Modern Physics

basics

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This material is part of the **basics-books project**. It is also available as a .pdf document.

Please check out the Github repo of the project, basics-book project.

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Part I

Special Relativity

CHAPTER ONE

SPECIAL RELATIVITY

Principles. (1) invariant nature of physical laws, (2) c as the maximum speed of propagation of information in space. Every inertial observer measures the same speed of light.

Lorentz's transformation as a special coordinate transformations for space-time description as seen by two inertial observers in relative motion with uniform velocity. Physical laws governing a physical process have the same expressions if seen by two inertial observers. In a homogeneous time-space the expression of equations of physics doesn't depend on translation of origin of time-space coordinates; in a isotropic space the expression of equations of physics doesn't depend on rotations space rotation; as two inertial observers may write the same expressions of the physical equations using their own coordinates, the expression of equations of physics doesn't depend on Lorentz's transformations.

- standard configuration: derivation of Lorentz's transformations from principles and symmetry considerations
- composition of transformations and general Lorentz's transformation

Some consequences and examples:

- **Finite speed of propagation and loss of simultaneity.**
- **Speed of light and causality.** Causality follows from the principle that c is the maximum speed of information: all the observers perceive causes before consequences.
- **Length contraction and time dilation**

Mechanics. Point kinematics: 4-velocity and 4-acceleration; dynamics of free particle: 4-momentum, energy-momentum relation, rest energy; dynamics of particles in external force filed: which field? For particles in EM field, Lorentz's force; Lagrangian approach

Electromagnetism. Special relativity and equations of electromagnetism. From classical electromagnetism to electromagnetism in special relativity theory; EM potentials, gauge condition, 4-current and EM field tensor; EM field equations; motion of a point charge in an EM field; energy balance equations. Relativity of electromagnetism: low-speed relativity for classical electromagnetism. Lagrangian approach for the equations of motion of charges in an EM field, and for the EM field equations.

CHAPTER

TWO

SPECIAL RELATIVITY - NOTES

An event is determined by spatio-temporal information together, $t, \vec{r}$. Absolute nature of physics needs vector algebra and calculus formalism

$$\mathbf{X} = ct\mathbf{E}_0 + \vec{r} = ct\mathbf{E}_0 + x^1\mathbf{E}_1 + x^2\mathbf{E}_2 + x^3\mathbf{E}_3 = X^\alpha \mathbf{E}_\alpha \ ,$$

having used Cartesian coordiantes for the space coordinate.

Geometry of time-space in special relativity. Minkowski metric reads

$$g_{\alpha\beta} = \mathbf{E}_\alpha \cdot \mathbf{E}_\beta = \text{diag}\{1, -1, -1, -1\}$$

Reciprocal basis

The reciprocal basis reads $\mathbf{E}_\alpha \cdot \mathbf{E}^\beta = \delta^\beta_\alpha$, $\mathbf{E}_\alpha = g_{\alpha\beta}\mathbf{E}^\beta$, s.t. the elementary interval between two events can be written as

$$d\mathbf{X} = dX^\alpha \, \mathbf{E}_\alpha = \underbrace{dX^\alpha \, g_{\alpha\beta}}_{=dX_\beta} \, \mathbf{E}^\beta = dX_\beta \, \mathbf{E}^\beta \ ,$$

having used Cartesian coordinates,

$$\begin{array}{llll} X^0 = ct & X^1 = x & X^2 = y & X^3 = z \\ X_0 = ct & X_1 = -x & X_2 = -y & X_3 = -z \end{array}$$

Elementary interval between two events

$$ds^2 = d\mathbf{X} \cdot d\mathbf{X} = (dX_\alpha \mathbf{E}^\alpha) \cdot \left(dX^\beta \mathbf{E}_\beta\right) = c^2 \, dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = c^2 \, dt^2 - |d\vec{r}|^2$$

2.1 Kinematics of point

Proper time. For a co-moving observer, $d\vec{r}\,' = \vec{0}$, and t' is commonly indicated with τ, and its differential is invariant itself, being the product of a constant (c is a universal constant in special relativity) and an invariant quantity.

$$ds^2 = c^2 dt'^2 - |d\vec{r}\,'|^2 = c^2 d\tau^2 \ .$$

Given the invariant nature of ds,

$$ds^2 = c^2 \, d\tau^2 = c^2 \, dt^2 - |d\vec{r}|^2 = c^2 \, dt^2 \left[1 - \frac{1}{c^2} \frac{|d\vec{r}|^2}{dt^2} \right] = c^2 \, dt^2 \left[1 - \frac{|\vec{v}|^2}{c^2} \right]$$

and thus

$$ds = c\,d\tau = \gamma^{-1}(v/c)\,c\,dt\ ,$$

with $\gamma(w) = \frac{1}{\sqrt{1-w^2}}$.

Note: ds is invariant **todo** prove it. And/or add a section about the role of invariance.

4-Velocity. Given the parametric representation of an event in space-time as a function of its proper time, $\mathbf{X}(\tau)$ or coordinate s, $\mathbf{X}(s)$ the derivative w.r.t. this parameter is defined as the 4-velocity of the event in space time. Using Cartesian coordinates inducing constant and uniform basis $\mathbf{E}_\alpha$, as a function of the observer time t, ct, $x^i(t)$, and the transformation of coordinates $t(\tau)$, with differential $dt = \frac{1}{\gamma}\,d\tau$

$$\mathbf{U}(\tau) := \mathbf{X}'(\tau) = \frac{d}{d\tau}\left(X^\alpha(\tau)\mathbf{E}_\alpha\right) = \frac{dt}{d\tau}(ct\mathbf{E}_0 + x^i(t)\mathbf{E}_i) = \gamma(v/c)\left(c\mathbf{E}_0 + \dot{x}^i(t)\mathbf{E}_i\right) = \gamma(v/c)\left(c\mathbf{E}_0 + \vec{v}\right)$$

or

$$\mathbf{U}(s) := \mathbf{X}'(s) = \frac{dt}{ds}\frac{d}{dt}\mathbf{X}(t) = \dots = \gamma(v/c)\left(\mathbf{E}_0 + \frac{\vec{v}}{c}\right)\ .$$

Note: Using s as the parameter, $\mathbf{U}$ is non-dimensional, and has pseudo-norm = 1,

$$\mathbf{U}(s)\cdot\mathbf{U}(s) = \gamma^2\underbrace{\left(1 - \frac{|\vec{v}|^2}{c^2}\right)}_{=\gamma^{-2}} = 1\ .$$

Using τ as the parameter, $\mathbf{U}$ has physical dimension of a velocity and pseudo-norm = c.

4-acceleration $\mathbf{X}''(\tau)$ or $\mathbf{X}''(s)$, todo

2.2 Dynamics of a point mass

4-momentum

$$\mathbf{P} = m\mathbf{U}$$

Using Cartesian coordinates and τ as independent variable,

$$\mathbf{P} = m\mathbf{U} = m\frac{d\mathbf{X}}{d\tau} = m\gamma(c,\vec{v})\ .$$

Low-speed limit - classical mechanics

The spatial component is γ times the 3-dimensional momentum $\vec{p} = m\vec{v}$; the time component reads

$$P^0 = m\gamma(w)c\ ,$$

and for small ratio $w := \frac{v}{c}$ it can be expanded in Taylor series around $w = 0$ as

$$\gamma(w) \sim \gamma(0) + w\,\gamma'(0) + \frac{1}{2}\,w^2\gamma''(0) + o(w^2)\ ,$$

with

$$\begin{aligned}
\gamma(w)|_{w=0} &= \left.\frac{1}{\sqrt{1-w^2}}\right|_{w=0} = 1\\
\gamma'(w)|_{w=0} &= \left.-\frac{1}{2}(1-w^2)^{-\frac{3}{2}}(-2w)\right|_{w=0} = w(1-w^2)^{-\frac{3}{2}} = 0\\
\gamma''(w)|_{w=0} &= \left.\left((1-w^2)^{-\frac{3}{2}} + w\left(-\frac{3}{2}\right)(1-w^2)^{-\frac{5}{2}}(-2w)\right)\right|_{w=0} =\\
&= \left.\left((1-w^2)^{-\frac{3}{2}} + 3w^2(1-w^2)^{-\frac{5}{2}}\right)\right|_{w=0} = 1
\end{aligned}$$

and thus

$$\gamma(w) = 1 + \frac{1}{2}w^2 + o(w^2)$$

and

$$\gamma(v/c)\, m\, c \sim m\, c\left(1 + \frac{v^2}{c^2}\right) = \frac{1}{c}\left(mc^2 + \frac{1}{2}m|\vec{v}|^2\right)$$

Energy-momentum relation

Recognizing energy ($E = \gamma mc^2$) and 3-momentum ($\vec{p} = m_3\vec{v}$, with $m_3 := \gamma m$)[1], the 4-momentum can be written as

$$\mathbf{P} = m\mathbf{U} = \gamma m\left(1, \frac{\vec{v}}{c}\right) =: \frac{1}{c}\left(\frac{E}{c}, \vec{p}\right)$$

Its pseudo-norm reads

$$m^2 = \mathbf{P}\cdot\mathbf{P} = \frac{1}{c^4}\left(E^2 - c^2|\vec{p}|^2\right)$$

and thus the relation between E, $\vec{p}$, m and c,

$$E^2 = m^2c^4 + c^2|\vec{p}|^2 \ ,$$

from which, for $\vec{v} = \vec{0} \rightarrow \vec{p} = \vec{0}$,

$$E^2 = m^2c^4 \ ,$$

and keeping only the solution with positive energy (**todo** *reference to Dirac's equation and anti-matter?*)

$$E = mc^2 \ .$$

4-force and dynamical equation for a point mass. The dynamical equation of a point mass subject to a 4-force **f** reads

$$\frac{d\mathbf{P}}{d\tau} = \mathbf{f} \ . \tag{2.1}$$

Examples:

- **Free particle** is subject to no net force, **f**,
- Point charge q in EM field is subject to **Lorentz force**, $\mathbf{f} = q\mathbf{F}\cdot\mathbf{U}$, being **F** the EM field tensor, (2.4),

$$\mathbf{F} = \nabla\mathbf{A} - (\nabla\mathbf{A})^T \ .$$

[1] This definitions naturally arise in Lagrange approach.

2.2.1 Lagrangian approach

Here variational approach to the motion of point particle in a force field $\mathbf{f}$ is derived from the weak form of the dynamical equation (2.1). In particular, the derivation is performed for a particle moving in a known EM field. The motion of a free particle immediately follows if the EM field is zero.

$$m\frac{d\mathbf{U}}{d\tau} = q\mathbf{F}\cdot\mathbf{U}$$

The equation of motion can be manipulated to get Lagrange equations

$$\begin{aligned}
\mathbf{0} &= m\frac{d\mathbf{U}}{d\tau} - q\nabla\mathbf{A}\cdot\mathbf{U} + q\left(\nabla\mathbf{A}\right)^T\cdot\mathbf{U} = \\
&= \frac{d}{d\tau}\left[m\mathbf{U} + q\mathbf{A}(\mathbf{X}(\tau))\right] - \nabla\left(q\mathbf{A}\cdot\mathbf{U}\right) = (1) \\
&= \frac{d}{d\tau}\left[\mathbf{E}^\alpha\frac{\partial}{\partial U^\alpha}\left(mc\sqrt{\mathbf{U}\cdot\mathbf{U}} + q\mathbf{A}(\mathbf{X}(\tau))\cdot\mathbf{U}\right)\right] - \mathbf{E}^\alpha\frac{\partial}{\partial X^\alpha}\left(mc\sqrt{\mathbf{U}\cdot\mathbf{U}} + q\mathbf{A}\cdot\mathbf{U}\right) = (2) \\
&= \mathbf{E}^\alpha\left[\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial X'^\alpha} - \frac{\partial\mathcal{L}}{\partial X^\alpha}\right] = \\
&= \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\mathbf{X}'} - \frac{\partial\mathcal{L}}{\partial\mathbf{X}}
\end{aligned} \tag{2.2}$$

since (1) $\mathbf{U} = \mathbf{E}^\alpha U_\alpha = c\mathbf{E}^\alpha\frac{\partial}{\partial U^\alpha}\sqrt{U^\beta U_\beta}$, see (2.3), and $\sqrt{\mathbf{U}\cdot\mathbf{U}}$ independent from $\mathbf{X}$, and equal to c^2 and (2) with the definition of the **Lagrangian function**

$$\mathcal{L}(\mathbf{X}'(\tau),\mathbf{X}(\tau),\tau) := -mc\sqrt{\mathbf{X}'(\tau)\cdot\mathbf{X}'(\tau)} - q\mathbf{A}(\mathbf{X}(\tau))\cdot\mathbf{X}'(\tau)$$

Proof of the relation used in (1)

$$\begin{aligned}
\mathbf{E}^\alpha\frac{\partial}{\partial U^\alpha}\sqrt{U^\beta U_\beta} &= \mathbf{E}^\alpha\frac{\partial}{\partial U^\alpha}\sqrt{g_{\beta\gamma}U^\beta U^\gamma} = \\
&= \mathbf{E}^\alpha\frac{1}{2\sqrt{U^\beta U_\beta}}\left(g_{\beta\gamma}\delta^\beta_\alpha U^\gamma + g_{\beta\gamma}U^\beta\delta^\gamma_\alpha\right) = \\
&= \mathbf{E}^\alpha\frac{1}{2\sqrt{U^\beta U_\beta}}\left(2U_\alpha\right) = \\
&= \mathbf{E}^\alpha\frac{U_\alpha}{\sqrt{U^\beta U_\beta}} = \\
&= \frac{\mathbf{U}}{\sqrt{\mathbf{U}\cdot\mathbf{U}}} = \frac{\mathbf{U}}{c}\,.
\end{aligned} \tag{2.3}$$

since $\mathbf{U}\cdot\mathbf{U} = c^2$ and the components of the metric tensor are independent from the velocity.

Weak form of the problem is derived from the Lagrange equation (2.2), with dot product with arbitrary 4-vector $\mathbf{W}$,

$$0 = \mathbf{W}\cdot\left[\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\mathbf{X}'} - \frac{\partial\mathcal{L}}{\partial\mathbf{X}}\right]\,.$$

and integrating over an interval of proper time and using integration by parts,

$$\begin{aligned}
0 &= \int_{\tau=\tau_0}^{\tau_1}\mathbf{W}\cdot\left[\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\mathbf{X}'} - \frac{\partial\mathcal{L}}{\partial\mathbf{X}}\right]d\tau = \\
&= \left[\mathbf{W}(\tau)\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{X}'}(\mathbf{X}'(\tau),\mathbf{X}(\tau),\tau)\right]\Big|_{\tau_0}^{\tau_1} - \int_{\tau=\tau_0}^{\tau_1}\left[\mathbf{W}'\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{X}'} + \mathbf{W}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{X}}\right]d\tau\,.
\end{aligned}$$

With the choice of the test function $\mathbf{W}(\tau)$ as the variation of the position of the particle, $\mathbf{W} = \delta\mathbf{X}$ just to remember its variational nature, with given values at τ_0, τ_1 and thus $\mathbf{W}(\tau_0) = \mathbf{W}(\tau_1) = 0$, the weak form of the dynamical equations is thus recast as the stationary condition of an action functional in a **variational principle**,

$$0 = -\delta S = -\delta \int_{\tau=\tau_0}^{\tau_1} \mathcal{L}\left(\mathbf{X}'(\tau), \mathbf{X}(\tau), \tau\right)\, d\tau\ .$$

todo *Repeat the process using generalized coordinates* $q^k(\tau)$, $\mathbf{X}(q^k(\tau), \tau)\dots$

Lagrange equations and variational principle with generalized coordinates

Choosing the test function as it's usually done in deriving Lagrange formulation of mechanics from the strong formulation, after writing the position and its velocity of the particle as a function of generalized coordinates q^k,

$$\begin{aligned}
&\mathbf{X}\left(q^k(\tau), \tau\right) \\
&\mathbf{U}\left(q^k(\tau), \tau\right) = \frac{d\mathbf{X}}{d\tau} = q^{k\prime}(\tau) \underbrace{\frac{\partial \mathbf{X}}{\partial q^k}}_{=\frac{\partial \mathbf{X}'}{\partial q^{k\prime}}} + \frac{\partial \mathbf{X}}{\partial \tau}
\end{aligned}$$

With the particular choice of the test function, $\mathbf{W} = w^k(\tau) \dfrac{\partial \mathbf{X}}{\partial q^k}(q^l(\tau), \tau)$

Variational principle as a function of time. Using the relation between proper time τ and the observer time t, $d\tau = \gamma^{-1}\, dt = \sqrt{1 - \frac{|\vec{v}|^2}{c^2}} dt$,

$$\begin{aligned}
S &= \int_{\tau=\tau_0}^{\tau_1} \mathcal{L}\left(\mathbf{X}'(\tau), \mathbf{X}(\tau), \tau\right)\, d\tau = \\
&= \int_{t=t_0}^{t_1} \mathcal{L}\left(\mathbf{X}'(\tau), \mathbf{X}(\tau), \tau\right)\, \gamma^{-1}\, dt = \\
&= \int_{t=t_0}^{t_1} \sqrt{1 - \frac{|\vec{v}|^2}{c^2}} \left(-mc^2 - \gamma q \phi(\vec{r}, t) + \gamma q \vec{a}(\vec{r}, t) \cdot \vec{v}(t)\right)\, dt = \\
&= \int_{t=t_0}^{t_1} \left(-\sqrt{1 - \frac{|\vec{v}|^2}{c^2}} mc^2 - q\phi(\vec{r}, t) + q\vec{a}(\vec{r}, t) \cdot \vec{v}(t)\right)\, dt = \\
&= \int_{t=t_0}^{t_1} L\left(\vec{v}(t), \vec{r}(t), t\right)\, dt
\end{aligned}$$

Three-dimensional momentum can be defined as the partial derivative of the Lagrangian function L w.r.t. the velocity $\vec{v}$,

$$\vec{p} := \frac{\partial L}{\partial \vec{v}} = \gamma \frac{\vec{v}}{c^2} mc^2 + q\vec{a} = \gamma m \vec{v} + q\vec{a}$$

The Hamiltoninan function reads

$$\begin{aligned}
H &= \vec{v} \cdot \frac{\partial L}{\partial \vec{v}} - L = \\
&= \gamma m |\vec{v}|^2 + q\vec{a} \cdot \vec{v} + \gamma^{-1} mc^2 + q\phi - q\vec{a} \cdot \vec{v} = \\
&= \gamma m |\vec{v}|^2 \left[1 + \left(1 - \frac{|\vec{v}|^2}{c^2}\right) \frac{c^2}{|\vec{v}|^2}\right] + q\phi = \\
&= \gamma mc^2 + q\phi\ .
\end{aligned}$$

The expressions of the Hamiltonian function (divided by the light speed) and the three-dimensional momentum can be recognized to be respectively the time and space components of 4-momentum,

$$\mathbf{P} = m\mathbf{U} + q\mathbf{A} \ ,$$

remembering the expression of the contravariant components of the 4-velocity and the 4-EM potential,

$$[\mathbf{P}]^\alpha = \left(\gamma mc + q\frac{\phi}{c}, \, \gamma m\vec{v} + q\vec{a} \right) = \left(\frac{H}{c}, \, \vec{p} \right)$$

2.3 Electromagnetism

2.3.1 Classical electromagnetic theory

Maxwell equations

Maxwell equations read

$$\begin{cases} \nabla \cdot \vec{d} = \rho_f \\ \nabla \times \vec{e} + \partial_t \vec{b} = \vec{0} \\ \nabla \cdot \vec{b} = 0 \\ \nabla \times \vec{h} - \partial_t \vec{d} = \vec{j}_f \end{cases}$$

or in vacuum, with $\rho_f = \rho$, $\vec{j} = \vec{j}_f$, $\vec{d} = \varepsilon_0 \vec{e}$, $\vec{b} = \mu_0 \vec{h}$

$$\begin{cases} \nabla \cdot \vec{e} = \frac{\rho}{\varepsilon_0} \\ \nabla \times \vec{e} + \partial_t \vec{b} = \vec{0} \\ \nabla \cdot \vec{b} = 0 \\ \nabla \times \vec{b} - \mu_0 \varepsilon_0 \partial_t \vec{e} = \mu_0 \vec{j} \end{cases}$$

Electromagnetic potentials

The electromagnetic field can be written in terms of the electromagnetic potentials

$$\begin{cases} \vec{b} = \nabla \times \vec{a} \\ \vec{e} = -\partial_t \vec{a} - \nabla \varphi \end{cases}$$

Lorentz force

A particle in motion in a electromagnetic field is subject to Lorentz force. In classical electromagnetism, the expression of Lorentz force reads

$$\vec{F} = q \left(\vec{e} - \vec{b} \times \vec{v} \right) \ ,$$

whose power is

$$\vec{v} \cdot \vec{F} = \vec{v} \cdot q \left(\vec{e} - \vec{b} \times \vec{v} \right) = q \vec{v} \cdot \vec{e} \ .$$

2.3.2 Electromagnetic potential

$$\begin{cases} \vec{b} = \nabla \times \vec{a} \\ \vec{e} = -\partial_t \vec{a} - \nabla \varphi \end{cases}$$

$$\mathbf{A} = \mathbf{E}_\alpha A^\alpha = \frac{\varphi}{c} \mathbf{E}_0 + \vec{a}$$

$$\boldsymbol{\nabla} \mathbf{A} = \left(\mathbf{E}^\alpha \frac{\partial}{\partial X^\alpha} \right) \left(A^\beta \mathbf{E}_\beta \right) = \frac{\partial A^\beta}{\partial X^\alpha} \mathbf{E}^\alpha \otimes \mathbf{E}_\beta = g^{\alpha\gamma} \frac{\partial A^\beta}{\partial X^\alpha} \mathbf{E}_\gamma \otimes \mathbf{E}_\beta \, .$$

whose components may be collected in a 2-dimensional array (first index for rows, second index for columns),

$$(\nabla \mathbf{A})_\alpha{}^\beta = \frac{\partial A^\beta}{\partial X^\alpha} = \begin{bmatrix} c^{-2} \partial_t \varphi & c^{-1} \partial_x \varphi & c^{-1} \partial_y \varphi & c^{-1} \partial_z \varphi \\ c^{-1} \partial_t a_x & \partial_x a_x & \partial_y a_x & \partial_z a_x \\ c^{-1} \partial_t a_y & \partial_x a_y & \partial_y a_y & \partial_z a_y \\ c^{-1} \partial_t a_z & \partial_x a_z & \partial_y a_z & \partial_z a_z \end{bmatrix}$$

or covariant-covariant coomponents,

$$(\nabla \mathbf{A})_{\alpha\beta} = \frac{\partial A_\beta}{\partial X^\alpha} = \frac{\partial A^\gamma}{\partial X^\alpha} g_{\gamma\beta} = \begin{bmatrix} c^{-2} \partial_t \varphi & -c^{-1} \partial_x \varphi & -c^{-1} \partial_y \varphi & -c^{-1} \partial_z \varphi \\ c^{-1} \partial_t a_x & -\partial_x a_x & -\partial_y a_x & -\partial_z a_x \\ c^{-1} \partial_t a_y & -\partial_x a_y & -\partial_y a_y & -\partial_z a_y \\ c^{-1} \partial_t a_z & -\partial_x a_z & -\partial_y a_z & -\partial_z a_z \end{bmatrix}$$

or contravariant-contravariant coomponents,

$$(\nabla \mathbf{A})^{\alpha\beta} = \frac{\partial A^\beta}{\partial X_\alpha} = g_{\alpha\gamma} \frac{\partial A^\beta}{\partial X^\gamma} = \begin{bmatrix} c^{-2} \partial_t \varphi & c^{-1} \partial_x \varphi & c^{-1} \partial_y \varphi & c^{-1} \partial_z \varphi \\ -c^{-1} \partial_t a_x & -\partial_x a_x & -\partial_y a_x & -\partial_z a_x \\ -c^{-1} \partial_t a_y & -\partial_x a_y & -\partial_y a_y & -\partial_z a_y \\ -c^{-1} \partial_t a_z & -\partial_x a_z & -\partial_y a_z & -\partial_z a_z \end{bmatrix}$$

The electromagnetic field tensor is defined as the anti-symmetric part of the gradient of the 4-electromagnetic potential,

$$\mathbf{F} = \left[\boldsymbol{\nabla} \mathbf{A} - (\boldsymbol{\nabla} \mathbf{A})^T \right] \tag{2.4}$$

whose components may be collected in a 2-dimensional array (first index for rows, second index for columns),

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -\frac{\underline{\underline{e}}^T}{c} \\ \frac{\underline{\underline{e}}}{c} & \underline{b}_\times \end{bmatrix} \qquad , \qquad F_{\alpha\beta} = \begin{bmatrix} 0 & \frac{\underline{\underline{e}}^T}{c} \\ -\frac{\underline{\underline{e}}}{c} & \underline{b}_\times \end{bmatrix}$$

$$F_\alpha{}^\beta = \begin{bmatrix} 0 & -\frac{\underline{\underline{e}}^T}{c} \\ -\frac{\underline{\underline{e}}}{c} & -\underline{b}_\times \end{bmatrix} \qquad , \qquad F^\alpha{}_\beta = \begin{bmatrix} 0 & \frac{\underline{\underline{e}}^T}{c} \\ \frac{\underline{\underline{e}}}{c} & -\underline{b}_\times \end{bmatrix}$$

2.3.3 Electromagnetic field and electromagnetic field equations

The pair of Maxwell equations

$$\begin{cases} \rho_f = \nabla \cdot \vec{d} \\ \vec{j}_f = -\partial_t \vec{d} + \nabla \times \vec{h} \end{cases}$$

can be re-written in 4-formalism, using 4-gradient in Cartesian coordinates

$$\boldsymbol{\nabla} = \mathbf{E}^\alpha \frac{\partial}{\partial X^\alpha} = \mathbf{E}_0 \frac{\partial}{c \partial t} + \mathbf{E}_i \frac{\partial}{\partial x^i} = \mathbf{E}_0 \frac{\partial}{c \partial t} + \nabla \, ,$$

and the definition of the 4-current density vector

$$\mathbf{J} = J^\alpha \mathbf{E}_\alpha = c\rho\,\mathbf{E}_0 + \vec{j}$$

so that

$$c\rho\mathbf{E}_0 + \vec{j} = \boldsymbol{\nabla}\cdot\mathbf{F} = \boldsymbol{\nabla}\cdot\left[(0\,\mathbf{E}_0 + c\vec{d})\otimes\mathbf{E}_0 + (-\mathbf{E}_0 c\vec{d} + \vec{h}_\times)\right] ,$$

with the displacement field tensor,

$$\mathbf{D} = D^{\alpha\beta}\,\mathbf{E}_\alpha\,\mathbf{E}_\beta ,$$

with components (rows for the first index, columns for the second index)

$$D^{\alpha\beta} = \begin{bmatrix} 0 & -cd_x & -cd_y & -cd_z \\ cd_x & 0 & -h_z & h_y \\ cd_y & h_z & 0 & -h_x \\ cd_z & -h_y & h_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -c\underline{d}^T \\ c\underline{d} & \underline{h}_\times \end{bmatrix} .$$

The pair of Maxwell equations

$$\begin{cases} \nabla\cdot\vec{b} = 0 \\ \partial_t\vec{b} + \nabla\times\vec{e} = \vec{0} \end{cases}$$

can be re-written in 4-formalism as

$$0 = \partial_\mu F_{\eta\xi} + \partial_\eta F_{\xi\mu} + \partial_\xi F_{\mu\eta}$$

Among these $64 = 4^3$ equations, there are only 4 independent equations.

- If 2 indices are the same, the corresponding equation is the identity $0 = 0$. As an example, if $\mu = \eta$

  $$0 = \partial_\mu F_{\mu\xi} + \partial_\mu \underbrace{F_{\xi\mu}}_{-F_{\mu\xi}} + \partial_\xi \underbrace{F_{\mu\mu}}_{=0} = 0 ,$$

  thus only combinations with different indices may provide some information.

- Given the ordered set of indices (μ, η, ξ), switching a pair of indices provides the same equation. As an example, switching μ and η

  $$\begin{aligned} 0 &= \partial_\eta F_{\mu\xi} + \partial_\mu F_{\xi\eta} + \partial_\xi F_{\eta\mu} = \\ &= \partial_\eta(-F_{\xi\mu}) + \partial_\mu(-F_{\eta\xi}) + \partial_\xi(-F_{\mu\eta}) . \end{aligned}$$

- Thus, only 4 combination of different indices, without taking order into account, provide independent information

  $$\begin{aligned} (1,2,3): &\quad 0 = \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \partial_x(-b_x) + \partial_y(-b_y) + \partial_z(-b_z) \\ (2,3,0): &\quad 0 = \partial_2 F_{30} + \partial_3 F_{02} + \partial_0 F_{23} = \partial_y\left(-\frac{e_z}{c}\right) + \partial_z\left(\frac{e_y}{c}\right) + \partial_{ct}(-b_x) \\ (3,0,1): &\quad 0 = \partial_3 F_{01} + \partial_0 F_{13} + \partial_1 F_{30} = \partial_z\left(\frac{e_x}{c}\right) + \partial_{ct}(-b_y) + \partial_x\left(-\frac{e_z}{c}\right) \\ (0,1,2): &\quad 0 = \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = \partial_{ct}(-b_z) + \partial_x\left(-\frac{e_y}{c}\right) + \partial_y\left(\frac{e_x}{c}\right) \end{aligned}$$

  i.e.

  $$\begin{aligned} (1,2,3): &\quad 0 = -\nabla\cdot\vec{b} \\ (2,3,0): &\quad 0 = -\frac{1}{c}\left[\partial_t b_x + (\partial_y e_z - \partial_z e_y)\right] \\ (3,0,1): &\quad 0 = -\frac{1}{c}\left[\partial_t b_y + (\partial_z e_x - \partial_x e_z)\right] \\ (0,1,2): &\quad 0 = -\frac{1}{c}\left[\partial_t b_z + (\partial_x e_y - \partial_y e_x)\right] \end{aligned}$$

i.e.

$$\begin{cases} 0 = \nabla \cdot \vec{b} \\ \vec{0} = \partial_t \vec{b} + \nabla \times \vec{e} \end{cases}$$

2.3.4 Point particle in electromagnetic field

Lorentz 4-force acting on a point charge of electric charge charge q reads

$$\mathbf{f} = \mathbf{F} \cdot \mathbf{J} = q \, \mathbf{F} \cdot \mathbf{U} \, .$$

so that the dynamical equation reads

$$m \, \mathbf{X}'' = q \, \mathbf{F} \cdot \mathbf{X}'$$

2.3.5 Energy balance

Continuity

$$\nabla \cdot \mathbf{J} = 0$$

Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \mathbf{J} \\ \nabla \cdot (\boldsymbol{\varepsilon} : \mathbf{F}) &= \mathbf{0} \end{aligned}$$

Energy and momentum balance equation of the EM field using 3-dimensional formalism,

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla \cdot \vec{s} &= -\vec{e} \cdot \vec{j}^f \\ \frac{\partial \vec{s}}{\partial t} + \nabla \cdot \left\{ c^2 \left[\frac{1}{2} \left(\vec{d} \cdot \vec{e} + \vec{h} \cdot \vec{b} \right) \mathbb{I} - \left(\vec{d} \otimes \vec{e} + \vec{h} \otimes \vec{b} \right) \right] \right\} &= -c^2 (\rho^f \vec{e} - \vec{b} \times \vec{j}^f) \\ \frac{1}{c} \frac{\partial u}{\partial t} + \nabla \cdot \left(\frac{\vec{s}}{c} \right) &= -\frac{\vec{e}}{c} \cdot \vec{j}^f \\ \frac{1}{c} \frac{\partial}{\partial t} \frac{\vec{s}}{c} + \nabla \cdot \left[\frac{1}{2} \left(\vec{d} \cdot \vec{e} + \vec{h} \cdot \vec{b} \right) \mathbb{I} - \left(\vec{d} \otimes \vec{e} + \vec{h} \otimes \vec{b} \right) \right] &= -\rho^f c \frac{\vec{e}}{c} + \vec{b} \times \vec{j}^f \end{aligned} \tag{2.5}$$

The governing equations of energy and momentum of the EM field can be recast in 4-dimensional formalism,

$$\boldsymbol{\nabla} \cdot \mathbf{T} = -\mathbf{F} \cdot \mathbf{J}$$

with the **energy-momentum-stress tensor**, **T**,

$$\begin{aligned} T^{\alpha\beta} &= \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} D^{\mu\nu} - F^{\alpha\nu} D^{\beta}{}_{\nu} = \\ &= \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} D^{\mu\nu} - F^{\alpha\nu} g^{\beta\mu} D_{\mu\nu} = \\ &= \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} D^{\mu\nu} + F^{\alpha\nu} D_{\nu\mu} g^{\mu\beta} \end{aligned} \tag{2.6}$$

Term $\mathbf{g} \; \mathbf{F} : \mathbf{D}$

$$\mathbf{F} : \mathbf{D} = F_{\mu\nu} D^{\mu\nu} = \begin{bmatrix} 0 & \frac{\underline{e}^T}{c} \\ -\frac{\underline{e}}{c} & \underline{b}_\times \end{bmatrix} : \begin{bmatrix} 0 & -c\underline{d}^T \\ c\underline{d} & \underline{h}_\times \end{bmatrix} = -2\underline{e}^T \underline{d} + 2\underline{b}^T \underline{h} = 2\left(-\vec{e} \cdot \vec{d} + \vec{b} \cdot \vec{h}\right) \ . \tag{2.7}$$

Term $F^{\alpha\nu} D^\beta{}_\nu$

$$F^{\alpha\nu} D^\beta{}_\nu = \begin{bmatrix} 0 & -\frac{\underline{e}^T}{c} \\ \frac{\underline{e}}{c} & \underline{b}_\times \end{bmatrix} \begin{bmatrix} 0 & c\underline{d}^T \\ c\underline{d} & -\underline{h}_\times^T \end{bmatrix} = \begin{bmatrix} -\underline{e}^T \underline{d} & \frac{(\underline{h}\times\underline{e})^T}{c} \\ c\underline{b} \times \underline{d} & \underline{e}\underline{d}^T + \underline{b}_\times \underline{h}_\times \end{bmatrix} \tag{2.8}$$

being

$$\underline{e}^T \underline{h}_\times^T = e_i \varepsilon_{kji} h_j = \left(\vec{h} \times \vec{e}\right)_k$$

$$[\underline{b}_\times \underline{h}_\times]_{im} = \varepsilon_{ijk} b_j \varepsilon_{klm} h_l = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, b_j h_l = h_i b_m - b_l h_l \delta_{im} = \underline{h}\underline{b}^T - \underline{b}^T \underline{h} \underline{\underline{I}}_3 \ .$$

Energy-stress tensor

Putting together the expression (2.7) of term $\frac{1}{4} g^{\alpha\beta} F_{\mu\nu} D^{\mu\nu}$ and (2.8) of term $F^{\alpha\nu} D^\beta{}_\nu$, with $\underline{d} = \varepsilon \underline{e}$, $\underline{b} = \mu \underline{h}$ (**todo** do energy balance for all charges and currents and spearating contributions of free and bound charges and currents)

$$\begin{aligned}
T^{\alpha\beta} &= \frac{1}{4} \begin{bmatrix} 2(-\underline{e}^T\underline{d} + \underline{b}^T\underline{h}) & \underline{0}^T \\ \underline{0} & -2(-\underline{e}^T\underline{d} + \underline{b}^T\underline{h})\underline{\underline{I}} \end{bmatrix} - \begin{bmatrix} -\underline{e}^T \underline{d} & \frac{(\underline{e}\times\underline{h})^T}{c} \\ c\underline{b} \times \underline{d} & \underline{e}\underline{d}^T + \underline{b}_\times \underline{h}_\times \end{bmatrix} = \\
&= \begin{bmatrix} \frac{1}{2}\left(\underline{e}^T\underline{d} + \underline{b}^T\underline{h}\right) & \frac{(\underline{e}\times\underline{h})^T}{c} \\ \frac{\underline{e}\times\underline{h}}{c} & \frac{1}{2}(\underline{e}^T\underline{d} + \underline{b}^T\underline{h})\underline{\underline{I}}_3 - \underline{e}\underline{d}^T - \underline{b}\underline{h}^T \end{bmatrix} = \\
&= \begin{bmatrix} u & \frac{\vec{s}}{c} \\ \frac{\vec{s}}{c} & u\mathbb{I} - \vec{e} \otimes \vec{d} - \vec{b} \otimes \vec{h} \end{bmatrix}
\end{aligned}$$

Energy-stress equation in 4-dimensional formalism

Energy and momentum balance equations (2.5) can be recast using 4-dimensional formalism,

$$\begin{gathered}
\nabla \cdot \mathbf{T} = -\mathbf{F} \cdot \mathbf{J} \\
\frac{\partial T^{\nu\mu}}{\partial X^\nu} = -F^{\mu\nu} J_\nu
\end{gathered}$$

as

$$F^{\mu\nu} J_\nu = \begin{bmatrix} 0 & -\frac{\underline{e}^T}{c} \\ \frac{\underline{e}}{c} & \underline{b}_\times \end{bmatrix} \begin{bmatrix} \rho c \\ -\underline{j} \end{bmatrix} = \begin{bmatrix} \frac{1}{c}\underline{e}^T \underline{j} \\ \rho \underline{e} - \underline{b}_\times \underline{j} \end{bmatrix}$$

$$\begin{aligned}
0 : \quad & \frac{\partial T^{\nu 0}}{\partial X^\nu} = -F^{0\nu} J_\nu \quad && \frac{1}{c}\frac{\partial}{\partial t} u + \frac{\partial}{\partial x_k} \frac{s_k}{c} = -\frac{1}{c} e_k j_k \\
1:3 : \quad & \frac{\partial T^{\nu i}}{\partial X^\nu} = -F^{i\nu} J_\nu \quad && \frac{1}{c}\frac{\partial}{\partial t} \frac{s_i}{c} + \frac{\partial \sigma_{ki}}{\partial x_k} = -\rho e_i + \varepsilon_{ijk} b_j j_k
\end{aligned}$$

Relativity under Lorentz's transformation. Components of the description of two inertial observers in relative motion, with aligned Cartesian space coordinates, are related by Lorentz's transformation

Energy-momentum-stress tensor

todo *Complete*

Contravariant (Cartesian in space) components of the energy-momentu-stress tensor can be collected in a symmetric matrix

$$\mathbf{T}' = \mathbf{\Lambda}\mathbf{T}\mathbf{\Lambda}^T$$

$$\mathbf{T} = \begin{bmatrix} u & \frac{\mathbf{s}^T}{c} \\ \frac{\mathbf{s}}{c} & \boldsymbol{\sigma} \end{bmatrix}$$

$$\mathbf{\Lambda} = \begin{bmatrix} \gamma & -\gamma\mathbf{v}^T \\ -\gamma\mathbf{v} & \mathbf{I}_3 + (\gamma - 1)\hat{\mathbf{v}}\hat{\mathbf{v}}^T \end{bmatrix} = \gamma \begin{bmatrix} 1 & -\mathbf{v}^T \\ -\mathbf{v} & \hat{\mathbf{v}}\hat{\mathbf{v}}^T \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{I} - \hat{\mathbf{v}}\hat{\mathbf{v}}^T \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\mathbf{v}^T \\ -\gamma\mathbf{v} & \mathbf{P} \end{bmatrix}$$

$$\begin{aligned} \mathbf{T}' &= \begin{bmatrix} \gamma & -\gamma\mathbf{v}^T \\ -\gamma\mathbf{v} & \mathbf{P} \end{bmatrix} \begin{bmatrix} u & \frac{\mathbf{s}^T}{c} \\ \frac{\mathbf{s}}{c} & \boldsymbol{\sigma} \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\mathbf{v}^T \\ -\gamma\mathbf{v} & \mathbf{P} \end{bmatrix} = \\ &= \begin{bmatrix} \gamma u - \frac{\gamma}{c}\mathbf{s}^T\mathbf{v} & \frac{\gamma}{c}\mathbf{s}^T - \gamma\mathbf{v}^T\boldsymbol{\sigma} \\ -\gamma u\mathbf{v} + \mathbf{P}\frac{\mathbf{s}}{c} & -\frac{\gamma}{c}\mathbf{v}\mathbf{s}^T + \mathbf{P}\boldsymbol{\sigma} \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\mathbf{v}^T \\ -\gamma\mathbf{v} & \mathbf{P} \end{bmatrix} = \\ &= \begin{bmatrix} \gamma^2 u - 2\frac{\gamma^2}{c}\mathbf{s}^T\mathbf{v} + \gamma^2\mathbf{v}^T\boldsymbol{\sigma}\mathbf{v} & -\gamma^2 u\mathbf{v}^T + \frac{\gamma^2}{c}\mathbf{s}^T\mathbf{v}\mathbf{v}^T + \frac{\gamma}{c}\mathbf{s}^T\mathbf{P} - \gamma\mathbf{v}^T\boldsymbol{\sigma}\mathbf{P} \\ -\gamma^2 u\mathbf{v} + \frac{\gamma}{c}\mathbf{P}\mathbf{s} + \frac{\gamma^2}{c}\mathbf{v}\mathbf{s}^T\mathbf{v} - \gamma\mathbf{P}\boldsymbol{\sigma}\mathbf{v} & \gamma^2 u\mathbf{v}\mathbf{v}^T - \frac{\gamma}{c}\left(\mathbf{P}\mathbf{s}\mathbf{v}^T + \mathbf{v}\mathbf{s}^T\mathbf{P}\right) + \mathbf{P}\boldsymbol{\sigma}\mathbf{P} \end{bmatrix} \end{aligned}$$

CHAPTER

THREE

INERTIAL REFERENCE FRAMES AND LORENTZ'S TRANSFORMATIONS

Equations of physics as seen by inertial reference frames (**todo** *but what's an inertial reference frame?*) have the same expressions. Change of observer can be related to change of coordinates (and choice of basis vectors, in special relativity) used to describe a physical phenomenon, that is invariant to this change for its very nature: reality doesn't change if observed from different point of view (if observations don't interact with reality itself, at least). Proper mathematical tools for dealing with invariance are vectors and tensors in general.

Using 2 different sets of coordinates associated with constant and uniform basis vectors $\mathbf{E}_\alpha$, $\mathbf{E}'_\beta$ a vector can be written as

$$\mathbf{V} = V^\alpha \mathbf{E}_\alpha = V'^\beta \mathbf{E}'_\beta \ .$$

Lorentz transformation describe the change of description between two inertial reference frames.

3.1 Lorentz's transformations

3.1.1 Lorentz's transformations in standard configuration

Standard configuration is defined for two observers with Cartesian bases for the space components, with the axes aligned, and with relative motion along x, x' axes.

…**todo** *derivation…see material for high school: Special Relativity:Relativity and Lorentz's transformations*

$$\begin{aligned} ct' &= \gamma(ct - vx) \\ x' &= \gamma(-vct + x) \\ y' &= y \\ z' &= z \end{aligned} \ ,$$

with $\gamma := \dfrac{1}{\sqrt{1-v^2}}$, so that the coordinates transformations can be written using matrix of change of coordinates

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$X'^\alpha = \Lambda^\alpha_\beta X^\beta \ .$$

This is the same transformation of the components of all the vectors of the 4-dimensional space as seen by two different inertial observers

3.1.2 Lorentz's transformation in general configuration

Beside change of origin of the coordinates, general transformation can be derived composing Lorentz's transformations in stanrdard configuration (the only one derived so far), and rotations of the space coordinates.

As an example, the general expression of Lorentz's transformation between two inertial reference frames with general relative (space) velocity $\vec{v}$ - and aligned axes - can be derived introducing two intermediate inertial reference frames: reference $Otxyz$ and $O't'x'y'z'$ are the two refence frames we'd like to link with a Lorentz transformation; $O_a t_a x_a y_a z_a$ is a reference frame at rest w.r.t. $Otxyz$ with the x_a axis with the same direction of the relative (space) velocity $\vec{v}$, $O_b t_b x_b y_a z_b$ is in standard configuration with O_a. The transformation of coordinates reads

$$\begin{aligned}
\begin{bmatrix} x'_0 \\ \mathbf{r}' \end{bmatrix} &= \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R}^T \end{bmatrix} \begin{bmatrix} x_{b,0} \\ \mathbf{x}_b \end{bmatrix} = \\
&= \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R}^T \end{bmatrix} \begin{bmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x_{a,0} \\ \mathbf{x}_a \end{bmatrix} = \\
&= \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R}^T \end{bmatrix} \left(\begin{bmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma - 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix} ,
\end{aligned}$$

and the two contributions give

$$\begin{aligned}
\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R}^T \end{bmatrix} \begin{bmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma - 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R} \end{bmatrix} &= \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R}^T \end{bmatrix} \begin{bmatrix} \gamma & -\gamma v \mathbf{R}_{1,:} \\ -\gamma v & (\gamma-1)\mathbf{R}_{1,:} \\ \mathbf{0}_{2,1} & \mathbf{0}_{2,4} \end{bmatrix} = \\
&= \begin{bmatrix} \gamma & -\gamma v \mathbf{R}_{1,:} \\ -\gamma v \mathbf{R}^T_{1,:} & (\gamma-1)\mathbf{R}^T_{1,:}\mathbf{R}_{1,:} \end{bmatrix} = \\
&= \begin{bmatrix} \gamma & -\gamma R_{11} v & -\gamma R_{12} v & -\gamma R_{13} v \\ -\gamma R_{11} v & (\gamma-1)R_{11}R_{11} & (\gamma-1)R_{11}R_{12} & (\gamma-1)R_{11}R_{13} \\ -\gamma R_{12} v & (\gamma-1)R_{12}R_{11} & (\gamma-1)R_{12}R_{12} & (\gamma-1)R_{12}R_{13} \\ -\gamma R_{13} v & (\gamma-1)R_{13}R_{11} & (\gamma-1)R_{13}R_{12} & (\gamma-1)R_{13}R_{13} \end{bmatrix} .
\end{aligned}$$

$$\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R}^T \end{bmatrix} \begin{bmatrix} 0 & \\ & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R} \end{bmatrix} = \begin{bmatrix} 0 & \\ & \mathbf{I}_3 \end{bmatrix}$$

Here, components $(R_{11}v, R_{12}v, R_{13}v)$ are the components of velocity $\vec{v}$ as seen by observer O, and thus the matrix for coordinate transformation reads

$$\begin{aligned}
\mathbf{\Lambda} &= \begin{bmatrix} 0 & \\ & \mathbf{I}_3 \end{bmatrix} + \begin{bmatrix} \gamma & -\gamma \mathbf{v}^T \\ -\gamma \mathbf{v} & (\gamma-1)\hat{\mathbf{v}}\hat{\mathbf{v}}^T \end{bmatrix} = \\
&= \begin{bmatrix} \gamma & -\gamma \mathbf{v}^T \\ -\gamma \mathbf{v} & \mathbf{I}_3 + (\gamma-1)\hat{\mathbf{v}}\hat{\mathbf{v}}^T \end{bmatrix}
\end{aligned} \tag{3.1}$$

and the transformation of coordinates can be written as

$$\begin{cases}
ct' & = \gamma(ct - \vec{v} \cdot \vec{r}) \\
\vec{r}' & = -\gamma ct \vec{v} + (\gamma - 1)\, \hat{v} \otimes \hat{v} \cdot \vec{r} + \vec{r} = \\
& = -\gamma ct \vec{v} + \underbrace{[\mathbb{I} - \hat{v} \otimes \hat{v}]}_{\mathbb{P}_{\perp \hat{v}}} \cdot \vec{r} + \gamma \underbrace{\hat{v} \otimes \hat{v}}_{\mathbb{P}_{\parallel \hat{v}}} \cdot \vec{r} \, ,
\end{cases} \tag{3.2}$$

having introduced the two orthogonal projectors in directions parallel and orthogonal to the direction of the velocity, $\hat{v}$.

Example (Standard configuration, as a special case.)

If $\vec{v} = v\hat{x}$, Lorentz's transformation for two inertial observers in starndard configuration is retrieved from the general expression of Lorentz's transformation

$$\begin{cases} ct' & = \gamma(ct - vx) \\ \vec{r}' & = -\gamma vct\hat{x} + y\hat{y} + z\hat{z} + \gamma vx\hat{x} \end{cases}$$

3.2 Transformation of vector components and vectors of the bases

$$\mathbf{V} = V^\alpha \mathbf{E}_\alpha = {V'}^\beta \mathbf{E}'_\beta$$
$${V'}^\beta = \Lambda^\beta_\alpha V^\alpha \ ,$$

being α and β the indice of columns and rows of matrix $\boldsymbol{\Lambda}$ of change of coordinates of the general Lorentz's transformation (3.1). In order to keep invariance, vectors of the bases transform with the transpose transformation, namely

$$\mathbf{V} = V^\alpha \mathbf{E}_\alpha = {V'}^\beta \mathbf{E}'_\beta = V^\alpha \underbrace{\Lambda^\beta_\alpha \mathbf{E}'_\beta}_{\mathbf{E}_\alpha}$$

$$\mathbf{E}_\alpha = \Lambda^\beta_\alpha \mathbf{E}'_\beta \ . \tag{3.3}$$

$$\left[\Lambda^{-1}\right]^\alpha_\phi \mathbf{E}_\alpha = \underbrace{[\Lambda^{-1}]^\alpha_\phi \Lambda^\beta_\alpha}_{=\delta^\beta_\phi} \mathbf{E}'_\beta$$

3.3 Transformation of tensor components

$$\mathbf{D} = D^{\alpha\beta} \mathbf{E}_\alpha \mathbf{E}_\beta = {D'}^{\phi\eta} \mathbf{E}'_\phi \mathbf{E}'_\eta =$$

and using the rule of transformation of vectors of the bases (3.3)

$${D'}^{\phi\eta} = D^{\alpha\beta} \Lambda^\phi_\alpha \Lambda^\eta_\beta \ ,$$

or using matrix notation (**todo** *avoid abuse of notation! Use underline for arrays of coomponents, bold for vectors and tensors*)

$$\mathbf{D}' = \boldsymbol{\Lambda} \mathbf{D} \boldsymbol{\Lambda}^T$$

Example (Electromagnetic field tensor)

Contravariant components of the electromagnetic field tensor

$$\begin{aligned}
\begin{bmatrix} 0 & -c\mathbf{d}'^T \\ c\mathbf{d}' & \mathbf{h}'_\times \end{bmatrix} &= \begin{bmatrix} \gamma & -\gamma\mathbf{v}^T \\ -\gamma\mathbf{v} & \mathbf{I}_3 + (\gamma-1)\hat{\mathbf{v}}\hat{\mathbf{v}}^T \end{bmatrix} \begin{bmatrix} 0 & -c\mathbf{d}^T \\ c\mathbf{d} & \mathbf{h}_\times \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\mathbf{v}^T \\ -\gamma\mathbf{v} & \mathbf{I}_3 + (\gamma-1)\hat{\mathbf{v}}\hat{\mathbf{v}}^T \end{bmatrix}^T = \\
&= \begin{bmatrix} \gamma & -\gamma\mathbf{v}^T \\ -\gamma\mathbf{v} & \mathbf{I}_3 + (\gamma-1)\hat{\mathbf{v}}\hat{\mathbf{v}}^T \end{bmatrix} \begin{bmatrix} \gamma c v d_v & -c\mathbf{d}^T - (\gamma-1)c d_v \hat{\mathbf{v}}^T \\ \gamma c\mathbf{d} - \gamma \mathbf{h}_\times \mathbf{v} & -\gamma c\mathbf{d}\mathbf{v}^T + \mathbf{h}_\times + (\gamma-1)\mathbf{h}_\times \hat{\mathbf{v}}\hat{\mathbf{v}}^T \end{bmatrix} = \\
&= \begin{bmatrix} 0 & \text{anti-sym} \\ c\mathbf{d}' & \mathbf{h}'_\times \end{bmatrix} =
\end{aligned}$$

with

$$\begin{aligned}
c\mathbf{d}' & = -\gamma^2 cv^2 \hat{\mathbf{v}} d_v + \gamma c\mathbf{d} - \gamma \mathbf{h}_\times \mathbf{v} + \gamma(\gamma-1)c\hat{\mathbf{v}}d_v = \\
& = c\gamma^2 \hat{\mathbf{v}} d_v \underbrace{(1-v^2)}_{=\gamma^{-2}} + \gamma c\mathbf{d} - \gamma\mathbf{h}\times\mathbf{v} - \gamma c\hat{\mathbf{v}} d_v = \\
& = \gamma c\mathbf{d} - \gamma\mathbf{h}\times\mathbf{v} + (1-\gamma)c\hat{\mathbf{v}}\hat{\mathbf{v}}\cdot\mathbf{d} \, ,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{h}'_\times & = & (1) \\
& = \gamma cv\hat{\mathbf{v}}\mathbf{d}^T + \gamma(\gamma-1)cvd_v\hat{\mathbf{v}}\hat{\mathbf{v}}^T + \\
& - \gamma cv\mathbf{d}\hat{\mathbf{v}} + \mathbf{h}_\times + (\gamma-1)(\mathbf{h}\times\hat{\mathbf{v}})\hat{\mathbf{v}}^T - \gamma(\gamma-1)cvd_v\hat{\mathbf{v}}\hat{\mathbf{v}}^T + (\gamma-1)\hat{\mathbf{v}}\mathbf{v}^T\mathbf{h}_\times + \mathbf{0} = & (2) \\
& = \gamma cv\left(\hat{\mathbf{v}}\mathbf{d}^T - \mathbf{d}\hat{\mathbf{v}}^T\right) + \mathbf{h}_\times + (\gamma-1)\left(\left(\mathbf{h}_\times\hat{\mathbf{v}}\right)\hat{\mathbf{v}}^T - \hat{\mathbf{v}}\left(\mathbf{h}_\times\hat{\mathbf{v}}\right)^T\right) = & (3) \\
& = \gamma cv\left(\mathbf{d}\times\hat{\mathbf{v}}\right)_\times + \mathbf{h}_\times + (\gamma-1)\left(\hat{\mathbf{v}}\times\left(\mathbf{h}\times\hat{\mathbf{v}}\right)\right)_\times
\end{aligned}$$

having (1) recognized that $\hat{\mathbf{v}}\hat{\mathbf{v}}^T\mathbf{h}_\times\hat{\mathbf{v}}\hat{\mathbf{v}}^T = \hat{\mathbf{v}}(\hat{\mathbf{v}}\cdot(\mathbf{h}\times\hat{\mathbf{v}}))\hat{\mathbf{v}} = \mathbf{0}$, (2) $\{\mathbf{v}^T\mathbf{h}_\times\}_k = v_i\varepsilon_{ijk}h_j = -\{\mathbf{h}\times\hat{\mathbf{v}}\}_k$ and (3)

$$\begin{aligned}
\left[(\mathbf{a}_\times\mathbf{b})_\times\right]_{ij} & = \varepsilon_{ikj}\varepsilon_{klm}a_l b_m = \\
& = \left(\delta_{jl}\delta_{im} - \delta_{jm}\delta_{il}\right)a_l b_m = \\
& = a_j b_i - a_i b_j = \left[\mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T\right]_{ij} \, .
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{h}' & = \mathbf{h} - \gamma c\mathbf{v}\times\mathbf{d} + (1-\gamma)(\mathbf{h}\times\hat{\mathbf{v}})\times\hat{\mathbf{v}} = \\
& = \mathbf{h} - \gamma c\mathbf{v}\times\mathbf{d} + (1-\gamma)\left(\hat{\mathbf{v}}\hat{\mathbf{v}}\cdot\mathbf{h} - \mathbf{h}\right) = \\
& = \gamma\mathbf{h} - \gamma c\mathbf{v}\times\mathbf{d} + (1-\gamma)\hat{\mathbf{v}}\hat{\mathbf{v}}\cdot\mathbf{h} \, .
\end{aligned}$$

Going back from non dimensional velocity to dimensional velocity $c\mathbf{v}\rightarrow\mathbf{v}$,

$$\begin{aligned}
\mathbf{d}' & = \gamma\left(\mathbf{d} - \frac{\mathbf{h}\times\mathbf{v}}{c^2}\right) + (1-\gamma)\hat{\mathbf{v}}\,\hat{\mathbf{v}}\cdot\mathbf{d} \\
\mathbf{h}' & = \gamma\left(\mathbf{h} - \mathbf{v}\times\mathbf{d}\right) + (1-\gamma)\hat{\mathbf{v}}\,\hat{\mathbf{v}}\cdot\mathbf{h}
\end{aligned}$$

Repeating the same process for the electromagnetic field tensor

$$[\mathbf{F}]^{\alpha\beta} = \begin{bmatrix} 0 & -\frac{\mathbf{e}^T}{c} \\ \frac{\mathbf{e}}{c} & \mathbf{b}_\times \end{bmatrix}$$

$$\begin{aligned}
\mathbf{e}' & = \gamma\left(\mathbf{e} - \mathbf{b}\times\mathbf{v}\right) + (1-\gamma)\hat{\mathbf{v}}\,\hat{\mathbf{v}}\cdot\mathbf{e} \\
\mathbf{b}' & = \gamma\left(\mathbf{b} - \frac{\mathbf{v}\times\mathbf{e}}{c^2}\right) + (1-\gamma)\hat{\mathbf{v}}\,\hat{\mathbf{v}}\cdot\mathbf{b}
\end{aligned}$$

Example (4-Current)

$$\mathbf{J}' = \mathbf{\Lambda}\mathbf{J}$$

$$\begin{aligned}
c\rho' & = \gamma c\rho - \gamma v\mathbf{j} \\
\mathbf{j}' & = -\gamma vc\rho + \gamma\mathbf{j}
\end{aligned}$$

Example (Constitutive equations)

Linear isotropic medium

$$\begin{aligned}\mathbf{d} &= \varepsilon_0 \mathbf{e} + \mathbf{p}\\ \mathbf{b} &= \mu_0 \mathbf{h} + \mu_0 \mathbf{m}\end{aligned}$$

with $c^2 = \dfrac{1}{\varepsilon_0 \mu_0}$

$$\begin{aligned}c\mathbf{d} &= \varepsilon_0 c^2 \frac{\mathbf{e}}{c} + c\mathbf{p} = \frac{1}{\mu_0}\frac{\mathbf{e}}{c} + c\mathbf{p}\\ \mathbf{h} &= \frac{1}{\mu_0}\mathbf{b} - \mathbf{m}\end{aligned}$$

the constitutive equations can be written in 4-dimensional form as

$$\mathbf{D} = \frac{1}{\mu_0}\mathbf{F} + \mathbf{P}$$

$$\begin{bmatrix} 0 & -c\mathbf{d}^T \\ c\mathbf{d} & \mathbf{h}_\times \end{bmatrix} = \frac{1}{\mu_0}\begin{bmatrix} 0 & -\mathbf{e}^T/c \\ \mathbf{e}/c & \mathbf{b}_\times \end{bmatrix} + \begin{bmatrix} 0 & -c\mathbf{p}^T \\ c\mathbf{p} & -\mathbf{m}_\times \end{bmatrix}$$

CHAPTER

FOUR

ELECTROMAGNETISM

From 3-dimensional to 4-dimensional formalism. In this section the 4-dimensional formalism is introduced, and the 4-dimensional version of governing equations of electromagnetism that naturally suits special relativity theory is derived starting from the governing equations of classical electromagnetism.

Definition of physical quantities

- 4-potential vector, $\mathbf{A} = \frac{\phi}{c}\mathbf{E}_0 + a^i\mathbf{E}_i$
- 4-current vector, $\mathbf{J} = c\rho\mathbf{E}_0 + j^i\mathbf{E}_i$
- EM field tensor,

$$\mathbf{F} = -\mathbf{E}_0 \otimes \mathbf{E}_i\frac{e_i}{c} + \frac{e_i}{c}\mathbf{E}_i \otimes \mathbf{E}_0 + \varepsilon_{ijk}b_j\mathbf{E}_i \otimes \mathbf{E}_k$$
$$\mathbf{D} = -\mathbf{E}_0 \otimes \mathbf{E}_i c d_i + d_i\mathbf{E}_i \otimes \mathbf{E}_0 + \varepsilon_{ijk}h_j\mathbf{E}_i \otimes \mathbf{E}_k$$

or

$$[F^{\alpha\beta}] = \begin{bmatrix} 0 & -\mathbf{e}^T/c \\ \mathbf{e}/c & \mathbf{b}_\times \end{bmatrix} \qquad , \qquad [D^{\alpha\beta}] = \begin{bmatrix} 0 & -c\mathbf{d}^T \\ c\mathbf{d} & \mathbf{h}_\times \end{bmatrix}$$

- 4-momentum-energy density tensor, $\mathbf{T}$

Definitions, relations and equations

- Lorentz's gauge, $\nabla \cdot \mathbf{A} = 0$
- EM field tensor, $\mathbf{F} = \nabla\mathbf{A} - (\nabla\mathbf{A})^T$
- 4-current continuity equation, $\nabla \cdot \mathbf{J} = 0$
- Maxwell's equations

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \mathbf{J}^f$$
$$\nabla \cdot (\boldsymbol{\epsilon} : \mathbf{F}) = \mathbf{0}$$

- Constitutive equations

$$\mathbf{D} = \frac{1}{\mu_0}\mathbf{F} + \mathbf{P}$$

- Wave equation for the pontential (what about wave equations for the EM field?)

$$\nabla \cdot \nabla\mathbf{A} = \mu_0\mathbf{J}$$

- Energy balance equation

$$\nabla \cdot \mathbf{T} = -\mathbf{F} \cdot \mathbf{J}$$

- Equation of motion of a point charge in a EM field,

$$m\mathbf{X}'' = q\mathbf{F} \cdot \mathbf{X}'$$

Einstein's special relativity and Lorentz's transformations for the EM quantities. After the equations are derived, Lorentz's transformations are used to discuss special relativity. Low-speed relations, used in classical electromagnetism for systems with charateristic speed $v \ll c$, are derived.

Lagrangian approach to electromagnetism in special relativity. Weak form of the equations are derived, and the Lagrangian approach to electromagnetism in special relativity is discussed: both field equations and dynamical equations of charges moving in an electromagnetic field are re-derived with a Lagrangian approach.

Part II

General Relativity

CHAPTER

FIVE

GENERAL RELATIVITY

CHAPTER

SIX

GENERAL RELATIVITY - NOTES

Part III

Statistical Mechanics

CHAPTER

SEVEN

STATISTICAL PHYSICS

CHAPTER

EIGHT

STATISTICAL PHYSICS - NOTES

8.1 Ensembles

8.2 Microcanonical ensemble

8.3 Canonical ensemble

8.4 Macrocanonical esemble

8.5 Statistics

Each of the N components of the system is in an **energy level** i. Energy level i has g_i sublevels with the same energy level.

- energy levels, E_i of each component
- occupation number N_i of level i
- **Central role of energy.** In a system macroscopically at rest, the energy of a system is the only macroscopic meaningful non-zero mechanical quantity, constant for closed and isolated systems
- **Principle of maximum uncertainty**, **maximum entropy**, **minimum information**: given a measurement of a macroscopic variable V, describing the macrostate of the system, the feasible un-observed/able microstates of the system are the microstates consistent with it: there's usually a sharp maximum of in the probability density of the micrsotates.

Given a macrostate, what's the number of ways $W(N_i; g_i)$ to get a consistent microstate? Once the expression is found, constrained optimization follows: optimization w.r.t. N_i is usually performed in the limit of $N_i \to +\infty$ (why in Fermi-Dirac distribution, obeying Pauli exclusion principle?), with the values of the macroscopic variables as constraints usually treated with Lagrange multiplier.

8.5.1 Maxwell-Boltzmann

Statistics of distinguishible components.

8.5.2 Bose-Einstein

Statistics of undistinguishable components that can be in the same (sub)level. Given the number of elementary components $\sum_i N_i = N$ and the energy $\sum_i N_i E_i = E$,

$$W_{BE,i} = \frac{(N_i + g_i - 1)!}{N_i!(g_i - 1)!} \qquad , \qquad W_{BE} = \prod_i W_{BE,i} \ . \tag{8.1}$$

Counting microstates

todo write page *Combinatorics* and add link

Most likely microstate. Instead of maximizing (8.1), the objective function is $\ln W_{BE}$, after using Stirling approximation in the limit of large N_i and g_i, $N_i! \sim \left(\frac{N_i}{e}\right)^{N_i}$. The approximate occupation number of one of the G_i sublevels of the i^{th} level of the most likely microstate is

$$n_i := \frac{N_i}{G_i} = \frac{1}{e^{\alpha + \beta E_i} - 1} \ .$$

Optimization

$$\begin{aligned}
J(N_i, \alpha, \beta) = \ln W_{BE} + \alpha \left(N - \sum_i N_i \right) + \beta \left(E - \sum_i N_i E_i \right) = \\
&= \sum_i \{ \ln(N_i + g_i - 1)! - \ln N_i! - \ln(g_i - 1)! \} + \alpha \left(N - \sum_i N_i \right) + \beta \left(E - \sum_i N_i E_i \right) \simeq \\
&\simeq \sum_i \{ (N_i + g_i - 1) \ln(N_i + g_i - 1) - N_i \ln N_i - (g_i - 1) \ln(g_i - 1) + N_i + g_i - 1 - N_i - (g_i - 1) \} + \alpha \left(N - \sum_i N \right. \\
&= \sum_i \{ (N_i + g_i - 1) \ln(N_i + g_i - 1) - N_i \ln N_i - (g_i - 1) \ln(g_i - 1) \} + \alpha \left(N - \sum_i N_i \right) + \beta \left(E - \sum_i N_i E_i \right)
\end{aligned}$$

Using $\partial_n (n + a) \ln(n + a) = \ln(n + a) + 1$,

$$0 = \partial_{N_k} J \simeq \{ \ln(N_k + g_k - 1) - \ln N_k \} - \alpha - \beta E_k \ ,$$

and thus

$$\ln \frac{N_k + g_k - 1}{N_k} = \alpha + \beta E_k \ ,$$

$$\frac{N_k + g_k - 1}{N_i} = e^{\alpha + \beta E_k}$$

$$N_k = \frac{g_k - 1}{e^{\alpha + \beta E_k} - 1} \simeq \frac{g_k}{e^{\alpha + \beta E_k} - 1} \ ,$$

Thus, in the limit of $g_k \gg 1$, the occupation number of the k level is

$$N_k = \frac{G_k}{e^{\alpha + \beta E_k} - 1} \ ,$$

and the average occupation number of one of the g_k sublevels in the k level is

$$n_k := \frac{N_k}{G_k} = \frac{1}{e^{\alpha+\beta E_k} - 1}$$

Meaning of α, β

Example 8.5.1 (Black-body radiation: Planck, Wien, and Stefan-Boltzmann laws)

Planck's law. Energy density w.r.t. frequency

$$u_f(f,T) = \frac{8\pi h f^3}{c^3} \frac{1}{e^{\frac{hf}{k_B T}} - 1}$$

Planck's law in a cubic box

Planck's law uses:

- relation between pulsation and wave vector, or frequency and wave number and the speed of light c for light waves

$$c = \frac{\omega}{|\vec{k}|} = \lambda f$$

$$f = \frac{\omega}{2\pi} = \frac{c|\vec{k}|}{2\pi}$$

- Planck assumption that the minimum non-zero energy of a mode with frequency f is $E = hf$, and all the possible values of the energy of the mode is

$$E_m = mhf \quad , \quad m \in \mathbb{N} \, .$$

Taking a cubic box with sides $L_x = L_y = L_z = L$, the possibile modes have (**todo** *why? Which boundary condition? Periodic? Some physical? Just fictitious discretization?*) in each direction wave-lengths $\lambda_n = \frac{L}{|\vec{n}|} = \frac{2\pi}{|\vec{k}|}$,

$$\vec{k} = \frac{2\pi}{L}\vec{n} \, .$$

Mode density in $\vec{n}$-domain is 2 mode per each volume of unit length (2 polarization), and thus the number of modes dN in an elementary volume is

$$dN = 2\, d^3\vec{n} \, ,$$

Changing variables, it's possible to find the mode density w.r.t. wave vector $\vec{k}$,

$$dN = 2\, d^3\vec{n} = 2\, \frac{L^3}{(2\pi)^3}\, d^3\vec{k} \, ,$$

or with its absolute value, exploiting the isotropy of the density function - and writing the elementary volume using "spherical coordinates" $d^3\vec{k} = 4\pi \left|\vec{k}\right|^2 d\left|\vec{k}\right|$,

$$\begin{aligned} dN &= \frac{V}{(2\pi)^3} 8\pi \left|\vec{k}\right|^2 d\left|\vec{k}\right| = \\ &= \frac{V}{(2\pi)^3} 8\pi \frac{8\pi^3}{c^3} f^2 df = \\ &= V \frac{8\pi}{c^3} f^2 df =: V g(f) df \, . \end{aligned}$$

Average energy of a mode

Using Boltzmann distribution (**why?**) for the energy distribution in a single mode,

$$P(E_r) = \frac{e^{-\beta E_r}}{Z} \ ,$$

with $E_r = rhf$, and the partition function

$$Z = \sum_s e^{-\beta E_s} = \sum_s e^{-\beta h f s} = \frac{1}{1 - e^{-\beta h f}} \ .$$

The average energy of the mode reads

$$\begin{aligned}
\langle E \rangle = \sum_r E_r P(E_r) = \\
= \sum_r r h f \frac{e^{-\beta h f r}}{Z} = \\
= h f (1 - e^{-\beta h f}) \sum_r r e^{-\beta h f r} = \\
= h f (1 - e^{-\beta h f}) \frac{e^{-\beta h f}}{(1 - e^{-\beta h f})^2} = \\
= \frac{h f}{e^{\beta h f} - 1} \ .
\end{aligned}$$

Putting together the mode number density and the average energy of a mode, the energy density per unit volume, per frequency reads

$$\begin{aligned}
u(f, T) = \langle E \rangle(f) \, g(f) = \\
= \frac{h f}{e^{\beta h f} - 1} \frac{8\pi}{c^3} f^2 = \\
= \frac{8 \pi h f^3}{c^3} \frac{1}{e^{\beta h f} - 1} \ .
\end{aligned}$$

Property of the series

$$\sum_{n=0}^{+\infty} n x^n = \frac{x}{(1-x)^2}$$

Proof. If the series is convergent (is this the required condition?)

$$\frac{d}{dx} \sum_{n=0}^{+\infty} x^n = \frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\frac{d}{dx} \sum_{n=0}^{+\infty} x^n = \sum_{n=0}^{+\infty} n x^{n-1}$$

$$x \frac{d}{dx} \sum_{n=0}^{+\infty} x^n = \sum_{n=0}^{+\infty} n x^n = \frac{x}{(1-x)^2}$$

Sperctral radiance, B_f, so that an infinitesimal amount of power radiated by a surface … is $dP = B_f(f, T) \cos\theta \, dA \, d\Omega \, df$

$$B_f(f, T) = \frac{2 h f^3}{c^2} \frac{1}{e^{\frac{hf}{k_B T}} - 1} \ .$$

This expression is obtained[1] assuming homogeneous radiation from a small hole cut into a wall of the box. Only half of the energy radiates through the hole - so factor $\frac{1}{2}$ in front of the energy density - through a solid angle 2π - and thus this process give the same result as a radiation of all the energy density in all the space directions, just providing the same factor $\frac{1}{4\pi}$. The flux of energy "has velocity" c and thus

$$B_f(f,T) = \frac{1}{4\pi} u_f(f,T) c \ .$$

Wien's law. Wien's law tells that the frequency f^* corresponding to the maximum of the spectral radiance of a black-body radiation described by Planck's law is proportional to its temperature.

From direct evaluation of the derivative of the spectral radiance as a function of f,

$$\begin{aligned}\partial_f B_f(f,T) &= \frac{2h}{c^2}\left[3f^2\frac{1}{e^{\frac{hf}{k_BT}}-1} + f^3\left(-\frac{\frac{h}{k_BT}e^{\frac{hf}{k_BT}}}{\left(e^{\frac{hf}{k_BT}}-1\right)^2}\right)\right] = \\ &= \frac{2hf^2e^{\frac{hf}{k_BT}}}{c^2\left(e^{\frac{hf}{k_BT}}-1\right)^2}\left[3\left(1-e^{-\frac{hf}{k_BT}}\right)-\frac{hf}{k_BT}\right] \ .\end{aligned}$$

Now, if $\partial_f B_f(f,T) = 0$ the frequency is either $f = 0$, or the solution of the nonlinear algebraic equation

$$0 = 3\left(1-e^{-\frac{hf}{k_BT}}\right) - \frac{hf}{k_BT} \ .$$

Defining $x := \frac{hf}{k_BT}$, this equation becomes

$$0 = 3(1-e^x) - x \ ,$$

whose solution $x^* \approx 2.82$ can be easily evaluated with an iterative method (or expressed in term of the Lambert's function W, so loved at Stanford and on Youtube: they'd probaly like to look at tabulated values, or pose). Once the solution x^* of this non-dimensional equation is found, the frequency where maximum energy density occurs reads

$$f^* = \frac{k_BT}{h}x^* \simeq 2.82\frac{k_B}{h}T \ .$$

Stefan-Boltzmann law.

$$\begin{aligned}\frac{P}{A} &= \int B_f(f,T)\cos\phi \, df \, d\Omega = \\ &= \int_{f=0}^{+\infty}\int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{2\pi} B_f(f,T)\cos\phi\sin\phi \, df \, d\phi \, d\theta = \\ &= \pi\int_{f=0}^{+\infty} B_f(f,T) \, df = \\ &= \frac{2\pi h}{c^2}\int_{f=0}^{+\infty}\frac{f^3}{e^{\frac{hf}{k_BT}}-1} \, df = \\ &= \frac{2\pi h}{c^2}\left(\frac{k_BT}{h}\right)^4\int_{u=0}^{+\infty}\frac{u^3}{e^u-1} \, du \ .\end{aligned}$$

The value of the integral is $\frac{\pi^4}{15}$ and thus

$$\frac{P}{A} = \sigma T^4 \qquad , \qquad \sigma = \frac{2\pi^5 k_B^4}{15c^2h^3} \ .$$

[1] Derivation of Planck's Law.

Example 8.5.2 (Energy density and radiance)

Radiance. The radiance $L_{e,\Omega}$ of a surface is the flux of energy per unit solid angle, per unit projected area of the source.

Spectral radiance in frequency is the radiance per unit frequency, $L_{e,\Omega,f} = \frac{\partial L_{e,\Omega}}{\partial f}$.

8.5.3 Fermi-Dirac

Statistics of undistinguishable components that can't be in the same (sub)level, obeying to the Pauli exclusion principle. Given the number of elementary components $\sum_i N_i = N$ and the energy $\sum_i N_i E_i = E$,

$$W_{FD,i} = \frac{G_i!}{(G_i - N_i)! N_i!} \qquad , \qquad W_{FD} = \prod_i W_{FD,i} \ . \tag{8.2}$$

Counting microstates

todo write page *Combinatorics* and add link

Most likely microstate. The approximate occupation number of the i^{th} level of the most likely microstate is

$$n_i := \frac{N_i}{G_i} = \frac{1}{1 + e^{\alpha + \beta E_i}} \ .$$

Optimization

$$\begin{aligned}
J(N_i, \alpha, \beta) = \ln W_{FD} + \alpha \left(N - \sum_i N_i \right) + \beta \left(E - \sum_i N_i E_i \right) = \\
= \sum_i \{ \ln G_i! - \ln(G_i - N_i)! - \ln N_i! \} + \alpha \left(N - \sum_i N_i \right) + \beta \left(E - \sum_i N_i E_i \right) = \\
= \sum_i \{ G_i \ln G_i - (G_i - N_i) \ln(G_i - N_i) - N_i \ln N_i \} + \alpha \left(N - \sum_i N_i \right) + \beta \left(E - \sum_i N_i E_i \right) =
\end{aligned}$$

Using $\partial_n (n+a) \ln(n+a) = \ln(n+a) + 1$,

$$0 = \partial_{N_k} J \simeq \{ \ln(G_k - N_k) - \ln N_k \} - \alpha - \beta E_k \ ,$$

and thus

$$\ln \frac{G_k - N_k}{N_k} = \alpha + \beta E_k \ ,$$

$$\frac{G_k}{N_k} - 1 = e^{\alpha + \beta E_k}$$

The occupation number of the k level is

$$N_k = \frac{G_k}{1 + e^{\alpha + \beta E_k}} \ .$$

The average occupation of the G_k sublevels of the k level is

$$n_k := \frac{N_k}{G_k} = \frac{1}{1 + e^{\alpha + \beta E_k}} \ .$$

Meaning of α, β

CHAPTER

NINE

STATISTICAL PHYSICS - STATISTICS MISCELLANEA

Information content and Entropy

Given a discrete random variable X with probability mass function $p_X(x)$, the self-information (**todo** *what about mutual information of random variables?*) is defined as the opposite of the logaritm of the mass function $p_X(x)$,

$$I_X(x) := -\ln\left(p_X(x)\right) \ .$$

Information content of indenpendent random variables is additive. Since $p_{X,Y}(x,y) = p_X(x)p_Y(y)$,

$$I_{X,Y}(x,y) = -\ln\left(p_{X,Y}(x,y)\right) = -\ln\left(p_X(x)p_Y(y)\right) = -\ln p_X(x) - \ln p_Y(y) \ .$$

Shannon entropy. Shannon entropy of a discrete random variable X is defined as the expected value of the information content,

$$H(X) := \mathbb{E}[I_X(X)] = \sum p_X(x) I_X(x) = -\sum p_X \ln p_X(x) \ .$$

Gibbs entropy. Gibbs entropy was defined by J.W.Gibbs in 1878,

$$S = -k_B \sum_i p_i \ln p_i \ .$$

Additivity holds for independent random variables.

Boltzmann entropy. Boltmann entropy holds for uniform distributions over Ω possible states, $p_i = \frac{1}{\Omega}$. Gibbs' entropy of this uniform distribution becomes

$$S = -k_B \Omega \frac{1}{\Omega} \ln \frac{1}{\Omega} = k_B \ln \Omega \ .$$

Entropy in Quantum Mechanics. todo

Boltzmann distribution

Given a set of discrete states with probability p_i, and the average measure as "macroscopic quantity" $E = \sum_i p_i E_i$, Boltzann distribution maximizes the entropy (**todo** *Link to min info, max uncertainty*)

$$S = -k_B \sum_i p_i \ln p_i \ .$$

The distribution follows from the constrained optimization

$$\widetilde{S} = S - \alpha\left(\sum_i p_i - 1\right) - \beta\left(\sum_i p_i E_i - E\right)$$

$$0 = \partial_\alpha \widetilde{S} = -\sum_i p_i - 1$$
$$0 = \partial_\beta \widetilde{S} = -\sum_i p_i E_i - E$$
$$0 = \partial_{p_k} \widetilde{S} = -k_B (\ln p_k + 1) - \alpha - \beta E_k$$

and thus

$$p_k = e^{-1-\frac{\alpha}{k_B}-\frac{\beta}{k_B}E_k} = e^{-\left(1+\frac{\alpha}{k_B}\right)} e^{-\frac{\beta}{k_B}E_k} = Ce^{-\frac{\beta}{k_B}E_k} \ ,$$

and the normalization constant C is determined by normalization condition

$$1 = \sum_k p_k = C \sum_k e^{-\frac{\beta E_k}{k_B}}$$

The inverse $Z = C^{-1}$ is defined as the **partition function**,

$$Z = C^{-1} = \sum_k e^{-\frac{\beta E_k}{k_B}} \ ,$$

and the probability distribution becomes

$$p_k = \frac{e^{-\frac{\beta E_k}{k_B}}}{Z} = \frac{e^{-\frac{\beta E_k}{k_B}}}{\sum_i e^{-\frac{\beta E_i}{k_B}}} \ .$$

Properties.

$$\frac{p_k}{p_i} = e^{-\frac{\beta}{k_B}(E_k - E_i)} \ .$$

Thermodynamics. Comparison of statistics and classical thermodynamics

First principle of classical thermodynamics (for a monocomponent gas with no electric charge,...) reads

$$T\,dS = dE + P\,dV$$

Entropy for Boltzmann distribution reads

$$\begin{aligned} S &= -k_B \sum_i p_i \ln p_i = \\ &= -k_B \sum_i \left[p_i \left(-\frac{\beta E_i}{k_B} - \ln Z \right) \right] = \\ &= \beta \langle E \rangle + k_B \ln Z \end{aligned}$$

From classical thermodyamics, temperature T can be defined as the partial derivative of the entropy of a system w.r.t. its internal energy keeping constant all the other independent variables,

$$\begin{aligned}\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)\Big|_X = \\ &= \frac{\partial \beta}{\partial E} E + \beta + k_B \frac{\partial \ln Z}{\partial E} = \\ &= \frac{\partial \beta}{\partial E} E + \beta + k_B \frac{1}{Z}\frac{\partial Z}{\partial E} = \\ &= \frac{\partial \beta}{\partial E} E + \beta + k_B \frac{1}{Z}\frac{\partial Z}{\partial \beta}\frac{\partial \beta}{\partial E} = \\ &= \frac{\partial \beta}{\partial E} E + \beta + k_B \frac{1}{Z}\left(-\sum_i \frac{E_i}{k_B} e^{-\frac{\beta E_i}{k_B}}\right)\frac{\partial \beta}{\partial E} = \\ &= \frac{\partial \beta}{\partial E} E + \beta - \left(\sum_i E_i p_i\right)\frac{\partial \beta}{\partial E} = \\ &= \frac{\partial \beta}{\partial E} E + \beta - E\frac{\partial \beta}{\partial E} = \beta\,.\end{aligned}$$

todo

- write the derivative above clearly in terms of composite functions
- microscopical/statistical approach to the first principle of thermodynamics

$$dE = d\left(\sum_i p_i E_i\right) = \sum_i E_i\, dp_i + \sum_i p_i dE_i$$

Part IV

Quantum Mechanics

CHAPTER

TEN

QUANTUM MECHANICS

- Principles and postulates
  - statistics and measurements outcomes (Heisenberg built its matrix mechanics only on observables…)
  - CCR
- angluar momentum, spin, and atom

10.1 Mathematical tools for quantum mechanics

Definition 10.1.1 (Operator)

Definition 10.1.2 (Adjoint operator)

Given an operator $\hat{A} : U \rightarrow V$, its adjoint operator $\hat{A}^* : V \rightarrow U$ is the operator s.t.

$$(\mathbf{v},\ \hat{A}\mathbf{u})_V = (\mathbf{u}, \hat{A}^*\mathbf{v})_U$$

holds for $\forall \mathbf{u} \in U,\ \mathbf{v} \in V$.

Definition 10.1.3 (Hermitian (self-adjoint) operator)

The operator $\hat{A} : U \rightarrow U$ is a self-adjoint operator if

$$\hat{A}^* = \hat{A}\ .$$

Self-adjoint operators have real eigenvalues, and orthogonal eigenvectors (at least those associated to different eigenvalues; those associated with the same eigenvalues can be used to build an orthogonal set of vectors with orthogonalization process).

10.2 Postulates of Quantum Mechanics

- …
- Canonical Commutation Relation (CCR) *and Canonical Anti-Commutation Relation…*
- …

10.3 Non-relativistic Mechanics

10.3.1 Statistical Interpretation and Measurement

Wave function

The state of a system is described by a wave function $|\Psi\rangle$

todo

- properties: domain, image,…
- unitary $1 = \langle\Psi|\Psi\rangle = |\Psi|^2$, for statistical interpretation of $|\Psi|^2$ as a density probability function

Operators and Observables

Physical **observable** quantities are represented by *Hermitian operators*. Possible outcomes of measurement are the eigenvalues of the operator

Given $\hat{A}$ and the set of its eigenvectors $\{|A_i\rangle\}_i$ (**todo** *continuous or discrete spectrum…, need to treat this difference quite in details*), with associated eigenvalues $\{a_i\}_i$

$$\hat{A}|A_i\rangle = a_i|A_i\rangle$$

$$|\Psi\rangle = |A_i\rangle\langle A_i|\Psi\rangle = |A_i\rangle\Psi_i^A$$

$$\langle A_j|\Psi\rangle = \langle A_j|A_i\rangle\langle A_i|\Psi\rangle = \Psi_j^A$$

and thus

$$\Psi_j^A = \langle A_j|\Psi\rangle$$
$$\Psi_j^{A*} = \langle\Psi|A_j\rangle$$

- identity operator $\sum_i |A_i\rangle\langle A_i| = \mathbb{1}$, since

$$\sum_i |A_i\rangle\langle A_i|\Psi\rangle = \sum_i |A_i\rangle\langle A_i|\Psi_j^A A_j\rangle = \sum_i |A_i\rangle\delta_{ij}\Psi_j^A = \sum_i |A_i\rangle\Psi_i^A = |\Psi\rangle$$

- Normalization:

$$1 = \langle\Psi|\Psi\rangle = \Psi_j^{A*}\underbrace{\langle A_j|A_i\rangle}_{\delta_{ij}}\Psi_i^A = \sum_i \left|\Psi_i^A\right|^2$$

  with $|\Psi_i^A|^2$ that can be interpreted as the probability of finding the system in state $|\Psi_i^a\rangle$

- Expected value of the physical quantity in the a state $|\Psi\rangle$, with possible values a_i with probability $|\Psi_i^A|^2$

$$\begin{aligned}\bar{A}_\Psi &= \sum_i a_i |\Psi_i^A|^2 = \\ &= \sum_i a_i \Psi_i^{A*} \Psi_i^A = \\ &= \sum_i a_i \langle\Psi|A_i\rangle\langle A_i|\Psi\rangle = \\ &= \langle\Psi| \left(\sum_i a_i |A_i\rangle\langle A_i|\right) |\Psi\rangle = \\ &= \langle\Psi|\hat{A}|\Psi\rangle =\end{aligned}$$

since an operator $\hat{A}$ can be written as a function of its eigenvalues and eigenvectors

$$\begin{aligned}\left(\sum_i a_i |A_i\rangle\langle A_i|\right) \Psi\rangle = \left(\sum_i a_i |A_i\rangle\langle A_i|\right) c_k |A_k\rangle &= \\ = \sum_i a_i |A_i\rangle c_i &= \\ = \sum_i \hat{A}|A_i\rangle c_i &= \\ = \hat{A} \sum_i |A_i\rangle c_i &= \hat{A}|\Psi\rangle \ .\end{aligned}$$

Space Representation

Position operator $\hat{\mathbf{r}}$ has eigenvalues $\mathbf{r}$ identifying the possible measurements of the position

$$\hat{\mathbf{r}}|\mathbf{r}\rangle = \mathbf{r}|\mathbf{r}\rangle \ ,$$

being $\mathbf{r}$ the result of the measurement (position in space, mathematically it could be a vector), and $|\mathbf{r}\rangle$ the state function corresponding to the measurement $\mathbf{r}$ of the position.

- Result of measurement, $\mathbf{r}$, is a position in space. As an example, it could be a point in an Euclidean space $P \in E^n$. It could be written using properties of Dirac's delta "function"

$$\mathbf{r} = \int_{\mathbf{r}'} \delta(\mathbf{r}' - \mathbf{r})\, \mathbf{r}' d\mathbf{r}'$$

- Projection of wave function over eigenstates of position operator

$$\begin{aligned}\langle\mathbf{r}|\Psi\rangle(t) = \Psi(\mathbf{r}, t) &= \int_{\mathbf{r}'} \delta(\mathbf{r} - \mathbf{r}')\Psi(\mathbf{r}', t) d\mathbf{r}' = \\ &= \int_{\mathbf{r}'} \langle\mathbf{r}|\mathbf{r}'\rangle\Psi(\mathbf{r}', t) d\mathbf{r}' = \\ &= \int_{\mathbf{r}'} \langle\mathbf{r}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle(t) d\mathbf{r}' = \\ &= \langle\mathbf{r}| \underbrace{\left(\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'| d\mathbf{r}'\right)}_{=\hat{\mathbb{1}}} |\Psi\rangle(t) \ .\end{aligned}$$

- having used orthogonality (**todo** *why? provide definition and examples of operators with continuous spectrum*)

$$\langle\mathbf{r}'|\mathbf{r}\rangle = \delta(\mathbf{r}' - \mathbf{r})$$

- Expansion of a state function $|\Psi\rangle(t)$ over the basis of the position operator

$$|\Psi\rangle(t) = \hat{\mathbf{1}}|\Psi\rangle(t) = \left(\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'d\mathbf{r}' \right) |\Psi\rangle(t) = \int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle(t)\, d\mathbf{r}' \ .$$

- Unitariety and probability density

$$\begin{aligned} 1 = \langle\Psi|\Psi\rangle(t) = \langle\Psi| \left(\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'d\mathbf{r}' \right) |\Psi\rangle \\ = \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle\, d\mathbf{r}' \\ = \int_{\mathbf{r}'} \Psi^*(\mathbf{r}', t)\Psi(\mathbf{r}', t)\, d\mathbf{r}' \\ = \int_{\mathbf{r}'} |\Psi(\mathbf{r}', t)|^2\, d\mathbf{r}' \end{aligned}$$

  and thus $|\Psi(\mathbf{r}, t)|^2$ can be interpreted as the **probability density function** of measuring position of the system equal to $\mathbf{r}'$.

- Average value of the operator

$$\begin{aligned} \bar{\mathbf{r}} = \langle\Psi|\hat{\mathbf{r}}|\Psi\rangle = \\ = \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|d\mathbf{r}'\ |\hat{\mathbf{r}}|\ \int_{\mathbf{r}''} |\mathbf{r}''\rangle\langle\mathbf{r}''|\Psi\rangle\, d\mathbf{r}'' \\ = \int_{\mathbf{r}'}\int_{\mathbf{r}''} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|\hat{\mathbf{r}}|\mathbf{r}''\rangle\langle\mathbf{r}''|\Psi\rangle\, d\mathbf{r}'d\mathbf{r}'' = \\ = \int_{\mathbf{r}'}\int_{\mathbf{r}''} \langle\Psi|\mathbf{r}'\rangle\ \underbrace{\langle\mathbf{r}'|\mathbf{r}''\rangle}_{=\delta(\mathbf{r}'-\mathbf{r}'')}\ \mathbf{r}''\langle\mathbf{r}''|\Psi\rangle\, d\mathbf{r}'d\mathbf{r}'' = \\ = \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\mathbf{r}'\langle\mathbf{r}'|\Psi\rangle\, d\mathbf{r}' = \\ = \int_{\mathbf{r}'} \Psi^*(\mathbf{r}', t)\ \mathbf{r}'\ \Psi(\mathbf{r}', t)\, d\mathbf{r}' = \\ = \int_{\mathbf{r}'} |\Psi(\mathbf{r}', t)|^2\ \mathbf{r}'\, d\mathbf{r}' \ . \end{aligned}$$

Momentum Representation

Momentum operator as the limit of…**todo** *prove the expression of the momentum operator as the limit of the generator of translation*

$$\langle\mathbf{r}|\hat{\mathbf{p}} = -i\hbar\nabla\langle\mathbf{r}|$$

- Spectrum

$$\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$$

$$\langle\mathbf{r}|\hat{\mathbf{p}}|\mathbf{p}\rangle = -i\hbar\nabla\langle\mathbf{r}|\mathbf{p}\rangle = \mathbf{p}\langle\mathbf{r}|\mathbf{p}\rangle$$

  and thus the eigenvectors in space base $\mathbf{p}(\mathbf{r}) = \langle\mathbf{r}|\mathbf{p}\rangle$ are the solution of the differential equation

$$-i\hbar\nabla\mathbf{p}(\mathbf{r}) = \mathbf{p}\mathbf{p}(\mathbf{r}) \ ,$$

that in Cartesian coordinates reads

$$-i\hbar\partial_j p_k(\mathbf{r}) = p_j p_k(\mathbf{r})$$

$$p_k(\mathbf{r}) = p_{k,0}\exp\left[i\frac{p_j}{\hbar}r_j\right]$$

or

$$\langle\mathbf{r}|\mathbf{p}\rangle = \mathbf{p}(\mathbf{r}) = \mathbf{p}_0\exp\left[i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}\right]$$

todo

- normalization factor $\frac{1}{(2\pi)^{\frac{3}{2}}}$

$$\mathcal{F}\{\delta(x)\}(k) = \int_{-\infty}^{\infty}\delta(x)e^{-ikx}\,dx = 1$$

- Fourier transform and inverse Fourier transform: definitions and proofs (link to a math section)
- representation in basis of wave vector operator $\hat{\mathbf{k}}$, $\hat{\mathbf{p}} = \hbar\hat{\mathbf{k}}$

From position to momentum representation

Momentum and wave vector, $\mathbf{p} = \hbar\mathbf{k}$

$$\begin{aligned}
\langle\mathbf{p}|\Psi\rangle &= \langle\mathbf{p}|\int_{\mathbf{r}'}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\
&= \int_{\mathbf{r}'}\langle\mathbf{p}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\
&= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'}\exp\left[i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}\right]\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' =
\end{aligned}$$

Relation between position and wave-number representation can be represented with a Fourier transform

$$\begin{aligned}
\langle\mathbf{k}|\Psi\rangle &= \langle\mathbf{k}|\int_{\mathbf{r}'}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\
&= \int_{\mathbf{r}'}\langle\mathbf{k}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\
&= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'}\exp\left[i\mathbf{k}\cdot\mathbf{r}'\right]\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\
&= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'}\exp\left[i\mathbf{k}\cdot\mathbf{r}'\right]\Psi(\mathbf{r}')d\mathbf{r}' = \\
&= \mathcal{F}\{\Psi(\mathbf{r})\}(\mathbf{k})
\end{aligned}$$

10.3.2 Schrodinger Equation

$$i\hbar\frac{d}{dt}|\Psi\rangle = \hat{H}|\Psi\rangle$$

being $\hat{H}$ the Hamiltonian operator and $|\Psi\rangle$ the wave function, as a function of time t as an independent variable.

Stationary States

Eigenspace of the Hamiltonian operator

$$\hat{H}|\Psi_k\rangle = E_k|\Psi_k\rangle \ ,$$

with E_k possible values of energy measurements. *If no eigenstates with the same eigenvalue exists, then…otherwise… Without external influence* **todo** *be more detailed!*, energy values and eigenstates of the systems are constant in time.

Thus, exapnding the state of the system $|\Psi\rangle$ over the stationary states gives $|\Psi_k\rangle$, $|\Psi\rangle = |\Psi_k\rangle c_k(t)$, and inserting in Schrodinger equation

$$i\hbar\dot{c}_k|\Psi_k\rangle = c_k E_k|\Psi_k\rangle$$

and exploiting orthogonality of eigenstates, a diagonal system for the amplitudes of stationary states ariese,

$$i\hbar\dot{c}_k = c_k E_k \ .$$

whose solution reads

$$c_k(t) = c_{k,0} \exp\left[-i\frac{E_k}{\hbar}t\right]$$

Thus the state of the system evolves like a superposition of monochromatic waves with frequencies $\omega_k = \frac{E_k}{\hbar}$,

$$|\Psi\rangle = |\Psi_k\rangle c_k(t) = |\Psi_k\rangle c_{k,0} \exp\left[-i\frac{E_k}{\hbar}t\right] \ .$$

$$\begin{aligned}
\frac{d}{dt}\bar{A} = \frac{d}{dt}\left(\langle\Psi|\hat{A}|\Psi\rangle\right) = & \\
& = \frac{d}{dt}\langle\Psi|\hat{A}|\Psi\rangle + \langle\Psi|\frac{d\hat{A}}{dt}|\Psi\rangle + \langle\Psi|\hat{A}\frac{d}{dt}|\Psi\rangle = \\
& = \langle\Psi|\frac{d\hat{A}}{dt}|\Psi\rangle + \frac{i}{\hbar}\langle\Psi|\hat{H}\hat{A}|\Psi\rangle - \frac{i}{\hbar}\langle\Psi|\hat{A}\hat{H}|\Psi\rangle = \\
& = \langle\Psi|\left(\frac{i}{\hbar}[\hat{H},\hat{A}] + \frac{d\hat{A}}{dt}\right)|\Psi\rangle \ .
\end{aligned}$$

Pictures

- Schrodinger
- Heisenberg
- Interaction

Schrodinger

If $\hat{H}$ not function of time

$$|\Psi\rangle(t) = \exp\left[-i\frac{\hat{H}}{\hbar}(t-t_0)\right]|\Psi\rangle(t_0) = \hat{U}(t,t_0)|\Psi\rangle(t_0)$$

$$\bar{A} = \langle\Psi|\hat{A}|\Psi\rangle = \langle\Psi_0|\hat{U}^*(t,t_0)\hat{A}\hat{U}(t,t_0)|\Psi_0\rangle$$

Heisenberg

…

for $\hat{H}$ independent from time t,

$$\frac{d}{dt}\bar{\mathbf{r}} = \overline{\frac{i}{\hbar}\left[\hat{H}, \hat{\mathbf{r}}\right]}$$
$$\frac{d}{dt}\bar{\mathbf{p}} = \overline{\frac{i}{\hbar}\left[\hat{H}, \hat{\mathbf{p}}\right]}$$

Hamiltonian Mechanics

From Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q_q$$

q generalized coordinates, $p := \frac{\partial L}{\partial \dot{q}}$ generalized momenta.

Hamiltonian

$$H(p, q, t) = p\dot{q} - L(\dot{q}, q, t)$$

Increment of the Hamiltonian

$$dH = \partial_p H dp + \partial_q H dq + \partial_t H dt$$

$$\begin{aligned} dH &= \dot{q}dp + pd\dot{q} - \partial_{\dot{q}} L d\dot{q} - \partial_q L dq - \partial_t L dt = \\ &= \dot{q}dp - \partial_q L dq - \partial_t L dt = \\ &= \dot{q}dp - \left(\dot{p} + Q_q\right) dq - \partial_t L dt = \end{aligned}$$

$$\begin{cases} \frac{\partial H}{\partial p} = \dot{q} \\ \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -\dot{p} + Q_q \\ \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \end{cases}$$

Physical quantity $f(p(t), q(t), t)$. Its time derivative reads

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t} = \\ &= \frac{\partial f}{\partial p}\left[-\frac{\partial H}{\partial q} + Q_q\right] + \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial f}{\partial t} = \\ &= \{H, f\} + \partial_t f + Q_q \partial_p f \end{aligned}$$

If $Q_q = 0$, the correspondence between quantum mechanics and classical mechanics

$$\frac{df}{dt} = \{H, f\} + \partial_t f \qquad \leftrightarrow \qquad \frac{d}{dt}\bar{\hat{f}} = \overline{\frac{i}{\hbar}[\hat{H}, \hat{f}]} + \overline{\frac{\partial \hat{f}}{\partial t}}$$

$$\{H, f\} \qquad \leftrightarrow \qquad \frac{i}{\hbar}[\hat{H}, \hat{f}]$$

Interaction

10.3.3 Matrix Mechanics

Attualization of 1925 papers

…to find the canonical commutation relation,

$$[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = i\hbar\mathbb{I}\,\hat{\mathbf{1}}\ .$$

$$\begin{aligned}
[\hat{\mathbf{r}}, \hat{\mathbf{p}}] & = \hat{\mathbf{r}}\hat{\mathbf{p}} - \hat{\mathbf{p}}\hat{\mathbf{r}} = \\
& = \hat{\mathbf{r}}\int_{\mathbf{r}} |\mathbf{r}\rangle\langle\mathbf{r}|d\mathbf{r}\hat{\mathbf{p}} - \hat{\mathbf{p}}\int_{\mathbf{r}} |\mathbf{r}\rangle\langle\mathbf{r}|d\mathbf{r}\ \hat{\mathbf{r}}\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'|d\mathbf{r}' = \\
& = -\int_{\mathbf{r}} \mathbf{r}|\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\,d\mathbf{r} - \hat{\mathbf{p}}\int_{\mathbf{r}}\int_{\mathbf{r}'} |\mathbf{r}\rangle\mathbf{r}'\underbrace{\langle\mathbf{r}|\mathbf{r}'\rangle}_{\delta(\mathbf{r}-\mathbf{r}')}\langle\mathbf{r}'|d\mathbf{r}' = \\
& = -\int_{\mathbf{r}} \mathbf{r}|\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\,d\mathbf{r} - \hat{\mathbf{p}}\int_{\mathbf{r}} \mathbf{r}|\mathbf{r}\rangle\langle\mathbf{r}|d\mathbf{r} = \\
& = -\int_{\mathbf{r}} \mathbf{r}|\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\,d\mathbf{r} - \int_{\mathbf{r}'} |\mathbf{r}\rangle\langle\mathbf{r}|d\mathbf{r}\ \hat{\mathbf{p}}\int_{\mathbf{r}'} \mathbf{r}'|\mathbf{r}'\rangle\langle\mathbf{r}'|d\mathbf{r}' = \\
& = -\int_{\mathbf{r}} \mathbf{r}|\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\,d\mathbf{r} + \int_{\mathbf{r}} |\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|d\mathbf{r}\int_{\mathbf{r}'} \mathbf{r}'|\mathbf{r}'\rangle\langle\mathbf{r}'|d\mathbf{r}' = \ldots
\end{aligned}$$

$$\begin{aligned}
[\hat{\mathbf{r}}, \hat{\mathbf{p}}]\,|\Psi\rangle & = -\int_{\mathbf{r}} \mathbf{r}|\mathbf{r}\rangle i\hbar\nabla\Psi(\mathbf{r}, t) + \int_{\mathbf{r}} |\mathbf{r}\rangle i\hbar\nabla\left(\mathbf{r}\Psi(\mathbf{r}, t)\right) = \\
& = -\int_{\mathbf{r}} |\mathbf{r}\rangle i\hbar\left[\mathbf{r}\nabla\Psi(\mathbf{r}, t) + \mathbb{I}\Psi(\mathbf{r}, t) + \mathbf{r}\nabla\Psi(\mathbf{r}, t)\right] = \\
& = i\hbar\underbrace{\int_{\mathbf{r}} |\mathbf{r}\rangle\langle\mathbf{r}|d\mathbf{r}}_{\hat{\mathbf{1}}}\,|\Psi\rangle\ ,
\end{aligned}$$

and since $|\Psi\rangle$ is arbitrary

$$[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = i\hbar\mathbb{I}\hat{\mathbf{1}}\ .$$

$$[\hat{r}_a, \hat{p}_b] = i\hbar\delta_{ab}\ .$$

10.3.4 Heisenberg Uncertainty relation

Uncertainty principle is a relation that holds for "wave descriptions" as it can be proved in the generic framework of Fourier transform, see Fourier transform:Uncertainty relation.

- Heisenberg uncertainty relation is a relation between product of the variance of two physical quantities and their commutator,
- **todo** *relation with measurement process and outcomes. Commutation as a measurement process: first measure B and then A, or first measure A and then B*

$$\sigma_A\sigma_B \ge \frac{1}{2}\left|\overline{[\hat{A}, \hat{B}]}\right|\ .$$

Proof of Heisenberg uncertainty "principle"

$$
\begin{aligned}
\sigma_A^2 \sigma_B^2 &= \langle \Psi | \left(\hat{A} - \bar{A} \right)^2 | \Psi \rangle \langle \Psi | \left(\hat{B} - \bar{B} \right)^2 | \Psi \rangle = \\
&= \langle (\hat{A} - \bar{A}) \Psi | (\hat{A} - \bar{A}) \Psi \rangle \langle (\hat{B} - \bar{B}) \Psi | (\hat{B} - \bar{B}) \Psi \rangle = \\
&= \| (\hat{A} - \bar{A}) \Psi \|^2 \| (\hat{B} - \bar{B}) \Psi \|^2 = \\
&\ge \left| \langle (\hat{A} - \bar{A}) \Psi | (\hat{B} - \bar{B}) \Psi \rangle \right|^2 = \\
&= \left| \langle \Psi | (\hat{A} - \bar{A}) (\hat{B} - \bar{B}) \Psi \rangle \right|^2 = \\
&= \left| \langle \Psi | \hat{A} \hat{B} - \hat{A} \bar{B} - \bar{A} \hat{B} + \bar{A} \bar{B} | \Psi \rangle \right|^2 = \\
&= \left| \langle \Psi | \hat{A} \hat{B} - \bar{A} \bar{B} | \Psi \rangle \right|^2 \ge \qquad (1) \\
&= \left| \frac{\langle \Psi | \hat{A} \hat{B} - \hat{B} \hat{A} | \Psi \rangle}{2i} \right|^2 = \\
&= \frac{\left| \langle \Psi | [\hat{A}, \hat{B}] | \Psi \rangle \right|^2}{4} = \frac{1}{4} \left| \overline{[\hat{A}, \hat{B}]} \right|^2
\end{aligned}
$$

having used Cauchy-Schwartz triangle inequality in (1),

$$|z| \ge |\text{im}(z)| = \frac{z - z^*}{2i} \ .$$

Hesienberg uncertainty principles applied to position and momentum reads

$$\sigma_{r_a} \sigma_{p_b} \ge \frac{1}{2} \left| \overline{[\hat{r}_a, \hat{p}_b]} \right| = \frac{\hbar}{2} \delta_{ab} \ .$$

10.4 Many-body problem

Wave function with symmetries: Fermions and Bosons

CHAPTER

ELEVEN

QUANTUM MECHANICS - NOTES

PROOF INDEX