
electromagnetism

basics

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This material is part of the [basics-books project](#). It is also available as a .pdf document.

Introduzione.

L'elettromagnetismo si occupa dello studio dei fenomeni elettromagnetici prodotti da cariche e correnti elettriche o dalla struttura microscopica della materia (magnetismo naturale)

Breve storia. *Prime esperienze: cariche di 2 tipi diversi e legge di Coulomb;*

Argomenti.

Prime esperienze **TODO** *Prime esperienze; elettromagnetismo come teoria dei campi* **TODO** *aggiungere una sezione su first-experiments-revisited, dopo la presentazione dei principi dell'elettromagnetismo*

todo Aggiungere sezione su strumenti matematici necessari, per la formulazione di una teoria dei campi

Principi dell'elettromagnetismo **TODO** *Trattare prima regime stazionario - elettricità e magnetismo - e poi regime non-stazionario - elettromagnetismo**** **TODO** *Principi. Conservazione della carica, leggi di Maxwell, legge di Lorentz* **TODO** *Principi in forma integrale; principi in forma differenziale - le leggi di Maxwell*

Energia

Onde elettromagnetiche

Approssimazione circuitale **TODO** *Circuiti elettrici; circuiti elettromagnetici; sistemi elettro-meccanici. Regimi: stazionario, non-stazionario: regime transitorio e armonico*

Extra.

Ottica

Elettromagnetismo e relatività **todo** *Relatività a per $v \ll c$; crisi della relatività galileiana*

Parte I

Electromagnetism

CAPITOLO 1

Brief history of Electromagnetism

Principles of Classical Electromagnetism

The progress in the study of electromagnetic phenomena during the 19th century allowed James Clerk Maxwell to formulate what are now known as *Maxwell's equations*, which can be considered the first consistent formulation of the principles of classical electromagnetism, together with the charge conservation law and the expression for the Lorentz force on an electric charge immersed in an electromagnetic field.

2.1 Principles in Differential Form

The principles in differential form can be derived from the more general *integral form*, provided the fields satisfy the necessary minimal regularity conditions, which can be qualitatively stated as «all operations must make sense.»

Conservation of Electric Charge.

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0 .$$

Maxwell's Equations.

$$\begin{cases} \nabla \cdot \mathbf{d} = \rho \\ \nabla \times \mathbf{e} + \partial_t \mathbf{b} = \mathbf{0} \\ \nabla \cdot \mathbf{b} = 0 \\ \nabla \times \mathbf{h} - \partial_t \mathbf{d} = \mathbf{j} \end{cases}$$

with the need to define constitutive equations $\mathbf{d}(\mathbf{e}, \mathbf{b})$, $\mathbf{h}(\mathbf{e}, \mathbf{b})$.

Lorentz Force. The force per unit volume acting on the electric charge present at a point \mathbf{r} in space is

$$\begin{aligned} \mathbf{f}(\mathbf{r}, t) &= \rho(\mathbf{r}, t) \mathbf{e}(\mathbf{r}, t) + \mathbf{j}(\mathbf{r}, t) \times \mathbf{b}(\mathbf{r}, t) = \\ &= \rho(\mathbf{r}, t) [\mathbf{e}(\mathbf{r}) + \mathbf{v}(\mathbf{r}, t) \times \mathbf{b}(\mathbf{r}, t)] = \\ &= \rho(\mathbf{r}, t) \mathbf{e}^*(\mathbf{r}, t) \end{aligned}$$

having defined \mathbf{e}^* as the electric field **seen by the moving charge**.

2.2 Principles in Integral Form: Electromagnetic Equations and Galilean Relativity

2.2.1 Integral Form on Control Volumes

The integral form of the principles of electromagnetism for fixed volumes V and surfaces S in space is obtained by integrating the differential equations over the domains and using the divergence theorem to obtain flux terms, and Stokes' theorem to obtain circulation terms.

Continuity of Electric Charge.

$$\frac{d}{dt} \int_V \rho + \oint_{\partial V} \mathbf{j} \cdot \hat{\mathbf{n}} = 0$$

Gauss's Law for the Field $\mathbf{d}(\mathbf{r}, t)$.

$$\oint_{\partial V} \mathbf{d} \cdot \hat{\mathbf{n}} = \int_V \rho$$

Gauss's Law for the Field $\mathbf{b}(\mathbf{r}, t)$.

$$\oint_{\partial V} \mathbf{b} \cdot \hat{\mathbf{n}} = 0$$

Faraday–Neumann–Lenz Law for Electromagnetic Induction.

$$\oint_{\partial S} \mathbf{e} \cdot \hat{\mathbf{t}} + \frac{d}{dt} \int_S \mathbf{b} \cdot \hat{\mathbf{n}} = 0$$

Ampère–Maxwell Law.

$$\oint_{\partial S} \mathbf{h} \cdot \hat{\mathbf{t}} - \frac{d}{dt} \int_S \mathbf{d} \cdot \hat{\mathbf{n}} = \int_S \mathbf{j} \cdot \hat{\mathbf{n}}.$$

2.2.2 Integral Form on Arbitrary Volumes

Due to their importance in fundamental applications such as electric motors, and to avoid confusion or leaps in logic when dealing with electromagnetic induction, it is crucial to provide the correct expression of the electromagnetic principles when moving volumes are involved in space. Not only is the form of these principles shown, but also the correct procedure to derive them starting from the fixed-control-volume version. This is done using rules for [time derivative for fundamental integrals over moving domains](#), such as the integral of a density function over a volume, the flux of a vector field through a surface, or the circulation along a curve.

These three derivative rules are listed here and proved in the material about [Mathematics:Vector and Tensor Algebra and Calculus:Time derivatives of integrals over moving domains](#)

$$\begin{aligned} \frac{d}{dt} \int_{v_t} f &= \int_{v_t} \frac{\partial f}{\partial t} + \oint_{\partial v_t} f \mathbf{u}_b \cdot \hat{\mathbf{n}} \\ \frac{d}{dt} \int_{s_t} \mathbf{f} \cdot \hat{\mathbf{n}} &= \int_{s_t} \frac{\partial \mathbf{f}}{\partial t} \cdot \hat{\mathbf{n}} + \int_{s_t} \nabla \cdot \mathbf{f} \mathbf{u}_b \cdot \hat{\mathbf{n}} - \oint_{\partial s_t} \mathbf{u}_b \times \mathbf{f} \cdot \hat{\mathbf{t}} \\ \frac{d}{dt} \int_{\ell_t} \mathbf{f} \cdot \hat{\mathbf{t}} &= \int_{\ell_t} \frac{\partial \mathbf{f}}{\partial t} \cdot \hat{\mathbf{t}} + \int_{\ell_t} \nabla \times \mathbf{f} \cdot \mathbf{u}_b \times \hat{\mathbf{t}} + \mathbf{f}_B \cdot \mathbf{u}_B - \mathbf{f}_A \cdot \mathbf{u}_A \end{aligned}$$

Continuity of Electric Charge.

$$\begin{aligned}
 0 &= \frac{d}{dt} \int_V \rho + \oint_{\partial V} \mathbf{j} \cdot \hat{\mathbf{n}} = \\
 &= \frac{d}{dt} \int_{v_t} \rho - \oint_{\partial v_t} \rho \mathbf{u}_b \cdot \hat{\mathbf{n}} + \oint_{\partial v_t} \mathbf{j} \cdot \hat{\mathbf{n}} \\
 &= \frac{d}{dt} \int_{v_t} \rho + \oint_{\partial v_t} \underbrace{\rho(\mathbf{u} - \mathbf{u}_b)}_{\mathbf{j}^*} \cdot \hat{\mathbf{n}}
 \end{aligned}$$

Gauss's Law for the Field $\mathbf{d}(\mathbf{r}, t)$.

$$\oint_{\partial v_t} \mathbf{d} \cdot \hat{\mathbf{n}} = \int_{v_t} \rho$$

Gauss's Law for the Field $\mathbf{b}(\mathbf{r}, t)$.

$$\oint_{\partial v_t} \mathbf{b} \cdot \hat{\mathbf{n}} = 0$$

Faraday–Neumann–Lenz Law for Electromagnetic Induction.

$$\begin{aligned}
 \mathbf{0} &= \oint_{\partial S} \mathbf{e} \cdot \hat{\mathbf{t}} + \frac{d}{dt} \int_S \mathbf{b} \cdot \hat{\mathbf{n}} = \\
 &= \oint_{\partial s_t} \mathbf{e} \cdot \hat{\mathbf{t}} + \frac{d}{dt} \int_{s_t} \mathbf{b} \cdot \hat{\mathbf{n}} - \int_{s_t} \underbrace{\nabla \cdot \mathbf{b}}_{=0} \mathbf{u}_b \cdot \hat{\mathbf{n}} + \oint_{s_t} \mathbf{u}_b \times \mathbf{b} \cdot \hat{\mathbf{t}} = \\
 &= \oint_{\partial s_t} \mathbf{e}^* \cdot \hat{\mathbf{t}} + \frac{d}{dt} \int_{s_t} \mathbf{b} \cdot \hat{\mathbf{n}},
 \end{aligned}$$

with the definition $\mathbf{e}^* := \mathbf{e} + \mathbf{u}_b \cdot \mathbf{b}$, already used in the expression of the Lorentz force law.

Ampère–Maxwell Law.

$$\begin{aligned}
 \mathbf{0} &= \oint_{\partial s_t} \mathbf{h} \cdot \hat{\mathbf{t}} - \frac{d}{dt} \int_{s_t} \mathbf{d} \cdot \hat{\mathbf{n}} - \int_{s_t} \mathbf{j} \cdot \hat{\mathbf{n}} = \\
 &= \oint_{\partial s_t} \mathbf{h} \cdot \hat{\mathbf{t}} - \frac{d}{dt} \int_{s_t} \mathbf{d} \cdot \hat{\mathbf{n}} + \int_{s_t} \underbrace{\nabla \cdot \mathbf{d}}_{=\rho} \mathbf{u}_b \cdot \hat{\mathbf{n}} - \oint_{s_t} \mathbf{u}_b \times \mathbf{d} \cdot \hat{\mathbf{t}} - \int_{s_t} \mathbf{j} \cdot \hat{\mathbf{n}} = \\
 &= \oint_{\partial s_t} \mathbf{h}^* \cdot \hat{\mathbf{t}} - \frac{d}{dt} \int_{s_t} \mathbf{d} \cdot \hat{\mathbf{n}} = \int_{s_t} \mathbf{j}^* \cdot \hat{\mathbf{n}},
 \end{aligned}$$

having defined $\mathbf{h}^* := \mathbf{h} - \mathbf{u}_b \times \mathbf{d}$, and using the previously introduced definition $\mathbf{j}^* := \mathbf{j} - \rho \mathbf{u}_b$.

Adding the definitions:

$$\rho^* = \rho$$

$$\mathbf{d}^* = \mathbf{d}$$

$$\mathbf{b}^* = \mathbf{b}$$

one obtains equations having the same form as those written for stationary domains in space, but which can be applied to moving domains. The definitions:

$$\begin{aligned}
 \rho^* &= \rho & , & & \mathbf{j}^* &= \mathbf{j} - \rho \mathbf{u}_b \\
 \mathbf{d}^* &= \mathbf{d} & , & & \mathbf{e}^* &= \mathbf{e} + \mathbf{u}_b \times \mathbf{b} \\
 \mathbf{b}^* &= \mathbf{b} & , & & \mathbf{h}^* &= \mathbf{h} - \mathbf{u}_b \times \mathbf{d}
 \end{aligned}$$

are nothing more than the transformation of the fields for two observers in relative motion, and correspond to the low-speed limit of Lorentz transformations from special relativity for velocities $|\mathbf{u}_b| \ll c$: in this procedure, the transformations for low relative speeds are obtained, as no transformation of spatial and temporal dimensions has been considered, unlike Einstein's theory of relativity.

todo Reference Galilean and Lorentz transformations for relativity in electromagnetism.

Electromagnetic Potentials

It is possible to demonstrate that the system of Maxwell's equations and the charge continuity equation is overdetermined. Specifically, it can be shown that, given the distribution of charge and current density—considered as the generating causes of the electric field—and the constitutive laws of the material, four unknowns are sufficient to define the six unknowns (three components for two vector fields) of the problem. Therefore, the problem can be formulated in terms of a scalar potential φ and a vector potential \mathbf{a} , along with a gauge condition that eliminates the remaining two arbitrary factors (irrelevant for the calculation of physical fields).

3.1 Vector Potential and Scalar Potential

Starting from Maxwell's equations, the potentials of the electromagnetic field can be defined. Using Gauss's law for the magnetic field, the vector potential $\mathbf{a}(\mathbf{r}, t)$ can be introduced,

$$0 = \nabla \cdot \mathbf{b} \quad \rightarrow \quad \mathbf{b} = \nabla \times \mathbf{a} ,$$

since the divergence of a curl is identically zero. Introducing this relationship into the Faraday-Neumann-Lenz equation, assuming sufficient regularity of the fields to allow the inversion of the order of derivatives,

$$0 = \nabla \times \mathbf{e} + \partial_t \mathbf{b} = \nabla \times \mathbf{e} + \partial_t \nabla \times \mathbf{a} = \nabla \times (\mathbf{e} + \partial_t \mathbf{a}) \quad \rightarrow \quad \mathbf{e} + \partial_t \mathbf{a} = -\nabla \varphi ,$$

since the curl of a gradient is identically zero. The «physical» quantities of the electric field $\mathbf{e}(\mathbf{r}, t)$ and the magnetic field $\mathbf{b}(\mathbf{r}, t)$ can therefore be written using the electromagnetic potentials as

$$\begin{cases} \mathbf{e} &= -\nabla \varphi - \partial_t \mathbf{a} \\ \mathbf{b} &= \nabla \times \mathbf{a} \end{cases}$$

3.2 Gauge Conditions

The potentials are defined up to a gauge condition, an additional condition that eliminates any arbitrariness in the definition. For example, the vector potential is defined up to the gradient of a scalar function, since $\nabla \times \nabla f \equiv \mathbf{0}$, and thus the potential $\tilde{\mathbf{a}} = \mathbf{a} + \nabla f$ produces the same magnetic field \mathbf{b}

$$\nabla \times \tilde{\mathbf{a}} = \nabla \times (\mathbf{a} + \nabla f) = \nabla \times \mathbf{a} .$$

Lorentz Gauge Condition. For reasons that will become clearer in the section on *electromagnetic waves*, a convenient gauge condition is

$$\nabla \cdot \mathbf{a} + \frac{1}{c^2} \partial_t \varphi = 0$$

Coulomb Gauge Condition.

$$\nabla \cdot \mathbf{a} = 0$$

todo

4.1 Vacuum

Electromagnetic phenomena in a vacuum are governed by Maxwell's equations in a vacuum,

$$\begin{cases} \nabla \cdot \mathbf{e} = \frac{\rho}{\varepsilon_0} \\ \nabla \times \mathbf{e} + \partial_t \mathbf{b} = \mathbf{0} \\ \nabla \cdot \mathbf{b} = 0 \\ \nabla \times \mathbf{b} - \mu_0 \varepsilon_0 \partial_t \mathbf{e} = \mu_0 \mathbf{j} \end{cases}$$

and by the continuity equation of electric charge,

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0 .$$

4.2 Continuous Media

In general, some materials respond to an imposed «external» electromagnetic field with polarization and magnetization. In particular, the electric polarization of a material corresponds to a local separation of electric charges, macroscopically equivalent to a volume density of dipoles, $\mathbf{p}(\mathbf{r}_0)$; magnetization corresponds to an orientation of the axes of Amperian current loops, macroscopically equivalent to a density of magnetic moment $\mathbf{m}(\mathbf{r}_0)$.

4.3 Polarization

4.3.1 Single Electric Dipole

A discrete electric dipole is formed by two equal and opposite electric charges $q, -q$, at points $P_+, P_- = P_+ \mathbf{l}$, in the limit $q \rightarrow +\infty, |\mathbf{l}| \rightarrow 0$ with $q|\mathbf{l}|$ finite.

The electric field (stationary **todo** check what happens in the non-stationary case. Perhaps after deriving the general solution to the problem, as a solution to the wave equations in terms of EM potentials) generated at the point in space \mathbf{r} by an electric dipole at the point \mathbf{r}_0 is calculated as the limit of the electric field generated by two equal and opposite charges q^\mp at the points $\mathbf{r}_0 \mp \frac{\mathbf{l}}{2}$,

$$\mathbf{e}(\mathbf{r}) = -\frac{q}{4\pi\epsilon_0} \frac{\mathbf{r} - (\mathbf{r}_0 - \frac{\mathbf{l}}{2})}{|\mathbf{r} - (\mathbf{r}_0 - \frac{\mathbf{l}}{2})|^3} + \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r} - (\mathbf{r}_0 + \frac{\mathbf{l}}{2})}{|\mathbf{r} - (\mathbf{r}_0 + \frac{\mathbf{l}}{2})|^3}.$$

Using the formula for the derivative of the terms

$$\begin{aligned} \partial_{\ell_k} \frac{x_i \pm \frac{\ell_i}{2}}{|\mathbf{x} \pm \frac{\mathbf{l}}{2}|^3} &= \frac{1}{2} \left[\pm \frac{\delta_{ik}}{r^3} - 3r^{-4} \left(\pm \frac{x_k \pm \frac{\ell_k}{2}}{r} \right) \right] \\ \partial_{\ell_k} \frac{x_i \pm \frac{\ell_i}{2}}{|\mathbf{x} \pm \frac{\mathbf{l}}{2}|^3} \Big|_{\mathbf{l}=\mathbf{0}} &= \mp \frac{1}{2} \left[-\frac{\delta_{ik}}{|\mathbf{x}|^3} + 3 \left(\frac{x_k}{r^5} \right) \right] = \mp \frac{1}{2} \partial_{r_{0k}} \frac{r_i - r_{0i}}{|\mathbf{r} - \mathbf{r}_0|^3} = \mp \frac{1}{2} \nabla_{\mathbf{r}_0} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \end{aligned}$$

we derive the first-order approximation in \mathbf{l} of the two terms

$$\frac{\mathbf{r} - (\mathbf{r}_0 \mp \frac{\mathbf{l}}{2})}{|\mathbf{r} - (\mathbf{r}_0 \mp \frac{\mathbf{l}}{2})|^3} = \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \pm \mathbf{l} \cdot \frac{1}{2} \nabla_{\mathbf{r}_0} \left(\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \right) + o(|\mathbf{l}|)$$

and, defining the dipole intensity $\mathbf{P}_0 := q\mathbf{l}$ and taking the quantities to the desired limit, that of the electric field

$$\begin{aligned} \mathbf{e}(\mathbf{r}) &= -\frac{1}{4\pi\epsilon_0} \mathbf{P}_0 \cdot \nabla_{\mathbf{r}_0} \left(\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \right) = \\ &= -\frac{1}{4\pi\epsilon_0} \left[\frac{(\mathbf{r} - \mathbf{r}_0)(\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^5} \cdot \mathbf{P}_0 - \frac{\mathbf{P}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \right] = \\ &= -\frac{1}{4\pi\epsilon_0} \left[\frac{(\mathbf{r} - \mathbf{r}_0) \otimes (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^5} - \frac{\mathbb{I}}{|\mathbf{r} - \mathbf{r}_0|^3} \right] \cdot \mathbf{P}_0. \end{aligned}$$

todo In the general case, it would be necessary to pay attention to the order of the factors in the product between vectors and tensors, but in this case, the symmetry of the second-order tensor (or of the operations) can be exploited.

4.3.2 Continuous Distribution of Dipoles

A distribution of dipoles with volume density $\mathbf{p}(\mathbf{r}_0)$, which produces the elementary dipole $\Delta\mathbf{P}(\mathbf{r}_0) = \mathbf{p}(\mathbf{r}_0)dV_0$ in the volume dV_0 , produces the electric field

$$\mathbf{e}(\mathbf{r}) = \int_{\mathbf{r}_0 \in V_0} \frac{1}{4\pi\epsilon_0} \mathbf{p}(\mathbf{r}_0) \cdot \nabla_{\mathbf{r}_0} \left(\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \right),$$

whose expression can be rewritten using the rules of integration by parts

$$\begin{aligned} \mathbf{e}(\mathbf{r}) &= \int_{\mathbf{r}_0 \in V_0} \frac{1}{4\pi\epsilon_0} \mathbf{p}(\mathbf{r}_0) \cdot \nabla_{\mathbf{r}_0} \left(\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \right) = \\ &= \oint_{\mathbf{r}_0 \in \partial V_0} \frac{1}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \underbrace{\hat{\mathbf{n}}(\mathbf{r}_0) \cdot \mathbf{p}(\mathbf{r}_0)}_{=:\sigma_P(\mathbf{r}_0)} + \int_{\mathbf{r}_0 \in V_0} \frac{1}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \underbrace{(-\nabla_{\mathbf{r}_0} \cdot \mathbf{p}(\mathbf{r}_0))}_{=:\rho_P(\mathbf{r}_0)}, \end{aligned}$$

having defined the surface polarization charge density σ_P and the volume polarization charge density ρ_P as the intensities of the distributed sources of the electric field, in analogy with the expression of Coulomb's law.

4.3.3 Reformulation of Maxwell's Equations and Charge Continuity

Gauss's equation determines the volume flux density of the electric field \mathbf{e} ,

$$\nabla \cdot \mathbf{e} = \frac{\rho}{\varepsilon_0}.$$

By decomposing the charge density as the sum of **free charges** ρ_f and **polarization charges** $\rho_P := -\nabla \cdot \mathbf{p}$, we can rework Gauss's equation,

$$\begin{aligned}\nabla \cdot \mathbf{e} &= \frac{\rho_f + \rho_P}{\varepsilon_0} \\ \nabla \cdot (\varepsilon_0 \mathbf{e} + \mathbf{p}) &= \rho_f\end{aligned}$$

$$\nabla \cdot \mathbf{d} = \rho_f,$$

having introduced the **displacement field**, $\mathbf{d} := \varepsilon_0 \mathbf{e} + \mathbf{p}$.

The decomposition of the electric current as the sum $\mathbf{j} = \mathbf{j}_f + \mathbf{j}_P$ of the free current \mathbf{j}_f and the polarization current \mathbf{j}_P , allows us to rework the continuity equation of electric charge

$$\begin{aligned}0 &= \partial_t \rho + \nabla \cdot \mathbf{j} = \\ &= \partial_t (\rho_f + \rho_P) + \nabla \cdot (\mathbf{j}_f + \mathbf{j}_P) = \\ &= \partial_t \rho_f + \nabla \cdot \mathbf{j}_f + \partial_t \rho_P + \nabla \cdot \mathbf{j}_P,\end{aligned}$$

and write the continuity equations for the two charge distributions (of different nature, it is assumed that both must satisfy charge continuity independently, if free charges remain free and polarization charges remain polarization charges),

$$\begin{aligned}\partial_t \rho_f + \nabla \cdot \mathbf{j}_f &= 0 \\ \partial_t \rho_P + \nabla \cdot \mathbf{j}_P &= 0 \quad \rightarrow \quad 0 = \nabla \cdot (-\partial_t \mathbf{p} + \mathbf{j}_P) \quad \rightarrow \quad \mathbf{j}_P = \partial_t \mathbf{p}\end{aligned}$$

todo justify absence of constant field

4.4 Magnetization

4.4.1 Single Magnetic Moment (Limit of an Elementary Loop)

Using the Biot-Savart law, specialized for a conductor carrying current $i(\mathbf{r}_0)$

$$\begin{aligned}d\mathbf{b}(\mathbf{r}) &= -\frac{\mu}{4\pi} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \times \mathbf{j}(\mathbf{r}_0) dV_0 = \\ &= -\frac{\mu}{4\pi} i(\mathbf{r}_0) \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \times \hat{\mathbf{t}}(\mathbf{r}_0) d\ell_0,\end{aligned}$$

we can calculate the magnetic field generated by a loop with path $\ell_0 = \partial S_0$ using the PSCE

$$\begin{aligned}\mathbf{b}(\mathbf{r}) &= \oint_{\ell_0} d\mathbf{b}(\mathbf{r}_0) = \\ &= -\frac{\mu}{4\pi} i_0 \oint_{\mathbf{r}_0 \in \ell_0} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \times \hat{\mathbf{t}}(\mathbf{r}_0) = \\ &= \frac{\mu}{4\pi} i_0 \int_{\mathbf{r}_0 \in S_0} \hat{\mathbf{n}}(\mathbf{r}_0) \cdot \nabla_{\mathbf{r}_0} \left(\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \right)\end{aligned}$$

The field generated by an elementary loop of surface S_0 with normal $\hat{\mathbf{n}}_0$, using the mean value theorem, is

$$\mathbf{b}(\mathbf{r}) = \frac{\mu}{4\pi} i_0 S_0 \hat{\mathbf{n}}_0 \cdot \nabla_{\mathbf{r}_0} \left(\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \right) + o(S_0)$$

and as $i_0 \rightarrow \infty$, $S_0 \rightarrow 0$ such that $\mathbf{M}_0 := i_0 S_0 \hat{\mathbf{n}}_0$

$$\begin{aligned} \mathbf{b}(\mathbf{r}) &= \frac{\mu}{4\pi} \mathbf{M}_0 \cdot \nabla_{\mathbf{r}_0} \left(\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \right) \\ &= -\frac{\mu_0}{4\pi} \left[\frac{(\mathbf{r} - \mathbf{r}_0)(\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^5} \cdot \mathbf{M}_0 - \frac{\mathbf{M}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \right] = \\ &= -\frac{\mu_0}{4\pi} \left[\frac{(\mathbf{r} - \mathbf{r}_0) \otimes (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^5} - \frac{\mathbb{I}}{|\mathbf{r} - \mathbf{r}_0|^3} \right] \cdot \mathbf{M}_0 . \end{aligned}$$

todo Analogy with the electric field produced by a distribution of dipoles.

Details

$$\begin{aligned} \oint_{\partial S} A t_i &= \int_S \varepsilon_{ijk} n_j \partial_k A \quad , \quad \oint_{\partial S} A \hat{\mathbf{t}} = \int_S \hat{\mathbf{n}} \times \nabla A \\ \oint_{\mathbf{r}_0 \in \ell_0} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \times \hat{\mathbf{t}}(\mathbf{r}_0) d\ell_0 &= \oint_{\mathbf{r}_0 \in \ell_0} \varepsilon_{ijk} \frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} t_k = \\ &= \int_{\mathbf{r}_0 \in S_0} \varepsilon_{krs} n_r \partial_s^0 \left(\varepsilon_{ijk} \frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} \right) = \\ &= \int_{\mathbf{r}_0 \in S_0} (\delta_{ir} \delta_{js} - \delta_{is} \delta_{jr}) n_r \partial_s^0 \left(\frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} \right) = \\ &= \int_{\mathbf{r}_0 \in S_0} \left\{ \underbrace{n_i \partial_j^0 \left(\frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} \right)}_{=0} - n_j \partial_i^0 \left(\frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} \right) \right\} = \\ &= - \int_{\mathbf{r}_0 \in S_0} n_j \partial_i^0 \left(\frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} \right) . \end{aligned}$$

4.4.2 Continuous Distribution of Magnetic Moment

To calculate the magnetic field generated by a volume distribution of magnetic moment, we can proceed in analogy with what was done to calculate the electric field generated by a distribution of dipoles

$$\begin{aligned} \mathbf{b}(\mathbf{r}) &= \int_{\mathbf{r}_0 \in V_0} \frac{\mu_0}{4\pi} \mathbf{m}(\mathbf{r}_0) \cdot \nabla_{\mathbf{r}_0} \left(\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \right) = \\ &= \oint_{\mathbf{r}_0 \in \partial V_0} \frac{\mu_0}{4\pi} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \hat{\mathbf{n}}(\mathbf{r}_0) \cdot \mathbf{m}(\mathbf{r}_0) + \int_{\mathbf{r}_0 \in V_0} \frac{\mu_0}{4\pi} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} (-\nabla_{\mathbf{r}_0} \cdot \mathbf{m}(\mathbf{r}_0)) , \end{aligned}$$

but without obtaining an analogy with the expression of the Biot-Savart law, which involves the cross product between the term $\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3}$ and a current density $\mathbf{j}(\mathbf{r}_0)$.

Details

We can rewrite

$$\begin{aligned}
& \oint_{\mathbf{r}_0 \in \partial V_0} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \times (\hat{\mathbf{n}}(\mathbf{r}_0) \times \mathbf{m}(\mathbf{r}_0)) \\
&= \oint_{\mathbf{r}_0 \in \partial V_0} \varepsilon_{ijk} \frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} \varepsilon_{krs} n_r m_s = \\
&= \int_{\mathbf{r}_0 \in V_0} (\delta_{ir} \delta_{js} - \delta_{is} \delta_{jr}) \partial_r^0 \left(\frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} m_s \right) = \\
&= \int_{\mathbf{r}_0 \in V_0} \left\{ \partial_i^0 \left(\frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} m_j \right) - \partial_j^0 \left(\frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} m_i \right) \right\} = \\
&= \int_{\mathbf{r}_0 \in V_0} \left\{ \partial_i^0 \frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} m_j + \frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} \partial_i^0 m_j - \frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} \partial_j^0 m_i - \underbrace{\partial_j^0 \frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3}}_{=0} m_i \right\} = \\
&= \int_{\mathbf{r}_0 \in V_0} \left\{ \partial_i^0 \frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} m_j + \varepsilon_{ijk} \varepsilon_{krs} \frac{r_j - r_{0,j}}{|\mathbf{r} - \mathbf{r}_0|^3} \partial_r^0 m_s \right\} = \\
&= \int_{\mathbf{r}_0 \in V_0} \left\{ \nabla_{\mathbf{r}_0} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \cdot \mathbf{m}(\mathbf{r}_0) + \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \times (\nabla_{\mathbf{r}_0} \times \mathbf{m}(\mathbf{r}_0)) \right\} =
\end{aligned}$$

using vector calculus identities,

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \varepsilon_{ijk} a_j \varepsilon_{krs} b_r c_s = \\
&= (\delta_{ir} \delta_{js} - \delta_{is} \delta_{jr}) a_j b_r c_s = \\
&= a_j b_i c_j - c_i b_j a_j = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \\
a_j \partial_i m_j - a_j \partial_j m_i &= (\delta_{ir} \delta_{js} - \delta_{is} \delta_{jr}) a_j \partial_r m_s = \\
&= \varepsilon_{ijk} \varepsilon_{krs} a_j \partial_r m_s = \\
&= \mathbf{a} \times (\nabla \times \mathbf{m})
\end{aligned}$$

The magnetic field generated by a distribution of magnetic moment can therefore be rewritten as

$$\begin{aligned}
\mathbf{b}(\mathbf{r}) &= \int_{\mathbf{r}_0 \in V_0} \frac{\mu_0}{4\pi} \mathbf{m}(\mathbf{r}_0) \cdot \nabla_{\mathbf{r}_0} \left(\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \right) = \\
&= -\frac{\mu_0}{4\pi} \oint_{\mathbf{r}_0 \in \partial V_0} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \times \underbrace{(-\hat{\mathbf{n}}(\mathbf{r}_0) \times \mathbf{m}(\mathbf{r}_0))}_{\mathbf{j}_M^s} - \frac{\mu_0}{4\pi} \int_{\mathbf{r}_0 \in V_0} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \times \underbrace{(\nabla_{\mathbf{r}_0} \times \mathbf{m}(\mathbf{r}_0))}_{\mathbf{j}_M},
\end{aligned}$$

having defined the surface magnetization current density \mathbf{j}_M^s and the volume magnetization current density \mathbf{j}_M as the intensities of the distributed singularities, in analogy with the expression of the Biot-Savart law.

4.4.3 Reformulation of Maxwell's Equations and Charge Continuity

The Ampère-Maxwell law can be rewritten

$$\begin{aligned}
\nabla \times \mathbf{b} - \mu_0 \varepsilon_0 \partial_t \mathbf{e} &= \mu_0 \mathbf{j} \\
\nabla \times \mathbf{b} - \mu_0 \partial_t (\mathbf{d} - \mathbf{p}) &= \mu_0 (\mathbf{j}_f + \mathbf{j}_P + \mathbf{j}_M) \\
\nabla \times \underbrace{(\mathbf{b} - \mu_0 \mathbf{m})}_{=:\mu_0 \mathbf{h}} - \mu_0 \partial_t \mathbf{d} + \mu_0 \underbrace{(\partial_t \mathbf{p} - \mathbf{j}_P)}_{=0} &= \mu_0 \mathbf{j}_f \\
\nabla \times \mathbf{h} - \partial_t \mathbf{d} &= \mathbf{j}_f
\end{aligned}$$

From the continuity equation of electric current,

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0 ,$$

we derive the continuity equation for magnetization charges

$$\begin{aligned} 0 &= \partial_t \rho_M + \nabla \cdot \mathbf{j}_M = \\ &= \partial_t \rho_M + \underbrace{\nabla \cdot \nabla \times \mathbf{m}}_{\equiv 0} . \end{aligned}$$

4.5 Examples

- conductors
- ferromagnetic and weak magnetism (para-, dia-, anti-)

4.6 Jump Conditions

Differential form of Maxwell's equations

$$\begin{cases} \nabla \cdot \mathbf{d} = \rho_f \\ \nabla \times \mathbf{e} + \partial_t \mathbf{b} = \mathbf{0} \\ \nabla \cdot \mathbf{b} = 0 \\ \nabla \times \mathbf{h} - \partial_t \mathbf{d} = \mathbf{j}_f \end{cases}$$

Integral form of Maxwell's equations

$$\begin{cases} \oint_{\partial V} \mathbf{d} \cdot \hat{\mathbf{n}} = \int_V \rho_f \\ \oint_{\partial S} \mathbf{e} \cdot \hat{\mathbf{t}} + \frac{d}{dt} \int_S \mathbf{b} \cdot \hat{\mathbf{n}} = 0 \\ \oint_{\partial V} \mathbf{b} \cdot \hat{\mathbf{n}} = 0 \\ \oint_{\partial S} \mathbf{h} \cdot \hat{\mathbf{t}} - \frac{d}{dt} \int_S \mathbf{d} \cdot \hat{\mathbf{n}} = \int_S \mathbf{j}_f \cdot \hat{\mathbf{n}} \end{cases}$$

Letting V and S «collapse on a discontinuity»...

$$\begin{cases} [d_n] = \sigma_f \\ [e_t] = 0 \\ [b_n] = 0 \\ [h_t] = \iota_f , \end{cases} \quad (4.1)$$

being σ_f and ι_f surface charge and current density, with physical dimension $\frac{\text{charge}}{\text{surface}}$, and $\frac{\text{current}}{\text{surface}}$ respectively. These contributions can be thought of as Dirac delta contributions in volume density, namely

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}, t) + \sigma(\mathbf{r}_s, t) \delta_1(\mathbf{r} - \mathbf{r}_s) ,$$

being $\rho(\mathbf{r}, t)$ the regular part of the volume density in all the points of the domain $\mathbf{r} \in V$, $\sigma(\mathbf{r}_s, t)$ the surface density on 2-dimensional surfaces $\mathbf{r}_s \in S$, $\delta_1()$ the Dirac's delta with physical dimension $\frac{1}{\text{length}}$.

If there's no free surface charge and currents, jump conditions for linear media become

$$\left\{ \begin{array}{l} [d_n] = 0 \\ [e_t] = 0 \\ [b_n] = 0 \\ [h_t] = 0 \end{array} \right. \rightarrow \left\{ \begin{array}{l} d_{n,1} = d_{n,2} \rightarrow \varepsilon_1 e_{n,1} = \varepsilon_2 e_{n,2} \\ e_{t,1} = e_{t,2} \\ b_{n,1} = b_{n,2} \\ h_{t,1} = h_{t,2} \rightarrow \frac{1}{\mu_1} b_{t,1} = \frac{1}{\mu_2} b_{t,2} \end{array} \right. \quad (4.2)$$

5.1 Zero electric field inside a conductor

Studying the transient of the electric charge distribution inside a conductor,

$$\vec{e} = \rho_R \vec{j} ,$$

whose constitutive equation is

$$\vec{d} = \varepsilon \vec{e} ,$$

with free electric charge continuity equation

$$\partial_t \rho_f + \nabla \cdot \vec{j}_f = 0 ,$$

and Gauss equation for the displacement field

$$\begin{aligned} \nabla \cdot \vec{d} &= \rho_f . \\ \partial_t \rho_f &= -\nabla \cdot \vec{j}_f = \\ &= -\nabla \cdot \left(\frac{1}{\rho_R} \vec{e} \right) = \\ &= -\frac{1}{\rho_R \varepsilon} \nabla \cdot \vec{d} = \\ &= -\frac{1}{\rho_R \varepsilon} \rho_f , \end{aligned}$$

having assumed uniform properties. The differential equation in the volume of the conductor provides the evolution of the electric charge in the volume $\rho(\mathbf{r}, t)$, given the initial condition $\rho(\mathbf{r}, 0) = \rho_{f,0}(\mathbf{r})$

$$\begin{aligned} \partial_t \rho_f &= -\frac{1}{\rho_R \varepsilon} \rho_f \\ \rho_f(\mathbf{r}, t) &= \rho_{f,0}(\mathbf{r}) \exp \left[-\frac{t}{\rho_R \varepsilon} \right] . \end{aligned}$$

For a conductor:

- $\varepsilon \sim \varepsilon_0 = 8.85 \cdot 10^{-12} \text{Fm}^{-1}$
- $\rho_R \sim 10^{-7} \Omega \text{m}$

so that the time constant (that can be thought as a characteristic time) of the process is

$$\tau = \rho_R \varepsilon \sim 8.85 \cdot 10^{-19} \text{s} ,$$

and thus, after a very short period of time the volume charge density is approximately zero everywhere in the volume: it accumulates in a very thin surface layer.

Proof

$$\partial_t \left(\rho_f e^{\frac{t}{\rho_R \varepsilon}} \right) = 0$$

$$\rho_f(\mathbf{r}, t) e^{\frac{t}{\rho_R \varepsilon}} = a(\mathbf{r})$$

and applying initial conditions in all the points of the domain, $\rho_f(\mathbf{r}, 0) = \rho_{f,0}(\mathbf{r})$, function $a(\mathbf{r})$ must be equal to $\rho_{f,0}(\mathbf{r})$ and the solution reads

$$\rho_f(\mathbf{r}, t) = \rho_{f,0}(\mathbf{r}) \exp \left[-\frac{t}{\rho_R \varepsilon} \right]$$

Energy balance in electromagnetism

6.1 Force, moment, and power on elementary components

6.1.1 Force, moment and power on a point electric charge

Point electric charge with charge q in a point $\vec{r}_P(t)$ at time t where electromagnetic field is $\vec{e}(\vec{r}, t), \vec{b}(\vec{r}, t)$:

- Lorentz's force

$$\vec{F} = q \left(\vec{e}(\vec{r}_P(t), t) - \vec{b}(\vec{r}_P(t), t) \times \vec{v}_P(t) \right) ,$$

- zero moment, since it has no dimension (and assumed uniform or symmetric or... distribution of electric charge)
- power

$$\begin{aligned} P &= \vec{v}_P(t) \cdot \vec{F} = \\ &= \vec{v}_P(t) \cdot q \left(\vec{e}(\vec{r}_P(t), t) - \vec{b}(\vec{r}_P(t), t) \times \vec{v}_P(t) \right) = q \vec{v}_P(t) \cdot \vec{e}(\vec{r}_P(t), t) . \end{aligned}$$

6.1.2 Force, moment and power on a electric dipole

Electric dipole with center $\vec{r}_C(t)$, axis $\vec{\ell}$, so that the positive charge q is in $P_+ = C + \frac{\vec{\ell}}{2}$ and the negative charge is in $P_- = C - \frac{\vec{\ell}}{2}$, with $q \rightarrow +\infty, |\vec{\ell}| \rightarrow 0$, s.t. $q|\vec{\ell}| = |\vec{d}|$ finite.

Kinematics and expansion of the field

$$\begin{aligned} \vec{v}_{\pm} &= \vec{v}_C \pm \vec{\omega} \times \frac{\vec{\ell}}{2} \\ \vec{e}(P_{\pm}) &= \vec{e} \left(C \pm \frac{\vec{\ell}}{2} \right) = \vec{e}(C) \pm \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}(C) + o(|\vec{\ell}|) \end{aligned}$$

$$\vec{b}(P_{\pm}) = \vec{b} \left(C \pm \frac{\vec{\ell}}{2} \right) = \vec{b}(C) \pm \frac{\vec{\ell}}{2} \cdot \nabla \vec{b}(C) + o(|\vec{\ell}|)$$

Net force.

$$\begin{aligned} \vec{F} &= \vec{F}_+ + \vec{F}_- = \\ &= q [\vec{e}(P_+) - \vec{b}(P_+) \times \vec{v}_+] - q [\vec{e}(P_-) - \vec{b}(P_-) \times \vec{v}_-] = \\ &= q \left[\vec{e}_C + \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}_C - \left(\vec{b}_C + \frac{\vec{\ell}}{2} \cdot \nabla \vec{b}_C \right) \times \left(\vec{v}_C + \vec{\omega} \times \frac{\vec{\ell}}{2} \right) \right] + \\ &\quad - q \left[\vec{e}_C - \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}_C - \left(\vec{b}_C - \frac{\vec{\ell}}{2} \cdot \nabla \vec{b}_C \right) \times \left(\vec{v}_C - \vec{\omega} \times \frac{\vec{\ell}}{2} \right) \right] = \\ &= q\vec{\ell} \cdot \nabla \vec{e}(C) - (q\vec{\ell} \cdot \nabla \vec{b}(C)) \times \vec{v}_C + \vec{b}(C) \times (\vec{\omega} \times q\vec{\ell}) + o(|\vec{\ell}|) \end{aligned}$$

Net moment, w.r.t. C .

$$\begin{aligned} \vec{M}_C &= \frac{\vec{\ell}}{2} \times \vec{F}_+ - \frac{\vec{\ell}}{2} \times \vec{F}_- = \\ &= q\frac{\vec{\ell}}{2} \times [\vec{e}(P_+) - \vec{b}(P_+) \times \vec{v}_+] + q\frac{\vec{\ell}}{2} \times [\vec{e}(P_-) - \vec{b}(P_-) \times \vec{v}_-] = \\ &= q\frac{\vec{\ell}}{2} \times \left[\vec{e}_C + \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}_C - \left(\vec{b}_C + \frac{\vec{\ell}}{2} \cdot \nabla \vec{b}_C \right) \times \left(\vec{v}_C + \vec{\omega} \times \frac{\vec{\ell}}{2} \right) \right] + \\ &\quad + q\frac{\vec{\ell}}{2} \times \left[\vec{e}_C - \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}_C - \left(\vec{b}_C - \frac{\vec{\ell}}{2} \cdot \nabla \vec{b}_C \right) \times \left(\vec{v}_C - \vec{\omega} \times \frac{\vec{\ell}}{2} \right) \right] = \\ &= q\vec{\ell} \times [\vec{e}_C - \vec{b}_C \times \vec{v}_C] + o(|\vec{\ell}|). \end{aligned}$$

Power.

$$\begin{aligned} P &= P_+ + P_- = \\ &= \vec{F}_+ \cdot \vec{v}_+ + \vec{F}_- \cdot \vec{v}_- = \\ &= q [\vec{e}(P_+) - \vec{b}(P_+) \times \vec{v}_+] \cdot \vec{v}_+ - q [\vec{e}(P_-) - \vec{b}(P_-) \times \vec{v}_-] \cdot \vec{v}_- = \\ &= q \vec{e}(P_+) \cdot \vec{v}_+ - q \vec{e}(P_-) \cdot \vec{v}_- = \\ &= q \left[\vec{e}_C + \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}_C \right] \cdot \left[\vec{v}_C + \vec{\omega} \times \frac{\vec{\ell}}{2} \right] - q \left[\vec{e}_C - \frac{\vec{\ell}}{2} \cdot \nabla \vec{e}_C \right] \cdot \left[\vec{v}_C - \vec{\omega} \times \frac{\vec{\ell}}{2} \right] = \\ &= \vec{e}_C \cdot (\vec{\omega} \times q\vec{\ell}) + (q\vec{\ell} \cdot \nabla \vec{e}_C) \cdot \vec{v}_C + o(|\vec{\ell}|^2). \end{aligned}$$

6.1.3 Force, moment and power on a magnetic dipole

On an elementary magnetic dipole, modeled as a «small» circuit with current i enclosing area S and center C , with $S \rightarrow 0$, $i \rightarrow +\infty$ so that $iS\hat{n} := \vec{m}$ finite

Force.

...

$$\vec{F} = \nabla \vec{b}(C) \cdot \vec{m}$$

Moment.

...

$$\vec{M}_C = \vec{m} \times \vec{b}(C)$$

Power.

$$P = \vec{v}_C \cdot \nabla \vec{b}(C) \cdot \vec{m} + \vec{\omega} \cdot \vec{m} \times \vec{b}(C) .$$

6.2 Energy balance

todo Check and put charges, currents, and dipoles together with the electromagnetic field

Ispirati dalle dimensioni fisiche dei campi elettromagnetici,

$$\begin{aligned} [\mathbf{e}] &= \frac{\text{force}}{\text{charge}} & , & & [\mathbf{d}] &= \frac{\text{charge}}{\text{length}^2} \\ [\mathbf{b}] &= \frac{\text{force} \cdot \text{time}}{\text{charge} \cdot \text{length}} & , & & [\mathbf{h}] &= \frac{\text{charge}}{\text{time} \cdot \text{length}} \end{aligned}$$

$$\begin{aligned} [\mathbf{e} \cdot \mathbf{d}] &= \frac{\text{force}}{\text{length}^2} = \frac{\text{energy}}{\text{length}^3} = [u] \\ [\mathbf{b} \cdot \mathbf{h}] &= \frac{\text{force}}{\text{length}^2} = \frac{\text{energy}}{\text{length}^3} = [u] \end{aligned}$$

si può costruire la densità di volume di energia (**todo** trovare motivazioni più convincenti, non basandosi solo sull'analisi dimensionale ma sul lavoro)

$$u = \frac{1}{2} (\mathbf{e} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{h}) .$$

Si può calcolare la derivata parziale nel tempo della densità di energia, u , e usare le equazioni di Maxwell per ottenere un'equazione di bilancio dell'energia del campo elettromagnetico. Per un mezzo isotropo lineare, per il quale valgono le equazioni costitutive $\mathbf{d} = \varepsilon \mathbf{e}$, $\mathbf{b} = \mu \mathbf{h}$, la derivata parziale nel tempo dell'energia elettromagnetica può essere riscritta sfruttando la regola di derivazione del prodotto e le equazioni di Faraday-Lenz-Neumann e Ampère-Maxwell,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{1}{2} \mathbf{e} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{h} \right) = \quad (\dots) \\ &= \mathbf{e} \cdot \partial_t \mathbf{d} + \mathbf{h} \cdot \partial_t \mathbf{b} = \\ &= \mathbf{e} \cdot (\nabla \times \mathbf{h} - \mathbf{j}) - \mathbf{h} \cdot \nabla \times \mathbf{e} . \end{aligned}$$

L'ultimo termine può essere ulteriormente manipolato, usando l'identità vettoriale

$$\begin{aligned} \mathbf{e} \cdot \nabla \times \mathbf{h} - \mathbf{h} \cdot \nabla \times \mathbf{e} &= e_i \varepsilon_{ijk} \partial_j h_k - h_i \varepsilon_{ijk} \partial_j e_k = \quad (i \rightarrow k, k \rightarrow i) \\ &= e_i \varepsilon_{ijk} \partial_j h_k - h_k \varepsilon_{kji} \partial_j e_i = \\ &= e_i \varepsilon_{ijk} \partial_j h_k + h_k \varepsilon_{ijk} \partial_j e_i = \\ &= \partial_j (\varepsilon_{ijk} e_i h_k) = \\ &= \partial_j (\varepsilon_{jki} e_i h_k) = \\ &= \nabla \cdot (\mathbf{h} \times \mathbf{e}) = -\nabla \cdot (\mathbf{e} \times \mathbf{h}) \end{aligned}$$

che permette di scrivere l'equazione del bilancio di energia elettromagnetica come,

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{s} = -\mathbf{e} \cdot \mathbf{j} ,$$

dove è stato definito il **vettore di Poynting**, o meglio il campo vettoriale di Poynting,

$$\mathbf{s}(\mathbf{r}, t) := \mathbf{e}(\mathbf{r}, t) \times \mathbf{h}(\mathbf{r}, t) ,$$

che può essere identificato come un flusso di potenza per unità di superficie, comparando sotto l'operatore di divergenza nel bilancio di energia.

todo. Rimandare a una sezione in cui si mostra questa ultima affermazione passando dal bilancio differenziale al bilancio integrale e si usa il teorema della divergenza, $\int_V \nabla \cdot \mathbf{s} = \oint_{\partial V} \mathbf{s} \cdot \hat{\mathbf{n}}$.

Bilancio di energia di cariche nel vuoto, o i materiali senza polarizzazione o magnetizzazione

Moto di cariche puntiformi. L'equazione del moto di carica puntiforme q_k nella posizione $\mathbf{r}_k(t)$ al tempo t è

$$m_k \ddot{\mathbf{r}}_k = \mathbf{f}_k + \mathbf{f}_k^{em} ,$$

avendo riconosciuto i contributi di forza dovuti al campo elettromagnetico come \mathbf{f}_k^{em} dagli altri. L'espressione della forza dovuta al campo elettromagnetico sulla carica k è data dalla forza di Lorentz,

$$\mathbf{f}_k^{em}(t) = q_k [\mathbf{e}(\mathbf{r}_k(t), t) - \mathbf{b}(\mathbf{r}_k(t), t) \times \dot{\mathbf{r}}_k(t)]$$

Continuità della carica elettrica. La densità di carica e di corrente elettrica di un insieme di cariche libere puntiformi macroscopiche può essere scritta come

$$\begin{aligned} \rho(\mathbf{r}, t) &= \sum_k q_k \delta(\mathbf{r} - \mathbf{r}_k(t)) \\ \mathbf{j}(\mathbf{r}, t) &= \sum_k q_k \dot{\mathbf{r}}_k(t) \delta(\mathbf{r} - \mathbf{r}_k(t)) . \end{aligned}$$

L'equazione di continuità della carica, $\partial_t \rho + \nabla \cdot \mathbf{j} = 0$, risulta quindi soddisfatta,

$$\begin{aligned} \partial_t \rho &= - \sum_k q_k \partial_i \delta(\mathbf{r} - \mathbf{r}_k(t)) \dot{r}_{k,i} \\ \partial_i j_i &= \sum_k q_k \dot{r}_{k,i} \partial_i \delta(\mathbf{r} - \mathbf{r}_k(t)) \end{aligned}$$

Procedimento alternativo (e più generale?)

todo In caso questo procedimento sia più generale, o più corretto, sostituire il procedimento precedente.

La carica elementare in un volumetto ΔV è data da dal prodotto tra il volume e la densità volumetrica di carica, $\rho \Delta V$; la velocità media locale della carica elettrica è \mathbf{v} ; la forza agente sulla carica elementare immersa in un campo elettromagnetico è determinata dalla formula di Lorentz, $\mathbf{f} \Delta V = \Delta V \rho (\mathbf{e} - \mathbf{b} \times \mathbf{v})$. La potenza di questa forza è il prodotto scalare con la velocità media delle cariche, $\Delta V \mathbf{f} \cdot \mathbf{v}$

La potenza del campo elettromagnetico sul moto della carica elettrica per unità di volume è quindi

$$\mathbf{v} \cdot \mathbf{f} = \rho \mathbf{v} \cdot (\mathbf{e} - \mathbf{b} \times \mathbf{v}) = \rho \mathbf{v} \cdot \mathbf{e} = \mathbf{j} \cdot \mathbf{e} .$$

todo

- discutere questo termine del bilancio di energia cinetica nel moto della carica elettrica
- questo termine compare con segno opposto nel bilancio dell'energia elettromagnetica del sistema
- dove compare la non-conservatività del problema in presenza di materiali dissipativi (come resistenza elettrica con $\mathbf{e} = \rho_R \mathbf{j}$?

Il termine $\mathbf{e} \cdot \mathbf{j}$ può essere manipolato usando le equazioni di Maxwell, e le relazioni

$$\begin{cases} \mathbf{d} = \varepsilon_0 \mathbf{e} + \mathbf{p} \\ \mathbf{h} = \frac{\mathbf{b}}{\mu_0} - \mathbf{m} \end{cases}$$

$$\begin{aligned} \mathbf{e} \cdot \mathbf{j} &= \mathbf{e} \cdot (\nabla \times \mathbf{h} - \partial_t \mathbf{d}) = \\ &= -\nabla \cdot (\mathbf{e} \times \mathbf{h}) + \mathbf{h} \cdot \nabla \times \mathbf{e} - \mathbf{e} \cdot \partial_t \mathbf{d} = \\ &= -\nabla \cdot (\mathbf{e} \times \mathbf{h}) - \mathbf{h} \cdot \partial_t \mathbf{b} - \mathbf{e} \cdot \partial_t \mathbf{d} \end{aligned}$$

Gli ultimi due termini possono essere manipolati in diverse maniere,

$$\begin{aligned} \mathbf{e} \cdot \partial_t \mathbf{d} &= \mathbf{e} \cdot \partial_t (\varepsilon_0 \mathbf{e} + \mathbf{p}) = \partial_t \left(\frac{1}{2} \varepsilon_0 \mathbf{e} \cdot \mathbf{e} \right) + \mathbf{e} \cdot \partial_t \mathbf{p} \\ &= \partial_t \left(\frac{1}{2} \mathbf{e} \cdot \mathbf{d} \right) + \frac{1}{2} (\mathbf{e} \cdot \partial_t \mathbf{p} - \mathbf{p} \cdot \partial_t \mathbf{e}) \\ &= \partial_t \left(\frac{1}{2\varepsilon_0} \mathbf{d} \cdot \mathbf{d} \right) - \frac{\mathbf{p}}{\varepsilon_0} \cdot \partial_t \mathbf{d} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \cdot \partial_t \mathbf{b} &= \mathbf{h} \cdot \partial_t (\mu_0 \mathbf{h} + \mu_0 \mathbf{m}) = \partial_t \left(\frac{1}{2} \mu_0 \mathbf{h} \cdot \mathbf{h} \right) + \mu_0 \mathbf{h} \cdot \partial_t \mathbf{m} \\ &= \partial_t \left(\frac{1}{2} \mathbf{b} \cdot \mathbf{h} \right) + \frac{1}{2} \mu_0 (\mathbf{h} \cdot \partial_t \mathbf{m} - \mathbf{m} \cdot \partial_t \mathbf{h}) \\ &= \partial_t \left(\frac{1}{2\mu_0} \mathbf{b} \cdot \mathbf{b} \right) - \mathbf{m} \cdot \partial_t \mathbf{b} \end{aligned}$$

Nel vuoto o in mezzi lineari $\mathbf{e} \cdot \partial_t \mathbf{p} - \mathbf{p} \cdot \partial_t \mathbf{e} = 0$, $\mathbf{h} \cdot \partial_t \mathbf{m} - \mathbf{m} \cdot \partial_t \mathbf{h} = 0$. Usando le seconde espressioni, si può riscrivere l'equazione dell'energia del campo elettromagnetico come

$$\begin{aligned} \partial_t \left(\frac{1}{2} \mathbf{e} \cdot \mathbf{d} + \frac{1}{2} \mathbf{b} \cdot \mathbf{h} \right) + \nabla \cdot (\mathbf{e} \times \mathbf{h}) &= -\mathbf{e} \cdot \mathbf{j} + \\ &- \frac{1}{2} [\mathbf{e} \cdot \partial_t \mathbf{p} - \mathbf{p} \cdot \partial_t \mathbf{e} + \mu_0 (\mathbf{h} \cdot \partial_t \mathbf{m} - \mathbf{m} \cdot \partial_t \mathbf{h})] \end{aligned}$$

o, usando le definizioni di densità di energia elettromagnetica u e vettore di Poynting \mathbf{s} ,

$$\partial_t u + \nabla \cdot \mathbf{s} = -\mathbf{e} \cdot \mathbf{j} - \frac{1}{2} [\mathbf{e} \cdot \partial_t \mathbf{p} - \mathbf{p} \cdot \partial_t \mathbf{e} + \mu_0 (\mathbf{h} \cdot \partial_t \mathbf{m} - \mathbf{m} \cdot \partial_t \mathbf{h})]$$

Energy and momentum balance in linear, local, isotropic, non-dispersive media

$$\begin{cases} \mathbf{d} = \varepsilon_0 \mathbf{e} + \mathbf{p} \\ \mathbf{h} = \frac{1}{\mu_0} \mathbf{b} - \mathbf{m} . \end{cases}$$

with

$$\mathbf{d} = \varepsilon \mathbf{e} \quad , \quad \mathbf{h} = \frac{\mathbf{b}}{\mu}$$

Let r be mass density, and \vec{v} be charge velocity field, the equation of motion - momentum equation - of electric charges reads

$$r \frac{D\mathbf{v}}{Dt} = \mathbf{f} ,$$

and the kinetic energy equation becomes

$$\mathbf{v} \cdot \mathbf{f} = r \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} = r \frac{D}{Dt} \frac{|\mathbf{v}|^2}{2} ,$$

or using continuity equation for r , it can be recast in conservative form. The same term can be recast using the expression of Lorentz's force on electric charges in electromagnetic field (**is this the right way to evaluate power of bounded charges and currents? check it!**)

$$\mathbf{v} \cdot \mathbf{f} = \mathbf{v} \cdot [\rho(\mathbf{e} - \mathbf{b} \times \mathbf{v})] = \rho \mathbf{v} \cdot \mathbf{e} = \mathbf{e} \cdot \mathbf{j} ,$$

and furthered manipulated writing $\mathbf{j} = \mathbf{j}_f + \mathbf{j}_p + \mathbf{j}_m$ and using Maxwell's equations

$$\begin{aligned} \mathbf{e} \cdot \mathbf{j} &= \mathbf{e} \cdot (\mathbf{j}_f + \mathbf{j}_p + \mathbf{j}_m) = \\ &= \mathbf{e} \cdot (\nabla \times \mathbf{h} - \partial_t \mathbf{d}) + \mathbf{e} \cdot \partial_t \mathbf{p} + \mathbf{e} \cdot \nabla \times \mathbf{m} = \\ &= \mathbf{e} \cdot \nabla \times (\mathbf{h} + \mathbf{m}) - \mathbf{e} \cdot \partial_t (\mathbf{d} - \mathbf{p}) = \\ &= \frac{1}{\mu_0} \mathbf{e} \cdot \nabla \times \mathbf{b} - \varepsilon_0 \mathbf{e} \cdot \partial_t \mathbf{e} = \\ &= -\frac{1}{\mu_0} \nabla \cdot (\mathbf{e} \times \mathbf{b}) - \frac{\mathbf{b}}{\mu_0} \cdot \partial_t \mathbf{b} - \varepsilon_0 \mathbf{e} \cdot \partial_t \mathbf{e} = \end{aligned}$$

$$\begin{aligned}\mathbf{e} \cdot \nabla \times \mathbf{h} &= e_i \varepsilon_{ijk} \partial_j h_k = \partial_j (\varepsilon_{jki} h_k e_i) - \varepsilon_{ijk} h_k \partial_j e_i = -\nabla \cdot (\mathbf{e} \times \mathbf{h}) + \mathbf{h} \cdot \nabla \times \mathbf{e} = -\nabla \cdot (\mathbf{e} \times \mathbf{h}) - \mathbf{h} \cdot \partial_t \mathbf{b} \\ \mathbf{e} \cdot \mathbf{j}_f &= -\nabla \cdot (\mathbf{e} \times \mathbf{h}) - \mathbf{e} \cdot \partial_t \mathbf{d} - \mathbf{h} \cdot \partial_t \mathbf{b}\end{aligned}$$

7.1 Linear media - energy

For linear media, the energy of the electromagnetic field per unit volume reads

$$u = \frac{1}{2} (\mathbf{e} \cdot \mathbf{d} + \mathbf{h} \cdot \mathbf{b})$$

so that the differential balance equation for the energy of the electromagnetic field becomes

$$\partial_t u + \nabla \cdot \mathbf{s} = -\mathbf{e} \cdot \mathbf{j},$$

with Poynting vector $\mathbf{s} := \mathbf{e} \times \mathbf{h}$, namely the momentum density of the electromagnetic field.

7.2 Linear media - momentum

$$\begin{aligned}\partial_t \mathbf{s} &= \partial_t s_i = \partial_t (\varepsilon_{ijk} e_j h_k) \\ \varepsilon_{ijk} \partial_t e_j h_k &= \frac{1}{\varepsilon} \varepsilon_{ijk} \partial_t d_j h_k \\ &= \frac{1}{\varepsilon} \varepsilon_{ijk} (\varepsilon_{jlm} \partial_l h_m - j_j^f) h_k \\ &= -\frac{1}{\varepsilon} \varepsilon_{ijk} j_j^f h_k + \frac{1}{\varepsilon} \varepsilon_{ijk} \varepsilon_{jlm} h_k \partial_l h_m \\ &= -\frac{1}{\varepsilon} \varepsilon_{ijk} j_j^f h_k + \frac{1}{\varepsilon} (\delta_{im} \delta_{kl} - \delta_{il} \delta_{km}) h_k \partial_l h_m = \\ &= -\frac{1}{\varepsilon} \varepsilon_{ijk} j_j^f h_k + \frac{1}{\varepsilon} (h_m \partial_m h_i - h_m \partial_i h_m) = \\ &= -\frac{1}{\varepsilon} \varepsilon_{ijk} j_j^f h_k + \frac{1}{\varepsilon} \left[\partial_m (h_m h_i) - \partial_m h_m h_i - \partial_i \left(\frac{h_m h_m}{2} \right) \right] = \\ &= \frac{1}{\varepsilon \mu} \varepsilon_{ijk} b_j j_k^f + \frac{1}{\varepsilon \mu} \left[\partial_m (b_m h_i) - \underbrace{\partial_m b_m}_{=0} h_i - \partial_i \left(\frac{h_m b_m}{2} \right) \right] = \\ \varepsilon_{ijk} e_j \partial_t h_k &= \frac{1}{\mu} \varepsilon_{ijk} e_j \partial_t b_k = \\ &= -\frac{1}{\mu} \varepsilon_{ijk} e_j (\varepsilon_{klm} \partial_l e_m) = \\ &= -\frac{1}{\mu} (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) e_j \partial_l e_m = \\ &= -\frac{1}{\mu} (e_m \partial_i e_m - e_m \partial_m e_i) = \\ &= -\frac{1}{\mu} \left[\partial_i \left(\frac{e_m e_m}{2} \right) - \partial_m (e_m e_i) + \partial_m e_m e_i \right] = \\ &= -\frac{1}{\varepsilon \mu} \left[\partial_i \left(\frac{d_m e_m}{2} \right) - \partial_m (d_m e_i) + \rho^f e_i \right].\end{aligned}$$

so that

$$\partial_t s_i + c^2 \partial_m \left[\frac{1}{2} (d_n e_n + h_n b_n) \delta_{mi} - (h_m b_i + d_m e_i) \right] = -c^2 \rho^f e_i + c^2 \varepsilon_{ijk} b_j j_k^f$$

or

$$\partial_t \mathbf{s} + c^2 \nabla \cdot \left[\frac{1}{2} (\mathbf{d} \cdot \mathbf{e} + \mathbf{h} \cdot \mathbf{b}) \mathbb{I} - (\mathbf{d} \otimes \mathbf{e} + \mathbf{h} \otimes \mathbf{b}) \right] = -c^2 (\rho^f \mathbf{e} - \mathbf{b} \times \mathbf{j}^f)$$

$$\begin{cases} \partial_t u + \nabla \cdot \mathbf{s} = -\mathbf{e} \cdot \mathbf{j}^f \\ \partial_t \mathbf{s} + c^2 \nabla \cdot [u \mathbb{I} - (\mathbf{d} \otimes \mathbf{e} + \mathbf{h} \otimes \mathbf{b})] = -c^2 (\mathbf{e} \rho^f - \mathbf{b} \times \mathbf{j}^f) \end{cases}$$

todo use this system to derive the 4-d formulation of special relativity in modern physics

CAPITOLO 8

Relativity in electromagnetism

9.1 Wave Equations in Electromagnetism

Vector Identity.

$$\Delta \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \times \nabla \times \mathbf{v}$$

Proof.

$$\begin{aligned} \nabla \times \nabla \times \mathbf{v} &= \varepsilon_{ijk} \partial_j (\varepsilon_{klm} \partial_l v_m) = \\ &= \varepsilon_{kij} \varepsilon_{klm} \partial_{jl} v_m = \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_{jl} v_m = \\ &= \partial_{ij} v_j - \partial_{jj} v_i = \\ &= \nabla(\nabla \cdot \mathbf{v}) - \Delta \mathbf{v}, \end{aligned}$$

using the identity

$$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

9.1.1 Electromagnetic Potentials

Starting from the definitions of electromagnetic potentials and Maxwell's equations, with the help of some vector identities, it is possible (**TODO** list the necessary assumptions for the derivation) to write wave equations for the vector potential and the scalar potential.

$$\begin{aligned} \mathbf{e} &= -\nabla \varphi - \partial_t \mathbf{a} \\ \mathbf{b} &= \nabla \times \mathbf{a} \end{aligned}$$

Using the constitutive equations

$$\mathbf{d} = \varepsilon \mathbf{e} \quad , \quad \mathbf{b} = \mu \mathbf{h}$$

Vector Potential.

$$\begin{aligned}
 \mathbf{b} &= \nabla \times \mathbf{a} \\
 \mathbf{0} &= \nabla \times \nabla \times \mathbf{a} - \nabla \times \mathbf{b} = \\
 &= -\Delta \mathbf{a} + \nabla(\nabla \cdot \mathbf{a}) - \mu \nabla \times \mathbf{h} = \\
 &= -\Delta \mathbf{a} + \nabla(\nabla \cdot \mathbf{a}) - \mu(\partial_t \mathbf{d} + \mathbf{j}) = \\
 &= -\Delta \mathbf{a} + \nabla(\nabla \cdot \mathbf{a}) - \mu(\varepsilon \partial_t \mathbf{e} + \mathbf{j}) = \\
 &= -\Delta \mathbf{a} + \nabla(\nabla \cdot \mathbf{a}) - \mu\varepsilon(-\partial_t \nabla \varphi - \partial_{tt} \mathbf{a}) + \mu \mathbf{j} = \\
 &= -\Delta \mathbf{a} + \nabla(\nabla \cdot \mathbf{a}) + \frac{1}{c^2} \partial_t \nabla \varphi + \frac{1}{c^2} \partial_{tt} \mathbf{a} - \mu \mathbf{j}
 \end{aligned}$$

Using the Lorentz gauge condition

$$\nabla \cdot \mathbf{a} + \frac{1}{c^2} \partial_t \varphi = 0 ,$$

we obtain a wave equation for the vector potential

$$\frac{1}{c^2} \partial_{tt} \mathbf{a} - \Delta \mathbf{a} = \mu \mathbf{j} .$$

Scalar Potential.

$$\mathbf{e} = \nabla \varphi - \partial_t \mathbf{a}$$

Taking the time derivative of the Lorentz gauge condition

$$\begin{aligned}
 0 &= \partial_t \left(\frac{1}{c^2} \partial_t \varphi + \nabla \cdot \mathbf{a} \right) = \\
 &= \frac{1}{c^2} \partial_{tt} \varphi + \nabla \cdot \partial_t \mathbf{a} = \\
 &= \frac{1}{c^2} \partial_{tt} \varphi - \nabla \cdot \nabla \varphi - \nabla \cdot \mathbf{e} = \\
 &= \frac{1}{c^2} \partial_{tt} \varphi - \Delta \varphi - \frac{\rho}{\varepsilon} =
 \end{aligned}$$

we arrive at the wave equation for the scalar potential,

$$\frac{1}{c^2} \partial_{tt} \varphi - \Delta \varphi = \frac{\rho}{\varepsilon} .$$

9.1.2 Electric Field and Magnetic Field

Using the definitions of the physical fields in terms of the electromagnetic potentials and the linearity (**TODO everything must be linear, including the constitutive laws**) of the operations, starting from the wave equations for the potentials, we can derive the wave equations for the physical fields. **TODO Assuming constant and uniform properties**

Electric Field.

$$\begin{aligned}
 \square \mathbf{e} &= \square(-\nabla \varphi - \partial_t \mathbf{a}) = \\
 &= -\nabla \square \varphi - \partial_t \square \mathbf{a} = \\
 &= -\nabla \frac{\rho}{\varepsilon} - \mu \partial_t \mathbf{j} .
 \end{aligned}$$

Magnetic Field.

$$\begin{aligned}
 \square \mathbf{b} &= \square \nabla \times \mathbf{a} = \\
 &= \nabla \times \square \mathbf{a} = \\
 &= \mu \nabla \times \mathbf{j}
 \end{aligned}$$

9.2 Plane Electromagnetic Waves

Harmonic decomposition of the electromagnetic field. The EM field can be written as the superposition of plane waves (Fourier decomposition)

$$\mathbf{e}(\mathbf{r}, t) = \mathbf{E}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\mathbf{b}(\mathbf{r}, t) = \mathbf{B}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

Introducing this decomposition into Maxwell's equations with no free charge and current

$$\begin{cases} \nabla \cdot \mathbf{d} = 0 \\ \nabla \times \mathbf{e} + \partial_t \mathbf{b} = \mathbf{0} \\ \nabla \cdot \mathbf{b} = 0 \\ \nabla \times \mathbf{h} - \partial_t \mathbf{d} = \mathbf{0} \end{cases}$$

we obtain

$$\begin{cases} i\mathbf{k} \cdot \mathbf{D} = 0 \\ i\mathbf{k} \times \mathbf{E} - i\omega \mathbf{B} = \mathbf{0} \\ i\mathbf{k} \cdot \mathbf{B} = 0 \\ i\mathbf{k} \times \mathbf{H} + i\omega \mathbf{D} = \mathbf{0} \end{cases} \rightarrow \begin{cases} i\varepsilon \mathbf{k} \cdot \mathbf{E} = 0 \\ i\mathbf{k} \times \mathbf{E} - i\omega \mathbf{B} = \mathbf{0} \\ i\mathbf{k} \cdot \mathbf{B} = 0 \\ i\frac{1}{\mu} \mathbf{k} \times \mathbf{B} + i\omega \varepsilon \mathbf{E} = \mathbf{0} \end{cases}$$

- From Gauss' equations for the electric and the magnetic field

$$\mathbf{k} \perp \mathbf{E} \quad , \quad \mathbf{k} \perp \mathbf{B}$$

- From Faraday and Ampère-Maxwell equations

$$\mathbf{B} = \frac{\mathbf{k}}{\omega} \times \mathbf{E}$$

$$\mathbf{E} = -\frac{1}{\mu\varepsilon} \frac{\mathbf{k}}{\omega} \times \mathbf{B}$$

It follows that:

- \mathbf{k} , \mathbf{E} , \mathbf{B} are orthogonal «RHS» set of vectors
- Relations between \mathbf{E} , \mathbf{B} , and \mathbf{k} and the speed of light

$$\mathbf{B} = \frac{1}{c} \hat{\mathbf{k}} \times \mathbf{E}$$

$$\mathbf{E} = -c \hat{\mathbf{k}} \times \mathbf{B}$$

hold, with speed of light $c = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{\omega}{|\mathbf{k}|}$, and unit vector $\hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}$.

Proof using vector algebra identity

Recalling $c^2 = \frac{1}{\mu\epsilon}$ and

$$\mathbf{B} = \frac{\mathbf{k}}{\omega} \times \mathbf{E} = \frac{\mathbf{k}}{\omega} \times \left[-c^2 \frac{\mathbf{k}}{\omega} \times \mathbf{B} \right] = -\frac{c^2 |\mathbf{k}|^2}{\omega^2} \hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{B})$$

Vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \epsilon_{ijk} a_j \epsilon_{klm} b_l c_m = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_j b_l c_m = b_i a_m c_m - c_i a_m b_m = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

applied to $\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{B})$ gives

$$\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{B}) = \underbrace{(\hat{\mathbf{k}} \mathbf{B})}_{=0 \text{ since } \mathbf{k} \perp \mathbf{B}} \hat{\mathbf{k}} - \underbrace{(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}})}_{=1} \mathbf{B} = -\mathbf{B},$$

and the original relation gives

$$\mathbf{B} = \mathbf{B} \frac{c^2 |\mathbf{k}|^2}{\omega^2},$$

and the relation between pulsation ω , wave vector \mathbf{k} and speed of light (EM radiation) c ,

$$c = \frac{\omega}{|\mathbf{k}|}.$$

9.2.1 Snell's Law at an Interface

Snell's law is derived here assuming isotropic linear media, so that

$$\begin{cases} \mathbf{d}(\mathbf{r}, t) = \epsilon \mathbf{e}(\mathbf{r}, t) \\ \mathbf{b}(\mathbf{r}, t) = \mu \mathbf{h}(\mathbf{r}, t) \end{cases}$$

and for harmonic plane EM waves

$$\begin{cases} \mathbf{e}(\mathbf{r}, t) = \mathbf{E}_a e^{i(\mathbf{k}_a \cdot \mathbf{r} - \omega t)} \\ \mathbf{b}(\mathbf{r}, t) = \mathbf{B}_a e^{i(\mathbf{k}_a \cdot \mathbf{r} - \omega t)} \end{cases}$$

$$\mathbf{B}_a = \frac{1}{c} \hat{\mathbf{k}}_a \times \mathbf{E}_a$$

$$\mathbf{E}_a = -c \hat{\mathbf{k}}_a \times \mathbf{B}_a$$

being index a representing the media involved: $a = 1$ for the medium with incident and reflected waves, $a = 2$ for the medium with the refracted wave.

Jump conditions of electromagnetic field at an interface with no charge or current surface density are given by conditions (4.2),

$$\begin{cases} \epsilon_1 e_{n,1} = \epsilon_2 e_{n,2} \\ e_{t_\alpha,1} = e_{t_\alpha,2} \\ b_{n,1} = b_{n,2} \\ \frac{1}{\mu_1} b_{t_\alpha,1} = \frac{1}{\mu_2} b_{t_\alpha,2} \end{cases}, \quad \alpha = 1 : 2$$

Definition of some vectors: $\hat{\mathbf{n}}$ unit normal vector, \mathbf{k} wave vector, $\hat{\mathbf{b}} = \frac{\hat{\mathbf{n}} \times \mathbf{k}}{|\hat{\mathbf{n}} \times \mathbf{k}|}$ (singular only for normal incident ray),

$$\hat{\mathbf{c}} = \frac{\hat{\mathbf{b}} \times \mathbf{k}}{|\hat{\mathbf{b}} \times \mathbf{k}|}, \quad \hat{\mathbf{t}} = \frac{\hat{\mathbf{b}} \times \hat{\mathbf{n}}}{|\hat{\mathbf{b}} \times \hat{\mathbf{n}}|}$$

Incident angle $\theta_{1,i}$ is the angle between $\hat{\mathbf{n}}$ and \mathbf{k} , s.t. $\hat{\mathbf{n}} \times \mathbf{k} = \hat{\mathbf{b}} k \sin \theta_{1,i}$.

$$\begin{cases} \hat{\mathbf{k}} = \cos \theta_{1,i} \hat{\mathbf{n}} + \sin \theta_{1,i} \hat{\mathbf{t}} \\ \hat{\mathbf{c}} = -\sin \theta_{1,i} \hat{\mathbf{n}} + \cos \theta_{1,i} \hat{\mathbf{t}} \end{cases}, \quad \begin{cases} \hat{\mathbf{n}} = \cos \theta_{1,i} \hat{\mathbf{k}} - \sin \theta_{1,i} \hat{\mathbf{c}} \\ \hat{\mathbf{t}} = \sin \theta_{1,i} \hat{\mathbf{k}} + \cos \theta_{1,i} \hat{\mathbf{c}} \end{cases}$$

The electromagnetic field can be written as

$$\begin{aligned} \mathbf{E} &= E_b \hat{\mathbf{b}} + E_c \hat{\mathbf{c}} = \\ &= E_b \hat{\mathbf{b}} - E_c \sin \theta_{1,i} \hat{\mathbf{n}} + E_c \cos \theta_{1,i} \hat{\mathbf{t}} \\ \mathbf{B} &= B_b \hat{\mathbf{b}} + B_c \hat{\mathbf{c}} = \\ &= \frac{E_c}{c} \hat{\mathbf{b}} - \frac{E_b}{c} \hat{\mathbf{c}} = \\ &= \frac{E_c}{c} \hat{\mathbf{b}} + \frac{E_b}{c} \sin \theta_{1,i} \hat{\mathbf{n}} - \frac{E_b}{c} \cos \theta_{1,i} \hat{\mathbf{t}}. \end{aligned}$$

so that jump relations become

$$\begin{cases} b : & E_{b,1} = E_{b,2} \\ n : & \dots \\ t : & \dots \end{cases}, \quad \begin{cases} b : & \dots \\ n : & \frac{E_{b,1}}{c_1} \sin \theta_{1,i} = \frac{E_{b,2}}{c_2} \sin \theta_{2,i} \\ t : & \dots \end{cases}$$

thus **Snell's law** follows

$$\frac{\sin \theta_{1,i}}{\sin \theta_{2,t}} = \frac{c_2}{c_1} = \frac{n_1}{n_2}.$$

Incident, Reflected, and Refracted Wave. The wave at the interface in medium 1 has the contribution of the incoming incident wave and the reflected one.

$$\begin{aligned} \mathbf{e}_1(\mathbf{r}, t) &= \mathbf{e}_i(\mathbf{r}, t) + \mathbf{e}_r(\mathbf{r}, t) = \\ &= \mathbf{E}_i e^{i(\mathbf{k}_i \cdot \mathbf{r} - \omega t)} + \mathbf{E}_r e^{i(\mathbf{k}_r \cdot \mathbf{r} - \omega t)} = \\ &= (\mathbf{E}_i e^{i\mathbf{k}_i \cdot \mathbf{r}} + \mathbf{E}_r e^{i\mathbf{k}_r \cdot \mathbf{r}}) e^{-i\omega t} \end{aligned}$$

with

$$\begin{aligned} \mathbf{k}_i &= k_{i,n} \hat{\mathbf{n}} + k_{i,t} \hat{\mathbf{t}} \\ \mathbf{k}_r &= k_{r,n} \hat{\mathbf{n}} + k_{r,t} \hat{\mathbf{t}} \end{aligned}$$

At the interface, $\mathbf{r}_s \cdot \hat{\mathbf{n}} = 0$, and thus

$$\begin{aligned} \mathbf{e}_1(\mathbf{r}_s, t) &= (\mathbf{E}_i e^{ik_{i,t} x_t} + \mathbf{E}_r e^{ik_{r,t} x_t}) e^{-i\omega t} \\ \mathbf{e}_2(\mathbf{r}_s, t) &= \mathbf{E}_t e^{ik_{t,t} x_t} e^{-i\omega t} \end{aligned}$$

In order for the boundary conditions to be satisfied at all the points of the interface at each time,

$$k_{i,t} = k_{r,t} = k_{t,t}.$$

Exploiting the relation between the pulsation, the wave-length, and the speed of light in media, $c_a = \frac{\omega}{|\mathbf{k}_a|} = \frac{c}{n_a}$,

$$|\mathbf{k}_i| = |\mathbf{k}_r| \quad \rightarrow \quad k_{r,n} = -k_{i,n}$$

$$\begin{aligned} \frac{|\mathbf{k}_2|}{|\mathbf{k}_1|} &= \frac{c_1}{c_2} \\ \frac{k_{t,t}^2 + k_{i,n}^2}{k_{i,t}^2 + k_{i,n}^2} &= \frac{c_1^2}{c_2^2} \end{aligned}$$

$$\begin{aligned}
 k_{i,n} &= |\mathbf{k}_i| \cos \theta_i & k_{i,t} &= |\mathbf{k}_i| \sin \theta_i \\
 k_{r,n} &= -|\mathbf{k}_r| \cos \theta_r & k_{r,t} &= |\mathbf{k}_r| \sin \theta_r \\
 k_{t,n} &= |\mathbf{k}_t| \cos \theta_t & k_{t,t} &= |\mathbf{k}_t| \sin \theta_t
 \end{aligned}$$

$$\begin{cases}
 E_n : & \varepsilon_1 (E_{i,c} \sin \theta_i + E_{r,c} \sin \theta_r) = \varepsilon_2 E_{t,c} \sin \theta_t \\
 E_t : & E_{i,c} \cos \theta_i - E_{r,c} \cos \theta_r = E_{t,c} \cos \theta_t \\
 E_b : & E_{i,b} + E_{r,b} = E_{t,b} \\
 B_n : & B_{i,c} \sin \theta_i + B_{r,c} \sin \theta_r = B_{t,c} \sin \theta_t \\
 B_t : & \frac{1}{\mu_1} (B_{i,c} \cos \theta_i - B_{r,c} \cos \theta_r) = \frac{1}{\mu_2} B_{t,c} \cos \theta_t \\
 B_b : & \frac{1}{\mu_1} (B_{i,b} + B_{r,b}) = \frac{1}{\mu_2} B_{t,b}
 \end{cases}$$

Writing the magnetic field as a function of the wave-vector and the magnetic field, it's possible to write 2 decoupled systems of equations

$$\begin{cases}
 E_n : & \varepsilon_1 (E_{i,c} \sin \theta_i + E_{r,c} \sin \theta_r) = \varepsilon_2 E_{t,c} \sin \theta_t \\
 E_t : & E_{i,c} \cos \theta_i - E_{r,c} \cos \theta_r = E_{t,c} \cos \theta_t \\
 B_b : & \frac{1}{\mu_1} \left(\frac{E_{i,c}}{c_1} + \frac{E_{r,c}}{c_1} \right) = \frac{1}{\mu_2} \frac{E_{t,c}}{c_2}
 \end{cases}$$

$$\begin{cases}
 E_b : & E_{i,b} + E_{r,b} = E_{t,b} \\
 B_n : & \frac{E_{i,b}}{c_1} \sin \theta_i + \frac{E_{r,b}}{c_1} \sin \theta_r = \frac{E_{t,b}}{c_2} \sin \theta_t \\
 B_t : & \frac{1}{\mu_1} \left(\frac{E_{i,b}}{c_1} \cos \theta_i - \frac{E_{r,b}}{c_1} \cos \theta_r \right) = \frac{1}{\mu_2} \frac{E_{t,b}}{c_2} \cos \theta_t
 \end{cases}$$

The equations E_n and B_b are equivalent; E_b and B_n are equivalent as well, because of Snell's law. Thus, defining

$$\begin{aligned}
 r_c &:= \frac{E_{r,c}}{E_{i,c}} & r_b &:= \frac{E_{r,b}}{E_{i,b}} \\
 t_c &:= \frac{E_{t,c}}{E_{i,c}} & t_b &:= \frac{E_{t,b}}{E_{i,b}}
 \end{aligned}$$

and $\alpha_i := \frac{1}{\mu_i c_i}$. These systems of equations can be written as two uncoupled linear systems of equations,

(for P-polarization **todo** *change index from c to p ; for S-polarization **todo** change index from b to s)

$$\begin{cases}
 E_t : & \cos \theta_i - \cos \theta_r r_c = \cos \theta_t t_c \\
 B_b : & \alpha_1 + \alpha_1 r_c = \alpha_2 t_c \\
 E_b : & 1 + r_b = t_b \\
 B_t : & \alpha_1 \cos \theta_i - \alpha_1 \cos \theta_r r_b = \alpha_2 \cos \theta_t t_b
 \end{cases}$$

Calling $\theta_i = \theta_r = \theta_1$, $\theta_t = \theta_2$, these linear systems can be written using matrix formalism,

$$\begin{bmatrix} -1 & 1 \\ 1 & \frac{\alpha_2 \cos \theta_2}{\alpha_1 \cos \theta_1} \end{bmatrix} \begin{bmatrix} r_b \\ t_b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{\cos \theta_2}{\cos \theta_1} \\ -1 & \frac{\alpha_2}{\alpha_1} \end{bmatrix} \begin{bmatrix} r_c \\ t_c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

todo Analysis of the total reflection, forcing $t_x = 0$. Check signs before

$$\begin{bmatrix} 1 & \frac{\cos \theta_2}{\cos \theta_1} \\ -1 & \frac{\alpha_2}{\alpha_1} \end{bmatrix} \begin{bmatrix} r_c \\ t_c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} r_c \\ t_c \end{bmatrix} = \frac{1}{\frac{\alpha_2}{\alpha_1} + \frac{\cos \theta_2}{\cos \theta_1}} \begin{bmatrix} \frac{\alpha_2}{\alpha_1} & -\frac{\cos \theta_2}{\cos \theta_1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\alpha_2 \cos \theta_1 - \alpha_1 \cos \theta_2}{\alpha_2 \cos \theta_1 + \alpha_1 \cos \theta_2} \\ \frac{2\alpha_1 \cos \theta_1}{\alpha_2 \cos \theta_1 + \alpha_1 \cos \theta_2} \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & \frac{\alpha_2 \cos \theta_2}{\alpha_1 \cos \theta_1} \end{bmatrix} \begin{bmatrix} r_b \\ t_b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} r_b \\ t_b \end{bmatrix} = \frac{1}{-\frac{\alpha_2 \cos \theta_2}{\alpha_1 \cos \theta_1} - 1} \begin{bmatrix} \frac{\alpha_2 \cos \theta_2}{\alpha_1 \cos \theta_1} & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\alpha_1 \cos \theta_1 - \alpha_2 \cos \theta_2}{\alpha_1 \cos \theta_1 + \alpha_2 \cos \theta_2} \\ \frac{2\alpha_1 \cos \theta_1}{\alpha_1 \cos \theta_1 + \alpha_2 \cos \theta_2} \end{bmatrix}$$

that can be recast with the wave impedance Z ,

$$\alpha_1 = \frac{1}{\mu_1 c_1} = \frac{\sqrt{\mu_1 \varepsilon_1}}{\mu_1} = \sqrt{\frac{\varepsilon_1}{\mu_1}} =: \frac{1}{Z_1} ,$$

$$\begin{bmatrix} r_c \\ t_c \end{bmatrix} = \begin{bmatrix} \frac{Z_1 \cos \theta_1 - Z_2 \cos \theta_2}{Z_1 \cos \theta_1 + Z_2 \cos \theta_2} \\ \frac{2Z_2 \cos \theta_1}{Z_1 \cos \theta_1 + Z_2 \cos \theta_2} \end{bmatrix}$$

$$\begin{bmatrix} r_b \\ t_b \end{bmatrix} = \begin{bmatrix} \frac{Z_2 \cos \theta_1 - Z_1 \cos \theta_2}{Z_2 \cos \theta_1 + Z_1 \cos \theta_2} \\ \frac{2Z_2 \cos \theta_1}{Z_2 \cos \theta_1 + Z_1 \cos \theta_2} \end{bmatrix}$$

Energy Balance and Transmission Coefficients. Energy balance for a domain collapsing on the interface reduces to power flux balance, namely

$$\oint_{\partial V} \mathbf{s} \cdot \hat{\mathbf{n}} = 0 ,$$

with $\mathbf{s} = \mathbf{e} \times \mathbf{h}$ the Poynting vector. For harmonic plane waves,

$$\begin{aligned} \mathbf{s}(\mathbf{r}, t) &= \mathbf{e}(\mathbf{r}, t) \times \mathbf{h}(\mathbf{r}, t) = \\ &= \frac{1}{\mu} [\mathbf{E} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \mathbf{E}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)}] \times [\mathbf{B} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \mathbf{B}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)}] = \\ &= \frac{1}{\mu} [\mathbf{E} \times \mathbf{B} e^{i2(\mathbf{k} \cdot \mathbf{r} - \omega t)} + c.c.] + \frac{1}{\mu} [\mathbf{E} \times \mathbf{B}^* + c.c.] = \\ &= \dots + \frac{1}{\mu} \mathbf{E} \times \left(\frac{1}{c} \hat{\mathbf{k}} \times \mathbf{E} \right)^* = \\ &= \dots + \frac{1}{\mu c} (\mathbf{E} \cdot \mathbf{E}^*) \hat{\mathbf{k}} = \\ &= \dots + \frac{1}{\mu c} |\mathbf{E}|^2 \hat{\mathbf{k}} . \end{aligned} \quad = \dots + \alpha |\mathbf{E}|^2 \hat{\mathbf{k}} .$$

For each one of the two polarizations, the following holds ($\cos \theta$ comes from the dot product $\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}$ appearing in the surface integral),

$$\alpha_1 \cos \theta_1 = \alpha_1 r_x^2 \cos \theta_1 + \alpha_2 t_x^2 \cos \theta_2 ,$$

i.e., the sum of reflected and transmitted power equals the incident power.

Proof of the power balance, for P-polarization

todo Here P is index c

Dividing by $\alpha_1 \cos \theta_1$

$$\begin{aligned} &\frac{1}{\alpha_1 \cos \theta_1} (\alpha_1 r_p^2 \cos \theta_1 + \alpha_2 t_p^2 \cos \theta_2) = \\ &= \frac{(\alpha_1 \cos \theta_1 - \alpha_2 \cos \theta_2)^2}{(\alpha_1 \cos \theta_1 + \alpha_2 \cos \theta_2)^2} + \frac{\alpha_2 \cos \theta_2}{\alpha_1 \cos \theta_1} \frac{(2\alpha_1 \cos \theta_1)^2}{(\alpha_1 \cos \theta_1 + \alpha_2 \cos \theta_2)^2} = \\ &= \frac{1}{(\alpha_1 \cos \theta_1 + \alpha_2 \cos \theta_2)^2} [\alpha_1^2 \cos^2 \theta_1 - 2\alpha_1 \alpha_2 \cos \theta_1 \cos \theta_2 + \alpha_2^2 \cos^2 \theta_2 + 4\alpha_1 \alpha_2 \cos \theta_1 \cos \theta_2] = \\ &= 1 . \end{aligned}$$

Parte II

Electric Engineering

Approssimazione circuitale

Circuiti elettrici. Condizioni per la validità dell'approssimazione circuitale; componenti elementari; regimi di utilizzo: stazionario, armonico (alternato), transitorio;

Circuiti elettromagnetici. Condizioni per la validità dell'approssimazione circuitale; es. trasformatori

Circuito elettro-magneto-meccanici. Es. semplici circuiti; motori elettrici e generatori

10.1 Circuiti elettrici

Se il sistema di interesse soddisfa alcune condizioni, è possibile ridurre la teoria di campo dell'elettromagnetismo a una teoria circuitale. Quando possibile, cioè quando capace di descrivere adeguatamente il comportamento del sistema di interesse, l'approccio circuitale semplifica di molto la descrizione del problema, non richiedendo la soluzione di un sistema di equazioni differenziali alle derivate parziali da risolvere nello spazio, ma la soluzione di equazioni differenziali ordinarie nelle incognite circuitali, che si riduce a un sistema algebrico, spesso lineare, in regime stazionario.

Giustificazione dell'approccio circuitale.

Componenti elementari di un circuito elettrico.

10.1.1 Validità dell'approccio circuitale

L'approccio circuitale consente di ridurre il problema elettromagnetico, in generale un problema di campo che richiede la soluzione di PDE, a un approccio «ai morsetti» **todo**, che richiede la soluzione di ODE.

Una rivisitazione dell'*equazione dell'energia* permette di valutare i regimi in cui è possibile usare un approccio circuitale a un sistema elettromagnetico.

In particolare, nell'equazione di bilancio dell'energia elettromagnetica

$$\frac{d}{dt} \int_V u = \oint_{\partial V} \mathbf{s} \cdot \hat{\mathbf{n}} - \int_V \mathbf{j} \cdot \mathbf{e} ,$$

viene indagato il termine di flusso alla frontiera, ricordando la definizione di vettore di Poynting $\mathbf{s} := \mathbf{e} \times \mathbf{h}$, e riscrivendo i campi elettrico e magnetico in funzione dei potenziali elettromagnetici, $\mathbf{b} = \nabla \times \mathbf{a}$, $\mathbf{e} = -\nabla\varphi - \partial_t \mathbf{a}$,

$$\begin{aligned}
 - \oint_{\partial V} \mathbf{s} \cdot \hat{\mathbf{n}} &= - \oint_{\partial V} (\mathbf{e} \times \mathbf{h}) \cdot \hat{\mathbf{n}} = \\
 &= \oint_{\partial V} (\nabla\varphi + \partial_t \mathbf{a}) \times \mathbf{h} \cdot \hat{\mathbf{n}} = \\
 &= \dots \\
 &= \underbrace{\oint_{\partial V} \hat{\mathbf{n}} \cdot \nabla \times (\varphi \mathbf{h})}_{=0 \text{ (Stokes' thm **todo** check)}} - \oint_{\partial V} \varphi \hat{\mathbf{n}} \cdot \frac{\nabla \times \mathbf{h}}{\partial_t \mathbf{d} + \mathbf{j}} + \oint_{\partial V} \hat{\mathbf{n}} \cdot \partial_t \mathbf{a} \times \mathbf{h} = \\
 &= - \oint_{\partial V} \varphi \mathbf{j} \cdot \hat{\mathbf{n}} - \oint_{\partial V} \hat{\mathbf{n}} \cdot (\partial_t \mathbf{d} + \mathbf{h} \times \partial_t \mathbf{a}) ,
 \end{aligned}$$

e assumendo che il flusso di carica elettrica avvenga solo in corrispondenza di un numero finito di sezioni $S_k \in \partial V$ equipotenziali a potenziale $v_k = -\varphi_k$, costante sulle sezioni, e riconoscendo il flusso di carica elettrica attraverso la sezione S_k come la corrente $i_k = \int_{S_k} \mathbf{j} \cdot \hat{\mathbf{n}}$, si può scrivere

$$- \oint_{\partial V} \mathbf{s} \cdot \hat{\mathbf{n}} = \sum_k v_k i_k - \oint_{\partial V} \hat{\mathbf{n}} \cdot (\partial_t \mathbf{d} + \mathbf{h} \times \partial_t \mathbf{a}) .$$

Il bilancio di energia elettromagnetica del sistema può quindi essere riscritto come

$$\frac{d}{dt} \int_V u = \sum_k v_k i_k - \int_V \mathbf{j} \cdot \mathbf{e} - \oint_{\partial V} \hat{\mathbf{n}} \cdot (\partial_t \mathbf{d} + \mathbf{h} \times \partial_t \mathbf{a}) .$$

Nelle condizioni in cui l'ultimo termine è nullo o trascurabile (**todo** quali? *Spendere due parole sulla validità dell'approssimazione, con analisi dimensionale? Fare esempio in cui l'approssimazione non funziona*), la variazione di energia interna al sistema è dovuta alla differenza della potenza in ingresso ai morsetti, e la dissipazione all'interno del volume (ad esempio dovuta alla conduzione non ideale in conduttori con resistività finita),

$$\dot{E}^{em} = P^{ext,vi} - \dot{D} ,$$

con $\dot{D} \geq 0$ per il secondo principio della termodinamica **todo** *aggiungere riferimento, e discussione*.

10.1.2 Induzione elettromagnetica nell'approssimazione circuitale

E' possibile applicare l'approssimazione circuitale anche in presenza di regioni in cui non è possibile trascurare il termine $\partial_t \mathbf{b}$, come ad esempio circuiti elettromagnetici che coinvolgono trasformatori e/o motori o generatori elettrici.

In queste situazioni, se è possibile identificare una regione V_0 dello spazio connessa nella quale il termine $\partial_t \mathbf{b} = \mathbf{0}$, e quindi $\nabla \times \mathbf{e} = \mathbf{0}$, in V_0 è possibile definire il campo elettrico in termini di un potenziale φ ,

$$\mathbf{e} = -\nabla\varphi , \quad \mathbf{r} \in V_0 .$$

E' possibile calcolare le differenze di potenziale ai morsetti di un sistema in cui $\delta_t \mathbf{b} \neq 0$, racchiuso nel volume V_k , con la legge di Faraday,

$$\oint_{\ell_k} \mathbf{e} \cdot \hat{\mathbf{t}} = - \frac{d}{dt} \int_{S_k} \mathbf{b} \cdot \hat{\mathbf{n}} ,$$

dove il percorso chiuso $\ell_k = \ell_k^{cond} \cup \ell_k^{mors}$ descrive il conduttore in V_k chiuso dalla linea geometrica tra i morsetti. Se si può trascurare la resistività del conduttore in V_k , $\int_{\ell_k^{cond}} \mathbf{e} \cdot \hat{\mathbf{t}} = 0$, la differenza di tensione ai morsetti vale

$$\Delta v_k = \int_{\ell_k^{mors}} \mathbf{e} \cdot \hat{\mathbf{t}} = - \frac{d}{dt} \int_{S_k} \mathbf{b} \cdot \hat{\mathbf{n}}$$

10.1.3 Componenti elementari dei circuiti elettrici

Resistore ohmico

Un resistore di Ohm risulta dall'approssimazione circuitale di un materiale con equazione costitutiva lineare

$$\mathbf{e} = \rho_R \mathbf{j} ,$$

tra il campo elettrico \mathbf{e} e la densità di corrente \mathbf{j} , tramite la costante di proporzionalità ρ_R , la **resistività** del materiale. La corrente elettrica attraverso una sezione del componente è definita come il flusso di carica attraverso una sua sezione

$$i = \int_S \mathbf{j} \cdot \hat{\mathbf{t}} \simeq j A ,$$

Nell'ipotesi che il vettore densità di corrente si allineato con l'asse del componente e uniforme sulla sezione A , «piccola». Se il materiale non è in grado di accumulare carica, il bilancio di carica elettrica si traduce nella continuità della corrente elettrica attraverso le sezioni del conduttore.

Utilizzando l'equazione costitutiva su un elemento di lunghezza elementare $d\mathbf{r} = \hat{\mathbf{t}} d\ell$, e assumendo che il campo elettrico sia allineato con l'asse del componente, $\mathbf{e} = e\hat{\mathbf{t}}$ si può scrivere il lavoro elementare per unità di carica come

$$\delta v = \mathbf{e} \cdot d\mathbf{r} = e d\ell = \rho_R j d\ell = \frac{\rho_R d\ell}{A} i .$$

Da questa ultima equazione seguono le due leggi di Ohm, per resistori lineari.

Prima legge di Ohm. La differenza di potenziale tra due sezioni di un resistore lineare è proporzionale alla corrente che passa attraverso di esso,

$$\delta v = dR i .$$

Seconda legge di Ohm. La costante di proporzionalità che lega la differenza di potenziale e la corrente all'interno di un resistore ohmico, la **resistenza** del resistore, è proporzionale alla resistività e alla lunghezza del resistore, e inversamente proporzionale alla sua sezione,

$$dR = \frac{\rho_R d\ell}{A} .$$

Se le proprietà sono uniformi nel resistore, si possono integrare le relazioni elementari per ottenere la relazione tra grandezze finite,

$$\Delta V = R i$$

$$R = \frac{\rho_R \ell}{A}$$

todo (perché si può usare il potenziale? Nelle mie note avevo usato il simbolo v^* , come se fosse una definizione leggermente diversa per incorporare movimento e instazionarietà, che si riduce a v nel caso stazionario).

Condensatore.

Induttore.

Generatore di tensione.

Generatore di corrente.

10.1.4 Regimi di funzionamento in circuiti elettrici

10.2 Circuiti elettromagnetici

Sotto opportune ipotesi è possibile usare un modello circuitale anche per sistemi elettromagnetici, come ad esempio i trasformatori, o i motori elettrici.

- legge di Gauss per il campo magnetico

$$\nabla \cdot \mathbf{b} = 0$$

- legge di Ampère-Maxwell

$$\nabla \times \mathbf{h} - \partial_t \mathbf{d} = \mathbf{j}$$

Si aggiungono le seguenti ipotesi:

- materiali lineari non-dissipativi e non-dispersivi $\mathbf{b} = \mu \mathbf{h}$ **todo** discutere questa ipotesi, insieme a isteresi materiali, cicli di magnetizzazione,....
- variazioni del campo \mathbf{d} nel tempo trascurabili, $\partial_t \mathbf{d} = \mathbf{0}$.

La legge di Gauss del campo magnetico in forma integrale permette di scrivere la **legge ai nodi** del flusso del campo magnetico per i circuiti magnetici,

$$0 = \oint_{\partial V} \mathbf{b} \cdot \hat{\mathbf{n}} = \sum_k \phi_k.$$

La legge di Ampère-Maxwell in forma integrale considerando:

- un percorso incatenato con il solo induttore

$$\int_{\ell_{ind}} \mathbf{h} \cdot \hat{\mathbf{t}} + \int_{\ell_{12}} \mathbf{h} \cdot \hat{\mathbf{t}} = \oint_{\ell_1} \mathbf{h} \cdot \hat{\mathbf{t}} = \int_{S^{ind}} \mathbf{j} \cdot \hat{\mathbf{n}} = Ni =: m$$

- un percorso incatenato con il traferro, aggirando l'induttore

$$0 = \int_{\ell_{traf}} \mathbf{h} \cdot \hat{\mathbf{t}} + \int_{\ell_{21}} \hat{h} \cdot \hat{\mathbf{t}} = \sum_k h_k \ell_k + \int_{\ell_{21}} \hat{h} \cdot \hat{\mathbf{t}}$$

e sommando le due equazioni, riconoscendo che i due integrali di linea sullo stesso percorso in versi opposti si annullano, si ottiene la **legge alle maglie** per i circuiti magnetici

$$\begin{aligned} m &= \int_{\ell_{ind}} \mathbf{h} \cdot \hat{\mathbf{t}} + \int_{\ell_{traf}} \mathbf{h} \cdot \hat{\mathbf{t}} = \\ &\approx \sum_{k \in \ell} h_k \ell_k = \sum_{k \in \ell} \frac{b_k}{\mu_k} \ell_k = \sum_{k \in \ell} \frac{\ell_k}{\mu_k A_k} \phi_k. \end{aligned}$$

Le leggi di Kirchhoff per i circuiti magnetici sono quindi

$$\begin{cases} \sum_{k \in N_j} \phi_k = 0 \\ m_{\ell_i} = \sum_{k \in \ell_i} \theta_k \phi_k, \end{cases}$$

avendo introdotto la riluttanza $\theta_k = \frac{\ell_k}{\mu_k A_k}$, l'inverso della permeanza $\Lambda_k = \theta_k^{-1}$.

10.2.1 Trasformatore

- flusso del campo magnetico, nell'ipotesi di campo uniforme, o in termini del campo medio

$$\phi = b A$$

- flusso del campo magnetico concatenato a N avvolgimenti

$$\psi = N \phi$$

- relazione tra tensione ai morsetti dell'induttore e flusso concatenato, applicando la *legge di Faraday solo in parte irrotazionali*

$$v = \dot{\psi}$$

Trasformatore ideale

In assenza di flussi dispersi e riluttanza nel traferro, la legge alle maglie nel traferro implica

$$0 = m_1 + m_2 = N_1 i_1 + N_2 i_2$$

Il flusso del campo magnetico può essere scritto in funzione del flusso concatenato agli avvolgimenti,

$$\phi = \frac{\psi_1}{N_1} = \frac{\psi_2}{N_2}$$

La derivata nel tempo di questa relazione, con numero di avvolgimenti costanti nel tempo, implica

$$\frac{v_2}{N_2} = \frac{v_1}{N_1}.$$

Trasformatore con flussi dispersi

$$\left\{ \begin{array}{l} \phi_1 - \phi_{1,d} = \phi \\ \phi_2 - \phi_{2,d} = \phi \\ m_1 = \theta_{1,d} \phi_{1,d} \\ m_2 = \theta_{2,d} \phi_{2,d} \\ m_1 + m_2 = 0 \end{array} \right.$$

$$\rightarrow 0 = m_1 + m_2 = N_1 i_1 + N_2 i_2$$

$$\begin{aligned} 0 &= \phi_2 - \phi_1 - \phi_{2,d} + \phi_{1,d} \\ &= \phi_2 - \phi_1 - \frac{m_2}{\theta_{2,d}} + \frac{m_1}{\theta_{1,d}} \end{aligned}$$

$$\rightarrow \frac{\psi_2}{N_2} - \frac{m_2}{\theta_{2,d}} = \frac{\psi_1}{N_1} - \frac{m_1}{\theta_{1,d}}.$$

$$\rightarrow \frac{1}{N_2} \left(v_2 - \frac{N_2^2}{\theta_{2,d}} \frac{di_2}{dt} \right) = \frac{1}{N_1} \left(v_1 - \frac{N_1^2}{\theta_{1,d}} \frac{di_1}{dt} \right).$$

Trasformatore con flussi dispersi e riluttanza θ_{Fe} nel traferro

$$\begin{cases} \phi_1 - \phi_{1,d} = \phi \\ \phi_2 - \phi_{2,d} = \phi \\ m_1 = \theta_{1,d} \phi_{1,d} \\ m_2 = \theta_{2,d} \phi_{2,d} \\ m_1 + m_2 = \theta_{Fe} \phi \end{cases}$$

todo finire e controllare i conti; disegnare circuito equivalente

10.3 Circuiti elettromeccanici

Alcuni sistemi di interesse e di enorme diffusione nella società moderna sfruttano le interazioni tra componenti fenomeni elettromagnetici e meccanici: un esempio fondamentale sono le macchine elettriche, alcune delle quali possono operare sia come motore (con la potenza fornita dal sistema elettrico e convertita in potenza meccanica) sia come generatore di energia elettrica (convertendo potenza meccanica in potenza elettrica).

In un sistema di induttori con mutua influenza, la differenza di tensione ai capi dell'induttore «potenziato» i è

$$v_i = \dot{\psi}_i = \frac{d}{dt} (N_i \phi_i) .$$

Il flusso concatenato dipende dall'effetto di tutti gli induttori del sistema (e del campo magnetico generato da eventuali cause esterne al sistema),

$$\phi_i = \sum_k \phi_{ik} = \sum_k \frac{1}{\theta_{ik}} m_k ,$$

avendo indicato con θ_{ik} la riluttanza del circuito tra l'induttore potenziante k e l'induttore potenziato i . Usando l'espressione della forza magneto-motrice $m_k = N_k i_k$, si può riscrivere l'espressione della differenza di tensione

$$v_i = \sum_k \frac{d}{dt} \left(\frac{N_i N_k}{\theta_{ik}} i_k \right) = \sum_k \frac{d}{dt} (L_{ik} i_k) .$$

In generale, in circuiti elettromeccanici le riluttanze non sono dei parametri costanti del sistema ma dipendono dallo stato «meccanico» del sistema, descritto qui dalle variabili \mathbf{x} ,

$$v_i = \sum_k \frac{d}{dt} \left(\frac{N_i N_k}{\theta_{ik}(\mathbf{x})} i_k \right) = \sum_k \frac{d}{dt} (L_{ik}(\mathbf{x}) i_k) .$$

$$\mathbf{v}(t) = \frac{d}{dt} \left(\mathbf{L}(\mathbf{x}(t)) \mathbf{i}(t) \right) .$$

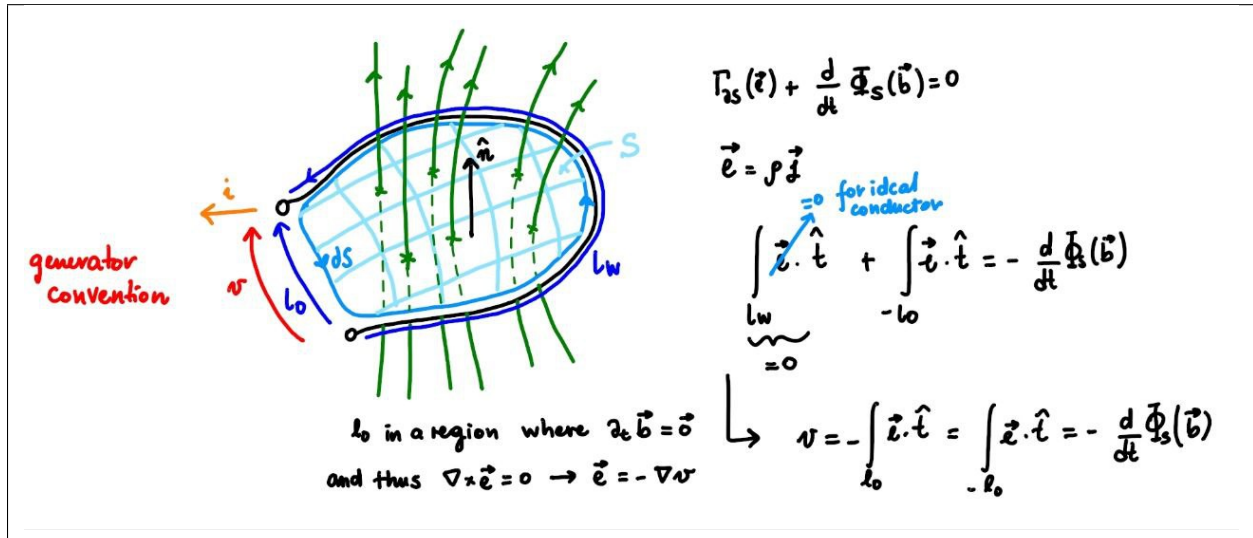
La matrice di induttanza \mathbf{L} è simmetrica **todo** Dimostrazione

Example 10.3.1

Given an constant and uniform magnetic field $\mathbf{b}(r) = \mathbf{B}$ in a region of space where a simple electric circuit is placed. The electric circuit consists in a simple circuit with a resistance R as a lumped load, and has a rectangular shape. Three sides are fixed, and the distance between the pair of parallel fixed sides is ℓ ; the fourth side can move and its distance between the parallel fixed side is x . The unit vector orthogonal to the rectangular surface enclosed in the circuit is $\hat{\mathbf{n}}$.

A mechanical system provides the prescribed motion $x(t) = x_0 + \Delta x \sin(\Omega t)$ to the moving side. It's asked to evaluate and discuss:

- voltage at the electric port of the load
- energy balance



Without considering the inductance of the simple circuit. Faraday's law

$$\Gamma_{\partial S_t}(\vec{e}) + \dot{\Phi}_{S_t}(\vec{b}) = 0 ,$$

provides the relation between the time derivative of the magnetic flux through two points of the electric circuit on opposite sides of the moving side of the circuit, corresponding to the voltage at the electric port of the load

$$v = - \int_{\ell_0}^{\ell_1} \vec{e} \cdot \hat{t} = - \dot{\Phi}_{S_t}(\vec{b}) = - \frac{d}{dt} (NBA) = -B\ell \dot{x} ,$$

being $N = 1$, and B constant and uniform if self-inductance is not considered. If the inductance of the circuit is neglected, from the constitutive equation of the resistance, $v = Ri$, and voltage Kirchhoff law, it follows that the current in the simple circuit is

$$i = \frac{v}{R} = - \dot{\Phi}_{S_t}(\vec{b}) = - \frac{B_n \dot{A}}{R} = - \frac{B_n \ell \dot{x}}{R} = - \frac{B_n \ell \Delta x}{R} \Omega \cos(\Omega t) .$$

The force acting on a wire conducting electric current i in a uniform magnetic field \vec{B} is

$$\vec{F} = -i\vec{B} \times \vec{l} .$$

Calling y the «positive» direction of the moving side, and assuming $\vec{B} = B\hat{z}$, with $\hat{z} = \hat{x} \times \hat{y}$,

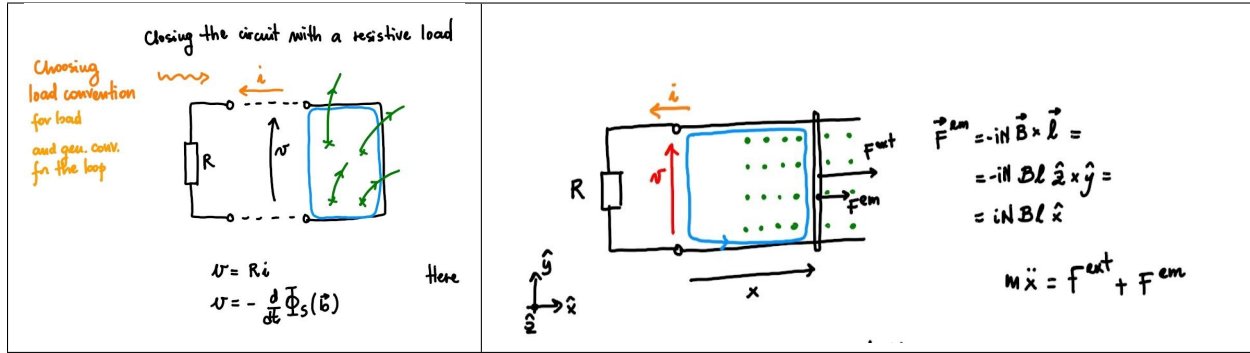
$$\vec{F} = iB\ell\hat{x} .$$

Assuming negligible mass of the moving wire, the second principle of dynamics reduces to force equilibrium, so that the external force provided to the wire must be opposite to the force acting on the wire due to the EM field

$$\vec{F}^e = -\vec{F} ,$$

and the external power reads

$$P^e = \dot{\vec{x}} \cdot \vec{F}^e = -iB\ell \dot{x} = \frac{B^2 \ell^2 \dot{x}^2}{R} = \frac{B^2 \ell^2 (\Delta x)^2}{R} \Omega^2 \cos^2(\Omega t) .$$



Considering the inductance of the circuit and inertia of the wire. Considering the self-induced magnetic flux ϕ ,

$$v = -\frac{d}{dt} (N(\phi + BA)) ,$$

with $\phi = \frac{m}{\theta} = \frac{N}{\theta} i$. The expression of the voltage at the port of the circuit can be recast as

$$v = -\frac{d}{dt} (NBA) - \frac{d}{dt} \left(\frac{N^2}{\theta} i \right) = -\frac{d}{dt} (NBlx) - \frac{d}{dt} (Li) .$$

Now, assuming everything constant except for the x and i , and connecting this circuit to the load with constitutive equation, $v = Ri$, the dynamical equation of the electric circuit becomes

$$L \frac{di}{dt} + Ri = -NBl \frac{dx}{dt} .$$

The dynamical equation of the wire is

$$\begin{aligned} m \frac{d^2 x}{dt^2} &= F^{ext} + F^{EM} = \\ &= F^{ext} + iBl . \end{aligned}$$

Energy balance immediately follows after multiplying the circuit equation by i , the dynamical equation by \dot{x} and summing,

$$\underbrace{\frac{d}{dt} \left(\frac{1}{2} m |\dot{x}|^2 + \frac{1}{2} Li^2 \right)}_{\text{energy: kin.+em.}} + \underbrace{Ri^2}_{\text{dissipation}} = \underbrace{F^{ext} \dot{x}}_{\text{ext. power done on the sys}} .$$

10.3.1 Sistemi elettromeccanici conservativi

Le equazioni che governano il sistema elettromeccanico, senza condensatori, in generale possono essere scritte come

$$\begin{cases} \mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}^{ext} + \mathbf{f}^{em} \\ \frac{d}{dt} (\mathbf{L}\mathbf{i}) + \mathbf{R}\mathbf{i} = \mathbf{e} \end{cases}$$

In termini di energia,

$$0 = \dot{\mathbf{x}}^T [\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} - \mathbf{f}^{ext} - \mathbf{f}^{em}] + \mathbf{i}^T \left[\frac{d}{dt} (\mathbf{L}\mathbf{i}) + \mathbf{R}\mathbf{i} - \mathbf{e} \right]$$

Nel caso di matrici di massa, smorzamento e rigidzza costanti, e usando la derivata del prodotto per ottenere un termine di derivata dell'energia degli induttori sfruttando la simmetria di \mathbf{L} ,

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} \dot{\mathbf{i}}^T \mathbf{L} \dot{\mathbf{i}} \right] &= \dot{\mathbf{i}}^T \frac{d}{dt} (\mathbf{L} \dot{\mathbf{i}}) + \frac{1}{2} \dot{\mathbf{i}}^T \frac{d\mathbf{L}}{dt} \dot{\mathbf{i}} = \\ &= \dot{\mathbf{i}}^T \frac{d}{dt} (\mathbf{L} \dot{\mathbf{i}}) + \sum_a \frac{1}{2} \dot{\mathbf{i}}^T \frac{\partial \mathbf{L}}{\partial x_a} \dot{\mathbf{i}} \dot{x}_a = \\ &= \dot{\mathbf{i}}^T \frac{d}{dt} (\mathbf{L} \dot{\mathbf{i}}) + \nabla \left(\frac{1}{2} \dot{\mathbf{i}}^T \mathbf{L} \dot{\mathbf{i}} \right) \dot{\mathbf{x}} . \end{aligned} \quad (10.1)$$

si può scrivere un'equazione di bilancio dell'energia meccanica macroscopica, $E^{mec,int}$

$$\begin{aligned} 0 &= \frac{d}{dt} \left[\frac{1}{2} \dot{\mathbf{x}}^T \mathbf{M} \dot{\mathbf{x}} + \frac{1}{2} \dot{\mathbf{x}}^T \mathbf{K} \mathbf{x} + \frac{1}{2} \dot{\mathbf{i}}^T \mathbf{L} \dot{\mathbf{i}} \right] - \dot{\mathbf{x}}^T (\mathbf{f}^{em} - \nabla E^{ind}(\mathbf{x}, \dot{\mathbf{i}})) + \\ &\quad - \dot{\mathbf{x}}^T \mathbf{f}^{ext} - \dot{\mathbf{i}}^T \mathbf{e} + \\ &\quad + \dot{\mathbf{x}}^T \mathbf{C} \dot{\mathbf{x}} + \dot{\mathbf{i}}^T \mathbf{R} \dot{\mathbf{i}} . \end{aligned}$$

Nell'ipotesi che il processo sia conservativo, si ricava la forma delle forze dovute ai fenomeni elettromagnetici,

$$\mathbf{f}^{em} = \nabla_{\mathbf{x}} E^{ind}(\mathbf{x}, \dot{\mathbf{i}}) . \quad (10.2)$$

10.3.2 Equazioni di governo

Usando l'espressione (10.2) delle azioni meccaniche dovute agli effetti elettromagnetici, del sistema sono

$$\begin{cases} \mathbf{M} \ddot{\mathbf{x}} + \mathbf{D} \dot{\mathbf{x}} + \mathbf{K} \mathbf{x} - \nabla_{\mathbf{x}} E^{ind}(\mathbf{x}, \dot{\mathbf{i}}) = \mathbf{f}^{ext} \\ \frac{d}{dt} (\mathbf{L}(\mathbf{x}) \dot{\mathbf{i}}) + \mathbf{R} \dot{\mathbf{i}} = \mathbf{e} \end{cases}$$

o nel caso generale

$$\begin{cases} \mathbf{M} \ddot{\mathbf{x}} - \nabla_{\mathbf{x}} E^{ind}(\mathbf{x}, \dot{\mathbf{i}}) = \mathbf{f}^{ext} \\ \frac{d}{dt} (\mathbf{L}(\mathbf{x}) \dot{\mathbf{i}}) + \mathbf{R} \dot{\mathbf{i}} = \mathbf{e} \end{cases}$$

10.3.3 Bilancio energetico

Energia meccanica macroscopica

Usando l'espressione (10.2) delle azioni meccaniche dovute ai fenomeni elettromagnetici, si può riscrivere la relazione (10.1), come un bilancio di energia meccanica macroscopica del sistema,

$$\frac{d}{dt} \left[\frac{1}{2} \dot{\mathbf{x}}^T \mathbf{M} \dot{\mathbf{x}} + \frac{1}{2} \dot{\mathbf{x}}^T \mathbf{K} \mathbf{x} + \frac{1}{2} \dot{\mathbf{i}}^T \mathbf{L} \dot{\mathbf{i}} \right] = \dot{\mathbf{x}}^T \mathbf{f}^{ext} + \dot{\mathbf{i}}^T \mathbf{e} - \dot{\mathbf{x}}^T \mathbf{D} \dot{\mathbf{x}} - \dot{\mathbf{i}}^T \mathbf{R} \dot{\mathbf{i}} ,$$

e quindi

$$\dot{E}^{mec} = P^{ext} - \dot{D} .$$

Energia cinetica

L'energia meccanica macroscopica può essere scritta come la somma dell'energia cinetica e dell'energia potenziale interna del sistema, $E^{mec} = K + V^{int}$. La derivata nel tempo dell'energia potenziale delle azioni interne è l'opposto della potenza delle azioni interne conservative, $P^{int,c} = -\dot{V}^{int}$; la dissipazione è l'opposto della potenza delle azioni interne non-conservative, $P^{int,nc} = -\dot{D}$. La potenza complessiva delle azioni interne può quindi essere scritta come

$$P^{int} = P^{int,c} + P^{int,nc} = -\dot{V}^{int} - \dot{D},$$

$$\dot{K} = \dot{E}^{mec} - \dot{V}^{int} = P^{ext} \underbrace{-\dot{D} - \dot{V}^{int}}_{=P^{int}}$$

Energia totale

Il primo principio della termodinamica fornisce l'equazione di bilancio dell'energia totale di un sistema chiuso,

$$\dot{E}^{tot} = P^{ext} + \dot{Q}^{ext}.$$

Energia interna

L'energia interna di un sistema è definita come la differenza dell'energia totale e dell'energia cinetica macroscopica, $E := E^{tot} - K$. L'equazione di bilancio dell'energia interna di un sistema chiuso è

$$\dot{E} = \dot{Q}^{ext} - P^{int}.$$

Energia interna termica (microscopica)

Se si definisce l'energia interna termica, corrispondente all'energia cinetica associata alle dinamiche microscopiche, come differenza tra energia interna e energia potenziale interna, o differenza di energia totale ed energia meccanica macroscopica,

$$E^{th} = E - V^{int} =$$

$$= E^{tot} - E^{mec},$$

l'equazione di bilancio dell'energia interna termica è

$$\dot{E}^{th} = \dot{Q}^{ext} + \dot{D}.$$

Dimostrazione

$$\dot{E}^{th} = \dot{E} - \dot{V}^{int} = \dot{Q}^{ext} - P^{int} - \dot{V}^{int} =$$

$$= \dot{Q}^{ext} + \dot{D} + \dot{V}^{int} - \dot{V}^{int} =$$

$$= \dot{Q}^{ext} + \dot{D}.$$

Con condensatori. todo

Equazioni

- Leggi ai nodi.

$$0 = \sum_{k \in B_j} \alpha_{jk} i_{jk}$$

$$\mathbf{A}\mathbf{i} = \mathbf{0}$$

- Differenza di potenziale nodi-lati.

$$\mathbf{A}^T \mathbf{v}_n = \mathbf{v}$$

- Nodo a terra.

$$\mathbf{v}_\perp = \mathbf{v}_0 \cdot$$

- Equazioni costitutive.

$$\mathbf{0} = \mathbf{v}_R - \mathbf{R}\mathbf{i}_R \quad \text{resistenze}$$

$$\mathbf{0} = \mathbf{v}_L - \frac{d}{dt} (\mathbf{L}\mathbf{i}_L) \quad \text{induttanze}$$

$$\mathbf{0} = \frac{d}{dt} (C\mathbf{v}_C) - \mathbf{i}_C \quad \text{condensatori}$$

10.4 Network analysis of linear circuits

Dynamical equations of a linear circuit can be written as a general linear state-space model

$$\begin{cases} \mathbf{M}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

The mathematical problem is a system of DAE (dynamical-algebraic equations), as it includes:

- constitutive equations of the linear components
- Kirchhoff laws for current at nodes and voltage in loops

Thus matrix \mathbf{M} is likely to be singular, here vector \mathbf{x} contains both dynamical (like voltage across a capacitor or current through an inductor) and algebraic grid variables, current and voltages whose time derivative doesn't appear explicitly in the system of DAE.

Different representations. Possible choices of the unknowns:

1. current through any side, voltage at any node
2. loop currents, voltage drops across any side.
3. ... *any other (linear) combination on the physical quantities*

10.4.1 Thevenin equivalent

One-port. Thevenin's theorem states that any linear circuit can be reduced to a single voltage source and a single impedance in series.

One-port circuit

As the goal of Thevenin's theorem is to find the constitutive equation of the network as $v(i)$, the network is connected to an external current generator that prescribes i and the voltage v at the port is evaluated.

The input of the extended network is

$$\mathbf{u} = (\mathbf{u}_{gen}, i),$$

while the output is, or at least contains, the voltage v

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}.$$

The linear system can be written in Laplace domain as

$$\begin{cases} s\mathbf{M}\mathbf{x} - \mathbf{M}\mathbf{x}_0 = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

The state and the output are the sum of the free response to non-zero initial conditions and forced response,

$$\begin{cases} \mathbf{x} = (s\mathbf{M} - \mathbf{A})^{-1}\mathbf{M}\mathbf{x}_0 + (s\mathbf{M} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}(s\mathbf{M} - \mathbf{A})^{-1}\mathbf{M}\mathbf{x}_0 + [\mathbf{C}(s\mathbf{M} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{u} \end{cases}$$

Forced response can be further manipulated exploiting PSCE, evaluating the effect of one input at a time, setting all the other inputs equal to zero.

- the effect of setting the input of the external current generator, $i = 0$, is equivalent to evaluate the system with an open circuit at the port
- the effect of setting equal to zero a tension generator, $e = 0$, is equivalent to a short-circuit on the same side
- the effect of setting equal to zero a current generator, $a = 0$, is equivalent to an open circuit on the same side

If the system is **asymptotically stable**, the free response is approximately zero when the **transient dynamics is over**, and the output equals the forced output. Introducing the transfer function

$$\mathbf{G}(s) = [\mathbf{G}_{gen}(s) \quad \mathbf{G}_i(s)],$$

the input-output relation reads

$$\begin{aligned} v = \mathbf{G}(s)\mathbf{u} &= \mathbf{G}_{gen}(s)\mathbf{u}_{gen} + G_i(s)i = \\ &= v_{Th}(s) - Z_{Th}(s)i(s), \end{aligned}$$

having recast it as Thevenin's theorem defining the voltage v_{Th} and the impedance Z_{Th} of the equivalent circuit,

$$\begin{aligned} v_{Th} &:= \mathbf{G}_{gen}(s)\mathbf{u}_{gen}(s) \\ Z_{Th}(s) &:= -G_i(s) \end{aligned}$$

Many-port circuit

$$\mathbf{v} = \mathbf{G}_{gen}(s)\mathbf{u}_{gen} + \mathbf{G}_i(s)\mathbf{i} = \mathbf{v}_{Th} - \mathbf{Z}_{Th}\mathbf{i}.$$

10.4.2 Norton equivalent

10.5 Network analysis of linear circuits - harmonic regime

The harmonic dynamics of a linear circuit can be evaluated in Fourier domain, or using complex numbers to represent harmonic functions,

$$\begin{aligned} v(t) &= V_{max} \cos(\Omega t + \varphi_v) = \operatorname{re}\{V_{max} e^{i(\Omega t + \varphi_v)}\} = \\ &= \sqrt{2}V \cos(\Omega t + \varphi_v) = \sqrt{2} \operatorname{re}\{V e^{i(\Omega t + \varphi_v)}\} = \sqrt{2} \operatorname{re}\{v e^{i\Omega t}\} \\ i(t) &= I_{max} \cos(\Omega t + \varphi_i) = \operatorname{re}\{I_{max} e^{j(\Omega t + \varphi_i)}\} = \\ &= \sqrt{2}I \cos(\Omega t + \varphi_i) = \sqrt{2} \operatorname{re}\{I e^{j(\Omega t + \varphi_i)}\} = \sqrt{2} \operatorname{re}\{i e^{j\Omega t}\} \end{aligned}$$

having anticipated the definition [Definition 10.5.1](#) of effective tension V and current I .

10.5.1 Power

Instantaneous power.

$$\begin{aligned} P(t) &= v(t)i(t) = \\ &= V_{max}I_{max} \cos(\Omega t) \cos(\Omega t - \varphi_i) = \\ &= \frac{1}{2}V_{max}I_{max} [\cos \varphi_i + \cos(2\Omega t)] \end{aligned} \tag{10.3}$$

having used [Werner's formula](#),

$$\cos x \cos y = \frac{1}{2} [\cos(x - y) + \cos(x + y)] .$$

and the property $\cos(-x) = \cos x$.

Average power on a period. Over a period $T = \frac{1}{f} = \frac{2\pi}{\Omega}$

$$\bar{P} = \frac{1}{T} \int_{t=t_0}^{t_0+T} P(t) dt = \frac{V_{max}I_{max}}{2} = VI ,$$

as the integral of the harmonic term with period $\frac{T}{2}$ of the instantaneous power (10.3) is identically zero, and with the definition of the **effective voltage and current**

Definition 10.5.1 (Effective voltage and current in AC)

Effective voltage and currents

$$V := \frac{V_{max}}{\sqrt{2}} , \quad I := \frac{I_{max}}{\sqrt{2}} ,$$

are defined as those voltage and current in DC providing the same value of average power.

Complex power. Complex power of a dipole with impedance Z , $v = Zi$

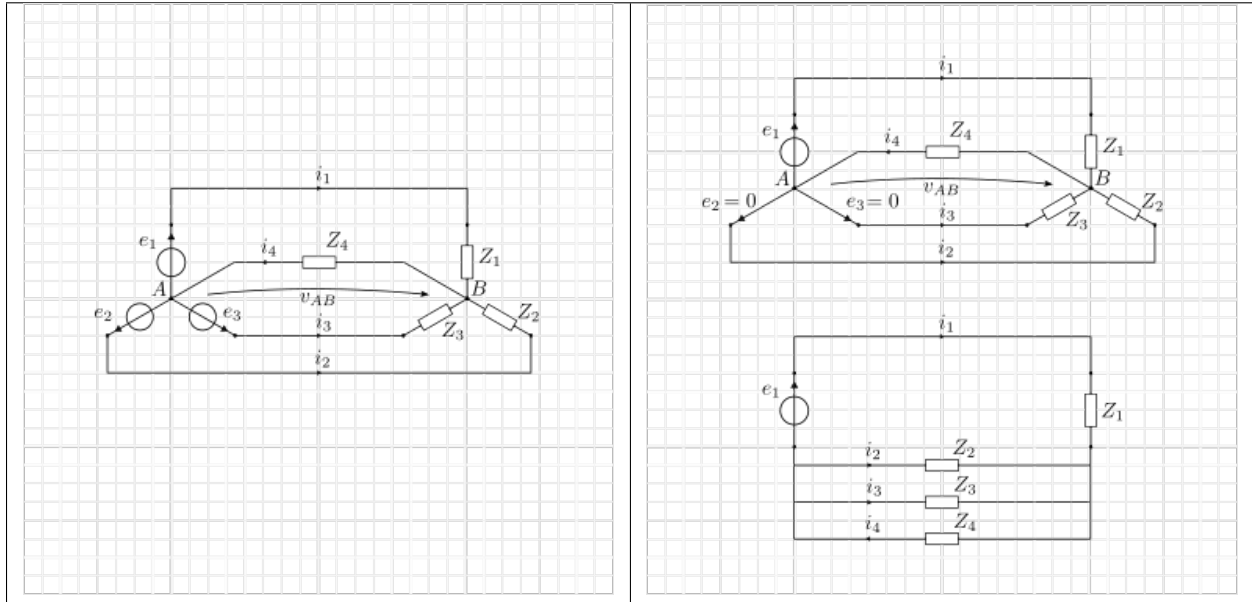
$$\begin{aligned} S &:= vi^* = |v|e^{j\varphi_v}|i|e^{-j\varphi_i} = |v||i|e^{j(\varphi_v - \varphi_i)} = \\ &= Zii^* = Z|i|^2 = (R + jX)|i|^2 = |Z||i|^2 e^{j\varphi_Z} = P + jQ, \end{aligned}$$

with the active power P and the reactive power Q

$$\begin{aligned} P &= \operatorname{re}\{S\} = |S| \cos \varphi_Z = \dots \\ Q &= \operatorname{im}\{S\} = |S| \sin \varphi_Z = \dots \end{aligned}$$

10.6 Three-phase circuits

10.6.1 Star-star network



General solution

Tension v_{AB} between the centers of the stars A, B

$$v_{AB} = \frac{\sum_{g=1}^3 Y_g e_g}{\sum_{i=1}^4 Y_i}.$$

Proof.

PSCE is used on the linear network, leaving only one tension generator on at a time, and then combining the results.

Tension generator e_1 on, $e_2 = e_3 = 0$ off. Leaving e_1 on, and switching off $e_2 = e_3 = 0$, tension generator sees an equivalent impedance

$$\begin{aligned} Z_{eq,1} &= Z_1 + (Z_2 \parallel Z_3 \parallel Z_4) \\ &= \frac{1}{Y_1} + \frac{1}{Y_2 + Y_3 + Y_4} = \frac{Y_{1234}}{Y_1 Y_{234}}, \end{aligned}$$

so that:

- the current through the generator reads

$$i_{1,1} = \frac{e_1}{Z_{eq,1}} = \frac{Y_1 Y_{234}}{Y_{1234}} e_1$$

- the currents through the other sides (acting as current dividers are):

$$\begin{aligned} i_{2,1} &= -\frac{Y_2}{Y_{234}} i_{1,1} = -\frac{Y_1 Y_2}{Y_{1234}} e_1 \\ i_{3,1} &= -\frac{Y_3}{Y_{234}} i_{1,1} = -\frac{Y_1 Y_3}{Y_{1234}} e_1 \\ i_{4,1} &= \frac{Y_4}{Y_{234}} i_{1,1} = \frac{Y_1 Y_4}{Y_{1234}} e_1 \end{aligned}$$

- tension v_{AB}

$$v_{AB,1} = e_1 - Z_1 i_{1,1} = \left(1 - \frac{Y_{234}}{Y_{1234}}\right) e_1 = \frac{Y_1 e_1}{\sum_{k=1}^4 Y_k}.$$

PSCE. Exploiting the PSCE and the symmetry of the system, the expressions of currents in the phases, in the neutral and the center-center voltage seamlessly follow

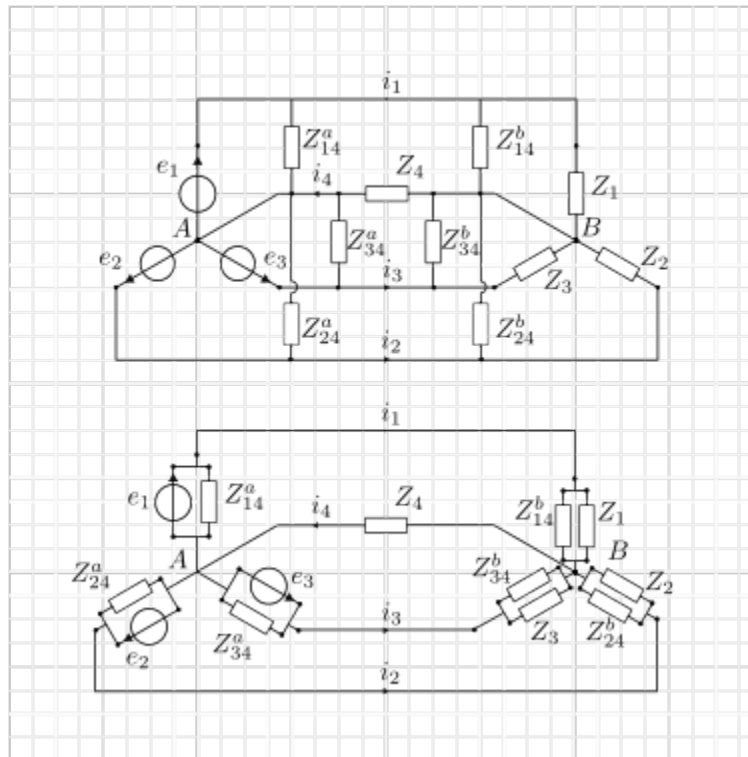
$$\begin{aligned} i_1 &= \frac{Y_1 Y_{234}}{Y_{1234}} e_1 - \frac{Y_1 Y_2}{Y_{1234}} e_2 - \frac{Y_1 Y_3}{Y_{1234}} e_3 = \\ &= Y_1 e_1 - \frac{Y_1}{Y_{1234}} \sum_{g=1}^3 Y_g e_g \\ i_2 &= Y_2 e_2 - \frac{Y_2}{Y_{1234}} \sum_{g=1}^3 Y_g e_g \\ i_3 &= Y_3 e_3 - \frac{Y_3}{Y_{1234}} \sum_{g=1}^3 Y_g e_g \\ i_4 &= \frac{Y_4}{Y_{1234}} \sum_{g=1}^3 Y_g e_g \\ v_{AB} &= \frac{\sum_{g=1}^3 Y_g e_g}{\sum_{k=1}^4 Y_k} \end{aligned}$$

Equilibrated generation and loads

Extra connections

Phase-neutral connections

Connections of a phase with the neutral result in parallel impedance with the generators and/or the loads



Phase-phase connections

Phase-phase connections don't influence the voltage v_{AB} between the centers A, B .

todo Write the proof.

10.7 Exercises

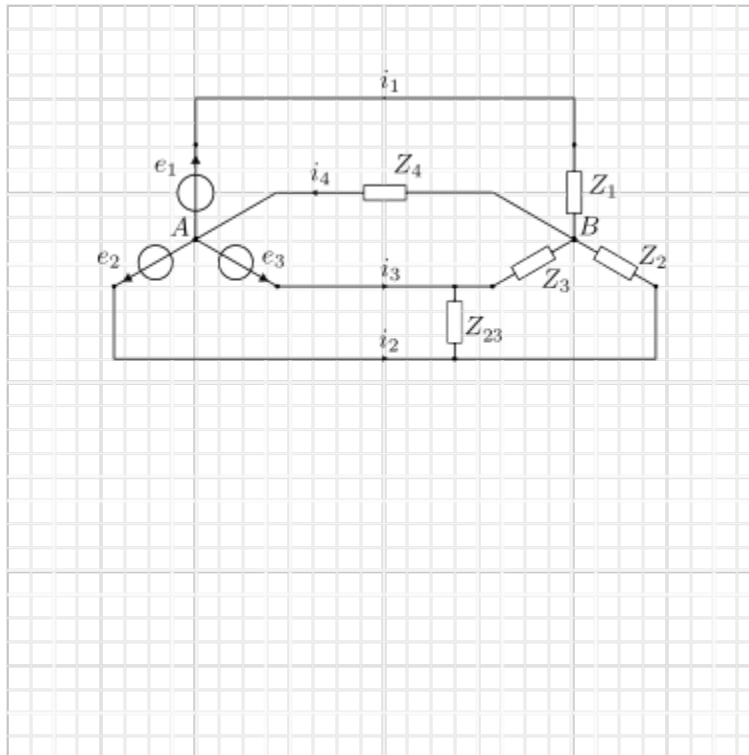
Topics: Thevenin and Norton equivalent;...

Electric circuits:

- Type a: transient dynamics of systems with 1 dynamic component (either capacitor or inductor);
- Type b: harmonic dynamics of linear systems: phasor algebra, complex power,...
- Type c: three-phase circuits, triangles and stars,...

Electromagnetic circuits:

- Type d: circuit approximation of magnetic circuit,...



Exams.

2025-02-11

1. Type a. Exercise ??
2. Type b. Exercise ??
3. Type b. Exercise ??
4. Theory: electrical line. Electro-thermal model of the cable,...

2025-01-22

1. Type a. Exercise ??
2. Type b. Exercise ??
3. Type d. Exercise ??
4. Theory: transformer

2024-09-06

1. Type a. Exercise ??
2. Type b. Exercise ??
3. Type c. Exercise ??
4. Theory: overload in cables

2024-07-22

1. Type a. Exercise ??
2. Type b. Exercise ??
3. Type c. Exercise ??

2024-06-19

1. Type c. Exercise ??
2. Type d. Exercise ??

2024-02-13

1. Type d.+a. Exercise ??
2. Type a. Exercise ??
3. Type c. Exercise ??

10.7.1 Transient dynamics of linear electrical grids with one dynamic component

Guidelines for solution

Breaking down the solution:

1. Find the **many-port equivalent** of the **linear algebraic part of the network** (resistor, and prescribed generators), using PSCE. Find the relation between port voltage and currents and all the required variables of the network,

$$\begin{aligned}\mathbf{v}_{port} &= \mathbf{v}_0(\mathbf{e}, \mathbf{a}) + \mathbf{R}\mathbf{i}_{port} \\ \mathbf{z} &= \mathbf{z}_0(\mathbf{e}, \mathbf{a}) + \mathbf{z}_{/i_{port}}\mathbf{i}_{port}\end{aligned}$$

If 2 ports exist and port A is connected to a dynamical linear component and port B is connected to an ideal switch, the equations become to

$$\begin{aligned}v_A &= v_{0,A}(\mathbf{e}, \mathbf{a}) + R_{AA}i_A + R_{AB}i_B \\ v_B &= v_{0,B}(\mathbf{e}, \mathbf{a}) + R_{BA}i_A + R_{BB}i_B \\ \mathbf{z} &= \mathbf{z}_0(\mathbf{e}, \mathbf{a}) + \mathbf{z}_{/i_{port}}\mathbf{i}_{port}\end{aligned}$$

2. Evaluate the **steady conditions** for $t \leq 0^-$, with the given state of the switch ($i_B = 0$ if it's open, $v_B = 0$ if it's closed), and using the constitutive equation of the dynamical element (a capacitor acts as an open circuit in steady conditions, $i_A = 0$ as $i_A = C \frac{dv_A}{dt}$; an inductor acts as a short-circuit in steady conditions, $v_A = 0$, as $v_A = L \frac{di_A}{dt}$).

In the first two equations of the system, two of the four variables $i_{A,B}$, $v_{A,B}$ are thus known, and this system can be solved to find the other two quantities. Once \mathbf{i}_{port} is known, grid variables \mathbf{z} can be evaluated.

3. **Transient dynamics** is then evaluated using the change of state in the switch

$$\text{open to close: } \begin{cases} v_A(t) = (1 - h(t)) v_{A,0^-} \\ i_A(t) = h(t) i_{A,t \geq 0}(t) \end{cases}$$

$$\text{close to open: } \begin{cases} v_A(t) = h(t) v_{A,t \geq 0}(t) \\ i_A(t) = (1 - h(t)) i_{A,0^-} \end{cases}$$

and using the conditions for $t \geq 0$ in the equations of the equivalent network to find the equivalent resistance R_{eq} of the algebraic part of the network to be used in the constitutive equations of the dynamical component,

$$\text{capacitor : } 0 = i_A + C \frac{dv_A}{dt} \rightarrow f(\mathbf{x}_B) = v_A + R_{eq} C \frac{dv_A}{dt}$$

$$\text{inductor : } 0 = v_A + L \frac{di_A}{dt} \rightarrow f(\mathbf{x}_B) = i_A + R_{eq} L \frac{di_A}{dt}$$

with $f(\mathbf{x}_B)$ a forcing term depending on the state of the switch, and the initial conditions for the state variable of the dynamical components equal to the steady conditions, as there's no jump in state variables without impulsive forces.

4. Once the state variables of the dynamical equations are known, it's possible to evaluate all the other required variables.

Exercise 10.7.1 (Exam 2025-02-11, Exercise 1.)

- 1) Il circuito di Figura 1, con ingressi stazionari, è così assegnato:

$$\begin{aligned} V_1 &= 5 \text{ V} \\ V_2 &= 8 \text{ V} \\ I_s &= 3 \text{ A} \\ R_1 &= 1 \Omega \\ R_2 &= 2 \Omega \\ R_3 &= 3 \Omega \\ R_4 &= 4 \Omega \\ C_1 &= 500 \text{ mF} \end{aligned}$$

L'interruttore S è aperto da tempo infinito e viene chiuso all'istante $t = 0$.

Determinare:

- l'andamento nel tempo della corrente $i_{cc}(t)$ sia in termini analitici che grafici (andamento qualitativo).
- l'energia immagazzinata nel capacitore nell'istante di tempo $t = 0$.

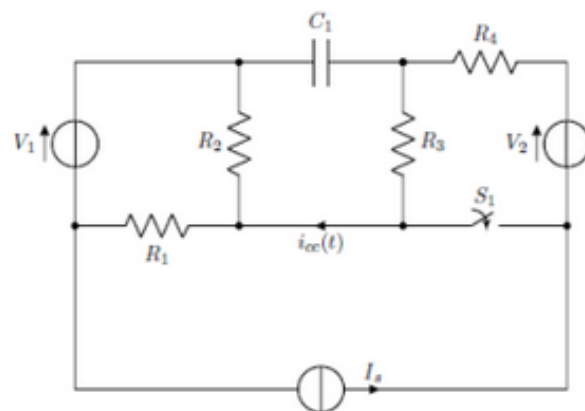
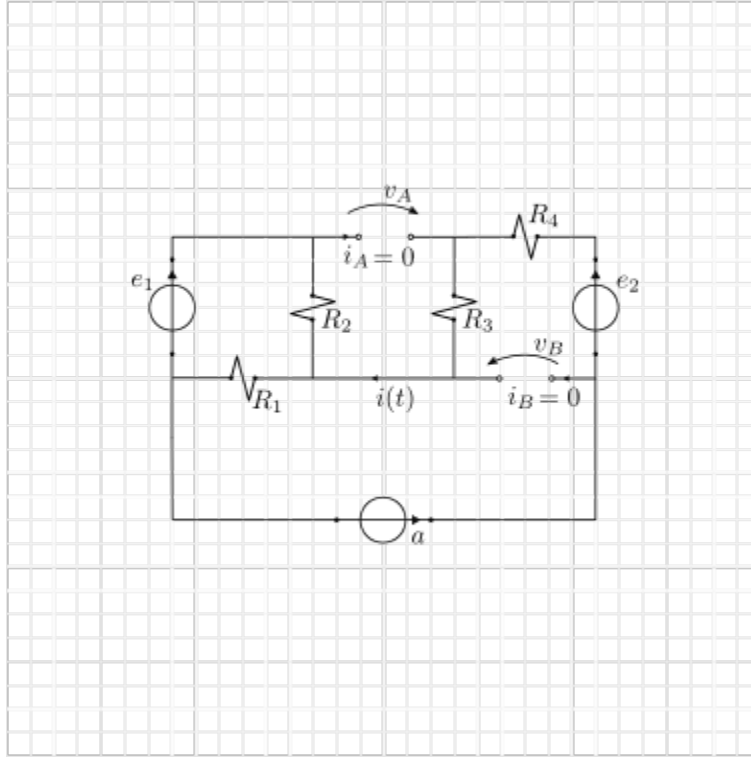


Fig. 1.

Solution

Following the **guidelines for the solution**, a *many-port Thevenin equivalent circuit* of the resistive part of the circuit is found, with two ports for interfacing with the capacitor (A) and with the switch. The dynamical equation of the system is written in state-space representation, writing the voltage at the ports and the unknown variable $i(t)$ as outputs; the capacitor constitutive equation is used to find the time evolution of the system once the switch is closed



Internal generators on, open circuit

Solution using two loop currents, i_1 in the upper part of the circuit and i_2 in the lower triangle. Using KVL

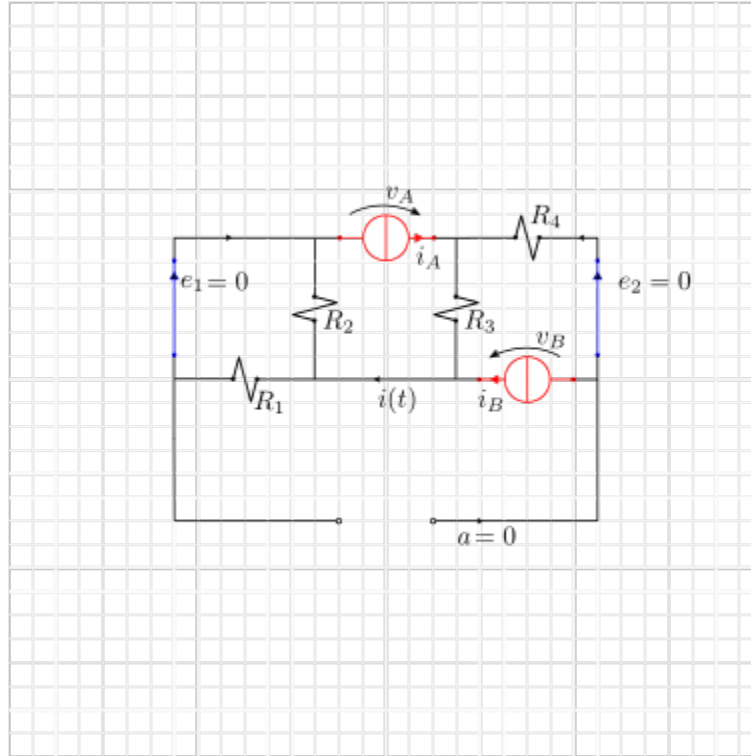
$$0 = e_1 - R_2 i_{1,0} - R_1(a + i_{1,0})$$

$$\rightarrow i_{1,0} = \frac{1}{R_1 + R_2} e_1 - \frac{R_1}{R_1 + R_2} a$$

so that the desired variables read

$$\begin{cases} v_{A,0} &= R_3 a - R_2 i_{1,0} = \left[R_3 + \frac{R_1 R_2}{R_1 + R_2} \right] a - \frac{R_2}{R_1 + R_2} e_1 \\ v_{B,0} &= e_2 - (R_3 + R_4) a \\ i_0 &= a \end{cases}$$

$$\begin{cases} v_{A,0} &= 7.67 \text{ V} \\ v_{B,0} &= -13.00 \text{ V} \\ i_0 &= 3.00 \text{ A} \end{cases}$$



Internal generators off, current generators at the ports

Calling i_A and i_B the current passing through the current generators connected at the ports. The solution is found powering one generation at a time and then exploiting PSCE

Powering A ...

Powering B. ...

Currents in the two parallel branches in the upper part of the circuit (current dividers) read

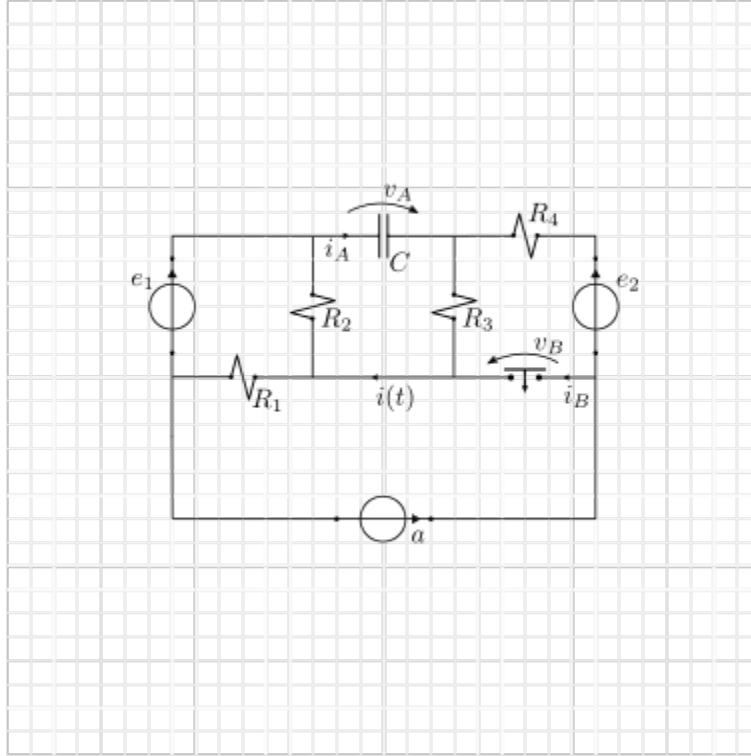
$$\begin{cases} i &= i_A \\ v_A &= \left[R_3 + \frac{R_1 R_2}{R_1 + R_2} \right] i_A - R_3 i_B \\ v_B &= -R_3 i_A + (R_3 + R_4) i_B \end{cases}$$

The equations of the equivalent algebraic system are

$$\begin{cases} v_A &= v_{A,0} + R_{AA} i_A + R_{AB} i_B \\ v_B &= v_{B,0} + R_{BA} i_A + R_{BB} i_B \\ i &= i_{,0} + i_{/i_A} i_A + i_{/i_B} i_B \end{cases}$$

$$\begin{bmatrix} v_A(t) \\ v_B(t) \end{bmatrix} = \begin{bmatrix} v_{A0} \\ v_{B0} \end{bmatrix} + \begin{bmatrix} R_3 + \frac{R_1 R_2}{R_1 + R_2} & -R_3 \\ -R_3 & R_3 + R_4 \end{bmatrix} \begin{bmatrix} i_A(t) \\ i_B(t) \end{bmatrix}$$

$$i(t) = i_0 + i_A(t)$$



$$\begin{aligned} \det \mathbf{R} &= \left(R_3 + \frac{R_1 R_2}{R_1 + R_2} \right) (R_3 + R_4) - R_3^2 = \\ &= (R_3 + R_4) \left(R_3 + \frac{R_1 R_2}{R_1 + R_2} - \frac{R_3^2}{R_3 + R_4} \right) = \\ &= (R_3 + R_4) \left(\frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4} \right). \end{aligned}$$

Steady solution for $t \leq 0^-$. With switch open $i_B = 0$ and steady conditions $i_A = C\dot{v}_A = 0$,

$$\begin{cases} v_A(0^-) = v_{A,0} = 7.67 \text{ V} \\ v_B(0^-) = v_{B,0} = -13.00 \text{ V} \\ i(0^-) = i_{,0} = 3.00 \text{ A} \end{cases}$$

Transient dynamics, when the switch closes $v_B(t \geq 0^+) = 0$,

$$i_A(t) = \frac{R_3 + R_4}{\det \mathbf{R}} \Delta v_A(t) + \frac{R_3}{\det \mathbf{R}} \Delta v_B(t)$$

- **Tension across the switch**

$$\begin{aligned} v_B(t) &= v_{B,0} h(-t) \\ \Delta v_B(t) &= v_B(t) - v_{B,0} = -v_{B,0} h(t). \end{aligned}$$

- **Tension across the capacitor.** The dynamical equation for the difference of the state variable reads

$$\begin{aligned} 0 &= i_A + C\dot{v}_A = \\ &= \frac{R_3 + R_4}{\det \mathbf{R}} \Delta v_A(t) + \frac{R_3}{\det \mathbf{R}} \Delta v_B(t) + C\dot{v}_A. \end{aligned}$$

As $v_A(t=0) = v_{A,0}$ (no jump in state variables without impulsive forcing), $\Delta v_A = v_A - v_{A,0}$, and $\frac{d}{dt}\Delta v_A = \frac{d}{dt}v_A$, the dynamical equation reads

$$\begin{cases} \frac{\det \mathbf{R}}{R_3 + R_4} C \frac{d}{dt} \Delta v_A + \Delta v_A = -\frac{R_3}{R_3 + R_4} \Delta v_B(t) = \frac{R_3}{R_3 + R_4} v_{B,0} h(t) \\ \Delta v_A(0^-) = 0. \end{cases}$$

$$\Delta v_A(t) = \frac{R_3}{R_3 + R_4} v_{B,0} \left[1 - \exp\left(-\frac{t}{\tau}\right) \right] h(t),$$

having defined the time constant and the equivalent resistance seen by the capacitor

$$R_{eq} := \frac{\det \mathbf{R}}{R_3 + R_4} = \frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4} = \frac{50}{21} \Omega = 2.381 \Omega$$

$$\tau := R_{eq} C = 1.1905 \text{ s}$$

Tension through the capacitor reads

$$\begin{aligned} v_A(t) &= v_{A,0} + \Delta v_A(t) = \\ &= v_{A,0} + \Delta v_{A,+ \infty} \left[1 - \exp\left(-\frac{t}{\tau}\right) \right] h(t), \end{aligned}$$

so that the values

$$\begin{aligned} v_A(0^+) &= v_{A,0} = 7.67 \text{ V} \\ v_A(+\infty) &= v_{A,0} + \Delta v_{A,+ \infty} = (7.667 - 5.571) \text{ V} = 2.095 \text{ V}. \end{aligned}$$

• **Current through the capacitor.**

$$\begin{aligned} i_A(t) &= \frac{R_3 + R_4}{\det \mathbf{R}} \Delta v_A(t) + \frac{R_3}{\det \mathbf{R}} \Delta v_B(t) = \\ &= \frac{R_3 + R_4}{\det \mathbf{R}} \frac{R_3}{R_3 + R_4} v_{B,0} \left[1 - \exp\left(-\frac{t}{\tau}\right) \right] h(t) - \frac{R_3}{\det \mathbf{R}} v_{B,0} h(t) = \\ &= -\frac{R_3}{\det \mathbf{R}} v_{B,0} \exp\left(-\frac{t}{\tau}\right) h(t) \\ &= 2.34 \text{ A} \exp\left(-\frac{t}{\tau}\right) h(t). \end{aligned}$$

so that the values

$$\begin{aligned} i_A(0^+) &= 2.34 \text{ A} \\ i_A(+\infty) &= 0.00 \text{ A} \end{aligned}$$

• **Current $i(t)$**

$$\begin{aligned} i(t) &= i_{,0} + i_A(t) = \\ &= a - \frac{R_3}{\det \mathbf{R}} v_{B,0} \exp\left(-\frac{t}{\tau}\right) h(t) \\ &= 3.00 \text{ A} + 2.34 \text{ A} e^{-\frac{t}{\tau}} h(t), \end{aligned}$$

so that the values

$$\begin{aligned} i(0^+) &= 5.35 \text{ A} \\ i(+\infty) &= 3.00 \text{ A} \end{aligned}$$

Energy stored in the capacitor at $t = 0$. Energy in the capacitor reads

$$E_C(t) = \frac{1}{2} C v_A^2(t).$$

At $t = 0$, $v_A(0) = 7.667 \text{ V}$ and $E_C(0) = 14.694 \text{ J}$.

Exercise 10.7.2 (Exam 2025-01-22, Exercise 1.)

1) Il circuito di Figura 1, con ingressi stazionari, è così assegnato:

$$\begin{array}{llll} E_1 = 30 \text{ V} & E_2 = 50 \text{ V} & R_1 = 4 \Omega & R_2 = 7 \Omega \\ R_3 = 10 \Omega & R_4 = 3 \Omega & R_6 = 8 \Omega & \\ R_7 = 2 \Omega & R_8 = 12 \Omega & C = 0.5 \text{ mF} & \end{array}$$

L'interruttore S è aperto da tempo infinito e viene chiuso all'istante $t = 0$.

Determinare:

- l'andamento nel tempo della corrente $i(t)$ sia in termini analitici che grafici (andamento qualitativo).
- l'energia immagazzinata nel capacitore nell'istante di tempo $t = \tau$, essendo τ la costante di tempo del circuito.

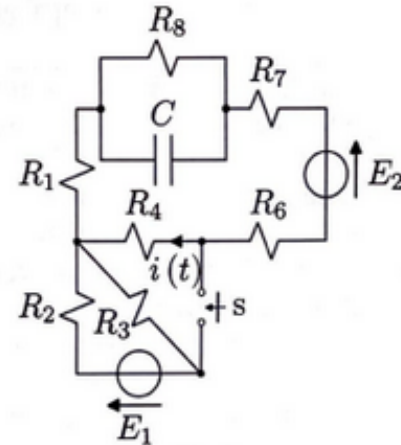


Fig. 1.

Solution

Following the **guidelines for the solution**, a *many-port Thevenin equivalent circuit* of the resistive part of the circuit is found, with two ports for interfacing with the capacitor (A) and with the switch. The dynamical equation of the system is written in state-space representation, writing the voltage at the ports and the unknown variable $i(t)$ as outputs; the capacitor constitutive equation is used to find the time evolution of the system once the switch is closed

Internal generators on, open circuit

Solution using two loop currents, i_1 in the upper part of the circuit and i_2 in the lower triangle. Using KVL

$$0 = e_2 - (R_7 + R_8 + R_1 + R_4 + R_6)i_{2,0}$$

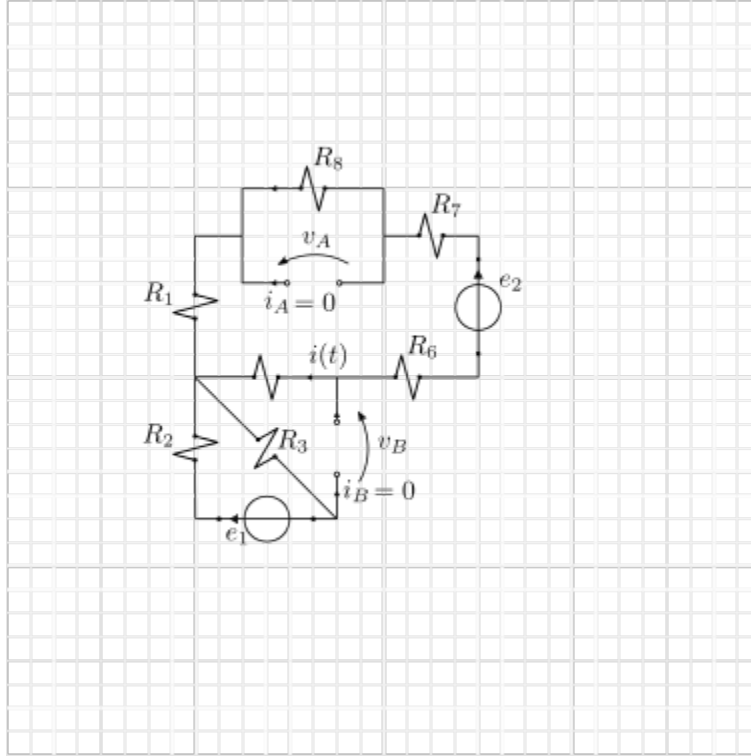
$$0 = e_1 - (R_2 + R_3)i_{1,0}$$

$$i_{2,0} = \frac{1}{R_{14678}} e_2$$

$$i_{1,0} = \frac{1}{R_{23}} e_1$$

with $R_{14678} = R_1 + R_4 + R_6 + R_7 + R_8$, and $R_{23} = R_2 + R_3$. The desired physical quantities are

$$\begin{cases} v_{A,0} = -R_8 i_{2,0} = -\frac{R_8}{R_{14678}} e_2 \\ v_{B,0} = -R_4 i_{2,0} + R_3 i_{1,0} = -\frac{R_4}{R_{14678}} e_2 + \frac{R_3}{R_{23}} e_1 \\ i_0 = -i_{2,0} = -\frac{1}{R_{14678}} e_2 \end{cases}$$



and their values

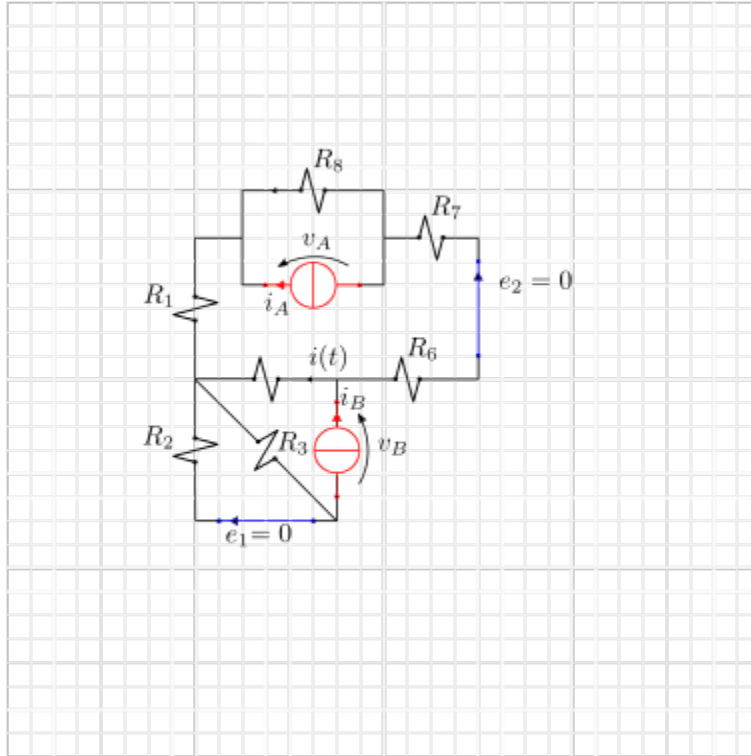
$$\begin{cases} v_{A,0} &= -20.6900 \text{ V} \\ v_{B,0} &= 12.4750 \text{ V} \\ i_0 &= -1.7241 \text{ A} \end{cases}$$

Internal generators off, current generators at the ports

Calling i_A and i_B the current passing through the current generators connected at the ports. The solution is found powering one generation at a time and then exploiting PSCE

Powering A

$$\begin{aligned} 0 &= (i_2 - i_A)R_8 + i_2(R_{14678}) \\ \rightarrow i_2 &= \frac{R_8}{R_{14678}} i_A \\ v_{A,A} &= -R_8(i_2 - i_A) = \frac{R_8 R_{14678}}{R_{14678}} i_A \\ v_{B,A} &= -R_4 i_2 = -\frac{R_4 R_8}{R_{14678}} i_A \\ i_{,A} &= -i_2 = -\frac{R_8}{R_{14678}} i_A \\ v_{A,A} &= R_{AA} i_A = 7.0345 \Omega i_A \\ v_{B,A} &= R_{BA} i_A = -1.2414 \Omega i_A \\ i_{,A} &= i_{/i_A} i_A = -0.4138 i_A \end{aligned}$$



Powering B .

Currents in the two parallel branches in the upper part of the circuit (current dividers) read

$$i_{2,B} = \frac{R_4}{R_{14678}} i_B$$

$$i_{3,B} = \frac{R_2}{R_{23}} i_B$$

and the desired variables

$$i_{,B} = i_{4,B} = \frac{R_{1678}}{R_{14678}} i_B$$

$$v_{A,B} = -R_8 i_{2,B} = -\frac{R_4 R_8}{R_{14678}} i_B$$

$$v_{B,B} = R_4 i_{4,B} + R_3 i_{3,B} = \left[\frac{R_4 (R_{1678})}{R_{14678}} + \frac{R_2 R_3}{R_{23}} \right] i_B$$

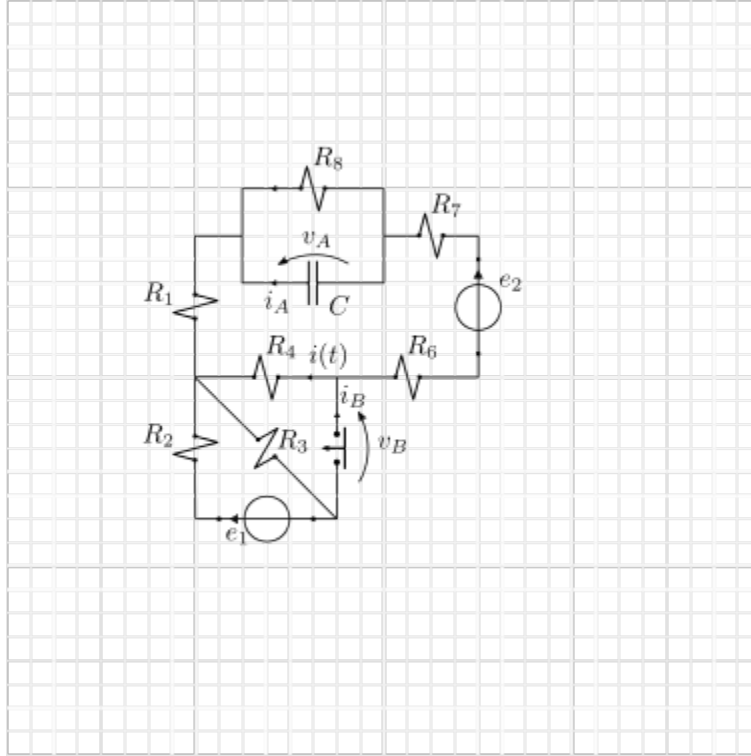
$$v_{A,B} = R_{AB} i_B = -1.2414 \Omega i_B$$

$$v_{B,B} = R_{BB} i_B = 6.8073 \Omega i_B$$

$$i_{,B} = i_{/i_B} i_B = 0.8966 i_B$$

The equations of the equivalent algebraic system are

$$\begin{cases} v_A = v_{A,0} + R_{AA} i_A + R_{AB} i_B \\ v_B = v_{B,0} + R_{BA} i_A + R_{BB} i_B \\ i = i_{,0} + i_{/i_A} i_A + i_{/i_B} i_B \end{cases}$$



and they can be used to write the currents as a function of the tensions

$$i_A = \frac{1}{\det \mathbf{R}} (R_{BB} \Delta v_A(t) - R_{AB} \Delta v_B(t))$$

$$i_B = \frac{1}{\det \mathbf{R}} (-R_{BA} \Delta v_A(t) + R_{AA} \Delta v_B(t))$$

The switch command is off for $t \leq 0^-$, on for $t > 0$,

$$i_B(t \leq 0^-) = 0 \quad , \quad v_B(t \geq 0^+) = 0 .$$

Steady solution for $t \leq 0^-$. With switch open $i_B = 0$ and steady conditions $i_A = C \dot{v}_A = 0$,

$$\begin{cases} v_A(0^-) = v_{A,0} = -20.6900 \text{ V} \\ v_B(0^-) = v_{B,0} = 12.4750 \text{ V} \\ i(0^-) = i_{,0} = -1.7241 \text{ A} \end{cases}$$

Transient dynamics. For $t \geq 0$, the switch is closed and thus $v_B(t \geq 0^+) = 0$.

- **Tension across the switch** as a function of time

$$v_B(t) = v_{B,0} h(-t) = v_{B,0} (1 - h(t))$$

$$\Delta v_B(t) = v_B(t) - v_{B,0} = -v_{B,0} h(t) .$$

- **Tension across the capacitor.** Writing i_A across the capacitor as a function of the tensions, the constitutive equation of the capacitor becomes

$$0 = C \frac{d\Delta v_A}{dt} + i_A =$$

$$= C \frac{d\Delta v_A}{dt} + \frac{1}{\det \mathbf{R}} (R_{BB} \Delta v_A - R_{AB} \Delta v_B)$$

$$\begin{cases} R_{eq} C \frac{d\Delta v_A}{dt} + \Delta v_A = \frac{R_{AB}}{R_{BB}} \Delta v_B(t) = -\frac{R_{AB}}{R_{BB}} v_{B,0} h(t) \\ \Delta v_A(0) = 0, \end{cases}$$

with

$$\begin{aligned} R_{eq} &= \frac{\det \mathbf{R}}{R_{BB}} = 6.8081 \, \Omega \\ \tau &= R_{eq} C = 3.4041 \cdot 10^{-3} \, s \\ \det \mathbf{R} &= 46.345 \, \Omega^2 \end{aligned}$$

The solution of the differential equation provides the difference of the tension through the capacitor w.r.t. the initial steady condition

$$\Delta v_A(t) = \Delta v_{A,+\infty} \left(1 - e^{-\frac{t}{\tau}}\right) h(t),$$

with $\Delta v_{A,+\infty} = -\frac{R_{AB}}{R_{BB}} v_{B,0} = 2.2742 \, V$. The voltage across the capacitor as a function of time t thus reads

$$\begin{aligned} v_A(t) &= v_{A,0} + \Delta v_A(t) = \\ &= v_{A,0} + \Delta v_{A,+\infty} \left(1 - e^{-\frac{t}{\tau}}\right) h(t), \end{aligned}$$

so that the values

$$\begin{aligned} v_A(0^+) &= v_{A,0} &= -20.69 \, V \\ v_A(+\infty) &= v_{A,0} + \Delta V = -20.69 \, V + 2.2742 \, V &= -18.4158 \, V \end{aligned}$$

• **Current through the capacitor.**

$$\begin{aligned} i_A(t) &= \frac{1}{\det \mathbf{R}} (R_{BB} \Delta v_A(t) - R_{AB} \Delta v_B(t)) = \\ &= \frac{1}{\det \mathbf{R}} \left[R_{BB} \left(-\frac{R_{AB}}{R_{BB}} v_{B,0} \right) \left(1 - e^{-\frac{t}{\tau}}\right) h(t) + R_{AB} v_{B,0} h(t) \right] = \\ &= \frac{R_{AB}}{\det \mathbf{R}} v_{B,0} e^{-\frac{t}{\tau}} h(t). \end{aligned}$$

so that the values

$$\begin{aligned} i_A(0^+) &= \frac{R_{AB}}{\det \mathbf{R}} v_{B,0} = \frac{-1.2414 \, \Omega}{46.908 \, \Omega^2} 12.475 \, V = -0.334 \, A \\ i_A(+\infty) &= v_{A,0} + \Delta V = -20.69 \, V + 2.2742 \, V &= 0.0 \, A \end{aligned}$$

or with $i_A = -C \frac{d\Delta v_A}{dt} \dots$

• **Current across the switch**

$$\begin{aligned} i_B(t) &= \frac{1}{R_{BB}} \left[v_B(t) - v_{B,0} - R_{BA} i_A(t) \right] = \\ &= \frac{1}{R_{BB}} \left[-v_{B,0} - R_{BA} \frac{R_{AB}}{\det \mathbf{R}} v_{B,0} e^{-\frac{t}{\tau}} \right] h(t) = \\ &= -\frac{v_{B,0}}{R_{BB}} \left[1 + \frac{R_{BA} R_{AB}}{\det \mathbf{R}} e^{-\frac{t}{\tau}} \right] h(t). \end{aligned}$$

so that the values

$$\begin{aligned} i_B(0^+) &= -\frac{v_{B,0}}{R_{BB}} \left[1 + \frac{R_{BA} R_{AB}}{\det \mathbf{R}} \right] = -\frac{v_{B,0} R_{AA}}{\det \mathbf{R}} = -\frac{7.0345 \, \Omega}{46.345 \, \Omega^2} 12.475 \, V = -1.8929 \, A \\ i_B(+\infty) &= -\frac{v_{B,0}}{R_{BB}} = -\frac{12.475 \, V}{6.8073 \, \Omega} = -1.8320 \, A. \end{aligned}$$

• **Current $i(t)$**

$$\begin{aligned} i(t) &= i_0 - 0.4138 i_A(t) + 0.8966 i_B(t) = \\ &= i_0 + \left[-0.4138 i_{A,0+} e^{-\frac{t}{\tau}} + 0.8966 \left(i_{B,+\infty} + (i_{B,0+} - i_{B,+\infty}) e^{-\frac{t}{\tau}} \right) \right] h(t), \end{aligned}$$

so that

$$\begin{aligned} i(0^+) &= i_0 - 0.4138 i_{A,0+} + 0.8966 i_{B,0+} = \\ &= -1.7214 \text{ A} - 0.4138 (-0.334 \text{ A}) + 0.8966 (-1.8929 \text{ A}) = -3.2831 \text{ A} \\ i(+\infty) &= i_0 + 0.8966 i_{B,+\infty} = \\ &= -1.7214 \text{ A} + 0.8966 (-1.8320 \text{ A}) = -3.3671 \text{ A} \end{aligned}$$

Energy stored in the capacitor.

$$E_C(t) = \frac{1}{2} C v_A^2(t),$$

and for $t = \tau$,

$$\begin{aligned} v_A(t) &= v_{A,0} + \Delta v_{A,+\infty} \left(1 - e^{-\frac{t}{\tau}} \right) h(t) = \\ &= -20.69 \text{ V} + 2.2742 \text{ V} \left(1 - e^{-\frac{t}{\tau}} \right) h(t), \end{aligned}$$

and thus $v_A(\tau) = -19.25 \text{ V}$

$$E_C(\tau) = 0.5 \cdot 5 \cdot 10^{-4} \text{ F} \cdot (19.25 \text{ V})^2 = 9.26 \cdot 10^{-2} \text{ J}.$$

Exercise 10.7.3 (Exam 2024-09-06, Exercise 1.)

1) Il circuito di Figura 1, con ingressi stazionari, è così assegnato:

$$\begin{aligned} V_s &= 5 \text{ V} \\ I_s &= 5 \text{ A} \\ R_1 &= 1 \Omega \\ R_2 &= 2 \Omega \\ R_3 &= 3 \Omega \\ R_4 &= 4 \Omega \\ L &= 100 \text{ mH} \end{aligned}$$

L'interruttore S è aperto da tempo infinito e viene chiuso all'istante $t = 0$.

Determinare:

- l'andamento nel tempo della corrente $i_{R4}(t)$ sia in termini analitici che grafici (andamento qualitativo).
- l'energia immagazzinata nell'induttore nell'istante di tempo $t = 0 \text{ s}$.

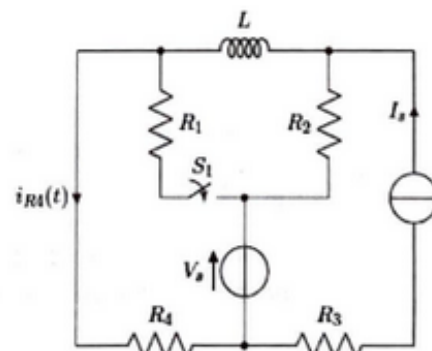


Fig. 1.

Solution

Exercise 10.7.4 (Exam 2024-07-22, Exercise 1.)

1) Il circuito di Figura 1, con ingressi stazionari, è così assegnato:

$$E_1 = 72 \text{ V}$$

$$E_2 = 95 \text{ V}$$

$$R_1 = 16 \text{ } \Omega$$

$$R_2 = 16 \text{ } \Omega$$

$$R_3 = 24 \text{ } \Omega$$

$$R_4 = 20 \text{ } \Omega$$

$$L = 44 \text{ mH}$$

L'interruttore S è aperto da tempo infinito e viene chiuso all'istante $t = 0$.

Determinare:

- l'andamento nel tempo della corrente $i(t)$.
- l'energia immagazzinata nell'induttore nell'istante di tempo $t = 3 \text{ ms}$.

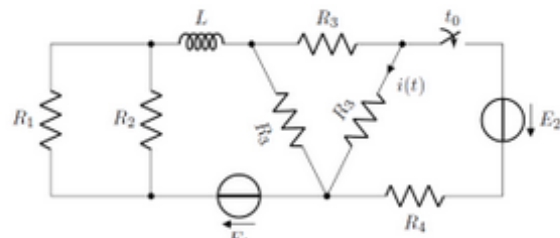


Fig. 1.

Solution

Exercise 10.7.5 (Exam 2024-02-13, Exercise 1.)

1b) SOLO GESTIONALI Il circuito di Figura 1, con ingressi stazionari, è così assegnato:

$$E_1 = 20 \text{ V} \quad R_1 = 5 \text{ } \Omega \quad R_4 = 15 \text{ } \Omega$$

$$E_2 = 15 \text{ V} \quad R_2 = 10 \text{ } \Omega \quad R_5 = 6 \text{ } \Omega$$

$$A = 10 \text{ A} \quad R_3 = 4 \text{ } \Omega$$

$$L = 1 \text{ mH}$$

L'interruttore S è aperto da tempo infinito e viene chiuso all'istante $t = 0$.

Determinare:

- l'andamento nel tempo della corrente $i_{R3}(t)$, formula e andamento grafico.

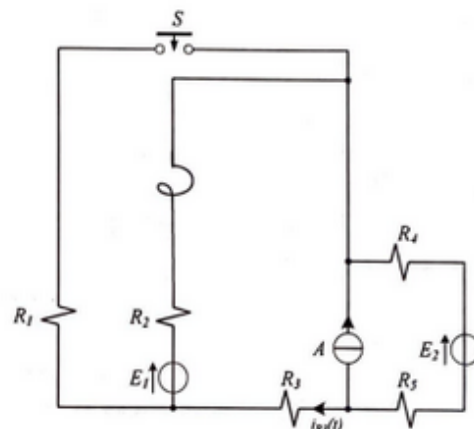


Fig. 1.

Solution

10.7.2 Harmonic regime of linear electrical grids

Exercise 10.7.6 (Exam 2025-02-11, Exercise 2.)

2) Il circuito di Figura 2, in regime alternato sinusoidale alla frequenza di 50Hz, è così assegnato:

$$L_1 = 250 \text{ mH}$$

$$C_1 = 350 \text{ } \mu\text{F}$$

$$R_1 = 20 \text{ } \Omega$$

$$\bar{Z}_1 = 1 + j2 \text{ } \Omega$$

$$\bar{Z}_2 = 2 - j2 \text{ } \Omega$$

$$\bar{Z}_3 = 3 + j4 \text{ } \Omega$$

$$e_1(t) = 50\sqrt{2} \cos(\omega t) \text{ V}$$

$$e_2(t) = 10\sqrt{2} \cos(\omega t + \pi) \text{ V}$$

$$a(t) = 3\sqrt{2} \sin(\omega t + \pi/2) \text{ A}$$

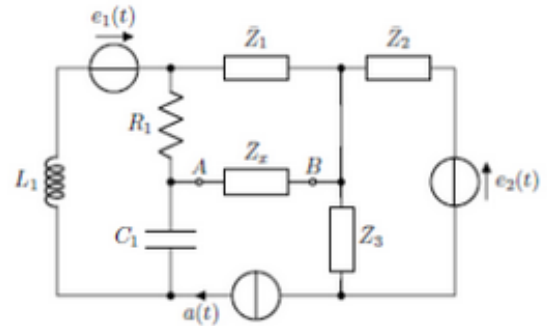


Fig. 2.

Determinare:

- il valore dell'impedenza \bar{Z}_x che garantisca il massimo trasferimento di potenza attiva
- le potenze attiva P_{Zx} e reattiva Q_{Zx} assorbite dall'impedenza \bar{Z}_x

Solution

First *one-port equivalent Thevenin circuit* of the circuit with port $A - B$ is evaluated, then *power flow in harmonic regime* is discussed.

Thevenin equivalent: voltage. With open circuit in $A - B$, current a flows in the lower branch and in impedance Z_1 . Clockwise loop currents i_1 and i_2 flows in the left and right loop respectively. Kirchhoff voltage laws in the left and right loops give

$$\begin{aligned} 0 &= e_1 - Z_L(i_1 + a) - (R_1 + Z_C)i_1 \\ 0 &= -e_2 - Z_2(i_2 + a) - Z_3i_2 \end{aligned} \quad \rightarrow \quad \begin{aligned} i_1 &= \frac{e_1 - Z_L a}{Z_L + Z_C + R_1} \\ i_2 &= -\frac{e_2 + Z_2 a}{Z_2 + Z_3} \end{aligned}$$

and thus using Kirchhoff voltage law on the loop with nodes $A - B$ and closing through Z_1 and R_1 ,

$$V_{Th} = R_1 i_1 + Z_1 a = \dots$$

Thevenin equivalent: impedance. Opening circuit at the current generator, and replace tension generators with short circuits, the equivalent impedance is

$$Z_{Th} = ((Z_C + Z_L) \parallel R_1) + Z_1.$$

Equivalent circuit. Kirchhoff voltage law on the equivalent circuit reads

$$0 = V_{Th} - Z_{Th} i - Z_x i = 0,$$

and thus

$$I = \frac{V_{Th}}{Z_{Th} + Z_x} = \dots$$

Power. Complex power reads

$$S = VI^* = Z_x |I|^2 = \frac{Z_x}{|Z_{Th} + Z_x|^2} |V_{Th}|^2 ,$$

Writing the impedance as $Z_x = R_x + iX_x$, the active power reads

$$P = \frac{R_x}{(R_{Th} + R_x)^2 + (X_{Th} + X_x)^2} |V_{Th}|^2 .$$

With the physical constraints $R \geq 0$, the problem is a constrained optimization problem of finding the maximum value of the function $P(R_x, X_x)$ subject to the constraint $R_x \geq 0$,

$$\text{find } \max_{R_x, X_x} P(R_x, X_x) \quad \text{s.t.} \quad R_x \geq 0 .$$

The denominator is the sum of two non negative terms, one function of R_x and one function of X_x . The independent variable X_x only appears in this term at the denominator, so that this term must vanish at the solution of the optimization problem, and thus

$$\tilde{X}_x = -X_{Th} .$$

The remaining term is a function of R_x only and proportional to

$$f(R_x) = \frac{R_x}{(R_{Th} + R_x)^2} .$$

Local extremes of this function is attained where

$$\begin{aligned} 0 = f'(R_x) &= \frac{(R_{Th} + R_x)^2 - 2R_x(R_{Th} + R_x)}{(R_{Th} + R_x)^4} = \\ &= \frac{R_{Th}^2 - R_x^2}{(R_{Th} + R_x)^4} \end{aligned}$$

and thus, within the physical limit of the problem, the local and global maximum of the function (check that $f''(\tilde{R}_x) < 0$), is attained for

$$\begin{aligned} \tilde{R}_x &= R_{Th} \\ \tilde{Z}_x &= R_{Th} - iX_{Th} \end{aligned}$$

and the maximum active power is

$$P_{max} = P(\tilde{Z}_x) = \frac{|V_{Th}|^2}{4R_{Th}} .$$

while the reactive power in this condition reads

$$Q = -\frac{X_{Th}}{4R_{Th}^2} |V_{Th}|^2 .$$

- 3) **SOLO ENERGETICI:** Il circuito di Figura 3, in regime sinusoidale alla $f=50\text{Hz}$, è così assegnato (tensione in valore efficace):

$$\begin{aligned} R_1 &= 1 \, \Omega \\ R_2 &= 10 \, \Omega \\ R_3 &= 2 \, \Omega \\ X_1 &= 400 \, \Omega \\ X_2 &= 100 \, \Omega \\ |\bar{V}_L| &= 400 \, \text{V} \\ A_L &= 3 \, \text{kVA} \\ \cos \phi_L &= 0.75 \, \text{ind.} \end{aligned}$$

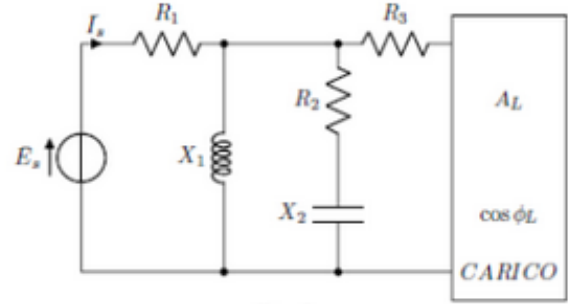


Fig. 3.

Determinare:

- il valore efficace della tensione del generatore \bar{E}_s
- il valore efficace della corrente \bar{I}_s
- il fattore di potenza associato al generatore E_s , cioè lo sfasamento tra E_s e I_s

Solution

First *power flow in harmonic regime* is used to calculate load impedance, then the electrical circuit is solved, and the power on the tension generator is computed.

Load impedance Z_L . Load impedance appears in the load constitutive equation $V_L = Z_L I_L$, and can be evaluated from data about complex power,

$$S_L = |S_L| e^{i\phi_L} = V_L I_L^* = Z_L |I|^2 = \frac{1}{Z_L^*} |V_L|^2$$

$$Z_L = \frac{|V_L|^2}{|S_L|} e^{i\phi_L}$$

Current I_s . From data of load power, it's possible to evaluate the current I_s . The current I_L through the load reads

$$S_L = V_L I_L^* \quad \rightarrow \quad I_L = \frac{S_L^*}{V_L^*} = \frac{|S_L|}{|V_L|} e^{i(-\phi_L + \phi_V)}$$

The three parallel sides act as current divider so that

$$I_L = \frac{(R_3 + Z_L)^{-1}}{(R_3 + Z_L)^{-1} + ((iX_1) \parallel (R_2 + iX_2))^{-1}} I_s$$

and thus

$$I_s = |I_s| e^{i\varphi_{I_s}} = \dots$$

Equivalent circuit. The impedance of the circuit powered by the tension generator is

$$Z_{eq} = R_1 + (iX_1 \parallel (R_2 + iX_2) \parallel (R_3 + Z_L)) .$$

Given the equivalent impedance, and the current I_s the voltage across the tension generator is

$$E_s = Z_{eq} I_s = |E_s| e^{i\varphi_{E_s}} \dots .$$

and the power factor is $\cos \varphi_s = \dots$, where

$$\varphi_s = \varphi_{E_s} - \varphi_{I_s} = \dots .$$

Exercise 10.7.8 (Exam 2025-01-22, Exercise 2.)

2) Il circuito di Figura 2, in regime alternato sinusoidale alla frequenza di 50Hz, è così assegnato:

$$\begin{array}{lll} \bar{E}_1 = 50e^{j\frac{\pi}{3}} \text{ V} & \bar{E}_2 = 100e^{j\frac{\pi}{6}} \text{ V} & \bar{A}_1 = 5e^{j\frac{\pi}{8}} \text{ A} \\ \bar{A}_2 = 10 \text{ A} & R_1 = 5 \Omega & L_1 = 50 \text{ mH} \\ C_1 = 0.1 \text{ mF} & L_2 = 15 \text{ mH} & R_2 = 10 \Omega \\ L_3 = 10 \text{ mH} & C_3 = 0.2 \text{ mF} & R_4 = 15 \Omega \\ L_4 = 20 \text{ mH} & L_5 = 30 \text{ mH} & C_5 = 0.3 \text{ mF} \end{array}$$

Determinare:

- l'espressione nel dominio del tempo della tensione ai capi del generatore di corrente \bar{A}_1
- le potenze complessa, apparente, attiva e reattiva messe in gioco da \bar{A}_1 (indicando esplicitamente: nome, simbolo e unità di misura)

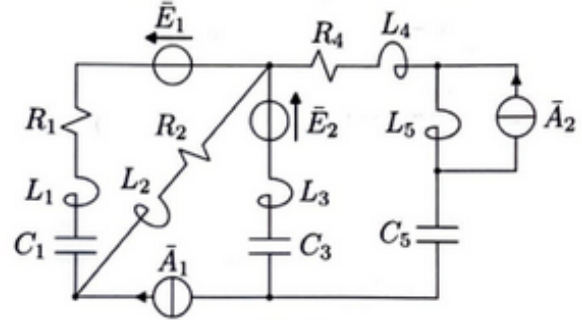


Fig. 2.

Solution

First *one-port equivalent Thevenin circuit* of the circuit with port $A - B$ is evaluated, then the equivalent circuit is solved to find the tension $v(t)$ across the current generator, and *power flow in harmonic regime* is discussed.

Thevenin equivalent: voltage. With an open circuit, the network can be split into two parts: the triangle in the upper-left side and the section in the right part.

In the triangular part, a current I_a flows in counter-clockwise direction, while current I_b flows in the right part in clockwise direction,

$$\begin{aligned} I_a &= \frac{E_1}{Z_1 + Z_2} \\ I_b &= \frac{E_2 + i\Omega L_5 A_2}{Z_4 + Z_5 + Z_3} \end{aligned}$$

as

$$E_2 + \left(Z_4 + Z_3 + \underbrace{-i\frac{1}{\Omega C_5} + i\Omega L_5}_{=Z_5} \right) I_b + i\Omega L_5 A_2 = 0.$$

with Z_k being the impedance of the k -th side. Thevenin voltage thus reads

$$V_{Th} = E_2 - Z_3 I_b + Z_2 I_a$$

Thevenin equivalent: impedance. Equivalent impedance reads

$$Z_{Th} = (Z_1 \parallel Z_2 + (Z_3 \parallel (Z_4 + Z_5)))$$

Equivalent circuit. Prescribed current A_1 flows in the equivalent circuit, and the voltage across the current generator is evaluated with Krichhoff voltage law

$$V_{A_1} - V_{Th} - Z_{Th} A_1 = 0,$$

$$V_{A_1} = V_{Th} + Z_{Th} A_1 = |V_{A_1}| e^{i\varphi_{V_{A_1}}}.$$

Signal in time is reconstructed using the relation between effective and maximum amplitude of the oscillation and evaluating the real part of the signal $|V_{A_1}| e^{i(\Omega t + \varphi_{V_{A_1}})}$

$$v_{A_1}(t) = \sqrt{2} |V_{A_1}| \cos(\Omega t + \varphi_{V_{A_1}}).$$

Poer. Using definitions of *power in circuits in harmonic regime*,

$$S_{A_1} = V_{A_1} I_{A_1}^*$$

$$|S_{A_1}| = |V_{A_1}| |I_{A_1}|$$

$$P_{A_1} = \text{re}\{S_{A_1}\}$$

$$Q_{A_1} = \text{im}\{S_{A_1}\}$$

Exercise 10.7.9 (Exam 2024-09-06, Exercise 2.)

2) Il circuito di Figura 2, in **regime alternato sinusoidale**, è così assegnato:

$$R = 10 \, \Omega$$

$$C = 550 \, \mu\text{F}$$

$$L = 350 \, \text{mH}$$

$$\bar{Z}_1 = 10 + j20 \, \Omega$$

$$\bar{Z}_2 = 5 - j5 \, \Omega$$

$$\bar{Z}_3 = 30 + j40 \, \Omega$$

$$e_1(t) = 150\sqrt{2} \cos(\omega t) \, \text{V}$$

$$e_2(t) = 100\sqrt{2} \cos(\omega t + \pi) \, \text{V}$$

$$a_1(t) = 5\sqrt{2} \sin(\omega t) \, \text{A}$$

$$a_2(t) = 3\sqrt{2} \sin(\omega t + \pi/2) \, \text{A}$$

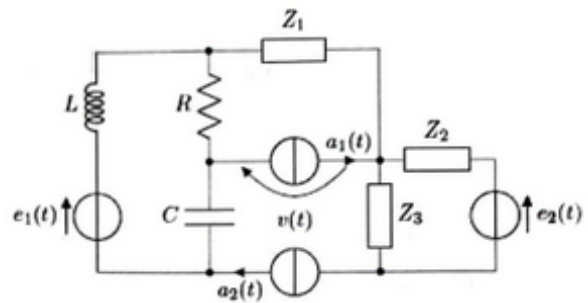


Fig. 2.

Determinare:

- l'espressione nel dominio del tempo della tensione $v(t)$
- le potenze complessa, apparente, attiva e reattiva messe in gioco dall'impedenza Z_1 (indicando esplicitamente: nome, simbolo e unità di misura)

Solution

Exercise 10.7.10 (Exam 2024-07-22, Exercise 2.)

Solution

2) Il circuito di Figura 2, in **regime alternato sinusoidale**, è così assegnato:

$$\begin{aligned} C_1 &= 5 \text{ mF} \\ R_2 &= 20 \, \Omega \\ L_3 &= 0,5 \text{ H} \\ e_8 &= 80 \cos(10t + \frac{3\pi}{4}) \text{ V} \\ e_9 &= 80 \cos(10t + \frac{\pi}{4}) \text{ V} \\ a_{10} &= 6 \cos(10t + \frac{\pi}{4}) \text{ A} \\ e_{11} &= 50\sqrt{2} \cos(10t + \pi) \text{ V} \\ R_4 &= 25 \, \Omega \\ L_5 &= 1 \text{ H} \\ R_6 &= 10 \, \Omega \\ a_{12} &= 3\sqrt{2} \cos(10t - \frac{\pi}{2}) \text{ A} \\ C_7 &= 10 \text{ mF} \end{aligned}$$

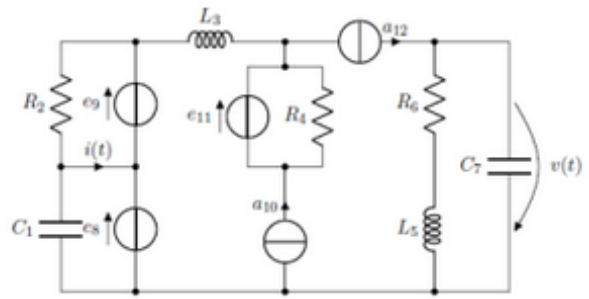


Fig. 2.

Determinare:

- l'espressione nel dominio del tempo della tensione $v(t)$
- il fasore associato alla corrente $i(t)$
- la potenza attiva generata da e_{11} ,
- la potenza complessa messa in gioco da a_{10}
- la potenza apparente elaborata da R_4 e dalle reattanze associate a L_3 e C_7

10.7.3 Three-phase electrical circuits in harmonic regime

Guidelines for solution

Analyse the network as a standard configuration of a three-phase network (*star-star*,...) and rely on results derived for *three-phase circuits*.

As an example, for a **star-star configuration**:

1. evaluate load impedances, impedances in parallel with the generators, interconnections between phases
2. evaluate voltage difference across the centers of the stars, v_{AB}
3. once v_{AB} is known, it should be easier to evaluate currents and voltages in the grid with KCL and KVL
4. use relations of *power in harmonic regime*, to answer the questions about power: just remember the difference between maximum and effective values, and that a wattmeter measures the active power

Exercise 10.7.11 (Exam 2024-09-06, Exercise 3.)

Solution

This network is a star-star connection with impedances

$$Z_g = (R_1 + sL_1) \parallel \frac{1}{sC_1} \quad g = 1 : 3$$

$$Z_4 = R_2 + \frac{1}{sC_2}$$

and inter-connection between phases 2 and 3 with impedance Z_4 .

Voltage v_{AB} .

$$v_{AB} = \frac{\sum_{g=1}^3 Y_g e_g}{\sum_{k=1}^4 Y_k}$$

3) Il circuito di Figura 3, in regime alternato sinusoidale alla frequenza di 50 Hz, è così assegnato:

$$e_1(t) = 220\sqrt{2} \cdot \cos(\omega t) \text{ V}$$

$$e_2(t) = 220\sqrt{2} \cdot \cos(\omega t + \frac{2}{3}\pi) \text{ V}$$

$$e_3(t) = 220\sqrt{2} \cdot \cos(\omega t + \frac{4}{3}\pi) \text{ V}$$

$$f = 50 \text{ Hz}$$

$$R_1 = 25\Omega$$

$$R_2 = 2k\Omega$$

$$C_1 = 100 \mu\text{F}$$

$$C_2 = 1kF$$

$$L_1 = 50 \text{ mH}$$

$$Z_4 = (10 - j5)\Omega$$

Determinare:

- Le correnti I_{Z_4} e I_{Z_2}
- La potenza complessa, apparente, attiva e reattiva elaborata dal generatore E_2 , indicando esplicitamente: nome, simbolo e unità di misura e discutendone il segno.

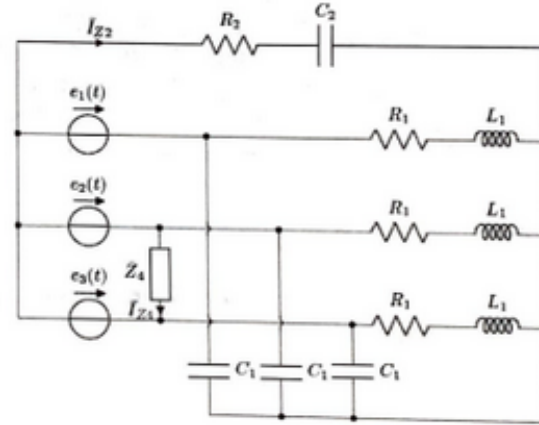


Fig. 3.

Generation and loads are equilibrated, and thus $\sum_{g=1}^3 Y_g e_g = 0$, and $v_{AB} = 0$.

Current i_{Z_2} . As $v_{AB} = 0$, then $i_{Z_2} = 0$, as in general it would be $i_{Z_2} = \frac{v_{AB}}{R_2 + \frac{1}{sC_2}}$.

Current i_{Z_4} . With KVL on the loop with the two tension generators e_2, e_3 closed with Z_4

$$0 = e_3 + Z_4 i_{Z_4} - e_2$$

$$\rightarrow i_{Z_4} = \frac{e_2 - e_3}{Z_4}$$

Currents i_{e_2} . Current i_{e_2} through the generator are evaluated through KVL between the centers of the stars,

$$0 = e_2 - \frac{1}{\frac{1}{R_1 + sL_1} + sC_1} i_{e_2} - v_{AB}$$

$$\rightarrow i_{e_2} = \left[\frac{1}{R_1 + sL_1} + sC_1 \right] e_2$$

Powers of generator 2.

$$S_2 = V_2 I_2^*$$

$$A_2 = |S_2|$$

$$P_2 = \text{re}\{S_2\}$$

$$Q_2 = \text{im}\{S_2\},$$

using the effective values of tension and current V_2, I_2 .

Exercise 10.7.12 (Exam 2024-07-22, Exercise 3.)

3) Il circuito di Figura 3, in **regime alternato sinusoidale alla frequenza di 50 Hz**, è così assegnato:

$E_1 = 200 \text{ V}$	$R_1 = 40 \Omega$	$C_1 = 100 \mu\text{F}$
$E_2 = 200 \text{ V}$	$R_2 = 50 \Omega$	$C_2 = 150 \mu\text{F}$
$E_3 = 200 \text{ V}$	$R_3 = 50 \Omega$	$L_1 = 15 \text{ mH}$
	$R_4 = 40 \Omega$	$L_2 = 10 \text{ mH}$

Determinare:

- l'indicazione del wattmetro
- La potenza complessa messa in gioco da C_2

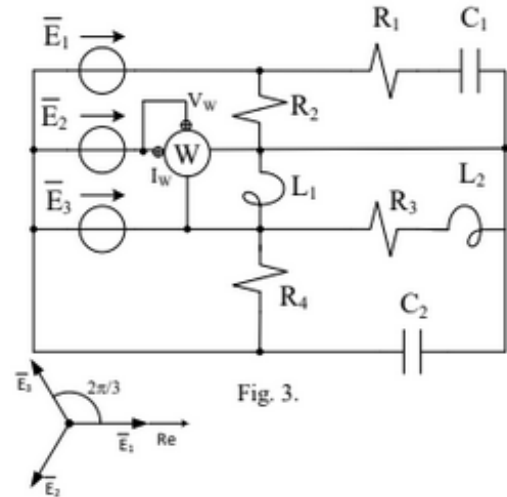


Fig. 3.

Solution

This network is a star-star connection with impedances

$$Z_1 = (R_1 + jX_{C_1}) \parallel R_2$$

$$Z_2 = 0$$

$$Z_3 = (R_3 + jX_{L_2}) \parallel jX_{L_1}$$

$$Z_4 = jX_{C_2}$$

and inter-connection between phase 3 and the neutral with **resistance** R_4 , before Z_4 , and thus **in parallel with the generator 3**.

Voltage v_{AB} . As $Z_2 = 0$, it's not possible to directly use

$$v_{AB} = \frac{\sum_{g=1}^3 Y_g e_g}{\sum_{k=1}^4 Y_k},$$

or this must be used with the limit $Y_2 \rightarrow +\infty$, and thus

$$v_{AB} = e_2.$$

Wattmeter tension v_W . KVL with the generators 2 and 3,

$$v_W = e_2 - e_3.$$

Wattmeter current $i_w = i_{e_2}$. KCL on the center of generation star, $0 = i_{e_1} + i_{e_2} + i_3 + i_4$, with

$$i_{e_1} = \frac{1}{Z_1}(e_1 - v_{AB})$$

$$i_3 = \frac{1}{Z_3}(e_3 - v_{AB})$$

$$i_4 = -\frac{1}{Z_4}v_{AB},$$

being $i_3 = i_{e_3} + i_{R_4}$ the sum of the current in the parallel connection on the branch 3 of the generation. Thus, current i_{e_2} reads

$$\begin{aligned} i_{e_2} &= -i_{e_1} - i_3 - i_4 = \\ &= -\frac{e_1}{Z_1} - \frac{e_3}{Z_3} + \left(\frac{1}{Z_1} + \frac{1}{Z_3} + \frac{1}{Z_4} \right) v_{AB} \end{aligned}$$

Wattmeter. Wattmeter reading provides the active power

$$P_w = \operatorname{re}\{S_w\} = \operatorname{re}\{v_w i_w^*\}.$$

Power on C_2 . Current and voltage across C_2 are

$$\begin{aligned} i_{C_2} &= i_4 \\ v_{C_2} &= Z_{C_2} i_{C_2} = \frac{1}{sC_2} i_{C_2}, \end{aligned}$$

and the complex power is

$$s = V_{C_2} I_{C_2}^*.$$

Exercise 10.7.13 (Exam 2024-06-19, Exercise 1.)

riguarda RIPROVATO e ORALE.

1) Il circuito di Figura 1, in **regime alternato sinusoidale alla frequenza di 50 Hz**, è così assegnato:

$E_1 = 230 \text{ V}$	$R_1 = 20 \Omega$	$C_1 = 100 \mu\text{F}$
$E_2 = 230 \text{ V}$	$R_2 = 30 \Omega$	$C_2 = 100 \mu\text{F}$
$E_3 = 230 \text{ V}$	$R_3 = 60 \Omega$	$L_1 = 20 \text{ mH}$
	$R_4 = 30 \Omega$	$L_2 = 15 \text{ mH}$
		$L_3 = 15 \text{ mH}$

Determinare:

- L'indicazione del wattmetro
- le potenze attiva, reattiva, apparente e complessa erogate dal generatore E_1 (esplicitando le unità di misura e discutendone il segno).

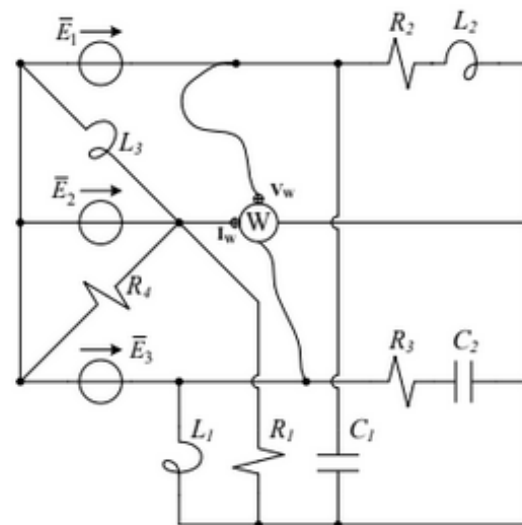
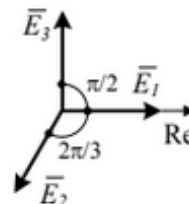


Fig. 1.



Solution

This network is a star-star connection with impedances

$$Z_1 = (R_2 + jX_{L_2}) \parallel (jX_{C_1})$$

$$Z_2 = (R_1 \parallel 0)$$

$$Z_3 = (R_3 + jX_{C_2}) \parallel jX_{L_1}$$

with L_2 and R_4 in parallel with generator e_2 . As R_1 is in parallel with a short-circuit in Z_2 , this impedance is zero and as it is the current through R_1 . There's no neutral.

Voltage v_{AB} . As $Z_2 = 0$ (see previous exercise), the voltage between the centers of the stars is

$$v_{AB} = e_2 .$$

Wattmeter tension v_W . KVL with the generators 2 and 3,

$$v_W = e_1 - e_3 .$$

Wattmeter current $i_w = i_2$. KCL on the center of generation star, $0 = i_{e_1} + i_2 + i_{e_3}$, with

$$i_{e_1} = \frac{1}{Z_1}(e_1 - e_2)$$

$$i_{e_3} = \frac{1}{Z_3}(e_3 - e_2)$$

being $i_2 = i_{e_2} + i_{L_1} + i_{R_4}$ the sum of the current in the parallel connection on the branch 2 of the generation. Thus, current i_w reads

$$\begin{aligned} i_w = i_2 &= -i_{e_1} - i_{e_3} = \\ &= \frac{1}{Z_1}(e_2 - e_1) + \frac{1}{Z_3}(e_2 - e_3) \end{aligned}$$

Wattmeter. Wattmeter reading provides the active power

$$P_w = \text{re}\{S_w\} = \text{re}\{v_w i_w^*\} .$$

Power of tension generator e_1 .

$$s_{e_1} = e_2 i_{e_2}^* .$$

...

Exercise 10.7.14 (Exam 2024-02-13, Exercise 2.)

Solution

1classical-electromagnetism:electrical-engineering-exercises:electromagnetic)=

2) Il circuito di Figura 2, in regime alternato sinusoidale alla frequenza di 50 Hz, è così assegnato:

$$\begin{array}{lll} E_1 = 230 \text{ V} & R_1 = 40 \, \Omega & C_1 = 100 \, \mu\text{F} \\ E_2 = 230 \text{ V} & R_2 = 50 \, \Omega & C_2 = 150 \, \mu\text{F} \\ E_3 = 230 \text{ V} & R_3 = 50 \, \Omega & L_3 = 15 \text{ mH} \\ & R_4 = 40 \, \Omega & \\ & R_5 = 30 \, \Omega & \\ & R_6 = 20 \, \Omega & \end{array}$$

La terna degli ingressi, assegnati in valore efficace, è simmetrica diretta.

Determinare:

- L'indicazione del wattmetro.

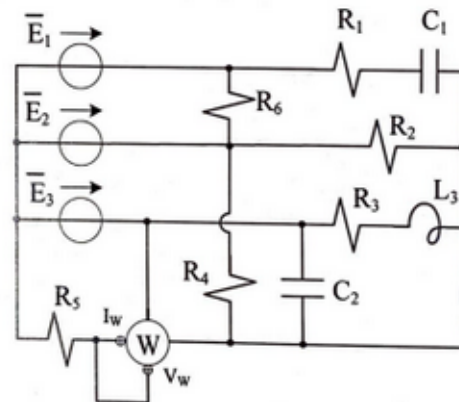


Fig. 2.

10.7.4 Electromagnetic circuits

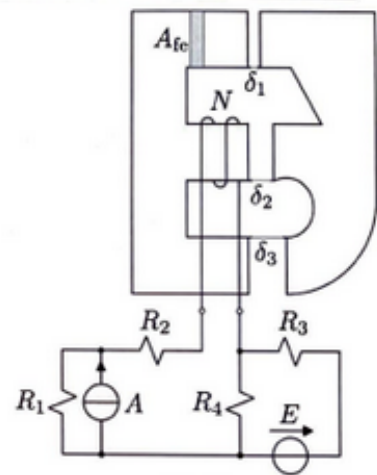
Exercise 10.7.15 (Exam 2025-01-22, Exercise 3.)

3) **SOLO ENERGETICI:** Il circuito di Figura 3, in regime stazionario, è così assegnato:

$$\begin{array}{llll} N = 100 & \delta_1 = 1 \text{ mm} & \delta_2 = 2 \text{ mm} & \delta_3 = 3 \text{ mm} \\ A_{fe} = 10 \text{ cm}^2 & R_1 = 5 \, \Omega & R_2 = 3 \, \Omega & R_3 = 2 \, \Omega \\ R_4 = 1 \, \Omega & A = 5 \text{ A} & E = 30 \text{ V} & \end{array}$$

Determinare:

- L'induttanza associata al circuito magnetico;
- l'energia accumulata complessivamente nel campo magnetico;
- le potenze associate ad ogni resistore;
- le potenze associate ad ogni generatore, discutendone il segno.



A) **SOLO ENERGETICI:** Trasformatore

Exercise 10.7.16 (Exam 2024-06-19, Exercise 2.)

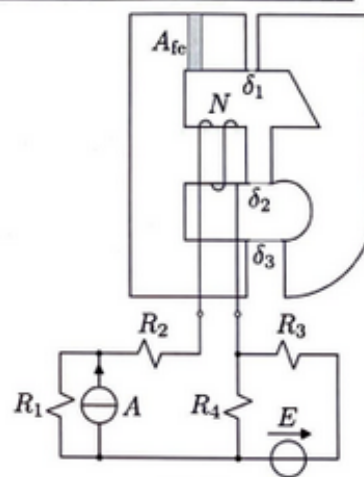
Exercise 10.7.17 (Exam 2024-02-13, Exercise 1a.)

3) **SOLO ENERGETICI:** Il circuito di Figura 3, in **regime stazionario**, è così assegnato:

$$\begin{array}{llll} N = 100 & \delta_1 = 1 \text{ mm} & \delta_2 = 2 \text{ mm} & \delta_3 = 3 \text{ mm} \\ A_{fe} = 10 \text{ cm}^2 & R_1 = 5 \, \Omega & R_2 = 3 \, \Omega & R_3 = 2 \, \Omega \\ R_4 = 1 \, \Omega & A = 5 \text{ A} & E = 30 \text{ V} & \end{array}$$

Determinare:

- L'induttanza associata al circuito magnetico;
- l'energia accumulata complessivamente nel campo magnetico;
- le potenze associate ad ogni resistore;
- le potenze associate ad ogni generatore, discutendone il segno.



4) **SOLO ENERGETICI:** Trasformatore

1) **SOLO ENERGETICI** Il circuito di Figura 1, con ingressi stazionari, è così assegnato:

$$\begin{array}{lll} E_1 = 20 \text{ V} & R_1 = 5 \, \Omega & R_4 = 15 \, \Omega \\ E_2 = 15 \text{ V} & R_2 = 10 \, \Omega & R_5 = 6 \, \Omega \\ A = 10 \text{ A} & R_3 = 4 \, \Omega & \\ A_{fe} = 1 \text{ cm}^2 & \delta_1 = 1 \text{ mm} & \\ N = 100 & \delta_2 = 2 \text{ mm} & \\ \mu_{fe} = \infty & & \end{array}$$

L'interruttore S è aperto da tempo infinito e viene chiuso all'istante $t = 0$.

Determinare:

- l'andamento nel tempo della corrente $i_{R3}(t)$, formula e andamento grafico.

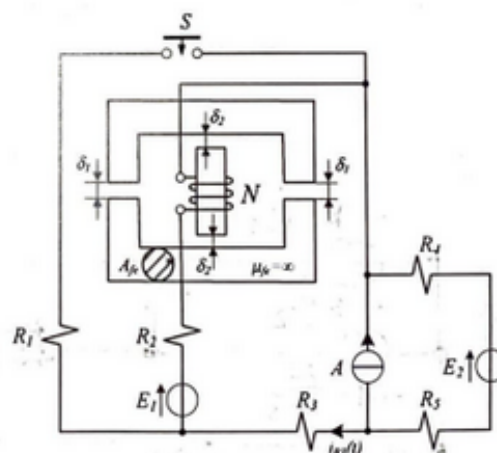


Fig. 1.

Parte III

Numerical Methods

basics

26 apr 2025

1 min read

11.1 Poisson equation

General Poisson's problem

$$\begin{cases} -\nabla^2 \mathbf{u}(\mathbf{r}, t) = \mathbf{f}(\mathbf{r}, t) \\ + \text{b.c.} \end{cases}$$

with common boundary conditions

$$\begin{cases} \mathbf{u} = \mathbf{g} & \text{on } S_D \\ \hat{\mathbf{n}} \cdot \nabla \mathbf{u} = \mathbf{h} & \text{on } S_N \end{cases}$$

over Dirichlet and Neumann regions of the boundary.

Poisson's problem for Green's function, in infinite domain

$$-\nabla_{\mathbf{r}}^2 G(\mathbf{r}; \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0)$$

Green's function method

$$\begin{aligned} E(\mathbf{r}_0, t) u_i(\mathbf{r}_0, t) &= \int_{\mathbf{r} \in \Omega} u_i(\mathbf{r}, t) \delta(\mathbf{r} - \mathbf{r}_0) = \\ &= - \int_{\mathbf{r} \in \Omega} u_i(\mathbf{r}, t) \nabla_{\mathbf{r}}^2 G(\mathbf{r} - \mathbf{r}_0) = \\ &= - \int_{\mathbf{r} \in \Omega} \nabla_{\mathbf{r}} \cdot (u_i \nabla_{\mathbf{r}} G - G \nabla_{\mathbf{r}} u_i) - \int_{\mathbf{r} \in \Omega} G \nabla^2 u_i = \\ &= - \oint_{\mathbf{r} \in \partial \Omega} \hat{\mathbf{n}} \cdot (u_i \nabla_{\mathbf{r}} G - G \nabla_{\mathbf{r}} u_i) + \int_{\mathbf{r} \in \Omega} G(\mathbf{r} - \mathbf{r}_0) f_i(\mathbf{r}, t). \end{aligned}$$

An integro-differential boundary problem can be written using boundary conditions. As an example, using Dirichlet and Neumann boundary conditions, the integro-differential problem reads

$$\begin{aligned} E(\mathbf{r}_0, t) \mathbf{u}(\mathbf{r}_0, t) + \int_{\mathbf{r} \in S_N} \mathbf{u}(\mathbf{r}, t) \hat{\mathbf{n}} \cdot \nabla_{\mathbf{r}} G(\mathbf{r} - \mathbf{r}_0) - \int_{\mathbf{r} \in S_D} G(\mathbf{r} - \mathbf{r}_0) \hat{\mathbf{n}} \cdot \nabla_{\mathbf{r}} \mathbf{u}(\mathbf{r}, t) = \\ = - \int_{\mathbf{r} \in S_D} \mathbf{g}(\mathbf{r}, t) \hat{\mathbf{n}} \cdot \nabla_{\mathbf{r}} G(\mathbf{r} - \mathbf{r}_0) + \int_{\mathbf{r} \in S_N} G(\mathbf{r} - \mathbf{r}_0) \mathbf{h}(\mathbf{r}, t) + \int_{\mathbf{r} \in \Omega} G(\mathbf{r} - \mathbf{r}_0) \mathbf{f}(\mathbf{r}, t). \end{aligned}$$

Green's function of the Poisson-Laplace equation reads

$$G(\mathbf{r}; \mathbf{r}_0) = \frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}_0|}.$$

Green's function of the Laplace equation

$$-\nabla^2 G = 0 \quad \text{for } \mathbf{r} \neq \mathbf{r}_0$$

Solutions with spherical symmetry,

$$0 = \nabla^2 G = \frac{1}{r^2} (r^2 G')' \rightarrow G'(r) = \frac{A}{r^2} \rightarrow G(r) = -\frac{A}{r} + B$$

Choosing $B = 0$ s.t. $G(r) \rightarrow 0$ as $r \rightarrow \infty$, and integrating over a sphere centered in $r = 0$ to get $A = -\frac{1}{4\pi}$,

$$1 = \int_V \delta(r) = - \int_V \nabla^2 G = - \oint_{\partial V} \hat{\mathbf{n}} \cdot \nabla G = - \oint_{\partial V} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \frac{A}{r^2} = -4\pi A$$

11.2 Helmholtz equation

todo from Fourier to Laplace transform in the first lines of this section

A Helmholtz's equation can be thought as the time Fourier transform of a wave equation,

$$\begin{cases} \frac{1}{c^2} \partial_{tt} \mathbf{u}(\mathbf{r}, t) - \nabla^2 \mathbf{u}(\mathbf{r}, t) = \mathbf{f}(\mathbf{r}, t) \\ + \text{b.c.} \\ + \text{i.c.} \end{cases}$$

Fourier transform in time of field $\mathbf{u}(\mathbf{r}, t)$ reads

$$\tilde{\mathbf{u}}(\mathbf{r}, \omega) = \mathcal{F}\{\mathbf{u}(\mathbf{r}, t)\} = \int_{t=-\infty}^{+\infty} \mathbf{u}(\mathbf{r}, t) e^{-i\omega t} d\omega$$

and, if $\mathbf{u}(\mathbf{r}, t)$ is compact in time, Fourier transform of its time partial derivatives read

$$\begin{aligned} \mathcal{F}\{\dot{\mathbf{u}}(\mathbf{r}, t)\} &= \int_{t=-\infty}^{+\infty} \dot{\mathbf{u}}(\mathbf{r}, t) e^{-i\omega t} d\omega = \\ &= \mathbf{u}(\mathbf{r}, t) e^{-i\omega t} \Big|_{t=-\infty}^{+\infty} + i\omega \int_{t=-\infty}^{+\infty} \mathbf{u}(\mathbf{r}, t) e^{-i\omega t} d\omega = \\ &= i\omega \mathcal{F}\{\mathbf{u}(\mathbf{r}, t)\} \\ \mathcal{F}\{\partial_t^n \mathbf{u}(\mathbf{r}, t)\} &= (i\omega)^n \tilde{\mathbf{u}}. \end{aligned}$$

The differential problem in the transformed domain thus reads

$$-\frac{\omega^2}{c^2} \tilde{\mathbf{u}} - \nabla^2 \tilde{\mathbf{u}} = \tilde{\mathbf{f}}$$

Green's function of Helmholtz's equation reads

$$G(\mathbf{r}, s) = \alpha^+ \frac{e^{\frac{s|\mathbf{r}-\mathbf{r}_0|}{c}}}{|\mathbf{r}-\mathbf{r}_0|} + \alpha^- \frac{e^{-\frac{s|\mathbf{r}-\mathbf{r}_0|}{c}}}{|\mathbf{r}-\mathbf{r}_0|}$$

with $\alpha^+ + \alpha^- = \frac{1}{4\pi}$.

Being the Laplace transform,

$$\mathcal{L}\{f(t)\} = \int_{t=0^-}^{+\infty} f(t) e^{-st} dt,$$

the Laplace transform of a causal function with time delay $\tau \geq 0$ reads

$$\mathcal{L}\{f(t-\tau)\} = \int_{t=0^-}^{+\infty} f(t-\tau) e^{-st} dt = \int_{z=-\tau}^{+\infty} f(z) e^{-s(z+\tau)} dz = e^{-s\tau} \int_{z=0}^{+\infty} f(z) e^{-sz} dz = e^{-s\tau} \mathcal{L}\{f(t)\}$$

having used causality $f(t) = 0$ for $t < 0$. Laplace transform of Dirac's delta $\delta(t)$ reads

$$\mathcal{L}\{\delta(t)\} = \int_{t=0^-}^{+\infty} \delta(t) dt = 1,$$

so that $e^{-s\tau} = e^{-s\tau} 1 = \mathcal{L}\{\delta(t-\tau)\}$.

Thus, Green's function for the wave equation reads

$$G(\mathbf{r}, t; \mathbf{r}_0, t_0) = \alpha^+ \frac{\delta\left(t - t_0 + \frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right)}{|\mathbf{r}-\mathbf{r}_0|} + \alpha^- \frac{\delta\left(t - t_0 - \frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right)}{|\mathbf{r}-\mathbf{r}_0|}$$

If $t \geq t_0$, and $G(\mathbf{r}, t; \mathbf{r}_0, t_0)$ connects the past t_0 with the future t , the first term is not causal, and thus $\alpha^+ = 0$ and

$$G(\mathbf{r}, t; \mathbf{r}_0, t_0) = \frac{1}{4\pi} \frac{\delta\left(t - t_0 - \frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right)}{|\mathbf{r}-\mathbf{r}_0|}.$$

Green's function of Helmholtz's equation

$$\frac{s^2}{c^2} G - \nabla^2 G = \delta(r)$$

$$G(r) = \frac{\alpha e^{kr} + \beta e^{-kr}}{r}$$

Proof:

- Gradient

$$\nabla G(r) = \hat{\mathbf{r}} \partial_r G = \hat{\mathbf{r}} \frac{\alpha(kr - 1)e^{kr} + \beta(-kr - 1)e^{-kr}}{r^2}$$

- Laplacian

$$\begin{aligned}
 \nabla^2 G(r) &= \frac{1}{r^2} (r^2 G'(r))' = \\
 &= \frac{1}{r^2} (\alpha(kr - 1)e^{kr} + \beta(-kr - 1)e^{-kr})' = \\
 &= \frac{1}{r^2} (\alpha ke^{kr} + \alpha k^2 r e^{kr} - \alpha k e^{kr} - \beta k e^{-kr} + \beta k^2 r e^{-kr} + \beta k e^{-kr}) = \\
 &= \frac{1}{r} (\alpha e^{kr} + \beta e^{-kr}) k^2 = k^2 G(r) .
 \end{aligned}$$

and thus $k^2 G(r) - \nabla^2 G = 0$, for $r \neq 0$;

- Unity

$$1 = \int_V \delta(r) = \int_V (k^2 G - \nabla^2 G) = \int_V k^2 G - \oint_{\partial V} \hat{\mathbf{n}} \cdot \nabla G$$

the second term is the sum of two contributions of the form

$$\oint_{\partial V} \hat{\mathbf{n}} \cdot \nabla G^\pm = \oint_{\partial V} \frac{\alpha^\pm (\pm kr - 1) e^{\pm kr}}{r^2} = 4\pi \alpha^\pm (\pm kr - 1) e^{\pm kr}$$

the first term is the sum of two contributions of the form

$$\begin{aligned}
 k^2 \int_V G(r) &= k^2 \int_V \frac{\alpha^\pm e^{\pm kr}}{r} = \\
 &= k^2 \alpha^\pm \int_{R=0}^r \int_{\phi=0}^\pi \int_{\theta=0}^{2\pi} \frac{e^{\pm kR}}{R} R^2 \sin \phi \, dR \, d\phi \, d\theta = \\
 &= k^2 \alpha^\pm 4\pi \int_{R=0}^r R e^{\pm kR} \, dR .
 \end{aligned}$$

the last integral can be evaluated with integration by parts

$$\begin{aligned}
 \int_{R=0}^r R e^{\pm kR} \, dR &= \left[\frac{1}{\pm k} e^{\pm kR} R \right]_{R=0}^r \mp \frac{1}{k} \int_{R=0}^r e^{\pm kR} \, dR = \\
 &= \frac{1}{\pm k} e^{\pm kr} r - \frac{1}{k^2} e^{\pm kR} + \frac{1}{k^2} =
 \end{aligned}$$

Thus summing everything together,

$$\begin{aligned}
 1 &= \alpha^+ \left[4\pi k^2 \left(\frac{r}{k} e^{kr} - \frac{1}{k^2} e^{kr} + \frac{1}{k^2} \right) - 4\pi (kr - 1) e^{kr} \right] + \alpha^- [\dots] = \\
 &= 4\pi (\alpha^+ + \alpha^-) .
 \end{aligned}$$

11.3 Wave equation

Wave equation general problem

$$\begin{cases} \frac{1}{c^2} \partial_{tt} \mathbf{u}(\mathbf{r}, t) - \nabla^2 \mathbf{u}(\mathbf{r}, t) = \mathbf{f}(\mathbf{r}, t) \\ + \text{b.c.} \\ + \text{i.c.} \end{cases}$$

Green's problem of the wave equation

$$\frac{1}{c^2} \partial_{tt} G(\mathbf{r}, t; \mathbf{r}_0, t_0) - \nabla_{\mathbf{r}}^2 G(\mathbf{r}, t; \mathbf{r}_0, t_0) = \delta(\mathbf{r} - \mathbf{r}_0) \delta(t - t_0)$$

Integration by parts

$$\begin{aligned}
 E(\mathbf{r}_\alpha, t_\alpha) \mathbf{u}(\mathbf{r}_\alpha, t_\alpha) &= \int_{t \in T} \int_{\mathbf{r} \in V} \delta(t - t_\alpha) \delta(\mathbf{r} - \mathbf{r}_\alpha) \mathbf{u}(\mathbf{r}, t) = \\
 &= \int_{t \in T} \int_{\mathbf{r} \in V} \left\{ \frac{1}{c^2} \partial_{tt} G - \nabla_{\mathbf{r}}^2 G \right\} \mathbf{u} = \\
 &= \int_{t \in T} \int_{\mathbf{r} \in V} \left\{ \frac{1}{c^2} [\partial_t (\mathbf{u} \partial_t G - G \partial_t \mathbf{u}) + G \partial_{tt} \mathbf{u}] - \nabla_{\mathbf{r}} \cdot (\nabla_{\mathbf{r}} G \mathbf{u} - G \nabla_{\mathbf{r}} \mathbf{u}) - G \nabla_{\mathbf{r}}^2 \mathbf{u} \right\} = \\
 &= \int_{\mathbf{r} \in V} \frac{1}{c^2} [\mathbf{u}(\mathbf{r}, t) \partial_t G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) - G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) \partial_t \mathbf{u}(\mathbf{r}, t)] \Big|_{t_0}^{t_1} + \\
 &\quad + \int_{t \in T} \oint_{\mathbf{r} \in \partial V} \{ -\hat{\mathbf{n}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{r}} G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) \mathbf{u}(\mathbf{r}, t) + G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) \hat{\mathbf{n}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{r}} \mathbf{u}(\mathbf{r}, t) \} + \\
 &\quad + \int_{t \in T} \int_{\mathbf{r} \in V} G(\mathbf{r}, t; \mathbf{r}_\alpha, t_\alpha) \underbrace{\left\{ \frac{1}{c^2} \partial_{tt} \mathbf{u}(\mathbf{r}, t) - \nabla_{\mathbf{r}}^2 \mathbf{u}(\mathbf{r}, t) \right\}}_{=\mathbf{f}(\mathbf{r}, t)} \\
 \int_{t \in T} \int_{\mathbf{r} \in V} \frac{1}{4\pi} \frac{\delta(t - t_\alpha + \frac{|\mathbf{r} - \mathbf{r}_\alpha|}{c})}{|\mathbf{r} - \mathbf{r}_\alpha|} \mathbf{f}(\mathbf{r}, t) &= \int_{\mathbf{r} \in V \cap B_{|\mathbf{r} - \mathbf{r}_\alpha| \leq c(t_\alpha - t)}} \frac{1}{4\pi |\mathbf{r} - \mathbf{r}_\alpha|} \mathbf{f}\left(\mathbf{r}, t_\alpha - \frac{|\mathbf{r} - \mathbf{r}_\alpha|}{c}\right)
 \end{aligned}$$

basics

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1 min read

12.1 Elettrostatica

I problemi dell'elettrostatica sono governate dalle due equazioni di Maxwell per i campi \mathbf{e} , \mathbf{d} ,

$$\begin{cases} \nabla \cdot \mathbf{d} = \rho \\ \nabla \times \mathbf{e} = \mathbf{0} \end{cases},$$

dotate delle opportune condizioni al contorno ed equazioni costitutive. Per un materiale lineare isotropo, ad esempio, $\mathbf{d} = \varepsilon \mathbf{e}$. La condizione di irrotazionalità del campo elettrico, permette di scriverlo come gradiente di un potenziale scalare, $\mathbf{e} = -\nabla v$, e di ottenere l'equazione di Poisson,

$$-\nabla \cdot (\varepsilon \nabla v) = \rho .$$

12.1.1 Sorgente

$$\mathbf{e}(r) = \frac{q_i}{4\pi\varepsilon} \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}$$

$$\mathbf{e}(\mathbf{r}) = -\nabla_{\mathbf{r}} v(\mathbf{r})$$

$$\varepsilon v(\mathbf{r}) = \frac{q_i}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}_i|}$$

12.1.2 Dipolo

Un dipolo è definito come due cariche di intensità uguale e contraria $-q_2 = q_1 = q > 0$, nei punti dello spazio $P_1, P_2 = P_1 + \mathbf{l}$, nelle condizioni limite $|\mathbf{l}| \rightarrow 0, q \rightarrow \infty$, in modo tale da avere $q|\mathbf{l}|$ finito, $\mathbf{p} = q\mathbf{l}$.

Il potenziale del dipolo è dato dal principio di sovrapposizione delle cause e degli effetti,

$$\begin{aligned} \varepsilon v(\mathbf{r}) &= -\frac{q}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}_0 + \frac{\mathbf{l}}{2}|} + \frac{q}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}_0 - \frac{\mathbf{l}}{2}|} = \\ &= \dots \\ &= \frac{q}{4\pi} \left(-\frac{1}{|\mathbf{r} - \mathbf{r}_0|} + \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \cdot \frac{\mathbf{l}}{2} + \frac{1}{|\mathbf{r} - \mathbf{r}_0|} + \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \cdot \frac{\mathbf{l}}{2} + o(|\mathbf{l}|) \right) = \\ &= \dots \\ &= \frac{1}{4\pi} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \cdot \mathbf{p}, \end{aligned}$$

avendo definito il vettore momento dipolo $\mathbf{P} = q\mathbf{l}$.

Polarizzazione - Potenziale generato da una distribuzione di dipoli.

$$d\mathbf{P} = \mathbf{p} \Delta V$$

$$\varepsilon v_P(\mathbf{r}) = \int_{\mathbf{r}_0 \in V_0} \frac{1}{4\pi} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \cdot \mathbf{p}(\mathbf{r}_0) dV_0$$

$$\begin{aligned} \partial_i |\mathbf{r}|^2 &= 2x_i \\ &= 2|\mathbf{r}| \partial_i |\mathbf{r}| \quad \rightarrow \quad \partial_i |\mathbf{r}| = \frac{x_i}{|\mathbf{r}|} \end{aligned}$$

$$\partial_i |\mathbf{r}|^n = n|\mathbf{r}|^{n-1} \partial_i |\mathbf{r}| = nx_i |\mathbf{r}|^{n-2}$$

$$\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} = \nabla_{\mathbf{r}_0} \frac{1}{|\mathbf{r} - \mathbf{r}_0|}$$

$$\begin{aligned} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \cdot \mathbf{p}(\mathbf{r}_0) &= \nabla_{\mathbf{r}_0} \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \cdot \mathbf{p}(\mathbf{r}_0) = \\ &= \nabla_{\mathbf{r}_0} \cdot \left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|} \mathbf{p}(\mathbf{r}_0) \right) - \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \nabla_{\mathbf{r}_0} \cdot \mathbf{p}(\mathbf{r}_0) = \end{aligned}$$

e quindi

$$4\pi \varepsilon v_P(\mathbf{r}) = \oint_{\mathbf{r}_0 \in \partial V_0} \frac{\hat{\mathbf{n}}(\mathbf{r}_0) \cdot \mathbf{p}(\mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|} - \oint_{\mathbf{r}_0 \in V_0} \frac{\nabla_{\mathbf{r}_0} \cdot \mathbf{p}(\mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|}$$

I due contributi hanno la forma di sorgenti, essendo termini proporzionali a $\frac{1}{|\mathbf{r} - \mathbf{r}_0|}$. Il potenziale dovuto alla densità di volume di dipoli equivale alla somma dei due contributi delle cariche di:

- polarizzazione di superficie $\sigma_p = \hat{\mathbf{n}} \cdot \mathbf{p}$
- polarizzazione di volume $\rho_p = -\nabla \cdot \mathbf{p}$

Oss. Se la polarizzazione è uniforme nel volume, il contributo della polarizzazione nel volume si annulla e rimane solo il contributo della polarizzazione sul contorno del volume.

Oss. Legge di Gauss per il campo elettrico,

$$\begin{aligned}\nabla \cdot \mathbf{e} &= \frac{1}{\varepsilon_0} \rho = \\ &= \frac{1}{\varepsilon_0} (\rho_l + \rho_p) = \\ &= \frac{1}{\varepsilon_0} (\rho_l - \nabla \cdot \mathbf{p}) \\ \nabla \cdot (\varepsilon_0 \mathbf{e} + \mathbf{p}) &= \rho_l \\ \nabla \cdot \mathbf{d} &= \rho_l\end{aligned}$$

Parte IV

Appendices

CAPITOLO 13

todo: appendices

- Optics
- Einstein's relativity
- Quantum
- ...

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