basics book template

basics

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If you want ot start a new basics-book, it could be a good idea to start from this template.

Please check out the Github repo of the project, basics-book project, and the landing page of the project.

- *Special Relativity*
- *General Relativity*
- *Statistical Physics*
- *Quantum Mechanics*

CAPITOLO 1

Special Relativity

CAPITOLO 2

General Relativity

CAPITOLO 3

Statistical Physics

CAPITOLO 4

Quantum Mechanics

4.1 Mathematical tools for quantum mechanics

Definition 1 (Operator)

Definition 2 (Adjoint operator)

Given an operator $\hat{A} : U \to V$, its self-adjoint $\hat{A}^* : V \to U$ is the operator s.t.

$$(\mathbf{v},\ \hat{A}\mathbf{u})_V = (\mathbf{u}, \hat{A}^*\mathbf{v})_U$$

holds for $\forall \mathbf{u} \in U,\ \mathbf{v} \in V$.

Definition 3 (Hermitian (self-adjoint) operator)

If $\hat{A} : U \to U$, it is a self-adjoint operator if $\hat{A}^* = \hat{A}$.

Self-adjoint operators have real eigenvalues, and orthogonal eigenvectors (at least those associated to different eigenvalues; those associated with the same eigenvalues can be used to build an orthogonal set of vectors with orthogonalization process).

4.2 Postulates of Quantum Mechanics

- …
- Canonical Commutation Relation (CCR) *and Canonical Anti-Commutation Relation…*

4.3 Non-relativistic Mechanics

4.3.1 Statistical Interpretation

Wave function

The state of a system is described by a wave function $|\Psi\rangle$

todo

- properties: domain, image,…
- unitary $1 = \langle\Psi|\Psi\rangle = |\Psi|^2$, for statistical interpretation of $|\Psi|^2$ as a density probability function

Operators and Observables

Physical **observable** quantities are represented by *Hermitian operators*. Possible outcomes of measurement are the eigenvalues of the operator

Given $\hat{A}$ and the set of its eigenvectors $\{|A_i\rangle\}_i$ (**todo** *continuous or discrete spectrum…, need to treat this difference quite in details*), with associated eigenvalues $\{a_i\}_i$

$$\hat{A}|A_i\rangle = a_i|A_i\rangle$$

$$|\Psi\rangle = |A_i\rangle\langle A_i|\Psi\rangle = |A_i\rangle\Psi_i^A$$

$$\langle A_j|\Psi\rangle = \langle A_j|A_i\rangle\langle A_i|\Psi\rangle = \Psi_j^A$$

and thus

$$\Psi_j^A = \langle A_j|\Psi\rangle$$
$$\Psi_j^{A*} = \langle\Psi|A_j\rangle$$

- identity operator $\sum_i |A_i\rangle\langle A_i| = \mathbb{I}$, since

$$\sum_i |A_i\rangle\langle A_i|\Psi\rangle = \sum_i |A_i\rangle\langle A_i|\Psi_j^A A_j\rangle = \sum_i |A_i\rangle\delta_{ij}\Psi_j^A = \sum_i |A_i\rangle\Psi_i^A = |\Psi\rangle$$

- Normalization:

$$1 = \langle\Psi|\Psi\rangle = \Psi_j^{A*}\underbrace{\langle A_j|A_i\rangle}_{\delta_{ij}}\Psi_i^A = \sum_i \left|\Psi_i^A\right|^2$$

with $|\Psi_i^A|^2$ that can be interpreted as the probability of finding the system in state $|\Psi_i^a\rangle$

- Expected value of the physical quantity in the a state $|\Psi\rangle$, with possible values a_i with probability $|\Psi_i^A|^2$

$$\begin{aligned}\bar{A}_\Psi &= \sum_i a_i |\Psi_i^A|^2 = \\ &= \sum_i a_i \Psi_i^{A*}\Psi_i^A = \\ &= \sum_i a_i \langle\Psi|A_i\rangle\langle A_i|\Psi\rangle = \\ &= \langle\Psi| \left(\sum_i a_i |A_i\rangle\langle A_i|\right)|\Psi\rangle = \\ &= \langle\Psi|\hat{A}|\Psi\rangle =\end{aligned}$$

since an operator $\hat{A}$ can be written as a function of its eigenvalues and eigenvectors

$$\begin{aligned}\left(\sum_i a_i |A_i\rangle\langle A_i|\right)\Psi\rangle &= \left(\sum_i a_i |A_i\rangle\langle A_i|\right) c_k|A_k\rangle = \\ &= \sum_i a_i |A_i\rangle c_i = \\ &= \sum_i \hat{A}|A_i\rangle c_i = \\ &= \hat{A}\sum_i |A_i\rangle c_i = \hat{A}|\Psi\rangle\ .\end{aligned}$$

Space Representation

Position operator $\hat{\mathbf{r}}$ has eigenvalues $\mathbf{r}$ identifying the possible measurements of the position

$$\hat{\mathbf{r}}|\mathbf{r}\rangle = \mathbf{r}|\mathbf{r}\rangle\ ,$$

being $\mathbf{r}$ the result of the measurement (position in space, mathematically it could be a vector), and $|\mathbf{r}\rangle$ the state function corresponding to the measurement $\mathbf{r}$ of the position.

- Result of measurement, $\mathbf{r}$, is a position in space. As an example, it could be a point in an Euclidean space $P \in E^n$. It could be written using properties of Dirac's delta «function»

$$\mathbf{r} = \int_{\mathbf{r}'} \delta(\mathbf{r}' - \mathbf{r})\,\mathbf{r}'\,d\mathbf{r}'$$

- Projection of wave function over eigenstates of position operator

$$\begin{aligned}\langle\mathbf{r}|\Psi\rangle(t) = \Psi(\mathbf{r},t) &= \int_{\mathbf{r}'} \delta(\mathbf{r} - \mathbf{r}')\Psi(\mathbf{r}',t)d\mathbf{r}' = \\ &= \int_{\mathbf{r}'} \langle\mathbf{r}|\mathbf{r}'\rangle\Psi(\mathbf{r}',t)d\mathbf{r}' = \\ &= \int_{\mathbf{r}'} \langle\mathbf{r}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle(t)d\mathbf{r}' = \\ &= \langle\mathbf{r}| \underbrace{\left(\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'|d\mathbf{r}'\right)}_{=\hat{\mathbb{1}}} |\Psi\rangle(t)\ .\end{aligned}$$

- having used orthogonality (**todo** *why? provide definition and examples of operators with continuous spectrum*)

$$\langle\mathbf{r}'|\mathbf{r}\rangle = \delta(\mathbf{r}' - \mathbf{r})$$

- Expansion of a state function $|\Psi\rangle(t)$ over the basis of the position operator

$$|\Psi\rangle(t) = \hat{\mathbf{1}}|\Psi\rangle(t) = \left(\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'d\mathbf{r}'\right)|\Psi\rangle(t) = \int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle(t)\, d\mathbf{r}' \ .$$

- Unitariety and probability density

$$\begin{aligned}
1 = \langle\Psi|\Psi\rangle(t) = \langle\Psi| &\left(\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'d\mathbf{r}'\right)|\Psi\rangle \\
&= \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle\, d\mathbf{r}' \\
&= \int_{\mathbf{r}'} \Psi^*(\mathbf{r}',t)\Psi(\mathbf{r}',t)\, d\mathbf{r}' \\
&= \int_{\mathbf{r}'} |\Psi(\mathbf{r}',t)|^2\, d\mathbf{r}'
\end{aligned}$$

and thus $|\Psi(\mathbf{r},t)|^2$ can be interpreted as the **probability density function** of measuring position of the system equal to $\mathbf{r}'$.

- Average value of the operator

$$\begin{aligned}
\bar{\mathbf{r}} &= \langle\Psi|\hat{\mathbf{r}}|\Psi\rangle = \\
&= \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|d\mathbf{r}'\ |\hat{\mathbf{r}}|\ \int_{\mathbf{r}''} |\mathbf{r}''\rangle\langle\mathbf{r}''|\Psi\rangle\, d\mathbf{r}'' \\
&= \int_{\mathbf{r}'}\int_{\mathbf{r}''} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|\hat{\mathbf{r}}|\mathbf{r}''\rangle\langle\mathbf{r}''|\Psi\rangle\, d\mathbf{r}'d\mathbf{r}'' = \\
&= \int_{\mathbf{r}'}\int_{\mathbf{r}''} \langle\Psi|\mathbf{r}'\rangle\ \underbrace{\langle\mathbf{r}'|\mathbf{r}''\rangle}_{=\delta(\mathbf{r}'-\mathbf{r}'')}\ \mathbf{r}''\langle\mathbf{r}''|\Psi\rangle\, d\mathbf{r}'d\mathbf{r}'' = \\
&= \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\mathbf{r}'\langle\mathbf{r}'|\Psi\rangle\, d\mathbf{r}' = \\
&= \int_{\mathbf{r}'} \Psi^*(\mathbf{r}',t)\ \mathbf{r}'\ \Psi(\mathbf{r}',t)\, d\mathbf{r}' = \\
&= \int_{\mathbf{r}'} |\Psi(\mathbf{r}',t)|^2\ \mathbf{r}'\, d\mathbf{r}' \ .
\end{aligned}$$

Momentum Representation

Momentum operator as the limit of…**todo** *prove the expression of the momentum operator as the limit of the generator of translation*

$$\langle\mathbf{r}|\hat{\mathbf{p}} = i\hbar\nabla\langle\mathbf{r}|$$

- Spectrum

$$\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$$

$$\langle\mathbf{r}|\hat{\mathbf{p}}|\mathbf{p}\rangle = i\hbar\nabla\langle\mathbf{r}|\mathbf{p}\rangle = \mathbf{p}\langle\mathbf{r}|\mathbf{p}\rangle$$

and thus the eigenvectors in space base $\mathbf{p}(\mathbf{r}) = \langle\mathbf{r}|\mathbf{p}\rangle$ are the solution of the differential equation

$$i\hbar\nabla\mathbf{p}(\mathbf{r}) = \mathbf{p}\mathbf{p}(\mathbf{r}) \ ,$$

that in Cartesian coordinates reads

$$i\hbar\partial_j p_k(\mathbf{r}) = p_j p_k(\mathbf{r})$$

$$p_k(\mathbf{r}) = p_{k,0}\exp\left[-i\frac{p_j}{\hbar}r_j\right]$$

or

$$\langle\mathbf{r}|\mathbf{p}\rangle = \mathbf{p}(\mathbf{r}) = \mathbf{p}_0\exp\left[i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}\right]$$

From position to momentum representation

Momentum and wave vector, $\mathbf{p} = \hbar\mathbf{k}$

$$\begin{aligned}\langle\mathbf{p}|\Psi\rangle &= \langle\mathbf{p}|\int_{\mathbf{r}'}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \int_{\mathbf{r}'}\langle\mathbf{p}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'}\exp\left[-i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}\right]\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \end{aligned}$$

$$\begin{aligned}\langle\mathbf{k}|\Psi\rangle &= \langle\mathbf{k}|\int_{\mathbf{r}'}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \int_{\mathbf{r}'}\langle\mathbf{k}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'}\exp\left[-i\mathbf{k}\cdot\mathbf{r}'\right]\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'}\exp\left[-i\mathbf{k}\cdot\mathbf{r}'\right]\Psi(\mathbf{r}')d\mathbf{r}' = \\ &= \mathcal{F}\{\Psi(\mathbf{r})\}(\mathbf{k})\end{aligned}$$

4.3.2 Schrodinger Equation

$$i\hbar\frac{d}{dt}|\Psi\rangle = \hat{H}|\Psi\rangle$$

being $\hat{H}$ the Hamiltonian operator and $|\Psi\rangle$ the wave function, as a function of time t as an independent variable.

Stationary States

Eigenspace of the Hamiltonian operator

$$\hat{H}|\Psi_k\rangle = E_k|\Psi_k\rangle\ ,$$

with E_k possible values of energy measurements. *If no eigenstates with the same eigenvalue exists, then…otherwise… Without external influence* **todo** *be more detailed!*, energy values and eigenstates of the systems are constant in time.

Thus, exapnding the state of the system $|\Psi\rangle$ over the stationary states gives $|\Psi_k\rangle$, $|\Psi\rangle = |\Psi_k\rangle c_k(t)$, and inserting in Schrodinger equation

$$i\hbar\dot{c}_k|\Psi_k\rangle = c_k E_k|\Psi_k\rangle$$

and exploiting orthogonality of eigenstates, a diagonal system for the amplitudes of stationary states ariese,

$$i\hbar\dot{c}_k = c_k E_k \ .$$

whose solution reads

$$c_k(t) = c_{k,0} \exp\left[-i\frac{E_k}{\hbar}t\right]$$

Thus the state of the system evolves like a superposition of monochromatic waves with frequencies $\omega_k = \frac{E_k}{\hbar}$,

$$|\Psi\rangle = |\Psi_k\rangle c_k(t) = |\Psi_k\rangle c_{k,0} \exp\left[-i\frac{E_k}{\hbar}t\right] \ .$$

Representations

Schrodinger

Heisenberg

...

4.3.3 Matrix Mechanics

Attualization of 1925 papers

$$|\Psi\rangle = \sum_k |\Psi_k\rangle c_{k,0} \exp\left[-i\frac{E_k}{\hbar}t\right]$$

$$\langle\Psi|\hat{\mathbf{X}}|\Psi\rangle$$

$$\langle\Psi|\Psi_m\rangle\langle\Psi_m|\hat{\mathbf{X}}|\Psi_n\rangle\langle\Psi_n|\Psi\rangle$$

$$X_{mn} = \langle\Psi_m|\hat{\mathbf{X}}|\Psi_n\rangle$$

$$\langle\Psi|\hat{\mathbf{P}}|\Psi\rangle$$

…to find the canonical commutation relation,

$$[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = i\hbar\,\hat{\mathbf{1}} \ .$$

4.4 Many-body problem

Wave function with symmetries: Fermions and Bosons

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