Modern Physics

basics

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This material is part of the **basics-books project**. It is also available as a .pdf document.

Please check out the Github repo of the project, basics-book project.

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Part I

Special Relativity

CHAPTER

ONE

SPECIAL RELATIVITY

- Electromagnetism and the need for new relativity
- Space-time, Lorentz transformations,…
- Mechanics: kinematics, dynamics,…
- Electromagnetism: Maxwell's equations, potentials, Lorentz force, energy balance

CHAPTER TWO

SPECIAL RELATIVITY - NOTES

An event is determined by spatio-temporal information together, $t, \vec{r}$. Absolute nature of physics needs vector algebra and calculus formalism

$$\mathbf{X} = c\,t\,\mathbf{e}_0 + \vec{r} = c\,t\,\mathbf{e}_0 + x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 = X^\alpha \mathbf{E}_\alpha \ ,$$

having used Cartesian coordiantes for the space coordinate.

Minkowski metric reads

$$g_{\alpha\beta} = \mathbf{E}_\alpha \cdot \mathbf{E}_\beta = \mathrm{diag}\{-1,1,1,1\}$$

The reciprocal basis reads $\mathbf{E}_\alpha \cdot \mathbf{E}^\beta = \delta_\alpha^\beta$, $\mathbf{E}_\alpha = g_{\alpha\beta}\mathbf{E}^\beta$, s.t. the elementary interval between two events can be written as

$$d\mathbf{X} = dX^\alpha\,\mathbf{E}_\alpha = \underbrace{dX^\alpha\,g_{\alpha\beta}}_{=dX_\beta}\,\mathbf{E}^\beta = dX_\beta\,\mathbf{E}^\beta \ ,$$

having used Cartesian coordinates,

$$\begin{array}{llll} X^0 = ct & X^1 = x & X^2 = y & X^3 = z \\ X_0 = ct & X_1 = -x & X_2 = -y & X_3 = -z \end{array}$$

Its "length", or better pseudo-norm with Minkowski metric, is invariant and reads

$$ds^2 = d\mathbf{X}\cdot d\mathbf{X} = (dX_\alpha \mathbf{E}^\alpha)\cdot\left(dX^\beta \mathbf{E}_\beta\right) = c^2\,dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = c^2\,dt^2 - |d\vec{r}|^2$$

Note: ds is invariant **todo** prove it. And/or add a section about the role of invariance.

For a co-moving observer, $d\vec{r}\,' = \vec{0}$, and t' is commonly indicated with τ, and its differential is invariant itself, being the product of a constant (c is a universal constant in special relativity) and an invariant quantity.

$$ds^2 = c^2 dt'^2 - |d\vec{r}\,'|^2 = c^2 d\tau^2 \ .$$

Given the invariant nature of ds,

$$ds^2 = c^2\,d\tau^2 = c^2\,dt^2 - |d\vec{r}|^2 = c^2\,dt^2\left[1 - \frac{1}{c^2}\frac{|d\vec{r}|^2}{dt^2}\right] = c^2\,dt^2\left[1 - \frac{|\vec{v}|^2}{c^2}\right]$$

and thus

$$ds = c\,d\tau = \gamma^{-1}(v/c)\,c\,dt \ ,$$

with $\gamma(w) = \frac{1}{\sqrt{1-w^2}}$.

4-Velocity Given the parametric representation of an event in space-time as a function of its proper time, $\mathbf{X}(\tau)$ or coordinate s, $\mathbf{X}(s)$ the derivative w.r.t. this parameter is defined as the 4-velocity of the event in space time. Using Cartesian coordinates inducing constant and uniform basis $\mathbf{E}_\alpha$, as a function of the observer time t, ct, $x^i(t)$, and the transformation of coordinates $t(\tau)$, with differential $dt = \frac{1}{\gamma}\,d\tau$

$$\mathbf{U}(\tau) := \mathbf{X}'(\tau) = \frac{d}{d\tau}\left(X^\alpha(\tau)\mathbf{E}_\alpha\right) = \frac{dt}{d\tau}(ct\mathbf{E}_0 + x^i(t)\mathbf{E}_i) = \gamma(v/c)\left(c\mathbf{E}_0 + \dot{x}^i(t)\mathbf{E}_i\right) = \gamma(v/c)\left(c\mathbf{E}_0 + \vec{v}\right)$$

or

$$\mathbf{U}(s) := \mathbf{X}'(s) = \frac{dt}{ds}\frac{d}{dt}\mathbf{X}(t) = \dots = \gamma(v/c)\left(\mathbf{E}_0 + \frac{\vec{v}}{c}\right) \ .$$

Note: Using s as the parameter, $\mathbf{U}$ is non-dimensional, and has pseudo-norm = 1,

$$\mathbf{U}(s)\cdot\mathbf{U}(s) = \gamma^2\underbrace{\left(1 - \frac{|\vec{v}|^2}{c^2}\right)}_{=\gamma^{-2}} = 1 \ .$$

Using τ as the parameter, $\mathbf{U}$ has physical dimension of a velocity and pseudo-norm = c.

4-acceleration $\mathbf{X}''(\tau)$ or $\mathbf{X}''(s)$, todo

2.1 Dynamics

4-momentum

$$\mathbf{P} = m\mathbf{U}$$

Using Cartesian coordinates and τ as independent variable,

$$\mathbf{P} = m\mathbf{U} = m\frac{d\mathbf{X}}{d\tau} = m\gamma(c,\vec{v}) \ .$$

The spatial component is γ times the 3-dimensional momentum $\vec{p} = m\vec{v}$; the time component reads

$$P^0 = m\gamma(w)c \ ,$$

and for small ratio $w := \frac{v}{c}$ it can be expanded in Taylor series around $w = 0$ as

$$\gamma(w) \sim \gamma(0) + w\,\gamma'(0) + \frac{1}{2}\,w^2\gamma''(0) + o(w^2) \ ,$$

with

$$\begin{aligned}
\gamma(w)|_{w=0} &= \left.\frac{1}{\sqrt{1-w^2}}\right|_{w=0} = 1 \\
\gamma'(w)|_{w=0} &= \left.-\frac{1}{2}(1-w^2)^{-\frac{3}{2}}(-2w)\right|_{w=0} = w(1-w^2)^{-\frac{3}{2}} = 0 \\
\gamma''(w)|_{w=0} &= \left.\left((1-w^2)^{-\frac{3}{2}} + w\left(-\frac{3}{2}\right)(1-w^2)^{-\frac{5}{2}}(-2w)\right)\right|_{w=0} = \\
&= \left.\left((1-w^2)^{-\frac{3}{2}} + 3w^2(1-w^2)^{-\frac{5}{2}}\right)\right|_{w=0} = 1
\end{aligned}$$

and thus

$$\gamma(w) = 1 + \frac{1}{2} w^2 + o(w^2)$$

and

$$\gamma(v/c)\, m\, c \sim m\, c \left(1 + \frac{v^2}{c^2} \right) = \frac{1}{c} \left(mc^2 + \frac{1}{2} m |\vec{v}|^2 \right)$$

Thus, recognizing energy ($E = \gamma m c^2$) and 3-momentum ($\vec{p} = m_3 \vec{v}$, with $m_3 := \gamma m$), the 4-momentum can be written as

$$\mathbf{P} = m\mathbf{U} = \gamma m \left(1, \frac{\vec{v}}{c} \right) =: \frac{1}{c} \left(\frac{E}{c}, \vec{p} \right)$$

Its pseudo-norm reads

$$m^2 = \mathbf{P} \cdot \mathbf{P} = \frac{1}{c^4} \left(E^2 - c^2 |\vec{p}|^2 \right)$$

and thus the relation between E, $\vec{p}$, m and c,

$$E^2 = m^2 c^4 + c^2 |\vec{p}|^2 \ ,$$

from which, for $\vec{v} = \vec{0} \rightarrow \vec{p} = \vec{0}$,

$$E^2 = m^2 c^4 \ ,$$

and keeping only the solution with positive energy (**todo** *reference to Dirac's equation and anti-matter?*)

$$E = mc^2 \ .$$

2.1.1 Lagrangian approach

Free particle.

$$\mathbf{0} = \frac{d\mathbf{P}}{ds} = \frac{d}{ds} \left(m\mathbf{X}'(s) \right)$$

Weak form

$$\begin{aligned} 0 &= \mathbf{W}(s) \cdot \frac{d}{ds} \left(m\mathbf{X}'(s) \right) = \\ &= \frac{d}{ds} \left[m\mathbf{W}(s) \cdot \mathbf{X}'(s) \right] - m\mathbf{W}'(s) \cdot \mathbf{X}'(s) = \end{aligned}$$

Using generalized coordinates $q^k(s)$, the event can be written in parametric form as $\mathbf{X}(q^k(s), s)$, while the velocity reads

$$\mathbf{U}(s) = \mathbf{X}'(s) = \frac{d}{ds} \mathbf{X}(q^k(s), s) = q^{k\prime}(s) \underbrace{\frac{\partial \mathbf{X}}{\partial q^k}(q^k(s), s)}_{= \frac{\partial \mathbf{X}'}{\partial q^{k\prime}}} + \frac{\partial \mathbf{X}}{\partial s}(q^k(s), s) = \mathbf{U}(q^{k\prime}(s), q^k(s), s)$$

Choosing $\mathbf{W} = \frac{\partial \mathbf{X}}{\partial q^k} = \frac{\partial \mathbf{X}'}{\partial q^{k\prime}}$ in the weak form,

$$\begin{aligned} 0 &= \frac{d}{ds} \left[m\mathbf{W} \cdot \mathbf{X}' \right] - m\mathbf{W}' \cdot \mathbf{X}' = \\ &= \frac{d}{ds} \left[m \frac{\partial \mathbf{X}'}{\partial q^{k\prime}} \cdot \mathbf{X}' \right] - m \frac{d}{ds} \frac{\partial \mathbf{X}}{\partial q^k} \cdot \mathbf{X}' = \\ &= \frac{1}{2} \left[\frac{d}{ds} \left(\frac{\partial}{\partial q^{k\prime}} \left(m\mathbf{X}' \cdot \mathbf{X}' \right) \right) - \frac{\partial}{\partial q^k} \left(m\mathbf{X}' \cdot \mathbf{X}' \right) \right] = \end{aligned}$$

Defining

$$f\left(q^{k\prime}(s), q^k(s), s\right) = -m\mathbf{X}'\left(q^{k\prime}(s), q^k(s), s\right) \cdot \mathbf{X}'\left(q^{k\prime}(s), q^k(s), s\right) = -m \ ,$$

multiplying by a “regular” generic function $w(s)$, neglecting factor $\frac{1}{2}$ and integrating by parts

$$\begin{aligned}
0 &= -\int_{s=s_a}^{s_b} w(s)\left[\frac{d}{ds}\frac{\partial f}{\partial q^{k\prime}} - \frac{\partial f}{\partial q^k}\right] ds = \\
&= -\left[w(s)\frac{\partial f}{\partial q^{k\prime}}\right]\Bigg|_{s=s_a}^{s_b} + \int_{s=s_a}^{s_b}\left[w'(s)\frac{\partial f}{\partial q^{k\prime}} + \frac{\partial f}{\partial q^k}\right] ds = \\
&= -\left[w(s)\frac{\partial f}{\partial q^{k\prime}}\right]\Bigg|_{s=s_a}^{s_b} + \delta\int_{s=s_a}^{s_b} f\left(q^{k\prime}(s), q^k(s), s\right) ds \ .
\end{aligned}$$

Thus, provided that $w(s_1) = w(s_2) = 0$, equation of motion of free particle implies stationariety of functional

$$\int_{s=s_a}^{s_b} f\left(q^{k\prime}(s), q^k(s), s\right) ds \ ,$$

i.e.

$$\delta\int_{s=s_a}^{s_b} f\left(q^{k\prime}(s), q^k(s), s\right) ds = 0$$

Using t as independent parameter, $ds = \gamma^{-1}\, c\, dt$, the functional can be recast as

$$\int_{t=t_a}^{t_b} -m\, c\sqrt{1 - \frac{|\vec{v}|^2}{c^2}}\, dt \ ,$$

to find the (3-dimensional) Lagrangian (multiply by c to get the right physical dimension; check if it’s required and wheter it’s possible to make c appear before),

$$\mathcal{L} = -\sqrt{1 - \frac{|\vec{v}|^2}{c^2}}\, m\, c^2 \ ,$$

and retrieve 3-momentum as (being $\vec{v} = \dot{\vec{r}}$)

$$\begin{aligned}
\vec{p} &:= \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = \\
&= -mc^2\frac{1}{2}\left(1 - \frac{|\vec{v}|^2}{c^2}\right)^{-\frac{1}{2}}\left(-2\frac{\vec{v}}{c^2}\right) = \\
&= m\left(1 - \frac{|\vec{v}|^2}{c^2}\right)^{-\frac{1}{2}}\vec{v} = \\
&= \gamma\, m\, \vec{v} \ ,
\end{aligned}$$

and energy as

$$\begin{aligned}
E &:= \vec{p}\cdot\vec{v} - \mathcal{L} = \\
&= \gamma\, m\, |\vec{v}|^2 + \gamma^{-1}\, m\, c^2 = \\
&= \gamma\, m\, c^2\left(\frac{|\vec{v}|^2}{c^2} + \gamma^{-2}\right) = \\
&= \gamma\, m\, c^2\left(\frac{|\vec{v}|^2}{c^2} + 1 - \frac{|\vec{v}|^2}{c^2}\right) = \\
&= \gamma\, m\, c^2 \ .
\end{aligned}$$

2.2 Electromagnetism

2.2.1 Classical electromagnetic theory

Maxwell equations

Maxwell equations read

$$\begin{cases}\nabla\cdot\vec{d}=\rho_f\\ \nabla\times\vec{e}+\partial_t\vec{b}=\vec{0}\\ \nabla\cdot\vec{b}=0\\ \nabla\times\vec{h}-\partial_t\vec{d}=\vec{j}_f\end{cases}$$

or in vacuum, with $\rho_f=\rho$, $\vec{j}=\vec{j}_f$, $\vec{d}=\varepsilon_0\vec{e}$, $\vec{b}=\mu_0\vec{h}$

$$\begin{cases}\nabla\cdot\vec{e}=\frac{\rho}{\varepsilon_0}\\ \nabla\times\vec{e}+\partial_t\vec{b}=\vec{0}\\ \nabla\cdot\vec{b}=0\\ \nabla\times\vec{b}-\mu_0\varepsilon_0\partial_t\vec{e}=\mu_0\vec{j}\end{cases}$$

Electromagnetic potentials

The electromagnetic field can be written in terms of the electromagnetic potentials

$$\begin{cases}\vec{b}=\nabla\times\vec{a}\\ \vec{e}=-\partial_t\vec{a}-\nabla\varphi\end{cases}$$

Lorentz force

A particle in motion in a electromagnetic field is subject to Lorentz force. In classical electromagnetism, the expression of Lorentz force reads

$$\vec{F}=q\left(\vec{e}-\vec{b}\times\vec{v}\right)\ ,$$

whose power is

$$\vec{v}\cdot\vec{F}=\vec{v}\cdot q\left(\vec{e}-\vec{b}\times\vec{v}\right)=q\vec{v}\cdot\vec{e}\ .$$

2.2.2 Electromagnetic potential

$$\begin{cases}\vec{b}=\nabla\times\vec{a}\\ \vec{e}=-\partial_t\vec{a}-\nabla\varphi\end{cases}$$

$$\mathbf{A}=\mathbf{E}_\alpha A^\alpha=\frac{\varphi}{c}\mathbf{E}_0+\vec{a}$$

$$\boldsymbol{\nabla}\mathbf{A}=\left(\mathbf{E}^\alpha\frac{\partial}{\partial X^\alpha}\right)\left(A^\beta\mathbf{E}_\beta\right)=\frac{\partial A^\beta}{\partial X^\alpha}\mathbf{E}^\alpha\otimes\mathbf{E}_\beta=g_{\alpha\gamma}\frac{\partial A^\beta}{\partial X^\alpha}\mathbf{E}_\gamma\otimes\mathbf{E}_\beta\ .$$

whose components may be collected in a 2-dimensional array (first index for rows, second index for columns),

$$(\nabla \mathbf{A})^{\beta}_{\alpha} = \frac{\partial A^{\beta}}{\partial X^{\alpha}} = \begin{bmatrix} c^{-2}\partial_t \varphi & c^{-1}\partial_x \varphi & c^{-1}\partial_y \varphi & c^{-1}\partial_z \varphi \\ c^{-1}\partial_t a_x & \partial_x a_x & \partial_y a_x & \partial_z a_x \\ c^{-1}\partial_t a_y & \partial_x a_y & \partial_y a_y & \partial_z a_y \\ c^{-1}\partial_t a_z & \partial_x a_z & \partial_y a_z & \partial_z a_z \end{bmatrix}$$

or covariant-covariant coomponents,

$$(\nabla \mathbf{A})_{\alpha\beta} = \frac{\partial A_{\beta}}{\partial X^{\alpha}} = g_{\beta\gamma}\frac{\partial A^{\gamma}}{\partial X^{\alpha}} = \begin{bmatrix} c^{-2}\partial_t \varphi & c^{-1}\partial_x \varphi & c^{-1}\partial_y \varphi & c^{-1}\partial_z \varphi \\ -c^{-1}\partial_t a_x & -\partial_x a_x & -\partial_y a_x & -\partial_z a_x \\ -c^{-1}\partial_t a_y & -\partial_x a_y & -\partial_y a_y & -\partial_z a_y \\ -c^{-1}\partial_t a_z & -\partial_x a_z & -\partial_y a_z & -\partial_z a_z \end{bmatrix}$$

or contravariant-contravariant coomponents,

$$(\nabla \mathbf{A})^{\alpha\beta} = \frac{\partial A^{\beta}}{\partial X_{\alpha}} = g_{\beta\gamma}\frac{\partial A^{\alpha}}{\partial X^{\gamma}} = \begin{bmatrix} c^{-2}\partial_t \varphi & -c^{-1}\partial_x \varphi & -c^{-1}\partial_y \varphi & -c^{-1}\partial_z \varphi \\ c^{-1}\partial_t a_x & -\partial_x a_x & -\partial_y a_x & -\partial_z a_x \\ c^{-1}\partial_t a_y & -\partial_x a_y & -\partial_y a_y & -\partial_z a_y \\ c^{-1}\partial_t a_z & -\partial_x a_z & -\partial_y a_z & -\partial_z a_z \end{bmatrix}$$

The electromagnetic field tensor is defined as the anti-symmetric part of the gradient of the 4-electromagnetic potential,

$$\mathbf{F} = \left[\boldsymbol{\nabla}\mathbf{A} - (\boldsymbol{\nabla}\mathbf{A})^T\right]$$

whose components may be collected in a 2-dimensional array (first index for rows, second index for columns),

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -\frac{\underline{e}^T}{c} \\ \frac{\underline{e}}{c} & \underline{b}_{\times} \end{bmatrix} \qquad , \qquad F_{\alpha\beta} = \begin{bmatrix} 0 & \frac{\underline{e}^T}{c} \\ -\frac{\underline{e}}{c} & \underline{b}_{\times} \end{bmatrix}$$

2.2.3 Electromagnetic field and electromagnetic field equations

The pair of Maxwell equations

$$\begin{cases} \rho_f = \nabla \cdot \vec{d} \\ \vec{j}_f = -\partial_t \vec{d} + \nabla \times \vec{h} \end{cases}$$

can be re-written in 4-formalism, using 4-gradient in Cartesian coordinates

$$\boldsymbol{\nabla} = \mathbf{E}^{\alpha}\frac{\partial}{\partial X^{\alpha}} = \mathbf{E}_0\frac{\partial}{c\partial t} + \mathbf{E}_i\frac{\partial}{\partial x^i} = \mathbf{E}_0\frac{\partial}{c\partial t} + \nabla \,,$$

and the definition of the 4-current density vector

$$\mathbf{J} = J^{\alpha}\mathbf{E}_{\alpha} = c\rho\,\mathbf{E}_0 + \vec{j}$$

so that

$$c\rho\mathbf{E}_0 + \vec{j} = \boldsymbol{\nabla}\cdot\mathbf{F} = \boldsymbol{\nabla}\cdot\left[(0\,\mathbf{E}_0 + c\vec{d})\otimes\mathbf{E}_0 + (-\mathbf{E}_0 c\vec{d} + \vec{h}_{\times})\right] ,$$

with the displacement field tensor,

$$\mathbf{D} = D^{\alpha\beta}\,\mathbf{E}_{\alpha}\,\mathbf{E}_{\beta} \,,$$

with components (rows for the first index, columns for the second index)

$$D^{\alpha\beta} = \begin{bmatrix} 0 & -cd_x & -cd_y & -cd_z \\ cd_x & 0 & -h_z & h_y \\ cd_y & h_z & 0 & -h_x \\ cd_z & -h_y & h_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -c\underline{d}^T \\ c\underline{d} & \underline{h}_{\times} \,. \end{bmatrix}$$

The pair of Maxwell equations

$$\begin{cases} \nabla \cdot \vec{b} = 0 \\ \partial_t \vec{b} + \nabla \times \vec{e} = \vec{0} \end{cases}$$

can be re-written in 4-formalism as

$$0 = \partial_\mu F_{\eta\xi} + \partial_\eta F_{\xi\mu} + \partial_\xi F_{\mu\eta}$$

Among these $64 = 4^3$ equations, there are only 4 independent equations.

- If 2 indices are the same, the corresponding equation is the identity $0 = 0$. As an example, if $\mu = \eta$

$$0 = \partial_\mu F_{\mu\xi} + \partial_\mu \underbrace{F_{\xi\mu}}_{-F_{\mu xi}} + \partial_\xi \underbrace{F_{\mu\mu}}_{=0} = 0 \ ,$$

  thus only combinations with different indices may provide some information.

- Given the ordered set of indices (μ, η, ξ), switching a pair of indices provides the same equation. As an example, switching μ and η

$$\begin{aligned} 0 &= \partial_\eta F_{\mu\xi} + \partial_\mu F_{\xi\eta} + \partial_\xi F_{\eta\mu} = \\ &= \partial_\eta (-F_{\xi\mu}) + \partial_\mu (-F_{\eta\xi}) + \partial_\xi (-F_{\mu\eta}) \ . \end{aligned}$$

- Thus, only 4 combination of different indices, without taking order into account, provide independent information

$$\begin{aligned} (1,2,3): &\quad 0 = \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \partial_x(-b_x) + \partial_y(-b_y) + \partial_z(-b_z) \\ (2,3,0): &\quad 0 = \partial_2 F_{30} + \partial_3 F_{02} + \partial_0 F_{23} = \partial_y\left(-\frac{e_z}{c}\right) + \partial_z\left(\frac{e_y}{c}\right) + \partial_{ct}(-b_x) \\ (3,0,1): &\quad 0 = \partial_3 F_{01} + \partial_0 F_{13} + \partial_1 F_{30} = \partial_z\left(\frac{e_x}{c}\right) + \partial_{ct}(-b_y) + \partial_x\left(-\frac{e_z}{c}\right) \\ (0,1,2): &\quad 0 = \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = \partial_{ct}(-b_z) + \partial_x\left(-\frac{e_y}{c}\right) + \partial_y\left(\frac{e_x}{c}\right) \end{aligned}$$

  i.e.

$$\begin{aligned} (1,2,3): &\quad 0 = -\nabla \cdot \vec{b} \\ (2,3,0): &\quad 0 = -\frac{1}{c}\left[\partial_t b_x + (\partial_y e_z - \partial_z e_y)\right] \\ (3,0,1): &\quad 0 = -\frac{1}{c}\left[\partial_t b_y + (\partial_z e_x - \partial_x e_z)\right] \\ (0,1,2): &\quad 0 = -\frac{1}{c}\left[\partial_t b_z + (\partial_x e_y - \partial_y e_x)\right] \end{aligned}$$

  i.e.

$$\begin{cases} 0 = \nabla \cdot \vec{b} \\ \vec{0} = \partial_t \vec{b} + \nabla \times \vec{e} \end{cases}$$

2.2.4 Point particle in electromagnetic field

Lorentz 4-force acting on a point charge of electric charge charge q reads

$$\mathbf{f} = \mathbf{F} \cdot \mathbf{J} = q\,\mathbf{F} \cdot \mathbf{U} \ .$$

so that the dynamical equation reads

$$m\,\mathbf{X}'' = q\,\mathbf{F} \cdot \mathbf{X}'$$

2.2.5 Energy balance

$$\frac{\partial u}{\partial t} =$$
$$\frac{\partial \vec{s}}{\partial t} =$$

…

$$\boldsymbol{\nabla} \cdot \mathbf{T} = -\mathbf{F} \cdot \mathbf{J}$$

Part II

General Relativity

CHAPTER THREE

GENERAL RELATIVITY

CHAPTER

FOUR

GENERAL RELATIVITY - NOTES

Part III

Statistical Mechanics

CHAPTER

FIVE

STATISTICAL PHYSICS

CHAPTER

SIX

STATISTICAL PHYSICS - NOTES

6.1 Ensembles

6.2 Microcanonical ensemble

6.3 Canonical ensemble

6.4 Macrocanonical esemble

6.5 Statistics

Each of the N components of the system is in an **energy level** i. Energy level i has g_i sublevels with the same energy level.

- energy levels, E_i of each component
- occupation number N_i of level i
- **Central role of energy.** In a system macroscopically at rest, the energy of a system is the only macroscopic meaningful non-zero mechanical quantity, constant for closed and isolated systems
- **Principle of maximum uncertainty**, **maximum entropy**, **minimum information**: given a measurement of a macroscopic variable V, describing the macrostate of the system, the feasible un-observed/able microstates of the system are the microstates consistent with it: there's usually a sharp maximum of in the probability density of the micrsotates.

Given a macrostate, what's the number of ways $W(N_i; g_i)$ to get a consistent microstate? Once the expression is found, constrained optimization follows: optimization w.r.t. N_i is usually performed in the limit of $N_i \rightarrow +\infty$ (why in Fermi-Dirac distribution, obeying Pauli exclusion principle?), with the values of the macroscopic variables as constraints usually treated with Lagrange multiplier.

6.5.1 Maxwell-Boltzmann

Statistics of distinguishible components.

6.5.2 Bose-Einstein

Statistics of undistinguishable components that can be in the same (sub)level. Given the number of elementary components $\sum_i N_i = N$ and the energy $\sum_i N_i E_i = E$,

$$W_{BE,i} = \frac{(N_i + g_i - 1)!}{N_i!(g_i - 1)!} \qquad , \qquad W_{BE} = \prod_i W_{BE,i} \ . \tag{6.1}$$

Counting microstates

todo write page *Combinatorics* and add link

Most likely microstate. Instead of maximizing (6.1), the objective function is $\ln W_{BE}$, after using Stirling approximation in the limit of large N_i and g_i, $N_i! \sim \left(\frac{N_i}{e}\right)^{N_i}$. The approximate occupation number of one of the G_i sublevels of the i^{th} level of the most likely microstate is

$$n_i := \frac{N_i}{G_i} = \frac{1}{e^{\alpha + \beta E_i} - 1} \ .$$

Optimization

$$\begin{aligned}
J(N_i, \alpha, \beta) = \ln W_{BE} + \alpha \left(N - \sum_i N_i \right) + \beta \left(E - \sum_i N_i E_i \right) = \\
&= \sum_i \left\{ \ln(N_i + g_i - 1)! - \ln N_i! - \ln(g_i - 1)! \right\} + \alpha \left(N - \sum_i N_i \right) + \beta \left(E - \sum_i N_i E_i \right) \simeq \\
&\simeq \sum_i \left\{ (N_i + g_i - 1) \ln(N_i + g_i - 1) - N_i \ln N_i - (g_i - 1) \ln(g_i - 1) + N_i + g_i - 1 - N_i - (g_i - 1) \right\} + \alpha \left(N - \sum_i N \right. \\
&= \sum_i \left\{ (N_i + g_i - 1) \ln(N_i + g_i - 1) - N_i \ln N_i - (g_i - 1) \ln(g_i - 1) \right\} + \alpha \left(N - \sum_i N_i \right) + \beta \left(E - \sum_i N_i E_i \right)
\end{aligned}$$

Using $\partial_n (n + a) \ln(n + a) = \ln(n + a) + 1$,

$$0 = \partial_{N_k} J \simeq \left\{ \ln(N_k + g_k - 1) - \ln N_k \right\} - \alpha - \beta E_k \ ,$$

and thus

$$\begin{aligned}
\ln \frac{N_k + g_k - 1}{N_k} &= \alpha + \beta E_k \ , \\
\frac{N_k + g_k - 1}{N_i} &= e^{\alpha + \beta E_k} \\
N_k = \frac{g_k - 1}{e^{\alpha + \beta E_k} - 1} &\simeq \frac{g_k}{e^{\alpha + \beta E_k} - 1} \ ,
\end{aligned}$$

Thus, in the limit of $g_k \gg 1$, the occupation number of the k level is

$$N_k = \frac{G_k}{e^{\alpha + \beta E_k} - 1} \ ,$$

and the average occupation number of one of the g_k sublevels in the k level is

$$n_k := \frac{N_k}{G_k} = \frac{1}{e^{\alpha+\beta E_k} - 1}$$

Meaning of α, β

Example 1 (Black-body radiation: Planck, Wien, and Stefan-Boltzmann laws)

Planck's law. Energy density w.r.t. frequency

$$u_f(f,T) = \frac{8\pi h f^3}{c^3} \frac{1}{e^{\frac{hf}{k_B T}} - 1}$$

Planck's law in a cubic box

Planck's law uses:

- relation between pulsation and wave vector, or frequency and wave number and the speed of light c for light waves

$$c = \frac{\omega}{|\vec{k}|} = \lambda f$$

$$f = \frac{\omega}{2\pi} = \frac{c|\vec{k}|}{2\pi}$$

- Planck assumption that the minimum non-zero energy of a mode with frequency f is $E = hf$, and all the possible values of the energy of the mode is

$$E_m = mhf \quad , \quad m \in \mathbb{N} \, .$$

Taking a cubic box with sides $L_x = L_y = L_z = L$, the possibile modes have (**todo** *why? Which boundary condition? Periodic? Some physical? Just fictitious discretization?*) in each direction wave-lengths $\lambda_n = \frac{L}{|\vec{n}|} = \frac{2\pi}{|\vec{k}|}$,

$$\vec{k} = \frac{2\pi}{L}\vec{n} \, .$$

Mode density in $\vec{n}$-domain is 2 mode per each volume of unit length (2 polarization), and thus the number of modes dN in an elementary volume is

$$dN = 2\, d^3\vec{n} \, ,$$

Changing variables, it's possible to find the mode density w.r.t. wave vector $\vec{k}$,

$$dN = 2\, d^3\vec{n} = 2\, \frac{L^3}{(2\pi)^3}\, d^3\vec{k} \, ,$$

or with its absolute value, exploiting the isotropy of the density function - and writing the elementary volume using "spherical coordinates" $d^3\vec{k} = 4\pi \left|\vec{k}\right|^2 d\left|\vec{k}\right|$,

$$\begin{aligned}
dN &= \frac{V}{(2\pi)^3} 8\pi \left|\vec{k}\right|^2 d\left|\vec{k}\right| = \\
&= \frac{V}{(2\pi)^3} 8\pi \frac{8\pi^3}{c^3} f^2 df = \\
&= V\frac{8\pi}{c^3} f^2 df =: V g(f) df \, .
\end{aligned}$$

Average energy of a mode

Using Boltzmann distribution (**why?**) for the energy distribution in a single mode,

$$P(E_r) = \frac{e^{-\beta E_r}}{Z} \ ,$$

with $E_r = rhf$, and the partition function

$$Z = \sum_s e^{-\beta E_s} = \sum_s e^{-\beta h f s} = \frac{1}{1 - e^{-\beta h f}} \ .$$

The average energy of the mode reads

$$\begin{aligned}
\langle E \rangle = \sum_r E_r P(E_r) = \\
& = \sum_r r h f \frac{e^{-\beta h f r}}{Z} = \\
& = h f (1 - e^{-\beta h f}) \sum_r r e^{-\beta h f r} = \\
& = h f (1 - e^{-\beta h f}) \frac{e^{-\beta h f}}{(1 - e^{-\beta h f})^2} = \\
& = \frac{h f}{e^{\beta h f} - 1} \ .
\end{aligned}$$

Putting together the mode number density and the average energy of a mode, the energy density per unit volume, per frequency reads

$$\begin{aligned}
u(f, T) = \langle E \rangle(f) \, g(f) = \\
& = \frac{h f}{e^{\beta h f} - 1} \frac{8\pi}{c^3} f^2 = \\
& = \frac{8 \pi h f^3}{c^3} \frac{1}{e^{\beta h f} - 1} \ .
\end{aligned}$$

Property of the series

$$\sum_{n=0}^{+\infty} n x^n = \frac{x}{(1-x)^2}$$

Proof. If the series is convergent (is this the required condition?)

$$\frac{d}{dx} \sum_{n=0}^{+\infty} x^n = \frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\frac{d}{dx} \sum_{n=0}^{+\infty} x^n = \sum_{n=0}^{+\infty} n x^{n-1}$$

$$x \frac{d}{dx} \sum_{n=0}^{+\infty} x^n = \sum_{n=0}^{+\infty} n x^n = \frac{x}{(1-x)^2}$$

Sperctral radiance, B_f, so that an infinitesimal amount of power radiated by a surface ... is $dP = B_f(f, T) \cos\theta \, dA \, d\Omega \, df$

$$B_f(f, T) = \frac{2 h f^3}{c^2} \frac{1}{e^{\frac{hf}{k_B T}} - 1} \ .$$

This expression is obtained[1] assuming homogeneous radiation from a small hole cut into a wall of the box. Only half of the energy radiates through the hole - so factor $\frac{1}{2}$ in front of the energy density - through a solid angle 2π - and thus this process give the same result as a radiation of all the energy density in all the space directions, just providing the same factor $\frac{1}{4\pi}$. The flux of energy "has velocity" c and thus

$$B_f(f,T) = \frac{1}{4\pi} u_f(f,T) c \ .$$

Wien's law. Wien's law tells that the frequency f^* corresponding to the maximum of the spectral radiance of a black-body radiation described by Planck's law is proportional to its temperature.

From direct evaluation of the derivative of the spectral radiance as a function of f,

$$\begin{aligned}
\partial_f B_f(f,T) &= \frac{2h}{c^2} \left[3f^2 \frac{1}{e^{\frac{hf}{k_B T}} - 1} + f^3 \left(- \frac{\frac{h}{k_B T} e^{\frac{hf}{k_B T}}}{\left(e^{\frac{hf}{k_B T}} - 1 \right)^2} \right) \right] = \\
&= \frac{2hf^2 e^{\frac{hf}{k_B T}}}{c^2 \left(e^{\frac{hf}{k_B T}} - 1 \right)^2} \left[3 \left(1 - e^{-\frac{hf}{k_B T}} \right) - \frac{hf}{k_B T} \right] \ .
\end{aligned}$$

Now, if $\partial_f B_f(f,T) = 0$ the frequency is either $f = 0$, or the solution of the nonlinear algebraic equation

$$0 = 3 \left(1 - e^{-\frac{hf}{k_B T}} \right) - \frac{hf}{k_B T} \ .$$

Defining $x := \frac{hf}{k_B T}$, this equation becomes

$$0 = 3(1 - e^x) - x \ ,$$

whose solution $x^* \approx 2.82$ can be easily evaluated with an iterative method (or expressed in term of the Lambert's function W, so loved at Stanford and on Youtube: they'd probaly like to look at tabulated values, or pose). Once the solution x^* of this non-dimensional equation is found, the frequency where maximum energy density occurs reads

$$f^* = \frac{k_B T}{h} x^* \simeq 2.82 \frac{k_B}{h} T \ .$$

Stefan-Boltzmann law.

$$\begin{aligned}
\frac{P}{A} &= \int B_f(f,T) \cos\phi \, df \, d\Omega = \\
&= \int_{f=0}^{+\infty} \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{2\pi} B_f(f,T) \cos\phi \sin\phi \, df \, d\phi \, d\theta = \\
&= \pi \int_{f=0}^{+\infty} B_f(f,T) \, df = \\
&= \frac{2\pi h}{c^2} \int_{f=0}^{+\infty} \frac{f^3}{e^{\frac{hf}{k_B T}} - 1} \, df = \\
&= \frac{2\pi h}{c^2} \left(\frac{k_B T}{h} \right)^4 \int_{u=0}^{+\infty} \frac{u^3}{e^u - 1} \, du \ .
\end{aligned}$$

The value of the integral is $\frac{\pi^4}{15}$ and thus

$$\frac{P}{A} = \sigma T^4 \qquad , \qquad \sigma = \frac{2\pi^5 k_B^4}{15 c^2 h^3} \ .$$

[1] Derivation of Planck's Law.

Example 2 (Energy density and radiance)

Radiance. The radiance $L_{e,\Omega}$ of a surface is the flux of energy per unit solid angle, per unit projected area of the source.

Spectral radiance in frequency is the radiance per unit frequency, $L_{e,\Omega,f} = \frac{\partial L_{e,\Omega}}{\partial f}$.

6.5.3 Fermi-Dirac

Statistics of undistinguishable components that can't be in the same (sub)level, obeying to the Pauli exclusion principle. Given the number of elementary components $\sum_i N_i = N$ and the energy $\sum_i N_i E_i = E$,

$$W_{FD,i} = \frac{G_i!}{(G_i - N_i)!N_i!} \qquad , \qquad W_{FD} = \prod_i W_{FD,i} \ . \tag{6.2}$$

Counting microstates

todo write page *Combinatorics* and add link

Most likely microstate. The approximate occupation number of the i^{th} level of the most likely microstate is

$$n_i := \frac{N_i}{G_i} = \frac{1}{1 + e^{\alpha + \beta E_i}} \ .$$

Optimization

$$\begin{aligned}
J(N_i, \alpha, \beta) = \ln W_{FD} + \alpha \left(N - \sum_i N_i \right) + \beta \left(E - \sum_i N_i E_i \right) = \\
= \sum_i \{ \ln G_i! - \ln (G_i - N_i)! - \ln N_i! \} + \alpha \left(N - \sum_i N_i \right) + \beta \left(E - \sum_i N_i E_i \right) = \\
= \sum_i \{ G_i \ln G_i - (G_i - N_i) \ln (G_i - N_i) - N_i \ln N_i \} + \alpha \left(N - \sum_i N_i \right) + \beta \left(E - \sum_i N_i E_i \right) =
\end{aligned}$$

Using $\partial_n (n+a) \ln (n+a) = \ln (n+a) + 1$,

$$0 = \partial_{N_k} J \simeq \{ \ln (G_k - N_k) - \ln N_k \} - \alpha - \beta E_k \ ,$$

and thus

$$\ln \frac{G_k - N_k}{N_k} = \alpha + \beta E_k \ ,$$

$$\frac{G_k}{N_k} - 1 = e^{\alpha + \beta E_k}$$

The occupation number of the k level is

$$N_k = \frac{G_k}{1 + e^{\alpha + \beta E_k}} \ .$$

The average occupation of the G_k sublevels of the k level is

$$n_k := \frac{N_k}{G_k} = \frac{1}{1 + e^{\alpha + \beta E_k}} \ .$$

Meaning of α, β

CHAPTER

SEVEN

STATISTICAL PHYSICS - STATISTICS MISCELLANEA

Information content and Entropy

Given a discrete random variable X with probability mass function $p_X(x)$, the self-information (**todo** *what about mutual information of random variables?*) is defined as the opposite of the logaritm of the mass function $p_X(x)$,

$$I_X(x) := -\ln\left(p_X(x)\right) \ .$$

Information content of indenpendent random variables is additive. Since $p_{X,Y}(x,y) = p_X(x) p_Y(y)$,

$$I_{X,Y}(x,y) = -\ln\left(p_{X,Y}(x,y)\right) = -\ln\left(p_X(x)p_Y(y)\right) = -\ln p_X(x) - \ln p_Y(y) \ .$$

Shannon entropy. Shannon entropy of a discrete random variable X is defined as the expected value of the information content,

$$H(X) := \mathbb{E}[I_X(X)] = \sum p_X(x) I_X(x) = -\sum p_X \ln p_X(x) \ .$$

Gibbs entropy. Gibbs entropy was defined by J.W.Gibbs in 1878,

$$S = -k_B \sum_i p_i \ln p_i \ .$$

Additivity holds for independent random variables.

Boltzmann entropy. Boltmann entropy holds for uniform distributions over Ω possible states, $p_i = \frac{1}{\Omega}$. Gibbs' entropy of this uniform distribution becomes

$$S = -k_B \Omega \frac{1}{\Omega} \ln \frac{1}{\Omega} = k_B \ln \Omega \ .$$

Entropy in Quantum Mechanics. todo

Boltzmann distribution

Given a set of discrete states with probability p_i, and the average measure as "macroscopic quantity" $E = \sum_i p_i E_i$, Boltzann distribution maximizes the entropy (**todo** *Link to min info, max uncertainty*)

$$S = -k_B \sum_i p_i \ln p_i \ .$$

The distribution follows from the constrained optimization

$$\widetilde{S} = S - \alpha \left(\sum_i p_i - 1 \right) - \beta \left(\sum_i p_i E_i - E \right)$$

$$0 = \partial_\alpha \widetilde{S} = -\sum_i p_i - 1$$
$$0 = \partial_\beta \widetilde{S} = -\sum_i p_i E_i - E$$
$$0 = \partial_{p_k} \widetilde{S} = -k_B \left(\ln p_k + 1\right) - \alpha - \beta E_k$$

and thus

$$p_k = e^{-1-\frac{\alpha}{k_B}-\frac{\beta}{k_B}E_k} = e^{-\left(1+\frac{\alpha}{k_B}\right)} e^{-\frac{\beta}{k_B}E_k} = C e^{-\frac{\beta}{k_B}E_k} \ ,$$

and the normalization constant C is determined by normalization condition

$$1 = \sum_k p_k = C \sum_k e^{-\frac{\beta E_k}{k_B}}$$

The inverse $Z = C^{-1}$ is defined as the **partition function**,

$$Z = C^{-1} = \sum_k e^{-\frac{\beta E_k}{k_B}} \ ,$$

and the probability distribution becomes

$$p_k = \frac{e^{-\frac{\beta E_k}{k_B}}}{Z} = \frac{e^{-\frac{\beta E_k}{k_B}}}{\sum_i e^{-\frac{\beta E_i}{k_B}}} \ .$$

Properties.

$$\frac{p_k}{p_i} = e^{-\frac{\beta}{k_B}(E_k - E_i)} \ .$$

Thermodynamics. Comparison of statistics and classical thermodynamics

First principle of classical thermodynamics (for a monocomponent gas with no electric charge,…) reads

$$T\,dS = dE + P\,dV$$

Entropy for Boltzmann distribution reads

$$\begin{aligned} S &= -k_B \sum_i p_i \ln p_i = \\ &= -k_B \sum_i \left[p_i \left(-\frac{\beta E_i}{k_B} - \ln Z \right) \right] = \\ &= \beta \langle E \rangle + k_B \ln Z \end{aligned}$$

From classical thermodyamics, temperature T can be defined as the partial derivative of the entropy of a system w.r.t. its internal energy keeping constant all the other independent variables,

$$\begin{aligned}
\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)\Big|_X & = \\
& = \frac{\partial \beta}{\partial E} E + \beta + k_B \frac{\partial \ln Z}{\partial E} = \\
& = \frac{\partial \beta}{\partial E} E + \beta + k_B \frac{1}{Z} \frac{\partial Z}{\partial E} = \\
& = \frac{\partial \beta}{\partial E} E + \beta + k_B \frac{1}{Z} \frac{\partial Z}{\partial \beta} \frac{\partial \beta}{\partial E} = \\
& = \frac{\partial \beta}{\partial E} E + \beta + k_B \frac{1}{Z} \left(-\sum_i \frac{E_i}{k_B} e^{-\frac{\beta E_i}{k_B}} \right) \frac{\partial \beta}{\partial E} = \\
& = \frac{\partial \beta}{\partial E} E + \beta - \left(\sum_i E_i p_i \right) \frac{\partial \beta}{\partial E} = \\
& = \frac{\partial \beta}{\partial E} E + \beta - E \frac{\partial \beta}{\partial E} = \beta \, .
\end{aligned}$$

todo

- write the derivative above clearly in terms of composite functions
- microscopical/statistical approach to the first principle of thermodynamics

$$dE = d\left(\sum_i p_i E_i\right) = \sum_i E_i \, dp_i + \sum_i p_i dE_i$$

Part IV

Quantum Mechanics

CHAPTER EIGHT

QUANTUM MECHANICS

- Principles and postulates
  - statistics and measurements outcomes (Heisenberg built its matrix mechanics only on observables…)
  - CCR
- angluar momentum, spin, and atom

8.1 Mathematical tools for quantum mechanics

Definition 1 (Operator)

Definition 2 (Adjoint operator)

Given an operator $\hat{A} : U \rightarrow V$, its self-adjoint $\hat{A}^* : V \rightarrow U$ is the operator s.t.

$$(\mathbf{v},\ \hat{A}\mathbf{u})_V = (\mathbf{u}, \hat{A}^*\mathbf{v})_U$$

holds for $\forall \mathbf{u} \in U,\ \mathbf{v} \in V$.

Definition 3 (Hermitian (self-adjoint) operator)

If $\hat{A} : U \rightarrow U$, it is a self-adjoint operator if $\hat{A}^* = \hat{A}$.

Self-adjoint operators have real eigenvalues, and orthogonal eigenvectors (at least those associated to different eigenvalues; those associated with the same eigenvalues can be used to build an orthogonal set of vectors with orthogonalization process).

8.2 Postulates of Quantum Mechanics

- …
- Canonical Commutation Relation (CCR) *and Canonical Anti-Commutation Relation…*
- …

8.3 Non-relativistic Mechanics

8.3.1 Statistical Interpretation and Measurement

Wave function

The state of a system is described by a wave function $|\Psi\rangle$

todo

- properties: domain, image,…
- unitary $1 = \langle\Psi|\Psi\rangle = |\Psi|^2$, for statistical interpretation of $|\Psi|^2$ as a density probability function

Operators and Observables

Physical **observable** quantities are represented by *Hermitian operators*. Possible outcomes of measurement are the eigenvalues of the operator

Given $\hat{A}$ and the set of its eigenvectors $\{|A_i\rangle\}_i$ (**todo** *continuous or discrete spectrum…, need to treat this difference quite in details*), with associated eigenvalues $\{a_i\}_i$

$$\hat{A}|A_i\rangle = a_i|A_i\rangle$$

$$|\Psi\rangle = |A_i\rangle\langle A_i|\Psi\rangle = |A_i\rangle\Psi_i^A$$

$$\langle A_j|\Psi\rangle = \langle A_j|A_i\rangle\langle A_i|\Psi\rangle = \Psi_j^A$$

and thus

$$\Psi_j^A = \langle A_j|\Psi\rangle$$
$$\Psi_j^{A*} = \langle\Psi|A_j\rangle$$

- identity operator $\sum_i |A_i\rangle\langle A_i| = \mathbb{1}$, since

$$\sum_i |A_i\rangle\langle A_i|\Psi\rangle = \sum_i |A_i\rangle\langle A_i|\Psi_j^A A_j\rangle = \sum_i |A_i\rangle\delta_{ij}\Psi_j^A = \sum_i |A_i\rangle\Psi_i^A = |\Psi\rangle$$

- Normalization:

$$1 = \langle\Psi|\Psi\rangle = \Psi_j^{A*}\underbrace{\langle A_j|A_i\rangle}_{\delta_{ij}}\Psi_i^A = \sum_i \left|\Psi_i^A\right|^2$$

  with $|\Psi_i^A|^2$ that can be interpreted as the probability of finding the system in state $|\Psi_i^a\rangle$

- Expected value of the physical quantity in the a state $|\Psi\rangle$, with possible values a_i with probability $|\Psi_i^A|^2$

$$\begin{aligned}
\bar{A}_\Psi &= \sum_i a_i |\Psi_i^A|^2 = \\
&= \sum_i a_i \Psi_i^{A*} \Psi_i^A = \\
&= \sum_i a_i \langle \Psi | A_i \rangle \langle A_i | \Psi \rangle = \\
&= \langle \Psi | \left(\sum_i a_i |A_i\rangle\langle A_i| \right) |\Psi\rangle = \\
&= \langle \Psi | \hat{A} | \Psi \rangle =
\end{aligned}$$

since an operator $\hat{A}$ can be written as a function of its eigenvalues and eigenvectors

$$\begin{aligned}
\left(\sum_i a_i |A_i\rangle\langle A_i| \right) \Psi\rangle &= \left(\sum_i a_i |A_i\rangle\langle A_i| \right) c_k |A_k\rangle = \\
&= \sum_i a_i |A_i\rangle c_i = \\
&= \sum_i \hat{A} |A_i\rangle c_i = \\
&= \hat{A} \sum_i |A_i\rangle c_i = \hat{A}|\Psi\rangle \ .
\end{aligned}$$

Space Representation

Position operator $\hat{\mathbf{r}}$ has eigenvalues $\mathbf{r}$ identifying the possible measurements of the position

$$\hat{\mathbf{r}}|\mathbf{r}\rangle = \mathbf{r}|\mathbf{r}\rangle \ ,$$

being $\mathbf{r}$ the result of the measurement (position in space, mathematically it could be a vector), and $|\mathbf{r}\rangle$ the state function corresponding to the measurement $\mathbf{r}$ of the position.

- Result of measurement, $\mathbf{r}$, is a position in space. As an example, it could be a point in an Euclidean space $P \in E^n$. It could be written using properties of Dirac's delta "function"

$$\mathbf{r} = \int_{\mathbf{r}'} \delta(\mathbf{r}' - \mathbf{r})\, \mathbf{r}' d\mathbf{r}'$$

- Projection of wave function over eigenstates of position operator

$$\begin{aligned}
\langle \mathbf{r} | \Psi \rangle (t) = \Psi(\mathbf{r}, t) &= \int_{\mathbf{r}'} \delta(\mathbf{r} - \mathbf{r}') \Psi(\mathbf{r}', t) d\mathbf{r}' = \\
&= \int_{\mathbf{r}'} \langle \mathbf{r} | \mathbf{r}' \rangle \Psi(\mathbf{r}', t) d\mathbf{r}' = \\
&= \int_{\mathbf{r}'} \langle \mathbf{r} | \mathbf{r}' \rangle \langle \mathbf{r}' | \Psi \rangle (t) d\mathbf{r}' = \\
&= \langle \mathbf{r} | \underbrace{\left(\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle \mathbf{r}'| d\mathbf{r}' \right)}_{=\hat{\mathbb{1}}} |\Psi\rangle(t) \ .
\end{aligned}$$

- having used orthogonality (**todo** *why? provide definition and examples of operators with continuous spectrum*)

$$\langle \mathbf{r}' | \mathbf{r} \rangle = \delta(\mathbf{r}' - \mathbf{r})$$

- Expansion of a state function $|\Psi\rangle(t)$ over the basis of the position operator

$$|\Psi\rangle(t) = \hat{\mathbf{1}}|\Psi\rangle(t) = \left(\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}' d\mathbf{r}' \right) |\Psi\rangle(t) = \int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle(t)\, d\mathbf{r}' \ .$$

- Unitariety and probability density

$$\begin{aligned}
1 = \langle\Psi|\Psi\rangle(t) = \langle\Psi| \left(\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}' d\mathbf{r}' \right) |\Psi\rangle \\
= \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle\, d\mathbf{r}' \\
= \int_{\mathbf{r}'} \Psi^*(\mathbf{r}', t)\Psi(\mathbf{r}', t)\, d\mathbf{r}' \\
= \int_{\mathbf{r}'} |\Psi(\mathbf{r}', t)|^2\, d\mathbf{r}'
\end{aligned}$$

  and thus $|\Psi(\mathbf{r}, t)|^2$ can be interpreted as the **probability density function** of measuring position of the system equal to $\mathbf{r}'$.

- Average value of the operator

$$\begin{aligned}
\bar{\mathbf{r}} &= \langle\Psi|\hat{\mathbf{r}}|\Psi\rangle = \\
&= \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|d\mathbf{r}'\ |\hat{\mathbf{r}}|\ \int_{\mathbf{r}''} |\mathbf{r}''\rangle\langle\mathbf{r}''|\Psi\rangle\, d\mathbf{r}'' \\
&= \int_{\mathbf{r}'}\int_{\mathbf{r}''} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|\hat{\mathbf{r}}|\mathbf{r}''\rangle\langle\mathbf{r}''|\Psi\rangle\, d\mathbf{r}' d\mathbf{r}'' = \\
&= \int_{\mathbf{r}'}\int_{\mathbf{r}''} \langle\Psi|\mathbf{r}'\rangle\ \underbrace{\langle\mathbf{r}'|\mathbf{r}''\rangle}_{=\delta(\mathbf{r}'-\mathbf{r}'')}\ \mathbf{r}''\langle\mathbf{r}''|\Psi\rangle\, d\mathbf{r}' d\mathbf{r}'' = \\
&= \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\mathbf{r}'\langle\mathbf{r}'|\Psi\rangle\, d\mathbf{r}' = \\
&= \int_{\mathbf{r}'} \Psi^*(\mathbf{r}', t)\ \mathbf{r}'\ \Psi(\mathbf{r}', t)\, d\mathbf{r}' = \\
&= \int_{\mathbf{r}'} |\Psi(\mathbf{r}', t)|^2\ \mathbf{r}'\, d\mathbf{r}' \ .
\end{aligned}$$

Momentum Representation

Momentum operator as the limit of…**todo** *prove the expression of the momentum operator as the limit of the generator of translation*

$$\langle\mathbf{r}|\hat{\mathbf{p}} = -i\hbar\nabla\langle\mathbf{r}|$$

- Spectrum

$$\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$$

$$\langle\mathbf{r}|\hat{\mathbf{p}}|\mathbf{p}\rangle = -i\hbar\nabla\langle\mathbf{r}|\mathbf{p}\rangle = \mathbf{p}\langle\mathbf{r}|\mathbf{p}\rangle$$

  and thus the eigenvectors in space base $\mathbf{p}(\mathbf{r}) = \langle\mathbf{r}|\mathbf{p}\rangle$ are the solution of the differential equation

$$-i\hbar\nabla\mathbf{p}(\mathbf{r}) = \mathbf{p}\mathbf{p}(\mathbf{r}) \ ,$$

that in Cartesian coordinates reads

$$-i\hbar\partial_j p_k(\mathbf{r}) = p_j p_k(\mathbf{r})$$

$$p_k(\mathbf{r}) = p_{k,0} \exp\left[i\frac{p_j}{\hbar}r_j\right]$$

or

$$\langle\mathbf{r}|\mathbf{p}\rangle = \mathbf{p}(\mathbf{r}) = \mathbf{p}_0 \exp\left[i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}\right]$$

todo

- normalization factor $\frac{1}{(2\pi)^{\frac{3}{2}}}$

$$\mathcal{F}\{\delta(x)\}(k) = \int_{-\infty}^{\infty} \delta(x)e^{-ikx}\,dx = 1$$

- Fourier transform and inverse Fourier transform: definitions and proofs (link to a math section)
- representation in basis of wave vector operator $\hat{\mathbf{k}}$, $\hat{\mathbf{p}} = \hbar\hat{\mathbf{k}}$

From position to momentum representation

Momentum and wave vector, $\mathbf{p} = \hbar\mathbf{k}$

$$\begin{aligned}\langle\mathbf{p}|\Psi\rangle &= \langle\mathbf{p}|\int_{\mathbf{r}'}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \int_{\mathbf{r}'}\langle\mathbf{p}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'}\exp\left[i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}\right]\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' =\end{aligned}$$

Relation between position and wave-number representation can be represented with a Fourier transform

$$\begin{aligned}\langle\mathbf{k}|\Psi\rangle &= \langle\mathbf{k}|\int_{\mathbf{r}'}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \int_{\mathbf{r}'}\langle\mathbf{k}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'}\exp\left[i\mathbf{k}\cdot\mathbf{r}'\right]\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'}\exp\left[i\mathbf{k}\cdot\mathbf{r}'\right]\Psi(\mathbf{r}') d\mathbf{r}' = \\ &= \mathcal{F}\{\Psi(\mathbf{r})\}(\mathbf{k})\end{aligned}$$

8.3.2 Schrodinger Equation

$$i\hbar\frac{d}{dt}|\Psi\rangle = \hat{H}|\Psi\rangle$$

being $\hat{H}$ the Hamiltonian operator and $|\Psi\rangle$ the wave function, as a function of time t as an independent variable.

Stationary States

Eigenspace of the Hamiltonian operator

$$\hat{H}|\Psi_k\rangle = E_k|\Psi_k\rangle \ ,$$

with E_k possible values of energy measurements. *If no eigenstates with the same eigenvalue exists, then…otherwise… Without external influence* **todo** *be more detailed!*, energy values and eigenstates of the systems are constant in time.

Thus, exapnding the state of the system $|\Psi\rangle$ over the stationary states gives $|\Psi_k\rangle$, $|\Psi\rangle = |\Psi_k\rangle c_k(t)$, and inserting in Schrodinger equation

$$i\hbar\dot{c}_k|\Psi_k\rangle = c_k E_k|\Psi_k\rangle$$

and exploiting orthogonality of eigenstates, a diagonal system for the amplitudes of stationary states ariese,

$$i\hbar\dot{c}_k = c_k E_k \ .$$

whose solution reads

$$c_k(t) = c_{k,0}\exp\left[-i\frac{E_k}{\hbar}t\right]$$

Thus the state of the system evolves like a superposition of monochromatic waves with frequencies $\omega_k = \frac{E_k}{\hbar}$,

$$|\Psi\rangle = |\Psi_k\rangle c_k(t) = |\Psi_k\rangle c_{k,0}\exp\left[-i\frac{E_k}{\hbar}t\right] \ .$$

$$\begin{aligned}\frac{d}{dt}\bar{A} = \frac{d}{dt}\left(\langle\Psi|\hat{A}|\Psi\rangle\right) = \\ &= \frac{d}{dt}\langle\Psi|\hat{A}|\Psi\rangle + \langle\Psi|\frac{d\hat{A}}{dt}|\Psi\rangle + \langle\Psi|\hat{A}\frac{d}{dt}|\Psi\rangle = \\ &= \langle\Psi|\frac{d\hat{A}}{dt}|\Psi\rangle + \frac{i}{\hbar}\langle\Psi|\hat{H}\hat{A}|\Psi\rangle - \frac{i}{\hbar}\langle\Psi|\hat{A}\hat{H}|\Psi\rangle = \\ &= \langle\Psi|\left(\frac{i}{\hbar}[\hat{H},\hat{A}] + \frac{d\hat{A}}{dt}\right)|\Psi\rangle \ .\end{aligned}$$

Pictures

- Schrodinger
- Heisenberg
- Interaction

Schrodinger

If $\hat{H}$ not function of time

$$|\Psi\rangle(t) = \exp\left[-i\frac{\hat{H}}{\hbar}(t-t_0)\right]|\Psi\rangle(t_0) = \hat{U}(t,t_0)|\Psi\rangle(t_0)$$

$$\bar{A} = \langle\Psi|\hat{A}|\Psi\rangle = \langle\Psi_0|\hat{U}^*(t,t_0)\hat{A}\hat{U}(t,t_0)|\Psi_0\rangle$$

Heisenberg

…

for $\hat{H}$ independent from time t,

$$\frac{d}{dt}\bar{\mathbf{r}} = \overline{\frac{i}{\hbar}\left[\hat{H}, \hat{\mathbf{r}}\right]}$$
$$\frac{d}{dt}\bar{\mathbf{p}} = \overline{\frac{i}{\hbar}\left[\hat{H}, \hat{\mathbf{p}}\right]}$$

Hamiltonian Mechanics

From Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q_q$$

q generalized coordinates, $p := \frac{\partial L}{\partial \dot{q}}$ generalized momenta.

Hamiltonian

$$H(p, q, t) = p\dot{q} - L(\dot{q}, q, t)$$

Increment of the Hamiltonian

$$dH = \partial_p H dp + \partial_q H dq + \partial_t H dt$$

$$\begin{aligned} dH &= \dot{q}dp + pd\dot{q} - \partial_{\dot{q}} L d\dot{q} - \partial_q L dq - \partial_t L dt = \\ &= \dot{q}dp - \partial_q L dq - \partial_t L dt = \\ &= \dot{q}dp - \left(\dot{p} + Q_q\right) dq - \partial_t L dt = \end{aligned}$$

$$\begin{cases} \frac{\partial H}{\partial p} = \dot{q} \\ \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -\dot{p} + Q_q \\ \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \end{cases}$$

Physical quantity $f(p(t), q(t), t)$. Its time derivative reads

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t} = \\ &= \frac{\partial f}{\partial p}\left[-\frac{\partial H}{\partial q} + Q_q\right] + \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial f}{\partial t} = \\ &= \{H, f\} + \partial_t f + Q_q \partial_p f \end{aligned}$$

If $Q_q = 0$, the correspondence between quantum mechanics and classical mechanics

$$\frac{df}{dt} = \{H, f\} + \partial_t f \qquad \leftrightarrow \qquad \frac{d}{dt}\bar{\hat{f}} = \overline{\frac{i}{\hbar}[\hat{H}, \hat{f}]} + \overline{\frac{\partial \hat{f}}{\partial t}}$$

$$\{H, f\} \qquad \leftrightarrow \qquad \frac{i}{\hbar}[\hat{H}, \hat{f}]$$

Interaction

8.3.3 Matrix Mechanics

Attualization of 1925 papers

…to find the canonical commutation relation,

$$[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = i\hbar \mathbb{I}\, \hat{\mathbf{1}} \ .$$

$$\begin{aligned}
[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = \hat{\mathbf{r}}\hat{\mathbf{p}} - \hat{\mathbf{p}}\hat{\mathbf{r}} = & \\
& = \hat{\mathbf{r}} \int_{\mathbf{r}} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r} \hat{\mathbf{p}} - \hat{\mathbf{p}} \int_{\mathbf{r}} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r}\, \hat{\mathbf{r}} \int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'| d\mathbf{r}' = \\
& = - \int_{\mathbf{r}} \mathbf{r} |\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\, d\mathbf{r} - \hat{\mathbf{p}} \int_{\mathbf{r}}\int_{\mathbf{r}'} |\mathbf{r}\rangle \mathbf{r}' \underbrace{\langle\mathbf{r}|\mathbf{r}'\rangle}_{\delta(\mathbf{r}-\mathbf{r}')} \langle\mathbf{r}'| d\mathbf{r}' = \\
& = - \int_{\mathbf{r}} \mathbf{r} |\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\, d\mathbf{r} - \hat{\mathbf{p}} \int_{\mathbf{r}} \mathbf{r} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r} = \\
& = - \int_{\mathbf{r}} \mathbf{r} |\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\, d\mathbf{r} - \int_{\mathbf{r}'} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r}\, \hat{\mathbf{p}} \int_{\mathbf{r}'} \mathbf{r}' |\mathbf{r}'\rangle\langle\mathbf{r}'| d\mathbf{r}' = \\
& = - \int_{\mathbf{r}} \mathbf{r} |\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\, d\mathbf{r} + \int_{\mathbf{r}} |\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}| d\mathbf{r} \int_{\mathbf{r}'} \mathbf{r}' |\mathbf{r}'\rangle\langle\mathbf{r}'| d\mathbf{r}' = \ldots
\end{aligned}$$

$$\begin{aligned}
[\hat{\mathbf{r}}, \hat{\mathbf{p}}]\, |\Psi\rangle & = - \int_{\mathbf{r}} \mathbf{r} |\mathbf{r}\rangle i\hbar\nabla\Psi(\mathbf{r}, t) + \int_{\mathbf{r}} |\mathbf{r}\rangle i\hbar\nabla \left(\mathbf{r}\Psi(\mathbf{r}, t)\right) = \\
& = - \int_{\mathbf{r}} |\mathbf{r}\rangle i\hbar \left[\mathbf{r}\nabla\Psi(\mathbf{r}, t) + \mathbb{I}\Psi(\mathbf{r}, t) + \mathbf{r}\nabla\Psi(\mathbf{r}, t)\right] = \\
& = i\hbar \underbrace{\int_{\mathbf{r}} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r}}_{\hat{\mathbf{1}}} |\Psi\rangle \ ,
\end{aligned}$$

and since $|\Psi\rangle$ is arbitrary

$$[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = i\hbar \mathbb{I} \hat{\mathbf{1}} \ .$$

$$[\hat{r}_a, \hat{p}_b] = i\hbar\delta_{ab} \ .$$

8.3.4 Heisenberg Uncertainty relation

Uncertainty principle is a relation that holds for "wave descriptions" as it can be proved in the generic framework of Fourier transform, see Fourier transform:Uncertainty relation.

- Heisenberg uncertainty relation is a relation between product of the variance of two physical quantities and their commutator,
- **todo** *relation with measurement process and outcomes. Commutation as a measurement process: first measure B and then A, or first measure A and then B*

$$\sigma_A \sigma_B \ge \frac{1}{2} \left| \overline{[\hat{A}, \hat{B}]} \right| \ .$$

Proof of Heisenberg uncertainty "principle"

$$
\begin{aligned}
\sigma_A^2 \sigma_B^2 &= \langle \Psi | \left(\hat{A} - \bar{A} \right)^2 | \Psi \rangle \langle \Psi | \left(\hat{B} - \bar{B} \right)^2 | \Psi \rangle = \\
&= \langle (\hat{A} - \bar{A}) \Psi | (\hat{A} - \bar{A}) \Psi \rangle \langle (\hat{B} - \bar{B}) \Psi | (\hat{B} - \bar{B}) \Psi \rangle = \\
&= \| (\hat{A} - \bar{A}) \Psi \|^2 \| (\hat{B} - \bar{B}) \Psi \|^2 = \\
&\ge \left| \langle (\hat{A} - \bar{A}) \Psi | (\hat{B} - \bar{B}) \Psi \rangle \right|^2 = \\
&= \left| \langle \Psi | (\hat{A} - \bar{A}) (\hat{B} - \bar{B}) \Psi \rangle \right|^2 = \\
&= \left| \langle \Psi | \hat{A}\hat{B} - \hat{A}\bar{B} - \bar{A}\hat{B} + \bar{A}\bar{B} | \Psi \rangle \right|^2 = \\
&= \left| \langle \Psi | \hat{A}\hat{B} - \bar{A}\bar{B} | \Psi \rangle \right|^2 \ge \qquad (1) \\
&= \left| \frac{\langle \Psi | \hat{A}\hat{B} - \hat{B}\hat{A} | \Psi \rangle}{2i} \right|^2 = \\
&= \frac{\left| \langle \Psi | [\hat{A}, \hat{B}] | \Psi \rangle \right|^2}{4} = \frac{1}{4} \left| \overline{[\hat{A}, \hat{B}]} \right|^2
\end{aligned}
$$

having used Cauchy-Schwartz triangle inequality in (1),

$$
|z| \ge |\mathrm{im}(z)| = \frac{z - z^*}{2i} \ .
$$

Hesienberg uncertainty principles applied to position and momentum reads

$$
\sigma_{r_a} \sigma_{p_b} \ge \frac{1}{2} \left| \overline{[\hat{r}_a, \hat{p}_b]} \right| = \frac{\hbar}{2} \delta_{ab} \ .
$$

8.4 Many-body problem

Wave function with symmetries: Fermions and Bosons

CHAPTER NINE

QUANTUM MECHANICS - NOTES

PROOF INDEX