Classical Mechanics

basics

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CONTENTS

This material is part of the **basics-books project**. It is also available as a .pdf document.

Classical mechanics deal with the motion of systems and its causes.

Different formulations of mechanics are available. Newton formulation was developed at the end of XVII century and starts from mass conservation and Newton's three principles of dynamics, summarised in invariance under Galileian transformations, the relation between force and the change of momentum of a system, and action/reaction principle. Analytical mechanics was developed in the following centuries by D'Alembert and Lagrange and starts from variational principles, leading to Lagrange or Hamiltonian equations of motion.

Newton Mechanics.

Kinematics

Actions

Inertia

Dynamics

Analytical Mechanics.

Lagrangian Mechanics

Hamiltonian Mechanics

Classical mechanics provides a reliable and useful theory for systems:

- much larger than atomic scales; at atomic scales, quantum mechanics is needed
- with velocity much slower than the speed of light or in domains where the finite value of finite speed of interactions can be neglected, as classical mechanics relies on istantaneous action at distance; if these assumptions fail, Einstein theory is needed either special relativity - as a consistent theory of mechanics and electromagnetism - or general relativity - as a theory of gravitation.
- with a small number of components, so that the integration of the governing equations of motion is feasible; continuous model of the systems are object of classical continuum mechanics, relying on the equations of classical mechanics and thermodynamics; systems with large number of components can be approached with the techniques developed in statistical mechanics.

Under these assumptions, mass conservation (Lavoisier principle) holds, inertial reference frames are related by Galileian relativity and the equations of motions are deterministic and can be solved with a reasonable effort - compared to the information and detail contained in the results - either analytically or numerically. Classical mechanics treats time and space as individually absolute physical quantities: this can be a good model whenever Einstein relativity effects are negligible.

Part I

Newton Mechanics

CHAPTER

ONE

KINEMATICS

Kinematics deals with the motion of mechanical systems, without taking into account the causes of motion.

1.1 Space and time

Classical mechanics relies on the concepts of **absolute 3-dimensional Euclidean space**, E^3, and **absolute time**.

Briefly, what is space? It's something you can measure with ruler (for distances) and square (for angles), or other space-measurment devices. What is time? It's something you can measure with a clock or other timekeeping devices.

1.2 Models

Different models of physical systems can be derived with an abstraction and modelling process, depending on the characteristics of the system under investigation and on the level of detail required by the analysis.

These models can be classified by:

- dimensions: 0: point; 1: line; 2: surfaces; 3: volume solid
- deformation: deformable or rigid components

A system can be composed of several components, either free or connected with constraints.

Here, the focus goes to the kinematics of *points* and *rigid bodies*, while deformable bodies are described in continuous mechanics - kinematics.

While space and time are absolute, the motion of a system is usually the **motion relative to an observer** or to a reference frame. After treating the kinematics of points and rigid bodies w.r.t. a given reference frame, *relative kinematics* provides the description of the motion of the same system w.r.t. 2 different observers/reference frames in relative motion.

1.3 Point

1.4 Rigid Body

1.4.1 Rigid motion

Rigid motion preserves distance between any pair of points, and thus angles. Rotations are described by rotation matrices.

The configuration of a material vector $\vec{a}$ undergoing a rotation is described by the product of the rotation tensor $\mathbb{R}$ by the reference configuration $\vec{a}^0$,

$$\vec{a} = \mathbb{R} \cdot \vec{a}^0 \ .$$

A rotation tensor is unitary to preserve distance,

$$\mathbb{I} = \mathbb{R} \cdot \mathbb{R}^T = \mathbb{R}^T \cdot \mathbb{R} \tag{1.1}$$

so that its time derivative reads,

$$\mathbb{0} = \frac{d}{dt} \left(\mathbb{R} \cdot \mathbb{R}^T \right) = \dot{\mathbb{R}} \cdot \mathbb{R}^T + \mathbb{R} \cdot \dot{\mathbb{R}}^T$$

It follows that the 2-nd order tensor $\dot{\mathbb{R}} \cdot \mathbb{R}^T = -\mathbb{R} \cdot \dot{\mathbb{R}}^T$ is anti-symmetric, and thus it can be written as

$$\dot{\mathbb{R}} \cdot \mathbb{R}^T =: \vec{\omega}_\times \ , \tag{1.2}$$

being the vector $\vec{\omega}$ the angular velocity. Since $\mathbb{R}$ is unitary by (1.1), multiplying (1.2) with the dot-product on the right by $\mathbb{R}$, it follows

$$\dot{\mathbb{R}} = \vec{\omega}_\times \cdot \mathbb{R} \ ,$$

and the expression of the time derivative of a material vector $\vec{a}$,

$$\frac{d\vec{a}}{dt} = \dot{\mathbb{R}} \cdot \vec{a}^0 = \vec{\omega}_\times \cdot \mathbb{R} \cdot \vec{a}^0 = \vec{\omega}_\times \cdot \vec{a} = \vec{\omega} \times \vec{a} \ .$$

1.4.2 Kinematics of rigid body

Rigid motion allows to describe the kinematics of a rigid body - determining position, velocity and acceleration of all of its points - given the position of a material point P and its orientation with respect to the reference frame, as an example using a rotation tensor $\mathbb{R}$.

Orientation.

Angular velocity.

Angular acceleration.

1.5 Rotations

1.5.1 Rotation tensor

Given 2 Cartesian bases $\{\hat{e}_i^0\}_{i=1:3}$, $\{\hat{e}_j^1\}_{j=1:3}$, the rotation tensor providing the transformation

$$\hat{e}_i^1 = \mathbb{R}^{0\to1} \cdot \hat{e}_i^0 \ ,$$

is

$$\mathbb{R}^{0\to1} = R_{ij}^{0\to1} \hat{e}_i^0 \otimes \hat{e}_j^0 = R_{ij}^{0\to1} \hat{e}_i^1 \otimes \hat{e}_j^1$$

with $R_{ij}^{0\to1} = \hat{e}_i^0 \cdot \hat{e}_j^1$.

Angular velocity.

$$\vec{\omega}^{01}_{\times} = \Omega^{01} = \dot{\mathbb{R}}^{01} \cdot \mathbb{R}^{01,T}$$

Using index notation

$$\varepsilon_{ijk}\omega_j = \dot{R}_{ij}R_{kj}$$

and the identities

$$\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

$$\varepsilon_{ijk}\varepsilon_{ljk} = \delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl} = 3\delta_{il} - \delta_{il} = 2\delta_{il}$$

it follows

$$\varepsilon_{ilk}\varepsilon_{ijk}\omega_j = \varepsilon_{ilk}\dot{R}_{ij}R_{kj}$$

$$2\delta_{lj}\omega_j = \varepsilon_{ilk}\dot{R}_{ij}R_{kj}$$

$$\omega_l = \frac{1}{2}\varepsilon_{ilk}\dot{R}_{ij}R_{kj} = -\frac{1}{2}\varepsilon_{lik}\dot{R}_{ij}R_{kj} = -\frac{1}{2}\varepsilon_{lij}\Omega_{ij}$$

Successive rotations

Given 3 Cartesian bases $\{\hat{e}^0_i\}_{i=1:3}$, $\{\hat{e}^1_j\}_{j=1:3}$, $\{\hat{e}^2_k\}_{k=1:3}$,

$$\begin{aligned}\hat{e}^2_i &= \mathbb{R}^{1\to 2} \cdot \hat{e}^1_i = \\ &= \mathbb{R}^{1\to 2} \cdot \mathbb{R}^{0\to 1} \cdot \hat{e}^0_i\end{aligned}$$

Time derivative w.r.t. reference frame 0 is indicated as the standard time derivative

$$\dot{a} = \frac{da}{dt} = \frac{{}^0da}{dt} \,,$$

$$\begin{aligned}\frac{d}{dt}\mathbb{R}^{21} &= \frac{d}{dt}\left[R^{21}_{ik}\hat{e}^1_i \otimes \hat{e}^1_k\right] = \\ &= \dot{R}^{21}_{ik}\hat{e}^1_i \otimes \hat{e}^1_k + \Omega^{10} \cdot \mathbb{R}^{21} - \mathbb{R}^{21} \cdot \Omega^{10} = \\ &= \frac{{}^1d}{dt}\mathbb{R}^{21} + \Omega^{10} \cdot \mathbb{R}^{21} - \mathbb{R}^{21} \cdot \Omega^{10} =\end{aligned}$$

$$\begin{aligned}\Omega^{20} &= \dot{\mathbb{R}}^{20} \cdot \mathbb{R}^{20,T} = \\ &= \frac{d}{dt}\left(\mathbb{R}^{21} \cdot \mathbb{R}^{10}\right) \cdot \mathbb{R}^{20,T} = \\ &= \left\{\left[\frac{{}^1d}{dt}\mathbb{R}^{21} + \Omega^{10} \cdot \mathbb{R}^{21} - \mathbb{R}^{21} \cdot \Omega^{10}\right] \cdot \mathbb{R}^{10} + \mathbb{R}^{21} \cdot \dot{\mathbb{R}}^{10}\right\} \cdot \mathbb{R}^{01} \cdot \mathbb{R}^{12} = \\ &= \frac{{}^1d}{dt}\mathbb{R}^{21} \cdot \mathbb{R}^{12} + \Omega^{10} = \\ &= \Omega^{21} + \Omega^{10} \,.\end{aligned}$$

Linearization of rotations

$$\mathbb{I} = \mathbb{R} \cdot \mathbb{R}^T$$

Increment

$$\mathbb{0} = \delta\mathbb{R} \cdot \mathbb{R}^T + \mathbb{R} \cdot \delta\mathbb{R}^T$$

and thus the antisymmetric tensor can be written as

$$\delta\theta_\times := \delta\mathbb{R} \cdot \mathbb{R}^T$$

so that

$$\delta\theta_l = -\frac{1}{2}\varepsilon_{lij}\delta R_{ik}\, R_{jk}$$

1.5.2 Quaternions

1.6 Continuous Medium

1.7 Relative Kinematics

Relative kinematics is discussed here using two Cartesian reference frames.

$$P - O_0 = x^{0i}_{P/O_0}\hat{e}^0_i$$

$$O_1 - O_0 = x^{0i}_{O_1/O_0}\hat{e}^0_i$$

$$P - O_1 = x^{1i}_{P/O_1}\hat{e}^1_i$$

$$\hat{e}^1_i = \hat{e}^1_i \cdot \hat{e}^0_k\, \hat{e}^0_k = \hat{e}^1_j \cdot \hat{e}^0_k\, \hat{e}^0_k \otimes \hat{e}^0_j \cdot \hat{e}^0_i = \\ = R^{0\to1}_{kj}\hat{e}^0_k \otimes \hat{e}^0_j \cdot \hat{e}^0_i = \mathbb{R}^{0\to1} \cdot \hat{e}^0_i \,.$$

1.7.1 Points

Position. Given two reference frames Ox^i, $O'x^{i'}$, for the position of a point P reads

$$(P - O) = (P - O') + (O' - O) \,,$$

i.e. the position vector $P-O$ of the point P w.r.t. point O - origin of the reference frame Ox^i - is the sum of the position vector $P-O'$ of the point P w.r.t. to the point O' - origin of the reference frame $O'x^{'i}$ - and the position vector $O'-O$, of the origin O' w.r.t. to O.

Velocity.

Acceleration.

1.7.2 Rigid bodies

Orientation.

Angular velocity.

Angular acceleration.

CHAPTER

TWO

ACTIONS

What's an action? Newton conceives the concept of an action, including both forces and moments, as the **causes of changes of the true motion** of mechanical systems, or **causes of difference of true motion w.r.t. a *general relative motion***.

Newton's concept of *true motion* is meant as the motion w.r.t. an inertial reference frame. So what is an **inertial reference frame**? In an inertial reference frame, dynamometers measure no force and moment associated with uniform motion.

2.1 Force, Moment of a Force, Distributed Actions

2.1.1 Concentrated Force

A (concentrated) force is a vector quantity with physical dimensions,

$$[\text{force}] = \frac{[\text{mass}][\text{length}]}{[\text{time}]^2}$$

which can be measured using a dynamometer, and whose effect can alter the equilibrium or motion conditions of a physical system.

In addition to the typical information of a vector quantity - magnitude, direction, and sense - contained in the force vector $\vec{F}$, it is often necessary to know the **point of application** or the line of application of the force.

2.1.2 Moment of a Concentrated Force

The moment of a force $\vec{F}$ applied at point P, or with a line of application passing through P, relative to point H is defined as the vector product,

$$\vec{M}_H = (P - H) \times \vec{F}$$

2.1.3 System of Forces, Resultant of Actions, and Equivalent Loads

Given a system of N forces $\left\{\vec{F}_n\right\}_{n=1:N}$, applied at points P_n, we define:

- **resultant** of the system of forces: the sum of the forces,

$$\vec{R} = \sum_{n=1}^{N} \vec{F}_n \ ,$$

- resultant of the moments with respect to a point H: the sum of the moments

$$\vec{M}_H = \sum_{n=1}^{N} (P_n - H) \times \vec{F}_n \,,$$

- an **equivalent load**: a system of forces that has the same resultant of forces and moments; for a system of forces, an equivalent load can be defined as a single force, the resultant of the forces $\vec{R}$ applied at point Q derived from the equivalence of moments

$$\vec{R} = \sum_{n=1}^{N} \vec{F}_n$$
$$(Q - H) \times \vec{R} = \sum_{n=1}^{N} (P_n - H) \times \vec{F}_n$$

2.1.4 Couple of Forces

A couple of forces is an equivalent load to two forces of equal magnitude and opposite sense, $\vec{F}_2 = -\vec{F}_1$, applied at two points P_1, P_2 not aligned along the line of application of the forces to have non-zero effects.

todo *image*

The resultant of the forces is zero,

$$\vec{R} = \vec{F}_1 + \vec{F}_2 = \vec{F}_1 - \vec{F}_1 = \vec{0} \,,$$

while the resultant of the moments does not depend on the moment pole,

$$\begin{aligned} \vec{M}_H &= (P_1 - H) \times \vec{F}_1 + (P_2 - H) \times \vec{F}_2 = \\ &= (P_1 - H) \times \vec{F}_1 - (P_2 - H) \times \vec{F}_1 = \\ &= (P_1 - P_2) \times \vec{F}_1 =: \vec{C} \,. \end{aligned}$$

2.1.5 Force Fields

todo

2.1.6 Distributed Actions

todo

2.2 Work and Power

In mechanics, as will become clearer later (**todo** add reference), the concept of work is linked to the concept of energy.
todo

2.2.1 Work and Power of a Force

Work. The elementary work of a force $\vec{F}$ applied at point P that undergoes an elementary displacement $d\vec{r}_P$ is defined as the dot product between the force and the displacement,

$$\delta W := \vec{F} \cdot d\vec{r}_P \, .$$

The work done by the force $\vec{F}$ applied at point P moving from point A to point B along the path ℓ_{AB} is the sum of all elementary contributions - and hence, in the limit for elementary displacements $\to 0$ for continuous variations, the line integral,

$$W_{\ell_{AB}} = \int_{\ell_{AB}} \delta W = \int_{\ell_{AB}} \vec{F} \cdot d\vec{r}_P \, .$$

In general, the work of a force or a field of forces depends on the path ℓ_{AB}. In cases where the work is independent of the path but depends only on the endpoints, we talk about *conservative actions*.

Power. The power of the force is defined as the time derivative of the work,

$$P := \frac{\delta W}{dt} = \vec{F} \cdot \frac{d\vec{r}_P}{dt} = \vec{F} \cdot \vec{v}_P \, ,$$

and coincides with the dot product between the force and the velocity of the point of application. Be cautious if a force is applied to geometric points rather than material points, such as in the case of a disk rolling without slipping on a surface: at every instant, the (new) material contact point has zero velocity, while the geometric contact point is the projection of the center of the disk and moves with the same velocity, $v = R\theta$

2.2.2 Work and Power of a System of Forces

Work. The work of a system of forces is the sum of the works of the individual forces,

$$\delta W = \sum_{n=1}^{N} \delta W_n = \sum_{n=1}^{N} \vec{F}_n \cdot d\vec{r}_n$$

Power. The power of a system of forces is the sum of the powers of the individual forces,

$$P = \sum_{n=1}^{N} P_n = \sum_{n=1}^{N} \vec{F}_n \cdot \vec{v}_n \, .$$

2.2.3 Work and Power of a Couple of Forces

Work. The elementary work of a couple of forces is the sum of the elementary works,

$$\begin{aligned} \delta W &= \vec{F}_1 \cdot d\vec{r}_1 + \vec{F}_2 \cdot d\vec{r}_2 = \\ &= \vec{F}_1 \cdot (d\vec{r}_1 - d\vec{r}_2) = \end{aligned}$$

Power. The power of a couple of forces,

$$P = \vec{F}_1 \cdot (\vec{v}_1 - \vec{v}_2)$$

can be rewritten if the points of application perform a rigid motion act (**todo** verify the definition of motion act and if it should be introduced),

$$\vec{v}_1 - \vec{v}_2 = \vec{\omega} \times (P_1 - P_2) \, ,$$

as

$$\begin{aligned} P &= \vec{F}_1 \cdot (\vec{v}_1 - \vec{v}_2) = \\ &= \vec{F}_1 \cdot [\vec{\omega} \times (P_1 - P_2)] = \\ &= \vec{\omega} \cdot \left[(P_1 - P_2) \times \vec{F}_1 \right] = \\ &= \vec{\omega} \cdot \vec{C} \, . \end{aligned}$$

2.3 Conservative Actions

A conservative force field is defined by the work it performs. In general, the work of a force field acting on a point P moving in space from point A to point B along a path ℓ_{AB} depends on the path. (**todo** add reference)

If the work of a force field does not depend on the path ℓ_{AB} but only on the endpoints A, B, for all pairs of points within a region of space Ω, the **force field** is said to be **conservative** in the region Ω of space.

In this case, the work done can be expressed as the difference of a scalar field, $U(P)$ or its opposite $V(P) := -U(P)$,

$$\begin{aligned} W_{AB} &= U(B) - U(A) = \Delta_{AB} U \\ &= V(A) - V(B) = -\Delta_{AB} V \end{aligned}$$

The functions U, V are respectively defined as the **potential** and **potential energy** of the force field.

The elementary work can thus be expressed in terms of the differential of these functions,

$$\begin{aligned} \delta W = \quad dU = \quad d\vec{r} \cdot \nabla U = \\ = -dV = -d\vec{r} \cdot \nabla V \end{aligned}$$

Comparing this relation with the definition of work $\delta W = d\vec{r} \cdot \vec{F}$, it is possible to identify the force field with the gradient of the potential function, and the opposite of the gradient of the potential energy,

$$\vec{F} = \nabla U = -\nabla V \, .$$

2.4 Examples of Forces

2.4.1 Gravitation

Universal Law of Gravitation

The force $\vec{F}_{12}$ exerted by a mass m_2 at P_2 on a mass m_1 at P_1 is described by **Newton's Universal Law of Gravitation**,

$$\vec{F}_{12} = G \, m_1 \, m_2 \frac{P_2 - P_1}{|P_2 - P_1|^3} \, ,$$

or

$$\vec{F}_{12} = G \, m_1 \, m_2 \frac{\hat{r}_{12}}{|\vec{r}_{12}|^2} \, ,$$

where $\vec{r}_{12} = (P_2 - P_1)$ is the vector pointing from point P_1 to point P_2, $r_{12} = |\vec{r}_{12}|$ is its magnitude, and $\hat{r}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|}$ is the unit vector in the same direction. The **universal gravitational constant** G is

$$G = 6.67 \cdot 10^{-11} \frac{N \, m^2}{kg^2}$$

and is considered a constant of nature.

Principle of Superposition of Causes and Effects (PSCE). Principle of superposition holds, i.e. the force acting on a mass m placed in P due to a set of N masses $\{m_k\}_{k=1:N}$ placed in P_k is the sum of individual forces $\vec{F}_k$,

$$\vec{F} = \sum_{k=1}^{N} \vec{F}_k = G\, m \sum_{k=1}^{N} m_k \frac{P_k - P}{|P_k - P|^3} \,. \tag{2.1}$$

Gravitational Field

The gravitational field generated by a set of masses $\{m_k\}_{k=1:N}$ located at P_k is a vector field associating a vector with physical dimensions $\frac{[\text{force}]}{[\text{mass}]}$ to each point in space P, that can be thought as the force per unit-mass acting on a **test mass** m placed in P, whose expression directly follows from (2.1)

$$\vec{g}(P) = \frac{\vec{F}}{m} = G \sum_{k=1}^{N} m_k \frac{P_k - P}{|P_k - P|^3} \,.$$

Given the gravitational field $\vec{g}(P)$, the gravitational force experienced by a system of mass m at P can be written as

$$\vec{F}_g = m\vec{g}(P)$$

Gravitational Potential Energy. Gravitational potential of a system of 2 masses reads

$$V(P) = -G\, m\, m_1 \frac{1}{|P - P_1|} \,,$$

as it can be easily shown evaluating its gradient,

$$\begin{aligned}
\nabla V(P) &= -G\, m\, m_1\, \hat{x}_k \frac{\partial}{\partial x_k} \frac{1}{|P - P_1|} = \\
&= -G\, m\, m_1\, \hat{x}_k \left(-\frac{1}{|P - P_1|^2} \right) \frac{\partial}{\partial x_k} |P - P_1| = \\
&= G\, m\, m_1\, \hat{x}_k \left(\frac{1}{|P - P_1|^2} \right) \frac{x_k - x_{1,k}}{|P - P_1|} = \\
&= G\, m\, m_1 \frac{x_k - x_{1,k}}{|P - P_1|^3}\, \hat{x}_k = \\
&= G\, m\, m_1 \frac{P - P_1}{|P - P_1|^3} \,.
\end{aligned}$$

Potential energy stored in a system of N point masses $\{m_k\}_{k=1:N}$ coincides with the work needed to build the system - a common choice to set the arbitrary additional constant of the energy is setting it equal to zero when masses are at infinite distances -, namely

$$V(P_k) = \sum_{\{i,k\}, i \neq k} G\, m_i\, m_k \frac{1}{|P_i - P_k|} \,,$$

summing over different unordered pairs, i.e. $\{1, 2\}$ and $\{2, 1\}$ are the same pair and thus considered only once, or

$$V(P_k) = \frac{1}{2} \sum_{(i,k), i \neq k} G\, m_i\, m_k \frac{1}{|P_i - P_k|} \,,$$

summing over different ordered pairs, i.e. $(1, 2)$ and $(2, 1)$ are different pairs.

Gravitational Field Near Earth's Surface

Within a limited domain near Earth's surface, it is common to approximate Earth's gravitational field as a uniform field, directed along the local vertical toward the center of the Earth, with intensity $g = G\frac{M_E}{R_E^2}$.

This model can be derived by approximating the position vector relative to the Earth's center $P - P_E \sim R_E\hat{r}$ and the unit vector identifying the direction from a point in the domain to the Earth's center with the local vertical $\hat{r}_{12} \sim -\hat{z}$,

$$\vec{g}(\vec{r}) = -G\frac{M_E}{R_E^2}\hat{z} = -g\hat{z}\,.$$

The gravitational force experienced by a body of mass m near Earth's surface is thus

$$\vec{F}_g = -mg\hat{z}\,,$$

commonly referred to as **weight**.

Gravitational Potential Energy. It can be shown that the gravitational potential near Earth's surface becomes

$$V(P) = m\,g\,z_P\,.$$

Proof.

With the series expansion, with $P - P_E = R_E\hat{r} + \vec{d}$, and $|\vec{d}| \ll R_E$,

$$\begin{aligned} V(P) &= -G\,m\,M_E\frac{1}{|P - P_E|} = \\ &\approx GM_E\,m\left[-\frac{1}{R_E} + \frac{R_E\hat{r}\cdot\vec{d}}{R_E^3}\right] = \\ &= \underbrace{-\,m\,\frac{GM_E}{R_E}}_{\text{const}} + m\,\underbrace{\frac{G\,M_E}{R_E^2}}_{=g}\,\underbrace{\hat{r}\cdot\vec{d}}_{=z} \end{aligned}$$

2.4.2 Elastic Actions: Linear Springs

According to Hooke's law, the behavior of an ideal linear elastic spring is described by the constitutive equation

$$F = k(\ell - \ell_0)\,,$$

where F is the absolute value of the force transmitted by the spring, k is the spring constant, ℓ_0 is the spring's rest length, and ℓ is the length in the considered configuration.

Potential Energy.

The elemental work done by the spring is given by

$$\delta W = Fd\ell = k(\ell - \ell_0)d\ell.$$

Integrating this from ℓ_1 to ℓ_2, we get

$$W = \int_{\ell_1}^{\ell_2} k(\ell - \ell_0)d\ell = \left[\frac{1}{2}k\ell^2 - k\ell_0\,\ell\right]\Bigg|_{\ell_1}^{\ell_2}.$$

The potential energy stored in the spring is

$$V = \frac{1}{2}k(\ell - \ell_0)^2.$$

This energy represents the work done to stretch or compress the spring from its rest position.

2.5 Constraint Reactions

Kinematic constraints act on a system by limiting its possible movements, exerting forces and moments, which are defined as constraint reactions.

In general, at an **ideal** constraint (**todo** provide definition of ideal constraint and discuss/mention/refer to friction), a constraint reaction corresponds to each constrained degree of freedom: for example, the constraint of translation of a point in a direction has a corresponding reaction force in that direction; the constraint of rotation around an axis has a corresponding moment aligned with that axis.

These conditions can be derived from the equations of dynamics for massless systems, as often considered in the ideal constraint model.

2.5.1 Contact Actions

Constraint Reactions of Ideal Constraints

Ideal constraints are models that **do not perform net work**, and are thus **conservative elements**. As should become evident in the subsequent sections from the expressions of relative velocities and exchanged actions,

$$\begin{aligned} P &= \vec{v}_1 \cdot \vec{F}_{21} + \vec{v}_2 \cdot \vec{F}_{12} + \vec{\omega}_1 \cdot \vec{M}_{21} + \vec{\omega}_2 \cdot \vec{M}_{12} = \\ &= (\vec{v}_1 - \vec{v}_2) \cdot \vec{F}_{21} + (\vec{\omega}_1 - \vec{\omega}_2) \cdot \vec{M}_{21} = \\ &= \vec{v}_{21}^{rel} \cdot \vec{F}_{21} + \vec{\omega}_{21}^{rel} \cdot \vec{M}_{21} \ , \end{aligned}$$

both terms are zero either because the relative motion is zero, or the actions act orthogonally to the relative motions.

Fixed Joint

The fixed joint constraint prevents both relative motion and relative rotation,

$$\begin{cases} \vec{0} = \vec{v}_{21}^{rel} = \vec{v}_2 - \vec{v}_1 \\ \vec{0} = \vec{\omega}_{21}^{rel} = \vec{\omega}_2 - \vec{\omega}_1 \end{cases} \quad , \quad \begin{cases} \vec{F}_{12} = -\vec{F}_{21} \\ \vec{M}_{12} = -\vec{M}_{21} \end{cases}$$

Slider

The slider constraint prevents relative motion in one direction and relative rotation.

$$\begin{cases} \forall \ \vec{v}_{\hat{t},21}^{rel} = \vec{v}_{\hat{t},2} - \vec{v}_{\hat{t},1} \\ 0 = v_{\hat{n},21}^{rel} = v_{\hat{n},2} - v_{\hat{n},1} \\ \vec{0} = \vec{\omega}_{21}^{rel} = \vec{\omega}_2 - \vec{\omega}_1 \end{cases} \quad , \quad \begin{cases} \vec{0} = \vec{F}_{\hat{t},12} = \vec{F}_{\hat{t},21} \\ F_{\hat{n},12} = -F_{\hat{n},21} \\ \vec{M}_{12} = -\vec{M}_{21} \end{cases}$$

Cylindrical Joint

The cylindrical joint constraint prevents relative motion and allows rotation around one axis.

$$\begin{cases} \vec{0} = \vec{v}_{21}^{rel} = \vec{v}_2 - \vec{v}_1 \\ \quad \forall\ \omega_{\hat{t},21}^{rel} = \omega_{\hat{t},2} - \omega_{\hat{t},1} \\ \vec{0} = \vec{\omega}_{\hat{n},21}^{rel} = \vec{\omega}_{\hat{n},2} - \vec{\omega}_{\hat{n},1} \end{cases} , \quad \begin{cases} \quad \vec{F}_{12} = -\vec{F}_{21} \\ 0 = M_{\hat{t},12} = M_{\hat{t},21} \\ \quad \vec{M}_{\hat{n},12} = -\vec{M}_{\hat{n},21} \end{cases}$$

Spherical Joint

The spherical joint constraint prevents relative motion but allows general rotation.

$$\begin{cases} \vec{0} = \vec{v}_{21}^{rel} = \vec{v}_2 - \vec{v}_1 \\ \quad \forall \vec{\omega}_{21}^{rel} = \vec{\omega}_2 - \vec{\omega}_1 \end{cases} , \quad \begin{cases} \quad \vec{F}_{12} = -\vec{F}_{21} \\ \vec{0} = \vec{M}_{12} = \vec{M}_{21} \end{cases}$$

Roller

The roller constraint can be thought of as a combination of a slider and a cylindrical joint.

$$\begin{cases} \quad \forall\ \vec{v}_{\hat{t},21}^{rel} = \vec{v}_{\hat{t},2} - \vec{v}_{\hat{t},1} \\ 0 = v_{\hat{n},21}^{rel} = v_{\hat{n},2} - v_{\hat{n},1} \\ \quad \forall\ \omega_{\hat{t},21}^{rel} = \omega_{\hat{t},2} - \omega_{\hat{t},1} \\ \vec{0} = \vec{\omega}_{\hat{n},21}^{rel} = \vec{\omega}_{\hat{n},2} - \vec{\omega}_{\hat{n},1} \end{cases} , \quad \begin{cases} \vec{0} = \vec{F}_{\hat{t},12} = \vec{F}_{\hat{t},21} \\ \quad F_{\hat{n},12} = -F_{\hat{n},21} \\ 0 = M_{\hat{t},12} = M_{\hat{t},21} \\ \quad \vec{M}_{\hat{n},12} = -\vec{M}_{\hat{n},21} \end{cases}$$

Support

The support constraint is a unilateral constraint **todo** *add description*

Friction

Static Friction

Static friction is the type of friction that occurs between two bodies when there is no relative motion between them, acting as a tangential force to the contact surface. The simplest model of static friction assumes that the maximum static friction force F_{max}^s that can be exerted between two bodies is proportional to the normal reaction between them, N,

$$F_{max}^s = \mu^s\, N\ .$$

The proportionality constant μ^s is defined as the **coefficient of static friction**. Generally, static friction forces are determined by the equilibrium conditions of the body, if these conditions can be met, and the relation

$$|F^s| \geq F_{max}^s\ .$$

Dynamic Friction

Dynamic friction occurs between two bodies in contact and in relative motion, acting as a tangential force to the contact surface. The simplest model of dynamic friction assumes that the dynamic friction force is proportional to the normal reaction between the two bodies and directed opposite to the relative velocity,

$$\vec{F}_{12} = -\mu^d N \frac{\vec{v}_{12}}{|\vec{v}_{12}|} \ ,$$

where $\vec{F}_{12}$ is the force acting on body 1 due to body 2, and $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$ is the velocity of body 1 relative to body 2.

Pure Rolling

todo *add description*

CHAPTER

THREE

INERTIA

Inertia deals with mass and mass distribution of systems.

But what is mass? Mass is a physical quantity, a property of the system, that manifests itself:

- in *gravitational attraction* (being both the origin of gravitational force and the property that makes a system sensible to gravitational attraction),
- in resistance to change of motion of a system under external *actions*, as it will be clear from principles and equations of motions in *dynamics*

In the range of application of classical mechanics **mass conservation** holds, as stated by **Lavoisier principle**: the mass of a closed system is constant.

CHAPTER

FOUR

DYNAMICS

Dynamics provides the link between the motion of a body and the forces causing that motion.

Newton's principles of dynamics and the cardinal equations of dynamics are the physical laws that govern the motion of mechanical systems: *Newton's principles* agree with the experimental observations (for systems with negligible quantum and Einstein relativity effects) and are the starting point - the principles - of Newton's formulation of mechanics; from these principles, *equations of motion* of mechanical systems are derived. These physical laws are formulated in terms of certain physical quantities, such as momentum, angular momentum, or the kinetic energy of the system - already discussed in the section about *inertia*. These dynamic quantities have the property of being additive (by definition), and making it particularly easy to write and interpret a general form of the equations governing motion. In general, these equations relate the time derivatives of these dynamic quantities to the causes of their variation. In the absence of net causes, conservation principles hold.

4.1 Principles of Newton mechanics

Primo principio della dinamica. Un corpo (o meglio, il baricentro di un corpo) sul quale agisce una forza netta nulla, persevera nel suo stato di quiete o di moto rettilineo uniforme rispetto a un sistema di riferimento inerziale.

Secondo principio della dinamica. Rispetto a un sistema di riferimento inerziale, la variazione della quantità di moto di un sistema è uguale all'impulso delle forze esterne agenti su di esso,

$$\Delta \mathbf{Q} = \mathbf{I}^e \ .$$

Nel caso di moto regolare, in cui la quantità di moto può essere rappresentata da una funzione continua e differenziabile in funzione del tempo, si può scrivere il secondo principio della dinamica in forma differenziale,

$$\dot{\mathbf{Q}} = \mathbf{R}^e \ ,$$

essendo la risultante delle forze esterne, $\mathbf{R}^e = \frac{d\mathbf{I}^e}{dt}$, la derivata nel tempo dell'impulso.

Terzo principio della dinamica. Se un sistema i esercita su un sistema j una forza $\mathbf{F}_{ji}$, allora il sistema j esercita sul sistema i una forza $\mathbf{F}_{ij}$ "uguale e contraria", con modulo uguale e verso opposto,

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji} \ .$$

4.2 Equations of motion

Starting from *principles of Newton mechanics*, it's possible to derive dynamical equations governing the motion of mechanical systems. The general form of these equations are easily expressed in terms of dynamical quantities discussed in the inertia section,

$$\begin{aligned} \frac{d\vec{Q}}{dt} &= \vec{R}^e && \text{(momentum balance equation)} \\ \frac{d\vec{\Gamma}_H}{dt} &= -\vec{r}_H \times \vec{Q} + \vec{M}^e_H && \text{(angular momentum balance equation)} \\ \frac{dK}{dt} &= P^{tot} && \text{(kinetic energy balance equation)} \end{aligned}$$

for every mechanical (closed) mechanical system. These equations will be derived for different systems in the following sections: point, system of points, rigid body.

Utilizzando i concetti di quantità di moto, momento della quantità di moto ed energia cinetica di un sistema, si possono scrivere le 3 equazioni cardinali della dinamica in una forma valida per **ogni sistema chiuso**. Nel caso siano soddisfatte alcune condizioni, e solo in questo caso, le equazioni cardinali della dinamica rappresentano dei principi di conservazione delle quantità dinamiche: osservando le espressioni delle equazioni cardinali, è facile intuire che la condizione da soddisfare per otenere un principio di conservazione è l'annullamento di tutti i termini ad eccezione della derivata temporale della quantità conservata.

4.2.1 Equazioni cardinali

Bilancio della quantità di moto. La derivata nel tempo della quantità di moto è uguale alla risultante delle forze esterne,

$$\dot{\mathbf{Q}} = \mathbf{R}^e \, . \tag{4.1}$$

Bilancio del momento della quantità di moto rispetto a un polo H. La derivata nel tempo del momento della quantità di moto rispetto a un punto H, a meno di "un termine di trasporto" è uguale alla risultate dei momenti esterni rispetto al polo H

$$\dot{\mathbf{L}}_H + \dot{\mathbf{x}}_H \times \mathbf{Q} = \mathbf{M}^e_H \, . \tag{4.2}$$

Bilancio dell'energia cinetica. La derivata nel tempo dell'energia cinetica è uguale alla potenza totale agente sul sistema, risultato della somma della potenza delle azioni esterne e della potenza delle azioni interne al sistema

$$\dot{K} = P^{tot} = P^e + P^i \tag{4.3}$$

4.2.2 Principi di conservazione

Conservazione della quantità di moto, in presenza di forze esterne con risultante nulla. Se la risultante delle forze esterne è nulla, $\mathbf{R}^e = \mathbf{0}$, dal bilancio della quantità di moto segue immediatamente

$$\dot{\mathbf{Q}} = \mathbf{0} \qquad \rightarrow \qquad \mathbf{Q} = \bar{\mathbf{Q}} = \text{cost.}$$

Conservazione del momento della quantità di moto, in presenza di momenti esterni con risultante nulla. Se la scelta del polo H rende nullo il termine di trasporto, $\dot{\mathbf{r}}_H \times \mathbf{Q} = \mathbf{0}$, se la risultante dei momenti esterni è nulla, $\mathbf{M}^e_H = \mathbf{0}$, dal bilancio del momento della quantità di moto segue immediatamente

$$\dot{\mathbf{L}}_H = \mathbf{0} \qquad \rightarrow \qquad \mathbf{L}_H = \bar{\mathbf{L}}_H = \text{cost.}$$

Conservazione dell'energia cinetica, in presenza di potenza totale nulla. Se la potenza totale delle azioni sul sistema è nulla, $P^{tot} = 0$, dal bilancio dell'energia cinetica segue immediatamente

$$\dot{K} = 0 \qquad \rightarrow \qquad K = \bar{K} = \text{cost.}$$

Conservazione dell'energia meccanica, in assenza di azioni non conservative. Ai tre principi di conservazione direttamente ottenibili dalle equazioni cardinali, si aggiunge il principio della conservazione dell'energia meccanica, somma dell'energia cinetica e dell'energia potenziale del sistema,

$$E^{mech} = K + V \ ,$$

in assenza di azioni non conservative. Se non ci sono forze non conervative, la potenza delle azioni agenti sul sistema può essere scritta come l'opposto della derivata nel tempo dell'energia potenziale del sistema,

$$P^{tot} = -\dot{V}$$

Dal bilancio dell'energia cinetica segue

$$\dot{K} = -\dot{V} \qquad \rightarrow \qquad \frac{d}{dt}(K+V) = 0 \qquad \rightarrow \qquad \dot{E}^{mech=0} \qquad \rightarrow \qquad E^{mech} = \bar{E}^{mech} = \text{const.}$$

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4.2.3 Equazioni cardinali della dinamica: punto materiale

Quantità dinamiche.

$$\begin{aligned} \mathbf{Q}_P &:= m_P \mathbf{v}_P \\ \mathbf{L}_{P,H} &:= (\mathbf{r}_P - \mathbf{r}_H) \times \mathbf{Q} = m_P (\mathbf{r}_P - \mathbf{r}_H) \times \mathbf{v}_P \\ K &:= \frac{1}{2} m_P \mathbf{v}_P \cdot \mathbf{v}_P = \frac{1}{2} m_P |\mathbf{v}_P|^2 \end{aligned}$$

Bilancio della quantità di moto. Il bilancio della quantità di moto di un punto materiale P, $\mathbf{Q}_P = m\mathbf{v}_P$ segue direttamente dal secondo principio della dinamica di Newton,

$$\dot{\mathbf{Q}}_P = \mathbf{R}^e_P$$

Bilancio del momento della quantità di moto. La derivata nel tempo del momento della quantità di moto viene calcolata usando la regola del prodotto,

$$\begin{aligned} \dot{\mathbf{L}}_{P,H} &= \frac{d}{dt} \left[m_P (\mathbf{r}_P - \mathbf{r}_H) \times \mathbf{v}_P \right] = \\ &= m \left[(\dot{\mathbf{r}}_P - \dot{\mathbf{r}}_H) \times \mathbf{v}_P + m_P (\mathbf{r}_P - \mathbf{r}_H) \times \dot{\mathbf{v}}_P \right] = \\ &= -m_P \dot{\mathbf{r}}_H \times \mathbf{v}_P + m_P (\mathbf{r}_P - \mathbf{r}_H) \times \dot{\mathbf{v}}_P = \\ &= -\dot{\mathbf{r}}_H \times \mathbf{Q} + \mathbf{M}^{ext}_H \ . \end{aligned}$$

Bilancio dell'energia cinetica.

$$\begin{aligned} \dot{K}_P &= \frac{d}{dt} \left(\frac{1}{2} m_P \mathbf{v}_P \cdot \mathbf{v}_P \right) = \\ &= m_P \dot{\mathbf{v}}_P \cdot \mathbf{v}_P = \\ &= \mathbf{R}^e \cdot \mathbf{v}_P = \\ &= \mathbf{R}^{tot} \cdot \mathbf{v}_P = P^{tot} \ . \end{aligned}$$

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4.2.4 Equazioni cardinali della dinamica: sistemi discreti di punti materiali

Partendo dalle equazioni dinamiche per un punto, si calcolano le equazioni dinamiche per un sistema di punti, sfruttando il terzo principio della dinamica di azione/reazione. Lo sviluppo delle equazioni permette di comprendere che la natura additiva delle grandezze dinamiche (quantità di moto, momento della quantità di moto, energia cinetica) segue direttamente dalla loro definizione.

Bilancio della quantità di moto. E' possibile scrivere il bilancio della quantità di moto per ogni punto i del sistema, scrivendo la risultante delle forze esterne agente sul punto come la somma delle forze esterne all'intero sistema agenti sul punto e le forze interne scambiate con gli altri punti del sistema,

$$\mathbf{R}_i^{ext,i} = \mathbf{F}_i^{ext} + \sum_{j \neq i} \mathbf{F}_{ij} \, .$$

L'equazione di bilancio per la i-esima massa diventa quindi

$$\dot{\mathbf{Q}}_i = \mathbf{R}_i^{ext,i} = \mathbf{F}_i^{ext} + \sum_{j \neq i} \mathbf{F}_{ij} \, .$$

Sommando le equazioni di bilancio di tutte le masse, si ottiene

$$\begin{aligned} \sum_i \dot{\mathbf{Q}}_i &= \sum_i \mathbf{F}_i^{ext} + \sum_i \sum_{j \neq i} \mathbf{F}_{ij} = \\ &= \sum_i \mathbf{F}_i^{ext} + \sum_{\{i,j\}} \underbrace{\left(\mathbf{F}_{ij} + \mathbf{F}_{ji}\right)}_{=\mathbf{0}} \end{aligned}$$

e definendo la quantità di moto di un sistema come la somma delle quantità di moto delle sue parti e la risultante delle forze esterne come somma delle forze esterne agenti sulle parti del sistema,

$$\mathbf{Q} := \sum_i \mathbf{Q}$$

$$\mathbf{R}^e := \sum_i \mathbf{F}_i^{ext}$$

si ritrova la forma generale del bilancio della quantità di moto,

$$\dot{\mathbf{Q}} = \mathbf{R}^e \, .$$

Bilancio del momento della quantità di moto. E' possibile scrivere il bilancio del momento della quantità di moto per ogni punto i del sistema, scrivendo la risultante dei momenti esterni agente sul punto come la somma dei momenti esterni all'intero sistema agenti sul punto e i momenti interni scambiati con gli altri punti del sistema,

$$\mathbf{M}_{H,i}^{ext,i} = \mathbf{M}_{H,i}^{ext} + \sum_{j \neq i} \mathbf{M}_{H,ij} \, .$$

Nel caso le parti del sistema interagiscano tramite forze, il momento rispetto al polo H generato dalla massa j sulla massa i vale

$$\mathbf{M}_{H,ij} = (\mathbf{r}_i - \mathbf{r}_H) \times \mathbf{F}_{ij} \, .$$

L'equazione di bilancio per la i-esima massa diventa quindi

$$\dot{\mathbf{L}}_{H,i} + \dot{\mathbf{r}}_H \times \mathbf{Q}_i = \mathbf{M}_{H,i}^{ext,i} = \mathbf{M}_{H,i}^{ext} + \sum_{j \neq i} \mathbf{M}_{H,ij} \, .$$

Sommando le equazioni di bilancio di tutte le masse, si ottiene

$$\begin{aligned} \sum_i \left(\dot{\mathbf{L}}_i + \dot{\mathbf{r}}_H \times \mathbf{Q}_i \right) &= \sum_i \mathbf{M}_{H,i}^{ext} + \sum_i \sum_{j \neq i} \mathbf{M}_{H,ij} = \\ &= \sum_i \mathbf{M}_{H,i}^{ext} + \sum_{\{i,j\}} \underbrace{\left(\mathbf{M}_{H,ij} + \mathbf{M}_{H,ji} \right)}_{=\mathbf{0}} \end{aligned}$$

e riconoscendo la quantità di moto del sistema e definendo il momento della quantità di moto di un sistema come la somma del momento della quantità di moto delle sue parti e la risultante dei momenti esterni come somma dei momenti esterni agenti sulle parti del sistema,

$$\mathbf{L}_H := \sum_i \mathbf{L}_{H,i}$$

$$\mathbf{M}_H^e := \sum_i \mathbf{M}_{H,i}^{ext}$$

si ritrova la forma generale del bilancio del momento della quantità di moto,

$$\dot{\mathbf{L}}_H + \dot{\mathbf{r}}_H \times \mathbf{Q} = \mathbf{M}_H^e \, .$$

Bilancio dell'energia cinetica. E' possibile ricavare il bilancio dell'energia cinetica del sistema, moltiplicando scalarmente il bilancio della quantità di moto di ogni punto,

$$\mathbf{v}_i \cdot m_i \dot{\mathbf{v}}_i = \mathbf{v}_i \cdot \left(\mathbf{F}_i^e + \sum_{j \neq i} \mathbf{F}_{ij} \right) \, ,$$

riconoscendo nel primo termine la derivata nel tempo dell'energia cinetica dell'i-esimo punto,

$$\dot{K}_i = \frac{d}{dt} \left(\frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i \right) = m_i \mathbf{v}_i \cdot \dot{\mathbf{v}}_i \, ,$$

e sommando queste equazioni di bilancio per ottenere

$$\sum_i \dot{K}_i = \sum_i \mathbf{v}_i \cdot \mathbf{F}_i^e + \sum_i \mathbf{v}_i \cdot \sum_{j \neq i} \mathbf{F}_{ij} \, .$$

Definendo l'energia cinetica di un sistema come la somma dell'energia cinetica delle sue parti, e definendo la potenza delle forze esterne/interne agenti sul sistema come la somma della potenza di tutte le forze esterne/interni al sistema,

$$K := \sum_i K_i$$

$$P^e := \sum_i P_i^{ext} = \sum_i \mathbf{v}_i \cdot \mathbf{F}_i^e$$

$$P^i := \sum_i P_i^{int} = \sum_i \mathbf{v}_i \cdot \sum_{j \neq i} \mathbf{F}_{ij}$$

si ritrova la forma generale del bilancio dell'energia cinetica,

$$\dot{K} = P^e + P^i = P^{tot} \, .$$

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4.2.5 Equazioni cardinali della dinamica: corpo rigido

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4.2.6 Equazioni cardinali della dinamica: mezzi continui

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4.3 Moti particolari

In questa sezione verranno studiati alcuni moti particolari, interessanti e utili da analizzare per motivi didattici, storici, e applicativi.

- moto rettilineo uniforme
- moto uniformemente accelerato
- moto circolare uniforme
- moti oscillatori e oscillatori smorzati:
  - oscillazioni libere:
    - sistema massa-molla(-smorzatore)
    - pendolo
  - oscillazioni forzate:
    - primo passo verso l'analisi strutturale e non solo ("ogni sistema fisico è un sistema di tanti oscillatori armonici")
    - concetti di risposta in frequenza e risonanza. **todo** video e/o script su risposta in frequenza di strutture e strutture antisismiche, mass-damper,…
- **Gravitazione**: partendo dalla legge di gravitazione universale fornita da Newton, si studia il moto dei corpi celesti in sistemi a due corpi, scoprendo che le loro traiettorie descrivono le sezioni coniche (cerchio, ellisse, parabola, iperbole), e dimostrando le leggi di Keplero
- rotazione di un corpo attorno a un punto fisso, moti di Poinsot

4.4 Equilibrium and Stability

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4.4.1 Gravitazione: problema dei due corpi

Part II

Analytical Mechanics

CHAPTER

FIVE

LAGRANGIAN MECHANICS

Classical mechanics can be re-formulated starting from variational principles, usually referred as **analytical mechanics**. Under some assumptions, that will be discussed during the derivation, analytical mechanics is equivalent to Newton mechanics.

Here, the equivalence of analytical mechanics and Newton mechanics is stressed, by means of the derivation of the principle of analytical mechanics starting from the equations of motions derived in Newtonian mechanics, relying on the conservation of mass and the three principles of Newton mechanics. The process is shown in the following sections for *point systems*, *systems of points*, *extended rigid bodies* and follows these steps:

- **strong form of equations.** Starting point is the dynamical equations of Newton mechanics, here also referred as the strong form of equations
- **weak form of equations.** Strong form are recast in weak form, also referred as **D'Alembert approach** or **virtual work formulation**, multiplying strong form of equations for arbitrary test functions
- **Lagrange equations.** A proper choice of test functions as a function of generalized coordinates, and some manipulation, leads to Lagrange equations. While the choice of test functions depends on the nature of the system, their expression always reads

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right) - \frac{\partial \mathcal{L}}{\partial q^k} = Q_{q^k} \ ,$$

  being $q^k(t)$ the generalized coordinates, $\mathcal{L}\left(\dot{q}^k(t), q?k(t), t\right) = K\left(\dot{q}^k(t), q^k(t), t\right) + U(q^k(t), t)$ the Lagrangian function of the system, defined as the sum of the kinetic energy K and the potential function $U = -V$, being V the potential energy - s.t. the conservative vector field reads $\vec{F} = -\nabla V$, and Q_q the generalized force.
- Lagrange equations can be interpreted as a result of a stationary principle of a functional, S, defined **action functional**, as it can be shown with the tools of calculus of variations. Here, **assuming** $Q_{q^k} = 0$, and multiplying by $w^k(t)$, integrating over time from t_0, t_1, and assuming that $w(t_0) = w(t_1) = 0$,

$$\begin{aligned} 0 &= \int_{t_0}^{t_1} w^k(t) \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right) - \frac{\partial \mathcal{L}}{\partial q}\right] dt = \\ &= w^k(t) \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left[\dot{w}^k(t) \frac{\partial \mathcal{L}}{\partial \dot{q}^k} + w^k(t) \frac{\partial \mathcal{L}}{\partial q^k}\right] dt \ . \end{aligned}$$

If $w^k(t)$ is equal to zero for t equal to t_0 and t_1, first term vanishes

$$
\begin{aligned}
0 &= -\int_{t_0}^{t_1} \left[\dot{w}^k(t) \, \frac{\partial \mathcal{L}}{\partial \dot{q}^k} + w^k(t) \, \frac{\partial \mathcal{L}}{\partial q^k} \right] \, dt \\
&= -\frac{1}{\varepsilon} \int_{t_0}^{t_1} \varepsilon \left[\dot{w}^k(t) \, \frac{\partial \mathcal{L}}{\partial \dot{q}^k} \left(\dot{q}^l(t), q^l(t), t \right) + w^k(t) \, \frac{\partial \mathcal{L}}{\partial q^k} \left(\dot{q}^l(t), q^l(t), t \right) \right] \, dt = \\
&= -\lim_{\varepsilon \to 0} \left\{ \frac{1}{\varepsilon} \int_{t_0}^{t_1} \varepsilon \left[\dot{w}^k(t) \, \frac{\partial \mathcal{L}}{\partial \dot{q}^k} \left(\dot{q}^l(t), q^l(t), t \right) + w^k(t) \, \frac{\partial \mathcal{L}}{\partial q^k} \left(\dot{q}^l(t), q^l(t), t \right) \right] \, dt \right\} = \\
&= -\lim_{\varepsilon \to 0} \left\{ \frac{1}{\varepsilon} \int_{t_0}^{t_1} \left[\mathcal{L} \left(\dot{q}^l(t) + \varepsilon \dot{w}^l(t), q^l(t) + \varepsilon w^l(t), t \right) - \mathcal{L} \left(\dot{q}^l(t), q^l(t), t \right) \right] \, dt + o(\varepsilon) \right\} = \\
&= -\delta \int_{t_0}^{t_1} \mathcal{L}(\dot{q}^l(t), q^l(t), t) \, dt =: -\delta S[q^k(t)] \, ,
\end{aligned}
$$

i.e. Lagrange equations are equivalent to the stationary condition of the action functional

$$
S[q^k(t)] := \int_{t_0}^{t_1} \mathcal{L} \left(\dot{q}^l(t), q^l(t), t \right) \, dt \, .
$$

5.1 Point

Newton dynamical equations - strong form. Dynamical equation governing the motion of a point P reads

$$
m \dot{\vec{v}}_P = \vec{R}^e \, ,
$$

being m the mass of the system, $\vec{v}_P$ the velocity of point P, $\vec{a}_P = \dot{\vec{v}}_P$ its acceleration and $\vec{R}^e$ the net external force acting on the system..

Weak form. Weak form of dynamical equations is derived with scalar multiplication of the strong form by an arbitrary test function $\vec{w}$,

$$
\vec{0} = \vec{w} \cdot \left(m \dot{\vec{v}} - \vec{R}^e \right) \qquad \forall \vec{w} \tag{5.1}
$$

Lagrange equations. Lagrange equations are derived from a proper choice of the test function. The position of the point P is written as a function of the generalized coordinates $q^k(t)$ and time t

$$
\vec{r}_P(t) = \vec{r}(q^k(t), t) \, ,
$$

so that its velocity can be written as

$$
\vec{v}_P(t) := \frac{d\vec{r}_P}{dt} = \dot{q}^k(t) \underbrace{\frac{\partial \vec{r}}{\partial q^k}}_{\frac{\partial \vec{v}}{\partial \dot{q}^k}} (q^l(t), t) + \frac{\partial \vec{r}}{\partial t} (q^l(t), t) = \vec{v} \left(\dot{q}^k(t), q^k(t), t \right) \, ,
$$

from which the relation between partial derivatives

$$
\frac{\partial \vec{r}}{\partial q^k} = \frac{\partial \vec{v}}{\partial \dot{q}^k} \, . \tag{5.2}
$$

follows. Choosing the test function $\vec{w}$ as

$$
\vec{w} = \frac{\partial \vec{r}}{\partial q^k} = \frac{\partial \vec{v}}{\partial \dot{q}^k} \, ,
$$

applying the rule of derivative of product, using Schwartz theorem to switch order of derivation, and exploiting relation (5.2) it's possible to recast weak form (5.1) as

$$\begin{aligned}\vec{0} &= \frac{\partial \vec{v}}{\partial \dot{q}^k} \cdot \left(m\dot{\vec{v}} - \vec{R}^e\right) = \\ &= \frac{d}{dt}\left(\frac{\partial \vec{v}}{\partial \dot{q}^k} \cdot m\vec{v}\right) - \frac{d}{dt}\frac{\partial \vec{r}}{\partial q^k} \cdot m\vec{v} - \frac{\partial \vec{r}}{\partial q^k} \cdot (\vec{R}^{e,c} + \vec{R}^{e,nc}) \\ &= \frac{d}{dt}\left(\frac{\partial \vec{v}}{\partial \dot{q}^k} \cdot m\vec{v}\right) - \frac{\partial \vec{v}}{\partial q^k} \cdot m\vec{v} - \frac{\partial \vec{r}}{\partial q^k} \cdot (\nabla U + \vec{R}^{e,nc}) \\ &= \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}^k}\right) - \frac{\partial K}{\partial q^k} - \frac{\partial U}{\partial q^k} - \underbrace{\frac{\partial \vec{r}}{\partial q^k} \cdot \vec{R}^{e,nc}}_{=:Q^k} .\end{aligned}$$

Introducing the **Lagrangian function**

$$\mathcal{L}(\dot{q}^k(t), q^k(t), t) := K(\dot{q}^k(t), q^k(t), t) + U(q^k(t), t) ,$$

and recalling that potential function U is not a function of velocity and thus of time derivatives of the generalized coordinates $\dot{q}^k$, it's possible to recast the dynamical equation as the **Lagrange equations**

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right) - \frac{\partial \mathcal{L}}{\partial q^k} = Q^k ,$$

being Q^k the **generalized force** not included in the gradient of the potential ∇U - usually a non conservative contribution -, $Q^k = \dfrac{\partial \vec{r}}{\partial q^k} \cdot \vec{R}^{e,nc}$.

5.2 System of points

Newton dynamical equations - strong form.

Weak form.

Lagrange equations.

5.3 Rigid Body

Newton dynamical equations - strong form. Dynamical equations governing the motion of a rigid body, referred to its center of mass G read

$$\begin{cases} \dot{\vec{Q}} = \vec{R}^e \\ \dot{\vec{\Gamma}}_G = \vec{M}^e_G , \end{cases}$$

with momentum $\vec{Q} = m\vec{v}_G$ and angular momentum $\vec{\Gamma}_G = \mathbb{I}_G \cdot \vec{\omega}$.

Weak form. Weak form of dynamical equations is derived with scalar multiplication of the strong form by an arbitrary test functions $\vec{w}_t$, $\vec{w}_r$

$$\vec{0} = \vec{w}_t \cdot \left(m\dot{\vec{v}}_G - \vec{R}^e\right) + \vec{w}_r \cdot \left(\dot{\vec{\Gamma}}_G - \vec{M}^e_G\right) \qquad \forall \vec{w}_t,\, \vec{w}_r \tag{5.3}$$

Lagrange equations. Lagrange equations are derived from the weak form, with a proper choice of the weak test functions. The "translational part" is recasted after choosing

$$\vec{w}_t = \frac{\partial \vec{r}}{\partial q^k} = \frac{\partial \vec{v}}{\partial \dot{q}^k} .$$

Following the same steps show to derive *Lagrange equations for a point system*, the translational part becomes

$$\frac{d}{dt}\frac{\partial K^{tr}}{\partial \dot{q}^k} - \frac{\partial K^{tr}}{\partial q^k} - \frac{\partial U^{tr}}{\partial q^k} = Q_k^{tr} \ ,$$

being $K^{tr} = \frac{1}{2} m |\vec{v}_G|^2$ the contribution to kinetic energy of the velocity of the center of mass G deriving from the momentum equation, U^{tr} the contribution to the potential energy U from the momentum equation, and Q_k^{tr} the contribution to the generalized force from the momentum equation.

The "rotational part" is recasted after choosing

$$\vec{w}_r = \frac{\partial \vec{\theta}}{\partial q^k} = \frac{\partial \vec{\omega}}{\partial \dot{q}^k}$$

Angular velocity $\vec{\omega}$ can be written w.r.t the inertial $\{\hat{e}_i\}$ or the material reference frame $\{\hat{E}_i\}$,

$$\vec{\omega} = \omega_i \hat{e}_i = \sigma_j \hat{E}_j \ ,$$

and the inertia tensor as

$$\mathbb{I}_G = I_{ij} \, \hat{E}_i \otimes \hat{E}_j \ ,$$

being the components I_{ij} constant.

$$0 = \frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \frac{d}{dt} \left(\mathbb{I}_G \cdot \vec{\omega} \right) - \frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \vec{M}_G^e = \frac{d}{dt} \left(\frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \mathbb{I}_G \cdot \vec{\omega} \right) - \frac{d}{dt} \frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \mathbb{I}_G \cdot \vec{\omega} - \frac{\partial \vec{\theta}}{\partial q^k} \cdot \vec{M}_G^e$$

The first term becomes

$$\frac{d}{dt} \left(\frac{\partial \vec{\omega}}{\partial \dot{q}^k} \cdot \mathbb{I}_G \cdot \vec{\omega} \right) = \frac{d}{dt} \left(\frac{\partial \sigma_a}{\partial \dot{q}^k} I_{ab} \sigma_b \right) = \frac{d}{dt} \frac{\partial}{\partial \dot{q}^k} \left(\frac{1}{2} \vec{\omega} \cdot \mathbb{I}_G \cdot \vec{\omega} \right) = \frac{d}{dt} \frac{\partial K^{rot}}{\partial \dot{q}^k}$$

The second term becomes

$$\begin{aligned}
\frac{d}{dt} \frac{\partial \vec{\theta}}{\partial q^k} \cdot \mathbb{I}_G \cdot \vec{\omega} &= \frac{\partial}{\partial q^k} \underbrace{\frac{d\vec{\theta}}{dt}}_{\vec{\omega}} \cdot \mathbb{I}_G \cdot \vec{\omega} = \\
&= \frac{\partial \vec{\omega}}{\partial q^k} \cdot \mathbb{I}_G \cdot \vec{\omega} = \\
&= \frac{\partial}{\partial q^k} \left(\sigma_a \hat{E}_a \right) \cdot \hat{E}_b \, I_{bc} \sigma_c = \\
&= \frac{\partial \sigma_a}{\partial q^k} \underbrace{\hat{E}_a \cdot \hat{E}_b}_{=\delta_{ab}} I_{bc} \sigma_c + \sigma_a \underbrace{\frac{\partial \hat{E}_a}{\partial q^k} \cdot \hat{E}_b}_{=0} I_{bc} \sigma_c = \\
&= \frac{\partial}{\partial q^k} \left(\frac{1}{2} \sigma_a \, I_{ab} \sigma_b \right) = \\
&= \frac{\partial}{\partial q^k} \left(\frac{1}{2} \vec{\omega} \cdot \mathbb{I}_G \cdot \vec{\omega} \right) = \frac{\partial K^{rot}}{\partial q^k} \ .
\end{aligned}$$

The third term can be written as the sum of the derivative of a potential function and a generalized force,

$$\frac{\partial \vec{\theta}}{\partial q^k} \cdot \vec{M}_G^e = \frac{\partial U^{rot}}{\partial q^k} + Q_{q^k}^{rot}$$

The rotational part of the wak form becomes

$$\frac{d}{dt}\frac{\partial K^{rot}}{\partial \dot{q}^k} - \frac{\partial K^{rot}}{\partial q^k} - \frac{\partial U^{rot}}{\partial q^k} = Q_{q^k}^{rot} \ ,$$

being $K^{rot} = \frac{1}{2}\vec{\omega} \cdot \mathbb{I}_G \cdot \vec{\omega}$ the contribution to kinetic energy of the rotation aroung the center of mass G deriving from the angular momentum equation, U^{rot} the contribution to the potential energy U from the angular momentum equation, and Q_k^{rot} the contribution to the generalized force from the angular momentum equation.

Adding together the contributions of the momentum and the angular momentum equations, the Lagrange equation can be formally written with the same expression found for the system of points,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right) - \frac{\partial \mathcal{L}}{\partial q^k} = Q_{q^k} \ ,$$

being $\mathcal{L} = K + U$ the Lagrangian function of the system, and $K = K^{tr} + K^{rot}$, $U = U^{tr} + U^{rot}$, $Q_{q^k} = Q_{q^k}^{tr} + Q_{q^k}^{rot}$ the kinetic energy the potential function and the generalized force of the system, defined as the sum of the contributions coming from the momentum and the angular momentum equations.

CHAPTER

SIX

HAMILTONIAN MECHANICS

Riformulazione ulteriore della meccanica di Newton, a partire dalla meccanica di Lagrange. Fornisce le basi per un approccio moderno anche in altre teorie della Fisica. **dots…**

Starting from Lagrange equations derived in *Lagrangian mechanics*,

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}}\Big) - \frac{\partial \mathcal{L}}{\partial q} = Q_q$$

the **generalized moment** is defined as

$$p_k \coloneqq \frac{\partial \mathcal{L}}{\partial \dot{q}^k} \, ,$$

and the **Hamiltonian function** as

$$\mathcal{H}(q^k(t), p_k(t), t) \coloneqq p_k \dot{q}^k - \mathcal{L}(\dot{q}^l(q^k, p_k, t), q^l(t), t) \, ,$$

its differential reads

$$\begin{aligned} d\mathcal{H} &= dq^k \, \frac{\partial \mathcal{H}}{\partial q^k} + dp_k \, \frac{\partial \mathcal{H}}{\partial p_k} + dt \, \frac{\partial \mathcal{H}}{\partial t} = \\ &= dp_k \, \dot{q}^k + \underbrace{p_k \, d\dot{q}^k - d\dot{q}^k \, \frac{\partial \mathcal{L}}{\partial \dot{q}^k}}_{=0} - dq^k \, \frac{\partial \mathcal{L}}{\partial q^k} - dt \, \frac{\partial \mathcal{L}}{\partial t} \end{aligned}$$

and thus it follows

$$\begin{cases} \dot{q}^k & = \dfrac{\partial \mathcal{H}}{\partial p_k} \\ \dfrac{\partial \mathcal{H}}{\partial q^k} & = -\dfrac{\partial \mathcal{L}}{\partial q^k} \\ \dfrac{\partial \mathcal{H}}{\partial t} & = -\dfrac{\partial \mathcal{L}}{\partial t} \, . \end{cases}$$

Recasting Lagrange equations as

$$\frac{\partial \mathcal{L}}{\partial q^k} = -Q_{q^k} + \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\Big) = -Q_{q^k} + \dot{p}_k$$

Hamilton equations follow

$$\begin{cases} \dot{q}^k & = \dfrac{\partial H}{\partial p_k} \\ \dot{p}_k & = -\dfrac{\partial H}{\partial q^k} + Q_{q^k} \, . \end{cases}$$