Modern Physics

basics

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CONTENTS

If you want ot start a new basics-book, it could be a good idea to start from this template.

Please check out the Github repo of the project, basics-book project, and the landing page of the project.

- *Special Relativity*
- *General Relativity*
- *Statistical Physics*
- *Quantum Mechanics*

CHAPTER

ONE

SPECIAL RELATIVITY

CHAPTER

TWO

GENERAL RELATIVITY

CHAPTER

THREE

STATISTICAL PHYSICS

CHAPTER

FOUR

QUANTUM MECHANICS

4.1 Mathematical tools for quantum mechanics

Definition 1 (Operator)

Definition 2 (Adjoint operator)

Given an operator $\hat{A} : U \to V$, its self-adjoint $\hat{A}^* : V \to U$ is the operator s.t.

$$(\mathbf{v},\ \hat{A}\mathbf{u})_V = (\mathbf{u}, \hat{A}^*\mathbf{v})_U$$

holds for $\forall \mathbf{u} \in U,\ \mathbf{v} \in V$.

Definition 3 (Hermitian (self-adjoint) operator)

If $\hat{A} : U \to U$, it is a self-adjoint operator if $\hat{A}^* = \hat{A}$.

Self-adjoint operators have real eigenvalues, and orthogonal eigenvectors (at least those associated to different eigenvalues; those associated with the same eigenvalues can be used to build an orthogonal set of vectors with orthogonalization process).

4.2 Postulates of Quantum Mechanics

- …
- Canonical Commutation Relation (CCR) *and Canonical Anti-Commutation Relation…*

4.3 Non-relativistic Mechanics

4.3.1 Statistical Interpretation

Wave function

The state of a system is described by a wave function $|\Psi\rangle$

todo

- properties: domain, image,…
- unitary $1 = \langle\Psi|\Psi\rangle = |\Psi|^2$, for statistical interpretation of $|\Psi|^2$ as a density probability function

Operators and Observables

Physical **observable** quantities are represented by *Hermitian operators*. Possible outcomes of measurement are the eigenvalues of the operator

Given $\hat{A}$ and the set of its eigenvectors $\{|A_i\rangle\}_i$ (**todo** *continuous or discrete spectrum…, need to treat this difference quite in details*), with associated eigenvalues $\{a_i\}_i$

$$\hat{A}|A_i\rangle = a_i|A_i\rangle$$

$$|\Psi\rangle = |A_i\rangle\langle A_i|\Psi\rangle = |A_i\rangle\Psi_i^A$$
$$\langle A_j|\Psi\rangle = \langle A_j|A_i\rangle\langle A_i|\Psi\rangle = \Psi_j^A$$

and thus

$$\Psi_j^A = \langle A_j|\Psi\rangle$$
$$\Psi_j^{A*} = \langle\Psi|A_j\rangle$$

- identity operator $\sum_i |A_i\rangle\langle A_i| = \mathbb{1}$, since

$$\sum_i |A_i\rangle\langle A_i|\Psi\rangle = \sum_i |A_i\rangle\langle A_i|\Psi_j^A A_j\rangle = \sum_i |A_i\rangle\delta_{ij}\Psi_j^A = \sum_i |A_i\rangle\Psi_i^A = |\Psi\rangle$$

- Normalization:

$$1 = \langle\Psi|\Psi\rangle = \Psi_j^{A*}\underbrace{\langle A_j|A_i\rangle}_{\delta_{ij}}\Psi_i^A = \sum_i \left|\Psi_i^A\right|^2$$

  with $|\Psi_i^A|^2$ that can be interpreted as the probability of finding the system in state $|\Psi_i^a\rangle$

- Expected value of the physical quantity in the a state $|\Psi\rangle$, with possible values a_i with probability $|\Psi_i^A|^2$

$$\begin{aligned}\bar{A}_\Psi &= \sum_i a_i|\Psi_i^A|^2 = \\ &= \sum_i a_i\Psi_i^{A*}\Psi_i^A = \\ &= \sum_i a_i\langle\Psi|A_i\rangle\langle A_i|\Psi\rangle = \\ &= \langle\Psi|\left(\sum_i a_i|A_i\rangle\langle A_i|\right)|\Psi\rangle = \\ &= \langle\Psi|\hat{A}|\Psi\rangle =\end{aligned}$$

since an operator $\hat{A}$ can be written as a function of its eigenvalues and eigenvectors

$$\begin{aligned}\left(\sum_i a_i|A_i\rangle\langle A_i|\right)\Psi\rangle = \left(\sum_i a_i|A_i\rangle\langle A_i|\right)c_k|A_k\rangle = \\ = \sum_i a_i|A_i\rangle c_i = \\ = \sum_i \hat{A}|A_i\rangle c_i = \\ = \hat{A}\sum_i |A_i\rangle c_i = \hat{A}|\Psi\rangle \ .\end{aligned}$$

Space Representation

Position operator $\hat{\mathbf{r}}$ has eigenvalues $\mathbf{r}$ identifying the possible measurements of the position

$$\hat{\mathbf{r}}|\mathbf{r}\rangle = \mathbf{r}|\mathbf{r}\rangle \ ,$$

being $\mathbf{r}$ the result of the measurement (position in space, mathematically it could be a vector), and $|\mathbf{r}\rangle$ the state function corresponding to the measurement $\mathbf{r}$ of the position.

- Result of measurement, $\mathbf{r}$, is a position in space. As an example, it could be a point in an Euclidean space $P \in E^n$. It could be written using properties of Dirac's delta "function"

$$\mathbf{r} = \int_{\mathbf{r}'} \delta(\mathbf{r}' - \mathbf{r})\,\mathbf{r}'\,d\mathbf{r}'$$

- Projection of wave function over eigenstates of position operator

$$\begin{aligned}\langle\mathbf{r}|\Psi\rangle(t) = \Psi(\mathbf{r},t) = \int_{\mathbf{r}'} \delta(\mathbf{r}-\mathbf{r}')\Psi(\mathbf{r}',t)d\mathbf{r}' = \\ = \int_{\mathbf{r}'} \langle\mathbf{r}|\mathbf{r}'\rangle\Psi(\mathbf{r}',t)d\mathbf{r}' = \\ = \int_{\mathbf{r}'} \langle\mathbf{r}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle(t)d\mathbf{r}' = \\ = \langle\mathbf{r}|\underbrace{\left(\int_{\mathbf{r}'}|\mathbf{r}'\rangle\langle\mathbf{r}'|d\mathbf{r}'\right)}_{=\hat{\mathbf{1}}}|\Psi\rangle(t) \ .\end{aligned}$$

- having used orthogonality (**todo** *why? provide definition and examples of operators with continuous spectrum*)

$$\langle\mathbf{r}'|\mathbf{r}\rangle = \delta(\mathbf{r}' - \mathbf{r})$$

- Expansion of a state function $|\Psi\rangle(t)$ over the basis of the position operator

$$|\Psi\rangle(t) = \hat{\mathbf{1}}|\Psi\rangle(t) = \left(\int_{\mathbf{r}'}|\mathbf{r}'\rangle\langle\mathbf{r}'d\mathbf{r}'\right)|\Psi\rangle(t) = \int_{\mathbf{r}'}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle(t)\,d\mathbf{r}' \ .$$

- Unitariety and probability density

$$\begin{aligned}1 = \langle\Psi|\Psi\rangle(t) &= \langle\Psi| \left(\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}' d\mathbf{r}'\right) |\Psi\rangle \\ &= \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle\, d\mathbf{r}' \\ &= \int_{\mathbf{r}'} \Psi^*(\mathbf{r}',t)\Psi(\mathbf{r}',t)\, d\mathbf{r}' \\ &= \int_{\mathbf{r}'} |\Psi(\mathbf{r}',t)|^2\, d\mathbf{r}'\end{aligned}$$

and thus $|\Psi(\mathbf{r},t)|^2$ can be interpreted as the **probability density function** of measuring position of the system equal to $\mathbf{r}'$.

- Average value of the operator

$$\begin{aligned}\bar{\mathbf{r}} &= \langle\Psi|\hat{\mathbf{r}}|\Psi\rangle = \\ &= \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|d\mathbf{r}'\ |\hat{\mathbf{r}}|\ \int_{\mathbf{r}''} |\mathbf{r}''\rangle\langle\mathbf{r}''|\Psi\rangle\, d\mathbf{r}'' \\ &= \int_{\mathbf{r}'}\int_{\mathbf{r}''} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|\hat{\mathbf{r}}|\mathbf{r}''\rangle\langle\mathbf{r}''|\Psi\rangle\, d\mathbf{r}' d\mathbf{r}'' = \\ &= \int_{\mathbf{r}'}\int_{\mathbf{r}''} \langle\Psi|\mathbf{r}'\rangle\ \underbrace{\langle\mathbf{r}'|\mathbf{r}''\rangle}_{=\delta(\mathbf{r}'-\mathbf{r}'')}\ \mathbf{r}''\langle\mathbf{r}''|\Psi\rangle\, d\mathbf{r}' d\mathbf{r}'' = \\ &= \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\mathbf{r}'\langle\mathbf{r}'|\Psi\rangle\, d\mathbf{r}' = \\ &= \int_{\mathbf{r}'} \Psi^*(\mathbf{r}',t)\ \mathbf{r}'\ \Psi(\mathbf{r}',t)\, d\mathbf{r}' = \\ &= \int_{\mathbf{r}'} |\Psi(\mathbf{r}',t)|^2\ \mathbf{r}'\ d\mathbf{r}'\ .\end{aligned}$$

Momentum Representation

Momentum operator as the limit of…**todo** *prove the expression of the momentum operator as the limit of the generator of translation*

$$\langle\mathbf{r}|\hat{\mathbf{p}} = -i\hbar\nabla\langle\mathbf{r}|$$

- Spectrum

$$\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$$

$$\langle\mathbf{r}|\hat{\mathbf{p}}|\mathbf{p}\rangle = -i\hbar\nabla\langle\mathbf{r}|\mathbf{p}\rangle = \mathbf{p}\langle\mathbf{r}|\mathbf{p}\rangle$$

and thus the eigenvectors in space base $\mathbf{p}(\mathbf{r}) = \langle\mathbf{r}|\mathbf{p}\rangle$ are the solution of the differential equation

$$-i\hbar\nabla\mathbf{p}(\mathbf{r}) = \mathbf{p}\mathbf{p}(\mathbf{r})\ ,$$

that in Cartesian coordinates reads

$$-i\hbar\partial_j p_k(\mathbf{r}) = p_j p_k(\mathbf{r})$$

$$p_k(\mathbf{r}) = p_{k,0}\exp\left[i\frac{p_j}{\hbar}r_j\right]$$

or

$$\langle \mathbf{r}|\mathbf{p}\rangle = \mathbf{p}(\mathbf{r}) = \mathbf{p}_0 \exp\left[i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}\right]$$

todo

- normalization factor $\frac{1}{(2\pi)^{\frac{3}{2}}}$

$$\mathcal{F}\{\delta(x)\}(k) = \int_{-\infty}^{\infty} \delta(x)e^{-ikx}\,dx = 1$$

- Fourier transform and inverse Fourier transform: definitions and proofs (link to a math section)
- representation in basis of wave vector operator $\hat{\mathbf{k}}$, $\hat{\mathbf{p}} = \hbar\hat{\mathbf{k}}$

From position to momentum representation

Momentum and wave vector, $\mathbf{p} = \hbar\mathbf{k}$

$$\begin{aligned}\langle \mathbf{p}|\Psi\rangle &= \langle \mathbf{p}|\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \int_{\mathbf{r}'} \langle\mathbf{p}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'} \exp\left[i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}\right]\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \end{aligned}$$

Relation between position and wave-number representation can be represented with a Fourier transform

$$\begin{aligned}\langle \mathbf{k}|\Psi\rangle &= \langle \mathbf{k}|\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \int_{\mathbf{r}'} \langle\mathbf{k}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'} \exp\left[i\mathbf{k}\cdot\mathbf{r}'\right]\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'} \exp\left[i\mathbf{k}\cdot\mathbf{r}'\right]\Psi(\mathbf{r}') d\mathbf{r}' = \\ &= \mathcal{F}\{\Psi(\mathbf{r})\}(\mathbf{k})\end{aligned}$$

4.3.2 Schrodinger Equation

$$i\hbar\frac{d}{dt}|\Psi\rangle = \hat{H}|\Psi\rangle$$

being $\hat{H}$ the Hamiltonian operator and $|\Psi\rangle$ the wave function, as a function of time t as an independent variable.

Stationary States

Eigenspace of the Hamiltonian operator

$$\hat{H}|\Psi_k\rangle = E_k|\Psi_k\rangle \ ,$$

with E_k possible values of energy measurements. *If no eigenstates with the same eigenvalue exists, then…otherwise… Without external influence* **todo** *be more detailed!*, energy values and eigenstates of the systems are constant in time.

Thus, exapnding the state of the system $|\Psi\rangle$ over the stationary states gives $|\Psi_k\rangle$, $|\Psi\rangle = |\Psi_k\rangle c_k(t)$, and inserting in Schrodinger equation

$$i\hbar\dot{c}_k|\Psi_k\rangle = c_k E_k|\Psi_k\rangle$$

and exploiting orthogonality of eigenstates, a diagonal system for the amplitudes of stationary states ariese,

$$i\hbar\dot{c}_k = c_k E_k \ .$$

whose solution reads

$$c_k(t) = c_{k,0}\exp\left[-i\frac{E_k}{\hbar}t\right]$$

Thus the state of the system evolves like a superposition of monochromatic waves with frequencies $\omega_k = \frac{E_k}{\hbar}$,

$$|\Psi\rangle = |\Psi_k\rangle c_k(t) = |\Psi_k\rangle c_{k,0}\exp\left[-i\frac{E_k}{\hbar}t\right] \ .$$

$$\begin{aligned}\frac{d}{dt}\bar{A} = \frac{d}{dt}\left(\langle\Psi|\hat{A}|\Psi\rangle\right) = \\ &= \frac{d}{dt}\langle\Psi|\hat{A}|\Psi\rangle + \langle\Psi|\frac{d\hat{A}}{dt}|\Psi\rangle + \langle\Psi|\hat{A}\frac{d}{dt}|\Psi\rangle = \\ &= \langle\Psi|\frac{d\hat{A}}{dt}|\Psi\rangle + \frac{i}{\hbar}\langle\Psi|\hat{H}\hat{A}|\Psi\rangle - \frac{i}{\hbar}\langle\Psi|\hat{A}\hat{H}|\Psi\rangle = \\ &= \langle\Psi|\left(\frac{i}{\hbar}[\hat{H},\hat{A}] + \frac{d\hat{A}}{dt}\right)|\Psi\rangle \ .\end{aligned}$$

Pictures

- Schrodinger
- Heisenberg
- Interaction

Schrodinger

If $\hat{H}$ not function of time

$$|\Psi\rangle(t) = \exp\left[-i\frac{\hat{H}}{\hbar}(t-t_0)\right]|\Psi\rangle(t_0) = \hat{U}(t,t_0)|\Psi\rangle(t_0)$$

$$\bar{A} = \langle\Psi|\hat{A}|\Psi\rangle = \langle\Psi_0|\hat{U}^*(t,t_0)\hat{A}\hat{U}(t,t_0)|\Psi_0\rangle$$

Heisenberg

…

for $\hat{H}$ independent from time t,

$$\begin{aligned}\frac{d}{dt}\bar{\mathbf{r}} &= \overline{\frac{i}{\hbar}\left[\hat{H},\hat{\mathbf{r}}\right]}\\ \frac{d}{dt}\bar{\mathbf{p}} &= \overline{\frac{i}{\hbar}\left[\hat{H},\hat{\mathbf{p}}\right]}\end{aligned}$$

Hamiltonian Mechanics

From Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q_q$$

q generalized coordinates, $p := \frac{\partial L}{\partial \dot{q}}$ generalized momenta.

Hamiltonian

$$H(p,q,t) = p\dot{q} - L(\dot{q},q,t)$$

Increment of the Hamiltonian

$$dH = \partial_p H dp + \partial_q H dq + \partial_t H dt$$

$$\begin{aligned}dH &= \dot{q}dp + pd\dot{q} - \partial_{\dot{q}}Ld\dot{q} - \partial_q L dq - \partial_t L dt = \\ &= \dot{q}dp - \partial_q L dq - \partial_t L dt = \\ &= \dot{q}dp - \left(\dot{p} + Q_q\right)dq - \partial_t L dt =\end{aligned}$$

$$\begin{cases}\frac{\partial H}{\partial p} = \dot{q}\\ \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -\dot{p} + Q_q\\ \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}\end{cases}$$

Physical quantity $f(p(t),q(t),t)$. Its time derivative reads

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t} = \\ &= \frac{\partial f}{\partial p}\left[-\frac{\partial H}{\partial q} + Q_q\right] + \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial f}{\partial t} = \\ &= \{H,f\} + \partial_t f + Q_q \partial_p f\end{aligned}$$

If $Q_q = 0$, the correspondence between quantum mechanics and classical mechanics

$$\begin{aligned}\frac{df}{dt} = \{H,f\} + \partial_t f \qquad &\leftrightarrow \qquad \frac{d}{dt}\bar{\hat{f}} = \overline{\frac{i}{\hbar}[\hat{H},\hat{f}]} + \overline{\frac{\partial \hat{f}}{\partial t}}\\ \{H,f\} \qquad &\leftrightarrow \qquad \frac{i}{\hbar}[\hat{H},\hat{f}]\end{aligned}$$

Interaction

4.3.3 Matrix Mechanics

Attualization of 1925 papers

…to find the canonical commutation relation,

$$[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = i\hbar \mathbb{I}\, \hat{\mathbf{1}} \ .$$

$$\begin{aligned}
[\hat{\mathbf{r}}, \hat{\mathbf{p}}] & = \hat{\mathbf{r}}\hat{\mathbf{p}} - \hat{\mathbf{p}}\hat{\mathbf{r}} = \\
& = \hat{\mathbf{r}} \int_{\mathbf{r}} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r} \hat{\mathbf{p}} - \hat{\mathbf{p}} \int_{\mathbf{r}} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r}\, \hat{\mathbf{r}} \int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'| d\mathbf{r}' = \\
& = - \int_{\mathbf{r}} \mathbf{r} |\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\, d\mathbf{r} - \hat{\mathbf{p}} \int_{\mathbf{r}}\int_{\mathbf{r}'} |\mathbf{r}\rangle \mathbf{r}' \underbrace{\langle\mathbf{r}|\mathbf{r}'\rangle}_{\delta(\mathbf{r}-\mathbf{r}')} \langle\mathbf{r}'| d\mathbf{r}' = \\
& = - \int_{\mathbf{r}} \mathbf{r} |\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\, d\mathbf{r} - \hat{\mathbf{p}} \int_{\mathbf{r}} \mathbf{r} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r} = \\
& = - \int_{\mathbf{r}} \mathbf{r} |\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\, d\mathbf{r} - \int_{\mathbf{r}'} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r}\, \hat{\mathbf{p}} \int_{\mathbf{r}'} \mathbf{r}' |\mathbf{r}'\rangle\langle\mathbf{r}'| d\mathbf{r}' = \\
& = - \int_{\mathbf{r}} \mathbf{r} |\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\, d\mathbf{r} + \int_{\mathbf{r}} |\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}| d\mathbf{r} \int_{\mathbf{r}'} \mathbf{r}' |\mathbf{r}'\rangle\langle\mathbf{r}'| d\mathbf{r}' = \dots \\
[\hat{\mathbf{r}}, \hat{\mathbf{p}}]\, |\Psi\rangle & = - \int_{\mathbf{r}} \mathbf{r} |\mathbf{r}\rangle i\hbar\nabla\Psi(\mathbf{r}, t) + \int_{\mathbf{r}} |\mathbf{r}\rangle i\hbar\nabla\left(\mathbf{r}\Psi(\mathbf{r}, t)\right) = \\
& = - \int_{\mathbf{r}} |\mathbf{r}\rangle i\hbar \left[\mathbf{r}\nabla\Psi(\mathbf{r}, t) + \mathbb{I}\Psi(\mathbf{r}, t) + \mathbf{r}\nabla\Psi(\mathbf{r}, t)\right] = \\
& = i\hbar \underbrace{\int_{\mathbf{r}} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r}}_{\hat{\mathbf{1}}}\, |\Psi\rangle \ ,
\end{aligned}$$

and since $|\Psi\rangle$ is arbitrary

$$[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = i\hbar \mathbb{I}\hat{\mathbf{1}} \ .$$

$$[\hat{r}_a, \hat{p}_b] = i\hbar\delta_{ab} \ .$$

4.3.4 Heisenberg Uncertainty "principle"

- Heisenberg uncertainty "principle" is a relation between product of variance of two physical quantities and their commutation,
- **todo** *relation with measurement process and outcomes. Commutation as a measurement process: first measure B and then A, or first measure A and then B*

$$\sigma_A\sigma_B \ge \frac{1}{2}\left|\overline{[\hat{A}, \hat{B}]}\right| \ .$$

Proof of Heisenberg uncertainty "principle"

$$
\begin{aligned}
\sigma_A^2 \sigma_B^2 &= \langle\Psi| \left(\hat{A}-\bar{A}\right)^2 |\Psi\rangle \langle\Psi| \left(\hat{B}-\bar{B}\right)^2 |\Psi\rangle = \\
&= \langle(\hat{A}-\bar{A})\Psi|(\hat{A}-\bar{A})\Psi\rangle\langle(\hat{B}-\bar{B})\Psi|(\hat{B}-\bar{B})\Psi\rangle = \\
&= \|(\hat{A}-\bar{A})\Psi\|^2 \|(\hat{B}-\bar{B})\Psi\|^2 = \\
&\geq \left|\langle(\hat{A}-\bar{A})\Psi|(\hat{B}-\bar{B})\Psi\rangle\right|^2 = \\
&= \left|\langle\Psi|(\hat{A}-\bar{A})(\hat{B}-\bar{B})\Psi\rangle\right|^2 = \\
&= \left|\langle\Psi|\hat{A}\hat{B}-\hat{A}\bar{B}-\bar{A}\hat{B}+\bar{A}\bar{B}|\Psi\rangle\right|^2 = \\
&= \left|\langle\Psi|\hat{A}\hat{B}-\bar{A}\bar{B}|\Psi\rangle\right|^2 = \\
&= \left|\frac{\langle\Psi|\hat{A}\hat{B}-\hat{B}\hat{A}|\Psi\rangle}{2i}\right|^2 = \\
&= \frac{\left|\langle\Psi|[\hat{A},\hat{B}]|\Psi\rangle\right|^2}{4} = \frac{1}{4}\left|\overline{[\hat{A},\hat{B}]}\right|^2
\end{aligned}
$$

having used Cauchy triangle inequality and

$$
|z| = \frac{\mathrm{re}\{z\}+\mathrm{re}\{z^*\}}{2} = \frac{\mathrm{im}\{z\}-\mathrm{im}\{z^*\}}{2i}
$$

Hesienberg uncertainty principles applied to position and momentum reads

$$
\sigma_{r_a}\sigma_{p_b} \geq \frac{1}{2}\left|\overline{[\hat{r}_a,\hat{p}_b]}\right| = \frac{\hbar}{2}\delta_{ab} \ .
$$

4.4 Many-body problem

Wave function with symmetries: Fermions and Bosons

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