Classical Mechanics

basics

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CONTENTS

This material is part of the **basics-books project**. It is also available as a .pdf document.

Classical mechanics deal with the motion of systems and its causes.

Different formulations of mechanics are available. Newton formulation was developed at the end of XVII century and starts from mass conservation and Newton's three principles of dynamics, summarised in invariance under Galileian transformations, the relation between force and the change of momentum of a system, and action/reaction principle. Analytical mechanics was developed in the following centuries by D'Alembert and Lagrange and starts from variational principles, leading to Lagrange or Hamiltonian equations of motion.

Newton Mechanics.

Kinematics

Actions

Inertia

Dynamics

Analytical Mechanics.

Lagrangian Mechanics

Hamiltonian Mechanics

Classical mechanics provides a reliable and useful theory for systems:

- much larger than atomic scales; at atomic scales, quantum mechanics is needed
- with velocity much slower than the speed of light or in domains where the finite value of finite speed of interactions can be neglected, as classical mechanics relies on istantaneous action at distance; if these assumptions fail, Einstein theory is needed either special relativity - as a consistent theory of mechanics and electromagnetism - or general relativity - as a theory of gravitation.
- with a small number of components, so that the integration of the governing equations of motion is feasible; continuous model of the systems are object of classical continuum mechanics, relying on the equations of classical mechanics and thermodynamics; systems with large number of components can be approached with the techniques developed in statistical mechanics.

Under these assumptions, mass conservation (Lavoisier principle) holds, inertial reference frames are related by Galileian relativity and the equations of motions are deterministic and can be solved with a reasonable effort - compared to the information and detail contained in the results - either analytically or numerically. Classical mechanics treats time and space as individually absolute physical quantities: this can be a good model whenever Einstein relativity effects are negligible.

Part I

Newton Mechanics

CHAPTER

ONE

KINEMATICS

Kinematics deals with the motion of mechanical systems, without taking into account the causes of motion.

1.1 Space and time

Classical mechanics relies on the concepts of **absolute 3-dimensional Euclidean space**, E^3, and **absolute time**.

Briefly, what is space? It's something you can measure with ruler (for distances) and square (for angles), or other space-measurment devices. What is time? It's something you can measure with a clock or other timekeeping devices.

1.2 Models

Different models of physical systems can be derived with an abstraction and modelling process, depending on the characteristics of the system under investigation and on the level of detail required by the analysis.

These models can be classified by:

- dimensions: 0: point; 1: line; 2: surfaces; 3: volume solid
- deformation: deformable or rigid components

A system can be composed of several components, either free or connected with constraints.

Here, the focus goes to the kinematics of *points* and *rigid bodies*, while deformable bodies are described in continuous mechanics - kinematics.

While space and time are absolute, the motion of a system is usually the **motion relative to an observer** or to a reference frame. After treating the kinematics of points and rigid bodies w.r.t. a given reference frame, *relative kinematics* provides the description of the motion of the same system w.r.t. 2 different observers/reference frames in relative motion.

Definition 1.2.1 (Configuration)

Definition 1.2.2 (State)

1.3 Point

The configuration of a point system is determined by its position in space, its state by its position and its velocity. Acceleration is usually required in mechanics, since equations of motions may be recast as a system of second-orde ordinary differential equations in the configuration of the system. These physical quantites are defined here w.r.t. a reference frame $O_o \hat{x}^0 \hat{y}^0 \hat{z}^0, t^0$, keeping constant the vectors of the base w.r.t. time t^0.

Position.

$$\vec{r}_P(t) = P - O_0 = x^0_{P,i}(t)\, \hat{e}^0_i$$

Velocity.

$$\vec{v}_P(t) = \frac{d\vec{r}_P}{dt} = \dot{x}^0_{P,i}\, \hat{e}^0_i$$

Acceleration.

$$\begin{aligned}\vec{a}_P(t) &= \frac{d\vec{v}_P}{dt} = \dot{v}^0_{P,i}\, \hat{e}^0_i = \\ &= \frac{d^2\vec{r}_P}{dt^2} = \ddot{x}^0_{P,i}\, \hat{e}^0_i\end{aligned}$$

1.4 Rigid Body

1.4.1 Rigid motion

Rigid motion preserves distance between any pair of points, and thus angles. The motion of two material points P, Q performing a rigid motion obeys

$$\vec{v}_P - \vec{v}_Q = \vec{\omega} \times (P - Q) \ , \tag{1.1}$$

being $\vec{v}_P$, $\vec{v}_Q$ the velocity of the points and $\vec{\omega}$ the **angular velocity** of the rigid motion. Taking a point Q as the reference point of the motion, the velocity of all other points can be found

$$\vec{v}_P = \vec{v}_Q + \vec{\omega} \times (P - Q) \ ,$$

as a function of the velocity of Q, the angular velocity of the rigid motion, and the relative position $P - Q$.

Proof.

Given 3 points $P(t)$, $Q(t)$, $R(t)$, the distance bewteen each pair of points is constant and thus its time derivative zero,

$$0 = \frac{d}{dt}|P(t) - Q(t)|^2 = 2\,(P-Q)\cdot\left(\vec{v}_P - \vec{v}_Q\right) \quad \rightarrow \quad \Delta\vec{v}_{QP} = \vec{\omega}_{QP} \times \Delta\vec{r}_{QP}$$

$$0 = \frac{d}{dt}|P(t) - R(t)|^2 = 2\,(P-R)\cdot\left(\vec{v}_P - \vec{v}_R\right) \quad \rightarrow \quad \Delta\vec{v}_{RP} = \vec{\omega}_{RP} \times \Delta\vec{r}_{RP}$$

$$\begin{aligned}0 = \frac{d}{dt}\left[(P-Q)\cdot(P-R)\right] &= \Delta\vec{v}_{QP}\cdot\Delta\vec{r}_{RP} + \Delta\vec{r}_{QP}\cdot\Delta\vec{v}_{RP} = \\ &= \vec{\omega}_{QP}\times\Delta\vec{r}_{QP}\cdot\Delta\vec{r}_{RP} + \Delta\vec{r}_{QP}\cdot\Delta\vec{\omega}_{RP}\times\Delta\vec{r}_{RP} = \\ &= \Delta\vec{r}_{QP}\times\Delta\vec{r}_{RP}\cdot(\vec{\omega}_{QP} - \vec{\omega}_{RP}) \ ,\end{aligned}$$

and since $\Delta\vec{r}_{QP}$, $\Delta\vec{r}_{RP}$ are arbitrary it follows that the vector $\vec{\omega} = \vec{\omega}_{QP} = \vec{\omega}_{RP}$ is unique for all the points performing a rigid motion.

The configuration of a material vector $\vec{a}$ undergoing a rotation is described by the product of the rotation tensor $\mathbb{R}$ by the reference configuration $\vec{a}^0$,

$$\vec{a} = \mathbb{R} \cdot \vec{a}^0 \qquad , \qquad \vec{b} = \mathbb{R} \cdot \vec{b}^0 \ .$$

In order to preserve distance, and angles

$$\begin{cases} |\vec{a}|^2 = & \vec{a} \cdot \vec{a} = \vec{a}^0 \cdot \mathbb{R}^T \cdot \mathbb{R} \cdot \vec{a}^0 = \vec{a}^0 \cdot \vec{a}^0 = |\vec{a}|^2 \\ & \vec{a} \cdot \vec{b} = \vec{a}^0 \cdot \mathbb{R}^T \cdot \mathbb{R} \cdot \vec{b}^0 = \vec{a}^0 \cdot \vec{a}^0 \end{cases} \qquad \rightarrow \qquad \mathbb{R}^T \cdot \mathbb{R} = \mathbb{I}$$

the **rotation tensor** is **unitary**

$$\mathbb{I} = \mathbb{R}^T \cdot \mathbb{R} = \mathbb{R} \cdot \mathbb{R}^T \tag{1.2}$$

Note: From relation (1.2), it follows that

$$1 = |\mathbb{I}| = |\mathbb{R}^T||\mathbb{R}| = |\mathbb{R}|^2 \ ,$$

and thus $|\mathbb{R}| = \mp 1$. If $|\mathbb{R}| = 1$, $\mathbb{R}$ represents a rotation, and implies conservation of orientation of space; if $|\mathbb{R}| = -1$ represents a reflection w.r.t. a plane, and implies inversion of orientation of space.

Orientation of space is determined by the transformation of a RHS triad of vectors: if the transform triad is RHS, then orientation of space is preserved; if it becomes LHS, then orientation of space is inverted.

Note: Rotation tensor $\mathbb{R}$ is not singular and its determinant equals $|\mathbb{R}| = 1$. Thus, $\mathbb{R}^T \cdot \mathbb{R} = \mathbb{I}$ implies $\mathbb{R} \cdot \mathbb{R}^T = \mathbb{I}$. Multiplying (1.2) by $\mathbb{R}$ on the left

$$\mathbb{0} = \mathbb{R} \cdot \mathbb{R}^T \cdot \mathbb{R} - \mathbb{R} \cdot \mathbb{I} = (\mathbb{R} \cdot \mathbb{R}^T - \mathbb{I}) \cdot \mathbb{R} \ ,$$

and since $\mathbb{R}$ is non-singular, it follows that $\mathbb{R} \cdot \mathbb{R}^T = \mathbb{I}$.

Time derivative of the relation (1.2) reads

$$\mathbb{0} = \frac{d}{dt} \left(\mathbb{R} \cdot \mathbb{R}^T \right) = \dot{\mathbb{R}} \cdot \mathbb{R}^T + \mathbb{R} \cdot \dot{\mathbb{R}}^T$$

It follows that the 2-nd order tensor $\dot{\mathbb{R}} \cdot \mathbb{R}^T = -\mathbb{R} \cdot \dot{\mathbb{R}}^T$ is anti-symmetric, and thus it can be written as

$$\dot{\mathbb{R}} \cdot \mathbb{R}^T =: \vec{\omega}_\times \ , \tag{1.3}$$

being the vector $\vec{\omega}$ the angular velocity. Since $\mathbb{R}$ is unitary by (1.2), multiplying (1.3) with the dot-product on the right by $\mathbb{R}$, it follows

$$\dot{\mathbb{R}} = \vec{\omega}_\times \cdot \mathbb{R} \ ,$$

and the expression of the time derivative of a material vector $\vec{a}$,

$$\frac{d\vec{a}}{dt} = \dot{\mathbb{R}} \cdot \vec{a}^0 = \vec{\omega}_\times \cdot \mathbb{R} \cdot \vec{a}^0 = \vec{\omega}_\times \cdot \vec{a} = \vec{\omega} \times \vec{a} \ . \tag{1.4}$$

Position and Orientation. The most general rigid motion is the combination of the translation of a reference point Q and the rotation w.r.t. this point of other material points,

$$\begin{aligned} \vec{r}_P & = \vec{r}_Q + (P-Q) = \\ & = \vec{r}_Q + \mathbb{R}\cdot(P-Q)^0 \end{aligned} \tag{1.5}$$

Velocity and Angular velocity. Time derivative of the relation (1.5) between positions of material points gives again (1.1)

$$\begin{aligned} \vec{v}_P & = \vec{v}_Q + \dot{\mathbb{R}}\cdot(P-Q)^0 = \\ & = \vec{v}_Q + \vec{\omega}_\times \mathbb{R}\cdot(P-Q)^0 = \\ & = \vec{v}_Q + \vec{\omega}\times(P-Q) \end{aligned} \tag{1.6}$$

Acceleration and Angular acceleration. Time derivatives of the relation {eq}`eq:rigid:vel} gives

$$\begin{aligned} \vec{a}_P & = \vec{a}_Q + \vec{\alpha}\times(P-Q) + \vec{\omega}\times\left(\vec{v}_P - \vec{Q}\right) = \\ & = \vec{a}_Q + \vec{\alpha}\times(P-Q) + \vec{\omega}\times\left[\vec{\omega}\times(P-Q)\right] \ . \end{aligned}$$

1.5 Continuous Medium

1.6 Relative Kinematics

Relative kinematics is discussed here using two Cartesian reference frames.

$$\begin{aligned} P - O_0 & = x^{0i}_{P/O_0}\hat{e}^0_i \\ O_1 - O_0 & = x^{0i}_{O_1/O_0}\hat{e}^0_i \\ P - O_1 & = x^{1i}_{P/O_1}\hat{e}^1_i \\ \hat{e}^1_i & = \hat{e}^1_i\cdot\hat{e}^0_k\,\hat{e}^0_k = \hat{e}^1_j\cdot\hat{e}^0_k\,\hat{e}^0_k\otimes\hat{e}^0_j\cdot\hat{e}^0_i = \\ & = R^{0\to1}_{kj}\hat{e}^0_k\otimes\hat{e}^0_j\cdot\hat{e}^0_i = \mathbb{R}^{0\to1}\cdot\hat{e}^0_i \ . \end{aligned}$$

1.6.1 Points

Position. Given two reference frames Ox^i, $O'x^{i'}$, for the position of a point P reads

$$\begin{aligned} (P-O_0) & = (O_1-O_0) + (P-O_1)\ , \\ x^0_{P/O_0,i}\hat{e}^0_i & = x^0_{O_1/O_0,i}\hat{e}^0_i + x^1_{P/O_1,k}\hat{e}^1_k\ , \end{aligned} \tag{1.7}$$

i.e. the position vector $P-O$ of the point P w.r.t. point O - origin of the reference frame Ox^i - is the sum of the position vector $P-O'$ of the point P w.r.t. to the point O' - origin of the reference frame $O'x'^i$ - and the position vector $O'-O$, of the origin O' w.r.t. to O.

Velocity. Time derivative of relative position relation (1.7) w.r.t. to reference frame 0 is performed keeping $\hat{e}^0_i$ constant.

$$\begin{aligned} \frac{{}^0d}{dt}(P-O_0) & = \frac{{}^0d}{dt}\left[(O_1-O_0)+(P-O_1)\right] = \\ & = \frac{{}^0d}{dt}\left(x^0_{O_1/O_0,i}\hat{e}^0_i\right) + \frac{{}^0d}{dt}\left(x^1_{P/O_1,k}\hat{e}^1_k\right) = \\ & = \frac{{}^0d}{dt}x^0_{O_1/O_0,i}\,\hat{e}^0_i + \frac{{}^0d}{dt}x^1_{P/O_1,k}\,\hat{e}^1_k + x^1_{P/O_1,k}\,\frac{{}^0d}{dt}\hat{e}^1_k = \\ & = \vec{v}^0_{O_1/O_0} + \vec{v}^1_{P/O_1} + \vec{\omega}_{1/0}\times\hat{e}^1_k x^1_{P/O_1,k} = \\ & = \vec{v}^0_{O_1/O_0} + \vec{v}^1_{P/O_1} + \vec{\omega}_{1/0}\times(P-O_1)\ , \end{aligned}$$

so that

$$\vec{v}^0_{P/O} = \vec{v}^0_{O_1/O_0} + \vec{v}^1_{P/O_1} + \vec{\omega}_{1/0} \times (P - O_1) \ . \tag{1.8}$$

Acceleration. Time derivative of relative velocity relation (1.8) w.r.t. reference frame 0 reads

$$\begin{aligned}
\frac{{}^0d}{dt}\vec{v}^0_{P/O_0} &= \frac{{}^0d}{dt}\left[\vec{v}^0_{O_1/O_0} + \vec{v}^1_{P/O_1} + \vec{\omega}_{1/0} \times (P - O_1)\right] = \\
&= \dots \\
&= \vec{a}^0_{O_1/O_0} + \vec{a}^1_{P/O_1} + 2\vec{\omega}_{1/0} \times \vec{v}^1_{P/O_1} + \vec{\alpha}_{1/0} \times (P - O_1) + \vec{\omega}_{1/0} \times [\vec{\omega}_{1/0} \times (P - O_1)]
\end{aligned}$$

so that

$$\vec{a}^0_{P/O_0} = \vec{a}^0_{O_1/O_0} + \vec{a}^1_{P/O_1} + \underbrace{\vec{\alpha}_{1/0} \times (P - O_1)}_{\text{tangential}} + \underbrace{2\vec{\omega}_{1/0} \times \vec{v}^1_{P/O_1}}_{\text{Coriolis}} + \underbrace{\vec{\omega}_{1/0} \times [\vec{\omega}_{1/0} \times (P - O_1)]}_{\text{centripetal}} \ .$$

1.6.2 Rigid bodies

Orientation.

Angular velocity.

Angular acceleration.

1.7 Rotations

1.7.1 Rotation tensor

Given 2 Cartesian bases $\{\hat{e}^0_i\}_{i=1:3}$, $\{\hat{e}^1_j\}_{j=1:3}$, the rotation tensor providing the transformation

$$\hat{e}^1_i = \mathbb{R}^{0\to 1} \cdot \hat{e}^0_i \ ,$$

is

$$\mathbb{R}^{0\to 1} = R^{0\to 1}_{ij} \hat{e}^0_i \otimes \hat{e}^0_j = R^{0\to 1}_{ij} \hat{e}^1_i \otimes \hat{e}^1_j$$

with $R^{0\to 1}_{ij} = \hat{e}^0_i \cdot \hat{e}^1_j$.

Angular velocity.

$$\vec{\omega}^{01}_{\times} = \Omega^{01} = \dot{\mathbb{R}}^{01} \cdot \mathbb{R}^{01,T}$$

Using index notation

$$\varepsilon_{ijk}\omega_j = \dot{R}_{ij}R_{kj}$$

and the identities

$$\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

$$\varepsilon_{ijk}\varepsilon_{ljk} = \delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl} = 3\delta_{il} - \delta_{il} = 2\delta_{il}$$

it follows

$$\varepsilon_{ilk}\varepsilon_{ijk}\omega_j = \varepsilon_{ilk}\dot{R}_{ij}R_{kj}$$

$$2\delta_{lj}\omega_j = \varepsilon_{ilk}\dot{R}_{ij}R_{kj}$$
$$\omega_l = \frac{1}{2}\varepsilon_{ilk}\dot{R}_{ij}R_{kj} = -\frac{1}{2}\varepsilon_{lik}\dot{R}_{ij}R_{kj} = -\frac{1}{2}\varepsilon_{lij}\Omega_{ij}$$

Angular acceleration. Angular acceleration, $\vec{\alpha}$, is the time derivative of angular velocity, $\vec{\omega}$,

$$\vec{\alpha} = \dot{\vec{\omega}} \, .$$

1.7.2 Successive rotations

Orientation. Given 3 Cartesian bases $\{\hat{e}^0_i\}_{i=1:3}$, $\{\hat{e}^1_j\}_{j=1:3}$, $\{\hat{e}^2_k\}_{k=1:3}$,

$$\begin{aligned}\hat{e}^2_i &= \mathbb{R}^{1\to 2}\cdot\hat{e}^1_i = \\ &= \mathbb{R}^{1\to 2}\cdot\mathbb{R}^{0\to 1}\cdot\hat{e}^0_i\end{aligned} ,$$

i.e composition of rotations holds

$$\mathbb{R}^{0\to 2} = \mathbb{R}^{1\to 2}\cdot\mathbb{R}^{0\to 1} \, .$$

Angular velocity. Time derivative w.r.t. reference frame 0 is indicated as the standard time derivative

$$\dot{a} = \frac{da}{dt} = \frac{{}^0 da}{dt} = \frac{{}^1 d}{dt} + \vec{\omega}_{1/0}\times \, ,$$

$$\begin{aligned}\frac{d}{dt}\mathbb{R}^{21} &= \frac{d}{dt}\left[R^{21}_{ik}\hat{e}^1_i\otimes\hat{e}^1_k\right] = \\ &= \dot{R}^{21}_{ik}\hat{e}^1_i\otimes\hat{e}^1_k + \Omega^{10}\cdot\mathbb{R}^{21} - \mathbb{R}^{21}\cdot\Omega^{10} = \\ &= \frac{{}^1 d}{dt}\mathbb{R}^{21} + \Omega^{10}\cdot\mathbb{R}^{21} - \mathbb{R}^{21}\cdot\Omega^{10} =\end{aligned}$$

$$\begin{aligned}\Omega^{20} &= \dot{\mathbb{R}}^{20}\cdot\mathbb{R}^{20,T} = \\ &= \frac{d}{dt}\left(\mathbb{R}^{21}\cdot\mathbb{R}^{10}\right)\cdot\mathbb{R}^{20,T} = \\ &= \left\{\left[\frac{{}^1 d}{dt}\mathbb{R}^{21} + \Omega^{10}\cdot\mathbb{R}^{21} - \mathbb{R}^{21}\cdot\Omega^{10}\right]\cdot\mathbb{R}^{10} + \mathbb{R}^{21}\cdot\dot{\mathbb{R}}^{10}\right\}\cdot\mathbb{R}^{01}\cdot\mathbb{R}^{12} = \\ &= \frac{{}^1 d}{dt}\mathbb{R}^{21}\cdot\mathbb{R}^{12} + \Omega^{10} = \\ &= \Omega^{21} + \Omega^{10} \, .\end{aligned}$$

so that addition of relative angular velocity holds

$$\Omega^{20} = \Omega^{21} + \Omega^{10} \qquad , \qquad \vec{\omega}_{2/0} = \vec{\omega}_{2/1} + \vec{\omega}_{1/0} \, .$$

Angular acceleration. Time derivative of angular velocity composition provides the addition of relative angular accelerations

$$\frac{{}^0 d}{dt}\vec{\omega}_{2/0} = \frac{{}^0 d}{dt}\vec{\omega}_{2/1} + \frac{{}^0 d}{dt}\vec{\omega}_{1/0} \, ,$$

or

$$\vec{\alpha}_{2/0} = \vec{\alpha}_{2/1} + \vec{\alpha}_{1/0} \, .$$

1.7.3 Linearization of rotations

$$\mathbb{I} = \mathbb{R} \cdot \mathbb{R}^T$$

Increment

$$\mathbb{0} = \delta\mathbb{R} \cdot \mathbb{R}^T + \mathbb{R} \cdot \delta\mathbb{R}^T$$

and thus the antisymmetric tensor can be written as

$$\delta\theta_\times := \delta\mathbb{R} \cdot \mathbb{R}^T = \delta\Theta \ ,$$

so that

$$\delta\theta_l = -\frac{1}{2}\varepsilon_{lij}\delta R_{ik}\, R_{jk} = -\frac{1}{2}\varepsilon_{lij}\delta\Theta_{ij}$$

1.7.4 Parametrizations

Minimal sets of parameters to represent rotations have 3 parameters. However these sets of parameters are not regular over all the possible rotations, and the transformation becomes singular somewhere. *Quaternions* provide a set of 4 parameters for a regular parametrization of rotations

Euler angles

Axis and rotation angle

Quaternions

CHAPTER

TWO

ACTIONS

What's an action? Newton conceives the concept of an action, including both forces and moments, as the **causes of changes of the true motion** of mechanical systems, or **causes of difference of true motion w.r.t. a *general relative motion***.

Newton's concept of *true motion* is meant as the motion w.r.t. an inertial reference frame. So what is an **inertial reference frame**? In an inertial reference frame, dynamometers measure no force and moment associated with uniform motion.

2.1 Force, Moment of a Force, Distributed Actions

2.1.1 Concentrated Force

A (concentrated) force is a vector quantity with physical dimensions,

$$[\text{force}] = \frac{[\text{mass}][\text{length}]}{[\text{time}]^2}$$

which can be measured using a dynamometer, and whose effect can alter the equilibrium or motion conditions of a physical system.

In addition to the typical information of a vector quantity - magnitude, direction, and sense - contained in the force vector $\vec{F}$, it is often necessary to know the **point of application** or the line of application of the force.

2.1.2 Moment of a Concentrated Force

The moment of a force $\vec{F}$ applied at point P, or with a line of application passing through P, relative to point H is defined as the vector product,

$$\vec{M}_H = (P - H) \times \vec{F}$$

2.1.3 System of Forces, Resultant of Actions, and Equivalent Loads

Given a system of N forces $\left\{\vec{F}_n\right\}_{n=1:N}$, applied at points P_n, we define:

- **resultant** of the system of forces: the sum of the forces,

$$\vec{R} = \sum_{n=1}^{N} \vec{F}_n \ ,$$

- resultant of the moments with respect to a point H: the sum of the moments

$$\vec{M}_H = \sum_{n=1}^{N} (P_n - H) \times \vec{F}_n \ ,$$

- an **equivalent load**: a system of forces that has the same resultant of forces and moments; for a system of forces, an equivalent load can be defined as a single force, the resultant of the forces $\vec{R}$ applied at point Q derived from the equivalence of moments

$$\vec{R} = \sum_{n=1}^{N} \vec{F}_n$$
$$(Q - H) \times \vec{R} = \sum_{n=1}^{N} (P_n - H) \times \vec{F}_n$$

2.1.4 Couple of Forces

A couple of forces is an equivalent load to two forces of equal magnitude and opposite sense, $\vec{F}_2 = -\vec{F}_1$, applied at two points P_1, P_2 not aligned along the line of application of the forces to have non-zero effects.

todo *image*

The resultant of the forces is zero,

$$\vec{R} = \vec{F}_1 + \vec{F}_2 = \vec{F}_1 - \vec{F}_1 = \vec{0} \ ,$$

while the resultant of the moments does not depend on the moment pole,

$$\begin{aligned} \vec{M}_H &= (P_1 - H) \times \vec{F}_1 + (P_2 - H) \times \vec{F}_2 = \\ &= (P_1 - H) \times \vec{F}_1 - (P_2 - H) \times \vec{F}_1 = \\ &= (P_1 - P_2) \times \vec{F}_1 =: \vec{C} \ . \end{aligned}$$

2.1.5 Force Fields

todo

2.1.6 Distributed Actions

todo

2.2 Work and Power

In mechanics, as will become clearer later (**todo** add reference), the concept of work is linked to the concept of energy. **todo**

2.2.1 Work and Power of a Force

Work. The elementary work of a force $\vec{F}$ applied at point P that undergoes an elementary displacement $d\vec{r}_P$ is defined as the dot product between the force and the displacement,

$$\delta W := \vec{F} \cdot d\vec{r}_P \ . \tag{2.1}$$

The work done by the force $\vec{F}$ applied at point P moving from point A to point B along the path ℓ_{AB} is the sum of all elementary contributions - and hence, in the limit for elementary displacements $\rightarrow 0$ for continuous variations, the line integral,

$$W_{\ell_{AB}} = \int_{\ell_{AB}} \delta W = \int_{\ell_{AB}} \vec{F} \cdot d\vec{r}_P \ . \tag{2.2}$$

In general, the work of a force or a field of forces depends on the path ℓ_{AB}. In cases where the work is independent of the path but depends only on the endpoints, we talk about *conservative actions*.

Power. The power of the force is defined as the time derivative of the work,

$$P := \frac{\delta W}{dt} = \vec{F} \cdot \frac{d\vec{r}_P}{dt} = \vec{F} \cdot \vec{v}_P \ ,$$

and coincides with the dot product between the force and the velocity of the point of application. Be cautious if a force is applied to geometric points rather than material points, such as in the case of a disk rolling without slipping on a surface: at every instant, the (new) material contact point has zero velocity, while the geometric contact point is the projection of the center of the disk and moves with the same velocity, $v = R\theta$

2.2.2 Work and Power of a System of Forces

Work. The work of a system of forces is the sum of the works of the individual forces,

$$\delta W = \sum_{n=1}^{N} \delta W_n = \sum_{n=1}^{N} \vec{F}_n \cdot d\vec{r}_n$$

Power. The power of a system of forces is the sum of the powers of the individual forces,

$$P = \sum_{n=1}^{N} P_n = \sum_{n=1}^{N} \vec{F}_n \cdot \vec{v}_n \ .$$

2.2.3 Work and Power of a Couple of Forces

Work. The elementary work of a couple of forces is the sum of the elementary works,

$$\begin{aligned} \delta W &= \vec{F}_1 \cdot d\vec{r}_1 + \vec{F}_2 \cdot d\vec{r}_2 = \\ &= \vec{F}_1 \cdot (d\vec{r}_1 - d\vec{r}_2) = \end{aligned}$$

Power. The power of a couple of forces,

$$P = \vec{F}_1 \cdot (\vec{v}_1 - \vec{v}_2)$$

can be rewritten if the points of application perform a rigid motion act (**todo** verify the definition of motion act and if it should be introduced),

$$\vec{v}_1 - \vec{v}_2 = \vec{\omega} \times (P_1 - P_2) \ ,$$

as

$$\begin{aligned}
P &= \vec{F}_1 \cdot (\vec{v}_1 - \vec{v}_2) = \\
&= \vec{F}_1 \cdot [\vec{\omega} \times (P_1 - P_2)] = \\
&= \vec{\omega} \cdot \left[(P_1 - P_2) \times \vec{F}_1 \right] = \\
&= \vec{\omega} \cdot \vec{C} \ .
\end{aligned}$$

2.3 Conservative Actions

In general, the work of a force field acting on a point P moving in space from point A to point B along a path ℓ_{AB} represented by integral (2.2) depends on the path, and this dependence on the path is usually highlighted with the use of the symbol δ in the elementary work (2.1).

If the work of a force field does not depend on the path ℓ_{AB} but only on the endpoints A, B, for all pairs of points within a region of space Ω, the **force field** is said to be **conservative** in the region Ω of space. In this case, the work integral can be written as the difference of a scalar field, $U(P)$ or its opposite $V(P) := -U(P)$,

$$\begin{aligned}
W_{AB} &= \int_{\ell_{AB}} \vec{F} \cdot d\vec{r} = \\
&= \int_{\ell_{AB}} \delta W = \\
&= U(B) - U(A) = \Delta_{AB} U \\
&= V(A) - V(B) = -\Delta_{AB} V
\end{aligned}$$

The functions U, V are respectively defined as the **potential** and **potential energy** of the force field. From the definition of a conservative force field it readily follows that

$$\oint_\ell \vec{F} \cdot d\vec{r} = 0 \ .$$

The elementary work can thus be expressed in terms of the differential of these functions,

$$\begin{aligned}
\delta W = \quad dU = \quad d\vec{r} \cdot \nabla U = \\
= -dV = -d\vec{r} \cdot \nabla V
\end{aligned}$$

Comparing this relation with the definition of work $\delta W = d\vec{r} \cdot \vec{F}$, it is possible to identify the force field with the gradient of the potential function, and the opposite of the gradient of the potential energy,

$$\vec{F} = \nabla U = -\nabla V \ .$$

Since the force field can be written as the gradient of a scalar field, and the curl of a gradient is identically zero, the curl of a potential force field is identically zero,

$$\nabla \times \vec{F} = \vec{0} \ .$$

Note: The reverse logical process - $\nabla \times \vec{F} = \vec{0}$ implies $\vec{F} = \nabla U$ implies $\vec{F}$ conservative, i.e. independence of the work from the path - requires the domain continaing the integration path ℓ to be simply connected.

Example 2.3.1 (Force fields in non-simply connected domains)

In a region of E^2, described with Cartesian coordinates, containing the origin $O \equiv (x_0, y_0) \equiv (0,0)$ the vector field

$$\vec{F}_1(\vec{r}) = \frac{x}{x^2+y^2}\hat{x} + \frac{y}{x^2+y^2}\hat{y}$$

is conservative, while the vector field

$$\vec{F}_2(\vec{r}) = -\frac{y}{x^2+y^2}\hat{x} + \frac{x}{x^2+y^2}\hat{y}$$

is not conservative, even though their curl is zero in all the points of the domain where the field is defined - they're not defined in the origin.

2.4 Examples of Forces

2.4.1 Gravitation

Universal Law of Gravitation

The force $\vec{F}_{12}$ exerted by a mass m_2 at P_2 on a mass m_1 at P_1 is described by **Newton's Universal Law of Gravitation**,

$$\vec{F}_{12} = G\, m_1\, m_2 \frac{P_2 - P_1}{|P_2 - P_1|^3} \ ,$$

or

$$\vec{F}_{12} = G\, m_1\, m_2 \frac{\hat{r}_{12}}{|\vec{r}_{12}|^2} \ ,$$

where $\vec{r}_{12} = (P_2 - P_1)$ is the vector pointing from point P_1 to point P_2, $r_{12} = |\vec{r}_{12}|$ is its magnitude, and $\hat{r}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|}$ is the unit vector in the same direction. The **universal gravitational constant** G is

$$G = 6.67 \cdot 10^{-11} \frac{N\, m^2}{kg^2}$$

and is considered a constant of nature.

Principle of Superposition of Causes and Effects (PSCE). Principle of superposition holds, i.e. the force acting on a mass m placed in P due to a set of N masses $\{m_k\}_{k=1:N}$ placed in P_k is the sum of individual forces $\vec{F}_k$,

$$\vec{F} = \sum_{k=1}^{N} \vec{F}_k = G\, m \sum_{k=1}^{N} m_k \frac{P_k - P}{|P_k - P|^3} \ . \tag{2.3}$$

Gravitational Field

The gravitational field generated by a set of masses $\{m_k\}_{k=1:N}$ located at P_k is a vector field associating a vector with physical dimensions $\frac{[\text{force}]}{[\text{mass}]}$ to each point in space P, that can be thought as the force per unit-mass acting on a **test mass** m placed in P, whose expression directly follows from (2.3)

$$\vec{g}(P) = \frac{\vec{F}}{m} = G \sum_{k=1}^{N} m_k \frac{P_k - P}{|P_k - P|^3} \ .$$

Given the gravitational field $\vec{g}(P)$, the gravitational force experienced by a system of mass m at P can be written as

$$\vec{F}_g = m\vec{g}(P)$$

Gravitational Potential Energy. Gravitational potential of a system of 2 masses reads

$$V(P) = -G\, m\, m_1 \frac{1}{|P - P_1|} \ ,$$

as it can be easily shown evaluating its gradient,

$$\begin{aligned}
\nabla V(P) & = -G\, m\, m_1\, \hat{x}_k\, \frac{\partial}{\partial x_k} \frac{1}{|P-P_1|} = \\
& = -G\, m\, m_1\, \hat{x}_k \left(-\frac{1}{|P-P_1|^2} \right) \frac{\partial}{\partial x_k} |P - P_1| = \\
& = G\, m\, m_1\, \hat{x}_k \left(\frac{1}{|P-P_1|^2} \right) \frac{x_k - x_{1,k}}{|P-P_1|} = \\
& = G\, m\, m_1\, \frac{x_k - x_{1,k}}{|P-P_1|^3}\, \hat{x}_k = \\
& = G\, m\, m_1\, \frac{P - P_1}{|P-P_1|^3} \ .
\end{aligned}$$

Potential energy stored in a system of N point masses $\{m_k\}_{k=1:N}$ coincides with the work needed to build the system - a common choice to set the arbitrary additional constant of the energy is setting it equal to zero when masses are at infinite distances -, namely

$$V(P_k) = \sum_{\{i,k\}, i \ne k} G\, m_i\, m_k \frac{1}{|P_i - P_k|} \ ,$$

summing over different unordered pairs, i.e. $\{1, 2\}$ and $\{2, 1\}$ are the same pair and thus considered only once, or

$$V(P_k) = \frac{1}{2} \sum_{(i,k), i \ne k} G\, m_i\, m_k \frac{1}{|P_i - P_k|} \ ,$$

summing over different ordered pairs, i.e. $(1, 2)$ and $(2, 1)$ are different pairs.

Gravitational Field Near Earth's Surface

Within a limited domain near Earth's surface, it is common to approximate Earth's gravitational field as a uniform field, directed along the local vertical toward the center of the Earth, with intensity $g = G\frac{M_E}{R_E^2}$.

This model can be derived by approximating the position vector relative to the Earth's center $P - P_E \sim R_E \hat{r}$ and the unit vector identifying the direction from a point in the domain to the Earth's center with the local vertical $\hat{r}_{12} \sim -\hat{z}$,

$$\vec{g}(\vec{r}) = -G\frac{M_E}{R_E^2}\hat{z} = -g\hat{z} \ .$$

The gravitational force experienced by a body of mass m near Earth's surface is thus

$$\vec{F}_g = -mg\hat{z} \ ,$$

commonly referred to as **weight**.

Gravitational Potential Energy. It can be shown that the gravitational potential near Earth's surface becomes

$$V(P) = m\, g\, z_P \ .$$

Proof.

With the series expansion, with $P - P_E = R_E \hat{r} + \vec{d}$, and $|\vec{d}| \ll R_E$,

$$\begin{aligned} V(P) = -G\, m\, M_E \frac{1}{|P-P_E|} & = \\ \approx G M_E\, m \left[-\frac{1}{R_E} + \frac{R_E \hat{r}\cdot\vec{d}}{R_E^3} \right] & = \\ = \underbrace{-\,m\,\frac{GM_E}{R_E}}_{\text{const}} + m \underbrace{\frac{G\,M_E}{R_E^2}}_{=g} \underbrace{\hat{r}\cdot\vec{d}}_{=z} & \end{aligned}$$

Gravitational field of a continuous mass distribution

Mass density field $\rho(\vec{r}_0)$ for $\vec{r}_0 \in V_0$ produces the **gravitational field** in $\vec{r}$,

$$\vec{g}(\vec{r}) = \int_{\vec{r}_0 \in V_0} d\vec{g}(\vec{r}, \vec{r}_0) = -\int_{\vec{r}_0 \in V_0} G \rho(\vec{r}_0) \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \,.$$

By direct computation, the **gravitational potential** $\phi(\vec{r})$, s.t. $\vec{g} = \nabla \phi$, reads

$$\phi(\vec{r}) = \int_{\vec{r}_0 \in V_0} G \rho(\vec{r}_0) \frac{1}{|\vec{r}-\vec{r}_0|}$$

Gauss' law for the gravitational field

The flux of the gravitational field produced by mass distribution $\rho(\vec{r}_0)$ in volume V_0 thorugh a closed surface ∂V reads

$$\begin{aligned} \oint_{\vec{r}\in\partial V} \vec{g}(\vec{r}) \cdot \hat{n}(\vec{r}) & = -G \oint_{\vec{r}\in\partial V} \int_{\vec{r}_0 \in V_0} \rho(\vec{r}_0) \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \cdot \hat{n}(\vec{r}) = \\ & = -G \int_{\vec{r}_0 \in V_0} \rho(\vec{r}_0) \oint_{\vec{r}\in\partial V} \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \cdot \hat{n}(\vec{r}) \end{aligned}$$

The inner integral can be written as the solid angle of the surface ∂V as seen by the point $\vec{r}_0$, whose value is

$$\oint_{\vec{r}\in\partial V} \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \cdot \hat{n}(\vec{r}) = 4\pi \begin{cases} 1 & \text{if } \vec{r}_0 \in V \\ \theta(\vec{r}_0, \partial V) & \text{if } \vec{r}_0 \in \partial V \\ 0 & \text{if } \vec{r}_0 \notin V \cup \partial V \end{cases}$$

Thus, net contributions of the flux of the gravitational field $\vec{g}(\vec{r})$ through ∂V come only from points $\vec{r}_0$ inside V, $\vec{r}_0 \in V$.[1] Thus the flux becomes

$$\oint_{\vec{r}\in\partial V} \vec{g}(\vec{r}) \cdot \hat{n}(\vec{r}) = -G \int_{\vec{r}_0 \in V_0 \cap V} 4\pi \rho(\vec{r}_0)$$

or, setting $\rho(\vec{r}_0) =$ in all the points $\vec{r} \in V$, $\vec{r} \notin V_0$, and changing the name of the dummy integration variable $\vec{r}_0 \to \vec{r}$,

$$\oint_{\vec{r}\in\partial V} \vec{g}(\vec{r}) \cdot \hat{n}(\vec{r}) = -G \int_{\vec{r} \in V} 4\pi \rho(\vec{r}) \,.$$

[1] Contributions from points outside ∂V are identically zero; contributions from surface ∂V are zero if volume mass density $\rho(\vec{r}_0)$ is regular enough, i.e. it contains Dirac's δ representing surface distribution that would have non-negligible contributions in integration over V.

If the gravitational field $\vec{g}(\vec{r})$ is regular enough for the divergence theorem to hold, it follows

$$\oint_{\vec{r}\in\partial V} \vec{g}(\vec{r}) \cdot \hat{n}(\vec{r}) = \int_{\vec{r}\in V} \nabla \cdot \vec{g}(\vec{r}) = -G \int_{\vec{r}\in V} 4\pi\rho(\vec{r}) \ , \qquad (2.4)$$

or, for the arbitrariety of the volume V,

$$-\nabla \cdot \vec{g} = 4\pi G\rho \ .$$

Introducing the gravitational potential $\phi(\vec{r})$, whose gradient equals the gravitational field $\nabla\phi = \vec{g}$ by definition, a **Poisson equation for the gravitational potential** follows

$$-\nabla^2\phi = 4\pi G\rho \ . \qquad (2.5)$$

2.4.2 Elastic Actions: Linear Springs

todo

2.5 Constraint Reactions

Kinematic constraints act on a system by limiting its possible movements, exerting forces and moments, which are defined as constraint reactions.

In general, at an **ideal** constraint (**todo** provide definition of ideal constraint and discuss/mention/refer to friction), a constraint reaction corresponds to each constrained degree of freedom: for example, the constraint of translation of a point in a direction has a corresponding reaction force in that direction; the constraint of rotation around an axis has a corresponding moment aligned with that axis.

These conditions can be derived from the equations of dynamics for massless systems, as often considered in the ideal constraint model.

2.5.1 Contact Actions

Constraint Reactions of Ideal Constraints

Ideal constraints are models that **do not perform net work**, and are thus **conservative elements**. As should become evident in the subsequent sections from the expressions of relative velocities and exchanged actions,

$$\begin{aligned}
P &= \vec{v}_1 \cdot \vec{F}_{21} + \vec{v}_2 \cdot \vec{F}_{12} + \vec{\omega}_1 \cdot \vec{M}_{21} + \vec{\omega}_2 \cdot \vec{M}_{12} = \\
&= (\vec{v}_1 - \vec{v}_2) \cdot \vec{F}_{21} + (\vec{\omega}_1 - \vec{\omega}_2) \cdot \vec{M}_{21} = \\
&= \vec{v}^{rel}_{21} \cdot \vec{F}_{21} + \vec{\omega}^{rel}_{21} \cdot \vec{M}_{21} \ ,
\end{aligned}$$

both terms are zero either because the relative motion is zero, or the actions act orthogonally to the relative motions.

Fixed Joint

The fixed joint constraint prevents both relative motion and relative rotation,

$$\begin{cases} \vec{0} = \vec{v}_{21}^{rel} = \vec{v}_2 - \vec{v}_1 \\ \vec{0} = \vec{\omega}_{21}^{rel} = \vec{\omega}_2 - \vec{\omega}_1 \end{cases} , \qquad \begin{cases} \vec{F}_{12} = -\vec{F}_{21} \\ \vec{M}_{12} = -\vec{M}_{21} \end{cases}$$

Slider

The slider constraint prevents relative motion in one direction and relative rotation.

$$\begin{cases} \forall\ \vec{v}_{\hat{t},21}^{rel} = \vec{v}_{\hat{t},2} - \vec{v}_{\hat{t},1} \\ 0 = v_{\hat{n},21}^{rel} = v_{\hat{n},2} - v_{\hat{n},1} \\ \vec{0} = \vec{\omega}_{21}^{rel} = \vec{\omega}_2 - \vec{\omega}_1 \end{cases} , \qquad \begin{cases} \vec{0} = \vec{F}_{\hat{t},12} = \vec{F}_{\hat{t},21} \\ F_{\hat{n},12} = -F_{\hat{n},21} \\ \vec{M}_{12} = -\vec{M}_{21} \end{cases}$$

Cylindrical Joint

The cylindrical joint constraint prevents relative motion and allows rotation around one axis.

$$\begin{cases} \vec{0} = \vec{v}_{21}^{rel} = \vec{v}_2 - \vec{v}_1 \\ \forall\ \omega_{\hat{t},21}^{rel} = \omega_{\hat{t},2} - \omega_{\hat{t},1} \\ \vec{0} = \vec{\omega}_{\hat{n},21}^{rel} = \vec{\omega}_{\hat{n},2} - \vec{\omega}_{\hat{n},1} \end{cases} , \qquad \begin{cases} \vec{F}_{12} = -\vec{F}_{21} \\ 0 = M_{\hat{t},12} = M_{\hat{t},21} \\ \vec{M}_{\hat{n},12} = -\vec{M}_{\hat{n},21} \end{cases}$$

Spherical Joint

The spherical joint constraint prevents relative motion but allows general rotation.

$$\begin{cases} \vec{0} = \vec{v}_{21}^{rel} = \vec{v}_2 - \vec{v}_1 \\ \forall \vec{\omega}_{21}^{rel} = \vec{\omega}_2 - \vec{\omega}_1 \end{cases} , \qquad \begin{cases} \vec{F}_{12} = -\vec{F}_{21} \\ \vec{0} = \vec{M}_{12} = \vec{M}_{21} \end{cases}$$

Roller

The roller constraint can be thought of as a combination of a slider and a cylindrical joint.

$$\begin{cases} \forall\ \vec{v}_{\hat{t},21}^{rel} = \vec{v}_{\hat{t},2} - \vec{v}_{\hat{t},1} \\ 0 = v_{\hat{n},21}^{rel} = v_{\hat{n},2} - v_{\hat{n},1} \\ \forall\ \omega_{\hat{t},21}^{rel} = \omega_{\hat{t},2} - \omega_{\hat{t},1} \\ \vec{0} = \vec{\omega}_{\hat{n},21}^{rel} = \vec{\omega}_{\hat{n},2} - \vec{\omega}_{\hat{n},1} \end{cases} , \qquad \begin{cases} \vec{0} = \vec{F}_{\hat{t},12} = \vec{F}_{\hat{t},21} \\ F_{\hat{n},12} = -F_{\hat{n},21} \\ 0 = M_{\hat{t},12} = M_{\hat{t},21} \\ \vec{M}_{\hat{n},12} = -\vec{M}_{\hat{n},21} \end{cases}$$

Support

The support constraint is a unilateral constraint **todo** *add description*

Friction

Static Friction

Static friction is the type of friction that occurs between two bodies when there is no relative motion between them, acting as a tangential force to the contact surface. The simplest model of static friction assumes that the maximum static friction force F^s_{max} that can be exerted between two bodies is proportional to the normal reaction between them, N,

$$F^s_{max} = \mu^s \, N \ .$$

The proportionality constant μ^s is defined as the **coefficient of static friction**. Generally, static friction forces are determined by the equilibrium conditions of the body, if these conditions can be met, and the relation

$$|F^s| \geq F^s_{max} \ .$$

Dynamic Friction

Dynamic friction occurs between two bodies in contact and in relative motion, acting as a tangential force to the contact surface. The simplest model of dynamic friction assumes that the dynamic friction force is proportional to the normal reaction between the two bodies and directed opposite to the relative velocity,

$$\vec{F}_{12} = -\mu^d N \frac{\vec{v}_{12}}{|\vec{v}_{12}|} \ ,$$

where $\vec{F}_{12}$ is the force acting on body 1 due to body 2, and $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$ is the velocity of body 1 relative to body 2.

Pure Rolling

todo *add description*

CHAPTER

THREE

INERTIA

Inertia deals with mass and mass distribution of systems.

But what is mass? Mass is a physical quantity, a property of the system, that manifests itself:

- in *gravitational attraction* (being both the origin of gravitational force and the property that makes a system sensible to gravitational attraction),
- in resistance to change of motion of a system under external *actions*, as it will be clear from principles and equations of motions in *dynamics*

In the range of application of classical mechanics **mass conservation** holds, as stated by **Lavoisier principle**: the mass of a closed system is constant.

Beside mass, three main **additive dynamical quantities** are introduced: **momentum**, **angular momentum**, and **kinetic energy**. Even though their meaning could not be clear in this chapter, it would be clear in the following chapters, in the derivation of *equations of motion* of mechanical systems, like *point mass*, *system of point masses* and *rigid bodies*,…

3.1 Point mass

3.2 Discrete masses

Momentum.

$$\vec{Q} := \sum_k m_k \vec{v}_k$$

Angular momentum.

$$\vec{L}_H := \int_{V_t} (P_k - H) \times m_k \vec{v}_k$$

Kinetic energy.

$$K := \sum_k \frac{1}{2} m_k |\vec{v}_k|^2$$

3.3 Continuous systems

Momentum.

$$\vec{Q} := \int_{V_t} \rho \vec{v}$$

Angular momentum.

$$\vec{L}_H := \int_{V_t} (P - H) \times \rho \vec{v}$$

Kinetic energy.

$$K := \int_{V_t} \frac{1}{2} \rho |\vec{v}|^2$$

3.4 Rigid systems

The expression of dynamical quantities for rigid bodies can be written in terms of the velocity $\vec{v}_Q$ of a point Q of the rigid body and its angular velocity $\vec{\omega}$, exploiting the law of rigid motion (1.1) to write the velocity of each points of the rigid system as functions of $\vec{v}_Q$ and $\vec{\omega}$,

$$\vec{v}_P = \vec{v}_Q + \vec{\omega} \times (P - Q) \ .$$

3.4.1 Discrete systems

Momentum.

$$\begin{aligned}
\vec{Q} = \sum_k m_k \vec{v}_k &= \sum_k m_k \left(\vec{v}_Q + \vec{\omega} \times (P_k - Q) \right) = \\
&= m \vec{v}_Q - \sum_k m_k (P_k - Q) \times \vec{\omega} = \\
&= m \vec{v}_Q + \mathbb{S}_Q \cdot \vec{\omega} \ ,
\end{aligned}$$

having defined the static moment of inertia (a 2^{nd}-order antisymmetric tensor)

$$\mathbb{S}_Q := \vec{s}_{P \times} := - \sum_k m_k (P_k - Q)_\times \ .$$

Angular momentum.

$$\vec{L}_H = \sum_k (P_k - H) \times m_k \vec{v}_k = \underbrace{\sum_k (P_k - Q) \times m_k \vec{v}_k}_{\vec{L}_Q} + (Q - H) \times \vec{Q}$$

and

$$\begin{aligned}
\vec{L}_Q = \sum_k (P_k - Q) \times m_k \vec{v}_k &= \sum_k (P_k - Q) \times m_k \left(\vec{v}_Q - (P_k - Q) \times \vec{\omega} \right) = \\
&= \mathbb{S}_Q^T \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega} \ ,
\end{aligned}$$

having recognized the transpose of the static moment of inertia, and introduced the tensor of inertia w.r.t. reference point Q

$$\mathbb{I}_Q := -\sum_k m_k \left(P_k - Q\right)_\times \left(P_k - Q\right)_\times \ .$$

Kinetic energy.

$$\begin{aligned}
K = \sum_k \frac{1}{2} m_k |\vec{v}_k|^2 & = \sum_k \frac{1}{2} m_k \left(\vec{v}_Q + \vec{\omega} \times (P_k - Q)\right) \cdot \left(\vec{v}_Q + \vec{\omega} \times (P_k - Q)\right) = \\
& = \sum_k \frac{1}{2} m_k |\vec{v}_Q|^2 + \frac{1}{2} \sum_k 2 m_k \vec{v}_Q \cdot (-(P_k - Q) \times \vec{\omega}) + \frac{1}{2} \sum_k \vec{\omega} \cdot (P_k - Q)_\times (P_k - Q)_\times \cdot \vec{\omega} = \\
& = \frac{1}{2} \left[\sum_k m_k\right] |\vec{v}_Q|^2 + \frac{1}{2} \vec{v}_Q \cdot \left[-\sum_k m_k (P_k - Q)_\times\right] \cdot \vec{\omega} + \\
& + \frac{1}{2} \vec{\omega} \cdot \left[\sum_k m_k (P_k - Q)_\times\right] \cdot \vec{v}_Q + \frac{1}{2} \vec{\omega} \cdot \left[\sum_k m_k (P_k - Q)_\times (P_k - Q)_\times\right] \cdot \vec{\omega} = \\
& = \frac{1}{2} m |\vec{v}_Q|^2 + \frac{1}{2} \vec{v}_Q \cdot \mathbb{S}_Q \cdot \vec{\omega} + \frac{1}{2} \vec{\omega} \cdot \mathbb{S}^T \cdot \vec{v}_Q + \frac{1}{2} \vec{\omega} \cdot \mathbb{I}_Q \cdot \vec{\omega} = \\
& = \frac{1}{2} \vec{v}_Q \cdot \left[m \vec{v}_Q + \mathbb{S}_Q \cdot \vec{\omega}\right] + \frac{1}{2} \vec{\omega} \cdot \left[\mathbb{S}_Q^T \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega}\right] = \\
& = \frac{1}{2} \vec{v}_Q \cdot \vec{Q} + \frac{1}{2} \vec{\omega} \cdot \vec{L}_Q \ .
\end{aligned}$$

having used the vector identities

$$\vec{a} \cdot \vec{b} \times \vec{c} = \vec{b} \cdot \vec{c} \times \vec{a}$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

and having introduced the expression of momentum and angular momentum in the last step.

Note: The components of static moment of inertia and tensor of inertia in a material basis - following the motion of the system - are constant.

3.4.2 Continuous systems

Momentum.

$$\begin{aligned}
\vec{Q} = \int_{V_t} \rho \vec{v} = \int_{V_t} \rho \left(\vec{v}_Q + \vec{\omega} \times (P - Q)\right) = \\
= m \vec{v}_Q - \int_{V_t} \rho (P_k - Q) \times \vec{\omega} = \\
= m \vec{v}_Q + \mathbb{S}_Q \cdot \vec{\omega} \ ,
\end{aligned}$$

having defined the static moment of inertia (a 2^{nd}-order antisymmetric tensor) for continuous systems,

$$\mathbb{S}_Q := \vec{s}_{P\times} := -\int_{V_t} \rho (P - Q)_\times \ .$$

Note: Using the definition of the center of mass G,

$$G = \frac{1}{m} \int_{V_t} \rho P \ ,$$

the static moment of inertia can be written as

$$\mathbb{S}_Q = -\int_{V_t} \rho (P-Q)_\times = -m(G-Q)_\times \ .$$

Note: The components of static moment of inertia w.r.t. a material reference frame are constant. Using a material Carteisan reference frame the tensor reads

$$\begin{aligned}
\mathbb{S}_Q & = -\int_{V_t} \rho (P-Q)_\times = \\
& = -\int_{V_t} \rho \left[(x^0 - x^0_Q)\hat{x}^0 + (y^0 - y^0_Q)\hat{y}^0 + (z^0 - z^0_Q)\hat{z}^0 \right]_\times = S_{ij} \hat{e}^0_i \hat{e}^0_j \ ,
\end{aligned}$$

whose components can be collected in the **antisymmetric matrix**

$$\underline{\underline{S}}_Q = \left[S_{Q,ij} \right] = -\int_{V_0} \rho \begin{bmatrix} 0 & -(z^0 - z^0_Q) & (y^0 - y^0_Q) \\ (z^0 - z^0_Q) & 0 & -(x^0 - x^0_Q) \\ -(y^0 - y^0_Q) & (x^0 - x^0_Q) & 0 \end{bmatrix} \ ,$$

so that the vector product between $-\int_V \rho (P-Q)$ and a vector $\vec{a}$ reads

$$\begin{aligned}
-\int_V \rho (P-Q) \times a & = -\int_V \rho \left[\hat{x}^0 \left(\Delta y^0 a_z - \Delta z^0 a_y \right) + \hat{y}^0 \left(\Delta z^0 a_x - \Delta x^0 a_z \right) + \hat{z}^0 \left(\Delta x^0 a_y - \Delta y^0 a_x \right) \right] \\
& = \mathbb{S} \cdot \vec{a} \ .
\end{aligned}$$

Angular momentum.

$$\vec{L}_H = \int_{V_t} (P-H) \times \rho \vec{v} = \underbrace{\int_{V_t} (P-Q) \times \rho \vec{v}_k}_{\vec{L}_Q} + (Q-H) \times \vec{Q}$$

and

$$\begin{aligned}
\vec{L}_Q = \int_{V_t} (P-Q) \times \rho \vec{v} & = \int_{V_t} (P-Q) \times \rho \left(\vec{v}_Q - (P-Q) \times \vec{\omega} \right) = \\
& = \mathbb{S}^T_Q \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega} \ ,
\end{aligned}$$

having recognized the transpose of the static moment of inertia, and introduced the tensor of inertia w.r.t. reference point Q

$$\begin{aligned}
\mathbb{I}_Q & := -\int_{V_t} \rho \, (P-Q)_\times \, (P-Q)_\times = \\
& := \int_{V_t} \rho \left[|P-Q|^2 \mathbb{I} - (P-Q) \otimes (P-Q) \right] \ ,
\end{aligned}$$

having used the tensor identity

$$-\vec{a}_\times \cdot \vec{a}_\times = |\vec{a}|^2 \mathbb{I} - \vec{a} \otimes \vec{a}$$

Note: The components of tensor of inertia w.r.t. a material reference frame are constant. Using a material Carteisan reference frame the tensor reads

$$\begin{aligned}\mathbb{I}_Q &= -\int_{V_t} \rho (P-Q)_\times (P-Q)_\times = \\ &= -\int_{V_t} \rho \left[|P-Q|^2 \mathbb{I} - (P-Q) \otimes (P-Q) \right] = \\ &= I^0_{Q,ij} \hat{e}^0_i \hat{e}^0_j \ ,\end{aligned}$$

whose components can be collected in the **symmetric matrix**

$$\underline{\underline{I}}^0_Q = \left[I^0_{Q,ij} \right] = \int_{V_0} \rho \begin{bmatrix} \Delta y_0^2 + \Delta z_0^2 & -\Delta x_0 \Delta y_0 & -\Delta x_0 \Delta z_0 \\ -\Delta y_0 \Delta x_0 & \Delta x_0^2 + \Delta y_0^2 & -\Delta y_0 \Delta z_0 \\ -\Delta z_0 \Delta x_0 & -\Delta z_0 \Delta y_0 & \Delta x_0^2 + \Delta z_0^2 \end{bmatrix} \ ,$$

being $\Delta x^0 := x^0_P - x^0_Q$.

3.4.3 Properties of inertia tensors of rigid bodies

Static inertia

Center of mass, G. Center of mass of of a rigid body is defined as the point G for which $\mathbb{S}_G \equiv \mathbb{0}$, whose coordinates are given by

$$G = \frac{1}{m} \int_{V_t} \rho \, P$$

Anti-symmetric. From the definition of the static inertia tensor

$$\mathbb{S}_Q \cdot \vec{a} = \int_{V_t} \rho (P-Q) \times \vec{a} = -\vec{a} \times \int_{V_t} \rho (P-Q) = -\vec{a} \cdot \mathbb{S}_Q = -\mathbb{S}^T \cdot \vec{a} \ .$$

Transport.

$$\begin{aligned}\mathbb{S}_Q &= -\int_{V_t} \rho (P-Q)_\times = \\ &= -\int_{V_t} \rho (P-R)_\times - \int_{V_t} \rho (R-Q)_\times = \\ &= \mathbb{S}_R - m(R-Q)_\times \ ,\end{aligned}$$

or w.r.t. the center of mass G,

$$\mathbb{S}_Q = \mathbb{S}_G - m(G-Q)_\times \ .$$

Tensor of inertia

Symmetric (semi)-definite positive. Inertia tensor is symmetric

$$\vec{v} \cdot \mathbb{I}_Q \cdot \vec{w} = \vec{v} \cdot \int_{V_t} \rho \left[|\Delta \vec{r}|^2 \mathbb{I} - \Delta \vec{r} \otimes \Delta \vec{r} \right] \cdot \vec{w} = \vec{w} \cdot \mathbb{I}_Q \cdot \vec{v} \ .$$

for all $\forall \vec{v}, \vec{w}$, and semi-definite positive

$$\begin{aligned}
\vec{v} \cdot \mathbb{I}_Q \cdot \vec{v} &= -\vec{v} \cdot \int_{V_t} \rho \Delta\vec{r}_\times \Delta\vec{r}_\times \cdot \vec{v} = \\
&= -\int_{V_t} \rho \vec{v} \cdot \Delta\vec{r}_\times \Delta\vec{r}_\times \cdot \vec{v} = \\
&= -\int_{V_t} \rho \vec{v} \cdot [\Delta\vec{r} \times (\Delta\vec{r} \times \cdot \vec{v})] = \\
&= -\int_{V_t} \rho \, (\Delta\vec{r} \times \vec{v}) \cdot (\vec{v} \times \Delta\vec{r}) = \\
&= \int_{V_t} \rho \, (\Delta\vec{r} \times \vec{v}) \cdot (\Delta\vec{r} \times \vec{v}) = \\
&= \int_{V_t} \rho |\Delta\vec{r} \times \vec{v}|^2 \geq 0
\end{aligned}$$

Principal axes of inertia. As the tensor of inertia is symmetric and definite positive, a set of orthogonal vectors $\hat{E}_i^0$ so that it can be written in diagonal form,

$$\mathbb{I}_Q = I_{XX}^0 \, \hat{E}_X^0 \otimes \hat{E}_X^0 + I_{YY}^0 \, \hat{E}_Y^0 \otimes \hat{E}_Y^0 + I_{ZZ}^0 \, \hat{E}_Z^0 \otimes \hat{E}_Z^0 \ ,$$

with $I_{ii}^0 \geq 0$ (no sum).

Theorem 3.4.1 (Transport - Huygens' theorem.)

$$\begin{aligned}
\mathbb{I}_Q &= -\int_{V_t} \rho (P-Q)_\times (P-Q)_\times = \\
&= -\int_{V_t} \rho (P-R)_\times (P-R)_\times - \int_{V_t} \rho (P-R)_\times (R-Q)_\times \\
&\quad -\int_{V_t} \rho (R-Q)_\times (P-R)_\times - \int_{V_t} \rho (R-Q)_\times (R-Q)_\times = \\
&= \mathbb{I}_R + \mathbb{S}_R \cdot (R-Q)_\times + (R-Q)_\times \cdot \mathbb{S}_R - m (R-Q)_\times (R-Q)_\times
\end{aligned}$$

or w.r.t. the center of mass G,

$$\mathbb{I}_Q = \mathbb{I}_G - m (Q-G)_\times (Q-G)_\times \ .$$

3.4.4 Time derivatives of dynamical quantities

Time derivatives of dynamical quantities are easily evaluated using a Cartesian material reference frame.

Momentum.

$$\begin{aligned}
\frac{d}{dt}\vec{Q} &= \frac{d}{dt}\left(m\vec{v}_Q + \mathbb{S}_Q \cdot \vec{\omega}\right) = \\
&= m\dot{\vec{v}}_Q + \frac{d}{dt}\left(\vec{E}_i^0 S_{ij}^0 \omega_j^0\right) = \\
&= m\dot{\vec{v}}_Q + \frac{d\vec{E}_i^0}{dt} S_{ij}^0 \omega_j^0 + \vec{E}_i^0 S_{ij}^0 \frac{d}{\omega_j}^0 dt = \\
&= m\dot{\vec{v}}_Q + \vec{\omega} \times \vec{E}_i^0 S_{ij}^0 \omega_j^0 + \vec{E}_i^0 S_{ij}^0 \frac{d}{dt}\omega_j^0 = \\
&= m\dot{\vec{v}}_Q + \vec{\omega} \times \left(\mathbb{S}_Q \cdot \vec{\omega}\right) + \mathbb{S}_Q \cdot \dot{\vec{\omega}}\,.
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\vec{\omega} &= \frac{d}{dt}\left(\hat{E}_i^0 \omega_i^0\right) = \\
&= \vec{\omega} \times \hat{E}_i^0 \omega_i^0 + \hat{E}_i^0 \frac{d\omega_i^0}{dt} = \\
&= \underbrace{\vec{\omega} \times \vec{\omega}}_{=\vec{0}} + \hat{E}_i^0 \frac{d\omega_i^0}{dt}\,.
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\vec{v} &= \frac{d}{dt}\left(\hat{E}_i^0 v_i^0\right) = \\
&= \vec{\omega} \times \hat{E}_i^0 v_i^0 + \hat{E}_i^0 \frac{dv_i^0}{dt} = \\
&= \vec{\omega} \times \vec{v} + \frac{{}^0 d}{dt}\vec{v}
\end{aligned}$$

Angular momentum.

$$\frac{d}{dt}\vec{L}_H = \frac{d}{dt}\left((Q-H) \times \vec{Q} + \vec{L}_Q\right)\,,$$

and

$$\frac{d}{dt}\left((Q-H) \times \vec{Q}\right) = (\vec{v}_Q - \dot{\vec{x}}_H) \times \vec{Q} + (Q-H) \times \dot{\vec{Q}}$$

and

$$\begin{aligned}
\frac{d\vec{L}_Q}{dt} &= \frac{d}{dt}\left(\mathbb{S}_Q^T \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega}\right) = \\
&= \frac{d}{dt}\left[\hat{E}_i^0\left(S_{ji}^0 v_{Q,j}^0 + I_{ij}^0 \omega_j^0\right)\right] = \\
&= \vec{\omega} \times \hat{E}_i^0\left(S_{ji}^0 v_{Q,j}^0 + I_{ij}^0 \omega_j^0\right) + \hat{E}_i^0\left(S_{ji}^0 \dot{v}_{Q,j}^0 + I_{ij}^0 \dot{\omega}_j^0\right) = \\
&= \vec{\omega} \times (\mathbb{S}_Q^T \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega}) + \left(\mathbb{S}_Q^T \cdot \frac{{}^0 d}{dt}\vec{v}_Q + \mathbb{I} \cdot \dot{\vec{\omega}}\right) = \\
&= \vec{\omega} \times (\mathbb{S}_Q^T \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega}) + \left(\mathbb{S}_Q^T \cdot \left(\dot{v}_Q - \vec{\omega} \times \vec{v}_Q\right) + \mathbb{I} \cdot \dot{\vec{\omega}}\right) \\
&= \left(\mathbb{S}_Q^T \cdot \left(\dot{v}_Q - \vec{\omega} \times \vec{v}_Q\right) + \mathbb{I} \cdot \dot{\vec{\omega}}\right) + \vec{\omega} \times (\mathbb{S}_Q^T \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega})
\end{aligned}$$

Dynamical quantities and time derivatives with G as reference point

$$\begin{cases} \vec{Q} = m\vec{v}_G \\ \vec{L}_G = \mathbb{I}_G \cdot \vec{\omega} \end{cases} \qquad , \qquad \begin{cases} \dot{\vec{Q}} = m\dot{\vec{v}}_G \\ \dot{\vec{L}}_G = \mathbb{I}_G \cdot \dot{\vec{\omega}} + \vec{\omega} \times \mathbb{I}_G \cdot \vec{\omega} \end{cases}$$

CHAPTER

FOUR

DYNAMICS

Dynamics provides the link between the motion of a body, described by *kinematics*, and the actions causing that motion.

Newton's principles of dynamics and the cardinal equations of dynamics are the physical laws that govern the motion of mechanical systems: *Newton's principles* agree with the experimental observations (for systems with negligible quantum and Einstein relativity effects) and are the starting point - the principles - of Newton's formulation of mechanics; from these principles, *equations of motion* of mechanical systems are derived. These physical laws are formulated in terms of certain physical quantities, such as momentum, angular momentum, or the kinetic energy of the system - already discussed in the section about *inertia*. These dynamic quantities have the property of being additive (by definition), and making it particularly easy to write and interpret a general form of the equations governing motion. In general, these equations relate the time derivatives of these dynamic quantities to the causes of their variation. In the absence of net causes, conservation principles hold.

4.1 Principles of Newtonian Mechanics

First Principle of Dynamics (Newton's First Law): inertia and Galileian invariance. A body (more precisely, the center of mass of a body) on which no net force acts remains in its state of rest or uniform rectilinear motion relative to an inertial reference frame.

todo discuss the role/definition of **inertial reference frames** in classical mechanics and **Galileian invariance** (equations of motions are invariant under Galileian transformations)

Second Principle of Dynamics (Newton's Second Law): momentum balance. Relative to an inertial reference frame, the change in momentum of a system is equal to the impulse of the external forces acting on it,

$$\Delta \vec{Q} = \vec{I}^e \ .$$

In the case of smooth motion, where the momentum can be represented as a continuous and differentiable function of time, the second principle of dynamics can be expressed in differential form,

$$\dot{\vec{Q}} = \vec{R}^e \ ,$$

where the resultant of the external forces, $\vec{R}^e = \frac{d\vec{I}^e}{dt}$, is the time derivative of the impulse.

Third Principle of Dynamics (Newton's Third Law): action-reaction. If a system i exerts a force $\vec{F}_{ji}$ on a system j, then system j exerts an "equal and opposite" force $\vec{F}_{ij}$ on system i, with equal magnitude and opposite direction,

$$\vec{F}_{ij} = -\vec{F}_{ji} \ .$$

4.1.1 Inertial reference frame

Force sensors in an inertial reference frame measure only "true forces". But what's a "true force"? In classical mechanics, true forces are due to gravitational interactions and electromagnetic interactions. Electromangetic interactions between charged systems; electromangetic interactions between electrically neutral systems as contact forces…

4.2 Equations of Motion and Conservation Principles

Starting from the *principles of Newtonian mechanics*, it is possible to derive the dynamical equations governing the motion of mechanical systems. These equations governs the change of dynamical quantities, **momentum**, **angular momentum**, **kinetic energy**, linking them to (external) **forces**, (external) **moments** and (total) **power**. Under certain conditions, and only in these cases, the *cardinal equations* of dynamics become*principles of conservation of dynamic quantities*: by observing the expressions of the cardinal equations, it is easy to infer that the condition to obtain a conservation principle is the vanishing of all terms except for the time derivative of the conserved quantity.

4.2.1 Equations of Motion

The general form of these equations is easily expressed in terms of the dynamical quantities discussed in the section about inertia. Cardinal equations, or equations of motion, are collected here in their most general form for closed systems, and derived in the following sectinos for different systems: *point mass*, *systems of point masses*, *rigid body*,…

Momentum Balance. The time derivative of the momentum is equal to the resultant of the external forces,

$$\dot{\vec{Q}} = \vec{R}^e \ . \tag{4.1}$$

Angular Momentum Balance with respect to a point H. The time derivative of the angular momentum with respect to a point H, minus the "transport term," is equal to the resultant of the external moments with respect to the point H,

$$\dot{\vec{L}}_H + \dot{\vec{x}}_H \times \vec{Q} = \vec{M}^e_H \ . \tag{4.2}$$

Kinetic Energy Balance. The time derivative of the kinetic energy is equal to the total power acting on the system, which is the sum of the power of the external actions and the power of the internal actions within the system,

$$\dot{K} = P^{tot} = P^e + P^i \tag{4.3}$$

4.2.2 Conservation Principles

Under certain condiitons, equations of motion become principles of conservation of dynamical quantities. These conditions are easily derived by inspection of the equations of motion, nullyfing the causes of change of the dynamical quantities. Beside the conservation of momentum, angular momentum and kinetic energy, a **principle of conservation of mechanical energy** arises when actions acting on the system are *conservative*, so that its power can be written as a time derivative of a potential energy.

Conservation of Momentum in the presence of zero net external forces. If the resultant of the external forces is zero, $\vec{R}^e = \vec{0}$, from the momentum balance, we immediately obtain

$$\dot{\vec{Q}} = \vec{0} \qquad \rightarrow \qquad \vec{Q} = \bar{\vec{Q}} = \text{const.}$$

Conservation of Angular Momentum in the presence of zero net external moments. If the choice of the point H nullifies the transport term, $\dot{\vec{r}}_H \times \vec{Q} = \vec{0}$, and if the resultant of the external moments is zero, $\vec{M}^e_H = \vec{0}$, from the angular momentum balance, we immediately obtain

$$\dot{\vec{L}}_H = \vec{0} \qquad \rightarrow \qquad \vec{L}_H = \bar{\vec{L}}_H = \text{const.}$$

Conservation of Kinetic Energy in the presence of zero total power. If the total power of the actions on the system is zero, $P^{tot} = 0$, from the kinetic energy balance, we immediately obtain

$$\dot{K} = 0 \qquad \rightarrow \qquad K = \bar{K} = \text{const.}$$

Conservation of Mechanical Energy in the absence of non-conservative forces. In addition to the three conservation principles directly derived from the cardinal equations, we add the principle of the conservation of mechanical energy, which is the sum of the system's kinetic and potential energy,

$$E^{mech} = K + V \ ,$$

in the absence of non-conservative actions. If there are no non-conservative forces, the power of the actions on the system can be written as the negative of the time derivative of the system's potential energy,

$$P^{tot} = -\dot{V}$$

From the kinetic energy balance, we get

$$\dot{K} = -\dot{V} \qquad \rightarrow \qquad \frac{d}{dt}(K + V) = 0 \qquad \rightarrow \qquad \dot{E}^{mech} = 0 \qquad \rightarrow \qquad E^{mech} = \bar{E}^{mech} = \text{const.}$$

4.3 Equations of motion of a point mass

Dynamic quantities.

$$\begin{aligned}
\vec{Q}_P &:= m_P \vec{v}_P \\
\vec{L}_{P,H} &:= (\vec{r}_P - \vec{r}_H) \times \vec{Q} = m_P (\vec{r}_P - \vec{r}_H) \times \vec{v}_P \\
K &:= \frac{1}{2} m_P \vec{v}_P \cdot \vec{v}_P = \frac{1}{2} m_P |\vec{v}_P|^2
\end{aligned}$$

Momentum balance equation. The balance equation of momentum of a point P with mass m, $\vec{Q}_P = m\vec{v}_P$ readily follows the second principle of dynamics,

$$\dot{\vec{Q}}_P = \vec{R}^e_P$$

Angular momentum balance equation. Time derivative of the angular momentum is evaluated with the rule of derivative of product,

$$\begin{aligned}
\dot{\vec{L}}_{P,H} &= \frac{d}{dt} \left[m_P (\vec{r}_P - \vec{r}_H) \times \vec{v}_P \right] = \\
&= m \left[(\dot{\vec{r}}_P - \dot{\vec{r}}_H) \times \vec{v}_P + m_P (\vec{r}_P - \vec{r}_H) \times \dot{\vec{v}}_P \right] = \\
&= -m_P \dot{\vec{r}}_H \times \vec{v}_P + m_P (\vec{r}_P - \vec{r}_H) \times \dot{\vec{v}}_P = \\
&= -\dot{\vec{r}}_H \times \vec{Q} + \vec{M}^{ext}_H \ .
\end{aligned}$$

Kinetic energy blanace equation.

$$\begin{aligned}
\dot{K}_P &= \frac{d}{dt} \left(\frac{1}{2} m_P \vec{v}_P \cdot \vec{v}_P \right) = \\
&= m_P \dot{\vec{v}}_P \cdot \vec{v}_P = \\
&= \vec{R}^e \cdot \vec{v}_P = P^e = P^{tot}
\end{aligned}$$

being the power of external actions P^e equal to the total power acting on the system, assuming there is no internal action in the point system, or at least they have zero net power.

4.4 Equations of motion of a discrete system of point masses

Starting from the dynamic equations for a single point, the dynamic equations for a system of particles can be derived using the third principle of dynamics, action/reaction. The development of these equations helps us understand that the additive nature of dynamical quantities (momentum, angular momentum, kinetic energy) directly follows from their definition.

Momentum Balance. The momentum balance for each point i in the system can be written by expressing the resultant of the external forces acting on the point as the sum of the external forces acting on the entire system and the internal forces exchanged with the other points of the system,

$$\vec{R}_i^{ext,i} = \vec{F}_i^{ext} + \sum_{j \neq i} \vec{F}_{ij} \ .$$

The momentum balance equation for the i-th mass thus becomes

$$\dot{\vec{Q}}_i = \vec{R}_i^{ext,i} = \vec{F}_i^{ext} + \sum_{j \neq i} \vec{F}_{ij} \ .$$

By summing the momentum balance equations for all masses, we obtain

$$\begin{aligned} \sum_i \dot{\vec{Q}}_i & = \sum_i \vec{F}_i^{ext} + \sum_i \sum_{j \neq i} \vec{F}_{ij} = \\ & = \sum_i \vec{F}_i^{ext} + \sum_{\{i,j\}} \underbrace{\left(\vec{F}_{ij} + \vec{F}_{ji} \right)}_{=\vec{0}} \end{aligned}$$

Defining the momentum of the system as the sum of the momenta of its parts, and the resultant of the external forces as the sum of the external forces acting on the parts of the system,

$$\begin{aligned} \vec{Q} & := \sum_i \vec{Q}_i \\ \vec{R}^e & := \sum_i \vec{F}_i^{ext} \end{aligned}$$

we recover the general form of the momentum balance equation,

$$\dot{\vec{Q}} = \vec{R}^e \ .$$

Angular Momentum Balance. The angular momentum balance for each point i in the system can be written by expressing the resultant of the external moments acting on the point as the sum of the external moments acting on the entire system and the internal moments exchanged with the other points of the system,

$$\vec{M}_{H,i}^{ext,i} = \vec{M}_{H,i}^{ext} + \sum_{j \neq i} \vec{M}_{H,ij} \ .$$

In the case where parts of the system interact via forces, the moment with respect to a point H generated by mass j on mass i is given by

$$\vec{M}_{H,ij} = (\vec{r}_i - \vec{r}_H) \times \vec{F}_{ij} \ .$$

The angular momentum balance equation for the i-th mass thus becomes

$$\dot{\vec{L}}_{H,i} + \dot{\vec{r}}_H \times \vec{Q}_i = \vec{M}_{H,i}^{ext,i} = \vec{M}_{H,i}^{ext} + \sum_{j \neq i} \vec{M}_{H,ij} \ .$$

By summing the angular momentum balance equations for all masses, we obtain

$$\begin{aligned} \sum_i \left(\dot{\vec{L}}_i + \dot{\vec{r}}_H \times \vec{Q}_i \right) & = \sum_i \vec{M}_{H,i}^{ext} + \sum_i \sum_{j \neq i} \vec{M}_{H,ij} = \\ & = \sum_i \vec{M}_{H,i}^{ext} + \sum_{\{i,j\}} \underbrace{\left(\vec{M}_{H,ij} + \vec{M}_{H,ji} \right)}_{=\vec{0}} \end{aligned}$$

Recognizing the total momentum of the system, and *defining the angular momentum of the system as the sum of the angular momentum of its parts*, and the resultant of the external moments as the sum of the external moments acting on the parts of the system,

$$\vec{L}_H := \sum_i \vec{L}_{H,i}$$

$$\vec{M}^e_H := \sum_i \vec{M}^{ext}_{H,i}$$

we recover the general form of the angular momentum balance equation,

$$\dot{\vec{L}}_H + \dot{\vec{r}}_H \times \vec{Q} = \vec{M}^e_H \ .$$

Kinetic Energy Balance. The kinetic energy balance of the system can be derived by taking the scalar product of the momentum balance equation for each point,

$$\vec{v}_i \cdot m_i \dot{\vec{v}}_i = \vec{v}_i \cdot \left(\vec{F}^e_i + \sum_{j \neq i} \vec{F}_{ij} \right) \ ,$$

recognizing in the first term the time derivative of the kinetic energy of the i-th point,

$$\dot{K}_i = \frac{d}{dt}\left(\frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i\right) = m_i \vec{v}_i \cdot \dot{\vec{v}}_i \ ,$$

and summing these equations to obtain

$$\sum_i \dot{K}_i = \sum_i \vec{v}_i \cdot \vec{F}^e_i + \sum_i \vec{v}_i \cdot \sum_{j \neq i} \vec{F}_{ij} \ .$$

Defining the kinetic energy of the system as the sum of the kinetic energies of its parts, and defining the power of the external/internal forces acting on the system as the sum of the power of all external/internal forces in the system,

$$K := \sum_i K_i$$

$$P^e := \sum_i P^{ext}_i = \sum_i \vec{v}_i \cdot \vec{F}^e_i$$

$$P^i := \sum_i P^{int}_i = \sum_i \vec{v}_i \cdot \sum_{j \neq i} \vec{F}_{ij}$$

we recover the general form of the kinetic energy balance equation,

$$\dot{K} = P^e + P^i = P^{tot} \ .$$

Note: While internal forces and moments have zero net resultant in momentum and angular momentum balance, this is not true for power of internal actions in kinetic energy equation.

4.5 Equations of motion of a rigid body

With different choices of the reference point H, the general expression of dynamical equations may have different, but equivalent, forms.

4.5.1 General equations

Momentum balance equation.

$$\frac{d}{dt}\vec{Q} = \vec{R}^e$$

Angular momentum balance equation.

$$\frac{d}{dt}\vec{L}_H + \dot{\vec{x}}_H \times \vec{Q} = \vec{M}^e_H$$

Kinetic energy balance equation.

$$\frac{d}{dt}K = P^{tot}$$

4.5.2 Dynamical equations w.r.t. the center of mass G

Momentum, angular momentum and kinetic energy

$$\begin{cases} \vec{Q} = m\vec{v}_G \\ \vec{L}_G = \mathbb{I}_G \cdot \vec{\omega} \\ K = \frac{1}{2}m|\vec{v}_G|^2 + \frac{1}{2}\vec{\omega}\cdot\mathbb{I}_G\cdot\vec{\omega} \end{cases} \quad , \quad \begin{cases} \dot{\vec{Q}} = m\dot{\vec{v}}_G \\ \dot{\vec{L}}_G = \mathbb{I}_G\cdot\dot{\vec{\omega}} + \vec{\omega}\times\mathbb{I}_G\cdot\vec{\omega} \\ \dot{K} = m\vec{v}_G\cdot\dot{\vec{v}}_G + \vec{\omega}\cdot\mathbb{I}_G\cdot\dot{\vec{\omega}} \end{cases}$$

Equations of motion.

$$\begin{cases} \dot{\vec{Q}} = \vec{R}^e \\ \dot{\vec{L}}_G = \vec{M}^e_G \end{cases} \quad , \quad \begin{cases} m\dot{\vec{v}}_G = \vec{R}^e \\ \mathbb{I}_G\cdot\dot{\vec{\omega}} + \vec{\omega}\times\mathbb{I}_G\cdot\vec{\omega} = \vec{M}^e_G \end{cases}$$

Time derivative of kinetic energy

$$\begin{aligned} \frac{dK}{dt} &= \frac{1}{2}\frac{d}{dt}\left(\vec{v}_G\cdot\vec{Q} + \vec{\omega}_G\cdot\vec{L}_G\right) = \\ &= \frac{1}{2}\left(\dot{\vec{v}}_G\cdot m\vec{v}_G + \vec{v}_G\cdot m\dot{\vec{v}}_G + \dot{\vec{\omega}}\cdot\mathbb{I}_G\cdot\vec{\omega} + \vec{\omega}\cdot\left(\mathbb{I}\cdot\dot{\vec{\omega}} + \vec{\omega}\times\mathbb{I}_G\cdot\vec{\omega}\right)\right) \\ &= \vec{v}_G\cdot m\dot{\vec{v}}_G + \vec{\omega}\cdot\mathbb{I}_G\cdot\dot{\vec{\omega}} \\ &= \vec{v}_G\cdot\dot{\vec{Q}} + \vec{\omega}\cdot\dot{\vec{L}}_G \\ &= \dot{\vec{v}}_G\cdot\vec{Q} + \dot{\vec{\omega}}\cdot\vec{L}_G\,. \end{aligned}$$

4.5.3 Dynamical equations w.r.t. a material point Q

Momentum, angular momentum and kinetic energy

$$\begin{cases} \vec{Q} = m\vec{v}_Q + \mathbb{S}_Q\cdot\vec{\omega} \\ \vec{L}_Q = \mathbb{S}^T_Q\cdot\vec{v}_Q + \mathbb{I}_Q\cdot\vec{\omega} \\ K = \dfrac{1}{2}\vec{v}_Q\cdot\vec{Q} + \dfrac{1}{2}\vec{\omega}\cdot\vec{L}_Q \end{cases} \quad , \quad \begin{cases} \dot{\vec{Q}} = m\dot{\vec{v}}_Q + \mathbb{S}_Q\cdot\dot{\vec{\omega}} + \vec{\omega}\times\mathbb{S}_Q\cdot\vec{\omega} \\ \dot{\vec{L}}_Q = \left[\mathbb{S}^T_Q\cdot\left(\dot{v}_Q - \vec{\omega}\times\vec{v}_Q\right) + \mathbb{I}\cdot\dot{\vec{\omega}}\right] + \vec{\omega}\times\left(\mathbb{S}^T_Q\cdot\vec{v}_Q + \mathbb{I}_Q\cdot\vec{\omega}\right) \\ \dot{K} = \dots \end{cases}$$

Equations of motion.

$$\begin{cases} \dot{\vec{Q}} = \vec{R}^e \\ \dot{\vec{L}}_Q + \vec{v}_Q\times\vec{Q} = \vec{M}^e_Q \end{cases}$$

$$\begin{cases} m\dot{\vec{v}}_Q + \mathbb{S}_Q \cdot \dot{\vec{\omega}} + \vec{\omega} \times \mathbb{S}_Q \cdot \vec{\omega} = \vec{R}^e \\ \left[\mathbb{S}_Q^T \cdot (\dot{v}_Q - \vec{\omega} \times \vec{v}_Q) + \mathbb{I} \cdot \dot{\vec{\omega}}\right] + \vec{\omega} \times (\mathbb{S}_Q^T \cdot \vec{v}_Q + \mathbb{I}_Q \cdot \vec{\omega}) + \vec{v}_Q \times \left[m\vec{v}_Q + \mathbb{S}_Q \cdot \vec{\omega}\right] = \vec{M}_Q^e \end{cases}$$

or using the "material time derivative"

$$\frac{^0 d}{dt}\_\_ = \frac{d}{dt}\_\_ - \vec{\omega} \times \_\_ \,,$$

and matrix formalism to write these two vector equations

$$\begin{bmatrix} m\mathbb{I} & \mathbb{S}_Q \\ \mathbb{S}_Q^T & \mathbb{I}_Q \end{bmatrix} \frac{^0 d}{dt} \begin{bmatrix} \vec{v}_Q \\ \vec{\omega} \end{bmatrix} + \begin{bmatrix} \vec{\omega}_\times & \vec{0} \\ \vec{v}_{Q\times} & \vec{\omega}_\times \end{bmatrix} \begin{bmatrix} m\mathbb{I} & \mathbb{S}_Q \\ \mathbb{S}_Q^T & \mathbb{I}_Q \end{bmatrix} \begin{bmatrix} \vec{v}_Q \\ \vec{\omega} \end{bmatrix} = \begin{bmatrix} \vec{R}^e \\ \vec{M}_Q^e \end{bmatrix} \,.$$

Starting from differential equations

todo

4.6 Equations of motion for continuous media

4.7 Particular Motions

In this section, we will study certain particular motions that are interesting and useful to analyze for educational, historical, and practical reasons.

- Uniform rectilinear motion
- Uniformly accelerated motion
- Uniform circular motion
- Oscillatory and damped oscillatory motions:
  - Free oscillations:
    - Mass-spring(-damper) system
    - Pendulum
  - Forced oscillations:
    - A first step towards structural analysis and beyond ("every physical system is a system of many harmonic oscillators")
    - Concepts of frequency response and resonance. **todo** video and/or script on frequency response of structures and seismic structures, mass-damper,…
- **Gravitation**: Starting from Newton's universal law of gravitation, we study the motion of celestial bodies in two-body systems, discovering that their trajectories describe conic sections (circle, ellipse, parabola, hyperbola), and demonstrating Kepler's laws.
- Rotation of a body around a fixed point, Poinsot's motions

4.8 Equilibrium and Stability

A system is in equilibrium is all its components are in equilibrium, and thus there exists a reference frame w.r.t. which the momentum and the angular momentum of all its components are equal to zero.

4.8.1 Eigenvalue stability

…

$$\delta\vec{Q} = m\delta\vec{v}_G$$

4.8.2

Part II

Analytical Mechanics

CHAPTER

FIVE

LAGRANGIAN MECHANICS

Classical mechanics can be re-formulated starting principles of calculus of variations, usually referred as **analytical mechanics**. Under some assumptions, that will be discussed during the derivation, analytical mechanics is equivalent to Newton mechanics.

Symmetric system. Lagrange mechanics provides a symmetric form of the (linearised?) governing equations, without any additional effort. This could be quite useful, especially for exploiting numerical methods for symmetric (and definite positive, sometimes) matrices.

Here, the equivalence of analytical mechanics and Newton mechanics is stressed, by means of the derivation of the principle of analytical mechanics starting from the equations of motions derived in Newtonian mechanics, relying on the conservation of mass and the three principles of Newton mechanics. The process is shown in the following sections for *point systems*, *systems of points*, *extended rigid bodies* and follows these steps:

- **strong form of equations.** Starting point is the dynamical equations of Newton mechanics, here also referred as the strong form of equations
- **weak form of equations.** Strong form are recast in weak form, also referred as **D'Alembert approach** or **virtual work formulation**, multiplying strong form of equations for arbitrary test functions
- **Lagrange equations.** A proper choice of test functions as a function of generalized coordinates, and some manipulation, leads to Lagrange equations. While the choice of test functions depends on the nature of the system, their expression always reads

  $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right) - \frac{\partial \mathcal{L}}{\partial q^k} = Q_{q^k} \ , \tag{5.1}$$

  being $q^k(t)$ the generalized coordinates, $\mathcal{L}\left(\dot{q}^k(t), q?k(t), t\right) = K\left(\dot{q}^k(t), q^k(t), t\right) + U(q^k(t), t)$ the Lagrangian function of the system, defined as the sum of the kinetic energy K and the potential function $U = -V$, being V the potential energy - s.t. the conservative vector field reads $\vec{F} = -\nabla V$, and Q_q the generalized force.

- Lagrange equations can be interpreted as a result of a stationary principle of a functional, S, defined **action functional**, as it can be shown with the tools of calculus of variations.
  - **If** $Q_{q^k} = 0$, multiplying by $w^k(t)$, integrating over time from t_0, t_1, and assuming that $w(t_0) = w(t_1) = 0$,

$$\begin{aligned}
0 &= \int_{t_0}^{t_1} w^k(t) \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right) - \frac{\partial \mathcal{L}}{\partial q}\right] dt = \\
&= w^k(t) \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right)\Bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left[\dot{w}^k(t)\,\frac{\partial \mathcal{L}}{\partial \dot{q}^k} + w^k(t)\,\frac{\partial \mathcal{L}}{\partial q^k}\right] dt \ .
\end{aligned}$$

If $w^k(t)$ is equal to zero for t equal to t_0 and t_1, first term vanishes

$$\begin{aligned}
0 &= -\int_{t_0}^{t_1} \left[\dot{w}^k(t) \, \frac{\partial \mathcal{L}}{\partial \dot{q}^k} + w^k(t) \, \frac{\partial \mathcal{L}}{\partial q^k} \right] \, dt \\
&= -\frac{1}{\varepsilon} \int_{t_0}^{t_1} \varepsilon \left[\dot{w}^k(t) \, \frac{\partial \mathcal{L}}{\partial \dot{q}^k} \left(\dot{q}^l(t), q^l(t), t \right) + w^k(t) \, \frac{\partial \mathcal{L}}{\partial q^k} \left(\dot{q}^l(t), q^l(t), t \right) \right] \, dt = \\
&= -\lim_{\varepsilon \rightarrow 0} \left\{ \frac{1}{\varepsilon} \int_{t_0}^{t_1} \varepsilon \left[\dot{w}^k(t) \, \frac{\partial \mathcal{L}}{\partial \dot{q}^k} \left(\dot{q}^l(t), q^l(t), t \right) + w^k(t) \, \frac{\partial \mathcal{L}}{\partial q^k} \left(\dot{q}^l(t), q^l(t), t \right) \right] \, dt \right\} = \\
&= -\lim_{\varepsilon \rightarrow 0} \left\{ \frac{1}{\varepsilon} \int_{t_0}^{t_1} \left[\mathcal{L} \left(\dot{q}^l(t) + \varepsilon \dot{w}^l(t), q^l(t) + \varepsilon w^l(t), t \right) - \mathcal{L} \left(\dot{q}^l(t), q^l(t), t \right) \right] \, dt + o(\varepsilon) \right\} = \\
&= -\delta \int_{t_0}^{t_1} \mathcal{L}(\dot{q}^l(t), q^l(t), t) \, dt =: -\delta S[q^k(t)] \, ,
\end{aligned}$$

i.e. Lagrange equations are equivalent to the stationary condition of the action functional

$$S[q^k(t)] := \int_{t_0}^{t_1} \mathcal{L} \left(\dot{q}^l(t), q^l(t), t \right) \, dt \, .$$

– **If** $Q_k \neq 0$, the variational principle becomes

$$\begin{aligned}
0 &= \int_{t_0}^{t_1} w^k(t) \left[\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k} \right) - \frac{\partial \mathcal{L}}{\partial q^k} - Q_k \right] \, dt = \\
&= w^k(t) \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k} \right) \bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left[\dot{w}^k(t) \, \frac{\partial \mathcal{L}}{\partial \dot{q}^k} + w^k(t) \, \frac{\partial \mathcal{L}}{\partial q^k} \right] \, dt + \int_{t_0}^{t_1} w^k(t) Q_k \, dt \, . \\
&= \dots \\
&= -\delta \int_{t_0}^{t_1} \mathcal{L} \left(\dot{q}^l(t), q^l(t), t \right) \, dt + \int_{t_0}^{t_1} \delta q^k(t) Q_k \, dt \, ,
\end{aligned}$$

having written the arbitrary test function as $w^k(t) =: \delta q^k(t)$ to keep in mind that they're used as variations of functions $q^k(t)$.

The second contribution is usually defined **virtual work** of generalized forces Q_k - that is equal to the virtual work of actions not included in the potential $U\left(q^k(t), t\right)$. For the very nature of variation, it can be thought as the *infinitesimal work done by forces for small displacements compatible with constraints, keeping time constant*.

5.1 Lagrange Equations of the Second Kind

For ideal constraints…

5.1.1 Point

Newton dynamical equations - strong form. Dynamical equation governing the motion of a point P reads

$$m \dot{\vec{v}}_P = \vec{R}^e \, ,$$

being m the mass of the system, $\vec{v}_P$ the velocity of point P, $\vec{a}_P = \dot{\vec{v}}_P$ its acceleration and $\vec{R}^e$ the net external force acting on the system..

Weak form. Weak form of dynamical equations is derived with scalar multiplication of the strong form by an arbitrary test function $\vec{w}$,

$$\vec{0} = \vec{w} \cdot \left(m\dot{\vec{v}} - \vec{R}^e \right) \qquad \forall \vec{w} \tag{5.2}$$

Lagrange equations. Lagrange equations are derived from a proper choice of the test function. The position of the point P is written as a function of the generalized coordinates $q^k(t)$ and time t

$$\vec{r}_P(t) = \vec{r}(q^k(t), t) \ ,$$

so that its velocity can be written as

$$\vec{v}_P(t) := \frac{d\vec{r}_P}{dt} = \dot{q}^k(t) \underbrace{\frac{\partial \vec{r}}{\partial q^k}}_{\frac{\partial \vec{v}}{\partial \dot{q}^k}}(q^l(t), t) + \frac{\partial \vec{r}}{\partial t}(q^l(t), t) = \vec{v}\left(\dot{q}^k(t), q^k(t), t\right) \ ,$$

from which the relation between partial derivatives

$$\frac{\partial \vec{r}}{\partial q^k} = \frac{\partial \vec{v}}{\partial \dot{q}^k} \ . \tag{5.3}$$

follows. Choosing the test function $\vec{w}$ as

$$\vec{w} = \frac{\partial \vec{r}}{\partial q^k} = \frac{\partial \vec{v}}{\partial \dot{q}^k} \ ,$$

applying the rule of derivative of product, using Schwartz theorem to switch order of derivation, and exploiting relation (5.3) it's possible to recast weak form (5.2) as

$$\begin{aligned}
\vec{0} = \frac{\partial \vec{v}}{\partial \dot{q}^k} \cdot \left(m\dot{\vec{v}} - \vec{R}^e \right) = & \\
= \frac{d}{dt}\left(\frac{\partial \vec{v}}{\partial \dot{q}^k} \cdot m\vec{v} \right) - \frac{d}{dt}\frac{\partial \vec{r}}{\partial q^k} \cdot m\vec{v} - \frac{\partial \vec{r}}{\partial q^k} \cdot (\vec{R}^{e,c} + \vec{R}^{e,nc}) & \\
= \frac{d}{dt}\left(\frac{\partial \vec{v}}{\partial \dot{q}^k} \cdot m\vec{v} \right) - \frac{\partial \vec{v}}{\partial q^k} \cdot m\vec{v} - \frac{\partial \vec{r}}{\partial q^k} \cdot (\nabla U + \vec{R}^{e,nc}) & \\
= \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}^k} \right) - \frac{\partial K}{\partial q^k} - \frac{\partial U}{\partial q^k} - \underbrace{\frac{\partial \vec{r}}{\partial q^k} \cdot \vec{R}^{e,nc}}_{=:Q^k} \ . &
\end{aligned}$$

Introducing the **Lagrangian function**

$$\mathcal{L}(\dot{q}^k(t), q^k(t), t) := K(\dot{q}^k(t), q^k(t), t) + U(q^k(t), t) \ ,$$

and recalling that potential function U is not a function of velocity and thus of time derivatives of the generalized coordinates $\dot{q}^k$, it's possible to recast the dynamical equation as the **Lagrange equations**

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k} \right) - \frac{\partial \mathcal{L}}{\partial q^k} = Q^k \ ,$$

being Q^k the **generalized force** not included in the gradient of the potential ∇U - usually a non conservative contribution -, $Q^k = \dfrac{\partial \vec{r}}{\partial q^k} \cdot \vec{R}^{e,nc}$.

5.1.2 System of points

Newton dynamical equations - strong form.

Weak form.

Lagrange equations.

5.1.3 Rigid Body

Newton dynamical equations - strong form. Dynamical equations governing the motion of a rigid body, referred to its center of mass G read

$$\begin{cases} \dot{\vec{Q}} = \vec{R}^e \\ \dot{\vec{\Gamma}}_G = \vec{M}^e_G \ , \end{cases}$$

with momentum $\vec{Q} = m\vec{v}_G$ and angular momentum $\vec{\Gamma}_G = \mathbb{I}_G \cdot \vec{\omega}$.

Weak form. Weak form of dynamical equations is derived with scalar multiplication of the strong form by an arbitrary test functions $\vec{w}_t$, $\vec{w}_r$

$$\vec{0} = \vec{w}_t \cdot \left(m\dot{\vec{v}}_G - \vec{R}^e \right) + \vec{w}_r \cdot \left(\dot{\vec{\Gamma}}_G - \vec{M}^e_G \right) \qquad \forall \vec{w}_t, \, \vec{w}_r \tag{5.4}$$

Lagrange equations. Lagrange equations are derived from the weak form, with a proper choice of the weak test functions. The “translational part” is recasted after choosing

$$\vec{w}_t = \frac{\partial \vec{r}}{\partial q^k} = \frac{\partial \vec{v}}{\partial \dot{q}^k} \ .$$

Following the same steps show to derive *Lagrange equations for a point system*, the translational part becomes

$$\frac{d}{dt} \frac{\partial K^{tr}}{\partial \dot{q}^k} - \frac{\partial K^{tr}}{\partial q^k} - \frac{\partial U^{tr}}{\partial q^k} = Q^{tr}_k \ ,$$

being $K^{tr} = \frac{1}{2} m |\vec{v}_G|^2$ the contribution to kinetic energy of the velocity of the center of mass G deriving from the momentum equation, U^{tr} the contribution to the potential energy U from the momentum equation, and Q^{tr}_k the contribution to the generalized force from the momentum equation.

The “rotational part” is recasted after choosing

$$\vec{w}_r = \frac{\partial \vec{\theta}}{\partial q^k} = \frac{\partial \vec{\omega}}{\partial \dot{q}^k}$$

Angular velocity $\vec{\omega}$ can be written w.r.t the inertial $\{\hat{e}_i\}$ or the material reference frame $\{\hat{E}_i\}$,

$$\vec{\omega} = \omega_i \hat{e}_i = \sigma_j \hat{E}_j \ ,$$

and the inertia tensor as

$$\mathbb{I}_G = I_{ij} \, \hat{E}_i \otimes \hat{E}_j \ ,$$

being the components I_{ij} constant.

$$0 = \frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \frac{d}{dt} \left(\mathbb{I}_G \cdot \vec{\omega} \right) - \frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \vec{M}^e_G = \frac{d}{dt} \left(\frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \mathbb{I}_G \cdot \vec{\omega} \right) - \frac{d}{dt} \frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \mathbb{I}_G \cdot \vec{\omega} - \frac{\partial \vec{\theta}}{\partial q^k} \cdot \vec{M}^e_G$$

The first term becomes

$$\frac{d}{dt}\left(\frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \mathbb{I}_G \cdot \vec{\omega}\right) = \frac{d}{dt}\left(\frac{\partial \sigma_a}{\partial \dot{q}^k} I_{ab}\sigma_b\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{q}^k}\left(\frac{1}{2}\vec{\omega} \cdot \mathbb{I}_G \cdot \vec{\omega}\right) = \frac{d}{dt}\frac{\partial K^{rot}}{\partial \dot{q}^k}$$

The second term becomes

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \vec{\theta}}{\partial q^k} \cdot \mathbb{I}_G \cdot \vec{\omega} = \frac{\partial}{\partial q^k}\underbrace{\frac{d\vec{\theta}}{dt}}_{\vec{\omega}} \cdot \mathbb{I}_G \cdot \vec{\omega} = & \\
= \frac{\partial \vec{\omega}}{\partial q^k} \cdot \mathbb{I}_G \cdot \vec{\omega} = & \\
= \frac{\partial}{\partial q^k}\left(\sigma_a \hat{E}_a\right) \cdot \hat{E}_b \, I_{bc}\sigma_c = & \\
= \frac{\partial \sigma_a}{\partial q^k}\underbrace{\hat{E}_a \cdot \hat{E}_b}_{=\delta_{ab}} \, I_{bc}\sigma_c + \sigma_a \underbrace{\frac{\partial \hat{E}_a}{\partial q^k} \cdot \hat{E}_b}_{=0} \, I_{bc}\sigma_c = & \\
= \frac{\partial}{\partial q^k}\left(\frac{1}{2}\sigma_a \, I_{ab}\sigma_b\right) = & \\
= \frac{\partial}{\partial q^k}\left(\frac{1}{2}\vec{\omega} \cdot \mathbb{I}_G \cdot \vec{\omega}\right) = \frac{\partial K^{rot}}{\partial q^k} \ . &
\end{aligned}$$

The third term can be written as the sum of the derivative of a potential function and a generalized force,

$$\frac{\partial \vec{\theta}}{\partial q^k} \cdot \vec{M}^e_G = \frac{\partial U^{rot}}{\partial q^k} + Q^{rot}_{q^k}$$

The rotational part of the wak form becomes

$$\frac{d}{dt}\frac{\partial K^{rot}}{\partial \dot{q}^k} - \frac{\partial K^{rot}}{\partial q^k} - \frac{\partial U^{rot}}{\partial q^k} = Q^{rot}_{q^k} \ ,$$

being $K^{rot} = \frac{1}{2}\vec{\omega} \cdot \mathbb{I}_G \cdot \vec{\omega}$ the contribution to kinetic energy of the rotation aroung the center of mass G deriving from the angular momentum equation, U^{rot} the contribution to the potential energy U from the angular momentum equation, and Q^{rot}_k the contribution to the generalized force from the angular momentum equation.

Adding together the contributions of the momentum and the angular momentum equations, the Lagrange equation can be formally written with the same expression found for the system of points,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right) - \frac{\partial \mathcal{L}}{\partial q^k} = Q_{q^k} \ ,$$

being $\mathcal{L} = K + U$ the Lagrangian function of the system, and $K = K^{tr} + K^{rot}$, $U = U^{tr} + U^{rot}$, $Q_{q^k} = Q^{tr}_{q^k} + Q^{rot}_{q^k}$ the kinetic energy the potential function and the generalized force of the system, defined as the sum of the contributions coming from the momentum and the angular momentum equations.

5.2 Lagrange Equations of the First Kind

Explicitly making appear constraint forces, due to constraints

$$\begin{aligned}
& \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = Q^e_k + Q^c_k \\
& g^j\left(q^k(t), t\right) = 0
\end{aligned}$$

Example 5.2.1

Pendulum with point mass m and length ℓ, with hinge position $x_H(t)$ w.r.t. an inertial reference frame, in a gravitational field $\vec{g} = g\hat{y}$

Position, and velocity of the point mass in P

$$\begin{aligned}
\vec{r}_P(t) &= x_P(t)\,\hat{x} + y_P(t)\,\hat{y} = (x_H(t) + \ell\sin\theta(t))\;\hat{x} + \ell\cos\theta(t)\,\hat{y} \\
\vec{v}_P(t) &= \dot{x}_P(t)\,\hat{x} + \dot{y}_P(t)\,\hat{y} = (\dot{x}_H + \ell\dot{\theta}(t)\cos\theta(t))\,\hat{x} - \ell\dot{\theta}(t)\sin\theta(t)\,\hat{y}
\end{aligned}$$

Approach 1. LE of the II Kind. LE of the II Kind provides free equations of motion. The system has one degree of freedom. Here the angle $\theta(t)$ is chosen as the generalized dof. Kinetic energy K and potential function U,

$$\begin{aligned}
K &= \frac{1}{2}m\left(\dot{x}_H^2 + 2\ell\dot{x}_H\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2\right) \\
U &= mg\ell\cos\theta
\end{aligned}$$

and Lagrange equation of the II-kind provides a free equation of motion, that immediately follows from direct evaluation of the required derivatives

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} &= \frac{d}{dt}\left[m(\ell\dot{x}_H\cos\theta + \ell^2\dot{\theta})\right] = m\ell\ddot{x}_H\cos\theta - m\ell\dot{x}_H\dot{\theta}\sin\theta + m\ell^2\ddot{\theta} \\
\frac{\partial L}{\partial\theta} &= -m\ell\dot{x}_H\dot{\theta}\sin\theta - mg\ell\sin\theta
\end{aligned}$$

Thus, Lagrange equation reads

$$0 = \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = m\ell^2\ddot{\theta} + mg\ell\sin\theta - m\ell\ddot{x}_H(t)\cos\theta$$

Approach 2. LE of the I Kind.

5.3 Lagrangian functions and time dependence

Some problems may have a Lagrangian function with an explicit dependence on time,

$$\mathcal{L}(\dot{q}^k(t), q^k(t), t)\ .$$

Using the general form (5.1) of Lagrange equations, the time derivative of the Lagrange function reads

$$\begin{aligned}
\frac{d\mathcal{L}}{dt} &= \ddot{q}^k\frac{\partial\mathcal{L}}{\partial\dot{q}^k} + \dot{q}^k\frac{\partial\mathcal{L}}{\partial q^k} + \frac{\partial\mathcal{L}}{\partial t} = && \text{(IxP)} \\
&= \frac{d}{dt}\left(\dot{q}^k\frac{\partial\mathcal{L}}{\partial\dot{q}^k}\right) - \dot{q}^k\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}^k} + \dot{q}^k\frac{\partial\mathcal{L}}{\partial q^k} + \frac{\partial\mathcal{L}}{\partial t} = && \text{(Lagrange eq.)} \\
&= \frac{d}{dt}\left(\dot{q}^k\frac{\partial\mathcal{L}}{\partial\dot{q}^k}\right) - \dot{q}^k Q_k + \frac{\partial\mathcal{L}}{\partial t}\ .
\end{aligned}$$

This latter relation can be recast as

$$\frac{d}{dt}\left[\dot{q}^k\frac{\partial\mathcal{L}}{\partial\dot{q}^k} - \mathcal{L}\right] = \dot{q}^k Q_k - \frac{\partial\mathcal{L}}{\partial t}\ , \tag{5.5}$$

i.e. time derivative of a physical quantity equals the power of actions not included in the potential, $\dot{q}^k Q_k$ and a contribution of partial derivative of the Lagrangian function, $\partial_t\mathcal{L}$.

As it's discusseed below, **when the Lagrangian function of the system is not an explicit function of time todo** *discuss the cases when $\partial_t\mathcal{L} \neq 0$, the equation (5.5) is nothing but the balance equation of mechanical energy.

5.3.1 Lagrangian function with no explicit dependence on time

Let's analyse first some properties of systems, whose Lagrangian function are not an explicit function of time,

$$\mathcal{L}(\dot{q}^k(t), q^k(t)) = K(\dot{q}^k(t), q^k(t)) + U(q^k(t)) \ ,$$

and then go back to the most general case.

As the Lagrange equation is not an explicit function of time, Euler-Beltrami equations hold,

$$\mathcal{L} - \dot{q}^k \frac{\partial \mathcal{L}}{\partial \dot{q}^k} = C \quad \text{const.}$$

Since the Lagrangian doesn't expliclty depend on time, and potential is not a function of time, relation (5.6) gives $\dfrac{\partial \mathcal{L}}{\partial \dot{q}^k} = 2T$, and thus

$$C = \mathcal{L} - \dot{q}^k \frac{\partial \mathcal{L}}{\partial \dot{q}^k} = T + U - 2T = -T + U =: -E^{mec}$$

Properties of kinetic energy and potential

This section collects some properties of the kinetic energy and potential of systems, where physical coordinates of the system are written as a function of generalized coordinates only, $q^k(t)$. As an example, coordinates of point masses, material points of rigid bodies and the rotation tensor representing their orientation in space can be written as

$$\vec{r}_P\left(q^k(t)\right) \quad , \quad \mathbb{R}\left(q^k(t)\right) \ ,$$

so that velocities and angular velocities become

$$\begin{aligned}
\vec{v}_P\left(\dot{q}^k(t), q^k(t)\right) &= \frac{d\vec{r}_P}{dt} = \dot{q}^k \frac{\partial \vec{r}}{\partial q^k}\left(q^k(t)\right) \\
\vec{\omega}_\times\left(\dot{q}^k(t), q^k(t)\right) &= \frac{d\mathbb{R}}{dt} \cdot \mathbb{R}^T = \dot{q}^k(t)\, \frac{\partial \mathbb{R}}{\partial q^k} \cdot \mathbb{R}^T = \dot{q}^k(t)\, \frac{\partial \vec{\theta}_\times}{\partial q^k}\left(q^k(t)\right) \ ,
\end{aligned}$$

As the kinetic energy of a mechanical system is a quadratic function of velocity and angular velocity of its sub-systems, the kinetice energy can be written as

$$K\left(\dot{q}^k(t), q^k(t)\right) = \frac{1}{2} A_{ij}\left(q^k(t)\right)\, \dot{q}^i(t)\, \dot{q}^j(t) \ .$$

Since A_{ij} is symmetric w.r.t. the swap of indices (or it can be written in a symmetric form), partial derivative of the kinetic energy w.r.t. $\dot{q}^l$ reads

$$\frac{\partial K}{\partial \dot{q}^l} = A_{lj} \dot{q}^j \ ,$$

and

$$\dot{q}^l \frac{\partial K}{\partial \dot{q}^l} = \dot{q}^l A_{lj} \dot{q}^j = 2T \ . \tag{5.6}$$

Proofs

$$\frac{\partial K}{\partial \dot{q}^l} = \frac{\partial}{\partial \dot{q}^l}\left[\frac{1}{2}A_{ij}\dot{q}^i\dot{q}^j\right] = \frac{1}{2}A_{ij}\left[\underbrace{\frac{\partial \dot{q}^i}{\partial \dot{q}^l}}_{\delta^i_l}\dot{q}^j + \dot{q}^i\underbrace{\frac{\partial \dot{q}^j}{\partial \dot{q}^l}}_{\delta^j_l}\right] = \frac{1}{2}\left[A_{lj}\dot{q}^j + A_{il}\dot{q}^i\right] = A_{lj}\dot{q}^j$$

5.3.2 Lagrangian function with explicit dependence on time

CHAPTER

SIX

HAMILTONIAN MECHANICS

Riformulazione ulteriore della meccanica di Newton, a partire dalla meccanica di Lagrange. Fornisce le basi per un approccio moderno anche in altre teorie della Fisica. **dots…**

Starting from Lagrange equations derived in *Lagrangian mechanics*,

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}}\Big) - \frac{\partial \mathcal{L}}{\partial q} = Q_q$$

the **generalized moment** is defined as

$$p_k := \frac{\partial \mathcal{L}}{\partial \dot{q}^k} \ ,$$

and the **Hamiltonian function** as

$$\mathcal{H}(q^k(t), p_k(t), t) := p_k \dot{q}^k - \mathcal{L}(\dot{q}^l(q^k, p_k, t), q^l(t), t) \ ,$$

its differential reads

$$\begin{aligned} d\mathcal{H} &= dq^k \, \frac{\partial \mathcal{H}}{\partial q^k} + dp_k \, \frac{\partial \mathcal{H}}{\partial p_k} + dt \, \frac{\partial \mathcal{H}}{\partial t} = \\ &= dp_k \, \dot{q}^k + \underbrace{p_k \, d\dot{q}^k - d\dot{q}^k \, \frac{\partial \mathcal{L}}{\partial \dot{q}^k}}_{=0} - dq^k \, \frac{\partial \mathcal{L}}{\partial q^k} - dt \, \frac{\partial \mathcal{L}}{\partial t} \end{aligned}$$

and thus it follows

$$\begin{cases} \dot{q}^k & = \dfrac{\partial \mathcal{H}}{\partial p_k} \\ \dfrac{\partial \mathcal{H}}{\partial q^k} & = - \dfrac{\partial \mathcal{L}}{\partial q^k} \\ \dfrac{\partial \mathcal{H}}{\partial t} & = - \dfrac{\partial \mathcal{L}}{\partial t} \ . \end{cases}$$

Recasting Lagrange equations as

$$\frac{\partial \mathcal{L}}{\partial q^k} = -Q_{q^k} + \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\Big) = -Q_{q^k} + \dot{p}_k$$

Hamilton equations follow

$$\begin{cases} \dot{q}^k & = \dfrac{\partial H}{\partial p_k} \\ \dot{p}_k & = - \dfrac{\partial H}{\partial q^k} + Q_{q^k} \ . \end{cases}$$

Part III

Exercises

CHAPTER

SEVEN

DYNAMICS

Exercise 7.1

todo Add description of the problem and image

Find:

- the pure equations of motion (solution methods: dynamical equilibria, Lagrange II,...) **done**
- constraint reactions (solution methods: dynamical equilibria, Lagrange I,...) **todo**
- equilibria and their stability **todo**
- evolution of the linear(ized) dynamics of the system around stable equilibria **todo**

Solution.

Kinematics. Using x, θ as generalized coordinates,

$$\begin{cases} \vec{r}_A = x\hat{x} \\ \vec{r}_B = (x + \ell \sin\theta)\hat{x} + \ell\cos\theta\hat{y} \end{cases}$$

$$\begin{cases} \vec{v}_A = \dot{x}\hat{x} \\ \vec{v}_B = (\dot{x} + \ell\dot{\theta}\cos\theta)\hat{x} - \ell\dot{\theta}\sin\theta\hat{y} \end{cases}$$

Lagrangian function.

$$\mathcal{L} = K + U$$

$$\begin{aligned} K &= \frac{1}{2} m_A |\vec{v}_A|^2 + \frac{1}{2} m_B |\vec{v}_B|^2 = \\ &= \frac{1}{2} m_A \dot{x}^2 + \frac{1}{2} m_B \left(\dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta \right) \\ U &= -\frac{1}{2} k x_A^2 + m_B g y_B = \\ &= -\frac{1}{2} k x^2 + m_B g \ell \cos\theta \end{aligned}$$

Lagrange equations (II type) RHS of Lagrange equations read

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{d}{dt}\left(m_A\dot{x} + m_B\dot{x} + m_B\ell\dot{\theta}\cos\theta \right) = (m_A + m_B)\,\ddot{x} + m_B\ell\ddot{\theta}\cos\theta - m_B\ell\dot{\theta}^2\sin\theta \\ \frac{\partial \mathcal{L}}{\partial x} &= -kx \end{aligned}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) = \frac{d}{dt}\left(m_B\ell^2\dot\theta + m_B\ell\dot x\cos\theta\right) = m_B\ell^2\ddot\theta + m_B\ell\ddot x\cos\theta - m_B\ell\dot x\dot\theta\sin\theta$$
$$\frac{\partial\mathcal{L}}{\partial\theta} = -m_B\ell\dot x\dot\theta\sin\theta - m_B g\ell\sin\theta$$

Generalized forces read

$$Q_x = F$$

$$Q_\theta = C$$

so that the pure equations of motion follows from Lagrange equations $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot q}\right) - \frac{\partial\mathcal{L}}{\partial q} = Q_q$

$$\begin{cases}(m_A+m_B)\,\ddot x + m_B\ell\ddot\theta\cos\theta - m_B\ell\dot\theta^2\sin\theta + kx = F \\ m_B\ell^2\ddot\theta + m_B\ell\ddot x\cos\theta - m_B\ell\dot x\dot\theta\sin\theta + m_B\ell\dot x\dot\theta\sin\theta + m_B g\ell\sin\theta = C\end{cases}$$

and after the simplifications

$$\begin{cases}(m_A+m_B)\,\ddot x + m_B\ell\ddot\theta\cos\theta - m_B\ell\dot\theta^2\sin\theta + kx = F \\ m_B\ell\ddot x\cos\theta + m_B\ell^2\ddot\theta + m_B g\ell\sin\theta = C\end{cases}$$

Obs. The first equation is the x-component of the momentum equation of the whole system. The second equation is the angular momentum equation of the rod around the hinge in A.

Generalized forces on rigid bodies

Following the derivation of the *Lagrange equations for rigid bodies*, generalized forces are

$$Q_q = \frac{\partial\vec r_G}{\partial q}\cdot\vec R^e + \frac{\partial\vec\theta}{\partial q}\cdot\vec M^e_G$$

With the definition of θ_δ in the increment of the rigid body motion

$$d\vec r_A = d\vec r_B + \theta_\delta\times(A-B)\ ,$$

and writing the resultant of forces and moments as

$$\vec R^e = \sum_i \vec F_i$$
$$\vec M^e_G = \sum_i (\vec r_G - \vec r_i)\times\vec F_i + \sum_i \vec C_i$$

the generalized force can be recast as

$$\begin{aligned}Q_q &= \frac{\partial\vec r_G}{\partial q}\cdot\vec R^e + \frac{\partial\vec\theta}{\partial q}\cdot\vec M^e_G = \\ &= \frac{\partial\vec r_G}{\partial q}\cdot\sum_i\vec F_i + \frac{\partial\vec\theta}{\partial q}\cdot\left[\sum_i(\vec r_G-\vec r_i)\times\vec F_i + \sum_i\vec C_i\right] = \\ &= \sum_i\left[\frac{\partial\vec r_G}{\partial q} + \frac{\partial\vec\theta}{\partial q}\times(\vec r_G-\vec r_i)\right]\cdot\vec F_i + \sum_i\frac{\partial\vec\theta}{\partial q}\cdot\vec C_i = \\ &= \sum_i\frac{\partial\vec r_i}{\partial q}\cdot\vec F_i + \sum_i\frac{\partial\vec\theta}{\partial q}\cdot\vec C_i\ ,\end{aligned}$$

i.e. as the contribution of the single forces $\vec F_i$ acting on different points $\vec r_i$ the rigid body and the overall contribution of the couples of forces $\vec C_i$.

With $\vec{r}_C\,(q(t),t)$ and $\mathbb{R}\,(q(t),t)$,

$$\vec{r}_i(q(t),t) - \vec{r}_C(q(t),t) = \mathbb{R}(q(t),t)\cdot(\vec{r}_i^0 - \vec{r}_C^0)$$

Time derivative becomes

$$\begin{aligned}\vec{v}_i - \vec{v}_C &= \left[\dot{q}(t)\,\frac{\partial\mathbb{R}}{\partial q} + \frac{\partial\mathbb{R}}{\partial t}\right]\cdot(\vec{r}_i^0 - \vec{r}_C^0) = \\ &= \left[\dot{q}(t)\,\frac{\partial\mathbb{R}}{\partial q} + \frac{\partial\mathbb{R}}{\partial t}\right]\cdot\mathbb{R}^T\cdot\underbrace{\mathbb{R}\cdot(\vec{r}_i^0 - \vec{r}_C^0)}_{\vec{r}_i-\vec{r}_C}\end{aligned}$$

$$\begin{aligned}\vec{\omega}_\times &= \dot{\mathbb{R}}\cdot\mathbb{R}^T \\ \delta\vec{\theta}_\times &= \delta\mathbb{R}\cdot\mathbb{R}^T \\ \frac{\partial\vec{\theta}_\times}{\partial q} &= \frac{\partial\mathbb{R}}{\partial q}\cdot\mathbb{R}^T\end{aligned}$$

Exercise 7.2

todo Add description of the problem and image

Find:

- the pure equations of motion (solution methods: dynamical equilibria, Lagrange II, Kinetic energy theorem - energy conservation - since the problem has 1 dof,…) **done**
- constraint reactions (solution methods: dynamical equilibria, Lagrange I,…) **todo**
- equilibria and their stability **todo**
- evolution of the linear(ized) dynamics of the system around stable equilibria **todo**

Solution

Geometry.

$$\begin{aligned}R &= d\sin\alpha \\ b &= d\cos^2\alpha\end{aligned}$$

so that $\frac{b}{R} = \frac{\cos^2\alpha}{\sin\alpha}$.

Kinematics.

Position of the center of mass, C

$$\begin{aligned}\vec{r}_C &= b\cos\theta\hat{x} + b\sin\theta\hat{y} - h\hat{z} \\ \vec{v}_C &= -b\dot{\theta}\sin\theta\hat{x} + b\dot{\theta}\cos\theta\hat{y} = b\dot{\theta}\hat{y}_1\end{aligned}$$

Angular velocity $\vec{\omega}$ of the rigid body

$$\vec{\omega} = \dot{\theta}\hat{z} + \dot{\varphi}\hat{x}_1$$

Velocity of point contact point A is zero, $\vec{v}_A = \vec{0}$ for **pure rolling** constraint. Being $(A-C) = R\hat{z}_1$, the general expression of $\vec{v}_A$ as function of θ and φ reads

$$\begin{aligned}\vec{v}_A &= \vec{v}_C + \vec{\omega}\times(A-C) = \\ &= \hat{y}_1 b\dot{\theta} + \left(\dot{\theta}\hat{z} + \dot{\varphi}\hat{x}_1\right)\times R\hat{z}_1 = \\ &= \hat{y}_1\left(b\dot{\theta} + R\dot{\theta}\sin\alpha - R\dot{\varphi}\right) =\end{aligned}$$

so that the kinetatic constraint (integrable, with arbitrary initial condition) between θ and φ is

$$R\varphi = (R \sin\alpha + b)\ \theta\ .$$

Lagrangian function.

With

$$\mathbb{I}_C = I_x \hat{x}_1 \hat{x}_1 + I_y \hat{y}_1 \hat{y}_1 + I_z \hat{z}_1 \hat{z}_1\ ,$$
$$\vec{\omega} = \dot{\theta}\hat{z} + \dot{\varphi}\hat{x}_1 = \left(\dot{\varphi} - \dot{\theta}\sin\alpha\right)\hat{x}_1 + \dot{\theta}\cos\alpha\hat{z}_1$$

and using

$$\dot{\varphi} - \dot{\theta}\sin\alpha = \frac{b}{R}\dot{\theta}\ ,$$

the Lagrangian function becomes

$$\begin{aligned}
\mathcal{L} &= K + U = \\
&= \frac{1}{2}m|\vec{v}_C|^2 + \frac{1}{2}\vec{\omega}\cdot\mathbb{I}_C\cdot\vec{\omega} + mgx_C = \\
&= \frac{1}{2}mb^2\dot{\theta}^2 + \frac{1}{2}\left[I_x\left(\dot{\varphi} - \dot{\theta}\sin\alpha\right)^2 + I_z\dot{\theta}^2\cos^2\alpha\right] + mgb\cos\theta = \\
&= \frac{1}{2}mb^2\dot{\theta}^2 + \frac{1}{2}\left[I_x\left(\frac{b}{R}\right)^2 + I_z\cos^2\alpha\right]\dot{\theta}^2 + mgb\cos\theta = \\
&= \frac{1}{2}\tilde{I}\dot{\theta}^2 + mgb\cos\theta\ ,
\end{aligned}$$

with the equivalent inertia

$$\tilde{I} = mb^2 + I_x\left(\frac{b}{R}\right)^2 + I_z\cos^2\alpha\ .$$

Method 1. Lagrange equation (II). Legrange equation gives a pure equation of motion

$$0 = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = \tilde{I}\ddot{\theta} + mgb\sin\theta\ .$$

Method 2. Kinetic energy theorem - or energy conservation

Exercise 7.3

Solution

Kinematics. todo *check kinematic constraints. No influence of θ?*

$$\varphi_2 = \frac{R_1}{R_2}\varphi_1 = r\varphi_1$$

Lagrangian function.

$$\begin{aligned}
K &= \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}m_1|\vec{v}_1|^2 = \\
&= \frac{1}{2}I_1\dot{\varphi}_1^2 + \frac{1}{2}I_2\dot{\varphi}_2^2 + \frac{1}{2}m_1\ell^2\dot{\theta}^2 = \\
&= \frac{1}{2}\left(I_1 + I_2 r^2\right)\dot{\varphi}_1^2 + \frac{1}{2}m_1\ell^2\dot{\theta}^2 =
\end{aligned}$$

$$U = -\frac{1}{2}k\theta^2 - mg\ell\sin\theta$$

Generalized forces read

$$\begin{aligned} Q_{\varphi_1} &= C - R_1 F \\ Q_\theta &= C \end{aligned}$$

Lagrange functions (II).

$$\begin{cases} \left(I_1 + I_2 r^2\right)\ddot{\varphi} = C - R_1 F \\ m_1 \ell^2 \ddot{\theta} + k\theta + mg\ell\cos\theta = C \end{cases}$$

Exercise 7.4 (Inverted pendulum)

Solution

CHAPTER

EIGHT

GRAVITATION

Exercise 8.1 (Ball falling in a tunnel through a planet)

A ball of mass m moves through a tunnel drilled through a planet of radius R, mass M and uniform mass distribution. Neglecting the "mas defect" due to the tunnel in the planet,

- provide the expression of the force acting on the ball in the tunnel
- assuming no rotation of the planet, and zero initial velocity of the ball, provide the dynamical equation governing the motion of the ball and integrate it to find the lwa of motion

Uniform mass density reads $\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$. Exploiting symmetry, the gravitational field can be a function of the distance r of the center of the planet only, and have radial direction,

$$\vec{g} = -g(r)\hat{r} \ . \tag{8.1}$$

Formula (2.4) of the flux of the gravitational field,

$$\oint_{\vec{r}\in\partial V} \vec{g}(\vec{r}) \cdot \hat{n}(\vec{r}) = -G \int_{\vec{r}\in V} 4\pi\rho(\vec{r})$$

across the surface of a sphere of radius r - that has outward pointing unit normal vector $\hat{n}(\vec{r}) = \hat{r}(\vec{r})$ -, exploting expression (8.1) from symmetry, becomes for $r < R$

$$-g(r) 4\pi r^2 = -G\, 4\pi\rho \frac{4}{3}\pi r^3 = -4\pi G M \frac{r^3}{R^3}$$

and thus

$$g(r) = \frac{GM}{R^3} r \qquad \rightarrow \qquad \vec{g}(\vec{r}) = -m\frac{GM}{R^3} r\hat{r} = -m\frac{GM}{R^3}\vec{r} \ .$$

Force acting on the ball of mass m thus reads $\vec{F}(\vec{r}) = m\vec{g}(\vec{r})$. The equation of motion becomes

$$m\ddot{\vec{r}} = \vec{F}(\vec{r}) = -m\frac{GM}{R^3}\vec{r} \ ,$$

a linear second-order ODE with constant coefficients, whose solution provides an harmonic motion with pulsation $\Omega = \sqrt{\frac{GM}{R^3}}$. The solution of this equation, with initial conditions at rest on the surface of the planet,

$$\begin{cases} \vec{r}(0) = \vec{r}_0 = R\hat{r} \\ \dot{\vec{r}}(0) = \vec{0} \end{cases}$$

reads

$$\vec{r}(t) = \vec{r}_0 \cos\left(\sqrt{\frac{GM}{R^3}}\, t\right) = \hat{r}\, R \cos\left(\sqrt{\frac{GM}{R^3}}\, t\right) \ .$$

Exercise 8.2

Investigate the dynamics of the ball in the previous problem, if the rotational motion of the planet around its axis is considered and if the ball is thrown in the tunnel with non-zero velocity.

- Normal actions of the wall of the tunnel on the ball
- At the end of the tunnel, the ball moves above the planet surface while it's attracted "downwards". When the ball comes back to the planet surface, does it target the tunnel?

PROOF INDEX