial_t \vec{a} && (a) \\ \vec{b} &= \nabla \times \vec{a} && (b) \end{aligned} \tag{3.1}$$

3.1.2 Gauge Conditions

The potentials are defined up to a gauge condition, an additional condition that eliminates any arbitrariness in the definition. For example, the vector potential is defined up to the gradient of a scalar function, since $\nabla \times \nabla f \equiv \vec{0}$, and thus the potential $\tilde{\vec{a}} = \vec{a} + \nabla f$ produces the same magnetic field $\vec{b}$

$$\nabla \times \tilde{\vec{a}} = \nabla \times (\vec{a} + \nabla f) = \nabla \times \vec{a} \ .$$

Lorentz Gauge Condition. For reasons that will become clearer in the section on *electromagnetic waves*, a convenient gauge condition is

$$\nabla \cdot \vec{a} + \frac{1}{c^2} \partial_t \varphi = 0 \tag{3.2}$$

Coulomb Gauge Condition.

$$\nabla \cdot \vec{a} = 0$$

3.2 Wave Equations in Electromagnetism

Wave equations for physical quantities in electromagnetism are derived from the governing equations for linear local isotropic homogeneous (ε, μ uniform, not function of space) media with constitutive equations

$$\vec{d} = \varepsilon \vec{e} \qquad , \qquad \vec{b} = \mu \vec{h} \ ,$$

using vector identity

$$\Delta \vec{v} = \nabla(\nabla \cdot \vec{v}) - \nabla \times \nabla \times \vec{v} \ .$$

If some of the assumptions made above is not true, slight modifications and extra terms in the equations are likely to appear during the manipulation of the equations done below.

3.2.1 Electromagnetic Potentials

Vector potential

Wave equation for the vector potential,

$$\vec{b} = \nabla \times \vec{a} \ ,$$

is derived taking the curl of its definition,

$$\begin{aligned}
\vec{0} &= \nabla \times \nabla \times \vec{a} - \nabla \times \vec{b} = && (1.a) \\
&= -\Delta \vec{a} + \nabla(\nabla \cdot \vec{a}) - \mu \nabla \times \vec{h} = && (2) \\
&= -\Delta \vec{a} + \nabla(\nabla \cdot \vec{a}) - \mu(\partial_t \vec{d} + \vec{j}_f) = && (1.b) \\
&= -\Delta \vec{a} + \nabla(\nabla \cdot \vec{a}) - \mu(\varepsilon \partial_t \vec{e} + \vec{j}_f) = && (3) \\
&= -\Delta \vec{a} + \nabla(\nabla \cdot \vec{a}) - \mu\varepsilon(-\partial_t \nabla \varphi - \partial_{tt} \vec{a}) + \mu \vec{j}_f = \\
&= -\Delta \vec{a} + \nabla(\nabla \cdot \vec{a}) + \frac{1}{c^2} \partial_t \nabla \varphi + \frac{1}{c^2} \partial_{tt} \vec{a} - \mu \vec{j}_f
\end{aligned}$$

and using (1) the constitutive law for homogeneous isotropic linear media, (2) Ampére-Maxwell's equation, (3), and (4) the definition of the electric field (3.1)(a) in terms of the potentials. Using the Lorentz gauge condition (3.2)

$$\nabla \cdot \vec{a} + \frac{1}{c^2} \partial_t \varphi = 0 \ ,$$

wave equation for the vector potential reads,

$$\frac{1}{c^2} \partial_{tt} \vec{a} - \Delta \vec{a} = \mu \vec{j} \ . \tag{3.3}$$

Scalar potential

Wave equation for the the scalar potential, $\varphi(\vec{r}, t)$, can be derived taking the time derivative of Lorentz's gauge condition,

$$\begin{aligned}
0 = \partial_t \left(\frac{1}{c^2} \partial_t \varphi + \nabla \cdot \vec{a} \right) = & \\
= \frac{1}{c^2} \partial_{tt} \varphi + \nabla \cdot \partial_t \vec{a} = & \qquad (1) \\
= \frac{1}{c^2} \partial_{tt} \varphi - \nabla \cdot \nabla \varphi - \nabla \cdot \vec{e} = & \qquad (2) \\
= \frac{1}{c^2} \partial_{tt} \varphi - \Delta \varphi - \frac{\rho_f}{\varepsilon} \ , &
\end{aligned}$$

using (1) the definition (3.1)(a) of the electric field as a function of the potentials, and (2) Gauss' law for the electric field,

$$\frac{1}{c^2} \partial_{tt} \varphi - \Delta \varphi = \frac{\rho_f}{\varepsilon} \ . \tag{3.4}$$

3.2.2 Electric Field and Magnetic Field

Exploiting the linearity - obviously, if the problem is linear - wave equations for the electric and the magnetic field can be readily derived from applying the wave operator

$$\Box := \frac{1}{c^2} \partial_{tt} - \Delta \ ,$$

to the (1) definitions (3.1) of the electric and the magnetic fields as functions of the potentials, (2) swapping the order of the operator $\Box$ with ∂_t and ∇[1], and (3) using the expressions of the wave equations for the vector potential (3.3) and the scalar potential (3.4).

Electric field

$$\begin{aligned}
\Box \vec{e} = & \qquad (1) \\
= \Box(-\nabla \varphi - \partial_t \vec{a}) = & \qquad (2) \\
= -\nabla \Box \varphi - \partial_t \Box \vec{a} = & \qquad (3) \\
= -\nabla \frac{\rho}{\varepsilon} - \mu \partial_t \vec{j} \ . &
\end{aligned}$$

Magnetic field

$$\begin{aligned}
\Box \vec{b} = & \qquad (1) \\
= \Box \nabla \times \vec{a} = & \qquad (2) \\
= \nabla \times \Box \vec{a} = & \qquad (3) \\
= \mu \nabla \times \vec{j} &
\end{aligned}$$

[1] $\Box \partial_k f = \left(\frac{1}{c^2} \partial_{tt} - \partial_{ii} \right) \partial_k f = \partial_k \left(\frac{1}{c^2} \partial_{tt} - \partial_{ii} \right) f = \partial_k \Box f$.

3.3 Plane Electromagnetic Waves

Harmonic decomposition of the electromagnetic field. The EM field can be written as the superposition of plane waves (Fourier decomposition)

$$\begin{aligned}\mathbf{e}(\mathbf{r},t) &= \mathbf{E}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\\ \mathbf{b}(\mathbf{r},t) &= \mathbf{B}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\end{aligned}$$

Introducing this decomposition into Maxwell's equations with no free charge and current

$$\begin{cases}\nabla\cdot\mathbf{d}=0\\ \nabla\times\mathbf{e}+\partial_t\mathbf{b}=\mathbf{0}\\ \nabla\cdot\mathbf{b}=0\\ \nabla\times\mathbf{h}-\partial_t\mathbf{d}=\mathbf{0}\end{cases}$$

we obtain

$$\begin{cases}i\mathbf{k}\cdot\mathbf{D}=0\\ i\mathbf{k}\times\mathbf{E}-i\omega\mathbf{B}=\mathbf{0}\\ i\mathbf{k}\cdot\mathbf{B}=0\\ i\mathbf{k}\times\mathbf{H}+i\omega\mathbf{D}=\mathbf{0}\end{cases}\qquad\rightarrow\qquad\begin{cases}i\varepsilon\mathbf{k}\cdot\mathbf{E}=0\\ i\mathbf{k}\times\mathbf{E}-i\omega\mathbf{B}=\mathbf{0}\\ i\mathbf{k}\cdot\mathbf{B}=0\\ i\dfrac{1}{\mu}\mathbf{k}\times\mathbf{B}+i\omega\varepsilon\mathbf{E}=\mathbf{0}\end{cases}$$

- From Gauss' equations for the electric and the magnetic field

$$\mathbf{k}\perp\mathbf{E}\quad,\quad\mathbf{k}\perp\mathbf{B}$$

- From Faraday and Ampère-Maxwell equations

$$\mathbf{B}=\frac{\mathbf{k}}{\omega}\times\mathbf{E}$$

$$\mathbf{E}=-\frac{1}{\mu\varepsilon}\frac{\mathbf{k}}{\omega}\times\mathbf{B}$$

It follows that:

- $\mathbf{k}$, $\mathbf{E}$, $\mathbf{B}$ are orthogonal "RHS" set of vectors
- Relations between $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{k}$ and the speed of light

$$\begin{aligned}\mathbf{B} &= \frac{1}{c}\,\hat{\mathbf{k}}\times\mathbf{E}\\ \mathbf{E} &= -c\,\hat{\mathbf{k}}\times\mathbf{B}\end{aligned}$$

hold, with speed of light $c=\dfrac{1}{\sqrt{\mu\varepsilon}}=\dfrac{\omega}{|\mathbf{k}|}$, and unit vector $\hat{\mathbf{k}}=\dfrac{\mathbf{k}}{|\mathbf{k}|}$.

Proof using vector algebra identity

Recalling $c^2=\frac{1}{\mu\varepsilon}$ and

$$\mathbf{B}=\frac{\mathbf{k}}{\omega}\times\mathbf{E}=\frac{\mathbf{k}}{\omega}\times\left[-c^2\frac{\mathbf{k}}{\omega}\times\mathbf{B}\right]=-\frac{c^2|\mathbf{k}|^2}{\omega^2}\hat{\mathbf{k}}\times\left(\hat{\mathbf{k}}\times\mathbf{B}\right)$$

Vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \varepsilon_{ijk} a_j \varepsilon_{klm} b_l c_m = \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) a_j \, b_l \, c_m = b_i a_m c_m - c_i a_m b_m = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

applied to $\hat{\mathbf{k}} \times \left(\hat{\mathbf{k}} \times \mathbf{B}\right)$ gives

$$\hat{\mathbf{k}} \times \left(\hat{\mathbf{k}} \times \mathbf{B}\right) = \underbrace{\left(\hat{\mathbf{k}}\dot{\mathbf{B}}\right)}_{=0 \text{ since } \mathbf{k} \perp \mathbf{B}} \hat{\mathbf{k}} - \underbrace{\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}\right)}_{=1} \mathbf{B} = -\mathbf{B} \, ,$$

and the original relation gives

$$\mathbf{B} = \mathbf{B}\frac{c^2|\mathbf{k}|^2}{\omega^2} \, ,$$

and the relation between pulsation ω, wave vector $\mathbf{k}$ and speed of light (EM radiation) c,

$$c = \frac{\omega}{|\mathbf{k}|} \, .$$

3.3.1 Snell's Law at an Interface

Snell's law is derived here assuming isotropic linear media, so that

$$\begin{cases} \mathbf{d}(\mathbf{r},t) = \varepsilon \mathbf{e}(\mathbf{r},t) \\ \mathbf{b}(\mathbf{r},t) = \mu \mathbf{h}(\mathbf{r},t) \end{cases}$$

and for harmonic plane EM waves

$$\begin{cases} \mathbf{e}(\mathbf{r},t) = \mathbf{E}_a \, e^{i(\mathbf{k}_a\cdot\mathbf{r} - \omega t)} \\ \mathbf{b}(\mathbf{r},t) = \mathbf{B}_a \, e^{i(\mathbf{k}_a\cdot\mathbf{r} - \omega t)} \end{cases}$$

$$\begin{aligned} \mathbf{B}_a &= \frac{1}{c}\hat{\mathbf{k}}_a \times \mathbf{E}_a \\ \mathbf{E}_a &= -c\,\hat{\mathbf{k}}_a \times \mathbf{B}_a \end{aligned}$$

being index a representing the media involved: $a = 1$ for the medium with incident and reflected waves, $a = 2$ for the medium with the refracted wave.

Jump conditions of electromagnetic field at an interface with no charge or current surface density are given by conditions (2.2),

$$\begin{cases} \varepsilon_1 e_{n,1} = \varepsilon_2 e_{n,2} \\ e_{t_\alpha,1} = e_{t_\alpha,2} & , \quad \alpha = 1:2 \\ b_{n,1} = b_{n,2} \\ \dfrac{1}{\mu_1} b_{t_\alpha,1} = \dfrac{1}{\mu_2} b_{t_\alpha,2} & , \quad \alpha = 1:2 \end{cases}$$

Definition of some vectors: $\hat{\mathbf{n}}$ unit normal vector, $\mathbf{k}$ wave vector, $\hat{\mathbf{b}} = \dfrac{\hat{\mathbf{n}} \times \mathbf{k}}{|\hat{\mathbf{n}} \times \mathbf{k}|}$ (singular only for normal incident ray), $\hat{\mathbf{c}} = \dfrac{\hat{\mathbf{b}} \times \mathbf{k}}{|\hat{\mathbf{b}} \times \mathbf{k}|}$, $\hat{\mathbf{t}} = \dfrac{\hat{\mathbf{b}} \times \hat{\mathbf{n}}}{|\hat{\mathbf{b}} \times \hat{\mathbf{n}}|}$

Incident angle $\theta_{1,i}$ is the angle between $\hat{\mathbf{n}}$ and $\mathbf{k}$, s.t. $\hat{\mathbf{n}} \times \mathbf{k} = \hat{\mathbf{b}}\, k \, \sin\theta_{1,i}$.

$$\begin{cases} \hat{\mathbf{k}} = \cos\theta_{1,i}\hat{\mathbf{n}} + \sin\theta_{1,i}\hat{\mathbf{t}} \\ \hat{\mathbf{c}} = -\sin\theta_{1,i}\hat{\mathbf{n}} + \cos\theta_{1,i}\hat{\mathbf{t}} \end{cases} \quad , \quad \begin{cases} \hat{\mathbf{n}} = \cos\theta_{1,i}\hat{\mathbf{k}} - \sin\theta_{1,i}\hat{\mathbf{c}} \\ \hat{\mathbf{t}} = \sin\theta_{1,i}\hat{\mathbf{k}} + \cos\theta_{1,i}\hat{\mathbf{c}} \end{cases}$$

The electromagnetic field can be written as

$$\begin{aligned}
\mathbf{E} &= E_b \hat{\mathbf{b}} + E_c \hat{\mathbf{c}} = \\
&= E_b \hat{\mathbf{b}} - E_c \sin\theta_{1,i} \hat{\mathbf{n}} + E_c \cos\theta_{1,i} \hat{\mathbf{t}} \\
\mathbf{B} &= B_b \hat{\mathbf{b}} + B_c \hat{\mathbf{c}} = \\
&= \frac{E_c}{c} \hat{\mathbf{b}} - \frac{E_b}{c} \hat{\mathbf{c}} = \\
&= \frac{E_c}{c} \hat{\mathbf{b}} + \frac{E_b}{c} \sin\theta_{1,i} \hat{\mathbf{n}} - \frac{E_b}{c} \cos\theta_{1,i} \hat{\mathbf{t}} \, .
\end{aligned}$$

so that jump relations become

$$\begin{cases} b: & E_{b,1} = E_{b,2} \\ n: & \dots \\ t: & \dots \end{cases} \quad , \quad \begin{cases} b: & \dots \\ n: & \frac{E_{b,1}}{c_1} \sin\theta_{1,i} = \frac{E_{b,2}}{c_2} \sin\theta_{2,i} \\ t: & \dots \end{cases}$$

thus **Snell's law** follows

$$\frac{\sin\theta_{1,i}}{\sin\theta_{2,t}} = \frac{c_2}{c_1} = \frac{n_1}{n_2} \, .$$

Incident, Reflected, and Refracted Wave. The wave at the interface in medium 1 has the contribution of the incoming incident wave and the reflected one.

$$\begin{aligned}
\mathbf{e}_1(\mathbf{r}, t) &= \mathbf{e}_i(\mathbf{r}, t) + \mathbf{e}_r(\mathbf{r}, t) = \\
&= \mathbf{E}_i e^{i(\mathbf{k}_i \cdot \mathbf{r} - \omega t)} + \mathbf{E}_r e^{i(\mathbf{k}_r \cdot \mathbf{r} - \omega t)} = \\
&= \left(\mathbf{E}_i e^{i\mathbf{k}_i \cdot \mathbf{r}} + \mathbf{E}_r e^{i\mathbf{k}_r \cdot \mathbf{r}} \right) e^{-i\omega t}
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{k}_i &= k_{i,n} \hat{\mathbf{n}} + k_{i,t} \hat{\mathbf{t}} \\
\mathbf{k}_r &= k_{r,n} \hat{\mathbf{n}} + k_{r,t} \hat{\mathbf{t}}
\end{aligned}$$

At the interface, $\mathbf{r}_s \cdot \hat{\mathbf{n}} = 0$, and thus

$$\begin{aligned}
\mathbf{e}_1(\mathbf{r}_s, t) &= \left(\mathbf{E}_i e^{i k_{i,t} x_t} + \mathbf{E}_r e^{i k_{r,t} x_t} \right) e^{-i\omega t} \\
\mathbf{e}_2(\mathbf{r}_s, t) &= \mathbf{E}_t e^{i k_{t,t} x_t} e^{-i\omega t}
\end{aligned}$$

In order for the boundary conditions to be satisfied at all the points of the interface at each time,

$$k_{i,t} = k_{r,t} = k_{t,t} \, .$$

Exploiting the relation between the pulsation, the wave-length, and the speed of light in media, $c_a = \frac{\omega}{|\mathbf{k}_a|} = \frac{c}{n_a}$,

$$|\mathbf{k}_i| = |\mathbf{k}_r| \qquad \rightarrow \qquad k_{r,n} = -k_{i,n}$$

$$\frac{|\mathbf{k}_2|}{|\mathbf{k}_1|} = \frac{c_1}{c_2}$$

$$\frac{k_{t,t}^2 + k_{t,n}^2}{k_{i,t}^2 + k_{i,n}^2} = \frac{c_1^2}{c_2^2}$$

$$\begin{aligned}
k_{i,n} &= \ \ |\mathbf{k}_i| \cos\theta_i \\
k_{r,n} &= -|\mathbf{k}_r| \cos\theta_r \\
k_{t,n} &= \ \ |\mathbf{k}_t| \cos\theta_t
\end{aligned} \quad , \quad \begin{aligned}
k_{i,t} &= |\mathbf{k}_i| \sin\theta_i \\
k_{r,t} &= |\mathbf{k}_r| \sin\theta_r \\
k_{t,t} &= |\mathbf{k}_t| \sin\theta_t
\end{aligned}$$

$$\begin{cases} E_n: & \varepsilon_1\left(E_{i,c}\sin\theta_i + E_{r,c}\sin\theta_r\right) = \varepsilon_2 E_{t,c}\sin\theta_t \\ E_t: & E_{i,c}\cos\theta_i - E_{r,c}\cos\theta_r = E_{t,c}\cos\theta_t \\ E_b: & E_{i,b} + E_{r,b} = E_{t,b} \\ B_n: & B_{i,c}\sin\theta_i + B_{r,c}\sin\theta_r = B_{t,c}\sin\theta_t \\ B_t: & \frac{1}{\mu_1}\left(B_{i,c}\cos\theta_i - B_{r,c}\cos\theta_r\right) = \frac{1}{\mu_2}B_{t,c}\cos\theta_t \\ B_b: & \frac{1}{\mu_1}\left(B_{i,b} + B_{r,b}\right) = \frac{1}{\mu_2}B_{t,b} \end{cases}$$

Writing the magnetic field as a function of the wave-vector and the magnetic field, it's possible to write 2 decoupled systems of equations

$$\begin{cases} E_n: & \varepsilon_1\left(E_{i,c}\sin\theta_i + E_{r,c}\sin\theta_r\right) = \varepsilon_2 E_{t,c}\sin\theta_t \\ E_t: & E_{i,c}\cos\theta_i - E_{r,c}\cos\theta_r = E_{t,c}\cos\theta_t \\ B_b: & \frac{1}{\mu_1}\left(\frac{E_{i,c}}{c_1} + \frac{E_{r,c}}{c_1}\right) = \frac{1}{\mu_2}\frac{E_{t,c}}{c_2} \end{cases}$$

$$\begin{cases} E_b: & E_{i,b} + E_{r,b} = E_{t,b} \\ B_n: & \frac{E_{i,b}}{c_1}\sin\theta_i + \frac{E_{r,b}}{c_1}\sin\theta_r = \frac{E_{t,b}}{c_2}\sin\theta_t \\ B_t: & \frac{1}{\mu_1}\left(\frac{E_{i,b}}{c_1}\cos\theta_i - \frac{E_{r,b}}{c_1}\cos\theta_r\right) = \frac{1}{\mu_2}\frac{E_{t,b}}{c_2}\cos\theta_t \end{cases}$$

The equations E_n and B_b are equivalent; E_b and B_n are equivalent as well, because of Snell's law. Thus, defining

$$\begin{aligned} r_c &:= \frac{E_{r,c}}{E_{i,c}} & r_b &:= \frac{E_{r,b}}{E_{i,b}} \\ t_c &:= \frac{E_{t,c}}{E_{i,c}} & t_b &:= \frac{E_{t,b}}{E_{i,b}} \end{aligned},$$

and $\alpha_i := \frac{1}{\mu_i c_i}$. These systems of equations can be written as two uncoupled linear systems of equations,

(for P-polarization **todo** *change index from c to p; for S-polarization **todo** *change index from b to s*)

$$\begin{cases} E_t: & \cos\theta_i - \cos\theta_r\, r_c = \cos\theta_t\, t_c \\ B_b: & \alpha_1 + \alpha_1\, r_c = \alpha_2\, t_c \end{cases}$$

$$\begin{cases} E_b: & 1 + r_b = t_b \\ B_t: & \alpha_1\, \cos\theta_i - \alpha_1\, \cos\theta_r\, r_b = \alpha_2\, \cos\theta_t\, t_b \end{cases}$$

Calling $\theta_i = \theta_r = \theta_1$, $\theta_2 = \theta_t$, these linear systems can be written using matrix formalism,

$$\begin{bmatrix} -1 & 1 \\ 1 & \frac{\alpha_2}{\alpha_1}\frac{\cos\theta_2}{\cos\theta_1} \end{bmatrix}\begin{bmatrix} r_b \\ t_b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{\cos\theta_2}{\cos\theta_1} \\ -1 & \frac{\alpha_2}{\alpha_1} \end{bmatrix}\begin{bmatrix} r_c \\ t_c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

todo *Analysis of the total reflection, forcing $t_x = 0$. Check signs before*

$$\begin{bmatrix} 1 & \frac{\cos\theta_2}{\cos\theta_1} \\ -1 & \frac{\alpha_2}{\alpha_1} \end{bmatrix}\begin{bmatrix} r_c \\ t_c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \rightarrow \qquad \begin{bmatrix} r_c \\ t_c \end{bmatrix} = \frac{1}{\frac{\alpha_2}{\alpha_1} + \frac{\cos\theta_2}{\cos\theta_1}}\begin{bmatrix} \frac{\alpha_2}{\alpha_1} & -\frac{\cos\theta_2}{\cos\theta_1} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\alpha_2\cos\theta_1 - \alpha_1\cos\theta_2}{\alpha_2\cos\theta_1 + \alpha_1\cos\theta_2} \\ \frac{2\alpha_1\cos\theta_1}{\alpha_2\cos\theta_1 + \alpha_1\cos\theta_2} \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & \frac{\alpha_2}{\alpha_1}\frac{\cos\theta_2}{\cos\theta_1} \end{bmatrix}\begin{bmatrix} r_b \\ t_b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \rightarrow \qquad \begin{bmatrix} r_b \\ t_b \end{bmatrix} = \frac{1}{-\frac{\alpha_2}{\alpha_1}\frac{\cos\theta_2}{\cos\theta_1} - 1}\begin{bmatrix} \frac{\alpha_2}{\alpha_1}\frac{\cos\theta_2}{\cos\theta_1} & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\alpha_1\cos\theta_1 - \alpha_2\cos\theta_2}{\alpha_1\cos\theta_1 + \alpha_2\cos\theta_2} \\ \frac{2\alpha_1\cos\theta_1}{\alpha_1\cos\theta_1 + \alpha_2\cos\theta_2} \end{bmatrix}$$

that can be recast with the wave impedance Z,

$$\alpha_1 = \frac{1}{\mu_1 c_1} = \frac{\sqrt{\mu_1\varepsilon_1}}{\mu_1} = \sqrt{\frac{\varepsilon_1}{\mu_1}} =: \frac{1}{Z_1}\,,$$

$$\begin{bmatrix} r_c \\ t_c \end{bmatrix} = \begin{bmatrix} \frac{Z_1 \cos\theta_1 - Z_2 \cos\theta_2}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2} \\ \frac{2 Z_2 \cos\theta_1}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2} \end{bmatrix}$$

$$\begin{bmatrix} r_b \\ t_b \end{bmatrix} = \begin{bmatrix} \frac{Z_2 \cos\theta_1 - Z_1 \cos\theta_2}{Z_2 \cos\theta_1 + Z_1 \cos\theta_2} \\ \frac{2 Z_2 \cos\theta_1}{Z_2 \cos\theta_1 + Z_1 \cos\theta_2} \end{bmatrix}$$

Energy Balance and Transmission Coefficients. Energy balance for a domain collapsing on the interface reduces to power flux balance, namely

$$\oint_{\partial V} \mathbf{s} \cdot \hat{\mathbf{n}} = 0 \ ,$$

with $\mathbf{s} = \mathbf{e} \times \mathbf{h}$ the Poynting vector. For harmonic plane waves,

$$\begin{aligned}
\mathbf{s}(\mathbf{r},t) & = \mathbf{e}(\mathbf{r},t) \times \mathbf{h}(\mathbf{r},t) = \\
& = \frac{1}{\mu} \left[\mathbf{E} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{E}^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \right] \times \left[\mathbf{B} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{B}^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \right] = \\
& = \frac{1}{\mu} \left[\mathbf{E} \times \mathbf{B} \, e^{i2(\mathbf{k}\cdot\mathbf{r}-\omega t)} + c.c. \right] + \frac{1}{\mu} \left[\mathbf{E} \times \mathbf{B}^* + c.c. \right] = \\
& = \dots + \frac{1}{\mu} \mathbf{E} \times \left(\frac{1}{c} \hat{\mathbf{k}} \times \mathbf{E} \right)^* = \\
& = \dots + \frac{1}{\mu c} \left(\mathbf{E} \cdot \mathbf{E}^* \right) \hat{\mathbf{k}} = \\
& = \dots + \frac{1}{\mu c} |\mathbf{E}|^2 \hat{\mathbf{k}} \ . & = \dots + \alpha |\mathbf{E}|^2 \hat{\mathbf{k}} \ .
\end{aligned}$$

For each one of the two polarizations, the following holds ($\cos\theta$ comes from the dot product $\hat{k} \cdot \hat{n}$ appearing in the surface integral),

$$\alpha_1 \cos\theta_1 = \alpha_1 r_x^2 \cos\theta_1 + \alpha_2 t_x^2 \cos\theta_2 \ ,$$

i.e., the sum of reflected and transmitted power equals the incident power.

Proof of the power balance, for P-polarization

todo Here P is index c

Dividing by $\alpha_1 \cos\theta_1$

$$\begin{aligned}
& \frac{1}{\alpha_1 \cos\theta_1} \left(\alpha_1 r_p^2 \cos\theta_1 + \alpha_2 t_p^2 \cos\theta_2 \right) = \\
& = \frac{\left(\alpha_1 \cos\theta_1 - \alpha_2 \cos\theta_2\right)^2}{\left(\alpha_1 \cos\theta_1 + \alpha_2 \cos\theta_2\right)^2} + \frac{\alpha_2 \cos\theta_2}{\alpha_1 \cos\theta_1} \frac{\left(2\alpha_1 \cos\theta_1\right)^2}{\left(\alpha_1 \cos\theta_1 + \alpha_2 \cos\theta_2\right)^2} = \\
& = \frac{1}{\left(\alpha_1 \cos\theta_1 + \alpha_2 \cos\theta_2\right)^2} \left[\alpha_1^2 \cos^2\theta_1 - 2\alpha_1\alpha_2 \cos\theta_1 \cos\theta_2 + \alpha_2^2 \cos^2\theta_2 + 4\alpha_1\alpha_2 \cos\theta_1 \cos\theta_2 \right] = \\
& = 1 \ .
\end{aligned}$$

CHAPTER

FOUR

FORCE, MOMENTS, ENERGY AND MOMENTUM IN ELECTROMAGNETISM

In this section, forces and moments on charges immersed in an electromagnetic field and the energy and momentum of the electromagnetic field are discussed.

Total energy and momentum of a system involving electromagnetic phenomena has contributions from charges, currents and the electromagnetic field.

Forces, moments and power. Forces and moments acting of elementary charge systems immersed in an electromagnetic field are evaluated and power of these actions are riscussed.

Energy and momentum balance equations of the electromagnetic field. Energy balance equation of the electromagnetic field

4.1 Force, moment, and power on elementary charge distributions

4.1.1 Force, moment and power on a point electric charge

Point electric charge with charge q in a point $\vec{r}_P(t)$ at time t where electromagnetic field is $\vec{e}(\vec{r},t)$, $\vec{b}(\vec{r},t)$:

- Lorentz's force

$$\vec{F} = q \left(\vec{e}(\vec{r}_P(t),t) - \vec{b}(\vec{r}_P(t),t) \times \vec{v}_P(t) \right) \ ,$$

- zero moment, since it has no dimension (and assumed uniform or symmetric or… distribution of electric charge)
- power

$$\begin{aligned} P & = \vec{v}_P(t) \cdot \vec{F} = \\ & = \vec{v}_P(t) \cdot q \left(\vec{e}(\vec{r}_P(t),t) - \vec{b}(\vec{r}_P(t),t) \times \vec{v}_P(t) \right) = q \, \vec{v}_P(t) \cdot \vec{e}(\vec{r}_P(t),t) \ . \end{aligned}$$

4.1.2 Force, moment and power on a electric dipole

Electric dipole with center $\vec{r}_C(t)$, axis $\vec{\ell}$, so that the positive charge q is in $P_+ = C + \dfrac{\vec{\ell}}{2}$ and the negative charge is in $P_- = C - \dfrac{\vec{\ell}}{2}$, with $q \rightarrow +\infty$, $|\vec{\ell}| \rightarrow 0$, s.t. $q|\vec{\ell}| = |\vec{d}|$ finite.

Kinematics and expansion of the field

$$\vec{v}_\pm = \vec{v}_C \pm \vec{\omega} \times \frac{\vec{\ell}}{2}$$

$$\vec{e}(P_\pm) = \vec{e}\left(C \pm \frac{\vec{\ell}}{2}\right) = \vec{e}(C) \pm \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}(C) + o(|\vec{\ell}|)$$
$$\vec{b}(P_\pm) = \vec{b}\left(C \pm \frac{\vec{\ell}}{2}\right) = \vec{b}(C) \pm \frac{\vec{\ell}}{2} \cdot \nabla \vec{b}(C) + o(|\vec{\ell}|)$$

Net force.

$$\begin{aligned}
\vec{F} = \vec{F}_+ + \vec{F}_- = \\
&= q\left[\vec{e}(P_+) - \vec{b}(P_+) \times \vec{v}_+\right] - q\left[\vec{e}(P_-) - \vec{b}(P_-) \times \vec{v}_-\right] = \\
&= q\left[\vec{e}_C + \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}_C - \left(\vec{b}_C + \frac{\vec{\ell}}{2} \cdot \nabla \vec{b}_C\right) \times \left(\vec{v}_C + \vec{\omega} \times \frac{\vec{\ell}}{2}\right)\right] + \\
&- q\left[\vec{e}_C - \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}_C - \left(\vec{b}_C - \frac{\vec{\ell}}{2} \cdot \nabla \vec{b}_C\right) \times \left(\vec{v}_C - \vec{\omega} \times \frac{\vec{\ell}}{2}\right)\right] = \\
&= q\vec{\ell} \cdot \nabla \vec{e}(C) - \left(q\vec{\ell} \cdot \nabla \vec{b}(C)\right) \times \vec{v}_C + \vec{b}(C) \times \left(\vec{\omega} \times q\vec{\ell}\right) + o(|\vec{\ell}|)
\end{aligned}$$

Net moment, w.r.t. C.

$$\begin{aligned}
\vec{M}_C = \frac{\vec{\ell}}{2} \times \vec{F}_+ - \frac{\vec{\ell}}{2} \times \vec{F}_- = \\
&= q\frac{\vec{\ell}}{2} \times \left[\vec{e}(P_+) - \vec{b}(P_+) \times \vec{v}_+\right] + q\frac{\vec{\ell}}{2} \times \left[\vec{e}(P_-) - \vec{b}(P_-) \times \vec{v}_-\right] = \\
&= q\frac{\vec{\ell}}{2} \times \left[\vec{e}_C + \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}_C - \left(\vec{b}_C + \frac{\vec{\ell}}{2} \cdot \nabla \vec{b}_C\right) \times \left(\vec{v}_C + \vec{\omega} \times \frac{\vec{\ell}}{2}\right)\right] + \\
&+ q\frac{\vec{\ell}}{2} \times \left[\vec{e}_C - \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}_C - \left(\vec{b}_C - \frac{\vec{\ell}}{2} \cdot \nabla \vec{b}_C\right) \times \left(\vec{v}_C - \vec{\omega} \times \frac{\vec{\ell}}{2}\right)\right] = \\
&= q\vec{\ell} \times \left[\vec{e}_C - \vec{b}_C \times \vec{v}_C\right] + o(|\vec{\ell}|) \ .
\end{aligned}$$

Power.

$$\begin{aligned}
P = P_+ + P_- = \\
&= \vec{F}_+ \cdot \vec{v}_+ + \vec{F}_- \cdot \vec{v}_- = \\
&= q\ \left[\vec{e}(P_+) - \vec{b}(P_+) \times \vec{v}_+\right] \cdot \vec{v}_+ - q\ \left[\vec{e}(P_-) - \vec{b}(P_-) \times \vec{v}_-\right] \cdot \vec{v}_- = \\
&= q\,\vec{e}(P_+) \cdot \vec{v}_+ - q\,\vec{e}(P_-) \cdot \vec{v}_- = \\
&= q\ \left[\vec{e}_C + \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}_C\right] \cdot \left[\vec{v}_C + \vec{\omega} \times \frac{\vec{\ell}}{2}\right] - q\ \left[\vec{e}_C - \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}_C\right] \cdot \left[\vec{v}_C - \vec{\omega} \times \frac{\vec{\ell}}{2}\right] = \\
&= \vec{e}_C \cdot \left(\vec{\omega} \times q\vec{\ell}\right) + \left(q\vec{\ell} \cdot \nabla \vec{e}_C\right) \cdot \vec{v}_C + o(|\vec{\ell}|^2) \ .
\end{aligned}$$

4.1.3 Force, moment and power on a magnetic dipole

On an elementary magnetic dipole, modeled as a “small” circuit with current i enclosing area S and center C, with $S \to 0$, $i \to +\infty$ so that $iS\hat{n} := \vec{m}$ finite

Force.

$$\dots$$

$$\vec{F} = \nabla \vec{b}(C) \cdot \vec{m}$$

Moment.

$$\dots$$

$$\vec{M}_C = \vec{m} \times \vec{b}(C)$$

Power.

$$P = \vec{v}_C \cdot \nabla \vec{b}(C) \cdot \vec{m} + \vec{\omega} \cdot \vec{m} \times \vec{b}(C) \ .$$

4.1.4 Energy balance

todo *Check and put charges, currents, and dipoles together with the electromagnetic field*

Ispirati dalle dimensioni fisiche dei campi elettromagnetici,

$$\begin{aligned}
[\mathbf{e}] &= \frac{\text{force}}{\text{charge}} \qquad , \qquad [\mathbf{d}] = \frac{\text{charge}}{\text{length}^2} \\
[\mathbf{b}] &= \frac{\text{force} \cdot \text{time}}{\text{charge} \cdot \text{length}} \qquad , \qquad [\mathbf{h}] = \frac{\text{charge}}{\text{time} \cdot \text{length}}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{e} \cdot \mathbf{d}] &= \frac{\text{force}}{\text{length}^2} = \frac{\text{energy}}{\text{length}^3} = [u] \\
[\mathbf{b} \cdot \mathbf{h}] &= \frac{\text{force}}{\text{length}^2} = \frac{\text{energy}}{\text{length}^3} = [u]
\end{aligned}$$

si può costruire la densità di volume di energia (**todo** trovare motivazioni più convincenti, non basandosi solo sull'analisi dimensionale ma sul lavoro)

$$u = \frac{1}{2} \left(\mathbf{e} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{h} \right) \ .$$

Si può calcolare la derivata parziale nel tempo della densità di energia, u, e usare le equazioni di Maxwell per ottenere un'equazione di bilancio dell'energia del campo elettromagnetico. Per un mezzo isotropo lineare, per il quale valgono le equazioni costitutive $\mathbf{d} = \varepsilon \mathbf{e}$, $\mathbf{b} = \mu \mathbf{h}$, la derivata parziale nel tempo dell'energia elettromagnetica può essere riscritta sfuttando la regola di derivazione del prodotto e le equazioni di Faraday-Lenz-Neumann e Ampére-Maxwell,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{1}{2} \mathbf{e} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{h} \right) = \qquad (\dots) \\
&= \mathbf{e} \cdot \partial_t \mathbf{d} + \mathbf{h} \cdot \partial_t \mathbf{b} = \\
&= \mathbf{e} \cdot (\nabla \times \mathbf{h} - \mathbf{j}) - \mathbf{h} \cdot \nabla \times \mathbf{e} \ .
\end{aligned}$$

L'ultimo termine può essere ulteriormente manipolato, usando l'identità vettoriale

$$\begin{aligned}
\mathbf{e} \cdot \nabla \times \mathbf{h} - \mathbf{h} \cdot \nabla \times \mathbf{e} &= e_i \varepsilon_{ijk} \partial_j h_k - h_i \varepsilon_{ijk} \partial_j e_k = \qquad (i \rightarrow k, k \rightarrow i) \\
&= e_i \varepsilon_{ijk} \partial_j h_k - h_k \varepsilon_{kji} \partial_j e_i = \\
&= e_i \varepsilon_{ijk} \partial_j h_k + h_k \varepsilon_{ijk} \partial_j e_i = \\
&= \partial_j (\varepsilon_{ijk} e_i h_k) = \\
&= \partial_j (\varepsilon_{jki} e_i h_k) = \\
&= \nabla \cdot (\mathbf{h} \times \mathbf{e}) = -\nabla \cdot (\mathbf{e} \times \mathbf{h})
\end{aligned}$$

che permette di scrivere l'equazione del bilancio di energia elettromagnetica come,

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{s} = -\mathbf{e} \cdot \mathbf{j} \ ,$$

dove è stato definito il **vettore di Poynting**, o meglio il campo vettoriale di Poynting,

$$\mathbf{s}(\mathbf{r},t) := \mathbf{e}(\mathbf{r},t) \times \mathbf{h}(\mathbf{r},t) \ ,$$

che può essere identificato come un flusso di potenza per unità di superficie, comparendo sotto l'operatore di divergenza nel bilnacio di energia.

todo. Rimandare a una sezione in cui si mostra questa ultima affermazione passando dal bilancio differenziale al bilancio integrale e si usa il teorema della divergenza, $\int_V \nabla \cdot \mathbf{s} = \oint_{\partial V} \mathbf{s} \cdot \hat{\mathbf{n}}$.

Bilancio di energia di cariche nel vuoto, o i materiali senza polarizzazione o magnetizzazione

Moto di cariche puntiformi. L'equazione del moto di carica puntiforme q_k nella posizione $\mathbf{r}_k(t)$ al tempo t è

$$m_k \ddot{\mathbf{r}}_k = \mathbf{f}_k + \mathbf{f}_k^{em} \ ,$$

avendo riconosciuto i contributi di forza dovuti al campo elettromagnetico come $\mathbf{f}_k^{em}$ dagli altri. L'espressione della forza dovuta al campo elettromagnetico sulla carica k è data dalla forza di Lorentz,

$$\mathbf{f}_k^{em}(t) = q_k \left[\mathbf{e}(\mathbf{r_k}(t),t) - \mathbf{b}(\mathbf{r}_k(t),t) \times \dot{\mathbf{r}}_k(t) \right]$$

Continuità della carica elettrica. La densità di carica e di corrente elettrica di un insieme di cariche libere puntiformi macroscopiche può essere scritta come

$$\begin{aligned}
\rho(\mathbf{r},t) &= \sum_k q_k \delta(\mathbf{r} - \mathbf{r}_k(t)) \\
\mathbf{j}(\mathbf{r},t) &= \sum_k q_k \dot{\mathbf{r}}_k(t) \delta(\mathbf{r} - \mathbf{r}_k(t)) \ .
\end{aligned}$$

L'equazione di continuità della carica, $\partial_t \rho + \nabla \cdot \mathbf{j} = 0$, risulta quindi soddisfatta,

$$\begin{aligned}
\partial_t \rho &= -\sum_k q_k \, \partial_i \delta(\mathbf{r} - \mathbf{r}_k(t)) \, \dot{r}_{k,i} \\
\partial_i j_i &= \sum_k q_k \, \dot{r}_{k,i} \, \partial_i \delta(\mathbf{r} - \mathbf{r}_k(t))
\end{aligned}$$

Procedimento alternativo (e più generale?)

todo *In caso questo procedimento sia più generale, o più corretto, sostituire il procedimento precedente.*

La carica elementare in un volumetto ΔV è data da dal prodotto tra il volume e la densità volumetrica di carica, $\rho \Delta V$; la velocità media locale della carica elettrica è $\mathbf{v}$; la forza agente sulla carica elementare immersa in un campo elettromagnetico è determinata dalla formula di Lorentz, $\mathbf{f} \Delta V = \Delta V \rho \left(\mathbf{e} - \mathbf{b} \times \mathbf{v} \right)$. La potenza di questa forza è il prodotto scalare con la velocità media delle cariche, $\Delta V \mathbf{f} \cdot \mathbf{v}$

La potenza del campo elettromagnetico sul moto della carica elettrica per unità di volume è quindi

$$\mathbf{v} \cdot \mathbf{f} = \rho \mathbf{v} \cdot (\mathbf{e} - \mathbf{b} \times \mathbf{v}) = \rho \mathbf{v} \cdot \mathbf{e} = \mathbf{j} \cdot \mathbf{e} \ .$$

todo

- discutere questo termine del bilancio di energia cinetica nel moto della carica elettrica
- questo termine compare con segno opposto nel bilancio dell'energia elettromagnetica del sistema
- dove compare la non-conservatività del problema in presenza di materiali dissipativi (come resistenza elettrica con $\mathbf{e} = \rho_R \mathbf{j}$?

Il termine $\mathbf{e}\cdot\mathbf{j}$ può essere manipolato usando le equazioni di Maxwell, e le relazioni

$$\begin{cases}\mathbf{d} = \varepsilon_0 \mathbf{e} + \mathbf{p} \\ \mathbf{h} = \frac{\mathbf{b}}{\mu_0} - \mathbf{m}\end{cases}$$

$$\begin{aligned}\mathbf{e}\cdot\mathbf{j} = \mathbf{e}\cdot(\nabla\times\mathbf{h} - \partial_t\mathbf{d}) = & \\ = -\nabla\cdot(\mathbf{e}\times\mathbf{h}) + \mathbf{h}\cdot\nabla\times\mathbf{e} - \mathbf{e}\cdot\partial_t\mathbf{d} = & \\ = -\nabla\cdot(\mathbf{e}\times\mathbf{h}) - \mathbf{h}\cdot\partial_t\mathbf{b} - \mathbf{e}\cdot\partial_t\mathbf{d} &\end{aligned}$$

Gli ultimi due termini possono essere manipolati in diverse maniere,

$$\begin{aligned}\mathbf{e}\cdot\partial_t\mathbf{d} = \mathbf{e}\cdot\partial_t\left(\varepsilon_0\mathbf{e}+\mathbf{p}\right) &= \partial_t\left(\frac{1}{2}\varepsilon_0\mathbf{e}\cdot\mathbf{e}\right) + \mathbf{e}\cdot\partial_t\mathbf{p} \\ &= \partial_t\left(\frac{1}{2}\mathbf{e}\cdot\mathbf{d}\right) + \frac{1}{2}\left(\mathbf{e}\cdot\partial_t\mathbf{p} - \mathbf{p}\cdot\partial_t\mathbf{e}\right) \\ &= \partial_t\left(\frac{1}{2\varepsilon_0}\mathbf{d}\cdot\mathbf{d}\right) - \frac{\mathbf{p}}{\varepsilon_0}\cdot\partial_t\mathbf{d}\end{aligned}$$

$$\begin{aligned}\mathbf{h}\cdot\partial_t\mathbf{b} = \mathbf{h}\cdot\partial_t\left(\mu_0\mathbf{h}+\mu_0\mathbf{m}\right) &= \partial_t\left(\frac{1}{2}\mu_0\mathbf{h}\cdot\mathbf{h}\right) + \mu_0\mathbf{h}\cdot\partial_t\mathbf{m} \\ &= \partial_t\left(\frac{1}{2}\mathbf{b}\cdot\mathbf{h}\right) + \frac{1}{2}\mu_0\left(\mathbf{h}\cdot\partial_t\mathbf{m} - \mathbf{m}\cdot\partial_t\mathbf{h}\right) \\ &= \partial_t\left(\frac{1}{2\mu_0}\mathbf{b}\cdot\mathbf{b}\right) - \mathbf{m}\cdot\partial_t\mathbf{b}\end{aligned}$$

Nel vuoto o in mezzi lineari $\mathbf{e}\cdot\partial_t\mathbf{p} - \mathbf{p}\cdot\partial_t\mathbf{e} = \mathbf{0}$, $\mathbf{h}\cdot\partial_t\mathbf{m} - \mathbf{m}\cdot\partial_t\mathbf{h} = \mathbf{0}$. Usando le seconde espressioni, si può riscrivere l'equazione dell'energia del campo elettromagnetico come

$$\begin{aligned}\partial_t\left(\frac{1}{2}\mathbf{e}\cdot\mathbf{d} + \frac{1}{2}\mathbf{b}\cdot\mathbf{h}\right) + \nabla\cdot(\mathbf{e}\times\mathbf{h}) = & - \mathbf{e}\cdot\mathbf{j} + \\ & - \frac{1}{2}\left[\mathbf{e}\cdot\partial_t\mathbf{p} - \mathbf{p}\cdot\partial_t\mathbf{e} + \mu_0\left(\mathbf{h}\cdot\partial_t\mathbf{m} - \mathbf{m}\cdot\partial_t\mathbf{h}\right)\right]\end{aligned}$$

o, usando le definizioni di densità di energia elettromagnetica u e vettore di Poynting $\mathbf{s}$,

$$\partial_t u + \nabla\cdot\mathbf{s} = -\,\mathbf{e}\cdot\mathbf{j}\; - \frac{1}{2}\left[\mathbf{e}\cdot\partial_t\mathbf{p} - \mathbf{p}\cdot\partial_t\mathbf{e} + \mu_0\left(\mathbf{h}\cdot\partial_t\mathbf{m} - \mathbf{m}\cdot\partial_t\mathbf{h}\right)\right]$$

4.2 Energy and momentum balance in linear, local, isotropic, non-dispersive media

4.2.1 Energy equation in differential form

In this section balance equations for the energy and the momentum of the system are derived for a linear, local, isotropic, homogeneous,… systems.

Power per unit volume of the Lorentz' force per unit volume acting on a charge distribution $\rho(\vec{r},t)$ with electric current density $\vec{j}(\vec{r},t)$ is

$$\begin{aligned}p(\vec{r},t) = \vec{f}(\vec{r},t)\cdot\vec{v}(\vec{r},t) = & \\ = \left[\rho(\vec{r},t)\,\vec{e}(\vec{r},t) - \vec{b}(\vec{r},t)\times\vec{v}(\vec{r},t)\right]\cdot\vec{v}(\vec{r},t) = & \\ = \rho(\vec{r},t)\,\vec{e}(\vec{r},t)\cdot\vec{v}(\vec{r},t) = & \\ = \vec{j}(\vec{r},t)\cdot\vec{e}(\vec{r},t)\;. &\end{aligned}$$

Total charge and current. Energy equation for **total charge and current**

$$\begin{aligned}
\vec{j}\cdot\vec{e} &= && (1)\\
&= \frac{1}{\mu_0}\left(\nabla\times\vec{b}-\varepsilon_0\partial_t\vec{e}\right)\cdot\vec{e} = && (2)\\
&= \nabla\cdot\left(\frac{\vec{b}\times\vec{e}}{\mu_0}\right)+\frac{1}{\mu_0}\vec{b}\cdot\nabla\times\vec{e}-\varepsilon_0\partial_t\vec{e}\cdot\vec{e} = && (3)\\
&= -\nabla\cdot\vec{s}-\frac{1}{\mu_0}\vec{b}\cdot\partial_t\vec{b}-\varepsilon_0\partial_t\vec{e}\cdot\vec{e}\,,
\end{aligned} \tag{4.1}$$

using (1) Ampère-Maxwell's equation, (2) identity $\nabla\times\vec{b}\cdot\vec{e}=\nabla\cdot\left(\vec{b}\times\vec{e}\right)+\vec{b}\cdot\nabla\times\vec{e}$[1], (3) Faraday's law, and introducing the definition of the Poynting vector

$$\vec{s} := \frac{\vec{e}\times\vec{b}}{\mu_0}\,. \tag{4.2}$$

Using the identity, $\vec{v}\cdot\partial_t\vec{v}=\partial_t\frac{|\vec{v}|^2}{2}$, energy equation (4.1) becomes

$$\partial_t u+\nabla\cdot\vec{s}=-\vec{j}\cdot\vec{e}\,, \tag{4.3}$$

with the energy volume density,

$$u := \frac{1}{2}\left(\varepsilon_0\vec{e}\cdot\vec{e}+\frac{1}{\mu_0}\vec{b}\cdot\vec{b}\right)\,. \tag{4.4}$$

Polarization current.

$$\begin{aligned}
\vec{j}_P\cdot\vec{e} &= \\
&= \partial_t\vec{p}\cdot\vec{e}
\end{aligned}$$

Magnetization current.

$$\begin{aligned}
\vec{j}_M\cdot\vec{e} &= \\
&= \nabla\times\vec{m}\cdot\vec{e}\\
&= \nabla\cdot\left(\vec{m}\times\vec{e}\right)+\vec{m}\cdot\nabla\times\vec{e}\\
&= \nabla\cdot\left(\vec{m}\times\vec{e}\right)-\vec{m}\cdot\partial_t\vec{b}
\end{aligned}$$

[1]

$$\begin{aligned}
\nabla\times\vec{h}\cdot\vec{e} &= e_i\varepsilon_{ijk}\partial_j h_k = \\
&= \varepsilon_{ijk}\partial_j\left(e_i h_k\right)-h_k\varepsilon_{ijk}\partial_j e_i = \\
&= \partial_j\left(\varepsilon_{jki}h_k e_i\right)+h_k\varepsilon_{kji}\partial_j e_i = \\
&= \nabla\cdot\left(\vec{h}\times\vec{e}\right)+\vec{h}\cdot\nabla\times\vec{e}\,.
\end{aligned}$$

Free current.

$$\begin{aligned}\vec{j}_f \cdot \vec{e} = & \\ & = \left(\nabla \times \vec{h} - \partial_t \vec{d}\right) \cdot \vec{e} \\ & = \nabla \cdot \left(\vec{h} \times \vec{e}\right) + \vec{h} \cdot \nabla \times \vec{e} - \partial_t \vec{d} \cdot \vec{e} = \\ & = -\nabla \cdot \vec{S} - \vec{h} \cdot \partial_t \vec{b} - \partial_t \vec{d} \cdot \vec{e}\end{aligned} \tag{4.5}$$

4.2.2 Energy equation in integral form - control volumes

Integral form of energy equation for a control volume V can be derived integrating the differential balance equation (4.3) over V,

$$\frac{d}{dt}\int_V u + \int_V \vec{e} \cdot \vec{j} = -\oint_{\partial V} \hat{n} \cdot \vec{s}\,, \tag{4.6}$$

having used the divergence theorem to transform volume integral of the divergence of Poynting vector into a flux integral across the boundary ∂V of the domain, and exploited the indepndence of V from time to take the time derivative outside the integral (see reuls for integration over time-depending domains).

Interpretation. This equation has an immediate interpretation in terms of energy of the system and power (dissipated? and exchanged with the external environemnt) **todo** *discuss*

This equation can be recast in different forms. One of them is particurarly useful later in this material to discuss energy balance in different regimes of electromagnetic systems and in circuit approximation and discuss the validity of the circuit approximation itself. Manipulating the surface contribution, the energy equation (4.6) can be recast as

$$\frac{d}{dt}\int_V u + \int_V \vec{e} \cdot \vec{j} = -\oint_{\partial V} \phi \vec{j} \cdot \hat{n} + \oint_{\partial V} \hat{n} \cdot \left[\frac{1}{\mu_0}\partial_t \vec{a} \times \vec{b} + \varepsilon_0\,\phi\,\partial_t \vec{e}\right]\,, \tag{4.7}$$

highlighting two contributions in the surface term:

- the first contribution can be recast as the common power flux at ports of circuits used in circuit approximations,

$$-\oint_{\partial V} \phi\,\vec{j} \cdot \hat{n} = \sum_{k \in \text{wires}} v_k i_k\,,$$

- the second contribution is often negligible in electromagnetic systems with **low characteristic frequencies** and **non-large-scale** systems, as it will be discussed **todo** *add link*

Boundary contribution to electromagnetic energy

$$\begin{aligned}\oint_{\partial V} \hat{n} \cdot \vec{s} = & \frac{1}{\mu_0}\oint_{\partial V} \hat{n} \cdot \vec{e} \times \vec{b} = \\ & = \frac{1}{\mu_0}\oint_{\partial V} \hat{n} \cdot (-\partial_t \vec{a} - \nabla \phi) \times \vec{b} = \\ & = -\frac{1}{\mu_0}\oint_{\partial V} \hat{n} \cdot \left(\partial_t \vec{a} \times \vec{b} + \nabla \times (\phi \vec{b}) - \phi \nabla \times \vec{b}\right) = \\ & = -\frac{1}{\mu_0}\oint_{\partial V} \hat{n} \cdot \left(\partial_t \vec{a} \times \vec{b} - \phi\,\left(\mu_0 \vec{j} + \varepsilon_0 \mu_0\,\partial_t \vec{e}\right)\right) = \\ & = \oint_{\partial V} \phi\,\vec{j} \cdot \hat{n} - \oint_{\partial V} \hat{n} \cdot \left[\frac{1}{\mu_0}\partial_t \vec{a} \times \vec{b} + \varepsilon_0\,\phi\,\partial_t \vec{e}\right]\,,\end{aligned}$$

where the integral of the flux of the curl across a closed surface goes to zero, assuming that curl theorem holds (**todo** does it hold?).

4.2.3 Energy equation in integral form - arbitrary domains

4.2.4 Linear isotropic media

Using constitutive equations of a linear isotropic medium

$$\begin{aligned}\vec{d} = \varepsilon_0 \vec{e} + \vec{p} \qquad & = \varepsilon\, \vec{e} \\ \vec{b} = \mu_0 \vec{h} - \mu_0 \vec{m} \quad & = \mu\, \vec{h}\ ,\end{aligned}$$

it's possible to derive dynamical equations for the energy density and momentum due to free current only,

$$\begin{cases} \partial_t U + \nabla \cdot \vec{S} = -\vec{e} \cdot \vec{j}^f \\ \partial_t \vec{S} + c^2 \nabla \cdot \left[U \mathbb{I} - \left(\vec{d} \otimes \vec{e} + \vec{h} \otimes \vec{b} \right) \right] = -c^2 \left(\vec{e}\, \rho^f - \vec{b} \times \vec{j}^f \right) \end{cases}$$

todo *use this system to derive the 4-d formulation of special relativity in modern physics*

Energy equation

The products in the power equation of free current (4.5) becomes

$$\begin{aligned}\vec{h} \cdot \partial_t \vec{b} + \partial_t \vec{d} \cdot \vec{e} = \frac{1}{\mu} \vec{b} \cdot \partial_t \vec{b} + \varepsilon \partial_t \vec{e} \cdot \vec{e} = & \\ = \partial_t \left[\frac{1}{2} \left(\frac{1}{\mu} \vec{b} \cdot \vec{b} + \varepsilon \vec{e} \cdot \vec{e} \right) \right] = & \\ = \partial_t \left[\frac{1}{2} \left(\vec{h} \cdot \vec{b} + \vec{e} \cdot \vec{d} \right) \right] = \partial_t U\ . &\end{aligned}$$

and $\vec{S} = \vec{e} \times \vec{h} = \frac{\vec{e} \times \vec{b}}{\mu}$. For linear media, the energy of the electromagnetic field per unit volume due to free current only thus reads

$$\partial_t U + \nabla \cdot \vec{S} = -\vec{e} \cdot \vec{j}_f\ .$$

Momentum

Taking the time derivative of the Poynting vector,

$$\begin{aligned}\partial_t \vec{S} = \partial_t S_i = \partial_t \left(\varepsilon_{ijk} e_j h_k \right) = & \\ = \varepsilon_{ijk}\, \partial_t e_j\, h_k + \varepsilon_{ijk}\, e_j\, \partial_t h_k\ , &\end{aligned}$$

and using the product rule to evaluate time derivative

$\varepsilon_{ijk}\,\partial_t e_j\,h_k$

$$\begin{aligned}
\varepsilon_{ijk}\partial_t e_j h_k &= \frac{1}{\varepsilon}\varepsilon_{ijk}\partial_t d_j h_k \\
&= \frac{1}{\varepsilon}\varepsilon_{ijk}\left(\varepsilon_{jlm}\partial_l h_m - j^f_j\right) h_k \\
&= -\frac{1}{\varepsilon}\varepsilon_{ijk}\, j^f_j\, h_k + \frac{1}{\varepsilon}\varepsilon_{ijk}\varepsilon_{jlm} h_k \partial_l h_m \\
&= -\frac{1}{\varepsilon}\varepsilon_{ijk}\, j^f_j\, h_k + \frac{1}{\varepsilon}\left(\delta_{im}\delta_{kl} - \delta_{il}\delta_{km}\right) h_k \partial_l h_m = \\
&= -\frac{1}{\varepsilon}\varepsilon_{ijk}\, j^f_j\, h_k + \frac{1}{\varepsilon}\left(h_m \partial_m h_i - h_m \partial_i h_m\right) = \\
&= -\frac{1}{\varepsilon}\varepsilon_{ijk}\, j^f_j\, h_k + \frac{1}{\varepsilon}\left[\partial_m (h_m h_i) - \partial_m h_m\, h_i - \partial_i\left(\frac{h_m h_m}{2}\right)\right] = \\
&= \frac{1}{\varepsilon\mu}\varepsilon_{ijk}\, b_j\, j^f_k + \frac{1}{\varepsilon\mu}\left[\partial_m (b_m h_i) - \underbrace{\partial_m b_m}_{=0}\, h_i - \partial_i\left(\frac{h_m b_m}{2}\right)\right] =
\end{aligned}$$

$\varepsilon_{ijk}\,e_j\,\partial_t h_k$

$$\begin{aligned}
\varepsilon_{ijk} e_j \partial_t h_k &= \frac{1}{\mu}\varepsilon_{ijk} e_j \partial_t b_k = \\
&= -\frac{1}{\mu}\varepsilon_{ijk} e_j \left(\varepsilon_{klm}\partial_l e_m\right) = \\
&= -\frac{1}{\mu}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) e_j \partial_l e_m = \\
&= -\frac{1}{\mu}\left(e_m \partial_i e_m - e_m \partial_m e_i\right) = \\
&= -\frac{1}{\mu}\left[\partial_i\left(\frac{e_m e_m}{2}\right) - \partial_m\left(e_m e_i\right) + \partial_m e_m\, e_i\right] = \\
&= -\frac{1}{\varepsilon\mu}\left[\partial_i\left(\frac{d_m e_m}{2}\right) - \partial_m\left(d_m e_i\right) + \rho^f\, e_i\right] \ .
\end{aligned}$$

the dynamical equation for the Poynting vector $\vec{S}$ reads

$$\partial_t S_i + c^2 \partial_m \left[\frac{1}{2}\left(d_n e_n + h_n b_n\right)\delta_{mi} - \left(h_m b_i + d_m e_i\right)\right] = -c^2 \rho^f e_i + c^2 \varepsilon_{ijk} b_j j^f_k$$

or with vector notation

$$\partial_t \vec{S} + c^2 \nabla \cdot \left[\frac{1}{2}\left(\vec{d}\cdot\vec{e} + \vec{h}\cdot\vec{b}\right)\mathbb{I} - \left(\vec{d}\otimes\vec{e} + \vec{h}\otimes\vec{b}\right)\right] = -c^2\left(\rho^f \vec{e} - \vec{b}\times\vec{j}^f\right) \ .$$

CHAPTER

FIVE

REGIMES IN ELECTROMAGNETIC SYSTEMS

Non-dimensional analysis allows to distinguish different regimes of electromagnetic systems.

5.1 Non-dimensional equations of electromagnetism

Continuity equation of electric charge.

$$\partial_t \rho + \nabla \cdot \vec{j} = 0$$

Maxwell's equations.

$$\begin{cases} \nabla \cdot \vec{e} = \frac{\rho}{\varepsilon_0} \\ \nabla \times \vec{e} + \partial_t \vec{b} = \vec{0} \\ \nabla \cdot \vec{b} = 0 \\ \nabla \times \vec{b} - \frac{1}{c_0^2} \partial_t \vec{e} = \mu_0 \vec{j} \end{cases}$$

Potentials. Gauge. Wave equations.

5.2 Electrostatics

Elextrostatics studies the electric phenomena in systems with stationary charges. Thus, current is identically zero $\vec{j} = \vec{0}$.

So far, random topics

- governing equations of electrostatics
- zero electric field insiede a conductor

5.2.1 Governing equation of electrostatics

Governing equations become

$$\partial_t \rho = 0$$

$$\begin{cases} \nabla \cdot \vec{e} = \frac{\rho}{\varepsilon_0} \\ \nabla \times \vec{e} = \vec{0} \\ \nabla \cdot \vec{b} = 0 \\ \nabla \times \vec{h} = \vec{0} \ . \end{cases}$$

In absence of magnetic field, the problem is fully determined by the Gauss' law for the electric field and the steady condition of the Faraday's law, implying that the irrotational electric field can be written as the gradient of a scalar potential,

$$\vec{e} = -\nabla \varphi \ .$$

Introducing this expression into Gauss' law for the electric field, electrostatics can be formulated as a problem governed by a Laplace equation for the scalar potential

$$-\Delta \varphi = \frac{\rho}{\varepsilon_0} \ ,$$

supplied with the proper boundary conditions. **todo** *discuss boundary conditions…*

5.2.2 Zero electric field inside a conductor

Studying the transient of the electric charge distribution inside a conductor,

$$\vec{e} = \rho_R \vec{j} \ ,$$

whose constitutive equation is

$$\vec{d} = \varepsilon \vec{e} \ ,$$

with free electric charge continuity equation

$$\partial_t \rho_f + \nabla \cdot \vec{j}_f = 0 \ ,$$

and Gauss equation for the displacement field

$$\nabla \cdot \vec{d} = \rho_f \ .$$

$$\begin{aligned}
\partial_t \rho_f & = -\nabla \cdot \vec{j}_f = \\
& = -\nabla \cdot \left(\frac{1}{\rho_R} \vec{e} \right) = \\
& = -\frac{1}{\rho_R \varepsilon} \nabla \cdot \vec{d} = \\
& = -\frac{1}{\rho_R \varepsilon} \rho_f \ ,
\end{aligned}$$

having assumed uniform properties. The differential equation in the volume of the conductor provides the evolution of the electric charge in the volume $\rho(\mathbf{r}, t)$, given the initial condition $\rho(\mathbf{r}, 0) = \rho_{f,0}(\mathbf{r})$

$$\partial_t \rho_f = -\frac{1}{\rho_R \varepsilon} \rho_f$$

$$\rho_f(\mathbf{r}, t) = \rho_{f,0}(\mathbf{r}) \exp\left[-\frac{t}{\rho_R \varepsilon} \right] \ .$$

For a conductor:

- $\varepsilon \sim \varepsilon_0 = 8.85 \cdot 10^{-12} \mathrm{Fm}^{-}1$
- $\rho_R \sim 10^{-7} \Omega \, \mathrm{m}$

so that the time constant (that can be thought as a characteristic time) of the process is

$$\tau = \rho_R \varepsilon \sim 8.85 \cdot 10^{-19} \, \mathrm{s} \ ,$$

and thus, after a very short period of time the volume charge density is approximately zero everywhere in the volume: it accumulates in a very thin surface layer.

Proof

$$\partial_t \left(\rho_f e^{\frac{t}{\rho_R \varepsilon}} \right) = 0$$

$$\rho_f(\mathbf{r}, t) e^{\frac{r}{\rho_R \varepsilon}} = a(\mathbf{r})$$

and appylying initial conditions in all the points of the domain, $\rho_f(\mathbf{r}, 0) = \rho_{f,0}(\mathbf{r})$, function $a(\mathbf{r})$ must be equal to $\rho_{f,0}(\mathbf{r})$ and the solution reads

$$\rho_f(\mathbf{r}, t) = \rho_{f,0}(\mathbf{r}) \exp\left[-\frac{t}{\rho_R \varepsilon} \right]$$

5.3 Steady regime

Steady regime - in a Eulerian description - allows for steady currents.

5.4 Slow...

leading to circuit approximation

Part II

Electrical Engineering

CHAPTER

SIX

CIRCUIT APPROXIMATION

Electric Circuits

- **Validity Conditions for Circuit Approximation:**
  - The dimensions of the circuit are much smaller than the wavelength of the electromagnetic waves involved.
  - The frequency of the signals is low enough that the propagation delay is negligible.
  - The electric and magnetic fields are confined within the circuit components.
- **Basic Components:** resistors (R); capacitors (C); inductors (L); voltage and current sources; switches; non-linear components: diodes, transistors,…
- **Operating Regimes:**
  - **Steady State (DC):** The circuit parameters (voltage, current) do not change over time.
  - **Harmonic (AC):** The circuit parameters vary sinusoidally with time.
  - **Transient:** The circuit parameters change over time, typically due to a change in input or initial conditions.

Electromagnetic Circuits

- **Validity Conditions for Circuit Approximation:**
  - term $\partial_t \vec{d}$ negligible,…**todo** *as an integral condition on a system? Discussion in terms of energy?*
- **Example: Transformers**
  - Transformers operate on the principle of electromagnetic induction.
  - They are used to step up or step down voltages and currents in AC circuits.
  - The circuit approximation is valid as long as the frequency of the AC signal is within the designed range of the transformer.

Electro-Magneto-Mechanical Circuits

- **Examples:**
  - **Simple Circuits:** Basic circuits involving electromagnetic and mechanical components, such as a moving coil in a magnetic field.
  - **Electric Motors:** Convert electrical energy into mechanical energy.
  - **Generators:** Convert mechanical energy into electrical energy.
- **Key Concepts:**
  - Interaction between electrical and mechanical components.
  - Energy conversion and efficiency.

– Dynamic behavior and control of electromechanical systems.

6.1 Circuiti elettrici

Se il sistema di interesse soddisfa alcune condizioni, è possibile ridurre la teoria di campo dell'elettromagnetismo a una teoria circuitale. Quando possibile, cioè quando capace di descrivere adeguatamente il comportamento del sistema di interesse, l'approccio circuitale semplifica di molto la descrizione del problema, non richiedendo la soluzione di un sistema di equazioni differenziali alle derivate parziali da risolvere nello spazio, ma la soluzione di equazioni differenziali ordinarie nelle incognite circuitali, che si riduce a un sistema algebrico, spesso lineare, in regime stazionario.

Giustificazione dell'approccio circuitale.

Componenti elementari di un circuito elettrico.

6.1.1 Validità dell'approccio circuitale

L'approccio circuitale consente di ridurre il problema elettromagnetico, in generale un problema di campo che richiede la soluzione di PDE, a un approccio "ai morsetti" **todo**, che richiede la soluzione di ODE.

Una rivisitazione dell'*equazione dell'energia* permette di valutare i regimi in cui è possibile usare un approccio circuitale a un sistema elettromagnetico.

In particolare, nell'equazione di bilancio dell'energia elettromagnetica

$$\frac{d}{dt}\int_V u = \oint_{\partial V} \mathbf{s}\cdot\hat{\mathbf{n}} - \int_V \mathbf{j}\cdot\mathbf{e}\,,$$

viene indagato il termine di flusso alla frontiera, ricordando la definizione di vettore di Poynting $\mathbf{s} := \mathbf{e}\times\mathbf{h}$, e riscrivendo i campi elettrico e magnetico in funzione dei potenziali elettromagnetici, $\mathbf{b} = \nabla\times\mathbf{a}$, $\mathbf{e} = -\nabla\varphi - \partial_t\mathbf{a}$,

$$\begin{aligned}
-\oint_{\partial V} \mathbf{s}\cdot\hat{\mathbf{n}} = & -\oint_{\partial V} (\mathbf{e}\times\mathbf{h})\cdot\hat{\mathbf{n}} = \\
& = \oint_{\partial V} (\nabla\varphi + \partial_t\mathbf{a})\times\mathbf{h}\cdot\hat{\mathbf{n}} = \\
& = \dots \\
& = \underbrace{\oint_{\partial V} \hat{\mathbf{n}}\cdot\nabla\times(\varphi\mathbf{h})}_{=0 \text{ (Stokes'thm **todo** check)}} - \oint_{\partial V} \varphi\hat{\mathbf{n}}\cdot\underbrace{\nabla\times\mathbf{h}}_{\partial_t\mathbf{d}+\mathbf{j}} + \oint_{\partial V} \hat{\mathbf{n}}\cdot\partial_t\mathbf{a}\times\mathbf{h} = \\
& = -\oint_{\partial V} \varphi\mathbf{j}\cdot\hat{\mathbf{n}} - \oint_{\partial V} \hat{\mathbf{n}}\cdot(\partial_t\mathbf{d} + \mathbf{h}\times\partial_t\mathbf{a})\ ,
\end{aligned}$$

e assumendo che il flusso di carica elettrica avvenga solo in corrispondenza di un numero finito di sezioni $S_k \in \partial V$ equipotenziali a potenziale $v_k = -\varphi_k$, costante sulle sezioni, e riconoscento il flusso di carica elettrica attraverso la sezione S_k come la corrente $i_k = \int_{S_k} \mathbf{j}\cdot\hat{\mathbf{n}}$, si può scrivere

$$-\oint_{\partial V} \mathbf{s}\cdot\hat{\mathbf{n}} = \sum_k v_k\, i_k - \oint_{\partial V} \hat{\mathbf{n}}\cdot(\partial_t\mathbf{d} + \mathbf{h}\times\partial_t\mathbf{a})\ .$$

Il bilancio di energia elettromagnetica del sistema può quindi essere riscritto come

$$\frac{d}{dt}\int_V u = \sum_k v_k\, i_k - \int_V \mathbf{j}\cdot\mathbf{e} - \oint_{\partial V} \hat{\mathbf{n}}\cdot(\partial_t\mathbf{d} + \mathbf{h}\times\partial_t\mathbf{a})\ .$$

Nelle condizioni in cui l'ultimo termine è nullo o trascurabile (**todo** *quali? Spendere due parole sulla validità dell'approssimazione, con analisi dimensionale? Fare esempio in cui l'approssimazione non funziona*), la variazione di energia interna al sistema è dovuta alla differenza della potenza in ingresso ai morsetti, e la dissipazione all'interno del volume (ad esempio dovuta alla conduzione non ideale in conduttori con resistività finita),

$$\dot{E}^{em} = P^{ext,vi} - \dot{D} \, ,$$

con $\dot{D} \geq 0$ per il secondo principio della termodinamica **todo** *aggiungere riferimento, e discussione.*

6.1.2 Induzione elettromagnetica nell'approssimazione circuitale

E' possibile applicare l'approssimazione circuitale anche in presenza di regioni in cui non è possibile trascurare il termine $\partial_t \mathbf{b}$, come ad esempio circuiti elettromagnetici che coinvolgono trasformatori e/o motori o generatori elettrici.

In queste situazioni, se è possibile identificare una regione V_0 dello spazio connessa nella quale il termine $\partial_t \mathbf{b} = \mathbf{0}$, e quindi $\nabla \times \mathbf{e} = \mathbf{0}$, in V_0 è possibile definire il campo elettrico in termini di un potenziale φ,

$$\mathbf{e} = -\nabla \varphi \qquad , \qquad \mathbf{r} \in V_0 \, .$$

E' possibile calcolare le differenze di potenziale ai morsetti di un sistema in cui $\delta_t \mathbf{b} \neq 0$, racchiuso nel volume V_k, con la legge di Faraday,

$$\oint_{\ell_k} \mathbf{e} \cdot \hat{\mathbf{t}} = -\frac{d}{dt} \int_{S_k} \mathbf{b} \cdot \hat{\mathbf{n}} \, ,$$

dove il percorso chiuso $\ell_k = \ell_k^{cond} \cup \ell_k^{mors}$ descrive il conduttore in V_k chiuso dalla linea geometrica tra i morsetti. Se si può trascurare la resistività del conduttore in V_k, $\int_{\ell_k^{cond}} \mathbf{e} \cdot \hat{\mathbf{t}} = 0$, la differenza di tensione ai morsetti vale

$$\Delta v_k = \int_{\ell_k^{mors}} \mathbf{e} \cdot \hat{\mathbf{t}} = -\frac{d}{dt} \int_{S_k} \mathbf{b} \cdot \hat{\mathbf{n}}$$

6.1.3 Componenti elementari dei circuiti elettrici

Resistore ohmico

Un resistore di Ohm risulta dall'approssimazione circuitale di un materiale con equazione costitutiva lineare

$$\mathbf{e} = \rho_R \, \mathbf{j} \, ,$$

tra il campo elettrico $\mathbf{e}$ e la densità di corrente $\mathbf{j}$, tramite la costante di proporzionalità ρ_R, la **resistività** del materiale. La corrente elettrica attraverso una sezione del componente è definita come il flusso di carica attraverso una sua sezione

$$i = \int_S \mathbf{j} \cdot \hat{\mathbf{t}} \simeq j \, A \, ,$$

Nell'ipotesi che il vettore densità di corrente si allineato con l'asse del componente e uniforme sulla sezione A, "piccola". Se il materiale non è in grado di accumulare carica, il bilancio di carica elettrica si traduce nella continuità della corrente elettrica attraverso le sezioni del conduttore.

Utilizzando l'equazione costitutiva su un elemento di lunghezza elementare $d\mathbf{r} = \hat{\mathbf{t}} \, d\ell$, e assumendo che il campo elettrico sia allineato con l'asse del componente, $\mathbf{e} = e\hat{\mathbf{t}}$ si può scrivere il lavoro elementare per unità di carica come

$$\delta v = \mathbf{e} \cdot d\mathbf{r} = e \, d\ell = \rho_R \, j \, d\ell = \frac{\rho_R \, d\ell}{A} i \, .$$

Da questa ultima equazione seguono le due leggi di Ohm, per resistori lineari.

Prima legge di Ohm. La differenza di potenziale tra due sezioni di un resistore lineare è proporzionale alla corrente che passa attraverso di esso,

$$\delta v = dR\, i\ .$$

Seconda legge di Ohm. La costante di proporzionalità che lega la differenza di potenziale e la corrente all'interno di un resistore ohmico, la **resistenza** del resistore, è proporzionale alla resistività e alla lunghezza del resistore, e inversamente proporzionale alla sua sezione,

$$dR = \frac{\rho_R\, d\ell}{A}\ .$$

Se le proprietà sono uniformi nel resistore, si possono integrare le relazioni elementari per ottenere la relazione tra grandezze finite,

$$\Delta V = R\, i$$

$$R = \frac{\rho_R\, \ell}{A}$$

todo (perché si può usare il potenziale? Nelle mie note avevo usato il simbolo v^*, come se fosse una definizione leggermente diversa per incorporare movimento e instazionarietà, che si riduce a v nel caso stazionario).

Condensatore.

Induttore.

Generatore di tensione.

Generatore di corrente.

6.1.4 Regimi di funzionamento in circuiti elettrici

6.2 Circuiti elettromagnetici

Sotto opportune ipotesi è possibile usare un modello circuitale anche per sistemi elettromagnetici, come ad esempio i trasformatori, o i motori elettrici.

- legge di Gauss per il campo magnetico

$$\nabla \cdot \mathbf{b} = 0$$

- legge di Ampére-Maxwell

$$\nabla \times \mathbf{h} - \partial_t \mathbf{d} = \mathbf{j}$$

Si aggiungono le seguenti ipotesi:

- materiali lineari non-dissipativi e non-dispersivi $\mathbf{b} = \mu \mathbf{h}$ **todo** discutere questa ipotesi, insieme a isteresi materiali, cicli di magnetizzazione,….
- variazioni del campo $\mathbf{d}$ nel tempo trascurabili, $\partial_t \mathbf{d} = \mathbf{0}$.

La legge di Gauss del campo magnetico in forma integrale permette di scrivere la **legge ai nodi** del flusso del campo magnetico per i circuiti magnetici,

$$0 = \oint_{\partial V} \mathbf{b} \cdot \hat{\mathbf{n}} = \sum_k \phi_k \ .$$

La legge di Ampére-Maxwell in forma integrale considerando:

- un percorso incatenato con il solo induttore

$$\int_{\ell_{ind}} \mathbf{h} \cdot \hat{\mathbf{t}} + \int_{\ell_{12}} \mathbf{h} \cdot \hat{\mathbf{t}} = \oint_{\ell_1} \mathbf{h} \cdot \hat{\mathbf{t}} = \int_{S^{ind}} \mathbf{j} \cdot \hat{\mathbf{n}} = Ni =: m$$

- un percorso incatenato con il traferro, aggirando l'induttore

$$0 = \int_{\ell_{traf}} \mathbf{h} \cdot \hat{\mathbf{t}} + \int_{\ell_{21}} \hat{h} \cdot \hat{\mathbf{t}} = \sum_k h_k \ell_k + \int_{\ell_{21}} \hat{h} \cdot \hat{\mathbf{t}}$$

e sommando le due equazioni, riconoscendo che i due integrali di linea sullo stesso percorsoin versi opposti si annullano, si ottiene la **legge alle maglie** per i circuiti magnetici

$$\begin{aligned} m & = \int_{\ell_{ind}} \mathbf{h} \cdot \hat{\mathbf{t}} + \int_{\ell_{traf}} \mathbf{h} \cdot \hat{\mathbf{t}} = \\ & \approx \sum_{k \in \ell} h_k \, \ell_k = \sum_{k \in \ell} \frac{b_k}{\mu_k} \, \ell_k = \sum_{k \in \ell} \frac{\ell_k}{\mu_k \, A_k} \, \phi_k \ . \end{aligned}$$

Le leggi di Kirchhoff per i circuiti magnetici sono quindi

$$\begin{cases} \sum_{k \in N_j} \phi_k = 0 \\ \\ m_{\ell_i} = \sum_{k \in \ell_i} \theta_k \phi_k \ , \end{cases}$$

avendo introdotto la riluttanza $\theta_k = \frac{\ell_k}{\mu_k A_k}$, l'inverso della permeanza $\Lambda_k = \theta_k^{-1}$.

6.2.1 Trasformatore

- flusso del campo magnetico, nell'ipotesi di campo uniforme, o in termini del campo medio

$$\phi = b \, A$$

- flusso del campo magnetico concatenato a N avvolgimenti

$$\psi = N \, \phi$$

- relazione tra tensione ai morsetti dell'induttore e flusso concatenato, applicando la *legge di Faraday solo in parte irrotazionali*

$$v = \dot{\psi}$$

Trasformatore ideale

In assenza di flussi dispersi e riluttanza nel traferro, la legge alle maglie nel traferro implica

$$0 = m_1 + m_2 = N_1\, i_1 + N_2\, i_2$$

Il flusso del campo magnetico può essere scritto in funzione del flusso concatenato agli avvolgimenti,

$$\phi = \frac{\psi_1}{N_1} = \frac{\psi_2}{N_2}$$

La derivata nel tempo di questa relazione, con numero di avvolgimenti costanti nel tempo, implica

$$\frac{v_2}{N_2} = \frac{v_1}{N_1} \ .$$

Trasformatore con flussi dispersi

$$\begin{cases} \phi_1 - \phi_{1,d} = \phi \\ \phi_2 - \phi_{2,d} = \phi \\ m_1 = \theta_{1,d}\phi_{1,d} \\ m_2 = \theta_{2,d}\phi_{2,d} \\ m_1 + m_2 = 0 \end{cases}$$

$$\begin{aligned} \rightarrow \qquad & 0 = m_1 + m_2 = N_1\, i_1 + N_2\, i_2 \\ & 0 = \phi_2 - \phi_1 - \phi_{2,d} + \phi_{1,d} \\ & \quad = \phi_2 - \phi_1 - \frac{m_2}{\theta_{2,d}} + \frac{m_1}{\theta_{1,d}} \\ \rightarrow \qquad & \frac{\psi_2}{N_2} - \frac{m_2}{\theta_{2,d}} = \frac{\psi_1}{N_1} - \frac{m_1}{\theta_{1,d}} \ . \\ \rightarrow \qquad & \frac{1}{N_2}\left(v_2 - \frac{N_2^2}{\theta_{2,d}}\frac{di_2}{dt} \right) = \frac{1}{N_1}\left(v_1 - \frac{N_1^2}{\theta_{1,d}}\frac{di_1}{dt} \right) \ . \end{aligned}$$

Trasformatore con flussi dispersi e riluttanza θ_{Fe} nel traferro

$$\begin{cases} \phi_1 - \phi_{1,d} = \phi \\ \phi_2 - \phi_{2,d} = \phi \\ m_1 = \theta_{1,d}\phi_{1,d} \\ m_2 = \theta_{2,d}\phi_{2,d} \\ m_1 + m_2 = \theta_{Fe}\, \phi \end{cases}$$

todo finire e controllare i conti; disegnare circuito equivalente

6.3 Electromechanical Circuits

Some systems of interest and widespread use in modern society exploit the interactions between electromagnetic and mechanical phenomena: a fundamental example is electric machines, some of which can operate both as motors (with power supplied by the electrical system and converted into mechanical power) and as generators of electrical energy (converting mechanical power into electrical power).

In a system of inductors with mutual influence, the voltage difference across the "enhanced" inductor i is

$$v_i = \dot{\psi}_i = \frac{d}{dt}\left(N_i\,\phi_i\right)\ .$$

The linked flux depends on the effect of all the inductors in the system (and the magnetic field generated by any causes external to the system),

$$\phi_i = \sum_k \phi_{ik} = \sum_k \frac{1}{\theta_{ik}}\,m_k\ ,$$

where θ_{ik} is the reluctance of the circuit between the enhancing inductor k and the enhanced inductor i. Using the expression for the magnetomotive force $m_k = N_k\,i_k$, the voltage difference expression can be rewritten as

$$v_i = \sum_k \frac{d}{dt}\left(\frac{N_i\,N_k}{\theta_{ik}} i_k\right) = \sum_k \frac{d}{dt}\left(L_{ik}\,i_k\right)\ .$$

In general, in electromechanical circuits, reluctances are not constant parameters of the system but depend on the "mechanical" state of the system, described here by the variables $\mathbf{x}$,

$$v_i = \sum_k \frac{d}{dt}\left(\frac{N_i\,N_k}{\theta_{ik}(\mathbf{x})} i_k\right) = \sum_k \frac{d}{dt}\left(L_{ik}(\mathbf{x})\,i_k\right)\ .$$

$$\mathbf{v}(t) = \frac{d}{dt}\Big(\mathbf{L}(\mathbf{x}(t))\,\mathbf{i}(t)\Big)\ .$$

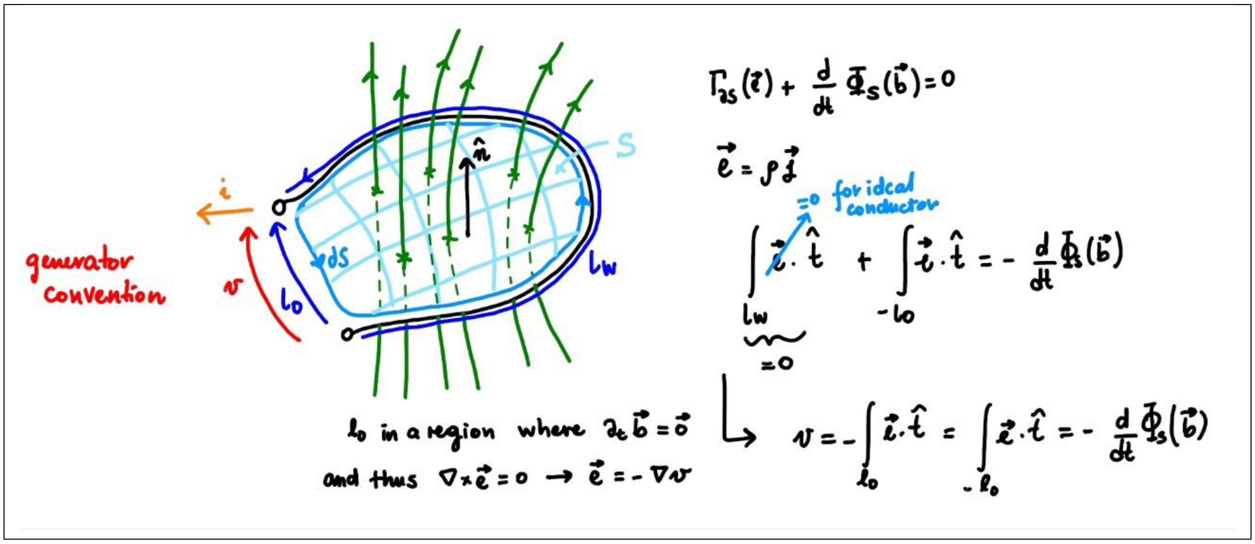
The inductance matrix $\mathbf{L}$ is symmetric **todo** *Proof*

Example 6.3.1

Given an constant and uniform magnetic field $\mathbf{b}(r) = \mathbf{B}$ in a region of space where a simple electric circuit is placed. The electric circuit consists in a simple circuit with a resistance R as a lumped load, and has a rectangular shape. Three sides are fixed, and the distance between the pair of parallel fixed sides is ℓ; the fourth side can move and its distance between the parallel fixed side is x. The unit vector orthogonal to the rectangular surface enclosed in the circuit is $\hat{\mathbf{n}}$.

A mechanical system provides the prescribed motion $x(t) = x_0 + \Delta x \sin(\Omega t)$ to the moving side. It's asked to evaluate and discuss:

- voltage at the electric port of the load
- energy balance



Without considering the inductance of the simple circuit. Faraday's law

$$\Gamma_{\partial s_t}(\mathbf{e}) + \dot{\Phi}_{s_t}(\mathbf{b}) = 0 \ ,$$

provides the relation between the time derivative of the magnetic flux though two points of the electric circuit on opposite sides of the moving side of the circuit, corresponding to the voltage at the electric port of the load

$$v = -\int_{\ell_0} \mathbf{e}\cdot\hat{t} = -\dot{\Phi}_{s_t}(\mathbf{b}) = -\frac{d}{dt}\left(NBA\right) = -B\ell\dot{x} \ ,$$

being $N = 1$, and B constant and uniform if self-inductance is not considered. If the inductance of the circuit is neglected, from the constitutive equation of the resistance, $v = Ri$, and voltage Kirchhoff law, it follows that the current in the simple circuit is

$$i = \frac{v}{R} = -\dot{\Phi}_{s_t}(\mathbf{b}) = -\frac{B_n \dot{A}}{R} = -\frac{B_n\,\ell\dot{x}}{R} = -\frac{B_n\,\ell\,\Delta x}{R}\,\Omega\cos(\Omega t) \ .$$

The force acting on a wire conducting electric current i in a uniform magnetic field $\mathbf{B}$ is

$$\mathbf{F} = -i\mathbf{B}\times\mathbf{l} \ .$$

Calling y the "positive" direction of the moving side, and assuming $\mathbf{B} = B\hat{\mathbf{z}}$, with $\hat{\mathbf{z}} = \hat{\mathbf{x}}\times\hat{\mathbf{y}}$,

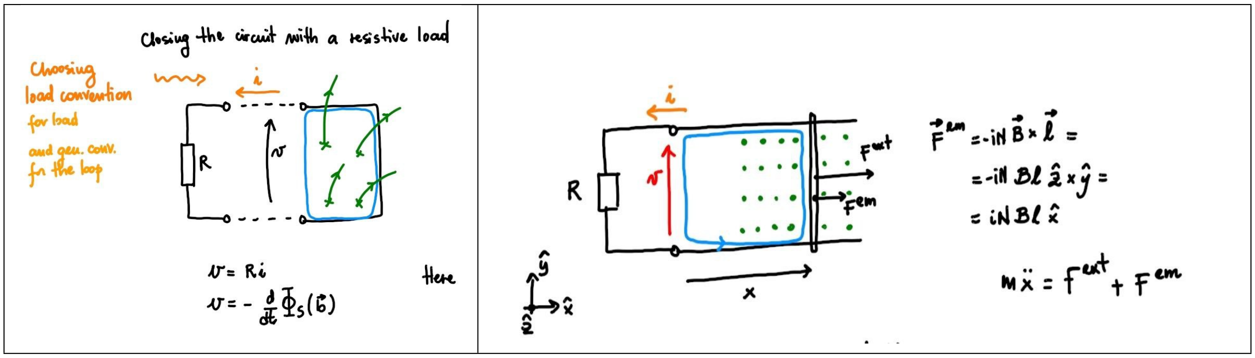
$$\mathbf{F} = iB\ell\hat{\mathbf{x}} \ .$$

Assuming negligible mass of the moving wire, the second principle of dynamics reduces to force equilibrium, so that the external force provided to the wire must be opposite to the force acting on the wire due to the EM field

$$\mathbf{F}^e = -\mathbf{F} \ ,$$

and the external power reads

$$P^e = \dot{\mathbf{x}}\cdot\mathbf{F}^e = -iB\ell\dot{x} = \frac{B^2\ell^2\dot{x}^2}{R} = \frac{B^2\ell^2\left(\Delta x\right)^2}{R}\Omega^2\cos^2(\Omega t) \ .$$



Considering the inductance of the circuit and inertia of the wire. Considering the self-induced magnetic flux ϕ,

$$v = -\frac{d}{dt}\left(N\left(\phi + BA\right)\right) \ ,$$

with $\phi = \dfrac{m}{\theta} = \dfrac{N}{\theta}i$. The expression of the voltage a the port of the circuit can be recast as

$$v = -\frac{d}{dt}\left(NBA\right) - \frac{d}{dt}\left(\frac{N^2}{\theta}i\right) = -\frac{d}{dt}\left(NB\ell x\right) - \frac{d}{dt}\left(Li\right) \ .$$

Now, assuming everything constant except for the x and i, and connecting this circuit to the load with constitutive equation, $v = Ri$, the dynamical equation of the electric circuit becomes

$$L\frac{di}{dt} + Ri = -NB\ell\frac{dx}{dt} \ .$$

The dynamical equation of the wire is

$$\begin{aligned} m\frac{d^2x}{dt^2} &= F^{ext} + F^{EM} = \\ &= F^{ext} + iB\ell \ . \end{aligned}$$

Energy balance immidiately follows after multiplying the circuit equation by i, the dynamical equation by $\dot{x}$ and summing,

$$\frac{d}{dt}\underbrace{\left(\frac{1}{2}m|\dot{x}|^2 + \frac{1}{2}Li^2\right)}_{\text{energy: kin.+em.}} + \underbrace{Ri^2}_{\text{dissipation}} = \underbrace{F^{ext}\dot{x}}_{\text{ext. power done on the sys}} \ .$$

6.3.1 Conservative Electromechanical Systems

The equations governing the electromechanical system, without capacitors, can generally be written as

$$\begin{cases} \mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}^{ext} + \mathbf{f}^{em} \\ \dfrac{d}{dt}\left(\mathbf{L}\mathbf{i}\right) + \mathbf{R}\mathbf{i} = \mathbf{e} \end{cases}$$

In terms of energy,

$$0 = \dot{\mathbf{x}}^T\left[\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} - \mathbf{f}^{ext} - \mathbf{f}^{em}\right] + \mathbf{i}^T\left[\frac{d}{dt}\left(\mathbf{L}\mathbf{i}\right) + \mathbf{R}\mathbf{i} - \mathbf{e}\right]$$

In the case of constant mass, damping, and stiffness matrices, and using the product rule to obtain a term of the derivative of the energy of the inductors exploiting the symmetry of $\mathbf{L}$,

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}\mathbf{i}^T\mathbf{L}\mathbf{i}\right] & = \mathbf{i}^T\frac{d}{dt}\left(\mathbf{L}\,\mathbf{i}\right) + \frac{1}{2}\mathbf{i}^T\frac{d\mathbf{L}}{dt}\mathbf{i} = \\ & = \mathbf{i}^T\frac{d}{dt}\left(\mathbf{L}\,\mathbf{i}\right) + \sum_a \frac{1}{2}\mathbf{i}^T\frac{\partial\mathbf{L}}{\partial x_a}\mathbf{i}\,\dot{x}_a = \\ & = \mathbf{i}^T\frac{d}{dt}\left(\mathbf{L}\,\mathbf{i}\right) + \nabla\left(\frac{1}{2}\mathbf{i}^T\mathbf{L}\mathbf{i}\right)\dot{\mathbf{x}}\,.\end{aligned} \tag{6.1}$$

one can write an equation of macroscopic mechanical energy balance, $E^{mec,int}$

$$\begin{aligned}0 = & \frac{d}{dt}\left[\frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} + \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x} + \frac{1}{2}\mathbf{i}^T\mathbf{L}\mathbf{i}\right] - \dot{\mathbf{x}}^T\left(\mathbf{f}^{em} - \nabla E^{ind}(\mathbf{x},\mathbf{i})\right) + \\ & - \dot{\mathbf{x}}^T\mathbf{f}^{ext} - \mathbf{i}^T\mathbf{e} + \\ & + \dot{\mathbf{x}}^T\mathbf{C}\dot{\mathbf{x}} + \mathbf{i}^T\mathbf{R}\mathbf{i}\,.\end{aligned}$$

Assuming the process is conservative, the form of the forces due to electromagnetic phenomena is derived,

$$\mathbf{f}^{em} = \nabla_{\mathbf{x}}E^{ind}(\mathbf{x},\mathbf{i})\,. \tag{6.2}$$

6.3.2 Governing Equations

Using the expression (6.2) of the mechanical actions due to electromagnetic effects, the system equations are

$$\begin{cases}\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} - \nabla_{\mathbf{x}}E^{ind}(\mathbf{x},\mathbf{i}) = \mathbf{f}^{ext} \\ \frac{d}{dt}\left(\mathbf{L}(\mathbf{x})\mathbf{i}\right) + \mathbf{R}\mathbf{i} = \mathbf{e}\end{cases}$$

or in the general case

$$\begin{cases}\mathbf{M}\ddot{\mathbf{x}} - \nabla_{\mathbf{x}}E^{ind}(\mathbf{x},\mathbf{i}) = \mathbf{f}^{ext} \\ \frac{d}{dt}\left(\mathbf{L}(\mathbf{x})\mathbf{i}\right) + \mathbf{R}\mathbf{i} = \mathbf{e}\end{cases}$$

6.3.3 Energy Balance

Macroscopic Mechanical Energy

Using the expression (6.2) of the mechanical actions due to electromagnetic phenomena, the relation (6.1) can be rewritten as a macroscopic mechanical energy balance of the system,

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} + \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x} + \frac{1}{2}\mathbf{i}^T\mathbf{L}\mathbf{i}\right] = \dot{\mathbf{x}}^T\mathbf{f}^{ext} + \mathbf{i}^T\mathbf{e} - \dot{\mathbf{x}}^T\mathbf{D}\dot{\mathbf{x}} - \mathbf{i}^T\mathbf{R}\mathbf{i}\,,$$

and therefore

$$\dot{E}^{mec} = P^{ext} - \dot{D}\,.$$

Kinetic Energy

The macroscopic mechanical energy can be written as the sum of the kinetic energy and the internal potential energy of the system, $E^{mec} = K + V^{int}$. The time derivative of the potential energy of the internal actions is the opposite of the power of the conservative internal actions, $P^{int,c} = -\dot{V}^{int}$; the dissipation is the opposite of the power of the non-conservative internal actions, $P^{int,nc} = -\dot{D}$. The total power of the internal actions can therefore be written as

$$P^{int} = P^{int,c} + P^{int,nc} = -\dot{V}^{int} - \dot{D} ,$$

$$\dot{K} = \dot{E}^{mec} - \dot{V}^{int} = P^{ext} \underbrace{- \dot{D} - \dot{V}^{int}}_{=P^{int}}$$

Total Energy

The first principle of thermodynamics provides the total energy balance equation of a closed system,

$$\dot{E}^{tot} = P^{ext} + \dot{Q}^{ext} .$$

Internal Energy

The internal energy of a system is defined as the difference between the total energy and the macroscopic kinetic energy, $E := E^{tot} - K$. The internal energy balance equation of a closed system is

$$\dot{E} = Q^{ext} - P^{int} .$$

Thermal (Microscopic) Internal Energy

If the thermal internal energy, corresponding to the kinetic energy associated with microscopic dynamics, is defined as the difference between internal energy and internal potential energy, or the difference between total energy and macroscopic mechanical energy,

$$\begin{aligned} E^{th} & = E - V^{int} = \\ & = E^{tot} - E^{mec} , \end{aligned}$$

the thermal internal energy balance equation is

$$\dot{E}^{th} = \dot{Q}^{ext} + \dot{D} .$$

Proof

$$\begin{aligned} \dot{E}^{th} = \dot{E} - V^{int} & = \dot{Q}^{ext} - P^{int} - V^{int} = \\ & = \dot{Q}^{ext} + \dot{D} + \dot{V}^{int} - \dot{V}^{int} = \\ & = \dot{Q}^{ext} + \dot{D} . \end{aligned}$$

Con condensatori. todo

Equazioni

- **Node laws.**

$$0 = \sum_{k \in B_j} \alpha_{jk}\, i_{jk}$$

$$\mathbf{A}\mathbf{i} = \mathbf{0}$$

- **Node-branch voltage difference.**

$$\mathbf{A}^T \mathbf{v}_n = \mathbf{v}$$

- **Ground node.**

$$\mathbf{v}_\perp = \mathbf{v}_0 \ .$$

- **Constitutive equations.**

$$\begin{aligned} \mathbf{0} &= \mathbf{v}_R - \mathbf{R}\mathbf{i}_R && \text{resistances} \\ \mathbf{0} &= \mathbf{v}_L - \frac{d}{dt}\left(\mathbf{L}\mathbf{i}_L\right) && \text{inductances} \\ \mathbf{0} &= \frac{d}{dt}\left(C\mathbf{v}_C\right) - \mathbf{i}_C && \text{capacitors} \end{aligned}$$

6.4 Network analysis of linear circuits

Dynamical equations of a linear circuit can be written as a general linear state-space model

$$\begin{cases} \mathbf{M}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

The mathematical problem is a system of DAE (dynamical-algebraic equations), as it includes:

- constitutive equations of the linear components
- Kirchhoff laws for current at nodes and voltage in loops

Thus matrix $\mathbf{M}$ is likely to be singular, here vector $\mathbf{x}$ contains both dynamical (like voltage across a capacitor or current through an inductor) and algebraic grid variables, current and voltages whose time derivative doesn't appear explicitly in the system of DAE.

Different representations. Possible choices of the unknowns:

1. current through any side, voltage at any node
2. loop currents, voltage drops across any side.
3. … *any other (linear) combination on the physical quantities*

6.4.1 Thevenin equivalent

One-port. Thevenin's theorem states that any linear circuit can be reduced to a single voltage source and a single impedance in series.

One-port circuit

As the goal of Thevenin's theorem is to find the constitutive equation of the network as $v(i)$, the network is connected to an external current generator that prescribes i and the voltage v at the port is evaluated.

The input of the extended network is

$$\mathbf{u} = (\mathbf{u}_{gen}, i) \ ,$$

while the output is, or at least contains, the voltage v

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \ .$$

The linear system can be written in Laplace domain as

$$\begin{cases} s\mathbf{M}\mathbf{x} - \mathbf{M}\mathbf{x}_0 = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

The state and the output are the sum of the free response to non-zero initial conditions and forced response,

$$\begin{cases} \mathbf{x} = (s\mathbf{M} - \mathbf{A})^{-1}\mathbf{M}\mathbf{x}_0 + (s\mathbf{M} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}(s\mathbf{M} - \mathbf{A})^{-1}\mathbf{M}\mathbf{x}_0 + \left[\mathbf{C}(s\mathbf{M} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right] \mathbf{u} \end{cases}$$

Forced response can be further manipulated exploiting PSCE, evaluating the effect of one input at a time, setting all the other inputs equal to zero.

- the effect of setting the input of the external current generator, $i = 0$, is equivalent to evaluate the system with an open circuit at the port
- the effect of setting equal to zero a tension generator, $e = 0$, is equivalent to a short-circuit on the same side
- the effect of setting equal to zero a current generator, $a = 0$, is equivalent to an open circuit on the same side

If the system is **asymptotically stable**, the free response is approximately zero when the **transient dynamics is over**, and the output equals the forced output. Introducing the transfer function

$$\mathbf{G}(s) = [\ \mathbf{G}_{gen}(s) \quad \mathbf{G}_i(s)\ ] \ ,$$

the input-output relation reads

$$\begin{aligned} v = \mathbf{G}(s)\mathbf{u} &= \mathbf{G}_{gen}(s)\mathbf{u}_{gen} + G_i(s)\, i = \\ &= v_{Th}(s) - Z_{Th}(s) i(s) \ , \end{aligned}$$

having recast it as Thevenin's theorem defining the voltage v_{Th} and the impedance Z_{Th} of the equivalent circuit,

$$\begin{aligned} v_{Th} &:= \mathbf{G}_{gen}(s)\mathbf{u}_{gen}(s) \\ Z_{Th}(s) &:= -G_i(s) \end{aligned}$$

Many-port circuit

$$\mathbf{v} = \mathbf{G}_{gen}(s)\mathbf{u}_{gen} + \mathbf{G}_i(s)\mathbf{i} = \mathbf{v}_{Th} - \mathbf{Z}_{Th}\mathbf{i} \ .$$

6.4.2 Norton equivalent

6.5 Network analysis of linear circuits - harmonic regime

The harmonic dynamics of a linear circuit can be evaluated in Fourier domain, or using complex numbers to represent harmonic functions,

$$\begin{aligned}
v(t) &= V_{max}\cos(\Omega t + \varphi_v) = \text{re}\{V_{max}e^{i(\Omega t + \varphi_v)}\} = \\
&= \sqrt{2}V\cos(\Omega t + \varphi_v) = \sqrt{2}\,\text{re}\{Ve^{i(\Omega t + \varphi_v)}\} = \sqrt{2}\,\text{re}\{ve^{i\Omega t}\} \\
i(t) &= I_{max}\cos(\Omega t + \varphi_i) = \text{re}\{I_{max}e^{j(\Omega t + \varphi_i)}\} = \\
&= \sqrt{2}I\cos(\Omega t + \varphi_i) = \sqrt{2}\,\text{re}\{Ie^{j(\Omega t + \varphi_i)}\} = \sqrt{2}\,\text{re}\{ie^{j\Omega t}\}
\end{aligned}$$

having anticipated the definition *Definition 6.5.1* of effective tension V and current I.

6.5.1 Power

Instantaneous power.

$$\begin{aligned}
P(t) &= v(t)i(t) = \\
&= V_{max}I_{max}\cos(\Omega t)\cos(\Omega t - \varphi_i) = \\
&= \frac{1}{2}V_{max}I_{max}\left[\cos\varphi_i + \cos(2\Omega t)\right]
\end{aligned} \tag{6.3}$$

having used Werner's formula,

$$\cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right] \ .$$

and the property $\cos(-x) = \cos x$.

Average power on a period. Over a period $T = \frac{1}{f} = \frac{2\pi}{\Omega}$

$$\overline{P} = \frac{1}{T}\int_{t=t_0}^{t_0+T} P(t)\,dt = \frac{V_{max}I_{max}}{2} = VI \ ,$$

as the integral of the harmonic term with period $\frac{T}{2}$ of the instantaneous power (6.3) is identically zero, and with the definition of the **effective voltage and current**

Definition 6.5.1 (Effective voltage and current in AC)

Effective voltage and currents

$$V := \frac{V_{max}}{\sqrt{2}} \quad , \quad I := \frac{I_{max}}{\sqrt{2}} \ ,$$

are defined as those voltage and current in DC providing the same value of average power.

Complex power. Complex power of a dipole with impedence Z, $v = Zi$

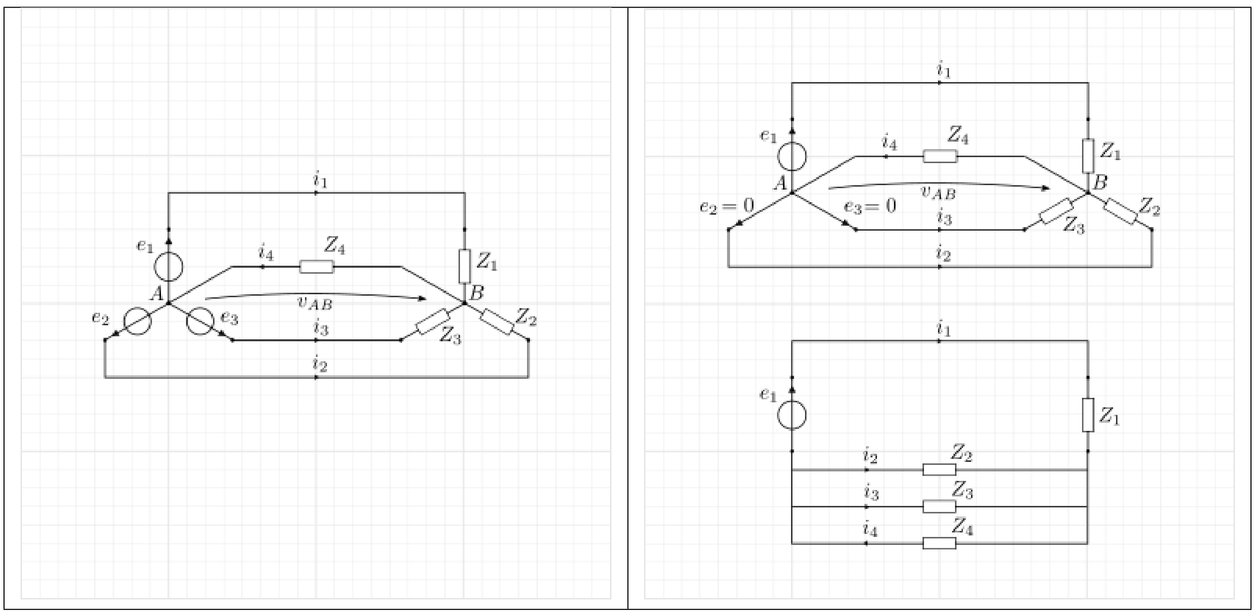
$$\begin{aligned} S := vi^* = |v|e^{j\varphi_v}|i|e^{-j\varphi_i} = |v||i|e^{j(\varphi_v - \varphi_i)} = \\ = Zii^* = Z|i|^2 = (R + jX)|i|^2 = |Z||i|^2 e^{j\varphi_Z} = P + jQ \ , \end{aligned}$$

with the active power P and the reactive power Q

$$\begin{aligned} P = \text{re}\{S\} \quad &= \quad |S|\cos\varphi_Z \quad = \quad \dots \\ Q = \text{im}\{S\} \quad &= \quad |S|\sin\varphi_Z \quad = \quad \dots \end{aligned}$$

6.6 Three-phase circuits

6.6.1 Star-star network



General solution

Tension v_{AB} between the centers of the stars A, B

$$v_{AB} = \frac{\sum_{g=1}^{3} Y_g e_g}{\sum_{i=1}^{4} Y_i} \ .$$

Proof.

PSCE is used on the linear network, leaving only one tension generator on at a time, and then combining the results.

Tension generator e_1 on, $e_2 = e_3 = 0$ off. Leaving e_1 on, and switching off $e_2 = e_3 = 0$, tension generator sees an equivalent impedance

$$\begin{aligned}
Z_{eq,1} & = Z_1 + (Z_2 \parallel Z_3 \parallel Z_4) \\
& = \frac{1}{Y_1} + \frac{1}{Y_2+Y_3+Y_4} = \frac{Y_{1234}}{Y_1 Y_{234}} \ ,
\end{aligned}$$

so that:

- the current through the generator reads

$$i_{1,1} = \frac{e_1}{Z_{eq,1}} = \frac{Y_1 Y_{234}}{Y_{1234}} e_1$$

- the currents through the other sides (acting as current dividers are):

$$\begin{aligned}
i_{2,1} & = -\frac{Y_2}{Y_{234}}\, i_{1,1} = -\frac{Y_1 Y_2}{Y_{1234}} e_1 \\
i_{3,1} & = -\frac{Y_3}{Y_{234}}\, i_{1,1} = -\frac{Y_1 Y_3}{Y_{1234}} e_1 \\
i_{4,1} & = \quad \frac{Y_4}{Y_{234}}\, i_{1,1} = \quad \frac{Y_1 Y_4}{Y_{1234}} e_1
\end{aligned}$$

- tension v_{AB}

$$v_{AB,1} = e_1 - Z_1 i_{1,1} = \left(1 - \frac{Y_{234}}{Y_{1234}}\right) e_1 = \frac{Y_1 e_1}{\sum_{k=1}^4 Y_k} \ .$$

PSCE. Exploiting the PSCE and the symmetry of the system, the expressions of currents in the phases, in the neutral and the center-center voltage seamlessly follow

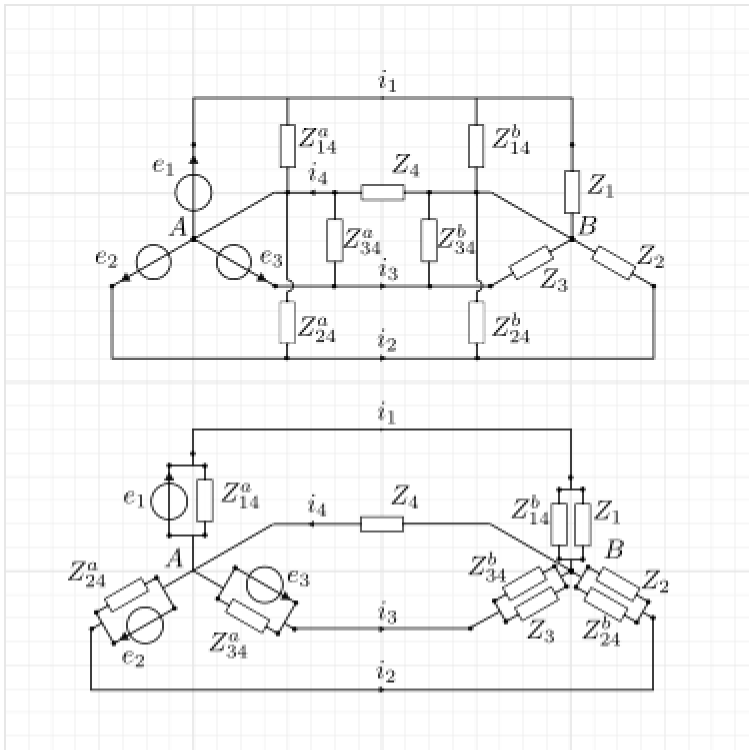
$$\begin{aligned}
i_1 & = \frac{Y_1 Y_{234}}{Y_{1234}} e_1 - \frac{Y_1 Y_2}{Y_{1234}} e_2 - \frac{Y_1 Y_3}{Y_{1234}} e_3 = \\
& = Y_1 e_1 - \frac{Y_1}{Y_{1234}} \sum_{g=1}^3 Y_g\, e_g \\
i_2 & = Y_2 e_2 - \frac{Y_2}{Y_{1234}} \sum_{g=1}^3 Y_g\, e_g \\
i_3 & = Y_3 e_3 - \frac{Y_3}{Y_{1234}} \sum_{g=1}^3 Y_g\, e_g \\
i_4 & = \frac{Y_4}{Y_{1234}} \sum_{g=1}^3 Y_g\, e_g \\
v_{AB} & = \frac{\sum_{g=1}^3 Y_g\, e_g}{\sum_{k=1}^4 Y_k}
\end{aligned}$$

Equilibrated generation and loads

Extra connections

Phase-neutral connections

Connections of a phase with the neutral result in parallel impedence with the generators and/or the loads



Phase-phase connections

Phase-phase connections don't influence the voltage v_{AB} between the centers A, B.

todo *Write the proof.*

6.7 Exercises

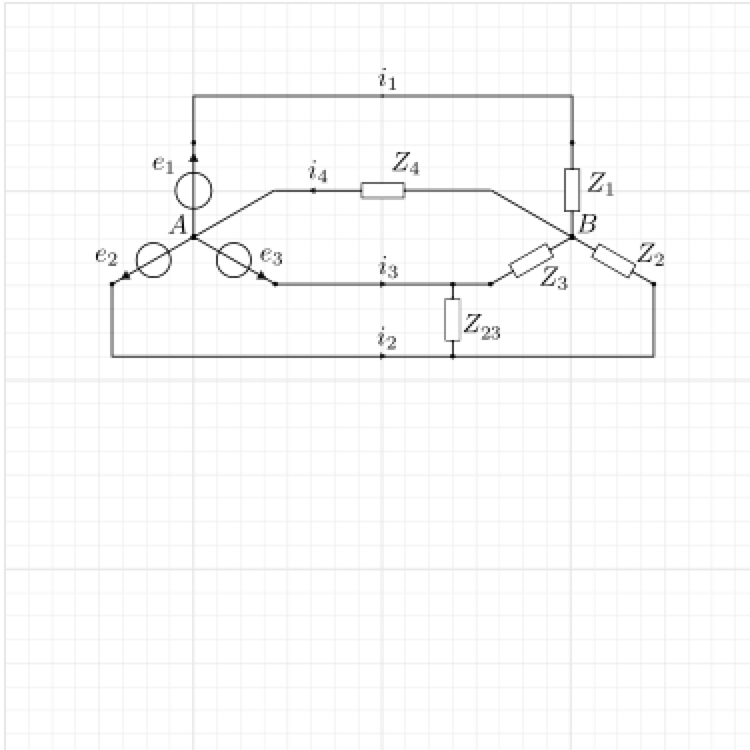
Topics: Thevenin and Norton equivalent;…

Electric circuits:

- Type a: transient dynamics of systems with 1 dynamic component (either capacitor or inductor);
- Type b: harmonic dynamics of linear systems: phasor algebra, complex power,…
- Type c: three-phase circuits, triangles and stars,…

Electromagnetic circuits:

- Type d: circuit approximation of magnetic circuit,…



Exams.

2025-02-11

1. Type a. Exercise **??**
2. Type b. Exercise **??**
3. Type b. Exercise **??**
4. Theory: electrical line. Electro-thermal model of the cable,…

2025-01-22

1. Type a. Exercise **??**
2. Type b. Exercise **??**
3. Type d. Exercise **??**
4. Theory: transformer

2024-09-06

1. Type a. Exercise **??**
2. Type b. Exercise **??**
3. Type c. Exercise **??**
4. Theory: overload in cables

2024-07-22

1. Type a. Exercise **??**
2. Type b. Exercise **??**
3. Type c. Exercise **??**

2024-06-19

1. Type c. Exercise **??**
2. Type d. Exercise **??**

2024-02-13

1. Type d.+a. Exercise **??**
2. Type a. Exercise **??**
3. Type c. Exercise **??**

6.7.1 Transient dynamics of linear electrical grids with one dynamic component

Guidelines for solution

Breaking down the solution:

1. Find the **many-port equivalent** of the **linear algebraic part of the network** (resistor, and prescribed generators), using PSCE. Find the relation between port voltage and currents and all the required variables of the network,

$$\begin{aligned}\mathbf{v}_{port} &= \mathbf{v}_0(\mathbf{e},\mathbf{a}) + \mathbf{R}\mathbf{i}_{port}\\ \mathbf{z} &= \mathbf{z}_0(\mathbf{e},\mathbf{a}) + \mathbf{z}_{/i_{port}}\mathbf{i}_{port}\end{aligned}$$

If 2 ports exist and port A is connected to a dynamical linear component and port B is connected to an ideal switch, the equations become to

$$\begin{aligned}v_A &= v_{0,A}(\mathbf{e},\mathbf{a}) + R_{AA}i_A + R_{AB}i_B\\ v_B &= v_{0,B}(\mathbf{e},\mathbf{a}) + R_{BA}i_A + R_{BB}i_B\\ \mathbf{z} &= \mathbf{z}_0(\mathbf{e},\mathbf{a}) + \mathbf{z}_{/i_{port}}\mathbf{i}_{port}\end{aligned}$$

2. Evaluate the **steady conditions** for $t \le 0^-$, with the given state of the switch ($i_B = 0$ if it's open, $v_B = 0$ if it's closed), and using the constitutive equation of the dynamical element (a capacitor acts as an open circuit in steady conditions, $i_A = 0$ as $i_A = C\frac{dv_A}{dt}$; an inductor acts as a short-circuit in steady conditions, $v_A = 0$, as $v_A = L\frac{di_A}{dt}$).

   In the first two equations of the system, two of the four varaibles $i_{A,B}$, $v_{A,B}$ are thus known, and this system can be solved to find the other two quantities. Once $\mathbf{i}_{port}$ is known, grid variables $\mathbf{z}$ can be evaluated.

3. **Transient dynamics** is then evaluated using the change of state in the switch

$$\text{open to close:} \quad \begin{cases} v_A(t) = (1 - h(t))\, v_{A,0^-} \\ i_A(t) = h(t)\, i_{A,t\ge 0}(t) \end{cases}$$

$$\text{close to open:} \quad \begin{cases} v_A(t) = h(t)\, v_{A,t\ge 0}(t) \\ i_A(t) = (1 - h(t))\, i_{A,0^-} \end{cases}$$

   and using the conditions for $t \ge 0$ in the equations of the equivalent network to find the equivalent resistance R_{eq} of the algebraic part of the network to be used in the constitutive equations of the dynamical component,

$$\begin{aligned} \text{capacitor}: &\quad 0 = i_A + C\frac{dv_A}{dt} &\quad \rightarrow \quad & f(\mathbf{x}_B) = v_A + R_{eq}C\frac{dv_A}{dt} \\ \text{inductor}: &\quad 0 = v_A + L\frac{di_A}{dt} &\quad \rightarrow \quad & f(\mathbf{x}_B) = i_A + R_{eq}L\frac{di_A}{dt} \end{aligned}$$

   with $f(\mathbf{x}_B)$ a forcing term depending on the state of the switch, and the initial conditions for the state variable of the dynamical components equal to the steady conditions, as there's no jump in state variables without impulsive forces.

4. Once the state variables of the dynamical equations are known. it's possible to evaluate all the other required variables.

Exercise 6.7.1 (Exam 2025-02-11, Exercise 1.)

1) Il circuito di Figura 1, con ingressi stazionari, è così assegnato:

   $V_1 = 5$ V
   $V_2 = 8$ V
   $I_s = 3$ A
   $R_1 = 1\ \Omega$
   $R_2 = 2\ \Omega$
   $R_3 = 3\ \Omega$
   $R_4 = 4\ \Omega$
   $C_1 = 500$ mF

L'interruttore S è aperto da tempo infinito e viene chiuso all'istante $t = 0$.

<u>Determinare:</u>

- l'andamento nel tempo della corrente $i_{cc}(t)$ sia in termini analitici che grafici (andamento qualitativo).
- l'energia immagazzinata nel capacitore nell'istante di tempo t = 0.

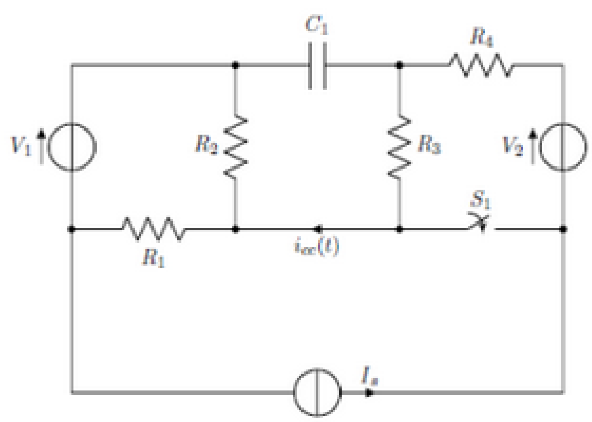
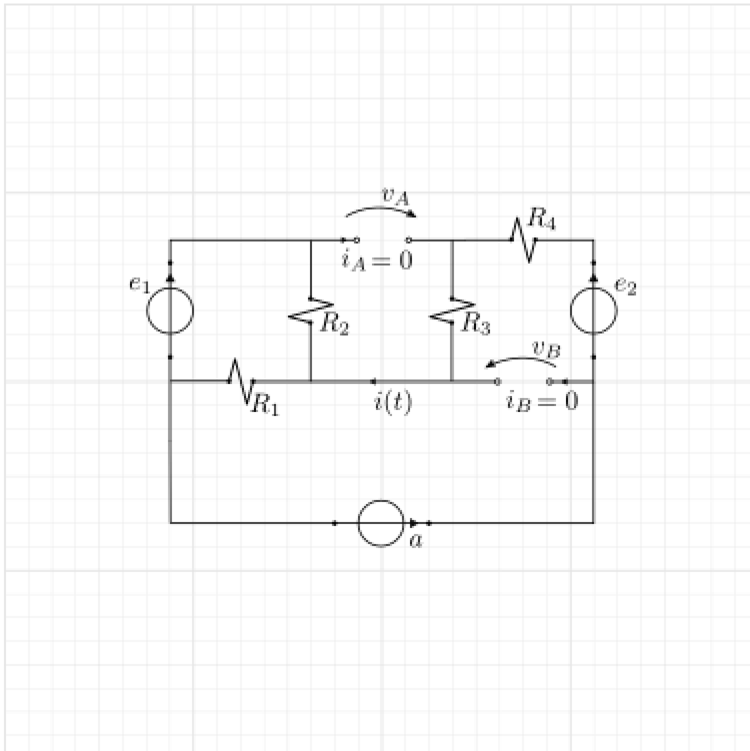


Fig. 1.

Solution

Following the **guidelines for the solution**, a *many-port Thevenin equivalent circuit* of the resistive part of the circuit is found, with two ports for interfacing with the capacitor (A) and with the switch. The dynamical equation of the system is written in state-space representation, writing the voltage at the ports and the unknown variable $i(t)$ as outputs; the capacitor contitutive equation is used to find the time evolution of the system once the switch is closed



Internal generators on, open circuit

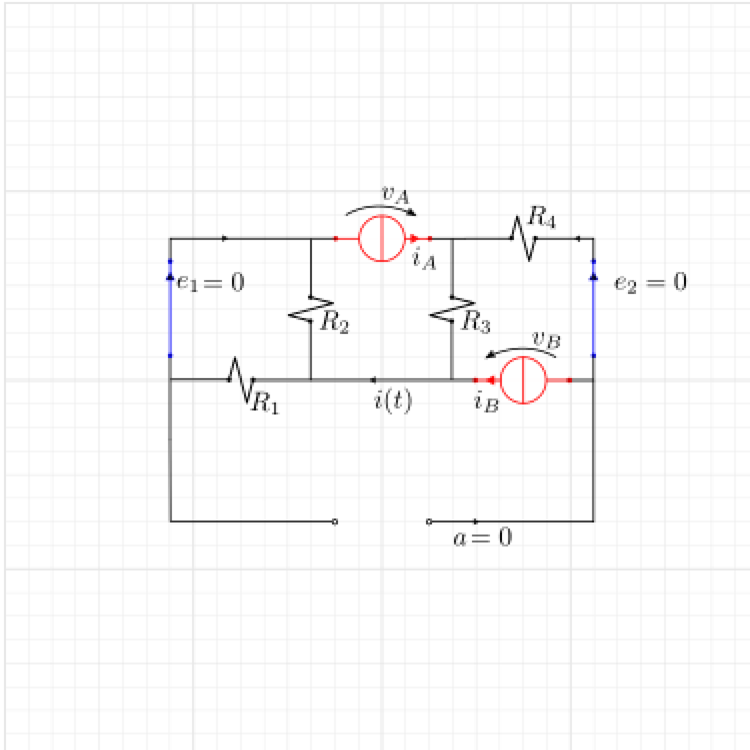
Solution using two loop currents, i_1 in the upper part of the circuit and i_2 in the lower triangle. Using KVL

$$\begin{aligned}
& 0 = e_1 - R_2 i_{1,0} - R_1(a + i_{1,0}) \\
\rightarrow \quad & i_{1,0} = \frac{1}{R_1 + R_2} e_1 - \frac{R_1}{R_1 + R_2} a
\end{aligned}$$

so that the desired variables read

$$\begin{cases}
v_{A,0} & = R_3 a - R_2 i_{1,0} = \left[R_3 + \dfrac{R_1 R_2}{R_1 + R_2} \right] a - \dfrac{R_2}{R_1 + R_2} e_1 \\
v_{B,0} & = e_2 - (R_3 + R_4) a \\
i_0 & = a
\end{cases}$$

$$\begin{cases}
v_{A,0} & = \quad 7.67\,V \\
v_{B,0} & = -13.00\,V \\
i_0 & = \quad 3.00\,A
\end{cases}$$



Internal generators off, current generators at the ports

Callling i_A and i_B the current passing through the current generators connected at the ports. The solution is found powering one generation at a time and then exploiting PSCE

Powering A …

Powering B. …

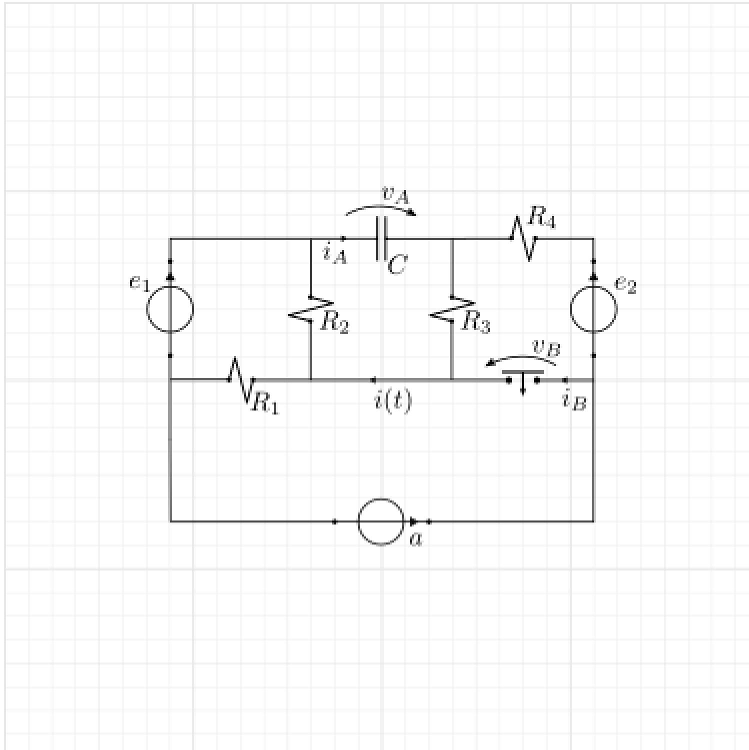
Currents in the two parallel branches in the upper part of the circuit (current dividers) read

$$\begin{cases} i & = i_A \\ v_A & = \left[R_3 + \dfrac{R_1 R_2}{R_1 + R_2} \right] i_A - R_3 \, i_B \\ v_B & = -R_3 \, i_A + (R_3 + R_4) \, i_B \end{cases}$$

The equations of the equivalent algebraic system are

$$\begin{cases} v_A & = v_{A,0} + R_{AA} \, i_A + R_{AB} \, i_B \\ v_B & = v_{B,0} + R_{BA} \, i_A + R_{BB} \, i_B \\ i & = i_{,0} + i_{/i_A} \, i_A + i_{/i_B} \, i_B \end{cases}$$

$$\begin{bmatrix} v_A(t) \\ v_B(t) \end{bmatrix} = \begin{bmatrix} v_{A0} \\ v_{B0} \end{bmatrix} + \begin{bmatrix} R_3 + \frac{R_1 R_2}{R_1 + R_2} & -R_3 \\ -R_3 & R_3 + R_4 \end{bmatrix} \begin{bmatrix} i_A(t) \\ i_B(t) \end{bmatrix}$$
$$i(t) = i_0 + i_A(t)$$



$$\det \mathbf{R} = \left(R_3 + \frac{R_1 R_2}{R_1 + R_2}\right)(R_3 + R_4) - R_3^2 =$$
$$= (R_3 + R_4)\left(R_3 + \frac{R_1 R_2}{R_1 + R_2} - \frac{R_3^2}{R_3 + R_4}\right) =$$
$$= (R_3 + R_4)\left(\frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4}\right) .$$

Steady solution for $t \le 0^-$. With switch open $i_B = 0$ and steady conditions $i_A = C\dot{v}_A = 0$,

$$\begin{cases} v_A(0^-) & = v_{A,0} = \quad 7.67\,V \\ v_B(0^-) & = v_{B,0} = -13.00\,V \\ i(0^-) & = i_{,0} = \quad 3.00\,A \end{cases}$$

Transient dynamics, when the switch closes $v_B(t \ge 0^+) = 0$,

$$i_A(t) = \frac{R_3 + R_4}{\det \mathbf{R}}\Delta v_A(t) + \frac{R_3}{\det \mathbf{R}}\Delta v_B(t)$$

- **Tension across the switch**

$$v_B(t) = v_{B,0}\, h(-t)$$
$$\Delta v_B(t) = v_B(t) - v_{B,0} = -v_{B,0}\, h(t) .$$

- **Tension across the capacitor.** The dynamical equation for the difference of the state variable reads

$$0 = i_A + C\dot{v}_A =$$
$$= \frac{R_3 + R_4}{\det \mathbf{R}}\Delta v_A(t) + \frac{R_3}{\det \mathbf{R}}\Delta v_B(t) + C\dot{v}_A .$$

As $v_A(t=0) = v_{A,0}$ (no jump in state variables without impulsive forcing), $\Delta v_A = v_A - v_{A,0}$, and $\frac{d}{dt}\Delta v_A = \frac{d}{dt} v_A$, the dynamical equation reads

$$\begin{cases} \dfrac{\det \mathbf{R}}{R_3+R_4} C \dfrac{d}{dt}\Delta v_A + \Delta v_A = -\dfrac{R_3}{R_3+R_4}\Delta v_B(t) = \dfrac{R_3}{R_3+R_4}\, v_{B,0}\, h(t) \\ \\ \Delta v_A(0^-) = 0 \,. \end{cases}$$

$$\Delta v_A(t) = \frac{R_3}{R_3+R_4} v_{B,0} \left[1 - \exp\left(-\frac{t}{\tau}\right)\right]\, h(t) \,,$$

having defined the time constant and the equivalent reistence seen by the capacitor

$$R_{eq} := \frac{\det \mathbf{R}}{R_3+R_4} = \frac{R_1 R_2}{R_1+R_2} + \frac{R_3 R_4}{R_3+R_4} = \frac{50}{21} V = 2.381\, V$$

$$\tau := R_{eq} C = 1.1905\, s$$

Tension through the capacitor reads

$$\begin{aligned} v_A(t) &= v_{A,0} + \delta v_A(t) = \\ &= v_{A,0} + \Delta v_{A,+\infty} \left[1 - \exp\left(-\frac{t}{\tau}\right)\right]\, h(t) \,, \end{aligned}$$

so that the values

$$\begin{aligned} v_A(0^+) &= v_{A,0} = 7.67\, V \\ v_A(+\infty) &= v_{A,0} + \Delta v_{A,+\infty} = (7.667 - 5.571)\, V = 2.095\, V \,. \end{aligned}$$

- **Current through the capacitor.**

$$\begin{aligned} i_A(t) &= \frac{R_3+R_4}{\det \mathbf{R}} \Delta v_A(t) + \frac{R_3}{\det \mathbf{R}} \Delta v_B(t) = \\ &= \frac{R_3+R_4}{\det \mathbf{R}} \frac{R_3}{R_3+R_4} v_{B,0} \left[1 - \exp\left(-\frac{t}{\tau}\right)\right]\, h(t) - \frac{R_3}{\det \mathbf{R}} v_{B,0}\, h(t) = \\ &= -\frac{R_3}{\det \mathbf{R}} v_{B,0} \exp\left(-\frac{t}{\tau}\right)\, h(t) \\ &= 2.34\, A\, \exp\left(-\frac{t}{\tau}\right)\, h(t) \,. \end{aligned}$$

so that the values

$$\begin{aligned} i_A(0^+) &= 2.34\, A \\ i_A(+\infty) &= 0.00\, A \end{aligned}$$

- **Current** $i(t)$

$$\begin{aligned} i(t) &= i_{,0} + i_A(t) = \\ &= a - \frac{R_3}{\det \mathbf{R}} v_{B,0} \exp\left(-\frac{t}{\tau}\right)\, h(t) \\ &= 3.00\, A + 2.34\, A\, e^{-\frac{t}{\tau}} h(t) \,, \end{aligned}$$

so that the values

$$\begin{aligned} i(0^+) &= 5.35\, A \\ i(+\infty) &= 3.00\, A \end{aligned}$$

Energy stored in the capacitor at $t = 0$. Energy in the capacitor reads

$$E_C(t) = \frac{1}{2} C v_A^2(t) \ .$$

At $t = 0$, $v_A(0) = 7.667\,V$ and $E_C(0) = 14.694\,J$.

Exercise 6.7.2 (Exam 2025-01-22, Exercise 1.)

1) Il circuito di Figura 1, con ingressi stazionari, è così assegnato:

$E_1 = 30\ \mathrm{V}$ $\quad E_2 = 50\ \mathrm{V}$ $\quad R_1 = 4\ \Omega$ $\quad R_2 = 7\ \Omega$

$R_3 = 10\ \Omega$ $\quad R_4 = 3\ \Omega$ $\quad R_6 = 8\ \Omega$

$R_7 = 2\ \Omega$ $\quad R_8 = 12\ \Omega$ $\quad C = 0.5\ \mathrm{mF}$

L'interruttore S è aperto da tempo infinito e viene chiuso all'istante $t = 0$.

Determinare:

- l'andamento nel tempo della corrente $i(t)$ sia in termini analitici che grafici (andamento qualitativo).
- l'energia immagazzinata nel capacitore nell'istante di tempo $t = \tau$, essendo τ la costante di tempo del circuito.

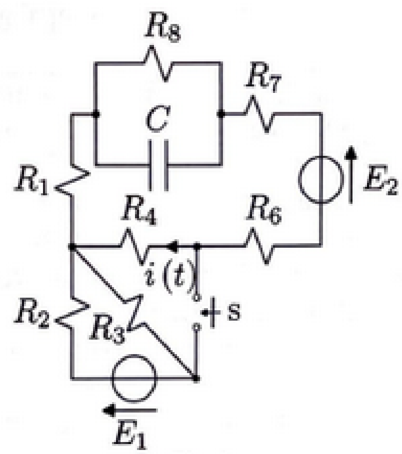


Fig. 1.

Solution

Following the **guidelines for the solution**, a *many-port Thevenin equivalent circuit* of the resistive part of the circuit is found, with two ports for interfacing with the capacitor (A) and with the switch. The dynamical equation of the system is written in state-space representation, writing the voltage at the ports and the unknown variable $i(t)$ as outputs; the capacitor contitutive equation is used to find the time evolution of the system once the switch is closed

Internal generators on, open circuit

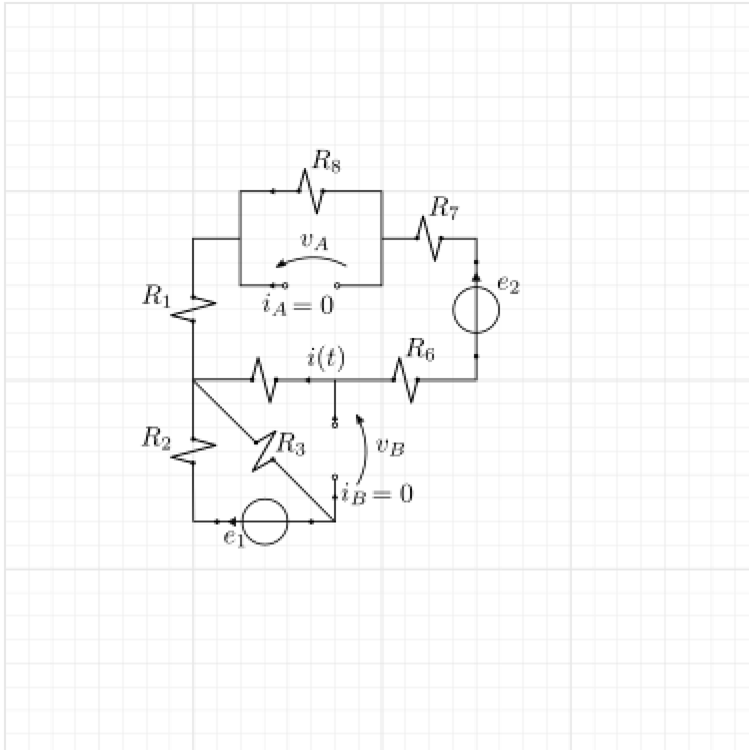
Solution using two loop currents, i_1 in the upper part of the circuit and i_2 in the lower triangle. Using KVL

$$\begin{aligned}
0 &= e_2 - (R_7 + R_8 + R_1 + R_4 + R_6) i_{2,0} \\
0 &= e_1 - (R_2 + R_3) i_{1,0}
\end{aligned}$$

$$\begin{aligned}
i_{2,0} &= \frac{1}{R_{14678}} e_2 \\
i_{1,0} &= \frac{1}{R_{23}} e_1
\end{aligned}$$

with $R_{14678} = R_1 + R_4 + R_6 + R_7 + R_8$, and $R_{23} = R_2 + R_3$. The desired physical quantities are

$$\begin{cases}
v_{A,0} &= -R_8 i_{2,0} = -\dfrac{R_8}{R_{14678}} e_2 \\
v_{B,0} &= -R_4 i_{2,0} + R_3 i_{1,0} = -\dfrac{R_4}{R_{14678}} e_2 + \dfrac{R_3}{R_{23}} e_1 \\
i_0 &= -i_{2,0} = -\dfrac{1}{R_{14678}} e_2
\end{cases}$$



and their values

$$\begin{cases} v_{A,0} & = -20.6900\,V \\ v_{B,0} & = \;\;\,12.4750\,V \\ i_0 & = -1.7241\,A \end{cases}$$

Internal generators off, current generators at the ports

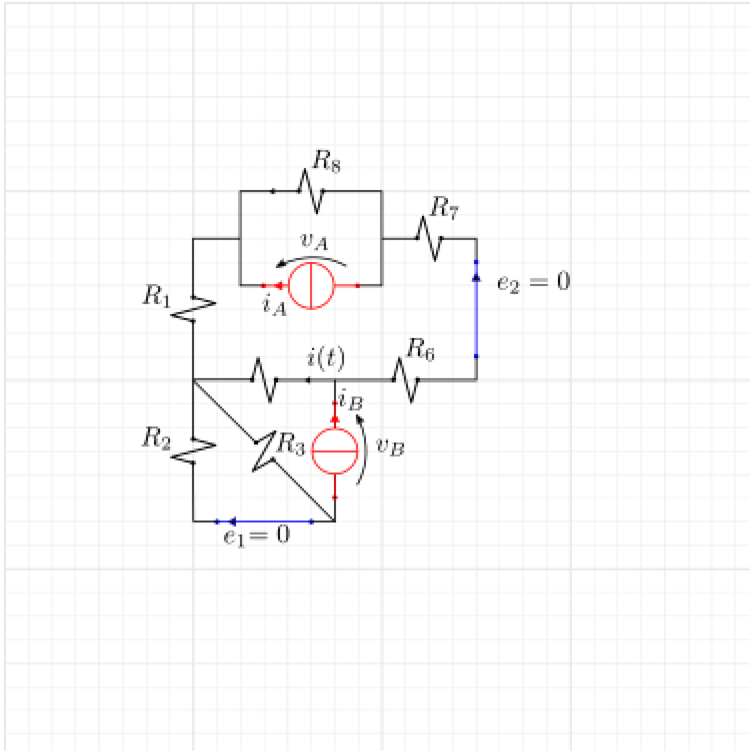
Callling i_A and i_B the current passing through the current generators connected at the ports. The solution is found powering one generation at a time and then exploiting PSCE

Powering A

$$\begin{aligned} & 0 = (i_2 - i_A)R_8 + i_2(R_{14678}) \\ \rightarrow \quad & i_2 = \frac{R_8}{R_{14678}} i_A \end{aligned}$$

$$\begin{aligned} v_{A,A} = -R_8(i_2 - i_A) \quad & = \frac{R_8 R_{1467}}{R_{14678}} i_A \\ v_{B,A} = -R_4 i_2 \quad & = -\frac{R_4 R_8}{R_{14678}} i_A \\ i_{,A} = -i_2 \quad & = -\frac{R_8}{R_{14678}} i_A \end{aligned}$$

$$\begin{aligned} v_{A,A} = R_{AA}\, i_A \quad & = \;\;\, 7.0345\,\Omega\, i_A \\ v_{B,A} = R_{BA}\, i_A \quad & = -1.2414\,\Omega\, i_A \\ i_{,A} = i_{/i_A}\, i_A \quad & = -0.4138\;\; i_A \end{aligned}$$



Powering B.

Currents in the two parallel branches in the upper part of the circuit (current dividers) read

$$
\begin{aligned}
i_{2,B} &= \frac{R_4}{R_{14678}} i_B \\
i_{3,B} &= \frac{R_2}{R_{23}} i_B
\end{aligned}
$$

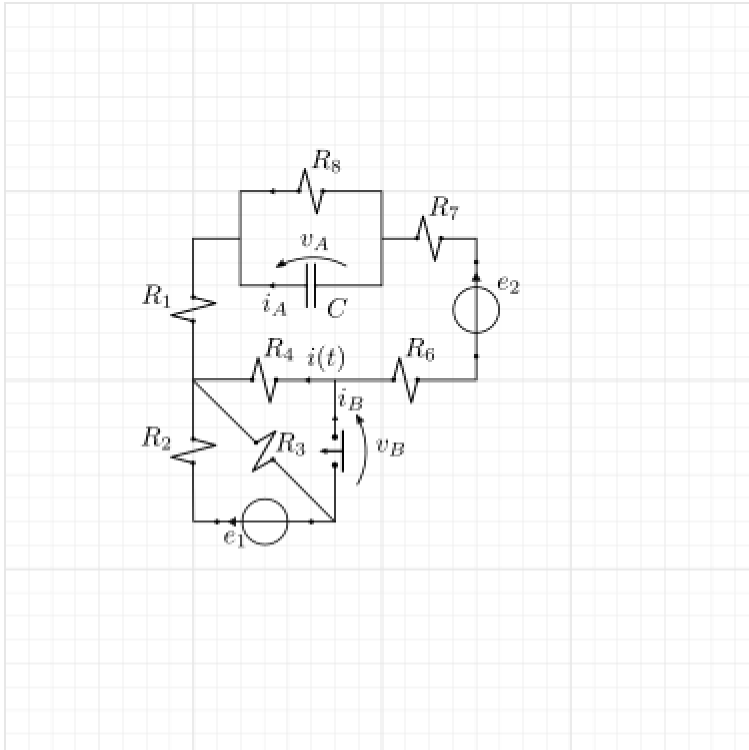
and the desired variables

$$
\begin{aligned}
i_{,B} &= i_{4,B} & &= \frac{R_{1678}}{R_{14678}} i_B \\
v_{A,B} &= -R_8 i_{2,B} & &= -\frac{R_4 R_8}{R_{14678}} i_B \\
v_{B,B} &= R_4 i_{4,B} + R_3 i_{3,B} & &= \left[\frac{R_4(R_{1678})}{R_{14678}} + \frac{R_2 R_3}{R_{23}}\right] i_B
\end{aligned}
$$

$$
\begin{aligned}
v_{A,B} = R_{AB}\, i_B &= -1.2414\,\Omega\, i_B \\
v_{B,B} = R_{BB}\, i_B &= 6.8073\,\Omega\, i_B \\
i_{,B} = i_{/i_B}\, i_B &= 0.8966\;\; i_B
\end{aligned}
$$

The equations of the equivalent algebraic system are

$$
\begin{cases}
v_A &= v_{A,0} + R_{AA}\, i_A + R_{AB}\, i_B \\
v_B &= v_{B,0} + R_{BA}\, i_A + R_{BB}\, i_B \\
i &= i_{,0} + i_{/i_A}\, i_A + i_{/i_B}\, i_B
\end{cases}
$$



and they can be used to write the currents as a function of the tensions

$$i_A = \frac{1}{\det \mathbf{R}} \left(R_{BB} \, \Delta v_A(t) - R_{AB} \, \Delta v_B(t) \right)$$
$$i_B = \frac{1}{\det \mathbf{R}} \left(-R_{BA} \, \Delta v_A(t) + R_{AA} \, \Delta v_B(t) \right)$$

The switch command is off for $t \le 0^-$, on for $t > 0$,

$$i_B(t \le 0^-) = 0 \qquad , \qquad v_B(t \ge 0^+) = 0 \, .$$

Steady solution for $t \le 0^-$. With switch open $i_B = 0$ and steady conditions $i_A = C \dot{v}_A = 0$,

$$\begin{cases} v_A(0^-) & = v_{A,0} = -20.6900 \, V \\ v_B(0^-) & = v_{B,0} = \quad 12.4750 \, V \\ i(0^-) & = i_{,0} = \ -1.7241 \, A \end{cases}$$

Transient dynamics. For $t \ge 0$, the switch is closed and thus $v_B(t \ge 0^+) = 0$.

- **Tension across the switch** as a function of time

$$v_B(t) = v_{B,0} \, h(-t) = v_{B,0} (1 - h(t))$$
$$\Delta v_B(t) = v_B(t) - v_{B,0} = -v_{B,0} \, h(t) \, .$$

- **Tension across the capacitor.** Writing i_A across the capacitor as a function of the tensions, the constitutive equation of the capacitor becomes

$$0 = C \frac{d \Delta v_A}{dt} + i_A =$$
$$= C \frac{d \Delta v_A}{dt} + \frac{1}{\det \mathbf{R}} \left(R_{BB} \, \Delta v_A - R_{AB} \, \Delta v_B \right)$$

$$\begin{cases} R_{eq} C \dfrac{d\Delta v_A}{dt} + \Delta v_A = \dfrac{R_{AB}}{R_{BB}} \, \Delta v_B(t) = -\dfrac{R_{AB}}{R_{BB}} v_{B,0} \, h(t) \\ \\ \Delta v_A(0) = 0 \ , \end{cases}$$

with

$$\begin{aligned} R_{eq} &= \frac{\det \mathbf{R}}{R_{BB}} = 6.8081 \, \Omega \\ \tau &= R_{eq} C = 3.4041 \cdot 10^{-3} \, s \\ \det \mathbf{R} &= 46.345 \, \Omega^2 \end{aligned}$$

The solution of the differential equation provides the difference of the tension through the capacitor w.r.t. the initial steady condition

$$\Delta v_A(t) = \Delta v_{A,+\infty} \left(1 - e^{-\frac{t}{\tau}} \right) \, h(t) \ ,$$

with $\Delta v_{A,+\infty} = -\frac{R_{AB}}{R_{BB}} v_{B,0} = 2.2742 \, V$. The voltage across the capacitor as a function of time t thus reads

$$\begin{aligned} v_A(t) &= v_{A,0} + \Delta v_A(t) = \\ &= v_{A,0} + \Delta v_{A,+\infty} \left(1 - e^{-\frac{t}{\tau}} \right) \, h(t) \ , \end{aligned}$$

so that the values

$$\begin{aligned} v_A(0^+) &= v_{A,0} && = -20.69 \, V \\ v_A(+\infty) &= v_{A,0} + \Delta V = -20.69 \, V + 2.2742 \, V && = -18.4158 \, V \end{aligned}$$

- **Current through the capacitor.**

$$\begin{aligned} i_A(t) &= \frac{1}{\det \mathbf{R}} \left(R_{BB} \, \Delta v_A(t) - R_{AB} \, \Delta v_B(t) \right) = \\ &= \frac{1}{\det \mathbf{R}} \left[R_{BB} \left(-\frac{R_{AB}}{R_{BB}} v_{B,0} \right) \left(1 - e^{-\frac{t}{\tau}} \right) \, h(t) + R_{AB} \, v_{B,0} \, h(t) \right] = \\ &= \frac{R_{AB}}{\det \mathbf{R}} v_{B,0} e^{-\frac{t}{\tau}} \, h(t) \ . \end{aligned}$$

so that the values

$$\begin{aligned} i_A(0^+) &= \frac{R_{AB}}{\det \mathbf{R}} v_{B,0} = \frac{-1.2414 \, \Omega}{46.908 \, \Omega^2} \, 12.475 \, V = -0.334 \, A \\ i_A(+\infty) &= v_{A,0} + \Delta V = -20.69 \, V + 2.2742 \, V && = 0.0 \, A \end{aligned}$$

or with $i_A = -C \frac{d\Delta v_A}{dt} \dots$

- **Current across the switch**

$$\begin{aligned} i_B(t) &= \frac{1}{R_{BB}} \Big[v_B(t) - v_{B,0} - R_{BA} i_A(t) \Big] = \\ &= \frac{1}{R_{BB}} \left[- v_{B,0} - R_{BA} \frac{R_{AB}}{\det \mathbf{R}} v_{B,0} e^{-\frac{t}{\tau}} \right] h(t) = \\ &= -\frac{v_{B,0}}{R_{BB}} \left[1 + \frac{R_{BA} R_{AB}}{\det \mathbf{R}} \, e^{-\frac{t}{\tau}} \right] h(t) \ . \end{aligned}$$

so that the values

$$\begin{aligned} i_B(0^+) &= -\frac{v_{B,0}}{R_{BB}} \left[1 + \frac{R_{BA} R_{AB}}{\det \mathbf{R}} \right] = -\frac{v_{B,0} R_{AA}}{\det \mathbf{R}} = -\frac{7.0345 \, \Omega}{46.345 \, \Omega^2} \, 12.475 \, V = -1.8929 \, A \\ i_B(+\infty) &= -\frac{v_{B,0}}{R_{BB}} = -\frac{12.475 \, V}{6.8073 \, \Omega} = -1.8320 \, A \ . \end{aligned}$$

- **Current** $i(t)$

$$
\begin{aligned}
i(t) &= i_0 - 0.4138\, i_A(t) + 0.8966\, i_B(t) = \\
&= i_0 + \left[-0.4138\, i_{A,0^+} e^{-\frac{t}{\tau}} + 0.8966\, \left(i_{B,+\infty} + (i_{B,0^+} - i_{B,+\infty}) e^{-\frac{t}{\tau}}\right)\right]\, h(t) \ ,
\end{aligned}
$$

so that

$$
\begin{aligned}
i(0^+) &= i_0 - 0.4138\, i_{A,0^+} + 0.8966\, i_{B,0^+} = \\
&= -1.7214\, A - 0.4138\, (-0.334\, A) + 0.8966\, (-1.8929\, A) = -3.2831\, A \\
i(+\infty) &= i_0 + 0.8966\, i_{B,+\infty} = \\
&= -1.7214\, A + 0.8966\, (-1.8320\, A) = -3.3671\, A
\end{aligned}
$$

Energy stored in the capacitor.

$$
E_C(t) = \frac{1}{2} C v_A^2(t) \ ,
$$

and for $t = \tau$,

$$
\begin{aligned}
v_A(t) &= v_{A,0} + \Delta v_{A,+\infty} \left(1 - e^{-\frac{t}{\tau}}\right)\, h(t) = \\
&= -20.69\, V + 2.2742\, V \left(1 - e^{-\frac{t}{\tau}}\right)\, h(t) \ ,
\end{aligned}
$$

and thus $v_A(\tau) = -19.25\, V$

$$
E_C(\tau) = 0.5 \cdot 5 \cdot 10^{-4}\, F \cdot (19.25\, V)^2 = 9.26 \cdot 10^{-2}\, J \ .
$$

Exercise 6.7.3 (Exam 2024-09-06, Exercise 1.)

1) Il circuito di Figura 1, con ingressi stazionari, è così assegnato:

$V_s = 5\,\mathrm{V}$
$I_s = 5\,\mathrm{A}$
$R_1 = 1\,\Omega$
$R_2 = 2\,\Omega$
$R_3 = 3\,\Omega$
$R_4 = 4\,\Omega$
$L = 100\,\mathrm{mH}$

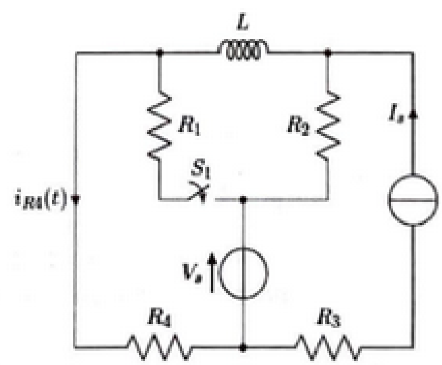


Fig. 1.

L'interruttore S è aperto da tempo infinito e viene chiuso all'istante $t = 0$.

Determinare:

- l'andamento nel tempo della corrente $i_{R4}(t)$ sia in termini analitici che grafici (andamento qualitativo).
- l'energia immagazzinata nell'induttore nell'istante di tempo t = 0 s.

Solution - todo

Exercise 6.7.4 (Exam 2024-07-22, Exercise 1.)

1) Il circuito di Figura 1, con ingressi stazionari, è così assegnato:

$E_1 = 72\ \mathrm{V}$
$E_2 = 95\ \mathrm{V}$
$R_1 = 16\ \Omega$
$R_2 = 16\ \Omega$
$R_3 = 24\ \Omega$
$R_4 = 20\ \Omega$
$L = 44\ \mathrm{mH}$

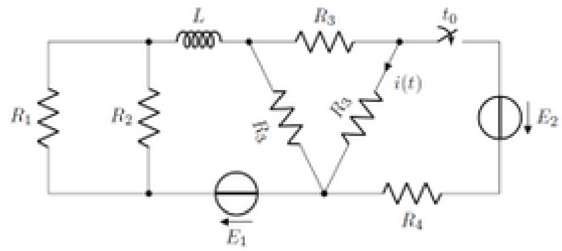


Fig. 1.

L'interruttore S è aperto da tempo infinito e viene chiuso all'istante $t = 0$.

Determinare:

- l'andamento nel tempo della corrente $i(t)$.
- l'energia immagazzinata nell'induttore nell'istante di tempo t = 3 ms.

Solution - todo

Exercise 6.7.5 (Exam 2024-02-13, Exercise 1.)

1b) SOLO GESTIONALI Il circuito di Figura 1, con ingressi stazionari, è così assegnato:

$E_1 = 20\ \mathrm{V}$ $\quad R_1 = 5\ \Omega$ $\quad R_4 = 15\ \Omega$
$E_2 = 15\ \mathrm{V}$ $\quad R_2 = 10\ \Omega$ $\quad R_5 = 6\ \Omega$
$A = 10\ \mathrm{A}$ $\quad R_3 = 4\ \Omega$
$L = 1\ \mathrm{mH}$

L'interruttore S è aperto da tempo infinito e viene chiuso all'istante $t = 0$.

Determinare:

- l'andamento nel tempo della corrente $i_{R3}(t)$, formula e andamento grafico.

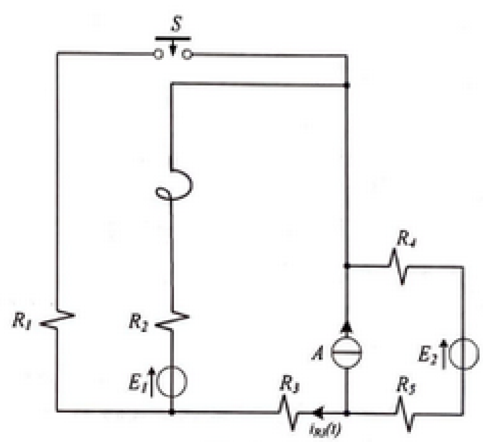


Fig. 1.

Solution - todo

6.7.2 Harmonic regime of linear electrical grids

Exercise 6.7.6 (Exam 2025-02-11, Exercise 2.)

2) Il circuito di Figura 2, in **regime alternato sinusoidale alla frequenza di 50Hz,** è così assegnato:

$L_1 = 250$ mH
$C_1 = 350\ \mu$F
$R_1 = 20\ \Omega$
$\bar{Z}_1 = 1 + j2\ \Omega$
$\bar{Z}_2 = 2 - j2\ \Omega$
$\bar{Z}_3 = 3 + j4\ \Omega$
$e_1(t) = 50\sqrt{2}\cos(\omega t)$ V
$e_2(t) = 10\sqrt{2}\cos(\omega t + \pi)$ V
$a(t) = 3\sqrt{2}\sin(\omega t + \pi/2)$ A

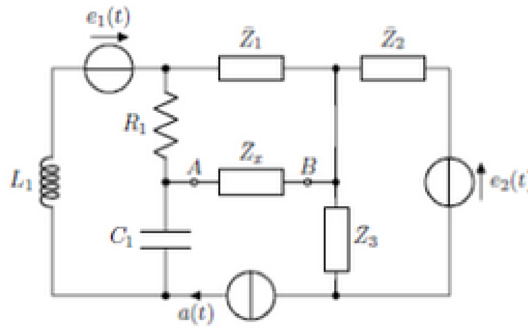


Fig. 2.

Determinare:

- il valore dell'impedenza $\bar{Z}_x$ che garantisca il massimo trasferimento di potenza attiva
- le potenze attiva P_{Zx} e reattiva Q_{Zx} assorbite dall'impedenza $\bar{Z}_x$

Solution

First *one-port equivalent Thevenin circuit* of the circuit with port $A - B$ is evaluated, then *power flow in harmonic regime* is discussed.

Thevenin equivalent: voltage. With open circuit in $A - B$, current a flows in the lower branch and in impedence Z_1. Clockwise loop currents i_1 and i_2 flows in the left and right loop respectively. Kirchhoff voltage laws in the left and right loops give

$$\begin{aligned} 0 &= e_1 - Z_L(i_1 + a) - (R_1 + Z_C)i_1 \\ 0 &= -e_2 - Z_2(i_2 + a) - Z_3 i_2 \end{aligned} \qquad \rightarrow \qquad \begin{aligned} i_1 &= \frac{e_1 - Z_L a}{Z_L + Z_C + R_1} \\ i_2 &= -\frac{e_2 + Z_2 a}{Z_2 + Z_3} \end{aligned}$$

and thus using Kirchhoff voltage law on the loop with nodes $A - B$ and closing through Z_1 and R_1,

$$V_{Th} = R_1 i_1 + Z_1 a = \dots$$

Thevenin equivalent: impedence. Opening circuit at the current generator, and replace tension generators with short circuits, the equivalent impedence is

$$Z_{Th} = ((Z_C + Z_L) \parallel R_1) + Z_1 \ .$$

Equivalent circuit. Kirchhoff voltage law on the equivalent circuit reads

$$0 = V_{Th} - Z_{Th} i - Z_x i = 0 \ ,$$

and thus

$$I = \frac{V_{Th}}{Z_{Th} + Z_x} = ...$$

Power. Complex power reads

$$S = VI^* = Z_x |I|^2 = \frac{Z_x}{|Z_{Th} + Z_x|^2} |V_{th}|^2 \ ,$$

Writing the impedence as $Z_x = R_x + iX_x$, the active power reads

$$P = \frac{R_x}{(R_{Th} + R_x)^2 + (X_{Th} + X_x)^2} |V_{Th}|^2 \ .$$

With the physical constraints $R \ge 0$, the problem is a constrained optimization problem of finding the maximum value of the function $P(R_x, X_x)$ subject to the constraint $R_x \ge 0$,

$$\text{find} \ \max_{R_x, X_x} P(R_x, X_x) \qquad \text{s.t.} \qquad R_x \ge 0 \ .$$

The denominator is the sum of two non negative terms, one function of R_x and one function of X_x. The independent variable X_x only appears in this term at the denominator, so that this term must vanish at the solution of the optimization problem, and thus

$$\widetilde{X}_x = -X_{Th} \ .$$

The remaining term is a function of R_x only and proportional to

$$f(R_x) = \frac{R_x}{(R_{Th} + R_x)^2} \ .$$

Local extremes of this function is attained where

$$\begin{aligned} 0 = f'(R_x) &= \frac{(R_{Th} + R_x)^2 - 2R_x(R_{Th} + R_x))}{(R_{Th} + R_x)^4} = \\ &= \frac{R_{Th}^2 - R_x^2}{(R_{Th} + R_x)^4} \end{aligned}$$

and thus, within the physical limit of the problem, the local and global maximum of the function (check that $f''(\widetilde{R}_x) < 0$), is attained for

$$\begin{aligned} \widetilde{R}_x &= R_{Th} \\ \widetilde{Z}_x &= R_{Th} - iX_{Th} \end{aligned}$$

and the maximum active power is

$$P_{max} = P(\widetilde{Z}_x) = \frac{|V_{Th}|^2}{4R_{Th}} \ .$$

while the reactive power in this condition reads

$$Q = -\frac{X_{Th}}{4R_{Th}^2} |V_{Th}|^2 \ .$$

Exercise 6.7.7 (Exam 2025-02-11, Exercise 3.)

3) **SOLO ENERGETICI:** Il circuito di Figura 3, in **regime sinusoidale alla f=50Hz**, è così assegnato (tensione in valore efficace):

$R_1 = 1\ \Omega$
$R_2 = 10\ \Omega$
$R_3 = 2\ \Omega$
$X_1 = 400\ \Omega$
$X_2 = 100\ \Omega$
$|\bar{V}_L| = 400\ \mathrm{V}$
$A_L = 3\,\mathrm{kVA}$
$\cos\phi_L = 0.75$ ind.

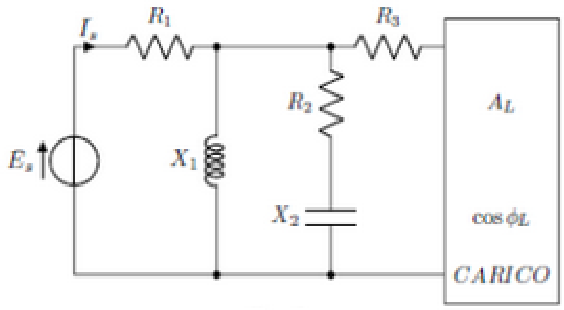


Fig. 3.

Determinare:

- il valore efficace della tensione del generatore $\bar{E}_S$
- il valore efficace della corrente $\bar{I}_S$
- il fattore di potenza associato al generatore $\bar{E}_S$, cioè lo sfasamento tra $\bar{E}_S$ e $\bar{I}_S$

Solution

First *power flow in harmonic regime* is used to calculate load impedence, then the electrical circuit is solved, and the power on the tension generator is computed.

Load impedence Z_L. Load impedence appears in the load constitutive equation $V_L = Z_L I_L$, and can be evalauted from data about complex power,

$$S_L = |S_L| e^{i\phi_L} = V_L I_L^* = Z_L |I|^2 = \frac{1}{Z_L^*} |V_L|^2$$

$$Z_L = \frac{|V_L|^2}{|S_L|} e^{i\varphi_L}$$

Current I_s**.** From data of load power, it's possible to evaluate the current I_s. The current I_L through the load reads

$$S_L = V_L I_L^* \qquad \rightarrow \qquad I_L = \frac{S_L^*}{V_L^*} = \frac{|S_L|}{|V_L|} e^{i(-\phi_L + \phi_V)}$$

The three parallel sides act as current divider so that

$$I_L = \frac{(R_3 + Z_L)^{-1}}{(R_3 + Z_L)^{-1} + ((iX_1) \parallel (R_2 + iX_2))^{-1}} I_s$$

and thus

$$I_s = |I_s| e^{i\varphi_{I_s}} = \ldots$$

Equivalent circuit. The impedence of the circuit powered by the tension generatore is

$$Z_{eq} = R_1 + (iX_1 \parallel (R_2 + iX_2) \parallel (R_3 + Z_L)) \ .$$

Given the equivalent impedance, and the current I_s the voltage across the tension generator is

$$E_s = Z_{eq} I_s = |E_s| e^{i\varphi_{E_s}} \ \ldots \ .$$

and the power factor is $\cos\varphi_s = \ldots$, where

$$\varphi_s = \varphi_{E_s} - \varphi_{I_s} = \ldots \ .$$

Exercise 6.7.8 (Exam 2025-01-22, Exercise 2.)

2) Il circuito di Figura 2, in **regime alternato sinusoidale alla frequenza di 50Hz,** è così assegnato:

$\bar{E}_1 = 50e^{j\frac{\pi}{3}}$ V $\quad \bar{E}_2 = 100e^{j\frac{\pi}{6}}$ V $\quad \bar{A}_1 = 5e^{j\frac{\pi}{8}}$ A
$\bar{A}_2 = 10$ A $\quad R_1 = 5\,\Omega \quad L_1 = 50$ mH
$C_1 = 0.1$ mF $\quad L_2 = 15$ mH $\quad R_2 = 10\,\Omega$
$L_3 = 10$ mH $\quad C_3 = 0.2$ mF $\quad R_4 = 15\,\Omega$
$L_4 = 20$ mH $\quad L_5 = 30$ mH $\quad C_5 = 0.3$ mF

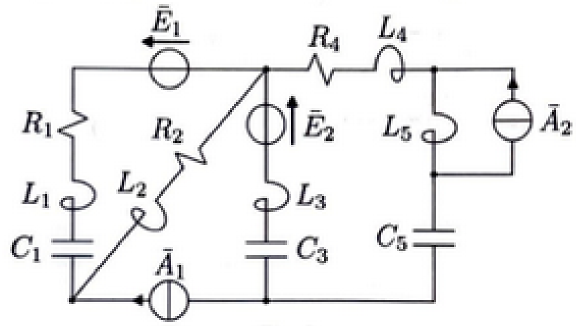


Fig. 2.

Determinare:

- l'espressione nel dominio del tempo della tensione ai capi del generatore di corrente $\bar{A}_1$
- le potenze complessa, apparente, attiva e reattiva messe in gioco da $\bar{A}_1$ (indicando esplicitamente: nome, simbolo e unità di misura)

Solution

First *one-port equivalent Thevenin circuit* of the circuit with port $A - B$ is evaluated, then the equivalent circuit is solved to find the tension $v(t)$ across the current generator, and *power flow in harmonic regime* is discussed.

Thevenin equivalent: voltage. With an open circuit, the network can be split into two parts: the triangle in the upper-left side and the section in the right part.

In the triangular part, a current I_a flows in counter-clockwise direction, while current I_b flows in the right part in clockwise direction,

$$\begin{aligned} I_a &= \frac{E_1}{Z_1 + Z_2} \\ I_b &= \frac{E_2 + i\Omega L_5 A_2}{Z_4 + Z_5 + Z_3} \end{aligned}$$

as

$$E_2 + \Bigg(Z_4 + Z_3 \underbrace{- i \frac{1}{\Omega C_5} + i\Omega L_5}_{=Z_5} \Bigg) I_b + i\Omega L_5 A_2 = 0 \ .$$

with Z_k being the impedence of the k-th side. Thevenin voltage thus reads

$$V_{Th} = E_2 - Z_3 I_b + Z_2 I_a$$

Thevenin equivalent: impedence. Equivalent impedence reads

$$Z_{Th} = (Z_1 \parallel Z_2 + (Z_3 \parallel (Z_4 + Z_5)))$$

Equivalent circuit. Prescribed current A_1 flows in the equivalent circuit, and the voltage across the current generator is evaluated with Krichhoff voltage law

$$V_{A_1} - V_{Th} - Z_{Th} A_1 = 0 \ ,$$

$$V_{A_1} = V_{Th} + Z_{Th} A_1 = |V_A| e^{i\varphi_{V_{A_1}}} \ .$$

Signal in time is reconstructed using using the relation between effective and maximum amplitude of the oscillation and evaluating the real part of the signal $|V_{A_1}| e^{i(\Omega t + \varphi_{V_{A_1}})}$

$$v_{A_1}(t) = \sqrt{2} |V_{A_1}| \cos(\Omega t + \varphi_{V_{A_1}}) \ .$$

Poer. Using definitions of *power in circuits in harmonic regime*,

$$\begin{aligned} S_{A_1} &= V_{A_1} I^*_{A_1} \\ |S_{A_1}| &= |V_{A_1}||I_{A_1}| \\ P_{A_1} &= \text{re}\{S_{A_1}\} \\ Q_{A_1} &= \text{im}\{S_{A_1}\} \end{aligned}$$

Exercise 6.7.9 (Exam 2024-09-06, Exercise 2.)

2) Il circuito di Figura 2, in **regime alternato sinusoidale,** è così assegnato:

$R = 10\ \Omega$
$C = 550\ \mu\text{F}$
$L = 350\ \text{mH}$
$\bar{Z}_1 = 10 + j20\ \Omega$
$\bar{Z}_2 = 5 - j5\ \Omega$
$\bar{Z}_3 = 30 + j40\ \Omega$
$e_1(t) = 150\sqrt{2}\cos(\omega t)\ \text{V}$
$e_2(t) = 100\sqrt{2}\cos(\omega t + \pi)\ \text{V}$
$a_1(t) = 5\sqrt{2}\sin(\omega t)\ \text{A}$
$a_2(t) = 3\sqrt{2}\sin(\omega t + \pi/2)\ \text{A}$

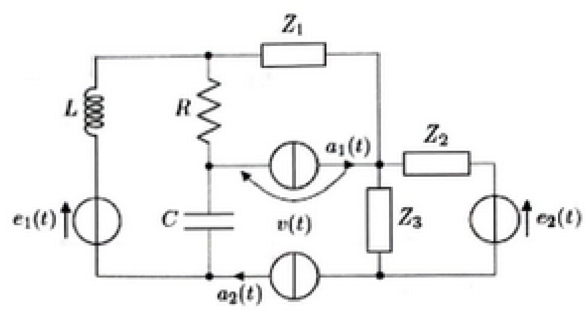


Fig. 2.

<u>Determinare:</u>

- l'espressione nel dominio del tempo della tensione $v(t)$
- le potenze complessa, apparente, attiva e reattiva messe in gioco dall'impedenza Z_1 (indicando esplicitamente: nome, simbolo e unità di misura)

Solution - todo

Exercise 6.7.10 (Exam 2024-07-22, Exercise 2.)

Solution - todo

2) Il circuito di Figura 2, in **regime alternato sinusoidale,** è così assegnato:

$e_8 = 80\cos(10t + \frac{3\pi}{4})$ V
$e_9 = 80\cos(10t + \frac{\pi}{4})$ V
$a_{10} = 6\cos(10t + \frac{\pi}{4})$ A
$e_{11} = 50\sqrt{2}\cos(10t + \pi)$ V
$a_{12} = 3\sqrt{2}\cos(10t - \frac{\pi}{2})$ A

$C_1 = 5$ mF
$R_2 = 20\ \Omega$
$L_3 = 0{,}5$ H
$R_4 = 25\ \Omega$
$L_5 = 1$ H
$R_6 = 10\ \Omega$
$C_7 = 10$ mF

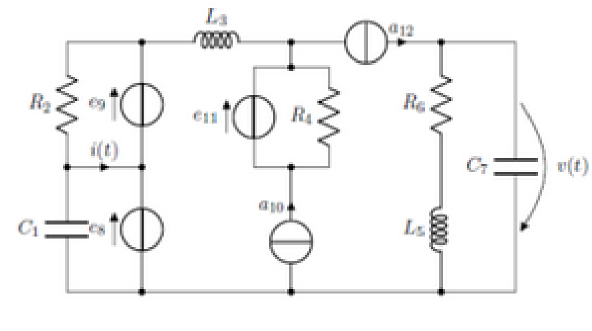


Fig. 2.

Determinare:

- l'espressione nel dominio del tempo della tensione $v(t)$
- il fasore associato alla corrente $i(t)$
- la potenza attiva generata da e_{11},
- la potenza complessa messa in gioco da a_{10}
- la potenza apparente elaborata da R_4 e dalle reattanze associate a L_3 e C_7

6.7.3 Three-phase electrical circuits in harmonic regime

Guidelines for solution

Analyse the network as a standard configuration of a three-phase network (*star-star*,…) and rely on results derived for *three-phase circuits*.

As an example, for a **star-star configuration**:

1. evaluate load impedances, impedances in parallel with the generators, interconnections between phases
2. evaluate voltage difference across the centers of the stars, v_{AB}
3. once v_{AB} is known, it should be easier to evaluate currents and voltages in the grid with KCL and KVL
4. use relations of *power in harmonic regime*, to answer the questions about power: just remember the difference between maximum and effective values, and that a wattmeter measures the active power

Exercise 6.7.11 (Exam 2024-09-06, Exercise 3.)

Solution

This network is a star-star connection with impedances

$$Z_g = (R_1 + sL_1) \parallel \frac{1}{sC_1} \qquad g = 1:3$$

$$Z_4 = R_2 + \frac{1}{sC_2}$$

and inter-connection between phases 2 and 3 with impedance Z_4.

Voltage v_{AB}**.**

$$v_{AB} = \frac{\sum_{g=1}^{3} Y_g\, e_g}{\sum_{k=1}^{4} Y_4}$$

3) Il circuito di Figura 3, in **regime alternato sinusoidale alla frequenza di 50 Hz**, è così assegnato:

$e_1(t) = 220\sqrt{2} \cdot \cos(\omega t)$ V
$e_2(t) = 220\sqrt{2} \cdot \cos(\omega t + \frac{2}{3}\pi)$ V
$e_3(t) = 220\sqrt{2} \cdot \cos(\omega t + \frac{4}{3}\pi)$ V
$f = 50$ Hz
$R_1 = 25\Omega$
$R_2 = 2k\Omega$
$C_1 = 100\ \mu$F
$C_2 = 1kF$
$L_1 = 50$ mH
$\bar{Z}_4 = (10 - j5)\Omega$

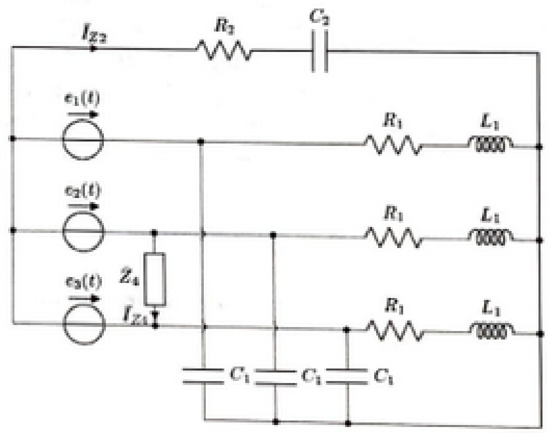


Fig. 3.

Determinare:

- Le correnti $\bar{I}_{Z4}$ e $\bar{I}_{Z2}$
- La potenza complessa, apparente, attiva e reattiva elaborate dal generatore E₂, indicando esplicitamente: nome, simbolo e unità di misura e discutendone il segno.

Generation and loads are equilibrated, and thus $\sum_{g=1}^{3} Y_g\, e_g = 0$, and $v_{AB} = 0$.

Current i_{Z_2}**.** As $v_{AB} = 0$, then $i_{Z_2} = 0$, as in general it whould be $i_{Z_2} = \frac{v_{AB}}{R_2 + \frac{1}{sC_2}}$.

Current i_{Z_4}**.** With KVL on the loop with the two tension generators e_2, e_3 closed with Z_4

$$\begin{aligned} 0 &= e_3 + Z_4 i_{Z_4} - e_2 \\ \rightarrow \quad i_{Z_4} &= \frac{e_2 - e_3}{Z_4} \end{aligned}$$

Currents i_{e_2}**.** Current i_{e_2} through the generator are evaluated through KVL between the centers of the stars,

$$\begin{aligned} 0 &= e_2 - \frac{1}{\frac{1}{R_1 + sL_1} + sC_1} i_{e_2} - v_{AB} \\ \rightarrow \quad i_{e_2} &= \left[\frac{1}{R_1 + sL_1} + sC_1\right] e_2 \end{aligned}$$

Powers of generator 2**.**

$$\begin{aligned} S_2 &= V_2 I_2^* \\ A_2 &= |S_2| \\ P_2 &= \mathrm{re}\{S_2\} \\ Q_2 &= \mathrm{im}\{S_2\} \ , \end{aligned}$$

using the effective values of tension and current V_2, I_2.

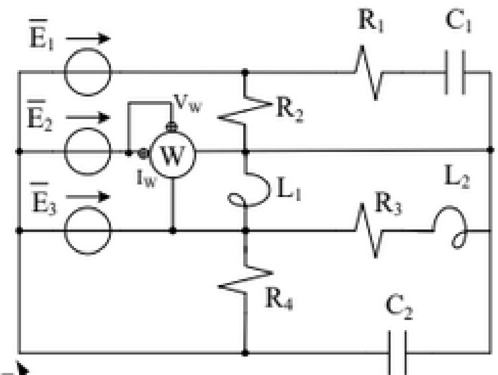
Exercise 6.7.12 (Exam 2024-07-22, Exercise 3.)

3) Il circuito di Figura 3, in **regime alternato sinusoidale alla frequenza di 50 Hz**, è così assegnato:

$E_1 = 200$ V	$R_1 = 40\ \Omega$	$C_1 = 100\ \mu F$
$E_2 = 200$ V	$R_2 = 50\ \Omega$	$C_2 = 150\ \mu F$
$E_3 = 200$ V	$R_3 = 50\ \Omega$	$L_1 = 15$ mH
	$R_4 = 40\ \Omega$	$L_2 = 10$ mH

Determinare:

- l'indicazione del wattmetro
- La potenza complessa messa in gioco da C_2



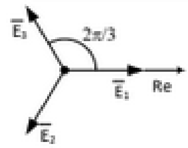


Fig. 3.

Solution

This network is a star-star connection with impedances

$$
\begin{aligned}
Z_1 &= (R_1 + jX_{C_1}) \parallel R_2 \\
Z_2 &= 0 \\
Z_3 &= (R_3 + jX_{L_2}) \parallel jX_{L_1} \\
Z_4 &= jX_{C_2}
\end{aligned}
$$

and inter-connection between phase 3 and the netural with **resistance** R_4, before Z_4, and thus **in parallel with the generator** 3.

Voltage v_{AB}**.** As $Z_2 = 0$, it's not possible to directly use

$$v_{AB} = \frac{\sum_{g=1}^{3} Y_g \, e_g}{\sum_{k=1}^{4} Y_4} \ ,$$

or this must be used with the limit $Y_2 \rightarrow +\infty$, and thus

$$v_{AB} = e_2 \ .$$

Wattmeter tension v_W**.** KVL with the generators 2 and 3,

$$v_W = e_2 - e_3 \ .$$

Wattmeter current $i_w = i_{e_2}$**.** KCL on the center of generation star, $0 = i_{e_1} + i_{e_2} + i_3 + i_4$, with

$$
\begin{aligned}
i_{e_1} &= \frac{1}{Z_1}(e_1 - v_{AB}) \\
i_3 &= \frac{1}{Z_3}(e_3 - v_{AB}) \\
i_4 &= -\frac{1}{Z_4} v_{AB} \ ,
\end{aligned}
$$

being $i_3 = i_{e_3} + i_{R_4}$ the sum of the current in the parallel connection on the branch 3 of the generation. Thus, current

i_{e_2} reads

$$\begin{aligned} i_{e_2} &= -i_{e_1} - i_3 - i_4 = \\ &= -\frac{e_1}{Z_1} - \frac{e_3}{Z_3} + \left(\frac{1}{Z_1} + \frac{1}{Z_3} + \frac{1}{Z_4}\right) v_{AB} \end{aligned}$$

Wattmeter. Wattmeter reading provides the active power

$$P_w = \mathrm{re}\{S_w\} = \mathrm{re}\{v_w i_w^*\} \ .$$

Power on C_2. Current and voltage across C_2 are

$$\begin{aligned} & i_{C_2} = i_4 \\ & v_{C_2} = Z_{C_2} i_{C_2} = \frac{1}{sC_2} i_{C_2} \ , \end{aligned}$$

and the complex power is

$$s = V_{C_2} I^*_{C_2} \ .$$

Exercise 6.7.13 (Exam 2024-06-19, Exercise 1.)

riguarda RIPROVATO e ORALE.

1) Il circuito di Figura 1, in **regime alternato sinusoidale alla frequenza di 50 Hz**, è così assegnato:

$E_1 = 230$ V	$R_1 = 20\ \Omega$	$C_1 = 100\ \mu$F
$E_2 = 230$ V	$R_2 = 30\ \Omega$	$C_2 = 100\ \mu$F
$E_3 = 230$ V	$R_3 = 60\ \Omega$	$L_1 = 20$ mH
	$R_4 = 30\ \Omega$	$L_2 = 15$ mH
		$L_3 = 15$ mH

Determinare:

- L'indicazione del wattmetro
- le potenze attiva, reattiva, apparente e complessa erogate dal generatore E_1 (esplicitando le unità di misura e discutendone il segno).

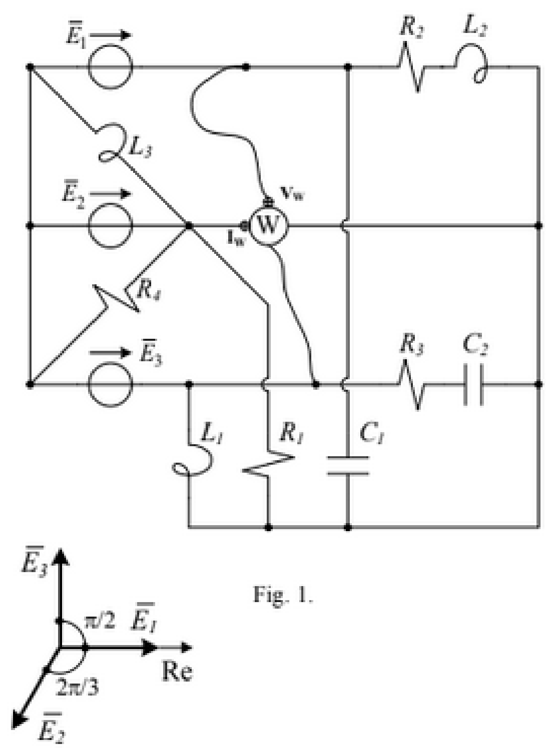


Fig. 1.

Solution

This network is a star-star connection with impedances

$$\begin{aligned}
Z_1 &= (R_2 + jX_{L_2}) \parallel (jX_{C_1}) \\
Z_2 &= (R_1 \parallel 0) \\
Z_3 &= (R_3 + jX_{C_2}) \parallel jX_{L_1}
\end{aligned}$$

with L_2 and R_4 in parallel with generator e_2. As R_1 is in parallel with a short-circuit in Z_2, this impedance is zero and as it is the current through R_1. There's no neutral.

Voltage v_{AB}. As $Z_2 = 0$ (see previous exercise), the voltage between the centers of the stars is

$$v_{AB} = e_2 \ .$$

Wattmeter tension v_W. KVL with the generators 2 and 3,

$$v_W = e_1 - e_3 \ .$$

Wattmeter current $i_w = i_2$. KCL on the center of generation star, $0 = i_{e_1} + i_2 + i_{e_3}$, with

$$\begin{aligned}
i_{e_1} &= \frac{1}{Z_1}(e_1 - e_2) \\
i_{e_3} &= \frac{1}{Z_3}(e_3 - e_2)
\end{aligned}$$

being $i_2 = i_{e_2} + i_{L_1} + i_{R_4}$ the sum of the current in the parallel connection on the branch 2 of the generation. Thus, current i_w reads

$$\begin{aligned}
i_w = i_2 = -i_{e_1} - i_{e_3} = \\
= \frac{1}{Z_1}(e_2 - e_1) + \frac{1}{Z_3}(e_2 - e_3)
\end{aligned}$$

Wattmeter. Wattmeter reading provides the active power

$$P_w = \text{re}\{S_w\} = \text{re}\{v_w i_w^*\} \ .$$

Power of tension generator e_1.

$$s_{e_1} = e_2 i_{e_2}^* \ .$$

…

Exercise 6.7.14 (Exam 2024-02-13, Exercise 2.)

Solution - todo

---

2) Il circuito di Figura 2, in regime alternato sinusoidale alla frequenza di 50 Hz, è così assegnato:

$E_1 = 230$ V	$R_1 = 40\ \Omega$	$C_1 = 100\ \mu F$
$E_2 = 230$ V	$R_2 = 50\ \Omega$	$C_2 = 150\ \mu F$
$E_3 = 230$ V	$R_3 = 50\ \Omega$	$L_3 = 15$ mH
	$R_4 = 40\ \Omega$	
	$R_5 = 30\ \Omega$	
	$R_6 = 20\ \Omega$	

La terna degli ingressi, assegnati in valore efficace, è simmetrica diretta.

Determinare:

- L'indicazione del wattmetro.

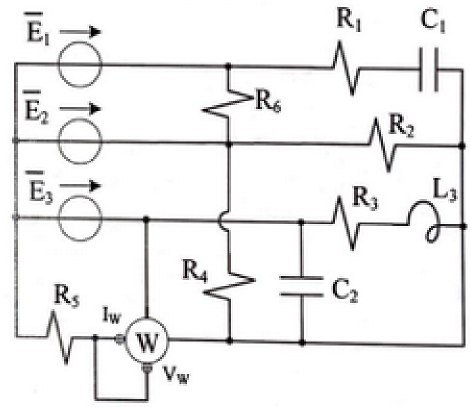


Fig. 2.

6.7.4 Electromagnetic circuits

Guidlines for solution

1. Find the equivalent magnetic network of the inductive part of the system to find the relation,

$$\mathbf{v}(t) = \dot{\boldsymbol{\psi}}(t) = \frac{d}{dt}\left(\mathbf{L}\,\mathbf{i}(t)\right) \ ,$$

   between the tensions and the currents at the ports of the electromagnetic system, usually under the assumpsions of

   - no dispersed fluxes,
   - linear constitutive equation of the ferromagnetic medium, $b = \mu_{\text{Fe}} h$, so that hysteresis is neglected
   - permeability of the ferromagnetic much larger than the permeability of free space, $\mu_{\text{Fe}} \gg \mu_0$, so that the reluctance of the ferrmagnetic medium is negligible if compared with the reluctance of the air gaps. Relucatnce of air gaps reads

$$\theta = \frac{\delta}{\mu_0 A} \ .$$

   In **stationary regime** $\frac{d}{dt} \equiv 0$, and thus inductors act as short-circuits.

2. Use the relation $\mathbf{v} = \frac{d}{dt}\left(\mathbf{L}\,\mathbf{i}\right)$ in the electric network to solve the electromagnetic system
3. Find all the other physical quantities needed, remembering that the volume density of electromagnetic energy in media, under the assumption of linear media, is

$$u = \frac{1}{2\mu}\left|\vec{b}(\vec{r},t)\right|^2 + \frac{1}{2}\varepsilon\left|\vec{e}(\vec{r},t)\right|^2 \ .$$

   Volume density must be integrated over the regions of space where it's not negligible, like air gaps.

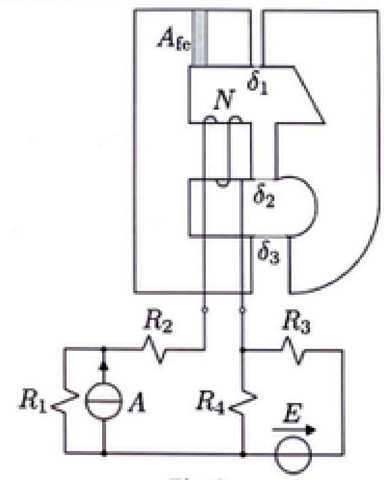
Exercise 6.7.15 (Exam 2025-01-22, Exercise 3.)

3) **SOLO ENERGETICI:** Il circuito di Figura 3, in **regime stazionario**, è così assegnato:

$N = 100$ $\delta_1 = 1\,mm$ $\delta_2 = 2\,mm$ $\delta_3 = 3\,mm$
$A_{fe} = 10\,cm^2$ $R_1 = 5\,\Omega$ $R_2 = 3\,\Omega$ $R_3 = 2\,\Omega$
$R_4 = 1\,\Omega$ $A = 5\,A$ $E = 30\,V$

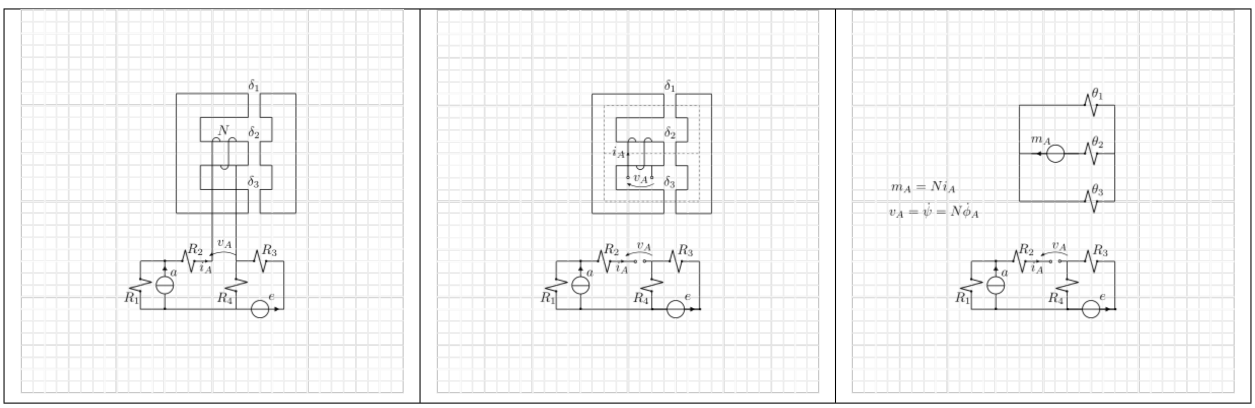
Determinare:

- L'induttanza associata al circuito magnetico;
- l'energia accumulata complessivamente nel campo magnetico;
- le potenze associate ad ogni resistore;
- le potenze associate ad ogni generatore, discutendone il segno.



4) SOLO ENERGETICI: Trasformatore

Solution



1. Equivalent magnetic network of the inductive part of the system. The equivalent reluctance seen by the magnetic flux generator $m_A = N i_A$ is

$$\theta_{eq} = \theta_2 + (\theta_1 \parallel \theta_3) \ .$$

and thus the flux through it reads

$$\phi_A = \frac{m_A}{\theta_{eq}} = \frac{N}{\theta_{eq}} i = \dots$$

The parallel part of the circuit acts as a current divider and thus magnetic fluxes through gaps 1 and 3 are

$$\begin{aligned} \phi_1 &= \frac{\theta_3}{\theta_1 + \theta_3}\phi_A = \frac{\theta_3}{\theta_1 + \theta_3}\frac{N}{\theta_{eq}} i_A = \dots \\ \phi_3 &= \frac{\theta_1}{\theta_1 + \theta_3}\phi_A = \frac{\theta_1}{\theta_1 + \theta_3}\frac{N}{\theta_{eq}} i_A = \dots \end{aligned} \tag{6.4}$$

Faraday's law provides the relation between the voltage and the concatenated flux,

$$v_A = \dot{\psi} = N\dot{\phi}_A = \frac{N^2}{\theta_{eq}}\frac{di_A}{dt} = L_{eq}\frac{di}{dt} \ ,$$

where the equivalent inductance of the magnetic circuit

$$L_{eq} = \dots$$

has been introduced. This relation becomes $v_A = 0$ in steady regime.

2. The electric network can be solved evaluating Thevenin equivalent network at the inductive port,

$$\begin{aligned}
v_{Th} &= \frac{R_3}{R_2+R_3} e + R_1 a \\
R_{Th} &= R_1 + R_2 + (R_3 \parallel R_4) \ ,
\end{aligned}$$

Thus the KVL on the equivalent complete network is

$$v_{Th} - R_{Th} i_A - L \frac{d i_A}{dt} = 0 \ .$$

In **steady regime**, $\frac{d}{dt} \equiv 0$, and thus

$$\bar{i}_A = \frac{v_{Th}}{R_{Th}} = ... \tag{6.5}$$

3. Energy stored in the magnetic field is the sum (integral) of the contribution $\frac{1}{2\mu}\left|\vec{b}\right|^2$ in electromagnetic energy density, u. With the assumption of negligible reluctance of the ferromagnetic medium,

$$\begin{aligned}
\int_V \frac{1}{2\mu}\left|\vec{b}\right|^2 &\sim \int_{V_{gaps}} \frac{1}{2\mu_0}\left|\vec{b}(\vec{r},t)\right|^2 = \\
&\sim \sum_{k\in\text{gaps}} \frac{1}{2\mu_0} b_k^2\, V_k = \\
&\sim \sum_{k\in\text{gaps}} \frac{1}{2\mu_0} \left(\frac{\phi_k}{A_k}\right)^2 A_k\, \delta_k = \\
&\sim \sum_{k\in\text{gaps}} \frac{1}{2}\frac{\delta_k}{\mu_0 A_k}\phi_k^2 = \\
&\sim \sum_{k\in\text{gaps}} \frac{1}{2}\theta_k \phi_k^2 = ... \ .
\end{aligned}$$

Fluxes can be evaluated with relations (6.4), once the current i_A is known, from (6.5).

4. After solving the electric circuit (e.g. introducing two loop currents in the left and right loops), powers through resistors and generators read

$$\begin{aligned}
P_{R_1} &= R_1\, i_1^2 = R_1 (i_A - a)^2 = ... \\
P_{R_2} &= R_2\, i_2^2 = R_2 i_A^2 = ... \\
P_{R_3} &= R_3\, i_3^2 = R_3 (i_A - i_{e,1})^2 = ... \\
P_{R_4} &= R_4\, i_4^2 = R_4 (i_A + i_{e,1})^2 = ...
\end{aligned}$$

$$\begin{aligned}
P_a &= v_a a = R_1 (i_A - a)\, a = ... \\
P_e &= e i_e = e(-i_A + i_{e,1}) = ...
\end{aligned}$$

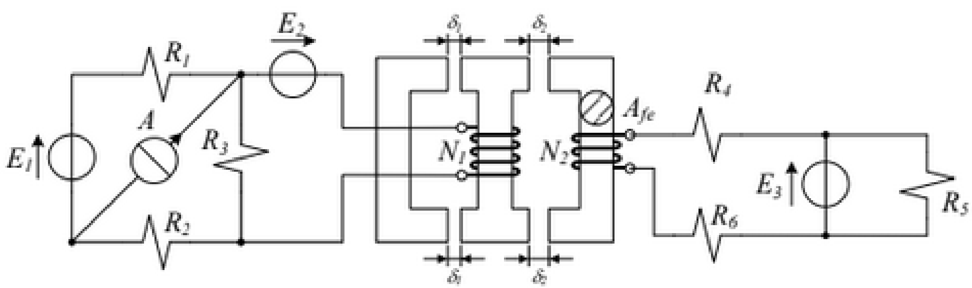
with $i_{e,1} = \frac{e}{R_3+R_4}$.

Exercise 6.7.16 (Exam 2024-06-19, Exercise 2.)

2) Il circuito di Figura 2, con ingressi stazionari, è così assegnato:

$E_1 = 20$ V	$R_1 = 5\ \Omega$	$\delta_1 = 2$ mm
$E_2 = 30$ V	$R_2 = 5\ \Omega$	$\delta_2 = 3$ mm
$E_3 = 10$ V	$R_3 = 20\ \Omega$	$A_{fe} = 16$ cm²
$A = 5$ A	$R_4 = 2\ \Omega$	$N_1 = 160$
	$R_5 = 10\ \Omega$	$N_2 = 200$
	$R_6 = 3\ \Omega$	$\mu_0 = 4\cdot\pi\cdot 10^{-7}$ H/m

Determinare:
- l'energia accumulata;
- le potenze messe in gioco dai generatori interpretandone il segno;
- le potenze messe in gioco dalle resistenze.



Solution - todo

Exercise 6.7.17 (Exam 2024-02-13, Exercise 1a.)

Solution - todo

1) ***SOLO ENERGETICI*** Il circuito di Figura 1, con ingressi stazionari, è così assegnato:

$E_1 = 20$ V $\quad R_1 = 5\ \Omega \quad R_4 = 15\ \Omega$
$E_2 = 15$ V $\quad R_2 = 10\ \Omega \quad R_5 = 6\ \Omega$
$A = 10$ A $\quad R_3 = 4\ \Omega$
$A_{fe} = 1\ \text{cm}^2 \quad \delta_1 = 1$ mm
$N = 100 \quad \delta_2 = 2$ mm
$\mu_{fe} = \infty$

L'interruttore S è aperto da tempo infinito e viene chiuso all'istante $t = 0$.

Determinare:

- l'andamento nel tempo della corrente $i_{R3}(t)$, formula e andamento grafico.

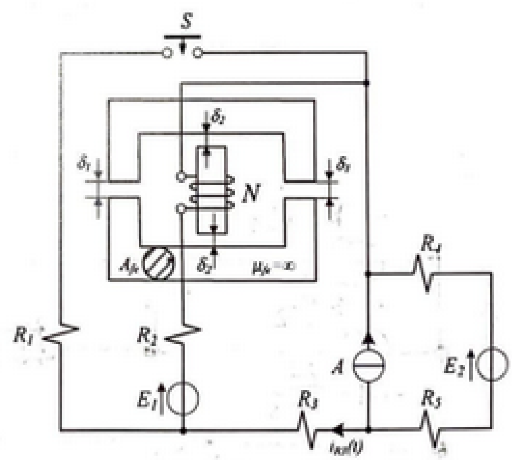


Fig. 1.

Part III

Numerical Methods

basics

Apr 29, 2025

1 min read

CHAPTER

SEVEN

GREEN'S FUNCTION METHOD

7.1 Poisson equation

General Poisson's problem

$$\begin{cases} -\nabla^2 \mathbf{u}(\mathbf{r},t) = \mathbf{f}(\mathbf{r},t) \\ + \text{b.c.} \end{cases}$$

with common boundary conditions

$$\begin{cases} \mathbf{u} = \mathbf{g} & \text{on } S_D \\ \hat{\mathbf{n}} \cdot \nabla \mathbf{u} = \mathbf{h} & \text{on } S_N \end{cases}$$

over Dirichlet and Neumann regions of the boundary.

Poisson's problem for Green's function, in infinite domain

$$-\nabla^2_{\mathbf{r}} G(\mathbf{r};\mathbf{r}_0) = \delta(\mathbf{r}-\mathbf{r}_0)$$

Green's function method

$$\begin{aligned} E(\mathbf{r}_0,t) u_i(\mathbf{r}_0,t) & = \int_{\mathbf{r}\in\Omega} u_i(\mathbf{r},t)\delta(\mathbf{r}-\mathbf{r}_0) = \\ & = -\int_{\mathbf{r}\in\Omega} u_i(\mathbf{r},t)\nabla^2_{\mathbf{r}} G(\mathbf{r}-\mathbf{r}_0) = \\ & = -\int_{\mathbf{r}\in\Omega} \nabla_{\mathbf{r}} \cdot (u_i \nabla_{\mathbf{r}} G - G\nabla_{\mathbf{r}} u_i) - \int_{\mathbf{r}\in\Omega} G\nabla^2 u_i = \\ & = -\oint_{\mathbf{r}\in\partial\Omega} \hat{\mathbf{n}} \cdot (u_i \nabla_{\mathbf{r}} G - G\nabla_{\mathbf{r}} u_i) + \int_{\mathbf{r}\in\Omega} G(\mathbf{r}-\mathbf{r}_0) f_i(\mathbf{r},t). \end{aligned}$$

An integro-differential boundary problem can be written using boundary conditions. As an example, using Dirichlet and Neumann boundary conditions, the integro-differential problem reads

$$\begin{aligned} & E(\mathbf{r}_0,t)\mathbf{u}(\mathbf{r}_0,t) + \int_{\mathbf{r}\in S_N} \mathbf{u}(\mathbf{r},t)\, \hat{\mathbf{n}} \cdot \nabla_{\mathbf{r}} G(\mathbf{r}-\mathbf{r}_0) - \int_{\mathbf{r}\in S_D} G(\mathbf{r}-\mathbf{r}_0)\, \hat{\mathbf{n}} \cdot \nabla_{\mathbf{r}} \mathbf{u}(\mathbf{r},t) = \\ & = -\int_{\mathbf{r}\in S_D} \mathbf{g}(\mathbf{r},t)\, \hat{\mathbf{n}} \cdot \nabla_{\mathbf{r}} G(\mathbf{r}-\mathbf{r}_0) + \int_{\mathbf{r}\in S_N} G(\mathbf{r}-\mathbf{r}_0)\, \mathbf{h}(\mathbf{r},t) + \int_{\mathbf{r}\in\Omega} G(\mathbf{r}-\mathbf{r}_0)\, \mathbf{f}(\mathbf{r},t). \end{aligned}$$

Green's function of the Poisson-Laplace equation reads

$$G(\mathbf{r};\mathbf{r}_0) = \frac{1}{4\pi} \frac{1}{|\mathbf{r}-\mathbf{r}_0|} \ .$$

Green's function of the Laplace equation

$$-\nabla^2 G = 0 \qquad \text{for } \mathbf{r} \neq \mathbf{r}_0$$

Solutions with spherical symmetry,

$$0 = \nabla^2 G = \frac{1}{r^2}\left(r^2 G'\right)' \quad \rightarrow \quad G'(r) = \frac{A}{r^2} \quad \rightarrow \quad G(r) = -\frac{A}{r} + B$$

Choosing $B = 0$ s.t. $G(r) \rightarrow 0$ as $r \rightarrow \infty$, and integrating over a sphere centered in $r = 0$ to get $A = -\frac{1}{4\pi}$,

$$1 = \int_V \delta(r) = -\int_V \nabla^2 G = -\oint_{\partial V} \hat{\mathbf{n}} \cdot \nabla G = -\oint_{\partial V} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \frac{A}{r^2} = -4\pi\, A$$

7.2 Helmholtz equation

todo from Fourier to Laplace trasnform in the first lines of this section

A Helmholtz's equation can be thouoght as the time Fourier transform of a wave equation,

$$\begin{cases} \dfrac{1}{c^2}\partial_{tt}\mathbf{u}(\mathbf{r},t) - \nabla^2\mathbf{u}(\mathbf{r},t) = \mathbf{f}(\mathbf{r},t) \\ + \text{b.c.} \\ + \text{i.c.}\ , \end{cases}$$

Fourier transform in time of field $\mathbf{u}(\mathbf{r}, t)$ reads

$$\tilde{\mathbf{u}}(\mathbf{r},\omega) = \mathcal{F}\left\{\mathbf{u}(\mathbf{r},t)\right\} = \int_{t=-\infty}^{+\infty} \mathbf{u}(\mathbf{r},t)e^{-i\omega t}\,d\omega$$

and, if $\mathbf{u}(\mathbf{r}, t)$ is compact in time, Fourier transform of its time partial derivatives read

$$\begin{aligned} \mathcal{F}\{\dot{\mathbf{u}}(\mathbf{r},t)\} &= \int_{t=-\infty}^{+\infty} \dot{\mathbf{u}}(\mathbf{r},t)e^{-i\omega t}\,d\omega = \\ &= \mathbf{u}(\mathbf{r},t)e^{-i\omega t}\Big|_{t=-\infty}^{+\infty} + i\omega \int_{t=-\infty}^{+\infty} \mathbf{u}(\mathbf{r},t)e^{-i\omega t}\,d\omega = \\ &= i\omega\mathcal{F}\left\{\mathbf{u}(\mathbf{r},t)\right\} \end{aligned}$$

$$\mathcal{F}\left\{\partial_t^n \mathbf{u}(\mathbf{r},t)\right\} = (i\omega)^n \tilde{\mathbf{u}}\ .$$

The differential problem in the transformed domain thus reads

$$-\frac{\omega^2}{c^2}\tilde{\mathbf{u}} - \nabla^2\tilde{\mathbf{u}} = \tilde{\mathbf{f}}$$

Green's function of Helmholtz'e equation reads

$$G(\mathbf{r},s) = \alpha^+ \frac{e^{\frac{s|\mathbf{r}-\mathbf{r}_0|}{c}}}{|\mathbf{r}-\mathbf{r}_0|} + \alpha^- \frac{e^{-\frac{s|\mathbf{r}-\mathbf{r}_0|}{c}}}{|\mathbf{r}-\mathbf{r}_0|}$$

with $\alpha^+ + \alpha^- = \frac{1}{4\pi}$.

Being the Laplace transform,

$$\mathcal{L}\{f(t)\} = \int_{t=0^-}^{+\infty} f(t)e^{-st}dt\ ,$$

the Laplace transform of a causal function with time delay $\tau \geq 0$ reads

$$\mathcal{L}\{f(t-\tau)\} = \int_{t=0^-}^{+\infty} f(t-\tau)e^{-st}dt = \int_{z=-\tau}^{+\infty} f(z)e^{-s(z+\tau)}\,dz = e^{-s\tau}\int_{z=0}^{+\infty} f(z)e^{-sz}\,dz = e^{-s\tau}\,\mathcal{L}\{f(t)\}$$

having used causality $f(t) = 0$ for $t < 0$. Laplace transform of Dirac's delta $\delta(t)$ reads

$$\mathcal{L}\{\delta(t)\} = \int_{t=0^-}^{+\infty} \delta(t)\,dt = 1 \ ,$$

so that $e^{-s\tau} = e^{-s\tau}\,1 = \mathcal{L}\{\delta(t-\tau)\}$.

Thus, Green's function for the wave equation reads

$$G(\mathbf{r},t;\mathbf{r}_0,t_0) = \alpha^+\frac{\delta\left(t-t_0+\frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right)}{|\mathbf{r}-\mathbf{r}_0|} + \alpha^-\frac{\delta\left(t-t_0-\frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right)}{|\mathbf{r}-\mathbf{r}_0|}$$

If $t \geq t_0$, and $G(\mathbf{r},t;\mathbf{r}_0,t_0)$ connects the past t_0 with the future t, the first term is not causal, and thus $\alpha^+ = 0$ and

$$G(\mathbf{r},t;\mathbf{r}_0,t_0) = \frac{1}{4\pi}\frac{\delta\left(t-t_0-\frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right)}{|\mathbf{r}-\mathbf{r}_0|} \ .$$

Green's function of Helmholtz's equation

$$\frac{s^2}{c^2}G - \nabla^2 G = \delta(r)$$

$$G(r) = \frac{\alpha e^{kr} + \beta e^{-kr}}{r}$$

Proof:

- Gradient

$$\nabla G(r) = \hat{\mathbf{r}}\partial_r G = \hat{\mathbf{r}}\frac{\alpha(kr-1)e^{kr} + \beta(-kr-1)e^{-kr}}{r^2}$$

- Laplacian

$$\begin{aligned}
\nabla^2 G(r) &= \frac{1}{r^2}\left(r^2 G'(r)\right)' = \\
&= \frac{1}{r^2}\left(\alpha(kr-1)e^{kr} + \beta(-kr-1)e^{-kr}\right)' = \\
&= \frac{1}{r^2}\left(\alpha k e^{kr} + \alpha k^2 r e^{kr} - \alpha k e^{kr} - \beta k e^{-kr} + \beta k^2 r e^{-kr} + \beta k e^{-kr}\right) = \\
&= \frac{1}{r}\left(\alpha e^{kr} + \beta e^{-kr}\right)k^2 = k^2 G(r) \ .
\end{aligned}$$

  and thus $k^2 G(r) - \nabla^2 G = 0$, for $r \neq 0$;

- Unity

$$1 = \int_V \delta(r) = \int_V (k^2 G - \nabla^2 G) = \int_V k^2 G - \oint_{\partial V} \hat{\mathbf{n}}\cdot\nabla G$$

  the second term is the sum of two contributions of the form

$$\oint_{\partial V} \hat{\mathbf{n}}\cdot\nabla G^{\pm} = \oint_{\partial V} \frac{\alpha^{\pm}(\pm kr-1)e^{\pm kr}}{r^2} = 4\pi\alpha^{\pm}(\pm kr-1)e^{\pm kr}$$

the first term is the sum of two contributions of the form

$$\begin{aligned}
k^2 \int_V G(r) = k^2 \int_V \frac{\alpha^\pm e^{\pm kr}}{r} = \\
& = k^2 \alpha^\pm \int_{R=0}^{r} \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \frac{e^{\pm kR}}{R} R^2 \sin\phi \, dR \, d\phi \, d\theta = \\
& = k^2 \alpha^\pm \, 4\pi \int_{R=0}^{r} R \, e^{\pm kR} \, dR \, .
\end{aligned}$$

the last integral can be evaluated with integration by parts

$$\begin{aligned}
\int_{R=0}^{r} R \, e^{\pm kR} \, dR = \left[\frac{1}{\pm k} e^{\pm kR} R \right] \Big|_{R=0}^{r} \mp \frac{1}{k} \int_{R=0}^{r} e^{\pm kR} \, dR = \\
= \frac{1}{\pm k} e^{\pm kr} r - \frac{1}{k^2} e^{\pm kR} + \frac{1}{k^2} =
\end{aligned}$$

Thus summing everything together,

$$\begin{aligned}
1 & = \alpha^+ \left[4\pi k^2 \left(\frac{r}{k} e^{kr} - \frac{1}{k^2} e^{kr} + \frac{1}{k^2} \right) - 4\pi \left(kr - 1 \right) e^{kr} \right] + \alpha^- \left[... \right] = \\
& = 4\pi \left(\alpha^+ + \alpha^- \right) \, .
\end{aligned}$$

7.3 Wave equation

Wave equation general problem

$$\begin{cases}
\frac{1}{c^2} \partial_{tt} \mathbf{u}(\mathbf{r}, t) - \nabla^2 \mathbf{u}(\mathbf{r}, t) = \mathbf{f}(\mathbf{r}, t) \\
+ \text{ b.c.} \\
+ \text{ i.c.}
\end{cases}$$

Green's problem of the wave equation

$$\frac{1}{c^2} \partial_{tt} G(\mathbf{r}, t; \mathbf{r}_0, t_0) - \nabla^2_{\mathbf{r}} G(\mathbf{r}, t; \mathbf{r}_0, t_0) = \delta(\mathbf{r} - \mathbf{r}_0) \delta(t - t_0)$$

Integration by parts

$$\begin{aligned}
E(\mathbf{r}_\alpha, t_\alpha) \mathbf{u}(\mathbf{r}_\alpha, t_\alpha) & = \int_{t \in T} \int_{\mathbf{r} \in V} \delta(t - t_\alpha) \delta(\mathbf{r} - \mathbf{r}_\alpha) \mathbf{u}(\mathbf{r}, t) = \\
& = \int_{t \in T} \int_{\mathbf{r} \in V} \left\{ \frac{1}{c^2} \partial_{tt} G - \nabla^2_{\mathbf{r}} G \right\} \mathbf{u} = \\
& = \int_{t \in T} \int_{\mathbf{r} \in V} \left\{ \frac{1}{c^2} \left[\partial_t \left(\mathbf{u} \partial_t G - G \partial_t \mathbf{u} \right) + G \partial_{tt} \mathbf{u} \right] - \nabla_{\mathbf{r}} \cdot \left(\nabla_{\mathbf{r}} G \, \mathbf{u} - G \nabla_{\mathbf{r}} \mathbf{u} \right) - G \, \nabla^2_{\mathbf{r}} \mathbf{u} \right\} = \\
& = \int_{\mathbf{r} \in V} \frac{1}{c^2} \left[\mathbf{u}(\mathbf{r}, t) \partial_t G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) - G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) \partial_t \mathbf{u}(\mathbf{r}, t) \right] \Big|_{t_0}^{t_1} + \\
& \quad + \int_{t \in T} \oint_{\mathbf{r} \in \partial V} \left\{ -\hat{\mathbf{n}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{r}} G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) \, \mathbf{u}(\mathbf{r}, t) + G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) \, \hat{\mathbf{n}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{r}} \mathbf{u}(\mathbf{r}, t) \right\} + \\
& \quad + \int_{t \in T} \int_{\mathbf{r} \in V} G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) \underbrace{\left\{ \frac{1}{c^2} \partial_{tt} \mathbf{u}(\mathbf{r}, t) - \nabla^2_{\mathbf{r}} \mathbf{u}(\mathbf{r}, t) \right\}}_{= \mathbf{f}(\mathbf{r}, t)}
\end{aligned}$$

$$\int_{t\in T}\int_{\mathbf{r}\in V}\frac{1}{4\pi}\frac{\delta\left(t-t_\alpha+\frac{|\mathbf{r}-\mathbf{r}_\alpha|}{c}\right)}{|\mathbf{r}-\mathbf{r}_\alpha|}\mathbf{f}(\mathbf{r},t)=\int_{\mathbf{r}\in V\cap B_{|\mathbf{r}-\mathbf{r}_\alpha|\leq c(t_\alpha-t)}}\frac{1}{4\pi|\mathbf{r}-\mathbf{r}_\alpha|}\mathbf{f}\left(\mathbf{r},t_\alpha-\frac{|\mathbf{r}-\mathbf{r}_\alpha|}{c}\right)$$

basics

Apr 29, 2025

1 min read

CHAPTER

EIGHT

METODI NUMERICI

8.1 Elettrostatica

I problemi dell'elettrostatica sono governate dalle due equazioni di Maxwell per i campi **e**, **d**,

$$\begin{cases} \nabla \cdot \mathbf{d} = \rho \\ \\ \nabla \times \mathbf{e} = \mathbf{0} \ , \end{cases}$$

dotate delle opportune condizioni al contorno ed equazioni costitutive. Per un materiale lineare isotropo, ad esempio, $\mathbf{d} = \varepsilon \mathbf{e}$. La condizione di irrotazionalità del campo elettrico, permette di scriverlo come gradiente di un potenziale scalare, $\mathbf{e} = -\nabla v$, e di ottenere l'equazione di Poisson,

$$-\nabla \cdot (\varepsilon \nabla v) = \rho \ .$$

8.1.1 Sorgente

$$\begin{aligned} \mathbf{e}(r) &= \frac{q_i}{4\pi\varepsilon} \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3} \\ \mathbf{e}(\mathbf{r}) &= -\nabla_{\mathbf{r}} v(\mathbf{r}) \\ \varepsilon \, v(\mathbf{r}) &= \frac{q_i}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}_i|} \end{aligned}$$

8.1.2 Dipolo

Un dipolo è definito come due cariche di intensità uguale e contraria $-q_2 = q_1 = q > 0$, nei punti dello spazio P_1, $P_2 = P_1 + \mathbf{l}$, nelle condizioni limite $|\mathbf{l}| \to 0$, $q \to \infty$, in modo tale da avere $q|\mathbf{l}|$ finito, $\mathbf{p} = q\mathbf{l}$.

Il potenziale del dipolo è dato dal principio di sovrapposizione delle cause e degli effetti,

$$\begin{aligned} \varepsilon \, v(\mathbf{r}) &= -\frac{q}{4\pi} \frac{1}{\left|\mathbf{r} - \mathbf{r}_0 + \frac{\mathbf{l}}{2}\right|} + \frac{q}{4\pi} \frac{1}{\left|\mathbf{r} - \mathbf{r}_0 - \frac{\mathbf{l}}{2}\right|} = \\ &= \dots \\ &= \frac{q}{4\pi} \left(-\frac{1}{|\mathbf{r} - \mathbf{r}_0|} + \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \cdot \frac{\mathbf{l}}{2} + \frac{1}{|\mathbf{r} - \mathbf{r}_0|} + \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \cdot \frac{\mathbf{l}}{2} + o(|\mathbf{l}|) \right) = \\ &= \dots \\ &= \frac{1}{4\pi} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \cdot \mathbf{P} \ , \end{aligned}$$

avendo definito il vettore momento dipolo $\mathbf{P} = q\mathbf{l}$.

Polariazazione - Potenziale generato da una distribuzione di dipoli.

$$d\mathbf{P} = \mathbf{p}\,\Delta V$$

$$\varepsilon v_P(\mathbf{r}) = \int_{\mathbf{r}_0 \in V_0} \frac{1}{4\pi} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} \cdot \mathbf{p}(\mathbf{r}_0)\, dV_0$$

$$\begin{aligned} \partial_i |\mathbf{r}|^2 &= 2x_i \\ &= 2|\mathbf{r}|\partial_i|\mathbf{r}| \end{aligned} \qquad \rightarrow \qquad \partial_i|\mathbf{r}| = \frac{x_i}{|\mathbf{r}|}$$

$$\partial_i |\mathbf{r}|^n = n|\mathbf{r}|^{n-1}\,\partial_i|\mathbf{r}| = n x_i |\mathbf{r}|^{n-2}$$

$$\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} = \nabla_{\mathbf{r}_0} \frac{1}{|\mathbf{r}-\mathbf{r}_0|}$$

$$\begin{aligned} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} \cdot \mathbf{p}(\mathbf{r}_0) &= \nabla_{\mathbf{r}_0} \frac{1}{|\mathbf{r}-\mathbf{r}_0|} \cdot \mathbf{p}(\mathbf{r}_0) = \\ &= \nabla_{\mathbf{r}_0} \cdot \left(\frac{1}{|\mathbf{r}-\mathbf{r}_0|} \mathbf{p}(\mathbf{r}_0) \right) - \frac{1}{|\mathbf{r}-\mathbf{r}_0|} \nabla_{\mathbf{r}_0} \cdot \mathbf{p}(\mathbf{r}_0) = \end{aligned}$$

e quindi

$$4\,\pi\,\varepsilon v_P(\mathbf{r}) = \oint_{\mathbf{r}_0 \in \partial V_0} \frac{\hat{\mathbf{n}}(\mathbf{r}_0) \cdot \mathbf{p}(\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|} - \oint_{\mathbf{r}_0 \in V_0} \frac{\nabla_{\mathbf{r}_0} \cdot \mathbf{p}(\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|}$$

I due contributi hanno la forma di sorgenti, essendo termini proporzionali a $\frac{1}{|\mathbf{r}-\mathbf{r}_0|}$. Il potenziale dovuto alla densità di volume di dipoli equivale alla somma dei due contributi delle cariche di:

- polarizzazione di superficie $\sigma_p = \hat{\mathbf{n}} \cdot \mathbf{p}$
- polarizzazione di volume $\rho_p = -\nabla \cdot \mathbf{p}$

Oss. Se la polarizzazione è uniforme nel volume, il contributo della polarizzazione nel volume si annulla e rimane solo il contributo della polarizzazione sul contorno del volume.

Oss. Legge di Gauss per il campo elettrico,

$$\begin{aligned} \nabla \cdot \mathbf{e} &= \frac{1}{\varepsilon_0} \rho = \\ &= \frac{1}{\varepsilon_0} \left(\rho_l + \rho_p \right) = \\ &= \frac{1}{\varepsilon_0} \left(\rho_l - \nabla \cdot \mathbf{p} \right) \\ \nabla \cdot (\varepsilon_0 \mathbf{e} + \mathbf{p}) &= \rho_l \\ \nabla \cdot \mathbf{d} &= \rho_l \end{aligned}$$

Part IV

Appendices

CHAPTER

NINE

TODO: APPENDICES

- Optics
- Einstein's relativity
- Quantum
- …

PROOF INDEX