Classical Mechanics

basics

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CONTENTS

This material is part of the **basics-books project**. It is also available as a .pdf document.

Classical mechanics deal with the motion of systems and its causes.

Different formulations of mechanics are available. Newton formulation was developed at the end of XVII century and starts from mass conservation and Newton's three principles of dynamics, summarised in invariance under Galileian transformations, the relation between force and the change of momentum of a system, and action/reaction principle. Analytical mechanics was developed in the following centuries by D'Alembert and Lagrange and starts from variational principles, leading to Lagrange or Hamiltonian equations of motion.

Newton Mechanics.

Kinematics

Actions

Inertia

Dynamics

Analytical Mechanics.

Lagrangian Mechanics

Hamiltonian Mechanics

Classical mechanics provides a reliable and useful theory for systems:

- much larger than atomic scales; at atomic scales, quantum mechanics is needed
- with velocity much slower than the speed of light or in domains where the finite value of finite speed of interactions can be neglected, as classical mechanics relies on istantaneous action at distance; if these assumptions fail, Einstein theory is needed either special relativity - as a consistent theory of mechanics and electromagnetism - or general relativity - as a theory of gravitation.
- with a small number of components, so that the integration of the governing equations of motion is feasible; continuous model of the systems are object of classical continuum mechanics, relying on the equations of classical mechanics and thermodynamics; systems with large number of components can be approached with the techniques developed in statistical mechanics.

Under these assumptions, mass conservation (Lavoisier principle) holds, inertial reference frames are related by Galileian relativity and the equations of motions are deterministic and can be solved with a reasonable effort - compared to the information and detail contained in the results - either analytically or numerically. Classical mechanics treats time and space as individually absolute physical quantities: this can be a good model whenever Einstein relativity effects are negligible.

Part I

Newton Mechanics

CHAPTER

ONE

KINEMATICS

Kinematics deals with the motion of mechanical systems, without taking into account the causes of motion.

Classical mechanics relies on the concepts of **absolute 3-dimensional Euclidean space**, E^3, and **absolute time**.

Space and time. Briefly, what is space? It's something you can measure with ruler (for distances) and square (for angles), or other space-measurment devices. Newton mechanics relies on space modeled as **Euclidean space**, a physical entity where the Euclidean geometry holds. What is time? It's something you can measure with a clock or other timekeeping devices, that can be related to order of events, and causality (cause comes before consequences).

Models. Different models of physical systems can be derived with an abstraction and modelling process, depending on the characteristics of the system under investigation and on the level of detail required by the analysis.

These models can be classified by:

- dimensions: 0: point; 1: line; 2: surfaces; 3: volume solid
- deformation: deformable or rigid components

A system can be composed of several components, either free or connected with constraints.

Here, the focus goes to the kinematics of *points* and *rigid bodies*, while deformable bodies are described in continuous mechanics - kinematics.

While space and time are absolute, the motion of a system is usually the **motion relative to an observer** or to a reference frame. After treating the kinematics of points and rigid bodies w.r.t. a given reference frame, *relative kinematics* provides the description of the motion of the same system w.r.t. 2 different observers/reference frames in relative motion.

Definition 1.1 (Configuration)

Definition 1.2 (State)

1.1 Point

The configuration of a point system is determined by its position in space, its state by its position and its velocity. Acceleration is usually required in mechanics, since equations of motions may be recast as a system of second-orde ordinary differential equations in the configuration of the system. These physical quantites are defined here w.r.t. a reference frame $O_o \hat{x}^0 \hat{y}^0 \hat{z}^0, t^0$, keeping constant the vectors of the base w.r.t. time t^0.

Position.

$$\vec{r}_P(t) = P - O_0 = x^0_{P,i}(t)\,\hat{e}^0_i$$

Velocity.

$$\vec{v}_P(t) = \frac{d\vec{r}_P}{dt} = \dot{x}^0_{P,i}\,\hat{e}^0_i$$

Acceleration.

$$\begin{aligned}\vec{a}_P(t) & = \frac{d\vec{v}_P}{dt} = \dot{v}^0_{P,i}\,\hat{e}^0_i = \\ & = \frac{d^2\vec{r}_P}{dt^2} = \ddot{x}^0_{P,i}\,\hat{e}^0_i\end{aligned}$$

1.2 Rigid Body

1.2.1 Rigid motion

Rigid motion preserves distance between any pair of points, and thus angles. The motion of two material points P, Q performing a rigid motion obeys

$$\vec{v}_P - \vec{v}_Q = \vec{\omega} \times (P - Q) \ , \tag{1.1}$$

being $\vec{v}_P$, $\vec{v}_Q$ the velocity of the points and $\vec{\omega}$ the **angular velocity** of the rigid motion. Taking a point Q as the reference point of the motion, the velocity of all other points can be found

$$\vec{v}_P = \vec{v}_Q + \vec{\omega} \times (P - Q) \ ,$$

as a function of the velocity of Q, the angular velocity of the rigid motion, and the relative position $P - Q$.

Proof.

Given 3 points $P(t)$, $Q(t)$, $R(t)$, the distance bewteen each pair of points is constant and thus its time derivative zero,

$$\begin{aligned}
0 = \frac{d}{dt}|P(t)-Q(t)|^2 = 2\,(P-Q)\cdot\left(\vec{v}_P-\vec{v}_Q\right) \quad & \rightarrow \quad \Delta\vec{v}_{QP} = \vec{\omega}_{QP}\times\Delta\vec{r}_{QP} \\
0 = \frac{d}{dt}|P(t)-R(t)|^2 = 2\,(P-R)\cdot\left(\vec{v}_P-\vec{v}_R\right) \quad & \rightarrow \quad \Delta\vec{v}_{RP} = \vec{\omega}_{RP}\times\Delta\vec{r}_{RP}
\end{aligned}$$

$$\begin{aligned}
0 = \frac{d}{dt}\left[(P-Q)\cdot(P-R)\right] & = \Delta\vec{v}_{QP}\cdot\Delta\vec{r}_{RP} + \Delta\vec{r}_{QP}\cdot\Delta\vec{v}_{RP} = \\
& = \vec{\omega}_{QP}\times\Delta\vec{r}_{QP}\cdot\Delta\vec{r}_{RP} + \Delta\vec{r}_{QP}\cdot\Delta\vec{\omega}_{RP}\times\Delta\vec{r}_{RP} = \\
& = \Delta\vec{r}_{QP}\times\Delta\vec{r}_{RP}\cdot\left(\vec{\omega}_{QP}-\vec{\omega}_{RP}\right) \ ,
\end{aligned}$$

and since $\Delta\vec{r}_{QP}$, $\Delta\vec{r}_{RP}$ are arbitrary it follows that the vector $\vec{\omega} = \vec{\omega}_{QP} = \vec{\omega}_{RP}$ is unique for all the points performing a rigid motion.

The configuration of a material vector $\vec{a}$ undergoing a rotation is described by the product of the rotation tensor $\mathbb{R}$ by the reference configuration $\vec{a}^0$,

$$\vec{a} = \mathbb{R}\cdot\vec{a}^0 \qquad , \qquad \vec{b} = \mathbb{R}\cdot\vec{b}^0 \ .$$

In order to preserve distance, and angles

$$\begin{cases} |\vec{a}|^2 = & \vec{a}\cdot\vec{a} = \vec{a}^0\cdot\mathbb{R}^T\cdot\mathbb{R}\cdot\vec{a}^0 = \vec{a}^0\cdot\vec{a}^0 = |\vec{a}|^2 \\ & \vec{a}\cdot\vec{b} = \vec{a}^0\cdot\mathbb{R}^T\cdot\mathbb{R}\cdot\vec{b}^0 = \vec{a}^0\cdot\vec{a}^0 \end{cases} \qquad \rightarrow \qquad \mathbb{R}^T\cdot\mathbb{R} = \mathbb{I}$$

the **rotation tensor** is **unitary**

$$\mathbb{I} = \mathbb{R}^T \cdot \mathbb{R} = \mathbb{R} \cdot \mathbb{R}^T \tag{1.2}$$

Note: From relation (1.2), it follows that

$$1 = |\mathbb{I}| = |\mathbb{R}^T||\mathbb{R}| = |\mathbb{R}|^2 \ ,$$

and thus $|\mathbb{R}| = \mp 1$. If $|\mathbb{R}| = 1$, $\mathbb{R}$ represents a rotation, and implies conservation of orientation of space; if $|\mathbb{R}| = -1$ represents a reflection w.r.t. a plane, and implies inversion of orientation of space.

Orientation of space is determined by the transformation of a RHS triad of vectors: if the transform triad is RHS, then orientation of space is preserved; if it becomes LHS, then orientation of space is inverted.

Note: Rotation tensor $\mathbb{R}$ is not singular and its determinant equals $|\mathbb{R}| = 1$. Thus, $\mathbb{R}^T \cdot \mathbb{R} = \mathbb{I}$ implies $\mathbb{R} \cdot \mathbb{R}^T = \mathbb{I}$. Multiplying (1.2) by $\mathbb{R}$ on the left

$$\mathbb{0} = \mathbb{R} \cdot \mathbb{R}^T \cdot \mathbb{R} - \mathbb{R} \cdot \mathbb{I} = (\mathbb{R} \cdot \mathbb{R}^T - \mathbb{I}) \cdot \mathbb{R} \ ,$$

and since $\mathbb{R}$ is non-singular, it follows that $\mathbb{R} \cdot \mathbb{R}^T = \mathbb{I}$.

Time derivative of the relation (1.2) reads

$$\mathbb{0} = \frac{d}{dt}\left(\mathbb{R} \cdot \mathbb{R}^T\right) = \dot{\mathbb{R}} \cdot \mathbb{R}^T + \mathbb{R} \cdot \dot{\mathbb{R}}^T$$

It follows that the 2-nd order tensor $\dot{\mathbb{R}} \cdot \mathbb{R}^T = -\mathbb{R} \cdot \dot{\mathbb{R}}^T$ is anti-symmetric, and thus it can be written as

$$\dot{\mathbb{R}} \cdot \mathbb{R}^T =: \vec{\omega}_\times \ , \tag{1.3}$$

being the vector $\vec{\omega}$ the angular velocity. Since $\mathbb{R}$ is unitary by (1.2), multiplying (1.3) with the dot-product on the right by $\mathbb{R}$, it follows

$$\dot{\mathbb{R}} = \vec{\omega}_\times \cdot \mathbb{R} \ ,$$

and the expression of the time derivative of a material vector $\vec{a}$,

$$\frac{d\vec{a}}{dt} = \dot{\mathbb{R}} \cdot \vec{a}^0 = \vec{\omega}_\times \cdot \mathbb{R} \cdot \vec{a}^0 = \vec{\omega}_\times \cdot \vec{a} = \vec{\omega} \times \vec{a} \ . \tag{1.4}$$

Position and Orientation. The most general rigid motion is the combination of the translation of a reference point Q and the rotation w.r.t. this point of other material points,

$$\begin{aligned} \vec{r}_P &= \vec{r}_Q + (P - Q) = \\ &= \vec{r}_Q + \mathbb{R} \cdot (P - Q)^0 \end{aligned} \tag{1.5}$$

Velocity and Angular velocity. Time derivative of the relation (1.5) between positions of material points gives again (1.1)

$$\begin{aligned} \vec{v}_P &= \vec{v}_Q + \dot{\mathbb{R}} \cdot (P - Q)^0 = \\ &= \vec{v}_Q + \vec{\omega}_\times \mathbb{R} \cdot (P - Q)^0 = \\ &= \vec{v}_Q + \vec{\omega} \times (P - Q) \end{aligned} \tag{1.6}$$

Acceleration and Angular acceleration. Time derivatives of the relation (1.6) gives

$$\begin{aligned} \vec{a}_P &= \vec{a}_Q + \vec{\alpha} \times (P - Q) + \vec{\omega} \times \left(\vec{v}_P - \vec{Q}\right) = \\ &= \vec{a}_Q + \vec{\alpha} \times (P - Q) + \vec{\omega} \times \left[\vec{\omega} \times (P - Q)\right] \ . \end{aligned}$$

1.3 Continuous Medium

1.4 Relative Kinematics

Relative kinematics is discussed here using two Cartesian reference frames.

$$P - O_0 = x^{0i}_{P/O_0} \hat{e}^0_i$$

$$O_1 - O_0 = x^{0i}_{O_1/O_0} \hat{e}^0_i$$

$$P - O_1 = x^{1i}_{P/O_1} \hat{e}^1_i$$

$$\begin{aligned}\hat{e}^1_i &= \hat{e}^1_i \cdot \hat{e}^0_k \, \hat{e}^0_k = \hat{e}^1_j \cdot \hat{e}^0_k \, \hat{e}^0_k \otimes \hat{e}^0_j \cdot \hat{e}^0_i = \\ &= R^{0\to 1}_{kj} \hat{e}^0_k \otimes \hat{e}^0_j \cdot \hat{e}^0_i = \mathbb{R}^{0\to 1} \cdot \hat{e}^0_i \ .\end{aligned}$$

1.4.1 Points

Position. Given two reference frames Ox^i, $O'x^{i'}$, for the position of a point P reads

$$\begin{aligned}(P - O_0) &= (O_1 - O_0) + (P - O_1) \ , \\ x^0_{P/O_0,i} \hat{e}^0_i &= x^0_{O_1/O_0,i} \hat{e}^0_i + x^1_{P/O_1,k} \hat{e}^1_k \ ,\end{aligned} \tag{1.7}$$

i.e. the position vector $P - O$ of the point P w.r.t. point O - origin of the reference frame Ox^i - is the sum of the position vector $P - O'$ of the point P w.r.t. to the point O' - origin of the reference frame $O'x'^i$ - and the position vector $O' - O$, of the origin O' w.r.t. to O.

Velocity. Time derivative of relative position relation (1.7) w.r.t. to reference frame 0 is performed keeping $\hat{e}^0_i$ constant.

$$\begin{aligned}\frac{{}^0 d}{dt}(P - O_0) &= \frac{{}^0 d}{dt} \left[(O_1 - O_0) + (P - O_1)\right] = \\ &= \frac{{}^0 d}{dt} \left(x^0_{O_1/O_0,i} \hat{e}^0_i \right) + \frac{{}^0 d}{dt} \left(x^1_{P/O_1,k} \hat{e}^1_k \right) = \\ &= \frac{{}^0 d}{dt} x^0_{O_1/O_0,i} \, \hat{e}^0_i + \frac{{}^0 d}{dt} x^1_{P/O_1,k} \, \hat{e}^1_k + x^1_{P/O_1,k} \, \frac{{}^0 d}{dt} \hat{e}^1_k = \\ &= \vec{v}^0_{O_1/O_0} + \vec{v}^1_{P/O_1} + \vec{\omega}_{1/0} \times \hat{e}^1_k x^1_{P/O_1,k} = \\ &= \vec{v}^0_{O_1/O_0} + \vec{v}^1_{P/O_1} + \vec{\omega}_{1/0} \times (P - O_1) \ ,\end{aligned}$$

so that

$$\vec{v}^0_{P/O} = \vec{v}^0_{O_1/O_0} + \vec{v}^1_{P/O_1} + \vec{\omega}_{1/0} \times (P - O_1) \ . \tag{1.8}$$

Acceleration. Time derivative of relative velocity relation (1.8) w.r.t. reference frame 0 reads

$$\begin{aligned}\frac{{}^0 d}{dt} \vec{v}^0_{P/O_0} &= \frac{{}^0 d}{dt} \left[\vec{v}^0_{O_1/O_0} + \vec{v}^1_{P/O_1} + \vec{\omega}_{1/0} \times (P - O_1) \right] = \\ &= \dots \\ &= \vec{a}^0_{O_1/O_0} + \vec{a}^1_{P/O_1} + 2\vec{\omega}_{1/0} \times \vec{v}^1_{P/O_1} + \vec{\alpha}_{1/0} \times (P - O_1) + \vec{\omega}_{1/0} \times \left[\vec{\omega}_{1/0} \times (P - O_1) \right]\end{aligned}$$

so that

$$\vec{a}^0_{P/O_0} = \vec{a}^0_{O_1/O_0} + \vec{a}^1_{P/O_1} + \underbrace{\vec{\alpha}_{1/0} \times (P - O_1)}_{\text{tangential}} + \underbrace{2\vec{\omega}_{1/0} \times \vec{v}^1_{P/O_1}}_{\text{Coriolis}} + \underbrace{\vec{\omega}_{1/0} \times \left[\vec{\omega}_{1/0} \times (P - O_1) \right]}_{\text{centripetal}} \ . \tag{1.9}$$

where:

- the "tangential component" is orthogonal to the instantaneous angular acceleration and radius,
- the "centripetal component" is orthogonal w.r.t. the instantaneous angular velocity

todo *tangent to what, centripetal w.r.t. what?* **state it clearly, otherwise delete this**

1.4.2 Rigid bodies

Orientation.

Angular velocity.

Angular acceleration.

1.5 Rotations

A rotation is a transformation preserving length, angles and orientation of space[1].

A rotation in 3-dimensional spaces can be represented by a **rotation tensor**, $\mathbb{R}$. A rotation tensor has some mathematical properties: it's a **unitary tensor**, $\mathbb{R}^T \cdot \mathbb{R} = \mathbb{R} \cdot \mathbb{R}^T = \mathbb{I}$, and unit determinant, $|\mathbb{R}| = 1$. From these properties, it immediately follows the definition of **angular velocity**, and angular acceleration. A rotation can be represented as **compositions of successive rotations**, $\mathbb{R}^{0\rightarrow 2} = \mathbb{R}^{1\rightarrow 2} \cdot \mathbb{R}^{0\rightarrow 1}$, while summation of relative angular velocity (and acceleration) of successive rotations holds, $\vec{\omega}^{0\rightarrow 2} = \vec{\omega}^{1\rightarrow 2} + \vec{\omega}^{0\rightarrow 1}$. For small rotations, **linearization** provides a small-amplitude approximation: in the linearization limit, composition of rotations reduces to summation, while incremental rotation in a small time interval reduces to the definition of angular velocity.

This section of kinematic deals with rotations in details, first in absolute tensor notation, and then introducing parametrizations to describe rotations like axis and angle $(\hat{n}, \theta)$, Euler's angles (ϕ, θ, ψ) for successive simple rotations, and unitary quaternions $q = q_0 + \vec{q}$, with $1 = q_0^2 + |\vec{q}|^2$.

Tensors.

Parametrizations: axis and angle.

Parametrizations: Euler's equations.

Parametrizations: quaternions.

1.5.1 Tensor formalism for rotations

This section deals with the tensor formalism to represent rotations, its increment and angular velocity, and composition. First the expression of the rotation tensor rotating a Cartesian basis $\{\hat{e}_i^0\}_i$ into another Cartesian basis $\{\hat{e}_j^1\}_j$ is given, then angular velocity and angular acceleration are defined and their role in the description of the motion of rigid bodies is discussed. Then successive rotations are discussed: a rotation can be represented as the composition of many rotations, via dot tensor product of the rotation tensors, while the corresponding angular velocity can be written as the sum of the angular velocity of the successive relative rotations.

[1] A reflection is a transformation preserving length and angles but reversing orientation of space. A reflection can be represented by a unitary tensor $\mathbb{R}$, with determinant $|\mathbb{R}| = -1$.

Rotation tensor

Given 2 right-handed Cartesian bases $\{\hat{e}^0_i\}_{i=1:3}$, $\{\hat{e}^1_j\}_{j=1:3}$, a rotation transforming the first basis $\{\hat{e}^0_i\}_i$ into the second basis $\{\hat{e}^1_j\}_j$ exists, and can be represented by unitary tensor with unit determinant, the rotation tensor $\mathbb{R}$. Given that for Cartesian bases (**todo** add link to Math:Vector Algebra)[1]

$$\hat{e}^1_k = \left(\hat{e}^1_k \cdot \hat{e}^0_i\right) \hat{e}^0_i \ ,$$

this expression can be recast using tensor formalism and introducing rotation tensor

$$\begin{aligned}\hat{e}^1_k &= \left(\hat{e}^1_k \cdot \hat{e}^0_i\right) \hat{e}^0_i = \\ &= \left(\hat{e}^1_j \cdot \hat{e}^0_i\right) \hat{e}^0_i \delta_{jk} = \\ &= \left(\hat{e}^1_j \cdot \hat{e}^0_i\right) \hat{e}^0_i \otimes \hat{e}^0_j \cdot \hat{e}^0_k = \\ &= \mathbb{R}^{0\rightarrow 1} \cdot \hat{e}^0_k \ ,\end{aligned}$$

having defined the rotation tensor $\mathbb{R}^{0\rightarrow 1}$ transforming the vectors of basis $\{\hat{e}^0_i\}_i$ into vectors of basis $\{\hat{e}^1_j\}_j$ as

$$\mathbb{R}^{0\rightarrow 1} = R^{0\rightarrow 1}_{ij} \hat{e}^0_i \otimes \hat{e}^0_j = \left(\hat{e}^0_i \cdot \hat{e}^1_j\right) \hat{e}^0_i \otimes \hat{e}^0_j$$

having defined the components of the rotation from 0 to 1 w.r.t. basis 0 as $R^{0\rightarrow 1}_{ij} = \hat{e}^0_i \cdot \hat{e}^1_j$.

Rotation tensor is unitary with unit determinant: component relation

As $\mathbb{R} \cdot \mathbb{R}^T = \mathbb{R}^T \cdot \mathbb{R} = \mathbb{I}$, it follows

$$R_{ij} R_{kj} = R_{ji} R_{jk} = \delta_{ik} \ .$$

Proof

Let's prove the first relation, explicitly writing rotation tensor using basis $\{\hat{e}^0_i\}_i$,

$$\begin{aligned}\delta_{ik} \hat{e}^0_i \otimes \hat{e}^0_k = \mathbb{I} = \mathbb{R} \cdot \mathbb{R}^T = \\ &= \left(R^{0\rightarrow 1}_{ij} \hat{e}^0_i \otimes \hat{e}^0_j\right) \cdot \left(R^{0\rightarrow 1}_{lk} \hat{e}^0_l \otimes \hat{e}^0_k\right)^T = \\ &= \left(R^{0\rightarrow 1}_{ij} \hat{e}^0_i \otimes \hat{e}^0_j\right) \cdot \left(R^{0\rightarrow 1}_{kl} \hat{e}^0_l \otimes \hat{e}^0_k\right) = \\ &= R^{0\rightarrow 1}_{ij} \, R^{0\rightarrow 1}_{kl} \, \hat{e}^0_i \otimes \underbrace{\hat{e}^0_j \cdot \hat{e}^0_l}_{\delta_{jl}} \otimes \hat{e}^0_k = \\ &= R^{0\rightarrow 1}_{ij} \, R^{0\rightarrow 1}_{kj} \, \hat{e}^0_i \otimes \hat{e}^0_k \ ,\end{aligned}$$

and thus $\delta_{ik} = R^{0\rightarrow 1}_{ij} \, R^{0\rightarrow 1}_{kj}$.

todo *Show that $\mathbb{I}$ has the same components in all the (Cartesian?) reference frames. Show the rule of transformation of Ricci tensor also.*

Components of a rotation tensor are the same in the original and transformed basis

[1] In general, basis transformation rely on the reciprocal basis, $\mathbf{b}^j$, defined as those set of vectors $\mathbf{b}^j \cdot \mathbf{b}_i = \delta^j_i$. A vector $\mathbf{w}$ can be written using two different bases $\{\mathbf{u}_i\}_i$ and $\{\mathbf{v}_j\}_j$, as $\mathbf{w} = u^i \mathbf{u}_i = v^j \mathbf{v}_j$. The vectors of a basis can be written as a linear combinations of the vectors of the other basis, namely $\mathbf{v}_j = T^i_j \mathbf{u}_i$, or $\mathbf{u}_i = \tilde{T}^j_i \mathbf{v}_j$, using the inverse transformation $T^{-1} := \tilde{T}$, defined as $\tilde{T}^k_i T^i_j = \delta^k_j$. Exploiting the definition of the reciprocal basis, it should be not hard to prove that $T^i_j = \mathbf{v}_j \cdot \mathbf{u}^i$ and $\tilde{T}^j_i = \mathbf{v}^j \cdot \mathbf{u}_i$. As the reciprocal basis of a Cartesian basis is the basis itself ($\mathbf{e}^j = g^{ij} \mathbf{e}_i$, with $g_{ij} = g^{ij} = \delta_{ij}$), the components of the transformation between two Cartesian bases read $T^i_j = \mathbf{u}_i \cdot \mathbf{v}_j$, while the inverse transform involves the transpose, $\tilde{T} = T^T$: $\mathbf{v}_j = T^i_j \mathbf{u}_i = \left(\mathbf{v}_j \cdot \mathbf{u}_i\right) \mathbf{u}_i$, or $\mathbf{u}_i = T^j_i \mathbf{v}_j = \left(\mathbf{u}_i \cdot \mathbf{v}_j\right) \mathbf{v}_j$.

Components of the rotation tensor are the same if referred to the original or to the rotated basis.

$$\begin{aligned}\mathbb{R}^{0\rightarrow 1} & = R^{0\rightarrow 1,00}_{ij}\hat{e}^0_i \otimes \hat{e}^0_j = \\ & = \underbrace{R^{0\rightarrow 1,00}_{ij}\left(R^{0\rightarrow 1,00}_{ik}\right.}_{=\delta_{jk}}\left.\hat{e}^1_k\right) \otimes \left(R^{0\rightarrow 1,00}_{jl}\hat{e}^1_l\right) = \\ & = R^{0\rightarrow 1,00}_{kl}\hat{e}^1_k \otimes \hat{e}^1_l \ ,\end{aligned}$$

and thus the components w.r.t. basis 1 equal the component w.r.t. basis 0, $R^{0\rightarrow 1,11} = R^{0\rightarrow 1,00}$. Transforming only one set of vectors of the rank-2 tensor, the rotation tensor can be further written as

$$\mathbb{R}^{0\rightarrow 1} = \delta_{jk}\hat{e}^1_k \otimes \hat{e}^0_j \ .$$

It's immediate to show the correctness of the last expression, showing the effect of tensor rotation on the vectors of the basis 0,

$$\mathbb{R}^{0\rightarrow 1} \cdot \hat{e}^0_i = \left(\delta_{jk}\,\hat{e}^1_k \otimes \hat{e}^0_j\right) \cdot \hat{e}^0_i = \hat{e}^1_k\,\delta_{jk}\,\delta_{ji} = \hat{e}^1_i \ .$$

Rotation of a vector

…

Rotation of basis, and change of components of given vectors and tensors

…

Angular velocity and acceleration

Angular velocity. Taken time derivative of the unitary relation **todo** *add ref*, the definition of angular velocity naturally arises. As the identity tensor is constant,

$$\mathbb{0} = \dot{\mathbb{R}} \cdot \mathbb{R}^T + \mathbb{R} \cdot \dot{\mathbb{R}}^T \ ,$$

and thus tensor $\dot{\mathbb{R}} \cdot \mathbb{R}^T$ is anti-symmetric,

$$\dot{\mathbb{R}} \cdot \mathbb{R}^T = -\mathbb{R} \cdot \dot{\mathbb{R}}^T = -\left(\dot{\mathbb{R}} \cdot \mathbb{R}^T\right)^T \ ,$$

and thus it can be written as the *cross* (**todo** *check the def*) of a vector,

$$\vec{\omega}_\times = \dot{\mathbb{R}} \cdot \mathbb{R}^T \ ,$$

defined angular velocity.

As it will be clear in the *section about successive rotations*, the angular velocity $\vec{\omega}^{1/0}$ of a reference frame 1 w.r.t. a reference frame 0 is defined by the relation

$$\vec{\omega}^{1/0}_\times := \Omega^{1/0} = \frac{{}^0 d}{dt}\mathbb{R}^{0\rightarrow 1} \cdot \mathbb{R}^{1\rightarrow 0} \ ,$$

having replaced the trasponse tensor with the tensor of the inverse rotation $1 \rightarrow 0$.

Using the properties of Dirac's delta and Ricci's tensor, it's possible to retrive the angular velocity vector $\vec{\omega}$,

$$\omega^0_l = \frac{1}{2}\varepsilon^0_{ilk}\dot{R}_{ij}R_{kj} = -\frac{1}{2}\varepsilon_{lik}\dot{R}_{ij}R_{kj} = -\frac{1}{2}\varepsilon_{lij}\Omega_{ij}$$

todo

- $$\omega_k^1 = \dots$$
- *Explicitly discuss the two common uses of rotation tensors: 1. rotate a given vector into a new vector; 2. write components of a given vector w.r.t. 2 different Cartesian bases.*

Proof

Using index notation

$$\varepsilon_{ijk}\omega_j^0 \hat{e}_i^0 \hat{e}_k^0 = \dot{R}_{ij} R_{kj} \hat{e}_i^0 \hat{e}_j^0 \ ,$$

and the identities

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{lmk} &= \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im} \\ \varepsilon_{ijk}\varepsilon_{ljk} &= \delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl} = 3\delta_{il} - \delta_{il} = 2\delta_{il}\end{aligned}$$

it follows

$$\begin{aligned}\varepsilon_{ilk}\varepsilon_{ijk}\omega_j^0 &= \varepsilon_{ilk}\dot{R}_{ij}R_{kj} \\ 2\delta_{lj}\omega_j^0 &= \varepsilon_{ilk}\dot{R}_{ij}R_{kj}\end{aligned}$$

and eventually

$$\omega_l^0 = \frac{1}{2}\varepsilon_{ilk}\dot{R}_{ij}R_{kj} = -\frac{1}{2}\varepsilon_{lik}\dot{R}_{ij}R_{kj} = -\frac{1}{2}\varepsilon_{lij}\Omega_{ij} \ .$$

Angular acceleration. Angular acceleration, $\vec{\alpha}$, is the time derivative of angular velocity, $\vec{\omega}$,

$$\vec{\alpha} = \dot{\vec{\omega}} \ .$$

Sequence of rotations

A rotation may be represented as the composition of successive rotations. The vectors of a Cartesian basis $\{\vec{e}_i^0\}_i$ can be transformed into the vectors of the Cartesian basis $\{\vec{e}_k^2\}_k$, first rotating 0 into an intermediate basis 1, and then rotating 1 into 2.

$$\mathbb{R}^{0\to 2} = \mathbb{R}^{1\to 2} \cdot \mathbb{R}^{0\to 1} \tag{1.10}$$

Angular velocity of the reference frame 2 w.r.t. 0 can be written as the sum of the relative angular velocity of the intermediate rotations,

$$\vec{\omega}^{2/0} = \vec{\omega}^{2/1} + \vec{\omega}^{1/0} \tag{1.11}$$

The same relation holds for angular acceleration

$$\vec{\alpha}^{2/0} = \vec{\alpha}^{2/1} + \vec{\alpha}^{1/0} \tag{1.12}$$

Orientation

Given 3 Cartesian bases $\{\hat{e}_i^0\}_{i=1:3}$, $\{\hat{e}_j^1\}_{j=1:3}$, $\{\hat{e}_k^2\}_{k=1:3}$,

$$\begin{aligned}\hat{e}_i^2 &= \mathbb{R}^{1\to 2} \cdot \hat{e}_i^1 = \\ &= \mathbb{R}^{1\to 2} \cdot \mathbb{R}^{0\to 1} \cdot \hat{e}_i^0\end{aligned} ,$$

i.e composition of rotations holds

$$\mathbb{R}^{0\to 2} = \mathbb{R}^{1\to 2} \cdot \mathbb{R}^{0\to 1} \ .$$

Angular velocity

Indicating with the apex a time derivative performed keeping constant the vectors of the basis, the following relations hold for vectors and tensors

$$\frac{d}{dt}\vec{a} = \frac{d}{dt}\left(a_i^1 \hat{e}_i^1\right) = \dot{a}_i^1 \hat{e}_i^1 + a_i^1 \vec{\omega}^{1/0} \times \hat{e}_i^1 = \frac{{}^1d}{dt}\vec{a} + \vec{\omega}^{1/0} \times \vec{a}$$

$$\begin{aligned}
\frac{d}{dt}\mathbb{R}^{1\to 2} & = \frac{d}{dt}\left[R_{ik}^{1\to 2}\hat{e}_i^1 \otimes \hat{e}_k^1\right] = \\
& = \dot{R}_{ik}^{1\to 2}\hat{e}_i^1 \otimes \hat{e}_k^1 + \Omega^{1/0}\cdot\mathbb{R}^{1\to 2} - \mathbb{R}^{1\to 2}\cdot\Omega^{1/0} = \\
& = \frac{{}^1d}{dt}\mathbb{R}^{1\to 2} + \Omega^{1/0}\cdot\mathbb{R}^{1\to 2} - \mathbb{R}^{1\to 2}\cdot\Omega^{1/0} \ .
\end{aligned}$$

todo *Show details of the evaluation of the last term, exploiting index notation and properties permutation properties of indices of Ricci's symbols, related to properties of vector product.*

Using the latter relation, it's possible to prove the angular velocity of a composite rotation is the sum of the relatvie angularvelocities of the intermediate rotations.

$$\begin{aligned}
\Omega^{2/0} & = \dot{\mathbb{R}}^{0\to 2}\cdot\mathbb{R}^{2\to 0} = \\
& = \frac{d}{dt}\left(\mathbb{R}^{1\to 2}\cdot\mathbb{R}^{0\to 1}\right)\cdot\mathbb{R}^{2\to 0} = \\
& = \left\{\left[\frac{{}^1d}{dt}\mathbb{R}^{1\to 2} + \Omega^{1/0}\cdot\mathbb{R}^{1\to 2} - \mathbb{R}^{1\to 2}\cdot\Omega^{1/0}\right]\cdot\mathbb{R}^{0\to 1} + \mathbb{R}^{1\to 2}\cdot\dot{\mathbb{R}}^{0\to 1}\right\}\cdot\mathbb{R}^{1\to 0}\cdot\mathbb{R}^{2\to 1} = \\
& = \left[\frac{{}^1d}{dt}\mathbb{R}^{1\to 2} + \Omega^{1/0}\cdot\mathbb{R}^{1\to 2} - \mathbb{R}^{1\to 2}\cdot\Omega^{1/0}\right]\cdot\mathbb{R}^{2\to 1} + \mathbb{R}^{1\to 2}\cdot\Omega^{1/0}\cdot\mathbb{R}^{2\to 1} = \\
& = \frac{{}^1d}{dt}\mathbb{R}^{1\to 2}\cdot\mathbb{R}^{2\to 1} + \Omega^{1/0} = \\
& = \Omega^{2/1} + \Omega^{1/0} \ .
\end{aligned}$$

so that addition of relative angular velocity holds

$$\Omega^{20} = \Omega^{21} + \Omega^{10} \qquad , \qquad \vec{\omega}_{2/0} = \vec{\omega}_{2/1} + \vec{\omega}_{1/0} \ .$$

Example 1.5.1 (Explicit form of time derivative of rotation tensor)

$$\begin{aligned}
\dot{\mathbb{R}}^{1\to 2} & = \frac{{}^1d}{dt}\mathbb{R}^{1\to 2} + \Omega^{1/0}\cdot\mathbb{R}^{1\to 2} - \mathbb{R}^{1\to 2}\cdot\Omega^{1/0} = \quad (1) \\
& = \left[\Omega^{2/1} + \Omega^{1/0}\right]\cdot\mathbb{R}^{1\to 2} - \mathbb{R}^{1\to 2}\cdot\Omega^{1/0} \ .
\end{aligned}$$

having used (1) $\Omega^{2/1} = \frac{{}^1d}{dt}\mathbb{R}^{1\to 2}\cdot\mathbb{R}^{2\to 1}$

Example 1.5.2 (Sequence of three rotations)

todo

Angular acceleration

Time derivative of angular velocity composition provides the addition of relative angular accelerations

$$\frac{{}^0d}{dt}\vec{\omega}_{2/0} = \frac{{}^0d}{dt}\vec{\omega}_{2/1} + \frac{{}^0d}{dt}\vec{\omega}_{1/0} \ ,$$

or

$$\vec{\alpha}_{2/0} = \vec{\alpha}_{2/1} + \vec{\alpha}_{1/0} \ .$$

Linearization of rotations

todo Take care of components. Here referred to 0 reference frame

- explicitly write components in the original and rotated reference frame
- deal with intermediate rotations: does increment of $\mathbb{R}^{i\to j}$ have simple expression with "relative variation", keeping constant the vectors of the original basis i, something like ${}^i\delta\vec{\theta}_\times = {}^i\delta\mathbb{R}\cdot\mathbb{R}$ General expression?

$$\mathbb{I} = \mathbb{R}\cdot\mathbb{R}^T$$

Increment

$$\mathbb{0} = \delta\mathbb{R}\cdot\mathbb{R}^T + \mathbb{R}\cdot\delta\mathbb{R}^T$$

and thus the antisymmetric tensor can be written as

$$\delta\theta_\times := \delta\mathbb{R}\cdot\mathbb{R}^T = \delta\Theta \ ,$$

so that

$$\delta\theta_l = -\frac{1}{2}\varepsilon_{lij}\delta R_{ik}\,R_{jk} = -\frac{1}{2}\varepsilon_{lij}\delta\Theta_{ij}$$

Parametrizations

Minimal sets of parameters to represent rotations have 3 parameters. However these sets of parameters are not regular over all the possible rotations, and the transformation becomes singular somewhere. *Quaternions* provide a set of 4 parameters for a regular parametrization of rotations

Euler angles

Axis and rotation angle

Quaternions

1.5.2 Rotation parametrization: axis and angle

Rotation tensor

$$\mathbb{R} = \mathbb{I} + \sin\theta\,\hat{n}_\times + (1-\cos\theta)\,\hat{n}_\times\hat{n}_\times \ , \tag{1.13}$$

Angular velocity

$$\vec{\omega}_\times = \dot{\mathbb{R}}\cdot\mathbb{R}^T =$$

Linearization

Rotation tensor

The expression of the rotation tensor $\mathbb{R}$ that rotates vector $\vec{v}^0$ into $\vec{v}$,

$$\vec{v} = \mathbb{R} \cdot \vec{v}^0 \ ,$$

comes from little geometry and vector algebra. The rotation of an angle θ around the axis determined by the unit vector $\hat{n}$ reads

$$\vec{v} = \hat{n} v^0_{\parallel} + \hat{x} v^0_{\perp} \cos\theta + \hat{y} v^0_{\perp} \sin\theta \ , \tag{1.14}$$

with

- $v^0_{\parallel} = \hat{n} \cdot \vec{v}^0$,
- $\hat{n} \times \vec{v}^0 = \hat{y} v^0_{\perp}$, and thus $\hat{y} = \frac{\hat{n} \times \vec{v}^0}{v^0_{\perp}}$,
- $\hat{x} = \hat{y} \times \hat{n}$, and thus $v^0_{\perp} \hat{x} = v^0_{\perp} \frac{(\hat{n} \times \vec{v}^0)}{v^0_{\perp}} \times \hat{n} = -\hat{n} \times (\hat{n} \times \vec{v}^0) = -\hat{n}_{\times} \hat{n}_{\times} \cdot \vec{v}^0$, where the dot product is meant between the $\hat{n}_{\times}$ tensors representing vector products.

Exploiting these expressions, the relation (1.14) between the vectors $\vec{v}$, $\vec{v}^0$ can be written in implicit tensor form. Different equivalent tensor expressions follows from the identity $\hat{v}_{\times} \hat{v}_{\times} = \vec{v} \otimes \vec{v} - |\vec{v}|^2 \mathbb{I}$,

$$\begin{aligned}
\vec{v} & = \hat{n} v^0_{\parallel} + \hat{x} v^0_{\perp} \cos\theta + \hat{y} v^0_{\perp} \sin\theta = \\
& = \hat{n}\hat{n} \cdot \vec{v}^0 - \cos\theta \hat{n}_{\times} \hat{n}_{\times} \cdot \vec{v}^0 + \sin\theta \hat{n}_{\times} \cdot \vec{v}^0 = \\
& = [\hat{n}\hat{n} - \cos\theta \hat{n}_{\times}\hat{n}_{\times} + \sin\theta \hat{n}_{\times}] \cdot \vec{v}^0 =
\end{aligned}$$

and thus

$$\begin{aligned}
\mathbb{R} & = \hat{n}\hat{n} - \cos\theta \hat{n}_{\times}\hat{n}_{\times} + \sin\theta \hat{n}_{\times} = \\
& = \mathbb{I} + \sin\theta \, \hat{n}_{\times} + (1 - \cos\theta) \, \hat{n}_{\times} \hat{n}_{\times} \ .
\end{aligned}$$

Proof of relation $\hat{v}_{\times} \hat{v}_{\times} = \vec{v} \otimes \vec{v} - |\vec{v}|^2 \mathbb{I}$

$$\begin{aligned}
\hat{v}_{\times} \hat{v}_{\times} & = \hat{e}^i \hat{e}_m \varepsilon_{ijk} v_j \varepsilon_{klm} v_l = \\
& = \hat{e}^i \hat{e}_m \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) v_j v_l \ , \quad = \hat{e}^i \hat{e}_m \left(v_i v_m - v_j v_j \delta_{im} \right) = \\
& = \vec{v} \otimes \vec{v} - |\vec{v}|^2 \mathbb{I}
\end{aligned}$$

Angular velocity

$$\vec{\omega} = \dot{\theta} \, \hat{n} + \sin\theta \, \dot{\hat{n}} - (1 - \cos\theta) \ \hat{n} \times \dot{\hat{n}}$$

Proof

Using the expression of the *angular velocity using unitary quaternion parametrization*,

$$\mathbf{q} = q_0 + \vec{q} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} \, \hat{n} \ .$$

and

$$\vec{\omega} = -2 \, \vec{q} \, \dot{q}_0 + 2 \, q_0 \, \dot{\vec{q}} - 2 \, \vec{q} \times \dot{\vec{q}}$$

with

$$\dot{q}_0 = -\frac{\dot{\theta}}{2} \sin\frac{\theta}{2}$$
$$\dot{\vec{q}} = \frac{\dot{\theta}}{2} \cos\frac{\theta}{2}\hat{n} + \sin\frac{\theta}{2}\dot{\hat{n}}$$

$$\begin{aligned}
\vec{\omega} &= -2\,\vec{q}\,\dot{q}_0 + 2\,q_0\,\dot{\vec{q}} - 2\,\vec{q}\times\dot{\vec{q}} = \\
&= -2\,\sin\frac{\theta}{2}\hat{n}\left(-\frac{\dot{\theta}}{2}\sin\frac{\theta}{2}\right) + 2\cos\frac{\theta}{2}\left(\frac{\dot{\theta}}{2}\cos\frac{\theta}{2}\hat{n} + \sin\frac{\theta}{2}\dot{\hat{n}}\right) - 2\,\sin\frac{\theta}{2}\hat{n}\times\left(\frac{\dot{\theta}}{2}\cos\frac{\theta}{2}\hat{n} + \sin\frac{\theta}{2}\dot{\hat{n}}\right) = \\
&= \dot{\theta}\,\hat{n} + \sin\theta\,\dot{\hat{n}} - (1-\cos\theta)\;\hat{n}\times\dot{\hat{n}}
\end{aligned}$$

having used $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 1 - 2\sin^2\frac{\theta}{2}$, to transform $\sin^2\frac{\theta}{2}$, and $\hat{n}\times\hat{n} = \vec{0}$.

1.5.3 Rotation parametrization: Euler's angles

A generic rotation tensor can be written as a combination of 3 successive rotations. **Euler's angle parametrizations** are built with three successive rotations around 1 axis of the intermediate reference frames. Two families of angles exist:

- proper Euler's angles, with the first and the last rotation around axes with the same labels (e.g. zxz, xyx, yzy, zyz, xzx, yxy)
- Tait-Bryan (or Cardan) angles, with three rotations around three axes with different labels (e.g. xyz, yzx, zxy, xzy, yxz, zyx)

Here, the expressions of rotation tensor and angular velocity are explicitly treated for the set of angles (ψ, θ, ϕ) around axes zyx.

Euler's angle parametrization is not regular: *gimbal lock*

Euler's angles can represent any rotation, but there this parametrization is not one-to-one: some rotations may be represented by many (infinite) set of parameters, as shown below. Thus, for some rotations it's not possible to recover a unique set of Euler's angles.

todo

- *add link*
- *add picture?*

Rotation tensor

Three successive rotations are

$$\begin{cases} \hat{x}_1 = \;\;\cos\psi\,\hat{x}_0 + \sin\psi\,\hat{y}_0 \\ \hat{y}_1 = -\sin\psi\,\hat{x}_0 + \cos\psi\,\hat{y}_0 \\ \hat{z}_1 = \hat{z}_0 \end{cases} \qquad \begin{cases} \hat{x}_0 = \;\;\cos\psi\,\hat{x}_1 - \sin\psi\,\hat{y}_1 \\ \hat{y}_0 = \;\;\sin\psi\,\hat{x}_1 + \cos\psi\,\hat{y}_1 \\ \hat{z}_0 = \hat{z}_1 \end{cases}$$
$$\begin{cases} \hat{x}_2 = \;\;\cos\theta\,\hat{x}_1 - \sin\theta\,\hat{z}_1 \\ \hat{y}_2 = \hat{y}_1 \\ \hat{z}_2 = \;\;\sin\theta\,\hat{x}_1 + \cos\theta\,\hat{z}_1 \end{cases} \quad , \quad \begin{cases} \hat{x}_1 = \;\;\cos\theta\,\hat{x}_2 + \sin\theta\,\hat{z}_2 \\ \hat{y}_1 = \hat{y}_2 \\ \hat{z}_1 = -\sin\theta\,\hat{x}_2 + \cos\theta\,\hat{z}_2 \end{cases}$$
$$\begin{cases} \hat{x}_3 = \hat{x}_2 \\ \hat{y}_3 = \;\;\cos\phi\,\hat{y}_2 + \sin\phi\,\hat{z}_2 \\ \hat{z}_3 = -\sin\phi\,\hat{y}_2 + \cos\phi\,\hat{z}_2 \end{cases} \qquad \begin{cases} \hat{x}_2 = \hat{x}_3 \\ \hat{y}_2 = \;\;\cos\phi\,\hat{y}_3 - \sin\phi\,\hat{z}_3 \\ \hat{z}_2 = \;\;\sin\phi\,\hat{y}_3 + \cos\phi\,\hat{z}_3 \end{cases}$$

Rotation of vector $\vec{v}^0 = v_x \hat{x}_0 + v_y \hat{y}_0 + v_z \hat{z}_0$ into vector $\vec{v} = v_x \hat{x}_3 + v_y \hat{y}_3 + v_z \hat{z}_3$

Vectors of basis 3 w.r.t. basis 0.

$$\begin{aligned}
\hat{x}_3 & = \hat{x}_2 = \\
& = \ \cos\theta\,\hat{x}_1 - \sin\theta\,\hat{z}_1 = \\
& = \ \cos\theta\,\left(\cos\psi\,\hat{x}_0 + \sin\psi\,\hat{y}_0\right) - \sin\theta\,\left(\hat{z}_0\right) = \\
\hat{y}_3 & = \cos\phi\,\hat{y}_2 + \sin\phi\,\hat{z}_2 = \\
& = \cos\phi\,\left(\hat{y}_1\right) + \sin\phi\,\left(\sin\theta\,\hat{x}_1 + \cos\theta\,\hat{z}_1\right) = \\
& = \hat{x}_0\left(-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi\right) + \hat{y}_0\left(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi\right) + \hat{z}_0\left(\sin\phi\cos\theta\right) \\
\hat{z}_3 & = -\sin\phi\,\hat{y}_2 + \cos\phi\,\hat{z}_2 = \\
& = -\sin\phi\,\left(\hat{y}_1\right) + \cos\phi\,\left(\sin\theta\,\hat{x}_1 + \cos\theta\,\hat{z}_1\right) = \\
& = \hat{x}_0\left(\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi\right) + \hat{y}_0\left(-\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi\right) + \hat{z}_0\left(\cos\phi\cos\theta\right)
\end{aligned}$$

The components $R^{0\rightarrow 3}_{ij} := \hat{e}^0 \cdot \hat{e}^3_j$ of the rotation tensor, transforming the basis 0 into the basis 3, referred either to basis 0 or basis 3,

$$\mathbb{R}^{0\rightarrow 3} = R^{0\rightarrow 3}_{ij} \hat{e}^0_i \otimes \hat{e}^0_j = R^{0\rightarrow 3}_{ij} \hat{e}^3_i \otimes \hat{e}^3_j$$

can be **collected in an array** $\mathbf{R}^{0\rightarrow 3}$, $[\mathbf{R}^{0\rightarrow 3}]_{ij} = R^{0\rightarrow 3}_{ij} = \hat{e}^0_i \cdot \hat{e}^3_j$,

$$\mathbf{R}^{0\rightarrow 3}_{321}(\psi,\theta,\phi) = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix}$$

This is the matrix of change of basis discussed in Math:Vector and Tensor Algebra:…

Gimbal lock

For $\theta = \mp\frac{\pi}{2}$, the components of the rotation tensor are

$$\begin{aligned}
\mathbf{R}^{0\rightarrow 3}_{321}(\psi, 0, \phi) & = \begin{bmatrix} 0 & 0 & \pm 1 \\ -\cos\phi\sin\psi \mp \sin\phi\cos\psi & \cos\phi\cos\psi \mp \sin\phi\sin\psi & 0 \\ \sin\phi\sin\psi \mp \cos\phi\cos\psi & -\sin\phi\cos\psi \mp \cos\phi\sin\psi & 0 \end{bmatrix} = \\
& = \begin{bmatrix} 0 & 0 & \pm 1 \\ -\sin(\psi\pm\phi) & \cos(\psi\pm\phi) & 0 \\ \mp\cos(\psi\pm\phi) & \mp\sin(\psi\pm\phi) & 0 \end{bmatrix} \ ,
\end{aligned}$$

i.e. orientations with $\theta = -\frac{\pi}{2}$ are determined by the sum $\psi + \phi$, while with $\theta = \frac{\pi}{2}$ by the difference $\psi - \phi$. In both situations, for a given orientation it's not possible to retrieve a unique pair of values (ψ, ϕ), but only their sum or difference.

Angular velocity

Angular velocity of simple rotation around a vector of the basis

Let $\mathbb{R}^{0\rightarrow 1}$ the rotation tensor representing a rotation of an angle ψ around the unit vector $\hat{z}^0 = \hat{z}^1$,

$$\mathbb{R}^{0\rightarrow 1} = \cos\psi \hat{x}^0\hat{x}^0 - \sin\psi \hat{x}^0\hat{y}^0 + \sin\psi \hat{y}^0\hat{x}^0 + \cos\psi \hat{y}^0\hat{y}^0 + 1\hat{z}^0\hat{z}^0 \ ,$$

Its time derivative at constant 0 reads

$$\frac{{}^0 d}{dt}\mathbb{R}^{0\rightarrow 1} = \dot{\psi}\left(-\sin\psi \hat{x}^0\hat{x}^0 - \cos\psi \hat{x}^0\hat{y}^0 + \cos\psi \hat{y}^0\hat{x}^0 - \sin\psi \hat{y}^0\hat{y}^0\right) \ ,$$

and thus the angular velocity reads

$$\begin{aligned}
\vec{\omega}_\times &:= \frac{{}^0 d}{dt}\mathbb{R}^{0\to 1}\cdot\mathbb{R}^{1\to 0} = \\
&:= \dot{\psi}\left(-\sin\psi\hat{x}^0\hat{x}^0 - \cos\psi\hat{x}^0\hat{y}^0 + \cos\psi\hat{y}^0\hat{x}^0 - \sin\psi\hat{y}^0\hat{y}^0\right)\cdot \\
&\quad\cdot\left(\cos\psi\hat{x}^0\hat{x}^0 - \sin\psi\hat{x}^0\hat{y}^0 + \sin\psi\hat{y}^0\hat{x}^0 + \cos\psi\hat{y}^0\hat{y}^0 + 1\hat{z}^0\hat{z}^0\right)^T = \\
&:= \dot{\psi}\left(-\sin\psi\hat{x}^0\hat{x}^0 - \cos\psi\hat{x}^0\hat{y}^0 + \cos\psi\hat{y}^0\hat{x}^0 - \sin\psi\hat{y}^0\hat{y}^0\right)\cdot \\
&\quad\cdot\left(\cos\psi\hat{x}^0\hat{x}^0 + \sin\psi\hat{x}^0\hat{y}^0 - \sin\psi\hat{y}^0\hat{x}^0 + \cos\psi\hat{y}^0\hat{y}^0 + 1\hat{z}^0\hat{z}^0\right) = \\
&= \dot{\psi}\left[\hat{x}^0\hat{x}^0\left(-\sin\psi\cos\psi + \cos\psi\sin\psi\right) + \hat{x}^0\hat{y}^0\left(-\sin^2\psi - \cos^2\psi\right) + \right. \\
&\left. + \hat{y}^0\hat{x}^0\left(\cos^2\psi + \sin^2\psi\right) + \hat{y}^0\hat{y}^0\left(\cos\psi\sin\psi - \sin\psi\cos\psi\right)\right] = \\
&= -\dot{\psi}\,\hat{x}^0\hat{y}^0 + \dot{\psi}\,\hat{y}^0\hat{x}^0 \ .
\end{aligned}$$

and thus

$$\vec{\omega} = \dot{\psi}\hat{z}^0 \ .$$

If the rotation occurs around the vector of the original basis, the components of the angular velocity don't change when referred to either bases, as $\hat{z}^1 = \mathbb{R}\cdot\hat{z}^0 = \hat{z}^0$, and thus

$$\vec{\omega} = \dot{\psi}\hat{z}^0 = \dot{\psi}\hat{z}^1 \ .$$

This is not true for generic rotations!

Relative angular velocities of intermediate Euler's rotations

As shown for the "yaw" angle ψ before, it's easy to prove that

$$\begin{aligned}
\vec{\omega}^{1/0} &= \dot{\psi}\,\hat{z}_0 = \dot{\psi}\,\hat{z}_1 \\
\vec{\omega}^{2/1} &= \dot{\theta}\,\hat{y}_1 = \dot{\theta}\,\hat{y}_2 \\
\vec{\omega}^{3/2} &= \dot{\phi}\,\hat{x}_2 = \dot{\phi}\,\hat{x}_3
\end{aligned}$$

Angular velocity as the sum of relative angular velocity of intermediate rotations

Using addition of relative angular velocity of intermediate rotations,

$$\vec{\omega}^{3/0} = \vec{\omega}^{3/2} + \vec{\omega}^{2/1} + \vec{\omega}^{1/0} \ ,$$

and it can be expressed w.r.t. vectors of reference frame 0

$$\begin{aligned}
\vec{\omega}^{3/0} &= \dot{\phi}\,\hat{x}_2 + \dot{\theta}\,\hat{y}_1 + \dot{\psi}\,\hat{z}_0 = \\
&= \dot{\phi}\,\left(\cos\theta\cos\psi\,\hat{x}^0 + \cos\theta\sin\psi\,\hat{y}^0 - \sin\theta\,\hat{z}^0\right) + \dot{\theta}\,\left(-\sin\psi\,\hat{x}^0 + \cos\psi\,\hat{y}^0\right) + \dot{\psi}\,\hat{z}_0 = \\
&= \left(-\dot{\theta}\,\sin\psi + \dot{\phi}\,\cos\theta\cos\psi\right)\hat{x}^0 + \left(\dot{\theta}\,\cos\psi + \dot{\phi}\,\cos\theta\sin\psi\right)\hat{y}^0 + \left(-\dot{\phi}\,\sin\theta + \dot{\psi}\right)\hat{z}^0
\end{aligned}$$

or w.r.t. vectors of reference frame 3

$$\begin{aligned}
\vec{\omega}^{3/0} &= \dot{\phi}\,\hat{x}_3 + \dot{\theta}\,\hat{y}_2 + \dot{\psi}\,\hat{z}_1 = \\
&= \dot{\phi}\,\hat{x}_3 + \dot{\theta}\,\left(\cos\phi\,\hat{y}_3 - \sin\phi\,\hat{z}_3\right) + \dot{\psi}\,\left(-\sin\theta\,\hat{x}_3 + \cos\theta\sin\phi\,\hat{y}_3 + \cos\theta\cos\phi\,\hat{z}_3\right) = \\
&= \left(\dot{\phi} - \dot{\psi}\,\sin\theta\right)\hat{x}_3 + \left(\dot{\theta}\,\cos\phi + \dot{\psi}\,\cos\theta\sin\phi\right)\hat{y}_3 + \left(-\dot{\theta}\,\sin\phi + \dot{\psi}\,\cos\theta\cos\phi\right)\hat{z}_3
\end{aligned}$$

Relation between components of angular velocity and rotation parameters and their time derivative

Let 3 a body reference frame, and 0 a global reference frame; let p, q, r the components of the angular velocity $\vec{\omega}$ w.r.t. the body reference frame

$$\vec{\omega} = p\,\hat{x}_3 + q\,\hat{y}_3 + r\,\hat{z}_3 \ .$$

Exploiting matrix formalism, the relation between the components of the angular velocity in the body reference frame, Euler's angles and thier time derivative is

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & \cdot & -\sin\theta \\ \cdot & \cos\phi & \cos\theta\sin\phi \\ \cdot & -\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

The inverse relation exists for $\cos\theta \neq 0$ (**gimbal lock**),

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \frac{1}{\cos\theta} \begin{bmatrix} 1 & \sin\theta\sin\phi & \sin\theta\cos\phi \\ \cdot & \cos\theta\cos\phi & -\cos\theta\sin\phi \\ \cdot & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Thus, collecting the body components of the angular velocity in $\omega = (p, q, r)^T$, calling $\mathbf{S}(\mathbf{q})$ the matrix collecing the coefficients depending on the rotation parameters $\mathbf{q} = (\phi, \theta, \psi)$, and $\dot{\mathbf{q}}$ their time derivatives, it remains proved that using Euler's angles the relation

$$\omega = \mathbf{S}(\mathbf{q})\,\dot{\mathbf{q}}$$

always exists, but the inverse relation exists only for $\theta \neq \mp\frac{\pi}{2}$.

Linearization

…

1.5.4 Rotation parametrization: quaternions

Here quaternion parametrization of rotations is introducing the unitary quaternion

$$\mathbf{q} = q_0 + \vec{q} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\vec{n} \ , \tag{1.15}$$

into *axis and angle parametrization*.

Normalization condition.

$$1 = q_0^2 + \vec{q}\cdot\vec{q}$$

Rotation tensor

$$\mathbb{R} = \mathbb{I} + 2q_0\vec{q}_\times + 2\vec{q}_\times\vec{q}_\times \ ,$$

Linearization

$$\begin{aligned} \Delta\mathbb{R} = 2\Delta q_0\vec{q}_\times + 2q_0\Delta\vec{q}_\times + 2\Delta\vec{q}_\times\vec{q}_\times + 2\vec{q}_\times\Delta\vec{q}_\times \\ \vec{\theta}_{\Delta,\times} := \Delta\mathbb{R}\cdot\mathbb{R}^T \\ \vec{\theta}_\Delta = -2\vec{q}\Delta q_0 + 2q_0\Delta\vec{q} - 2\vec{q}_\times\Delta\vec{q} \end{aligned}$$

Angular velocity

$$\begin{aligned} \vec{\omega}_\times := \dot{\mathbb{R}}\cdot\mathbb{R}^T \\ \vec{\omega} = -2\vec{q}\dot{q}_0 + 2q_0\dot{\vec{q}} - 2\vec{q}_\times\dot{\vec{q}} \end{aligned}$$

Rotation tensor

Introducing the expression (1.15) of the unitary quaternion into expression (1.13) of the rotation tensor written using *axis and angle parametrization*, and using trigonometric identities

$$\begin{aligned}
\sin\theta &= \sin\left(2\frac{\theta}{2}\right) = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\
\cos\theta &= \cos\left(2\frac{\theta}{2}\right) = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \\
&= 2\cos^2\frac{\theta}{2} - 1 = \\
&= 1 - 2\sin^2\frac{\theta}{2}
\end{aligned}$$

the expression of the rotation tensor becomes

$$\begin{aligned}
\mathbb{R} &= \mathbb{I} + \sin\theta\hat{n}_\times + (1-\cos\theta)\hat{n}_\times\hat{n}_\times = \\
&= \mathbb{I} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\hat{n}_\times + 2\sin^2\frac{\theta}{2}\hat{n}_\times\hat{n}_\times = \\
&= \mathbb{I} + 2q_0\vec{q}_\times + 2\vec{q}_\times\vec{q}_\times
\end{aligned}$$

Angular velocity

$$\begin{aligned}
\vec{\omega}_\times &= \dot{\mathbb{R}}\cdot\mathbb{R}^T = \\
&= \left(2\dot{q}_0\vec{q}_\times + 2q_0\dot{\vec{q}}_\times + 2\dot{\vec{q}}_\times\vec{q}_\times + 2\vec{q}_\times\dot{\vec{q}}_\times\right)\cdot\left(\mathbb{I} - 2q_0\vec{q}_\times + 2\vec{q}_\times\vec{q}_\times\right) = \\
&= 2\dot{q}_0\vec{q}_\times + 2q_0\dot{\vec{q}}_\times + 2\dot{\vec{q}}_\times\vec{q}_\times + 2\vec{q}_\times\dot{\vec{q}}_\times + \\
&\quad - 4\dot{q}_0 q_0\vec{q}_\times\vec{q}_\times - 4q_0^2\dot{\vec{q}}_\times\vec{q}_\times - \underbrace{4q_0\dot{\vec{q}}_\times\vec{q}_\times\vec{q}_\times}_{(a)} - \underbrace{4q_0\vec{q}_\times\dot{\vec{q}}_\times\vec{q}_\times}_{-4q_0\left(\dot{\vec{q}}\cdot\vec{q}\right)\vec{q}_\times} + \\
&\quad + 4\dot{q}_0\underbrace{\vec{q}_\times\vec{q}_\times\vec{q}_\times}_{-|\vec{q}|^2\vec{q}_\times} + \underbrace{4q_0\dot{\vec{q}}_\times\vec{q}_\times\vec{q}_\times}_{(a)} + 4\dot{\vec{q}}_\times\underbrace{\vec{q}_\times\vec{q}_\times\vec{q}_\times}_{-|\vec{q}|^2\vec{q}_\times} + \underbrace{4\vec{q}_\times\dot{\vec{q}}_\times\vec{q}_\times\vec{q}_\times}_{-\left(\dot{\vec{q}}\cdot\vec{q}\right)\vec{q}_\times\vec{q}_\times} = \\
&= \underbrace{2\dot{q}_0\vec{q}_\times}_{(e)} + 2q_0\dot{\vec{q}}_\times + \underbrace{2\dot{\vec{q}}_\times\vec{q}_\times}_{(c.1)} + 2\vec{q}_\times\dot{\vec{q}}_\times + \\
&\quad - 4\underbrace{\dot{q}_0 q_0\vec{q}_\times\vec{q}_\times}_{-(b)} - \underbrace{4q_0^2\dot{\vec{q}}_\times\vec{q}_\times}_{(c.2)} + \underbrace{4q_0\left(\dot{\vec{q}}\cdot\vec{q}\right)\vec{q}_\times}_{(d)} + \\
&\quad - \underbrace{4\dot{q}_0|\vec{q}|^2\vec{q}_\times}_{4\dot{q}_0\vec{q}_\times - 4\dot{q}_0 q_0^2\vec{q}_\times = 2(e)-(d)} - \underbrace{4|\vec{q}|^2\dot{\vec{q}}_\times\vec{q}_\times}_{(c.3)} - \underbrace{4\left(\dot{\vec{q}}\cdot\vec{q}\right)\vec{q}_\times\vec{q}_\times}_{(b)} = \\
&= -2\dot{q}_0\vec{q}_\times + 2q_0\dot{\vec{q}}_\times - 2\dot{\vec{q}}_\times\vec{q}_\times + 2\vec{q}_\times\dot{\vec{q}}_\times = \\
&= \left[-2\vec{q}\dot{q}_0 + 2q_0\dot{\vec{q}} - 2\vec{q}_\times\dot{\vec{q}}\right]_\times
\end{aligned}$$

having used

- $\vec{q}_\times^T = -\vec{q}_\times$, $\left(\vec{q}_\times\vec{q}_\times\right)^T = \vec{q}_\times\vec{q}_\times$
- $1 = q_0^2 + |\vec{q}|^2$, and thus $\vec{q}\cdot\dot{\vec{q}} = -q_0\dot{q}_0$
- $\vec{q}_\times\vec{q}_\times = \vec{q}\otimes\vec{q} - |\vec{q}|^2\mathbb{I}$
- $\vec{q}_\times\vec{q}_\times\vec{q} = \underbrace{\vec{q}_\times\vec{q}\otimes\vec{q}}_{=\mathbb{0}} - |\vec{q}|^2\vec{q}_\times = -|\vec{q}|^2\vec{q}_\times$
- $\dot{\vec{q}}_\times\vec{q}_\times = \vec{q}\,\dot{\vec{q}} - \left(\dot{\vec{q}}\cdot\vec{q}\right)\mathbb{I}$

- $\vec{q}_\times \dot{\vec{q}}_\times = \dot{\vec{q}}\,\vec{q} - \left(\dot{\vec{q}} \cdot \vec{q}\right) \mathbb{I}$

Proof - todo

- $(\vec{a}_\times \vec{b})_\times = \vec{a}_\times \vec{b}_\times - \vec{b}_\times \vec{a}_\times$

Proof $(\vec{a}_\times \vec{b})_\times = \vec{a}_\times \vec{b}_\times - \vec{b}_\times \vec{a}_\times$

Using

$$(\vec{a} \times \vec{b}) \times \vec{c} + (\vec{b} \times \vec{c}) \times \vec{a} + (\vec{c} \times \vec{a}) \times \vec{b} = \vec{0}$$

(todo, prove it with index notation) it immediately follows the desired relation

$$\begin{aligned}
(\vec{a}_\times \vec{b})_\times \vec{v} &= (\vec{a} \times \vec{b}) \times \vec{v} = \\
&= -(\vec{b} \times \vec{v}) \times \vec{a} - (\vec{v} \times \vec{a}) \times \vec{b} = \\
&= \vec{a} \times (\vec{b} \times \vec{v}) - \vec{b} \times (\vec{a} \times \vec{v}) = \\
&= \left(\vec{a}_\times \vec{b}_\times - \vec{b}_\times \vec{a}_\times\right) \vec{v}
\end{aligned}$$

CHAPTER

TWO

ACTIONS

What is an action?[1] Following the introduction of fundamental mechanics concepts in his *Principia Naturalis*, Newton conceived the concept of action—including both **forces** and **moments—**as the possible **causes of variation in the "true motion"** of a mechanical system or, equivalently, the causes of the **difference between true motion and a *generic relative motion***.

Definition 2.1 ("True motion")

Newton's concept of *true motion* is meant as the motion w.r.t. an inertial reference frame. So what is an ***inertial reference frame***? From an operational point of view, dynamometers measure no force and moment associated with uniform motion w.r.t. a inertial reference frame.

A force is a vectorial physical quantity that, from an operational point of view, can be measured with a **3-axis force sensor** (which measures its three components in a 3-dimensional space) or with a **dynamometer** (which measures its intensity), provided it is free to orient itself along the force direction or if the force direction is known and the instrument is aligned with it.

Definition 2.2 (True forces in classical mechanics)

Referring to the four fundamental interactions, the significant interactions in the realm of classical mechanics are only those of ***gravitational*** and **electromagnetic** nature. Electromagnetic interactions can manifest with bodies having a net charge or, more frequently in classical mechanics, between bodies with no net charge. Among the latter cases, in classical mechanics, it is common to observe the macroscopic manifestation of the microscopic electromagnetic interaction between the elementary components of matter in the form of:

- ***contact*** interactions, where it is possible to distinguish:
  - a **normal** component to the contact surfaces responsible for the impenetrability of bodies,
  - and a tangential component to the surfaces that manifests as ***friction***
- material response to stresses, such as in the elastic constitutive law for *springs*

A measured action that is not a result of the fundamental interactions is due to non-inertial motion of the dynamometer - or, very unlikely, to a new interaction you've just discover. If you experience this situation, please remember to send me an invitation for the Nobel cerimony.

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[1] The answer to the question "what is an action?" might imply a "true" knowledge—whatever that means—of the object-concept "action." Here too, as in other cases, the question "what is…?" can be replaced with "what do we mean by…?", and an "operational answer" can be considered satisfactory, as it reflects the mode of knowledge and formation of understanding in the scientific field: without delving into more abstract philosophical domains, in physics, we are content to define something through its interactions and effects on other systems, its properties, and a reliable process for its measurement.

2.1 Force, Moment of a Force, Distributed Actions

2.1.1 Concentrated Force

A (concentrated) force is a vector quantity with physical dimensions,

$$[\text{force}] = \frac{[\text{mass}][\text{length}]}{[\text{time}]^2}$$

which can be measured using a dynamometer, and whose effect can alter the equilibrium or motion conditions of a physical system.

In addition to the typical information of a vector quantity - magnitude, direction, and sense - contained in the force vector $\vec{F}$, it is often necessary to know the **point of application** or the line of application of the force.

2.1.2 Moment of a Concentrated Force

The moment of a force $\vec{F}$ applied at point P, or with a line of application passing through P, relative to point H is defined as the vector product,

$$\vec{M}_H = (P - H) \times \vec{F}$$

2.1.3 System of Forces, Resultant of Actions, and Equivalent Loads

Given a system of N forces $\left\{\vec{F}_n\right\}_{n=1:N}$, applied at points P_n, we define:

- **resultant** of the system of forces: the sum of the forces,

$$\vec{R} = \sum_{n=1}^{N} \vec{F}_n \ ,$$

- resultant of the moments with respect to a point H: the sum of the moments

$$\vec{M}_H = \sum_{n=1}^{N} (P_n - H) \times \vec{F}_n \ ,$$

- an **equivalent load**: a system of forces that has the same resultant of forces and moments; for a system of forces, an equivalent load can be defined as a single force, the resultant of the forces $\vec{R}$ applied at point Q derived from the equivalence of moments

$$\vec{R} = \sum_{n=1}^{N} \vec{F}_n$$
$$(Q - H) \times \vec{R} = \sum_{n=1}^{N} (P_n - H) \times \vec{F}_n$$

2.1.4 Couple of Forces

A couple of forces is an equivalent load to two forces of equal magnitude and opposite sense, $\vec{F}_2 = -\vec{F}_1$, applied at two points P_1, P_2 not aligned along the line of application of the forces to have non-zero effects.

todo *image*

The resultant of the forces is zero,

$$\vec{R} = \vec{F}_1 + \vec{F}_2 = \vec{F}_1 - \vec{F}_1 = \vec{0} \, ,$$

while the resultant of the moments does not depend on the moment pole,

$$\begin{aligned}
\vec{M}_H &= (P_1 - H) \times \vec{F}_1 + (P_2 - H) \times \vec{F}_2 = \\
&= (P_1 - H) \times \vec{F}_1 - (P_2 - H) \times \vec{F}_1 = \\
&= (P_1 - P_2) \times \vec{F}_1 =: \vec{C} \, .
\end{aligned}$$

2.1.5 Force Fields

todo

2.1.6 Distributed Actions

todo

2.2 Work and Power

In mechanics, as will become clearer later (**todo** add reference), the concept of work is linked to the concept of energy. **todo**

2.2.1 Work and Power of a Force

Work. The elementary work of a force $\vec{F}$ applied at point P that undergoes an elementary displacement $d\vec{r}_P$ is defined as the dot product between the force and the displacement,

$$\delta W := \vec{F} \cdot d\vec{r}_P \, . \tag{2.1}$$

The work done by the force $\vec{F}$ applied at point P moving from point A to point B along the path ℓ_{AB} is the sum of all elementary contributions - and hence, in the limit for elementary displacements $\rightarrow 0$ for continuous variations, the line integral,

$$W_{\ell_{AB}} = \int_{\ell_{AB}} \delta W = \int_{\ell_{AB}} \vec{F} \cdot d\vec{r}_P \, . \tag{2.2}$$

In general, the work of a force or a field of forces depends on the path ℓ_{AB}. In cases where the work is independent of the path but depends only on the endpoints, we talk about *conservative actions*.

Power. The power of the force is defined as the time derivative of the work,

$$P := \frac{\delta W}{dt} = \vec{F} \cdot \frac{d\vec{r}_P}{dt} = \vec{F} \cdot \vec{v}_P \, ,$$

and coincides with the dot product between the force and the velocity of the point of application. Be cautious if a force is applied to geometric points rather than material points, such as in the case of a disk rolling without slipping on a surface: at every instant, the (new) material contact point has zero velocity, while the geometric contact point is the projection of the center of the disk and moves with the same velocity, $v = R\theta$

2.2.2 Work and Power of a System of Forces

Work. The work of a system of forces is the sum of the works of the individual forces,

$$\delta W = \sum_{n=1}^{N} \delta W_n = \sum_{n=1}^{N} \vec{F}_n \cdot d\vec{r}_n$$

Power. The power of a system of forces is the sum of the powers of the individual forces,

$$P = \sum_{n=1}^{N} P_n = \sum_{n=1}^{N} \vec{F}_n \cdot \vec{v}_n \ .$$

2.2.3 Work and Power of a Couple of Forces

Work. The elementary work of a couple of forces is the sum of the elementary works,

$$\begin{aligned} \delta W &= \vec{F}_1 \cdot d\vec{r}_1 + \vec{F}_2 \cdot d\vec{r}_2 = \\ &= \vec{F}_1 \cdot (d\vec{r}_1 - d\vec{r}_2) = \end{aligned}$$

Power. The power of a couple of forces,

$$P = \vec{F}_1 \cdot (\vec{v}_1 - \vec{v}_2)$$

can be rewritten if the points of application perform a rigid motion act (**todo** verify the definition of motion act and if it should be introduced),

$$\vec{v}_1 - \vec{v}_2 = \vec{\omega} \times (P_1 - P_2) \ ,$$

as

$$\begin{aligned} P &= \vec{F}_1 \cdot (\vec{v}_1 - \vec{v}_2) = \\ &= \vec{F}_1 \cdot [\vec{\omega} \times (P_1 - P_2)] = \\ &= \vec{\omega} \cdot \left[(P_1 - P_2) \times \vec{F}_1 \right] = \\ &= \vec{\omega} \cdot \vec{C} \ . \end{aligned}$$

2.3 Conservative Actions

In general, the work of a force field acting on a point P moving in space from point A to point B along a path ℓ_{AB} represented by integral (2.2) depends on the path, and this dependence on the path is usually highlighted with the use of the symbol δ in the elementary work (2.1).

If the work of a force field does not depend on the path ℓ_{AB} but only on the endpoints A, B, for all pairs of points within a region of space Ω, the **force field** is said to be **conservative** in the region Ω of space. In this case, the work integral can be written as the difference of a scalar field, $U(P)$ or its opposite $V(P) := -U(P)$,

$$\begin{aligned} W_{AB} = \int_{\ell_{AB}} & \vec{F} \cdot d\vec{r} = \\ = \int_{\ell_{AB}} & \delta W = \\ & = U(B) - U(A) = \Delta_{AB} U \\ & = V(A) - V(B) = -\Delta_{AB} V \end{aligned}$$

The functions U, V are respectively defined as the **potential** and **potential energy** of the force field. From the definition of a conservative force field it readily follows that

$$\oint_{\ell} \vec{F} \cdot d\vec{r} = 0 \; .$$

The elementary work can thus be expressed in terms of the differential of these functions,

$$\begin{aligned} \delta W = \quad dU = \quad d\vec{r} \cdot \nabla U = \\ = -dV = -d\vec{r} \cdot \nabla V \end{aligned}$$

Comparing this relation with the definition of work $\delta W = d\vec{r} \cdot \vec{F}$, it is possible to identify the force field with the gradient of the potential function, and the opposite of the gradient of the potential energy,

$$\vec{F} = \nabla U = -\nabla V \; .$$

Since the force field can be written as the gradient of a scalar field, and the curl of a gradient is identically zero, the curl of a potential force field is identically zero,

$$\nabla \times \vec{F} = \vec{0} \; .$$

Note: The reverse logical process - $\nabla \times \vec{F} = \vec{0}$ implies $\vec{F} = \nabla U$ implies $\vec{F}$ conservative, i.e. independence of the work from the path - requires the domain continaing the integration path ℓ to be simply connected.

Example 2.3.1 (Force fields in non-simply connected domains)

In a region of E^2, described with Cartesian coordinates, containing the origin $O \equiv (x_0, y_0) \equiv (0, 0)$ the vector field

$$\vec{F}_1(\vec{r}) = \frac{x}{x^2 + y^2} \hat{x} + \frac{y}{x^2 + y^2} \hat{y}$$

is conservative, while the vector field

$$\vec{F}_2(\vec{r}) = -\frac{y}{x^2 + y^2} \hat{x} + \frac{x}{x^2 + y^2} \hat{y}$$

is not conservative, even though their curl is zero in all the points of the domain where the field is defined - they're not defined in the origin.

2.4 Examples of Forces

2.4.1 Gravitation

Universal Law of Gravitation

The force $\vec{F}_{12}$ exerted by a mass m_2 at P_2 on a mass m_1 at P_1 is described by **Newton's Universal Law of Gravitation**,

$$\vec{F}_{12} = G\, m_1\, m_2 \frac{P_2 - P_1}{|P_2 - P_1|^3} \ ,$$

or

$$\vec{F}_{12} = G\, m_1\, m_2 \frac{\hat{r}_{12}}{|\vec{r}_{12}|^2} \ ,$$

where $\vec{r}_{12} = (P_2 - P_1)$ is the vector pointing from point P_1 to point P_2, $r_{12} = |\vec{r}_{12}|$ is its magnitude, and $\hat{r}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|}$ is the unit vector in the same direction. The **universal gravitational constant** G is

$$G = 6.67 \cdot 10^{-11} \frac{N\, m^2}{kg^2}$$

and is considered a constant of nature.

Principle of Superposition of Causes and Effects (PSCE). Principle of superposition holds, i.e. the force acting on a mass m placed in P due to a set of N masses $\{m_k\}_{k=1:N}$ placed in P_k is the sum of individual forces $\vec{F}_k$,

$$\vec{F} = \sum_{k=1}^{N} \vec{F}_k = G\, m \sum_{k=1}^{N} m_k \frac{P_k - P}{|P_k - P|^3} \ . \tag{2.3}$$

Gravitational Field

The gravitational field generated by a set of masses $\{m_k\}_{k=1:N}$ located at P_k is a vector field associating a vector with physical dimensions $\frac{[\text{force}]}{[\text{mass}]}$ to each point in space P, that can be thought as the force per unit-mass acting on a **test mass** m placed in P, whose expression directly follows from (2.3)

$$\vec{g}(P) = \frac{\vec{F}}{m} = G \sum_{k=1}^{N} m_k \frac{P_k - P}{|P_k - P|^3} \ .$$

Given the gravitational field $\vec{g}(P)$, the gravitational force experienced by a system of mass m at P can be written as

$$\vec{F}_g = m \vec{g}(P)$$

Gravitational Potential Energy. Gravitational potential of a system of 2 masses reads

$$V(P) = -G\, m\, m_1 \frac{1}{|P - P_1|} \ ,$$

as it can be easily shown evaluating its gradient,

$$\begin{aligned}
\nabla V(P) & = - G \, m \, m_1 \, \hat{x}_k \, \frac{\partial}{\partial x_k} \frac{1}{|P - P_1|} = \\
& = - G \, m \, m_1 \, \hat{x}_k \, \left(- \frac{1}{|P - P_1|^2} \right) \frac{\partial}{\partial x_k} |P - P_1| = \\
& = G \, m \, m_1 \, \hat{x}_k \, \left(\frac{1}{|P - P_1|^2} \right) \frac{x_k - x_{1,k}}{|P - P_1|} = \\
& = G \, m \, m_1 \, \frac{x_k - x_{1,k}}{|P - P_1|^3} \, \hat{x}_k = \\
& = G \, m \, m_1 \, \frac{P - P_1}{|P - P_1|^3} \ .
\end{aligned}$$

Potential energy stored in a system of N point masses $\{m_k\}_{k=1:N}$ coincides with the work needed to build the system - a common choice to set the arbitrary additional constant of the energy is setting it equal to zero when masses are at infinite distances -, namely

$$V(P_k) = \sum_{\{i,k\}, i \neq k} G \, m_i \, m_k \frac{1}{|P_i - P_k|} \ ,$$

summing over different unordered pairs, i.e. $\{1, 2\}$ and $\{2, 1\}$ are the same pair and thus considered only once, or

$$V(P_k) = \frac{1}{2} \sum_{(i,k), i \neq k} G \, m_i \, m_k \frac{1}{|P_i - P_k|} \ ,$$

summing over different ordered pairs, i.e. $(1, 2)$ and $(2, 1)$ are different pairs.

Gravitational Field Near Earth's Surface

Within a limited domain near Earth's surface, it is common to approximate Earth's gravitational field as a uniform field, directed along the local vertical toward the center of the Earth, with intensity $g = G \frac{M_E}{R_E^2}$.

This model can be derived by approximating the position vector relative to the Earth's center $P - P_E \sim R_E \hat{r}$ and the unit vector identifying the direction from a point in the domain to the Earth's center with the local vertical $\hat{r}_{12} \sim -\hat{z}$,

$$\vec{g}(\vec{r}) = -G \frac{M_E}{R_E^2} \hat{z} = -g \hat{z} \ .$$

The gravitational force experienced by a body of mass m near Earth's surface is thus

$$\vec{F}_g = -mg\hat{z} \ ,$$

commonly referred to as **weight**.

Gravitational Potential Energy. It can be shown that the gravitational potential near Earth's surface becomes

$$V(P) = m \, g \, z_P \ .$$

Proof.

With the series expansion, with $P - P_E = R_E \hat{r} + \vec{d}$, and $|\vec{d}| \ll R_E$,

$$\begin{aligned}
V(P) &= -G\, m\, M_E \frac{1}{|P - P_E|} = \\
&\approx G M_E\, m \left[-\frac{1}{R_E} + \frac{R_E \hat{r} \cdot \vec{d}}{R_E^3} \right] = \\
&= \underbrace{-\, m \frac{G M_E}{R_E}}_{\text{const}} + m \underbrace{\frac{G\, M_E}{R_E^2}}_{=g} \underbrace{\hat{r} \cdot \vec{d}}_{=z}
\end{aligned}$$

Gravitational field of a continuous mass distribution

Mass density field $\rho(\vec{r}_0)$ for $\vec{r}_0 \in V_0$ produces the **gravitational field** in $\vec{r}$,

$$\vec{g}(\vec{r}) = \int_{\vec{r}_0 \in V_0} d\vec{g}(\vec{r}, \vec{r}_0) = -\int_{\vec{r}_0 \in V_0} G \rho(\vec{r}_0) \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3} \,.$$

By direct computation, the **gravitational potential** $\phi(\vec{r})$, s.t. $\vec{g} = \nabla \phi$, reads

$$\phi(\vec{r}) = \int_{\vec{r}_0 \in V_0} G \rho(\vec{r}_0) \frac{1}{|\vec{r} - \vec{r}_0|}$$

Gauss' law for the gravitational field

The flux of the gravitational field produced by mass distribution $\rho(\vec{r}_0)$ in volume V_0 thorugh a closed surface ∂V reads

$$\begin{aligned}
\oint_{\vec{r} \in \partial V} \vec{g}(\vec{r}) \cdot \hat{n}(\vec{r}) &= -G \oint_{\vec{r} \in \partial V} \int_{\vec{r}_0 \in V_0} \rho(\vec{r}_0) \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3} \cdot \hat{n}(\vec{r}) = \\
&= -G \int_{\vec{r}_0 \in V_0} \rho(\vec{r}_0) \oint_{\vec{r} \in \partial V} \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3} \cdot \hat{n}(\vec{r})
\end{aligned}$$

The inner integral can be written as the solid angle of the surface ∂V as seen by the point $\vec{r}_0$, whose value is

$$\oint_{\vec{r} \in \partial V} \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3} \cdot \hat{n}(\vec{r}) = 4\pi \begin{cases} 1 & \text{if } \vec{r}_0 \in V \\ \theta(\vec{r}_0, \partial V) & \text{if } \vec{r}_0 \in \partial V \\ 0 & \text{if } \vec{r}_0 \notin V \cup \partial V \end{cases}$$

Thus, net contributions of the flux of the gravitational field $\vec{g}(\vec{r})$ through ∂V come only from points $\vec{r}_0$ inside V, $\vec{r}_0 \in V$.[1] Thus the flux becomes

$$\oint_{\vec{r} \in \partial V} \vec{g}(\vec{r}) \cdot \hat{n}(\vec{r}) = -G \int_{\vec{r}_0 \in V_0 \cap V} 4\pi \rho(\vec{r}_0)$$

or, setting $\rho(\vec{r}_0) =$ in all the points $\vec{r} \in V$, $\vec{r} \notin V_0$, and changing the name of the dummy integration variable $\vec{r}_0 \to \vec{r}$,

$$\oint_{\vec{r} \in \partial V} \vec{g}(\vec{r}) \cdot \hat{n}(\vec{r}) = -G \int_{\vec{r} \in V} 4\pi \rho(\vec{r}) \,.$$

[1] Contributions from points outside ∂V are identically zero; contributions from surface ∂V are zero if volume mass density $\rho(\vec{r}_0)$ is regular enough, i.e. it contains Dirac's δ representing surface distribution that would have non-negligible contributions in integration over V.

If the gravitational field $\vec{g}(\vec{r})$ is regular enough for the divergence theorem to hold, it follows

$$\oint_{\vec{r}\in\partial V} \vec{g}(\vec{r}) \cdot \hat{n}(\vec{r}) = \int_{\vec{r}\in V} \nabla \cdot \vec{g}(\vec{r}) = -G \int_{\vec{r}\in V} 4\pi\rho(\vec{r}) \ , \tag{2.4}$$

or, for the arbitrariety of the volume V,

$$-\nabla \cdot \vec{g} = 4\pi G\rho \ .$$

Introducing the gravitational potential $\phi(\vec{r})$, whose gradient equals the gravitational field $\nabla\phi = \vec{g}$ by definition, a **Poisson equation for the gravitational potential** follows

$$-\nabla^2\phi = 4\pi G\rho \ . \tag{2.5}$$

2.4.2 Elastic Actions: Linear Springs

todo

2.5 Constraint Reactions

Kinematic constraints act on a system by limiting its possible movements, exerting forces and moments, which are defined as constraint reactions.

In general, at an **ideal** constraint (**todo** provide definition of ideal constraint and discuss/mention/refer to friction), a constraint reaction corresponds to each constrained degree of freedom: for example, the constraint of translation of a point in a direction has a corresponding reaction force in that direction; the constraint of rotation around an axis has a corresponding moment aligned with that axis.

These conditions can be derived from the equations of dynamics for massless systems, as often considered in the ideal constraint model.

2.5.1 Contact Actions

Constraint Reactions of Ideal Constraints

Ideal constraints are models that **do not perform net work**, and are thus **conservative elements**. As should become evident in the subsequent sections from the expressions of relative velocities and exchanged actions,

$$\begin{aligned}
P &= \vec{v}_1 \cdot \vec{F}_{21} + \vec{v}_2 \cdot \vec{F}_{12} + \vec{\omega}_1 \cdot \vec{M}_{21} + \vec{\omega}_2 \cdot \vec{M}_{12} = \\
&= (\vec{v}_1 - \vec{v}_2) \cdot \vec{F}_{21} + (\vec{\omega}_1 - \vec{\omega}_2) \cdot \vec{M}_{21} = \\
&= \vec{v}^{rel}_{21} \cdot \vec{F}_{21} + \vec{\omega}^{rel}_{21} \cdot \vec{M}_{21} \ ,
\end{aligned}$$

both terms are zero either because the relative motion is zero, or the actions act orthogonally to the relative motions.

Fixed Joint

The fixed joint constraint prevents both relative motion and relative rotation,

$$\begin{cases} \vec{0} = \vec{v}_{21}^{rel} = \vec{v}_2 - \vec{v}_1 \\ \vec{0} = \vec{\omega}_{21}^{rel} = \vec{\omega}_2 - \vec{\omega}_1 \end{cases} , \quad \begin{cases} \vec{F}_{12} = -\vec{F}_{21} \\ \vec{M}_{12} = -\vec{M}_{21} \end{cases}$$

Slider

The slider constraint prevents relative motion in one direction and relative rotation.

$$\begin{cases} \forall\ \vec{v}_{\hat{t},21}^{rel} = \vec{v}_{\hat{t},2} - \vec{v}_{\hat{t},1} \\ 0 = v_{\hat{n},21}^{rel} = v_{\hat{n},2} - v_{\hat{n},1} \\ \vec{0} = \vec{\omega}_{21}^{rel} = \vec{\omega}_2 - \vec{\omega}_1 \end{cases} , \quad \begin{cases} \vec{0} = \vec{F}_{\hat{t},12} = \vec{F}_{\hat{t},21} \\ F_{\hat{n},12} = -F_{\hat{n},21} \\ \vec{M}_{12} = -\vec{M}_{21} \end{cases}$$

Cylindrical Joint

The cylindrical joint constraint prevents relative motion and allows rotation around one axis.

$$\begin{cases} \vec{0} = \vec{v}_{21}^{rel} = \vec{v}_2 - \vec{v}_1 \\ \forall\ \omega_{\hat{t},21}^{rel} = \omega_{\hat{t},2} - \omega_{\hat{t},1} \\ \vec{0} = \vec{\omega}_{\hat{n},21}^{rel} = \vec{\omega}_{\hat{n},2} - \vec{\omega}_{\hat{n},1} \end{cases} , \quad \begin{cases} \vec{F}_{12} = -\vec{F}_{21} \\ 0 = M_{\hat{t},12} = M_{\hat{t},21} \\ \vec{M}_{\hat{n},12} = -\vec{M}_{\hat{n},21} \end{cases}$$

Spherical Joint

The spherical joint constraint prevents relative motion but allows general rotation.

$$\begin{cases} \vec{0} = \vec{v}_{21}^{rel} = \vec{v}_2 - \vec{v}_1 \\ \forall \vec{\omega}_{21}^{rel} = \vec{\omega}_2 - \vec{\omega}_1 \end{cases} , \quad \begin{cases} \vec{F}_{12} = -\vec{F}_{21} \\ \vec{0} = \vec{M}_{12} = \vec{M}_{21} \end{cases}$$

Roller

The roller constraint can be thought of as a combination of a slider and a cylindrical joint.

$$\begin{cases} \forall\ \vec{v}_{\hat{t},21}^{rel} = \vec{v}_{\hat{t},2} - \vec{v}_{\hat{t},1} \\ 0 = v_{\hat{n},21}^{rel} = v_{\hat{n},2} - v_{\hat{n},1} \\ \forall\ \omega_{\hat{t},21}^{rel} = \omega_{\hat{t},2} - \omega_{\hat{t},1} \\ \vec{0} = \vec{\omega}_{\hat{n},21}^{rel} = \vec{\omega}_{\hat{n},2} - \vec{\omega}_{\hat{n},1} \end{cases} , \quad \begin{cases} \vec{0} = \vec{F}_{\hat{t},12} = \vec{F}_{\hat{t},21} \\ F_{\hat{n},12} = -F_{\hat{n},21} \\ 0 = M_{\hat{t},12} = M_{\hat{t},21} \\ \vec{M}_{\hat{n},12} = -\vec{M}_{\hat{n},21} \end{cases}$$

Support

The support constraint is a unilateral constraint **todo** *add description*

Friction

Static Friction

Static friction is the type of friction that occurs between two bodies when there is no relative motion between them, acting as a tangential force to the contact surface. The simplest model of static friction assumes that the maximum static friction force F^s_{max} that can be exerted between two bodies is proportional to the normal reaction between them, N,

$$F^s_{max} = \mu^s \, N \ .$$

The proportionality constant μ^s is defined as the **coefficient of static friction**. Generally, static friction forces are determined by the equilibrium conditions of the body, if these conditions can be met, and the relation

$$|F^s| \geq F^s_{max} \ .$$

Dynamic Friction

Dynamic friction occurs between two bodies in contact and in relative motion, acting as a tangential force to the contact surface. The simplest model of dynamic friction assumes that the dynamic friction force is proportional to the normal reaction between the two bodies and directed opposite to the relative velocity,

$$\vec{F}_{12} = -\mu^d N \frac{\vec{v}_{12}}{|\vec{v}_{12}|} \ ,$$

where $\vec{F}_{12}$ is the force acting on body 1 due to body 2, and $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$ is the velocity of body 1 relative to body 2.

Pure Rolling

todo *add description*

CHAPTER

THREE

INERTIA

Inertia deals with mass and mass distribution of systems.

But what is mass? Mass is a physical quantity, a property of the system, that manifests itself:

- in *gravitational attraction* (being both the origin of gravitational force and the property that makes a system sensible to gravitational attraction),
- in resistance to change of motion of a system under external *actions*, as it will be clear from principles and equations of motions in *dynamics*

In the range of application of classical mechanics **mass conservation** holds, as stated by **Lavoisier principle**: the mass of a closed system is constant.

Beside mass, three main **additive dynamical quantities** are introduced: **momentum**, **angular momentum**, and **kinetic energy**. Even though their meaning could not be clear in this chapter, it would be clear in the following chapters, in the derivation of *equations of motion* of mechanical systems, like *point mass*, *system of point masses* and *rigid bodies*,…

3.1 Point mass

3.2 Discrete masses

Momentum.

$$\vec{Q} := \sum_k m_k \vec{v}_k$$

Angular momentum.

$$\vec{L}_H := \int_{V_t} (P_k - H) \times m_k \vec{v}_k$$

Kinetic energy.

$$K := \sum_k \frac{1}{2} m_k |\vec{v}_k|^2$$

3.3 Continuous systems

Momentum.

$$\vec{Q} := \int_{V_t} \rho \vec{v}$$

Angular momentum.

$$\vec{L}_H := \int_{V_t} (P - H) \times \rho \vec{v}$$

Kinetic energy.

$$K := \int_{V_t} \frac{1}{2} \rho |\vec{v}|^2$$

3.4 Rigid systems

The expression of dynamical quantities for rigid bodies can be written in terms of the velocity $\vec{v}_Q$ of a point Q of the rigid body and its angular velocity $\vec{\omega}$, exploiting the law of rigid motion (1.1) to write the velocity of each points of the rigid system as functions of $\vec{v}_Q$ and $\vec{\omega}$,

$$\vec{v}_P = \vec{v}_Q + \vec{\omega} \times (P - Q) \ .$$

3.4.1 Discrete systems

Momentum.

$$\begin{aligned}
\vec{Q} = \sum_k m_k \vec{v}_k &= \sum_k m_k \left(\vec{v}_Q + \vec{\omega} \times (P_k - Q) \right) = \\
&= m \vec{v}_Q - \sum_k m_k (P_k - Q) \times \vec{\omega} = \\
&= m \vec{v}_Q + \mathbb{S}_Q \cdot \vec{\omega} \ ,
\end{aligned}$$

having defined the static moment of inertia (a 2^{nd}-order antisymmetric tensor)

$$\mathbb{S}_Q := \vec{s}_{P \times} := - \sum_k m_k (P_k - Q)_\times \ .$$

Angular momentum.

$$\vec{L}_H = \sum_k (P_k - H) \times m_k \vec{v}_k = \underbrace{\sum_k (P_k - Q) \times m_k \vec{v}_k}_{\vec{L}_Q} + (Q - H) \times \vec{Q}$$

and

$$\begin{aligned}
\vec{L}_Q = \sum_k (P_k - Q) \times m_k \vec{v}_k &= \sum_k (P_k - Q) \times m_k \left(\vec{v}_Q - (P_k - Q) \times \vec{\omega} \right) = \\
&= \mathbb{S}_Q^T \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega} \ ,
\end{aligned}$$

having recognized the transpose of the static moment of inertia, and introduced the tensor of inertia w.r.t. reference point Q

$$\mathbb{I}_Q := -\sum_k m_k \left(P_k - Q\right)_\times \left(P_k - Q\right)_\times \ .$$

Kinetic energy.

$$\begin{aligned}
K = \sum_k \frac{1}{2} m_k |\vec{v}_k|^2 &= \sum_k \frac{1}{2} m_k \left(\vec{v}_Q + \vec{\omega} \times (P_k - Q)\right) \cdot \left(\vec{v}_Q + \vec{\omega} \times (P_k - Q)\right) = \\
&= \sum_k \frac{1}{2} m_k |\vec{v}_Q|^2 + \frac{1}{2} \sum_k 2 m_k \vec{v}_Q \cdot \left(-(P_k - Q) \times \vec{\omega}\right) + \frac{1}{2} \sum_k \vec{\omega} \cdot (P_k - Q)_\times (P_k - Q)_\times \cdot \vec{\omega} = \\
&= \frac{1}{2} \left[\sum_k m_k\right] |\vec{v}_Q|^2 + \frac{1}{2} \vec{v}_Q \cdot \left[-\sum_k m_k (P_k - Q)_\times\right] \cdot \vec{\omega} + \\
&+ \frac{1}{2} \vec{\omega} \cdot \left[\sum_k m_k (P_k - Q)_\times\right] \cdot \vec{v}_Q + \frac{1}{2} \vec{\omega} \cdot \left[\sum_k m_k (P_k - Q)_\times (P_k - Q)_\times\right] \cdot \vec{\omega} = \\
&= \frac{1}{2} m |\vec{v}_Q|^2 + \frac{1}{2} \vec{v}_Q \cdot \mathbb{S}_Q \cdot \vec{\omega} + \frac{1}{2} \vec{\omega} \cdot \mathbb{S}^T \cdot \vec{v}_Q + \frac{1}{2} \vec{\omega} \cdot \mathbb{I}_Q \cdot \vec{\omega} = \\
&= \frac{1}{2} \vec{v}_Q \cdot \left[m \vec{v}_Q + \mathbb{S}_Q \cdot \vec{\omega}\right] + \frac{1}{2} \vec{\omega} \cdot \left[\mathbb{S}_Q^T \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega}\right] = \\
&= \frac{1}{2} \vec{v}_Q \cdot \vec{Q} + \frac{1}{2} \vec{\omega} \cdot \vec{L}_Q \ .
\end{aligned} \tag{3.1}$$

having used the vector identities

$$\vec{a} \cdot \vec{b} \times \vec{c} = \vec{b} \cdot \vec{c} \times \vec{a}$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

and having introduced the expression of momentum and angular momentum in the last step.

Note: The components of static moment of inertia and tensor of inertia in a material basis - following the motion of the system - are constant.

3.4.2 Continuous systems

Momentum.

$$\begin{aligned}
\vec{Q} = \int_{V_t} \rho \vec{v} &= \int_{V_t} \rho \left(\vec{v}_Q + \vec{\omega} \times (P - Q)\right) = \\
&= m \vec{v}_Q - \int_{V_t} \rho (P_k - Q) \times \vec{\omega} = \\
&= m \vec{v}_Q + \mathbb{S}_Q \cdot \vec{\omega} \,,
\end{aligned}$$

having defined the static moment of inertia (a 2^{nd}-order antisymmetric tensor) for continuous systems,

$$\mathbb{S}_Q := \vec{s}_{P\times} := -\int_{V_t} \rho (P - Q)_\times \ .$$

Note: Using the definition of the center of mass G,

$$G = \frac{1}{m} \int_{V_t} \rho P \ ,$$

the static moment of inertia can be written as

$$\mathbb{S}_Q = -\int_{V_t} \rho (P-Q)_\times = -m(G-Q)_\times \ .$$

Note: The components of static moment of inertia w.r.t. a material reference frame are constant. Using a material Carteisan reference frame the tensor reads

$$\begin{aligned}
\mathbb{S}_Q &= -\int_{V_t} \rho (P-Q)_\times = \\
&= -\int_{V_t} \rho \left[(x^0 - x^0_Q)\hat{x}^0 + (y^0 - y^0_Q)\hat{y}^0 + (z^0 - z^0_Q)\hat{z}^0 \right]_\times = S_{ij} \hat{e}^0_i \hat{e}^0_j \ ,
\end{aligned}$$

whose components can be collected in the **antisymmetric matrix**

$$\underline{\underline{S}}_Q = [S_{Q,ij}] = -\int_{V_0} \rho \begin{bmatrix} 0 & -(z^0 - z^0_Q) & (y^0 - y^0_Q) \\ (z^0 - z^0_Q) & 0 & -(x^0 - x^0_Q) \\ -(y^0 - y^0_Q) & (x^0 - x^0_Q) & 0 \end{bmatrix} \ ,$$

so that the vector product between $-\int_V \rho(P-Q)$ and a vector $\vec{a}$ reads

$$\begin{aligned}
-\int_V \rho(P-Q) \times a &= -\int_V \rho \left[\hat{x}^0 \left(\Delta y^0 a_z - \Delta z^0 a_y \right) + \hat{y}^0 \left(\Delta z^0 a_x - \Delta x^0 a_z \right) + \hat{z}^0 \left(\Delta x^0 a_y - \Delta y^0 a_x \right) \right] \\
&= \mathbb{S} \cdot \vec{a} \ .
\end{aligned}$$

Angular momentum.

$$\vec{L}_H = \int_{V_t} (P-H) \times \rho \vec{v} = \underbrace{\int_{V_t} (P-Q) \times \rho \vec{v}_k}_{\vec{L}_Q} + (Q-H) \times \vec{Q}$$

and

$$\begin{aligned}
\vec{L}_Q = \int_{V_t} (P-Q) \times \rho \vec{v} &= \int_{V_t} (P-Q) \times \rho \left(\vec{v}_Q - (P-Q) \times \vec{\omega} \right) = \\
&= \mathbb{S}^T_Q \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega} \ ,
\end{aligned}$$

having recognized the transpose of the static moment of inertia, and introduced the tensor of inertia w.r.t. reference point Q

$$\begin{aligned}
\mathbb{I}_Q &:= -\int_{V_t} \rho \left(P-Q\right)_\times \left(P-Q\right)_\times = \\
&:= \int_{V_t} \rho \left[|P-Q|^2 \mathbb{I} - (P-Q) \otimes (P-Q) \right] \ ,
\end{aligned}$$

having used the tensor identity

$$-\vec{a}_\times \cdot \vec{a}_\times = |\vec{a}|^2 \mathbb{I} - \vec{a} \otimes \vec{a}$$

Note: The components of tensor of inertia w.r.t. a material reference frame are constant. Using a material Carteisan reference frame the tensor reads

$$\begin{aligned}\mathbb{I}_Q &= -\int_{V_t} \rho (P-Q)_\times (P-Q)_\times = \\ &= -\int_{V_t} \rho \left[|P-Q|^2 \mathbb{I} - (P-Q)\otimes(P-Q) \right] = \\ &= I^0_{Q,ij} \hat{e}^0_i \hat{e}^0_j \ ,\end{aligned}$$

whose components can be collected in the **symmetric matrix**

$$\underline{\underline{I}}^0_Q = \left[I^0_{Q,ij} \right] = \int_{V_0} \rho \begin{bmatrix} \Delta y_0^2 + \Delta z_0^2 & -\Delta x_0 \Delta y_0 & -\Delta x_0 \Delta z_0 \\ -\Delta y_0 \Delta x_0 & \Delta x_0^2 + \Delta y_0^2 & -\Delta y_0 \Delta z_0 \\ -\Delta z_0 \Delta x_0 & -\Delta z_0 \Delta y_0 & \Delta x_0^2 + \Delta z_0^2 \end{bmatrix} \ ,$$

being $\Delta x^0 := x^0_P - x^0_Q$.

3.4.3 Properties of inertia tensors of rigid bodies

Static inertia

Center of mass, G. Center of mass of of a rigid body is defined as the point G for which $\mathbb{S}_G \equiv \mathbb{0}$, whose coordinates are given by

$$G = \frac{1}{m} \int_{V_t} \rho \, P$$

Anti-symmetric. From the definition of the static inertia tensor

$$\mathbb{S}_Q \cdot \vec{a} = \int_{V_t} \rho (P-Q) \times \vec{a} = -\vec{a} \times \int_{V_t} \rho (P-Q) = -\vec{a} \cdot \mathbb{S}_Q = -\mathbb{S}^T \cdot \vec{a} \ .$$

Transport.

$$\begin{aligned}\mathbb{S}_Q &= -\int_{V_t} \rho (P-Q)_\times = \\ &= -\int_{V_t} \rho (P-R)_\times - \int_{V_t} \rho (R-Q)_\times = \\ &= \mathbb{S}_R - m (R-Q)_\times \ ,\end{aligned}$$

or w.r.t. the center of mass G,

$$\mathbb{S}_Q = \mathbb{S}_G - m (G-Q)_\times \ .$$

Tensor of inertia

Symmetric (semi)-definite positive. Inertia tensor is symmetric

$$\vec{v} \cdot \mathbb{I}_Q \cdot \vec{w} = \vec{v} \cdot \int_{V_t} \rho \left[|\Delta \vec{r}|^2 \mathbb{I} - \Delta \vec{r} \otimes \Delta \vec{r} \right] \cdot \vec{w} = \vec{w} \cdot \mathbb{I}_Q \cdot \vec{v} \ .$$

for all $\forall \vec{v}, \vec{w}$, and semi-definite positive

$$\begin{aligned}
\vec{v} \cdot \mathbb{I}_Q \cdot \vec{v} = -\vec{v} \cdot \int_{V_t} \rho \Delta\vec{r}_\times \Delta\vec{r}_\times \cdot \vec{v} = \\
= -\int_{V_t} \rho \vec{v} \cdot \Delta\vec{r}_\times \Delta\vec{r}_\times \cdot \vec{v} = \\
= -\int_{V_t} \rho \vec{v} \cdot [\Delta\vec{r} \times (\Delta\vec{r} \times \cdot\vec{v})] = \\
= -\int_{V_t} \rho \, (\Delta\vec{r} \times \vec{v}) \cdot (\vec{v} \times \Delta\vec{r}) = \\
= \int_{V_t} \rho \, (\Delta\vec{r} \times \vec{v}) \cdot (\Delta\vec{r} \times \vec{v}) = \\
= \int_{V_t} \rho |\Delta\vec{r} \times \vec{v}|^2 \geq 0
\end{aligned}$$

Principal axes of inertia. As the tensor of inertia is symmetric and definite positive, a set of orthogonal vectors $\hat{E}_i^0$ so that it can be written in diagonal form,

$$\mathbb{I}_Q = I^0_{XX} \, \hat{E}^0_X \otimes \hat{E}^0_X + I^0_{YY} \, \hat{E}^0_Y \otimes \hat{E}^0_Y + I^0_{ZZ} \, \hat{E}^0_Z \otimes \hat{E}^0_Z \, ,$$

with $I^0_{ii} \geq 0$ (no sum).

Theorem 3.4.1 (Transport - Huygens' theorem.)

$$\begin{aligned}
\mathbb{I}_Q &= -\int_{V_t} \rho (P-Q)_\times (P-Q)_\times = \\
&= -\int_{V_t} \rho (P-R)_\times (P-R)_\times - \int_{V_t} \rho (P-R)_\times (R-Q)_\times \\
&\quad - \int_{V_t} \rho (R-Q)_\times (P-R)_\times - \int_{V_t} \rho (R-Q)_\times (R-Q)_\times = \\
&= \mathbb{I}_R + \mathbb{S}_R \cdot (R-Q)_\times + (R-Q)_\times \cdot \mathbb{S}_R - m(R-Q)_\times (R-Q)_\times
\end{aligned}$$

or w.r.t. the center of mass G,

$$\mathbb{I}_Q = \mathbb{I}_G - m(Q-G)_\times (Q-G)_\times \, .$$

3.4.4 Time derivatives of dynamical quantities

Time derivatives of dynamical quantities are easily evaluated using a Cartesian material reference frame.

Momentum.

$$
\begin{aligned}
\frac{d}{dt}\vec{Q} &= \frac{d}{dt}\left(m\vec{v}_Q + \mathbb{S}_Q\cdot\vec{\omega}\right) = \\
&= m\dot{\vec{v}}_Q + \frac{d}{dt}\left(\vec{E}_i^0 S_{ij}^0\omega_j^0\right) = \\
&= m\dot{\vec{v}}_Q + \frac{d\vec{E}_i^0}{dt} S_{ij}^0\omega_j^0 + \vec{E}_i^0 S_{ij}^0 \frac{d}{\omega_j}^0 dt = \\
&= m\dot{\vec{v}}_Q + \vec{\omega}\times\vec{E}_i^0 S_{ij}^0\omega_j^0 + \vec{E}_i^0 S_{ij}^0\frac{d}{dt}\omega_j^0 = \\
&= m\dot{\vec{v}}_Q + \vec{\omega}\times\left(\mathbb{S}_Q\cdot\vec{\omega}\right) + \mathbb{S}_Q\cdot\dot{\vec{\omega}}\,.
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\vec{\omega} &= \frac{d}{dt}\left(\hat{E}_i^0\omega_i^0\right) = \\
&= \vec{\omega}\times\hat{E}_i^0\omega_i^0 + \hat{E}_i^0\frac{d\omega_i^0}{dt} = \\
&= \underbrace{\vec{\omega}\times\vec{\omega}}_{=\vec{0}} + \hat{E}_i^0\frac{d\omega_i^0}{dt}\,.
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\vec{v} &= \frac{d}{dt}\left(\hat{E}_i^0 v_i^0\right) = \\
&= \vec{\omega}\times\hat{E}_i^0 v_i^0 + \hat{E}_i^0\frac{dv_i^0}{dt} = \\
&= \vec{\omega}\times\vec{v} + \frac{{}^0 d}{dt}\vec{v}
\end{aligned}
$$

Angular momentum.

$$
\frac{d}{dt}\vec{L}_H = \frac{d}{dt}\left((Q-H)\times\vec{Q} + \vec{L}_Q\right)\;,
$$

and

$$
\frac{d}{dt}\left((Q-H)\times\vec{Q}\right) = \left(\vec{v}_Q - \dot{\vec{x}}_H\right)\times\vec{Q} + (Q-H)\times\dot{\vec{Q}}
$$

and

$$
\begin{aligned}
\frac{d\vec{L}_Q}{dt} &= \frac{d}{dt}\left(\mathbb{S}_Q^T\cdot\vec{v}_Q + \mathbb{I}_Q\cdot\vec{\omega}\right) = \\
&= \frac{d}{dt}\left[\hat{E}_i^0\left(S_{ji}^0 v_{Q,j}^0 + I_{ij}^0\omega_j^0\right)\right] = \\
&= \vec{\omega}\times\hat{E}_i^0\left(S_{ji}^0 v_{Q,j}^0 + I_{ij}^0\omega_j^0\right) + \hat{E}_i^0\left(S_{ji}^0\dot{v}_{Q,j}^0 + I_{ij}^0\dot{\omega}_j^0\right) = \\
&= \vec{\omega}\times\left(\mathbb{S}_Q^T\cdot\vec{v}_Q + \mathbb{I}_Q\cdot\vec{\omega}\right) + \left(\mathbb{S}_Q^T\cdot\frac{{}^0 d}{dt}\vec{v}_Q + \mathbb{I}\cdot\dot{\vec{\omega}}\right) = \\
&= \vec{\omega}\times\left(\mathbb{S}_Q^T\cdot\vec{v}_Q + \mathbb{I}_Q\cdot\vec{\omega}\right) + \left(\mathbb{S}_Q^T\cdot\left(\dot{v}_Q - \vec{\omega}\times\vec{v}_Q\right) + \mathbb{I}\cdot\dot{\vec{\omega}}\right) \\
&= \left(\mathbb{S}_Q^T\cdot\left(\dot{v}_Q - \vec{\omega}\times\vec{v}_Q\right) + \mathbb{I}\cdot\dot{\vec{\omega}}\right) + \vec{\omega}\times\left(\mathbb{S}_Q^T\cdot\vec{v}_Q + \mathbb{I}_Q\cdot\vec{\omega}\right)
\end{aligned}
$$

Dynamical quantities and time derivatives with G as reference point

$$\begin{cases} \vec{Q} = m\vec{v}_G \\ \vec{L}_G = \mathbb{I}_G \cdot \vec{\omega} \end{cases} \quad , \quad \begin{cases} \dot{\vec{Q}} = m\dot{\vec{v}}_G \\ \dot{\vec{L}}_G = \mathbb{I}_G \cdot \dot{\vec{\omega}} + \vec{\omega} \times \mathbb{I}_G \cdot \vec{\omega} \end{cases}$$

CHAPTER

FOUR

DYNAMICS

Dynamics provides the link between the motion of a body, described by *kinematics*, and the actions causing that motion.

Newton's principles of dynamics and the cardinal equations of dynamics are the physical laws that govern the motion of mechanical systems: *Newton's principles* agree with the experimental observations (for systems with negligible quantum and Einstein relativity effects) and are the starting point - the principles - of Newton's formulation of mechanics; from these principles, *equations of motion* of mechanical systems are derived. These physical laws are formulated in terms of certain physical quantities, such as momentum, angular momentum, or the kinetic energy of the system - already discussed in the section about *inertia*. These dynamic quantities have the property of being additive (by definition), and making it particularly easy to write and interpret a general form of the equations governing motion. In general, these equations relate the time derivatives of these dynamic quantities to the causes of their variation. In the absence of net causes, conservation principles hold.

4.1 Principles of Newtonian Mechanics

Newton's principle of dynamics are first established for **closed systems**, i.e. systems that don't exchange mass with the external environment and thus have constant mass. Balance equations for mass, momentum, angular momentum and other quantities for arbitary systems (closed or open) are derived in Continuum Mechanics:Governing Equations, exploiting Reynolds' transport theorem.[1]

Mass conservation. In the regime of classical mechanics, Lavoisier principle states that the mass of a closed system is constant. Roughly speaking "nothing is created nothing is destroyed".

First Principle of Dynamics (Newton's First Law): inertia and Galileian invariance. A body (more precisely, the center of mass of a body) on which no net force acts remains in its state of rest or uniform rectilinear motion relative to an *inertial reference frame*.

Second Principle of Dynamics (Newton's Second Law): momentum balance. Relative to an *inertial reference frame*, the change in momentum of a system is equal to the impulse of *true external forces* (see *Definition 2.2*) acting on it,

$$\Delta\vec{Q} = \vec{I}^e \ .$$

In the case of smooth motion, where the momentum can be represented as a continuous and differentiable function of time, the second principle of dynamics can be expressed in differential form,

$$\dot{\vec{Q}} = \vec{R}^e \ ,$$

where the resultant of the external forces, $\vec{R}^e = \frac{d\vec{I}^e}{dt}$, is the time derivative of the impulse.

[1] Reynolds's transport theorem provides the rules for time derivatives of integrals on arbitrary domains in Euclidean space, that allows to change representation to and from closed systems (material volume, a geometric volume moving with the physical system) and open systems (usually control volumes of some regions of the system that can exchage mass with the exterrnal environment).

Third Principle of Dynamics (Newton's Third Law): action-reaction. If a system i exerts a force $\vec{F}_{ji}$ on a system j, then system j exerts an "equal and opposite" force $\vec{F}_{ij}$ on system i, with equal magnitude and opposite direction,

$$\vec{F}_{ij} = -\vec{F}_{ji} \ .$$

4.1.1 Inertial reference frame

Force sensors in an inertial reference frame measure only *true forces*, as defined in *Definition 2.2*.

Here the effect of non-inertial reference frame in the description of the motion is shown on the **dynamics of a point system**, using 2^{nd} principle for the description of the dynamics for an inertial observer, and the rules for the *relative kinematics of a point*.

Here, an inertial reference frame is referred as 0, while the generic reference frame is referred as 1. The equation of motion of a point system P w.r.t. the inertial reference frame is governed by Newton's second principle,

$$m\vec{a}^0_{P/O_0} = \vec{F} \ ,$$

being $\vec{a}^0_{P/O_0}$ the acceleration of point P w.r.t. the origin O_0 of the inertial reference frame 0, as seen by the same reference frame 0 (the meaning of the apex). From relative kinematics, (1.9)

$$\begin{aligned} m\vec{a}^1_{P/O_1} & = \vec{F} + \\ & - m\left[\vec{a}^0_{O_1/O_0} + \vec{\alpha}_{1/0} \times (P - O_1) + 2\vec{\omega}_{1/0} \times \vec{v}^1_{P/O_1} + \vec{\omega}_{1/0} \times [\vec{\omega}_{1/0} \times (P - O_1)\right] \end{aligned}$$

In order for the reference frame 1 to agree with the acceleration of any point P with the inertial reference frame 0, the content of the square brackets must vanish, and thus:

- $\vec{a}^0_{O_1/O_0} = \vec{0}$, i.e. the acceleration of the origin of the reference frame O_1 w.r.t. to the reference frame 0 must be zero
- $\vec{\omega}_{1/0} = \vec{0}$, and thus implying $\vec{\alpha}_{1/0} = \vec{0}$, i.e. the angular velocity of the reference frame 1 w.r.t. the reference frame 0 are zero.

Thus, reference frame 1 agrees with the measurements of accelerations and forces of an inertial reference frame if:

- its origin O_1 is in uniform motion w.r.t. O_0
- its orientation $\mathbb{R}^{1/0}$ is constant in time, so that angular velocity and acceleration are identically zero.

Definition 4.1.1 (Class of inertial reference frames)

If these conditions hold, it's not possible to find which reference frame is "more absolute" than the other, but all the reference frames with constant relative orientation and origin in uniform relative motion are inertial reference frames, w.r.t. which the governing equations have the same expression.

4.2 Equations of Motion and Conservation Principles

Starting from the *principles of Newtonian mechanics*, it is possible to derive the dynamical equations governing the motion of mechanical systems. These equations governs the change of dynamical quantities, **momentum**, **angular momentum**, **kinetic energy**, linking them to (external) **forces**, (external) **moments** and (total) **power**. Under certain conditions, and only in these cases, the *cardinal equations* of dynamics become*principles of conservation of dynamic quantities*: by observing the expressions of the cardinal equations, it is easy to infer that the condition to obtain a conservation principle is the vanishing of all terms except for the time derivative of the conserved quantity.

4.2.1 Equations of Motion

The general form of these equations is easily expressed in terms of the dynamical quantities discussed in the section about inertia. Cardinal equations, or equations of motion, are collected here in their most general form for closed systems, and derived in the following sectinos for different systems: *point mass*, *systems of point masses*, *rigid body*,…

Momentum Balance. The time derivative of the momentum is equal to the resultant of the external forces,

$$\dot{\vec{Q}} = \vec{R}^e \ . \tag{4.1}$$

Angular Momentum Balance with respect to a point H. The time derivative of the angular momentum with respect to a point H, minus the "transport term," is equal to the resultant of the external moments with respect to the point H,

$$\dot{\vec{L}}_H + \dot{\vec{x}}_H \times \vec{Q} = \vec{M}^e_H \ . \tag{4.2}$$

Kinetic Energy Balance. The time derivative of the kinetic energy is equal to the total power acting on the system, which is the sum of the power of the external actions and the power of the internal actions within the system,

$$\dot{K} = P^{tot} = P^e + P^i \tag{4.3}$$

4.2.2 Conservation Principles

Under certain condiitons, equations of motion become principles of conservation of dynamical quantities. These conditions are easily derived by inspection of the equations of motion, nullyfing the causes of change of the dynamical quantities. Beside the conservation of momentum, angular momentum and kinetic energy, a **principle of conservation of mechanical energy** arises when actions acting on the system are *conservative*, so that its power can be written as a time derivative of a potential energy.

Conservation of Momentum in the presence of zero net external forces. If the resultant of the external forces is zero, $\vec{R}^e = \vec{0}$, from the momentum balance, we immediately obtain

$$\dot{\vec{Q}} = \vec{0} \qquad \rightarrow \qquad \vec{Q} = \bar{\vec{Q}} = \text{const.}$$

Conservation of Angular Momentum in the presence of zero net external moments. If the choice of the point H nullifies the transport term, $\dot{\vec{r}}_H \times \vec{Q} = \vec{0}$, and if the resultant of the external moments is zero, $\vec{M}^e_H = \vec{0}$, from the angular momentum balance, we immediately obtain

$$\dot{\vec{L}}_H = \vec{0} \qquad \rightarrow \qquad \vec{L}_H = \bar{\vec{L}}_H = \text{const.}$$

Conservation of Kinetic Energy in the presence of zero total power. If the total power of the actions on the system is zero, $P^{tot} = 0$, from the kinetic energy balance, we immediately obtain

$$\dot{K} = 0 \qquad \rightarrow \qquad K = \bar{K} = \text{const.}$$

Conservation of Mechanical Energy in the absence of non-conservative forces. In addition to the three conservation principles directly derived from the cardinal equations, we add the principle of the conservation of mechanical energy, which is the sum of the system's kinetic and potential energy,

$$E^{mech} = K + V \ ,$$

in the absence of non-conservative actions. If there are no non-conservative forces, the power of the actions on the system can be written as the negative of the time derivative of the system's potential energy,

$$P^{tot} = -\dot{V}$$

From the kinetic energy balance, we get

$$\dot{K} = -\dot{V} \qquad \rightarrow \qquad \frac{d}{dt}(K+V) = 0 \qquad \rightarrow \qquad \dot{E}^{mech} = 0 \qquad \rightarrow \qquad E^{mech} = \bar{E}^{mech} = \text{const.}$$

4.3 Equations of motion of a point mass

Dynamic quantities.

$$\begin{aligned}
\vec{Q}_P & := m_P \vec{v}_P \\
\vec{L}_{P,H} & := (\vec{r}_P - \vec{r}_H) \times \vec{Q} = m_P (\vec{r}_P - \vec{r}_H) \times \vec{v}_P \\
K & := \frac{1}{2} m_P \vec{v}_P \cdot \vec{v}_P = \frac{1}{2} m_P |\vec{v}_P|^2
\end{aligned}$$

Momentum balance equation. The balance equation of momentum of a point P with mass m, $\vec{Q}_P = m\vec{v}_P$ readily follows the second principle of dynamics,

$$\dot{\vec{Q}}_P = \vec{R}^e_P$$

Angular momentum balance equation. Time derivative of the angular momentum is evaluated with the rule of derivative of product,

$$\begin{aligned}
\dot{\vec{L}}_{P,H} & = \frac{d}{dt} \left[m_P (\vec{r}_P - \vec{r}_H) \times \vec{v}_P \right] = \\
& = m \left[(\dot{\vec{r}}_P - \dot{\vec{r}}_H) \times \vec{v}_P + m_P (\vec{r}_P - \vec{r}_H) \times \dot{\vec{v}}_P \right] = \\
& = -m_P \dot{\vec{r}}_H \times \vec{v}_P + m_P (\vec{r}_P - \vec{r}_H) \times \dot{\vec{v}}_P = \\
& = -\dot{\vec{r}}_H \times \vec{Q} + \vec{M}^{ext}_H \ .
\end{aligned}$$

Kinetic energy blanace equation.

$$\begin{aligned}
\dot{K}_P & = \frac{d}{dt} \left(\frac{1}{2} m_P \vec{v}_P \cdot \vec{v}_P \right) = \\
& = m_P \dot{\vec{v}}_P \cdot \vec{v}_P = \\
& = \vec{R}^e \cdot \vec{v}_P = P^e = P^{tot}
\end{aligned}$$

being the power of external actions P^e equal to the total power acting on the system, assuming there is no internal action in the point system, or at least they have zero net power.

4.4 Equations of motion of a discrete system of point masses

Starting from the dynamic equations for a single point, the dynamic equations for a system of particles can be derived using the third principle of dynamics, action/reaction. The development of these equations helps us understand that the additive nature of dynamical quantities (momentum, angular momentum, kinetic energy) directly follows from their definition.

Momentum Balance. The momentum balance for each point i in the system can be written by expressing the resultant of the external forces acting on the point as the sum of the external forces acting on the entire system and the internal forces exchanged with the other points of the system,

$$\vec{R}_i^{ext,i} = \vec{F}_i^{ext} + \sum_{j \neq i} \vec{F}_{ij} \ .$$

The momentum balance equation for the i-th mass thus becomes

$$\dot{\vec{Q}}_i = \vec{R}_i^{ext,i} = \vec{F}_i^{ext} + \sum_{j \neq i} \vec{F}_{ij} \ .$$

By summing the momentum balance equations for all masses, we obtain

$$\begin{aligned}
\sum_i \dot{\vec{Q}}_i & = \sum_i \vec{F}_i^{ext} + \sum_i \sum_{j \neq i} \vec{F}_{ij} = \\
& = \sum_i \vec{F}_i^{ext} + \sum_{\{i,j\}} \underbrace{\left(\vec{F}_{ij} + \vec{F}_{ji} \right)}_{=\vec{0}}
\end{aligned}$$

Defining the momentum of the system as the sum of the momenta of its parts, and the resultant of the external forces as the sum of the external forces acting on the parts of the system,

$$\begin{aligned}
\vec{Q} & := \sum_i \vec{Q}_i \\
\vec{R}^e & := \sum_i \vec{F}_i^{ext}
\end{aligned}$$

we recover the general form of the momentum balance equation,

$$\dot{\vec{Q}} = \vec{R}^e \ .$$

Angular Momentum Balance. The angular momentum balance for each point i in the system can be written by expressing the resultant of the external moments acting on the point as the sum of the external moments acting on the entire system and the internal moments exchanged with the other points of the system,

$$\vec{M}_{H,i}^{ext,i} = \vec{M}_{H,i}^{ext} + \sum_{j \neq i} \vec{M}_{H,ij} \ .$$

In the case where parts of the system interact via forces, the moment with respect to a point H generated by mass j on mass i is given by

$$\vec{M}_{H,ij} = (\vec{r}_i - \vec{r}_H) \times \vec{F}_{ij} \ .$$

The angular momentum balance equation for the i-th mass thus becomes

$$\dot{\vec{L}}_{H,i} + \dot{\vec{r}}_H \times \vec{Q}_i = \vec{M}_{H,i}^{ext,i} = \vec{M}_{H,i}^{ext} + \sum_{j \neq i} \vec{M}_{H,ij} \ .$$

By summing the angular momentum balance equations for all masses, we obtain

$$\begin{aligned}
\sum_i \left(\dot{\vec{L}}_i + \dot{\vec{r}}_H \times \vec{Q}_i \right) & = \sum_i \vec{M}_{H,i}^{ext} + \sum_i \sum_{j \neq i} \vec{M}_{H,ij} = \\
& = \sum_i \vec{M}_{H,i}^{ext} + \sum_{\{i,j\}} \underbrace{\left(\vec{M}_{H,ij} + \vec{M}_{H,ji} \right)}_{=\vec{0}}
\end{aligned}$$

Recognizing the total momentum of the system, and *defining the angular momentum of the system as the sum of the angular momentum of its parts*, and the resultant of the external moments as the sum of the external moments acting on the parts of the system,

$$\vec{L}_H := \sum_i \vec{L}_{H,i}$$

$$\vec{M}_H^e := \sum_i \vec{M}_{H,i}^{ext}$$

we recover the general form of the angular momentum balance equation,

$$\dot{\vec{L}}_H + \dot{\vec{r}}_H \times \vec{Q} = \vec{M}_H^e \ .$$

Kinetic Energy Balance. The kinetic energy balance of the system can be derived by taking the scalar product of the momentum balance equation for each point,

$$\vec{v}_i \cdot m_i \dot{\vec{v}}_i = \vec{v}_i \cdot \left(\vec{F}_i^e + \sum_{j \neq i} \vec{F}_{ij} \right) \ ,$$

recognizing in the first term the time derivative of the kinetic energy of the i-th point,

$$\dot{K}_i = \frac{d}{dt} \left(\frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i \right) = m_i \vec{v}_i \cdot \dot{\vec{v}}_i \ ,$$

and summing these equations to obtain

$$\sum_i \dot{K}_i = \sum_i \vec{v}_i \cdot \vec{F}_i^e + \sum_i \vec{v}_i \cdot \sum_{j \neq i} \vec{F}_{ij} \ .$$

Defining the kinetic energy of the system as the sum of the kinetic energies of its parts, and defining the power of the external/internal forces acting on the system as the sum of the power of all external/internal forces in the system,

$$K := \sum_i K_i$$

$$P^e := \sum_i P_i^{ext} = \sum_i \vec{v}_i \cdot \vec{F}_i^e$$

$$P^i := \sum_i P_i^{int} = \sum_i \vec{v}_i \cdot \sum_{j \neq i} \vec{F}_{ij}$$

we recover the general form of the kinetic energy balance equation,

$$\dot{K} = P^e + P^i = P^{tot} \ .$$

Note: While internal forces and moments have zero net resultant in momentum and angular momentum balance, this is not true for power of internal actions in kinetic energy equation.

4.5 Equations of motion of a rigid body

With different choices of the reference point H, the general expression of dynamical equations may have different, but equivalent, forms.

4.5.1 General equations

Momentum balance equation.

$$\frac{d}{dt}\vec{Q} = \vec{R}^e$$

Angular momentum balance equation.

$$\frac{d}{dt}\vec{L}_H + \dot{\vec{x}}_H \times \vec{Q} = \vec{M}^e_H$$

Kinetic energy balance equation.

$$\frac{d}{dt}K = P^{tot}$$

4.5.2 Dynamical equations w.r.t. the center of mass G

Momentum, angular momentum and kinetic energy

$$\begin{cases} \vec{Q} = m\vec{v}_G \\ \vec{L}_G = \mathbb{I}_G \cdot \vec{\omega} \\ K = \frac{1}{2}m|\vec{v}_G|^2 + \frac{1}{2}\vec{\omega} \cdot \mathbb{I}_G \cdot \vec{\omega} \end{cases} \quad , \quad \begin{cases} \dot{\vec{Q}} = m\dot{\vec{v}}_G \\ \dot{\vec{L}}_G = \mathbb{I}_G \cdot \dot{\vec{\omega}} + \vec{\omega} \times \mathbb{I}_G \cdot \vec{\omega} \\ \dot{K} = m\vec{v}_G \cdot \dot{\vec{v}}_G + \vec{\omega} \cdot \mathbb{I}_G \cdot \dot{\vec{\omega}} \end{cases}$$

Equations of motion.

$$\begin{cases} \dot{\vec{Q}} = \vec{R}^e \\ \dot{\vec{L}}_G = \vec{M}^e_G \end{cases} \quad , \quad \begin{cases} m\dot{\vec{v}}_G = \vec{R}^e \\ \mathbb{I}_G \cdot \dot{\vec{\omega}} + \vec{\omega} \times \mathbb{I}_G \cdot \vec{\omega} = \vec{M}^e_G \end{cases}$$

Time derivative of kinetic energy

$$\begin{aligned} \frac{dK}{dt} &= \frac{1}{2}\frac{d}{dt}\left(\vec{v}_G \cdot \vec{Q} + \vec{\omega}_G \cdot \vec{L}_G\right) = \\ &= \frac{1}{2}\left(\dot{\vec{v}}_G \cdot m\vec{v}_G + \vec{v}_G \cdot m\dot{\vec{v}}_G + \dot{\vec{\omega}} \cdot \mathbb{I}_G \cdot \vec{\omega} + \vec{\omega} \cdot \left(\mathbb{I} \cdot \dot{\vec{\omega}} + \vec{\omega} \times \mathbb{I}_G \cdot \vec{\omega}\right)\right) \\ &= \vec{v}_G \cdot m\dot{\vec{v}}_G + \vec{\omega} \cdot \mathbb{I}_G \cdot \dot{\vec{\omega}} \\ &= \vec{v}_G \cdot \dot{\vec{Q}} + \vec{\omega} \cdot \dot{\vec{L}}_G \\ &= \dot{\vec{v}}_G \cdot \vec{Q} + \dot{\vec{\omega}} \cdot \vec{L}_G \,. \end{aligned}$$

4.5.3 Dynamical equations w.r.t. a material point Q

Momentum, angular momentum and kinetic energy

$$\begin{cases} \vec{Q} = m\vec{v}_Q + \mathbb{S}_Q \cdot \vec{\omega} \\ \vec{L}_Q = \mathbb{S}_Q^T \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega} \\ K = \dfrac{1}{2}\vec{v}_Q \cdot \vec{Q} + \dfrac{1}{2}\vec{\omega} \cdot \vec{L}_Q \end{cases} \quad , \quad \begin{cases} \dot{\vec{Q}} = m\dot{\vec{v}}_Q + \mathbb{S}_Q \cdot \dot{\vec{\omega}} + \vec{\omega} \times \mathbb{S}_Q \cdot \vec{\omega} \\ \dot{\vec{L}}_Q = \left[\mathbb{S}_Q^T \cdot \left(\dot{v}_Q - \vec{\omega} \times \vec{v}_Q\right) + \mathbb{I} \cdot \dot{\vec{\omega}}\right] + \vec{\omega} \times \left(\mathbb{S}_Q^T \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega}\right) \\ \dot{K} = \dots \end{cases}$$

Equations of motion.

$$\begin{cases} \dot{\vec{Q}} = \vec{R}^e \\ \dot{\vec{L}}_Q + \vec{v}_Q \times \vec{Q} = \vec{M}^e_Q \end{cases}$$

$$\begin{cases} m\dot{\vec{v}}_Q + \mathbb{S}_Q \cdot \dot{\vec{\omega}} + \vec{\omega} \times \mathbb{S}_Q \cdot \vec{\omega} = \vec{R}^e \\ \left[\mathbb{S}_Q^T \cdot (\dot{v}_Q - \vec{\omega} \times \vec{v}_Q) + \mathbb{I} \cdot \dot{\vec{\omega}}\right] + \vec{\omega} \times (\mathbb{S}_Q^T \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega}) + \vec{v}_Q \times \left[m\vec{v}_Q + \mathbb{S}_Q \cdot \vec{\omega}\right] = \vec{M}_Q^e \end{cases}$$

or using the "material time derivative"

$$\frac{{}^0 d}{dt} \_\_ = \frac{d}{dt} \_\_ - \vec{\omega} \times \_\_ \,,$$

and matrix formalism to write these two vector equations

$$\begin{bmatrix} m\mathbb{I} & \mathbb{S}_Q \\ \mathbb{S}_Q^T & \mathbb{I}_Q \end{bmatrix} \frac{{}^0 d}{dt} \begin{bmatrix} \vec{v}_Q \\ \vec{\omega} \end{bmatrix} + \begin{bmatrix} \vec{\omega}_\times & \vec{0} \\ \vec{v}_{Q\times} & \vec{\omega}_\times \end{bmatrix} \begin{bmatrix} m\mathbb{I} & \mathbb{S}_Q \\ \mathbb{S}_Q^T & \mathbb{I}_Q \end{bmatrix} \begin{bmatrix} \vec{v}_Q \\ \vec{\omega} \end{bmatrix} = \begin{bmatrix} \vec{R}^e \\ \vec{M}_Q^e \end{bmatrix} \,.$$

Starting from differential equations

todo

4.6 Equations of motion for continuous media

4.7 Particular Motions

In this section, we will study certain particular motions that are interesting and useful to analyze for educational, historical, and practical reasons.

- Uniform rectilinear motion
- Uniformly accelerated motion
- Uniform circular motion
- Oscillatory and damped oscillatory motions:
  - Free oscillations:
    - Mass-spring(-damper) system
    - Pendulum
  - Forced oscillations:
    - A first step towards structural analysis and beyond ("every physical system is a system of many harmonic oscillators")
    - Concepts of frequency response and resonance. **todo** video and/or script on frequency response of structures and seismic structures, mass-damper,…
- **Gravitation**: Starting from Newton's universal law of gravitation, we study the motion of celestial bodies in two-body systems, discovering that their trajectories describe conic sections (circle, ellipse, parabola, hyperbola), and demonstrating Kepler's laws.
- Rotation of a body around a fixed point, Poinsot's motions

4.8 Equilibrium and Stability

A system is in equilibrium is all its components are in equilibrium, and thus there exists a reference frame w.r.t. which the momentum and the angular momentum of all its components are equal to zero.

4.8.1 Eigenvalue stability

…

$$\delta\vec{Q} = m\delta\vec{v}_G$$

4.8.2

Part II

Analytical Mechanics

CHAPTER

FIVE

LAGRANGIAN MECHANICS

Classical mechanics can be re-formulated starting principles of calculus of variations, usually referred as **analytical mechanics**. Under some assumptions, that will be discussed during the derivation, analytical mechanics is equivalent to Newton mechanics.

Lagrange equations - II kind. Lagrange equations of the II kind provides **pure equations of motion**, in which constraint forces do not appear. Given a system with N degrees of freedom, that can be described with N independent generalized coordinates $\{q^k\}_{k=1:N}$, Lagrange equations of the II kind are a set of N 2^{nd} **order ODEs** in the generalized coordinates.

The equivalence with Newton's approach to classical mechanics is discussed in detail for different kind of systems (points, rigid bodies,…): starting from Newton's dynamical equations of motion (strong form), D'Alembert approach (weak form) is derived, and Lagrange equations are derived from that with a proper choice of test functions. Lagrange equations are then recast as the stationariety condition of a functional, providing a variational approach to classical mechanics.

Tip: Lagrange equations of the II kind could be the best approach for small-dimensional problems, when there's no interest in evaluating constraint reactions.

Lagrange equations - I kind. Lagrange equations of the I kind provides a set of **DAEs**, explicitly including constraints as independent equations and adding their effects - their constraint reactions - in the dynamical equations of the degrees of freedom as **Lagrange multipliers**.

Tip: Lagrange equations of the I kind could be the best approach for numerical approach to generic mechanical systems, as it is a less problem-dependent approach, without any case-dependent manipulation.

Lagrangian mechanics, with expliicitly time-dependent Lagrangian function. Energy conservation is related to explicit time independent of its Lagrangian function, and can be related - beside the absence of non-conservative actions - to the absence of any time-dependent external input, or the choice of an inertial reference frame. Consequence of explicit time dependence in the Lagrangian functions are discussed, with examples.

Constraints in Lagrangian mechanics. Constraints in Lagrangian mechanics are discussed: holonomic and non-holonomic are discussed, and an approach to ideal non-holonomic (semi-linear?) constraints in Lagrange mechanics is outlined.

Properties. Some properties of Lagrangian approach to mechanics are discussed. As an example, Lagrange mechanics provides a **symmetric form** of the (linearised?) governing equations, without any additional effort. This could be quite useful, especially for exploiting numerical methods for symmetric (and definite positive, sometimes) matrices.

5.1 Lagrange Equations of the Second Kind

Here, the equivalence of analytical mechanics and Newton mechanics is stressed, by means of the derivation of the principle of analytical mechanics starting from the equations of motions derived in Newtonian mechanics, relying on the conservation of mass and the three principles of Newton mechanics. The process is shown in the following sections for *point systems*, *systems of points*, *extended rigid bodies* and follows these steps:

- **strong form of equations.** Starting point is the dynamical equations of Newton mechanics, here also referred as the strong form of equations
- **weak form of equations.** Strong form are recast in weak form, also referred as **D'Alembert approach** or **virtual work formulation**, multiplying strong form of equations for arbitrary test functions
- **Lagrange equations.** A proper choice of test functions as a function of generalized coordinates, and some manipulation, leads to Lagrange equations. While the choice of test functions depends on the nature of the system, their expression always reads

  $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right) - \frac{\partial \mathcal{L}}{\partial q^k} = Q_{q^k} \ , \tag{5.1}$$

  being $q^k(t)$ the generalized coordinates, $\mathcal{L}\left(\dot{q}^k(t), q?k(t), t\right) = K\left(\dot{q}^k(t), q^k(t), t\right) + U(q^k(t), t)$ the Lagrangian function of the system, defined as the sum of the kinetic energy K and the potential function $U = -V$, being V the potential energy - s.t. the conservative vector field reads $\vec{F} = -\nabla V$, and Q_q the generalized force.
- Lagrange equations can be interpreted as a result of a stationary principle of a functional, S, defined **action functional**, as it can be shown with the tools of calculus of variations.
  - **If** $Q_{q^k} = 0$, multiplying by $w^k(t)$, integrating over time from t_0, t_1, and assuming that $w(t_0) = w(t_1) = 0$,

    $$\begin{aligned}
    0 &= \int_{t_0}^{t_1} w^k(t) \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right) - \frac{\partial \mathcal{L}}{\partial q}\right] dt = \\
    &= w^k(t) \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right)\Bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left[\dot{w}^k(t) \frac{\partial \mathcal{L}}{\partial \dot{q}^k} + w^k(t) \frac{\partial \mathcal{L}}{\partial q^k}\right] dt \ .
    \end{aligned}$$

    If $w^k(t)$ is equal to zero for t equal to t_0 and t_1, first term vanishes

    $$\begin{aligned}
    0 &= -\int_{t_0}^{t_1} \left[\dot{w}^k(t) \frac{\partial \mathcal{L}}{\partial \dot{q}^k} + w^k(t) \frac{\partial \mathcal{L}}{\partial q^k}\right] dt \\
    &= -\frac{1}{\varepsilon}\int_{t_0}^{t_1} \varepsilon \left[\dot{w}^k(t) \frac{\partial \mathcal{L}}{\partial \dot{q}^k}\left(\dot{q}^l(t), q^l(t), t\right) + w^k(t) \frac{\partial \mathcal{L}}{\partial q^k}\left(\dot{q}^l(t), q^l(t), t\right)\right] dt = \\
    &= -\lim_{\varepsilon \to 0}\left\{\frac{1}{\varepsilon}\int_{t_0}^{t_1} \varepsilon \left[\dot{w}^k(t) \frac{\partial \mathcal{L}}{\partial \dot{q}^k}\left(\dot{q}^l(t), q^l(t), t\right) + w^k(t) \frac{\partial \mathcal{L}}{\partial q^k}\left(\dot{q}^l(t), q^l(t), t\right)\right] dt\right\} = \\
    &= -\lim_{\varepsilon \to 0}\left\{\frac{1}{\varepsilon}\int_{t_0}^{t_1} \left[\mathcal{L}\left(\dot{q}^l(t) + \varepsilon \dot{w}^l(t), q^l(t) + \varepsilon w^l(t), t\right) - \mathcal{L}\left(\dot{q}^l(t), q^l(t), t\right)\right] dt + o(\varepsilon)\right\} = \\
    &= -\delta \int_{t_0}^{t_1} \mathcal{L}(\dot{q}^l(t), q^l(t), t)\, dt =: -\delta S[q^k(t)] \ ,
    \end{aligned}$$

    i.e. Lagrange equations are equivalent to the stationary condition of the action functional

    $$S[q^k(t)] := \int_{t_0}^{t_1} \mathcal{L}\left(\dot{q}^l(t), q^l(t), t\right)\, dt \ .$$

– **If** $Q_k \neq 0$, the variational principle becomes

$$\begin{aligned}
0 & = \int_{t_0}^{t_1} w^k(t) \left[\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k} \right) - \frac{\partial \mathcal{L}}{\partial q^k} - Q_k \right] dt = \\
& = w^k(t) \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k} \right) \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left[\dot{w}^k(t) \frac{\partial \mathcal{L}}{\partial \dot{q}^k} + w^k(t) \frac{\partial \mathcal{L}}{\partial q^k} \right] dt + \int_{t_0}^{t_1} w^k(t) Q_k \, dt \, . \\
& = \dots \\
& = -\delta \int_{t_0}^{t_1} \mathcal{L}\left(\dot{q}^l(t), q^l(t), t \right) \, dt + \int_{t_0}^{t_1} \delta q^k(t) Q_k \, dt \; ,
\end{aligned}$$

having written the arbitrary test function as $w^k(t) =: \delta q^k(t)$ to keep in mind that they're used as variations of functions $q^k(t)$.

The second contribution is usually defined **virtual work** of generalized forces Q_k - that is equal to the virtual work of actions not included in the potential $U\left(q^k(t), t\right)$. For the very nature of variation, it can be thought as the *infinitesimal work done by forces for small displacements compatible with constraints, keeping time constant*.

5.1.1 Point

Newton dynamical equations - strong form. Dynamical equation governing the motion of a point P reads

$$m \dot{\vec{v}}_P = \vec{R}^e \; ,$$

being m the mass of the system, $\vec{v}_P$ the velocity of point P, $\vec{a}_P = \dot{\vec{v}}_P$ its acceleration and $\vec{R}^e$ the net external force acting on the system..

Weak form. Weak form of dynamical equations is derived with scalar multiplication of the strong form by an arbitrary test function $\vec{w}$,

$$\vec{0} = \vec{w} \cdot \left(m \dot{\vec{v}} - \vec{R}^e \right) \qquad \forall \vec{w} \tag{5.2}$$

Lagrange equations. Lagrange equations are derived from a proper choice of the test function. The position of the point P is written as a function of the generalized coordinates $q^k(t)$ and time t

$$\vec{r}_P(t) = \vec{r}(q^k(t), t) \; ,$$

so that its velocity can be written as

$$\vec{v}_P(t) := \frac{d\vec{r}_P}{dt} = \dot{q}^k(t) \underbrace{\frac{\partial \vec{r}}{\partial q^k}}_{\frac{\partial \vec{v}}{\partial \dot{q}^k}} (q^l(t), t) + \frac{\partial \vec{r}}{\partial t} (q^l(t), t) = \vec{v}\left(\dot{q}^k(t), q^k(t), t \right) \; ,$$

from which the relation between partial derivatives

$$\frac{\partial \vec{r}}{\partial q^k} = \frac{\partial \vec{v}}{\partial \dot{q}^k} \; . \tag{5.3}$$

follows. Choosing the test function $\vec{w}$ as

$$\vec{w} = \frac{\partial \vec{r}}{\partial q^k} = \frac{\partial \vec{v}}{\partial \dot{q}^k} \; ,$$

applying the rule of derivative of product, using Schwartz theorem to switch order of derivation, and exploiting relation (5.3) it's possible to recast weak form (5.2) as

$$\begin{aligned}
\vec{0} &= \frac{\partial \vec{v}}{\partial \dot{q}^k} \cdot \left(m\dot{\vec{v}} - \vec{R}^e \right) = \\
&= \frac{d}{dt} \left(\frac{\partial \vec{v}}{\partial \dot{q}^k} \cdot m\vec{v} \right) - \frac{d}{dt} \frac{\partial \vec{r}}{\partial q^k} \cdot m\vec{v} - \frac{\partial \vec{r}}{\partial q^k} \cdot (\vec{R}^{e,c} + \vec{R}^{e,nc}) \\
&= \frac{d}{dt} \left(\frac{\partial \vec{v}}{\partial \dot{q}^k} \cdot m\vec{v} \right) - \frac{\partial \vec{v}}{\partial q^k} \cdot m\vec{v} - \frac{\partial \vec{r}}{\partial q^k} \cdot (\nabla U + \vec{R}^{e,nc}) \\
&= \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{q}^k} \right) - \frac{\partial K}{\partial q^k} - \frac{\partial U}{\partial q^k} - \underbrace{\frac{\partial \vec{r}}{\partial q^k} \cdot \vec{R}^{e,nc}}_{=:Q^k} \ .
\end{aligned}$$

Introducing the **Lagrangian function**

$$\mathcal{L}(\dot{q}^k(t), q^k(t), t) := K(\dot{q}^k(t), q^k(t), t) + U(q^k(t), t) \ ,$$

and recalling that potential function U is not a function of velocity and thus of time derivatives of the generalized coordinates $\dot{q}^k$, it's possible to recast the dynamical equation as the **Lagrange equations**

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k} \right) - \frac{\partial \mathcal{L}}{\partial q^k} = Q^k \ ,$$

being Q^k the **generalized force** not included in the gradient of the potential ∇U - usually a non conservative contribution -, $Q^k = \dfrac{\partial \vec{r}}{\partial q^k} \cdot \vec{R}^{e,nc}$.

5.1.2 System of points

Newton dynamical equations - strong form.

Weak form.

Lagrange equations.

5.1.3 Rigid Body

Newton dynamical equations - strong form. Dynamical equations governing the motion of a rigid body, referred to its center of mass G read

$$\begin{cases} \dot{\vec{Q}} = \vec{R}^e \\ \dot{\vec{\Gamma}}_G = \vec{M}^e_G \ , \end{cases}$$

with momentum $\vec{Q} = m\vec{v}_G$ and angular momentum $\vec{\Gamma}_G = \mathbb{I}_G \cdot \vec{\omega}$.

Weak form. Weak form of dynamical equations is derived with scalar multiplication of the strong form by an arbitrary test functions $\vec{w}_t$, $\vec{w}_r$

$$\vec{0} = \vec{w}_t \cdot \left(m\dot{\vec{v}}_G - \vec{R}^e \right) + \vec{w}_r \cdot \left(\dot{\vec{\Gamma}}_G - \vec{M}^e_G \right) \qquad \forall \vec{w}_t, \, \vec{w}_r \tag{5.4}$$

Lagrange equations. Lagrange equations are derived from the weak form, with a proper choice of the weak test functions. The "translational part" is recasted after choosing

$$\vec{w}_t = \frac{\partial \vec{r}}{\partial q^k} = \frac{\partial \vec{v}}{\partial \dot{q}^k} \ .$$

Following the same steps show to derive *Lagrange equations for a point system*, the translational part becomes

$$\frac{d}{dt}\frac{\partial K^{tr}}{\partial \dot{q}^k} - \frac{\partial K^{tr}}{\partial q^k} - \frac{\partial U^{tr}}{\partial q^k} = Q^{tr}_k \ ,$$

being $K^{tr} = \frac{1}{2}m|\vec{v}_G|^2$ the contribution to kinetic energy of the velocity of the center of mass G deriving from the momentum equation, U^{tr} the contribution to the potential energy U from the momentum equation, and Q^{tr}_k the contribution to the generalized force from the momentum equation.

The "rotational part" is recasted after choosing

$$\vec{w}_r = \frac{\partial \vec{\theta}}{\partial q^k} = \frac{\partial \vec{\omega}}{\partial \dot{q}^k}$$

Angular velocity $\vec{\omega}$ can be written w.r.t the inertial $\{\hat{e}_i\}$ or the material reference frame $\{\hat{E}_i\}$,

$$\vec{\omega} = \omega_i \hat{e}_i = \sigma_j \hat{E}_j \ ,$$

and the inertia tensor as

$$\mathbb{I}_G = I_{ij}\, \hat{E}_i \otimes \hat{E}_j \ ,$$

being the components I_{ij} constant.

$$0 = \frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \frac{d}{dt}\left(\mathbb{I}_G \cdot \vec{\omega}\right) - \frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \vec{M}^e_G = \frac{d}{dt}\left(\frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \mathbb{I}_G \cdot \vec{\omega}\right) - \frac{d}{dt}\frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \mathbb{I}_G \cdot \vec{\omega} - \frac{\partial \vec{\theta}}{\partial q^k} \cdot \vec{M}^e_G$$

The first term becomes

$$\frac{d}{dt}\left(\frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \mathbb{I}_G \cdot \vec{\omega}\right) = \frac{d}{dt}\left(\frac{\partial \sigma_a}{\partial \dot{q}^k} I_{ab}\sigma_b\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{q}^k}\left(\frac{1}{2}\vec{\omega} \cdot \mathbb{I}_G \cdot \vec{\omega}\right) = \frac{d}{dt}\frac{\partial K^{rot}}{\partial \dot{q}^k}$$

The second term becomes

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \vec{\theta}}{\partial q^k} \cdot \mathbb{I}_G \cdot \vec{\omega} &= \frac{\partial}{\partial q^k}\underbrace{\frac{d\vec{\theta}}{dt}}_{\vec{\omega}} \cdot \mathbb{I}_G \cdot \vec{\omega} = \\
&= \frac{\partial \vec{\omega}}{\partial q^k} \cdot \mathbb{I}_G \cdot \vec{\omega} = \\
&= \frac{\partial}{\partial q^k}\left(\sigma_a \hat{E}_a\right) \cdot \hat{E}_b\, I_{bc}\sigma_c = \\
&= \frac{\partial \sigma_a}{\partial q^k}\underbrace{\hat{E}_a \cdot \hat{E}_b}_{=\delta_{ab}}\, I_{bc}\sigma_c + \sigma_a \underbrace{\frac{\partial \hat{E}_a}{\partial q^k} \cdot \hat{E}_b}_{=0}\, I_{bc}\sigma_c = \\
&= \frac{\partial}{\partial q^k}\left(\frac{1}{2}\sigma_a\, I_{ab}\sigma_b\right) = \\
&= \frac{\partial}{\partial q^k}\left(\frac{1}{2}\vec{\omega} \cdot \mathbb{I}_G \cdot \vec{\omega}\right) = \frac{\partial K^{rot}}{\partial q^k} \ .
\end{aligned}$$

The third term can be written as the sum of the derivative of a potential function and a generalized force,

$$\frac{\partial \vec{\theta}}{\partial q^k} \cdot \vec{M}^e_G = \frac{\partial U^{rot}}{\partial q^k} + Q^{rot}_{q^k}$$

The rotational part of the wak form becomes

$$\frac{d}{dt}\frac{\partial K^{rot}}{\partial \dot{q}^k} - \frac{\partial K^{rot}}{\partial q^k} - \frac{\partial U^{rot}}{\partial q^k} = Q^{rot}_{q^k} \ ,$$

being $K^{rot} = \frac{1}{2}\vec{\omega} \cdot \mathbb{I}_G \cdot \vec{\omega}$ the contribution to kinetic energy of the rotation aroung the center of mass G deriving from the angular momentum equation, U^{rot} the contribution to the potential energy U from the angular momentum equation, and Q_k^{rot} the contribution to the generalized force from the angular momentum equation.

Adding together the contributions of the momentum and the angular momentum equations, the Lagrange equation can be formally written with the same expression found for the system of points,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right) - \frac{\partial \mathcal{L}}{\partial q^k} = Q_{q^k} \ ,$$

being $\mathcal{L} = K + U$ the Lagrangian function of the system, and $K = K^{tr} + K^{rot}$, $U = U^{tr} + U^{rot}$, $Q_{q^k} = Q_{q^k}^{tr} + Q_{q^k}^{rot}$ the kinetic energy the potential function and the generalized force of the system, defined as the sum of the contributions coming from the momentum and the angular momentum equations.

5.2 Lagrange Equations of the First Kind

Explicitly making appear constraint forces, due to constraints

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = Q_k^e + Q_k^c \\
&g^j\left(q^k(t), t\right) = 0
\end{aligned}$$

Example 5.2.1

Pendulum with point mass m and length ℓ, with hinge position $x_H(t)$ w.r.t. an inertial reference frame, in a gravitational field $\vec{g} = g\hat{y}$

Position, and velocity of the point mass in P

$$\begin{aligned}
\vec{r}_P(t) &= x_P(t)\,\hat{x} + y_P(t)\,\hat{y} = \left(x_H(t) + \ell\sin\theta(t)\right)\,\hat{x} + \ell\cos\theta(t)\,\hat{y} \\
\vec{v}_P(t) &= \dot{x}_P(t)\,\hat{x} + \dot{y}_P(t)\,\hat{y} = \left(\dot{x}_H + \ell\dot{\theta}(t)\cos\theta(t)\right)\hat{x} - \ell\dot{\theta}(t)\sin\theta(t)\,\hat{y}
\end{aligned}$$

Approach 1. LE of the II Kind. LE of the II Kind provides free equations of motion. The system has one degree of freedom. Here the angle $\theta(t)$ is chosen as the generalized dof. Kinetic energy K and potential function U,

$$\begin{aligned}
K &= \frac{1}{2}m\left(\dot{x}_H^2 + 2\ell\dot{x}_H\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2\right) \\
U &= mg\ell\cos\theta
\end{aligned}$$

and Lagrange equation of the II-kind provides a free equation of motion, that immediately follows from direct evaluation of the required derivatives

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= \frac{d}{dt}\left[m(\ell\dot{x}_H\cos\theta + \ell^2\dot{\theta})\right] = m\ell\ddot{x}_H\cos\theta - m\ell\dot{x}_H\dot{\theta}\sin\theta + m\ell^2\ddot{\theta} \\
\frac{\partial L}{\partial \theta} &= -m\ell\dot{x}_H\dot{\theta}\sin\theta - mg\ell\sin\theta
\end{aligned}$$

Thus, Lagrange equation reads

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = m\ell^2\ddot{\theta} + mg\ell\sin\theta - m\ell\ddot{x}_H(t)\cos\theta$$

Approach 2. LE of the I Kind.

5.3 Lagrangian functions and time dependence

Some problems may have a Lagrangian function with an explicit dependence on time,

$$\mathcal{L}(\dot{q}^k(t), q^k(t), t) \ .$$

Using the general form (5.1) of Lagrange equations, the time derivative of the Lagrange function reads

$$\begin{aligned}
\frac{d\mathcal{L}}{dt} & = \ddot{q}^k \frac{\partial \mathcal{L}}{\partial \dot{q}^k} + \dot{q}^k \frac{\partial \mathcal{L}}{\partial q^k} + \frac{\partial \mathcal{L}}{\partial t} = && \text{(IxP)} \\
& = \frac{d}{dt}\left(\dot{q}^k \frac{\partial \mathcal{L}}{\partial \dot{q}^k} \right) - \dot{q}^k \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^k} + \dot{q}^k \frac{\partial \mathcal{L}}{\partial q^k} + \frac{\partial \mathcal{L}}{\partial t} = && \text{(Lagrange eq.)} \\
& = \frac{d}{dt}\left(\dot{q}^k \frac{\partial \mathcal{L}}{\partial \dot{q}^k} \right) - \dot{q}^k Q_k + \frac{\partial \mathcal{L}}{\partial t} \ .
\end{aligned}$$

This latter relation can be recast as

$$\frac{d}{dt}\left[\dot{q}^k \frac{\partial \mathcal{L}}{\partial \dot{q}^k} - \mathcal{L} \right] = \dot{q}^k Q_k - \frac{\partial \mathcal{L}}{\partial t} \ , \tag{5.5}$$

i.e. time derivative of a physical quantity equals the power of actions not included in the potential, $\dot{q}^k Q_k$ and a contribution of partial derivative of the Lagrangian function, $\partial_t \mathcal{L}$.

As it's discusseed in the *section for systems with Lagrangian function with no explicit dependence on time* **when the Lagrangian function of the system is not an explicit function of time todo** *discuss the cases when $\partial_t \mathcal{L} \neq 0$, the equation (5.5) is nothing but the **balance equation of mechanical energy**.

When there is no generalized force, that can't be included in the potential, $Q_k = 0$, and no explicit dependence of Lagrangian function on time, $\partial_t \mathcal{L} = 0$, the equation (5.5) can be recast as an Euler-Beltrami equation,

$$\dot{q}^k \frac{\partial \mathcal{L}}{\partial \dot{q}^k} - \mathcal{L} = \overline{E} \quad \text{const.}$$

describing the conservation of mechanical energy w.r.t. an inertial reference frame, in absence of non-conservative forces, as discussed in the .

5.3.1 Lagrangian function with no explicit dependence on time

Let's analyse first some properties of systems, whose Lagrangian function are not an explicit function of time,

$$\mathcal{L}(\dot{q}^k(t), q^k(t)) = K(\dot{q}^k(t), q^k(t)) + U(q^k(t)) \ ,$$

and then go back to the most general case. As the Lagrange equation is not an explicit function of time, relation (5.5) reads

$$\frac{d}{dt}\left[\dot{q}^k \frac{\partial \mathcal{L}}{\partial \dot{q}^k} - \mathcal{L} \right] = \dot{q}^k Q_k \ .$$

Since the Lagrangian doesn't expliclty depend on time, and potential is not a function of time, relation (5.6) gives $\dot{q}^k \frac{\partial \mathcal{L}}{\partial \dot{q}^k} = 2K$, and thus the content of the braces is the mechanical energy of the system,

$$\dot{q}^k \frac{\partial \mathcal{L}}{\partial \dot{q}^k} - \mathcal{L} = 2K - \mathcal{L} = 2K - K - U = K - U = E^{mec} \ ,$$

and it becomes clear that the relation is nothhing but the balance equation of mechanical energy

$$\frac{dE^{mec}}{dt} = \dot{q}^k Q_k \ ,$$

that becomes conservation of mechanical energy, in absence of non-conservative actions, $Q_k = 0$,

$$E^{mec} = \overline{E}^{mec} \quad \text{const.}$$

Properties of kinetic energy and potential

This section collects some properties of the kinetic energy and potential of systems, where physical coordinates of the system are written as a function of generalized coordinates only, $q^k(t)$. As an example, coordinates of point masses, material points of rigid bodies and the rotation tensor representing their orientation in space can be written as

$$\vec{r}_P\left(q^k(t)\right) \quad , \quad \mathbb{R}\left(q^k(t)\right) \ ,$$

so that velocities and angular velocities become

$$\begin{aligned}
\vec{v}_P\left(\dot{q}^k(t), q^k(t)\right) &= \frac{d\vec{r}_P}{dt} = \dot{q}^k \frac{\partial \vec{r}}{\partial q^k}\left(q^k(t)\right) \\
\vec{\omega}_\times\left(\dot{q}^k(t), q^k(t)\right) &= \frac{d\mathbb{R}}{dt} \cdot \mathbb{R}^T = \dot{q}^k(t)\, \frac{\partial \mathbb{R}}{\partial q^k} \cdot \mathbb{R}^T = \dot{q}^k(t)\, \frac{\partial \vec{\theta}_\times}{\partial q^k}\left(q^k(t)\right) \ ,
\end{aligned}$$

As the kinetic energy of a mechanical system is a quadratic function of velocity and angular velocity of its sub-systems, the kinetic energy can be written as

$$K\left(\dot{q}^k(t), q^k(t)\right) = \frac{1}{2} A_{ij}\left(q^k(t)\right)\, \dot{q}^i(t)\, \dot{q}^j(t) \ .$$

Since A_{ij} is symmetric w.r.t. the swap of indices (or it can be written in a symmetric form), partial derivative of the kinetic energy w.r.t. $\dot{q}^l$ reads

$$\frac{\partial K}{\partial \dot{q}^l} = A_{lj} \dot{q}^j \ ,$$

and

$$\dot{q}^l \frac{\partial K}{\partial \dot{q}^l} = \dot{q}^l A_{lj} \dot{q}^j = 2K \ . \tag{5.6}$$

Proofs

$$\frac{\partial K}{\partial \dot{q}^l} = \frac{\partial}{\partial \dot{q}^l}\left[\frac{1}{2} A_{ij} \dot{q}^i \dot{q}^j\right] = \frac{1}{2} A_{ij}\left[\underbrace{\frac{\partial \dot{q}^i}{\partial \dot{q}^l}}_{\delta^i_l} \dot{q}^j + \dot{q}^i \underbrace{\frac{\partial \dot{q}^j}{\partial \dot{q}^l}}_{\delta^j_l}\right] = \frac{1}{2}\left[A_{lj} \dot{q}^j + A_{il} \dot{q}^i\right] = A_{lj} \dot{q}^j$$

5.3.2 Lagrangian function with explicit dependence on time

Some problems may have a Lagrangian function with an explicit dependence on time,

$$\mathcal{L}\left(\dot{q}^k(t), q^k(t), t\right) \ ,$$

in some cases like:

1. time-dependent contraints, whose motion is prescribed
2. time-dependent forces that can be included in the potential energy
3. choice of coordinates $q^k(t)$ that gives an explicit dependence on time of the physical coordinates, like positions and orientations of rigid bodies

$$\vec{r}_P\left(q^k(t), t\right) \quad , \quad \mathbb{R}_P\left(q^k(t), t\right)$$

and leading to an explicit dependence on time of the velocities and angular velocities

$$\vec{v}_P\left(\dot{q}^k(t), q^k(t), t\right) = \dot{q}^i(t)\frac{\partial \vec{r}_P}{\partial q^i}\left(q^k(t), t\right) + \frac{\partial \vec{r}_P}{\partial t}\left(q^k(t), t\right)$$
$$\vec{\omega}_P\left(\dot{q}^k(t), q^k(t), t\right) = \dot{q}^i(t)\frac{\partial \vec{\theta}_P}{\partial q^i}\left(q^k(t), t\right) + \frac{\partial \vec{\theta}_P}{\partial t}\left(q^k(t), t\right)$$

and thus of the kinetic energy.

In general, external actions with net power are required for conditions 1., 2., i.e. for moving constraints or for changing potential fields acting on the system: even if variable constraints or external forcing acting on a system are prescribed and thus add no degree of freedom to the system, they're not free in general but requires external actions and power, as it can be realized looking at the balance equation of mechanica energy.

Condition 3. is usually a result of a choice of coordinates in a non-inertial reference frame, whose motion is not described by the coordinates themselves.

Energy balance

$$\begin{aligned}
\frac{dE}{dt} = \frac{dK}{dt} - \frac{dU}{dt} = \\
& = 2\frac{dK}{dt} - \frac{d\mathcal{L}}{dt} = \\
& = 2\frac{dK}{dt} - \ddot{q}^k\frac{\partial \mathcal{L}}{\partial \dot{q}^k} - \dot{q}^k\frac{\partial \mathcal{L}}{\partial q^k} - \frac{\partial \mathcal{L}}{\partial t} = \\
& = 2\frac{dK}{dt} - \frac{d}{dt}\left(\dot{q}^k\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right) + \dot{q}^k\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^k} - \dot{q}^k\frac{\partial \mathcal{L}}{\partial q^k} - \frac{\partial \mathcal{L}}{\partial t} = \\
& = \frac{d}{dt}\left[2K - \dot{q}^k\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right] + \dot{q}^k\underbrace{\left[\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^k} - \frac{\partial \mathcal{L}}{\partial q^k}\right]}_{=Q_k} - \frac{\partial \mathcal{L}}{\partial t} = \\
& = \dot{q}^k Q_k - \frac{\partial \mathcal{L}}{\partial t} + \frac{d}{dt}\left[\dot{q}^k B_k + 2C\right]
\end{aligned}$$

Properties of kinetic energy and potential

The kinetic energy has a general expression

$$K = \frac{1}{2}\dot{q}^i\dot{q}^j A_{ij}\left(q^k(t), t\right) + \dot{q}^i B_i\left(q^k(t), t\right) + C\left(q^k(t), t\right)$$

$$\dot{q}^k\frac{\partial K}{\partial \dot{q}^k} = \dot{q}^k\left(A_{kj}\dot{q}^j + B_k\right)$$

$$\begin{aligned}
2K - \dot{q}^k\frac{\partial K}{\partial \dot{q}^k} & = \dot{q}^k A_{kj}\dot{q}^j + 2\dot{q}^k B_k + 2C - \left[\dot{q}^k\left(A_{kj}\dot{q}^j + B_k\right)\right] = \\
& = \dot{q}^k B_k + 2C
\end{aligned}$$

Example 5.3.1

5.4 Constraints classification

5.4.1 Holonomic vs. non-holonomic constraints

Definition 5.4.1 (Holonomic constraint)

A holonomic constraints can be written in a form

$$g(q^k(t), t) = 0 \ .$$

Definition 5.4.2 (Non-holonomic constraint)

Every constraint that is not holonomic, is non-holonomic. (**wow!**)

todo *Add some examples…; even some constraints that looks like a non-holonomic constraint that are holonomic: "integrable constraints", Pfaffian method?…*

5.4.2 Ideal constraints

Definition 5.4.3 (Ideal constraint)

An ideal constraint produces no net power.

Given the generalized actions $\mathbf{f}_c$ introduced in the dynamical systems by constraints,

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{f} + \mathbf{f}_c \ ,$$

power of ideal constraints reads

$$\dot{\mathbf{q}}^T \mathbf{f}_c = 0 \ .$$

Ideal holonomic constraints

If constraints don't have explicit dependence (what happens if there's explicit *time dependence*? Treat in the proper section…) from time t

$$\begin{aligned}\mathbf{M}\ddot{\mathbf{q}} &= \mathbf{f} + \mathbf{f}_c \\ \mathbf{g}(\mathbf{q}(t)) &= \mathbf{0}\end{aligned}$$

with $\mathbf{q} \in \mathbb{R}^N$, $\mathbf{g} \in \mathbb{R}^C$.

Power of ideal constraint. Power of ideal constraints is zero, and this condition provides the most general form of constraint reactions $\mathbf{f}_c$ as a linear combination of the gradient of the constraints, and a set of well-defined (*extra conditions?*) and determined DAEs, with equal number of equations and unknowns. Power of contraint reactions of ideal constrains is zero,

$$0 = \dot{\mathbf{q}}^T \mathbf{f}_c$$

Time derivative of the constraint equation reads $0 = g_i(q^k) = \dot{q}^k \frac{\partial g_i}{\partial q^k}$, and thus for every $\mathbf{s} \in \mathbb{R}^C$,

$$0 = \dot{\mathbf{q}}^T \nabla_{\mathbf{q}} \mathbf{g} \, \mathbf{s} \ , \tag{5.7}$$

and thus constraint reactions can be written as a linear combination of the (columns of the) gradient of the constraint equation w.r.t. the generalized coordiantes,

$$\mathbf{f}_c = \nabla_{\mathbf{q}} \mathbf{g} \, \mathbf{s}$$

or

$$\mathbf{f}_c = \nabla_{\mathbf{q}} g_i \, s_i \qquad , \qquad f_{c,k} = \frac{\partial \mathbf{g}^T}{\partial q^k} \mathbf{s} = \frac{\partial g^i}{\partial q^k} \, s_i \ .$$

Determined set of DAEs. Introducing the expression (5.7) of the constraint reactions in the original set of DAEs, the set of equations governing the constrained system reads

$$\begin{aligned} & \mathbf{M}\ddot{\mathbf{q}} = \mathbf{f} + \nabla_{\mathbf{q}} \mathbf{g} \, \mathbf{s} \\ & \mathbf{g}(\mathbf{q}) = \mathbf{0} \end{aligned}$$

Ideal non-holonomic constraints

The equations of a constrained system with non-holonomic constraints in semi-linear form and no explicit time dependence reads

$$\begin{aligned} & \mathbf{M}\ddot{\mathbf{q}} = \mathbf{f} + \mathbf{f}_c \\ & \mathbf{a}(\mathbf{q}(t)) \, \dot{\mathbf{q}}(t) = \mathbf{0} \end{aligned}$$

Power of ideal constraints. As done in the *section about holonomic constraints*, the condition of zero power of ideal constraints provides the most general form of constraint reactions $\mathbf{f}_c$ as a linear combination of the gradient of the constraints, and thus a determined set of DAEs.

$$0 = \dot{\mathbf{q}}^T \, \mathbf{f}_c$$

compared with the (transpose of the) non-holonomic constraint,

$$0 = \dot{\mathbf{q}}^T \mathbf{a}^T(\mathbf{q}) \mathbf{s} \ , \qquad \forall \mathbf{s} \in \mathbb{R}^C$$

and thus constraint reactions can be written as the linear combination of the columns of the transpose of matrix $\mathbf{a}(\mathbf{q})$,

$$\mathbf{f} = \mathbf{a}^T(\mathbf{q}) \mathbf{s} \ . \tag{5.8}$$

Determined set of DAEs. Introducing the expression (5.8) of the constraint reactions in the original set of DAEs, the determined set of equations governing the constrained system reads

$$\begin{aligned} & \mathbf{M}\ddot{\mathbf{q}} = \mathbf{f} + \mathbf{a}^T(\mathbf{q}) \, \mathbf{s} \\ & \mathbf{a}(\mathbf{q}) \, \dot{\mathbf{q}}(t) = \mathbf{0} \end{aligned}$$

5.5 Properties of the Lagrangian approach to classical mechanics

5.5.1 Kinetic energy K and potential function U

Kinetic energy of each component is a symmetric function of the velocity of its reference point P and its angular velocity $\vec{\omega}$, as shown as an example by the expression of the kinetic energy of a rigid body,

$$\begin{aligned} K_P & = \frac{1}{2} \vec{v}_P \cdot \vec{Q} + \frac{1}{2} \vec{\omega} \cdot \vec{L}_P \\ & = \frac{1}{2} \vec{v}_P \cdot [m \vec{v}_P + \mathbb{S}_P \cdot \vec{\omega}] + \frac{1}{2} \vec{\omega} \cdot \left[\mathbb{S}_P^T \cdot \vec{v}_P + \mathbb{I}_P \cdot \vec{\omega} \right] \ , \end{aligned}$$

as the tensor of inertia $\mathbb{I}_P$ is symmetric, i.e. $\vec{a} \cdot \mathbb{I}_P \cdot \vec{b} = \vec{b} \cdot \mathbb{I}_P \cdot \vec{a}$, $\forall \vec{a}, \vec{b}$.

todo *Treat continuous deformable systems.*

Kinetic energy of a system, with no explicit dependence of time of the degrees of freedom is a symmetric positive definite quadratic function.

$$K = \frac{1}{2} \sum_{ik} M_{ik} \left(q^l(t)\right) \dot{q}^i \dot{q}^k = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} \ ,$$

with $M_{ik} = M_{ki}$, and $K \ge 0$.

Kinetic energy symmetric quadratic form of generalized coordinates.

If degrees of freedom of the system can be written as functions of the generalized coordinates $q^k(t)$, with no explicit dependence on time t,

$$\vec{r}_P \left(q^k(t)\right) \qquad , \qquad \mathbb{R} \left(q^k(t)\right) \ ,$$

velocities and angular velocities becomes

$$\begin{aligned}
\vec{v}_P \left(\dot{q}^k(t), q^k(t)\right) &= \dot{q}^k \frac{\partial \vec{r}_P}{\partial q^k} \left(q^k(t)\right) \\
\vec{\omega}_P \left(\dot{q}^k(t), q^k(t)\right) &= \dot{q}^k \frac{\partial \vec{\theta}_P}{\partial q^k} \left(q^k(t)\right)
\end{aligned}$$

The kinetic energy of the system is the sum of the kinetic energy of its parts and thus

$$\begin{aligned}
K = \sum_P K_P &= \sum_P \left\{ \frac{1}{2} \vec{v}_P \cdot \left[m \vec{v}_P + \mathbb{S}_P \cdot \vec{\omega} \right] + \frac{1}{2} \vec{\omega} \cdot \left[\mathbb{S}_P^T \cdot \vec{v}_P + \mathbb{I}_P \cdot \vec{\omega} \right] \right\} = \\
&= \sum_P \left\{ \frac{1}{2} \sum_i \dot{q}^i \partial_{q^i} \vec{r}_P \cdot \left[m \sum_k \dot{q}^k \partial_{q^k} \vec{r}_P + \mathbb{S}_P \cdot \sum_k \dot{q}^k \partial_{q^k} \vec{\theta}_P \right] + \right. \\
&\qquad \left. + \frac{1}{2} \sum_i \dot{q}^i \partial_{q^i} \vec{\theta}_P \cdot \left[\mathbb{S}_P^T \cdot \sum_k \dot{q}^k \partial_{q^k} \vec{r}_P + \mathbb{I}_P \cdot \sum_k \dot{q}^k \partial_{q^k} \vec{\theta}_P \right] \right\} = \\
&= \frac{1}{2} \sum_{i,k} \sum_P \left\{ m \partial_i \vec{r}_P \cdot \partial_k \vec{r}_P + \partial_i \vec{r}_P \cdot \mathbb{S}_P \cdot \partial_k \vec{\theta}_P + \partial_i \vec{\theta}_P \cdot \mathbb{S}_P^T \cdot \partial_k \vec{r}_P + \partial_i \vec{\theta}_P \cdot \mathbb{I}_P \cdot \partial_k \vec{\theta}_P \right\} \dot{q}^i \dot{q}^k = \\
&= \frac{1}{2} \sum_{ik} M_{ik} \dot{q}^i \, \dot{q}^k \ ,
\end{aligned}$$

with the coefficients $M_{ik}(q^l(t))$ symmetric w.r.t. a swap of the indices i, k by definition.

Potential function is a function of generalized coordinates only, $U(\mathbf{q})$.

5.5.2 Lagrange equations

Lagrangian function is

$$L(\dot{\mathbf{q}}, \mathbf{q}) = K(\dot{\mathbf{q}}, \mathbf{q}) + U(\mathbf{q}) = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} + U\left(\mathbf{q}\right)$$

Lagrange equations of the II kind

Lagrange equations of the II kind read

$$\begin{aligned}
\mathbf{Q}_{\mathbf{q}} & = \frac{d}{dt}\left(\partial_{\dot{\mathbf{q}}} L\right) - \partial_{\mathbf{q}} L = \\
& = \frac{d}{dt}\left(\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}\right) - \partial_{\mathbf{q}}\left(\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + U(\mathbf{q})\right) \ ,
\end{aligned}$$

$$\begin{aligned}
Q_i & = \frac{d}{dt}\left(M_{ij}(q^k)\dot{q}^j\right) - \partial_{q^i}\left(\frac{1}{2}\dot{q}^a M_{ab}(q^k)\dot{q}^b + U(q^k)\right) = \\
& = \partial_{q^k} M_{ij}(q^l)\dot{q}^k\dot{q}^j + M_{ij}\ddot{q}^j - \frac{1}{2}\partial_{q^i} M_{kj}(q^l)\dot{q}^k\dot{q}^j - \partial_{q^i} U(q^l)
\end{aligned}$$

$$\begin{aligned}
M_{ij}(q^l)\ddot{q}^j - \partial_{q^i} U(q^l) + \partial_{q^k} M_{ij}(q^l)\dot{q}^k\dot{q}^j - \frac{1}{2}\partial_{q^i} M_{kj}(q^l)\dot{q}^k\dot{q}^j = Q_i \\
\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} - \nabla_{\mathbf{q}} U(\mathbf{q}) + \dot{\mathbf{q}}^T\nabla_{\mathbf{q}}\left(\mathbf{M}\dot{\mathbf{q}}\right) - \frac{1}{2}\nabla_{\mathbf{q}}\left(\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}}\right) = \mathbf{Q}
\end{aligned}$$

Equilibrium conditions correspond to steady solutions of the equations of motion,

$$-\nabla_{\mathbf{q}} U(\overline{\mathbf{q}}) = \mathbf{Q}(\overline{\mathbf{q}}) \ .$$

Linearized system around a (stable) equilibrium follows from the approximation

$$\begin{aligned}
\mathbf{q} & = \overline{\mathbf{q}} + \mathbf{q}' \\
\dot{\mathbf{q}} & = \dot{\mathbf{q}}' \\
\ddot{\mathbf{q}} & = \ddot{\mathbf{q}}' \ ,
\end{aligned}$$

and neglecting higher-order contributions in $\mathbf{q}'$,

$$\begin{aligned}
Q_i(\mathbf{q}) & = M_{ij}(\mathbf{q})\ddot{q}^j - \partial_i U(\mathbf{q}) + \dot{q}^k\partial_k M_{ij}(\mathbf{q})\dot{q}^j - \frac{1}{2}\partial_i M_{jk}(\mathbf{q})\dot{q}^j\dot{q}^k \\
Q_i(\overline{\mathbf{q}}) + q^{k'}\partial_{q^k} Q_i(\overline{\mathbf{q}}) & \sim M_{ij}(\overline{\mathbf{q}})\ddot{q}^j - \partial_i U(\overline{\mathbf{q}}) - q^{j'}\partial_{ij} U(\overline{\mathbf{q}})
\end{aligned}$$

$$\begin{aligned}
M_{ij}(\overline{\mathbf{q}})\ddot{q}'_j + \left[-\partial_{ij} U(\overline{\mathbf{q}}) - \partial_j Q_i(\overline{\mathbf{q}})\right] q'_j = 0 \\
\mathbf{M}(\overline{\mathbf{q}})\ddot{\mathbf{q}}' + \left[-\nabla\nabla U(\overline{\mathbf{q}})\right]\mathbf{q}' = \nabla\mathbf{Q}^T(\overline{\mathbf{q}})\mathbf{q}' \\
\mathbf{M}(\overline{\mathbf{q}})\ddot{\mathbf{q}}' + \mathbf{K}(\overline{\mathbf{q}})\mathbf{q}' = \nabla\mathbf{Q}^T(\overline{\mathbf{q}})\mathbf{q}' \ .
\end{aligned}$$

Matrices are symmetric. Mass matrix is symmetric by construction, as already proved. Stiffness matrix is symmetric as well, due to Schwarz's theorem, as its the Hessian of a scalar function, $\left[\nabla\nabla U\right]_{ij} = \partial_{ij} U$

Matrices are (semi)-definite positive. Mass matrix is definite positive by definition, $\mathbf{M} > 0$, as already proved above as the kinetic energy is a non-negative scalar physical quantity. Stiffness matrix is definite positive if the **equilibrium $\overline{\mathbf{q}}$ is stable**: $\nabla\nabla U(\overline{\mathbf{q}}) < 0$ implies $\mathbf{K} > 0$.

Spectrum of a stable system. todo *add a link to structure mechanics in continuum mechanics, and modal approach*

Lagrange equations of the I kind

Recasting the expression of the dynamical equations of an "uncostrained" system (or with some constraints treated implicitly with Lagrange equations of II kind),

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) \ ,$$

and then adding constraints - at least holonomic or semi-linear non-holonomic constraints - and the costraint reactions, the system becomes

$$\begin{aligned}
&\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q},\dot{\mathbf{q}}) + \nabla_{\mathbf{q}}\mathbf{g}(\mathbf{q})\,\mathbf{s} \\
&\mathbf{g}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{0}
\end{aligned}$$

with either $\mathbf{g}(\mathbf{q}) = \mathbf{0}$ for holonomic time-independent constraints, or $\mathbf{g}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{a}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}$ for semi-linear non-holonomic time-independent constraints, with the forcing term $\mathbf{f}$ containing the non-conservative forces $\mathbf{Q}$ (or at least contributions not included in the potential $U(\mathbf{q})$), the conservative forces $\nabla_{\mathbf{q}}U(\mathbf{q})$, and the (*quadratic!* Important for neglecting them in linearization) contributions from time derivative of kinetic energy,

$$\mathbf{f}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{Q} + \nabla_{\mathbf{q}}U(\mathbf{q}) - \dot{\mathbf{q}}^T\nabla_{\mathbf{q}}\left(\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}\right) + \frac{1}{2}\nabla_{\mathbf{q}}\left(\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}\right) \ .$$

Linearized equations become

$$\begin{aligned}
&\mathbf{M}(\overline{\mathbf{q}})\ddot{\mathbf{q}}' + \mathbf{K}(\overline{\mathbf{q}})\mathbf{q}' = \mathbf{Q}' + \nabla_{\mathbf{q}}\mathbf{g}(\overline{\mathbf{q}})\mathbf{s}' + \mathbf{q}'\nabla_{\mathbf{q}}\nabla_{\mathbf{q}}\mathbf{g}(\overline{\mathbf{q}})\mathbf{s} \\
&\nabla_{\dot{\mathbf{q}}}^T\mathbf{g}(\overline{\mathbf{q}},\mathbf{0})\,\dot{\mathbf{q}}' + \nabla_{\mathbf{q}}^T\mathbf{g}(\overline{\mathbf{q}},\mathbf{0})\,\mathbf{q}' = \mathbf{0}
\end{aligned}$$

CHAPTER

SIX

HAMILTONIAN MECHANICS

Riformulazione ulteriore della meccanica di Newton, a partire dalla meccanica di Lagrange. Fornisce le basi per un approccio moderno anche in altre teorie della Fisica. **dots…**

Starting from Lagrange equations derived in *Lagrangian mechanics*,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = Q_q$$

the **generalized moment** is defined as

$$p_k := \frac{\partial \mathcal{L}}{\partial \dot{q}^k} \ ,$$

and the **Hamiltonian function** as

$$\mathcal{H}(q^k(t), p_k(t), t) := p_k \dot{q}^k - \mathcal{L}(\dot{q}^l(q^k, p_k, t), q^l(t), t) \ ,$$

its differential reads

$$\begin{aligned} d\mathcal{H} &= dq^k \, \frac{\partial \mathcal{H}}{\partial q^k} + dp_k \, \frac{\partial \mathcal{H}}{\partial p_k} + dt \, \frac{\partial \mathcal{H}}{\partial t} = \\ &= dp_k \, \dot{q}^k + \underbrace{p_k \, d\dot{q}^k - d\dot{q}^k \, \frac{\partial \mathcal{L}}{\partial \dot{q}^k}}_{=0} - dq^k \, \frac{\partial \mathcal{L}}{\partial q^k} - dt \, \frac{\partial \mathcal{L}}{\partial t} \end{aligned}$$

and thus it follows

$$\begin{cases} \dot{q}^k & = \dfrac{\partial \mathcal{H}}{\partial p_k} \\ \dfrac{\partial \mathcal{H}}{\partial q^k} & = -\dfrac{\partial \mathcal{L}}{\partial q^k} \\ \dfrac{\partial \mathcal{H}}{\partial t} & = -\dfrac{\partial \mathcal{L}}{\partial t} \ . \end{cases}$$

Recasting Lagrange equations as

$$\frac{\partial \mathcal{L}}{\partial q^k} = -Q_{q^k} + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right) = -Q_{q^k} + \dot{p}_k$$

Hamilton equations follow

$$\begin{cases} \dot{q}^k & = \dfrac{\partial H}{\partial p_k} \\ \dot{p}_k & = -\dfrac{\partial H}{\partial q^k} + Q_{q^k} \ . \end{cases}$$

Part III

Exercises

CHAPTER SEVEN

DYNAMICS

Exercise 7.1

todo Add description of the problem and image

Find:

- the pure equations of motion (solution methods: dynamical equilibria, Lagrange II,…) **done**
- constraint reactions (solution methods: dynamical equilibria, Lagrange I,…) **todo**
- equilibria and their stability **todo**
- evolution of the linear(ized) dynamics of the system around stable equilibria **todo**

Solution.

Kinematics. Using x, θ as generalized coordinates,

$$\begin{cases} \vec{r}_A = x\hat{x} \\ \vec{r}_B = (x + \ell\sin\theta)\hat{x} + \ell\cos\theta\hat{y} \end{cases}$$

$$\begin{cases} \vec{v}_A = \dot{x}\hat{x} \\ \vec{v}_B = (\dot{x} + \ell\dot{\theta}\cos\theta)\hat{x} - \ell\dot{\theta}\sin\theta\hat{y} \end{cases}$$

Lagrangian function.

$$\mathcal{L} = K + U$$

$$\begin{aligned} K &= \frac{1}{2}m_A|\vec{v}_A|^2 + \frac{1}{2}m_B|\vec{v}_B|^2 = \\ &= \frac{1}{2}m_A\dot{x}^2 + \frac{1}{2}m_B\left(\dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta\right) \\ U &= -\frac{1}{2}kx_A^2 + m_B g y_B = \\ &= -\frac{1}{2}kx^2 + m_B g\ell\cos\theta \end{aligned}$$

Lagrange equations (II type) RHS of Lagrange equations read

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) &= \frac{d}{dt}\left(m_A\dot{x} + m_B\dot{x} + m_B\ell\dot{\theta}\cos\theta\right) = (m_A + m_B)\,\ddot{x} + m_B\ell\ddot{\theta}\cos\theta - m_B\ell\dot{\theta}^2\sin\theta \\ \frac{\partial\mathcal{L}}{\partial x} &= -kx \end{aligned}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(m_B \ell^2 \dot{\theta} + m_B \ell \dot{x} \cos\theta\right) = m_B \ell^2 \ddot{\theta} + m_B \ell \ddot{x} \cos\theta - m_B \ell \dot{x} \dot{\theta} \sin\theta$$
$$\frac{\partial \mathcal{L}}{\partial \theta} = -m_B \ell \dot{x} \dot{\theta} \sin\theta - m_B g \ell \sin\theta$$

Generalized forces read

$$Q_x = F$$

$$Q_\theta = C$$

so that the pure equations of motion follows from Lagrange equations $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = Q_q$

$$\begin{cases} (m_A + m_B)\,\ddot{x} + m_B \ell \ddot{\theta} \cos\theta - m_B \ell \dot{\theta}^2 \sin\theta + kx = F \\ m_B \ell^2 \ddot{\theta} + m_B \ell \ddot{x} \cos\theta - m_B \ell \dot{x} \dot{\theta} \sin\theta + m_B \ell \dot{x} \dot{\theta} \sin\theta + m_B g \ell \sin\theta = C \end{cases}$$

and after the simplifications

$$\begin{cases} (m_A + m_B)\,\ddot{x} + m_B \ell \ddot{\theta} \cos\theta - m_B \ell \dot{\theta}^2 \sin\theta + kx = F \\ m_B \ell \ddot{x} \cos\theta + m_B \ell^2 \ddot{\theta} + m_B g \ell \sin\theta = C \end{cases}$$

Obs. The first equation is the x-component of the momentum equation of the whole system. The second equation is the angular momentum equation of the rod around the hinge in A.

Generalized forces on rigid bodies

Following the derivation of the *Lagrange equations for rigid bodies*, generalized forces are

$$Q_q = \frac{\partial \vec{r}_G}{\partial q} \cdot \vec{R}^e + \frac{\partial \vec{\theta}}{\partial q} \cdot \vec{M}^e_G$$

With the definition of θ_δ in the increment of the rigid body motion

$$d\vec{r}_A = d\vec{r}_B + \theta_\delta \times (A - B)\,,$$

and writing the resultant of forces and moments as

$$\vec{R}^e = \sum_i \vec{F}_i$$
$$\vec{M}^e_G = \sum_i (\vec{r}_G - \vec{r}_i) \times \vec{F}_i + \sum_i \vec{C}_i$$

the generalized force can be recast as

$$\begin{aligned} Q_q &= \frac{\partial \vec{r}_G}{\partial q} \cdot \vec{R}^e + \frac{\partial \vec{\theta}}{\partial q} \cdot \vec{M}^e_G = \\ &= \frac{\partial \vec{r}_G}{\partial q} \cdot \sum_i \vec{F}_i + \frac{\partial \vec{\theta}}{\partial q} \cdot \left[\sum_i (\vec{r}_G - \vec{r}_i) \times \vec{F}_i + \sum_i \vec{C}_i\right] = \\ &= \sum_i \left[\frac{\partial \vec{r}_G}{\partial q} + \frac{\partial \vec{\theta}}{\partial q} \times (\vec{r}_G - \vec{r}_i)\right] \cdot \vec{F}_i + \sum_i \frac{\partial \vec{\theta}}{\partial q} \cdot \vec{C}_i = \\ &= \sum_i \frac{\partial \vec{r}_i}{\partial q} \cdot \vec{F}_i + \sum_i \frac{\partial \vec{\theta}}{\partial q} \cdot \vec{C}_i\,, \end{aligned}$$

i.e. as the contribution of the single forces $\vec{F}_i$ acting on different points $\vec{r}_i$ the rigid body and the overall contribution of the couples of forces $\vec{C}_i$.

With $\vec{r}_C\left(q(t),t\right)$ and $\mathbb{R}\left(q(t),t\right)$,

$$\vec{r}_i(q(t),t) - \vec{r}_C(q(t),t) = \mathbb{R}(q(t),t)\cdot(\vec{r}_i^0 - \vec{r}_C^0)$$

Time derivative becomes

$$\begin{aligned}\vec{v}_i - \vec{v}_C &= \left[\dot{q}(t)\,\frac{\partial\mathbb{R}}{\partial q} + \frac{\partial\mathbb{R}}{\partial t}\right]\cdot(\vec{r}_i^0 - \vec{r}_C^0) = \\ &= \left[\dot{q}(t)\,\frac{\partial\mathbb{R}}{\partial q} + \frac{\partial\mathbb{R}}{\partial t}\right]\cdot\mathbb{R}^T\cdot\underbrace{\mathbb{R}\cdot(\vec{r}_i^0 - \vec{r}_C^0)}_{\vec{r}_i-\vec{r}_C}\end{aligned}$$

$$\begin{aligned}\vec{\omega}_\times &= \dot{\mathbb{R}}\cdot\mathbb{R}^T \\ \delta\vec{\theta}_\times &= \delta\mathbb{R}\cdot\mathbb{R}^T \\ \frac{\partial\vec{\theta}_\times}{\partial q} &= \frac{\partial\mathbb{R}}{\partial q}\cdot\mathbb{R}^T\end{aligned}$$

Exercise 7.2

todo Add description of the problem and image

Find:

- the pure equations of motion (solution methods: dynamical equilibria, Lagrange II, Kinetic energy theorem - energy conservation - since the problem has 1 dof,...) **done**
- constraint reactions (solution methods: dynamical equilibria, Lagrange I,...) **todo**
- equilibria and their stability **todo**
- evolution of the linear(ized) dynamics of the system around stable equilibria **todo**

Solution

Geometry.

$$\begin{aligned}R &= d\sin\alpha \\ b &= d\cos^2\alpha\end{aligned}$$

so that $\frac{b}{R} = \frac{\cos^2\alpha}{\sin\alpha}$.

Kinematics.

Position of the center of mass, C

$$\begin{aligned}\vec{r}_C &= b\cos\theta\hat{x} + b\sin\theta\hat{y} - h\hat{z} \\ \vec{v}_C &= -b\dot{\theta}\sin\theta\hat{x} + b\dot{\theta}\cos\theta\hat{y} = b\dot{\theta}\hat{y}_1\end{aligned}$$

Angular velocity $\vec{\omega}$ of the rigid body

$$\vec{\omega} = \dot{\theta}\hat{z} + \dot{\varphi}\hat{x}_1$$

Velocity of point contact point A is zero, $\vec{v}_A = \vec{0}$ for **pure rolling** constraint. Being $(A - C) = R\hat{z}_1$, the general expression of $\vec{v}_A$ as function of θ and φ reads

$$\begin{aligned}\vec{v}_A &= \vec{v}_C + \vec{\omega}\times(A - C) = \\ &= \hat{y}_1 b\dot{\theta} + \left(\dot{\theta}\hat{z} + \dot{\varphi}\hat{x}_1\right)\times R\hat{z}_1 = \\ &= \hat{y}_1\left(b\dot{\theta} + R\dot{\theta}\sin\alpha - R\dot{\varphi}\right) =\end{aligned}$$

so that the kinetatic constraint (integrable, with arbitrary initial condition) between θ and φ is

$$R\varphi = (R\sin\alpha + b)\ \theta \ .$$

Lagrangian function.

With

$$\begin{aligned}
\mathbb{I}_C & = I_x \hat{x}_1\hat{x}_1 + I_y \hat{y}_1\hat{y}_1 + I_z \hat{z}_1\hat{z}_1 \ , \\
\vec{\omega} & = \dot{\theta}\hat{z} + \dot{\varphi}\hat{x}_1 = \left(\dot{\varphi} - \dot{\theta}\sin\alpha\right)\hat{x}_1 + \dot{\theta}\cos\alpha\hat{z}_1
\end{aligned}$$

and using

$$\dot{\varphi} - \dot{\theta}\sin\alpha = \frac{b}{R}\dot{\theta} \ ,$$

the Lagrangian function becomes

$$\begin{aligned}
\mathcal{L} & = K + U = \\
& = \frac{1}{2}m|\vec{v}_C|^2 + \frac{1}{2}\vec{\omega}\cdot\mathbb{I}_C\cdot\vec{\omega} + mgx_C = \\
& = \frac{1}{2}mb^2\dot{\theta}^2 + \frac{1}{2}\left[I_x\left(\dot{\varphi} - \dot{\theta}\sin\alpha\right)^2 + I_z\dot{\theta}^2\cos^2\alpha\right] + mgb\cos\theta = \\
& = \frac{1}{2}mb^2\dot{\theta}^2 + \frac{1}{2}\left[I_x\left(\frac{b}{R}\right)^2 + I_z\cos^2\alpha\right]\dot{\theta}^2 + mgb\cos\theta = \\
& = \frac{1}{2}\tilde{I}\dot{\theta}^2 + mgb\cos\theta \ ,
\end{aligned}$$

with the equivalent inertia

$$\tilde{I} = mb^2 + I_x\left(\frac{b}{R}\right)^2 + I_z\cos^2\alpha \ .$$

Method 1. Lagrange equation (II). Legrange equation gives a pure equation of motion

$$0 = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = \tilde{I}\ddot{\theta} + mgb\sin\theta \ .$$

Method 2. Kinetic energy theorem - or energy conservation

Exercise 7.3

Solution

Kinematics. todo *check kinematic constraints. No influence of θ?*

$$\varphi_2 = \frac{R_1}{R_2}\varphi_1 = r\varphi_1$$

Lagrangian function.

$$\begin{aligned}
K & = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}m_1|\vec{v}_1|^2 = \\
& = \frac{1}{2}I_1\dot{\varphi}_1^2 + \frac{1}{2}I_2\dot{\varphi}_2^2 + \frac{1}{2}m_1\ell^2\dot{\theta}^2 = \\
& = \frac{1}{2}\left(I_1 + I_2 r^2\right)\dot{\varphi}_1^2 + \frac{1}{2}m_1\ell^2\dot{\theta}^2 =
\end{aligned}$$

$$U = -\frac{1}{2}k\theta^2 - mg\ell\sin\theta$$

Generalized forces read

$$\begin{aligned} Q_{\varphi_1} &= C - R_1 F \\ Q_\theta &= C \end{aligned}$$

Lagrange functions (II).

$$\begin{cases} \left(I_1 + I_2 r^2\right)\ddot{\varphi} = C - R_1 F \\ m_1 \ell^2 \ddot{\theta} + k\theta + mg\ell\cos\theta = C \end{cases}$$

Exercise 7.4 (Inverted pendulum)

Solution

CHAPTER

EIGHT

GRAVITATION

Exercise 8.1 (Ball falling in a tunnel through a planet)

A ball of mass m moves through a tunnel drilled through a planet of radius R, mass M and uniform mass distribution. Neglecting the "mas defect" due to the tunnel in the planet,

- provide the expression of the force acting on the ball in the tunnel
- assuming no rotation of the planet, and zero initial velocity of the ball, provide the dynamical equation governing the motion of the ball and integrate it to find the lwa of motion

Uniform mass density reads $\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$. Exploiting symmetry, the gravitational field can be a function of the distance r of the center of the planet only, and have radial direction,

$$\vec{g} = -g(r)\hat{r} \ . \tag{8.1}$$

Formula (2.4) of the flux of the gravitational field,

$$\oint_{\vec{r}\in\partial V} \vec{g}(\vec{r}) \cdot \hat{n}(\vec{r}) = -G \int_{\vec{r}\in V} 4\pi\rho(\vec{r})$$

across the surface of a sphere of radius r - that has outward pointing unit normal vector $\hat{n}(\vec{r}) = \hat{r}(\vec{r})$ -, exploting expression (8.1) from symmetry, becomes for $r < R$

$$-g(r)4\pi r^2 = -G\,4\pi\rho\frac{4}{3}\pi r^3 = -4\pi GM\frac{r^3}{R^3}$$

and thus

$$g(r) = \frac{GM}{R^3}r \qquad \rightarrow \qquad \vec{g}(\vec{r}) = -m\frac{GM}{R^3}r\hat{r} = -m\frac{GM}{R^3}\vec{r} \ .$$

Force acting on the ball of mass m thus reads $\vec{F}(\vec{r}) = m\vec{g}(\vec{r})$. The equation of motion becomes

$$m\ddot{\vec{r}} = \vec{F}(\vec{r}) = -m\frac{GM}{R^3}\vec{r} \ ,$$

a linear second-order ODE with constant coefficients, whose solution provides an harmonic motion with pulsation $\Omega = \sqrt{\frac{GM}{R^3}}$. The solution of this equation, with initial conditions at rest on the surface of the planet,

$$\begin{cases} \vec{r}(0) = \vec{r}_0 = R\hat{r} \\ \dot{\vec{r}}(0) = \vec{0} \end{cases}$$

reads

$$\vec{r}(t) = \vec{r}_0 \cos\left(\sqrt{\frac{GM}{R^3}}\, t\right) = \hat{r}\, R \cos\left(\sqrt{\frac{GM}{R^3}}\, t\right) \ .$$

Exercise 8.2

Investigate the dynamics of the ball in the previous problem, if the rotational motion of the planet around its axis is considered and if the ball is thrown in the tunnel with non-zero velocity.

- Normal actions of the wall of the tunnel on the ball
- At the end of the tunnel, the ball moves above the planet surface while it's attracted "downwards". When the ball comes back to the planet surface, does it target the tunnel?

PROOF INDEX