# Modern Physics

## basics

Jan 11, 2025

# CONTENTS


I Special Relativity 3

1 Special Relativity 5

2 Special Relativity - Notes 7
- 2.1 Dynamics . . . 8
- 2.2 Electromagnetism . . . 11

II General Relativity 15

3 General Relativity 17

4 General Relativity - Notes 19

III Statistical Mechanics 21

5 Statistical Physics 23

6 Statistical Physics - Notes 25

IV Quantum Mechanics 27

7 Quantum Mechanics 29
- 7.1 Mathematical tools for quantum mechanics . . . 29
- 7.2 Postulates of Quantum Mechanics . . . 30
- 7.3 Non-relativistic Mechanics . . . 30
- 7.4 Many-body problem . . . 37

8 Quantum Mechanics - Notes 39

Proof Index 41

If you want ot start a new basics-book, it could be a good idea to start from this template.

Please check out the Github repo of the project, basics-book project, and the landing page of the project.

- Special Relativity
  - *Special Relativity*
  - *Special Relativity - Notes*
- General Relativity
  - *General Relativity*
  - *General Relativity - Notes*
- Statistical Mechanics
  - *Statistical Physics*
  - *Statistical Physics - Notes*
- Quantum Mechanics
  - *Quantum Mechanics*
  - *Quantum Mechanics - Notes*

# Part I

# Special Relativity

CHAPTER

ONE

# SPECIAL RELATIVITY

- Electromagnetism and the need for new relativity
- Space-time, Lorentz transformations,…
- Mechanics: kinematics, dynamics,…
- Electromagnetism: Maxwell's equations, potentials, Lorentz force, energy balance

# CHAPTER TWO

# SPECIAL RELATIVITY - NOTES

An event is determined by spatio-temporal information together, $t, \vec{r}$. Absolute nature of physics needs vector algebra and calculus formalism

$$\mathbf{X} = c\,t\,\mathbf{e}_0 + \vec{r} = c\,t\,\mathbf{e}_0 + x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 = X^\alpha \mathbf{E}_\alpha \ ,$$

having used Cartesian coordiantes for the space coordinate.

Minkowski metric reads

$$g_{\alpha\beta} = \mathbf{E}_\alpha \cdot \mathbf{E}_\beta = \mathrm{diag}\{-1, 1, 1, 1\}$$

The reciprocal basis reads $\mathbf{E}_\alpha \cdot \mathbf{E}^\beta = \delta^\beta_\alpha$, $\mathbf{E}_\alpha = g_{\alpha\beta}\mathbf{E}^\beta$, s.t. the elementary interval between two events can be written as

$$d\mathbf{X} = dX^\alpha\,\mathbf{E}_\alpha = \underbrace{dX^\alpha\,g_{\alpha\beta}}_{=dX_\beta}\,\mathbf{E}^\beta = dX_\beta\,\mathbf{E}^\beta \ ,$$

having used Cartesian coordinates,

$$\begin{array}{llll} X^0 = ct & X^1 = x & X^2 = y & X^3 = z \\ X_0 = ct & X_1 = -x & X_2 = -y & X_3 = -z \end{array}$$

Its “length”, or better pseudo-norm with Minkowski metric, is invariant and reads

$$ds^2 = d\mathbf{X} \cdot d\mathbf{X} = (dX_\alpha \mathbf{E}^\alpha) \cdot \left(dX^\beta \mathbf{E}_\beta\right) = c^2\,dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = c^2\,dt^2 - |d\vec{r}|^2$$

**Note:** $ds$ is invariant **todo** prove it. And/or add a section about the role of invariance.

For a co-moving observer, $d\vec{r}\,' = \vec{0}$, and $t'$ is commonly indicated with $\tau$, and its differential is invariant itself, being the product of a constant ($c$ is a universal constant in special relativity) and an invariant quantity.

$$ds^2 = c^2 dt'^2 - |d\vec{r}\,'|^2 = c^2 d\tau^2 \ .$$

Given the invariant nature of $ds$,

$$ds^2 = c^2\,d\tau^2 = c^2\,dt^2 - |d\vec{r}|^2 = c^2\,dt^2 \left[1 - \frac{1}{c^2}\frac{|d\vec{r}|^2}{dt^2}\right] = c^2\,dt^2 \left[1 - \frac{|\vec{v}|^2}{c^2}\right]$$

and thus

$$ds = c\,d\tau = \gamma^{-1}(v/c)\,c\,dt \ ,$$

with $\gamma(w) = \frac{1}{\sqrt{1-w^2}}$.

**4-Velocity** Given the parametric representation of an event in space-time as a function of its proper time, $\mathbf{X}(\tau)$ or coordinate $s$, $\mathbf{X}(s)$ the derivative w.r.t. this parameter is defined as the 4-velocity of the event in space time. Using Cartesian coordinates inducing constant and uniform basis $\mathbf{E}_\alpha$, as a function of the observer time $t$, $ct$, $x^i(t)$, and the transformation of coordinates $t(\tau)$, with differential $dt = \frac{1}{\gamma}\,d\tau$

$$\mathbf{U}(\tau) := \mathbf{X}'(\tau) = \frac{d}{d\tau}\left(X^\alpha(\tau)\mathbf{E}_\alpha\right) = \frac{dt}{d\tau}(ct\mathbf{E}_0 + x^i(t)\mathbf{E}_i) = \gamma(v/c)\left(c\mathbf{E}_0 + \dot{x}^i(t)\mathbf{E}_i\right) = \gamma(v/c)\left(c\mathbf{E}_0 + \vec{v}\right)$$

or

$$\mathbf{U}(s) := \mathbf{X}'(s) = \frac{dt}{ds}\frac{d}{dt}\mathbf{X}(t) = \dots = \gamma(v/c)\left(\mathbf{E}_0 + \frac{\vec{v}}{c}\right) \ .$$

---

**Note:** Using $s$ as the parameter, $\mathbf{U}$ is non-dimensional, and has pseudo-norm = 1,

$$\mathbf{U}(s)\cdot\mathbf{U}(s) = \gamma^2 \underbrace{\left(1 - \frac{|\vec{v}|^2}{c^2}\right)}_{=\gamma^{-2}} = 1 \ .$$

Using $\tau$ as the parameter, $\mathbf{U}$ has physical dimension of a velocity and pseudo-norm = $c$.

---

**4-acceleration $\mathbf{X}''(\tau)$** or $\mathbf{X}''(s)$, **todo**

## 2.1 Dynamics

**4-momentum**

$$\mathbf{P} = m\mathbf{U}$$

Using Cartesian coordinates and $\tau$ as independent variable,

$$\mathbf{P} = m\mathbf{U} = m\frac{d\mathbf{X}}{d\tau} = m\gamma(c, \vec{v}) \ .$$

The spatial component is $\gamma$ times the 3-dimensional momentum $\vec{p} = m\vec{v}$; the time component reads

$$P^0 = m\gamma(w)c \ ,$$

and for small ratio $w := \frac{v}{c}$ it can be expanded in Taylor series around $w = 0$ as

$$\gamma(w) \sim \gamma(0) + w\,\gamma'(0) + \frac{1}{2}\,w^2\gamma''(0) + o(w^2) \ ,$$

with

$$\begin{aligned}
\gamma(w)|_{w=0} &= \left.\frac{1}{\sqrt{1-w^2}}\right|_{w=0} = 1 \\
\gamma'(w)|_{w=0} &= \left.-\frac{1}{2}(1-w^2)^{-\frac{3}{2}}(-2w)\right|_{w=0} = w(1-w^2)^{-\frac{3}{2}} = 0 \\
\gamma''(w)|_{w=0} &= \left.\left((1-w^2)^{-\frac{3}{2}} + w\left(-\frac{3}{2}\right)(1-w^2)^{-\frac{5}{2}}(-2w)\right)\right|_{w=0} = \\
&= \left.\left((1-w^2)^{-\frac{3}{2}} + 3w^2(1-w^2)^{-\frac{5}{2}}\right)\right|_{w=0} = 1
\end{aligned}$$

and thus

$$\gamma(w) = 1 + \frac{1}{2} w^2 + o(w^2)$$

and

$$\gamma(v/c)\, m\, c \sim m\, c \left( 1 + \frac{v^2}{c^2} \right) = \frac{1}{c} \left( mc^2 + \frac{1}{2} m |\vec{v}|^2 \right)$$

Thus, recognizing energy ($E = \gamma m c^2$) and 3-momentum ($\vec{p} = m_3 \vec{v}$, with $m_3 := \gamma m$), the 4-momentum can be written as

$$\mathbf{P} = m\mathbf{U} = \gamma m \left( 1, \frac{\vec{v}}{c} \right) =: \frac{1}{c} \left( \frac{E}{c}, \vec{p} \right)$$

Its pseudo-norm reads

$$m^2 = \mathbf{P} \cdot \mathbf{P} = \frac{1}{c^4} \left( E^2 - c^2 |\vec{p}|^2 \right)$$

and thus the relation between $E$, $\vec{p}$, $m$ and $c$,

$$E^2 = m^2 c^4 + c^2 |\vec{p}|^2 \ ,$$

from which, for $\vec{v} = \vec{0} \rightarrow \vec{p} = \vec{0}$,

$$E^2 = m^2 c^4 \ ,$$

and keeping only the solution with positive energy (**todo** *reference to Dirac's equation and anti-matter?*)

$$E = mc^2 \ .$$

### 2.1.1 Lagrangian approach

**Free particle.**

$$\mathbf{0} = \frac{d\mathbf{P}}{ds} = \frac{d}{ds} \left( m \mathbf{X}'(s) \right)$$

Weak form

$$\begin{aligned} 0 &= \mathbf{W}(s) \cdot \frac{d}{ds} \left( m \mathbf{X}'(s) \right) = \\ &= \frac{d}{ds} \left[ m \mathbf{W}(s) \cdot \mathbf{X}'(s) \right] - m \mathbf{W}'(s) \cdot \mathbf{X}'(s) = \end{aligned}$$

Using generalized coordinates $q^k(s)$, the event can be written in parametric form as $\mathbf{X}(q^k(s), s)$, while the velocity reads

$$\mathbf{U}(s) = \mathbf{X}'(s) = \frac{d}{ds} \mathbf{X}(q^k(s), s) = q^{k\prime}(s) \underbrace{\frac{\partial \mathbf{X}}{\partial q^k}(q^k(s), s)}_{= \frac{\partial \mathbf{X}'}{\partial q^{k\prime}}} + \frac{\partial \mathbf{X}}{\partial s}(q^k(s), s) = \mathbf{U}(q^{k\prime}(s), q^k(s), s)$$

Choosing $\mathbf{W} = \frac{\partial \mathbf{X}}{\partial q^k} = \frac{\partial \mathbf{X}'}{\partial q^{k\prime}}$ in the weak form,

$$\begin{aligned} 0 &= \frac{d}{ds} \left[ m \mathbf{W} \cdot \mathbf{X}' \right] - m \mathbf{W}' \cdot \mathbf{X}' = \\ &= \frac{d}{ds} \left[ m \frac{\partial \mathbf{X}'}{\partial q^{k\prime}} \cdot \mathbf{X}' \right] - m \frac{d}{ds} \frac{\partial \mathbf{X}}{\partial q^k} \cdot \mathbf{X}' = \\ &= \frac{1}{2} \left[ \frac{d}{ds} \left( \frac{\partial}{\partial q^{k\prime}} \left( m \mathbf{X}' \cdot \mathbf{X}' \right) \right) - \frac{\partial}{\partial q^k} \left( m \mathbf{X}' \cdot \mathbf{X}' \right) \right] = \end{aligned}$$

Defining

$$f\left(q^{k\prime}(s), q^k(s), s\right) = -m\mathbf{X}'\left(q^{k\prime}(s), q^k(s), s\right) \cdot \mathbf{X}'\left(q^{k\prime}(s), q^k(s), s\right) = -m\ ,$$

multiplying by a "regular" generic function $w(s)$, neglecting factor $\frac{1}{2}$ and integrating by parts

$$\begin{aligned}
0 &= -\int_{s=s_a}^{s_b} w(s)\left[\frac{d}{ds}\frac{\partial f}{\partial q^{k\prime}} - \frac{\partial f}{\partial q^k}\right] ds = \\
&= -\left[w(s)\frac{\partial f}{\partial q^{k\prime}}\right]\Bigg|_{s=s_a}^{s_b} + \int_{s=s_a}^{s_b}\left[w'(s)\frac{\partial f}{\partial q^{k\prime}} + \frac{\partial f}{\partial q^k}\right] ds = \\
&= -\left[w(s)\frac{\partial f}{\partial q^{k\prime}}\right]\Bigg|_{s=s_a}^{s_b} + \delta\int_{s=s_a}^{s_b} f\left(q^{k\prime}(s), q^k(s), s\right) ds\ .
\end{aligned}$$

Thus, provided that $w(s_1) = w(s_2) = 0$, equation of motion of free particle implies stationariety of functional

$$\int_{s=s_a}^{s_b} f\left(q^{k\prime}(s), q^k(s), s\right) ds\ ,$$

i.e.

$$\delta\int_{s=s_a}^{s_b} f\left(q^{k\prime}(s), q^k(s), s\right) ds = 0$$

Using $t$ as independent parameter, $ds = \gamma^{-1}\, c\, dt$, the functional can be recast as

$$\int_{t=t_a}^{t_b} -m\, c\sqrt{1 - \frac{|\vec{v}|^2}{c^2}}\, dt\ ,$$

to find the (3-dimensional) Lagrangian (multiply by $c$ to get the right physical dimension; check if it's required and wheter it's possible to make $c$ appear before),

$$\mathcal{L} = -\sqrt{1 - \frac{|\vec{v}|^2}{c^2}}\, m\, c^2\ ,$$

and retrieve 3-momentum as (being $\vec{v} = \dot{\vec{r}}$)

$$\begin{aligned}
\vec{p} &:= \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}} = \\
&= -mc^2\frac{1}{2}\left(1 - \frac{|\vec{v}|^2}{c^2}\right)^{-\frac{1}{2}}\left(-2\frac{\vec{v}}{c^2}\right) = \\
&= m\left(1 - \frac{|\vec{v}|^2}{c^2}\right)^{-\frac{1}{2}}\vec{v} = \\
&= \gamma\, m\, \vec{v}\ ,
\end{aligned}$$

and energy as

$$\begin{aligned}
E &:= \vec{p}\cdot\vec{v} - \mathcal{L} = \\
&= \gamma\, m\, |\vec{v}|^2 + \gamma^{-1}\, m\, c^2 = \\
&= \gamma\, m\, c^2\left(\frac{|\vec{v}|^2}{c^2} + \gamma^{-2}\right) = \\
&= \gamma\, m\, c^2\left(\frac{|\vec{v}|^2}{c^2} + 1 - \frac{|\vec{v}|^2}{c^2}\right) = \\
&= \gamma\, m\, c^2\ .
\end{aligned}$$

## 2.2 Electromagnetism

### 2.2.1 Classical electromagnetic theory

#### Maxwell equations

Maxwell equations read

$$\begin{cases} \nabla \cdot \vec{d} = \rho_f \\ \nabla \times \vec{e} + \partial_t \vec{b} = \vec{0} \\ \nabla \cdot \vec{b} = 0 \\ \nabla \times \vec{h} - \partial_t \vec{d} = \vec{j}_f \end{cases}$$

or in vacuum, with $\rho_f = \rho$, $\vec{j} = \vec{j}_f$, $\vec{d} = \varepsilon_0 \vec{e}$, $\vec{b} = \mu_0 \vec{h}$

$$\begin{cases} \nabla \cdot \vec{e} = \frac{\rho}{\varepsilon_0} \\ \nabla \times \vec{e} + \partial_t \vec{b} = \vec{0} \\ \nabla \cdot \vec{b} = 0 \\ \nabla \times \vec{b} - \mu_0 \varepsilon_0 \partial_t \vec{e} = \mu_0 \vec{j} \end{cases}$$

#### Electromagnetic potentials

The electromagnetic field can be written in terms of the electromagnetic potentials

$$\begin{cases} \vec{b} = \nabla \times \vec{a} \\ \vec{e} = -\partial_t \vec{a} - \nabla \varphi \end{cases}$$

#### Lorentz force

A particle in motion in a electromagnetic field is subject to Lorentz force. In classical electromagnetism, the expression of Lorentz force reads

$$\vec{F} = q \left( \vec{e} - \vec{b} \times \vec{v} \right) \ ,$$

whose power is

$$\vec{v} \cdot \vec{F} = \vec{v} \cdot q \left( \vec{e} - \vec{b} \times \vec{v} \right) = q \vec{v} \cdot \vec{e} \ .$$

### 2.2.2 Electromagnetic potential

$$\begin{cases} \vec{b} = \nabla \times \vec{a} \\ \vec{e} = -\partial_t \vec{a} - \nabla \varphi \end{cases}$$

$$\mathbf{A} = \mathbf{E}_\alpha A^\alpha = \frac{\varphi}{c} \mathbf{E}_0 + \vec{a}$$

$$\boldsymbol{\nabla} \mathbf{A} = \left( \mathbf{E}^\alpha \frac{\partial}{\partial X^\alpha} \right) \left( A^\beta \mathbf{E}_\beta \right) = \frac{\partial A^\beta}{\partial X^\alpha} \mathbf{E}^\alpha \otimes \mathbf{E}_\beta = g_{\alpha\gamma} \frac{\partial A^\beta}{\partial X^\alpha} \mathbf{E}_\gamma \otimes \mathbf{E}_\beta \ .$$

whose components may be collected in a 2-dimensional array (first index for rows, second index for columns),

$$(\nabla \mathbf{A})_\alpha^{\ \beta} = \frac{\partial A^\beta}{\partial X^\alpha} = \begin{bmatrix} c^{-2}\partial_t \varphi & c^{-1}\partial_x \varphi & c^{-1}\partial_y \varphi & c^{-1}\partial_z \varphi \\ c^{-1}\partial_t a_x & \partial_x a_x & \partial_y a_x & \partial_z a_x \\ c^{-1}\partial_t a_y & \partial_x a_y & \partial_y a_y & \partial_z a_y \\ c^{-1}\partial_t a_z & \partial_x a_z & \partial_y a_z & \partial_z a_z \end{bmatrix}$$

or covariant-covariant coomponents,

$$(\nabla \mathbf{A})_{\alpha\beta} = \frac{\partial A_\beta}{\partial X^\alpha} = g_{\beta\gamma}\frac{\partial A^\gamma}{\partial X^\alpha} = \begin{bmatrix} c^{-2}\partial_t \varphi & c^{-1}\partial_x \varphi & c^{-1}\partial_y \varphi & c^{-1}\partial_z \varphi \\ -c^{-1}\partial_t a_x & -\partial_x a_x & -\partial_y a_x & -\partial_z a_x \\ -c^{-1}\partial_t a_y & -\partial_x a_y & -\partial_y a_y & -\partial_z a_y \\ -c^{-1}\partial_t a_z & -\partial_x a_z & -\partial_y a_z & -\partial_z a_z \end{bmatrix}$$

or contravariant-contravariant coomponents,

$$(\nabla \mathbf{A})^{\alpha\beta} = \frac{\partial A^\beta}{\partial X_\alpha} = g_{\beta\gamma}\frac{\partial A^\alpha}{\partial X^\gamma} = \begin{bmatrix} c^{-2}\partial_t \varphi & -c^{-1}\partial_x \varphi & -c^{-1}\partial_y \varphi & -c^{-1}\partial_z \varphi \\ c^{-1}\partial_t a_x & -\partial_x a_x & -\partial_y a_x & -\partial_z a_x \\ c^{-1}\partial_t a_y & -\partial_x a_y & -\partial_y a_y & -\partial_z a_y \\ c^{-1}\partial_t a_z & -\partial_x a_z & -\partial_y a_z & -\partial_z a_z \end{bmatrix}$$

The electromagnetic field tensor is defined as the anti-symmetric part of the gradient of the 4-electromagnetic potential,

$$\mathbf{F} = \left[\boldsymbol{\nabla}\mathbf{A} - (\boldsymbol{\nabla}\mathbf{A})^T\right]$$

whose components may be collected in a 2-dimensional array (first index for rows, second index for columns),

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -\frac{\underline{e}^T}{c} \\ \frac{\underline{e}}{c} & \underline{b}_\times \end{bmatrix} \qquad , \qquad F_{\alpha\beta} = \begin{bmatrix} 0 & \frac{\underline{e}^T}{c} \\ -\frac{\underline{e}}{c} & \underline{b}_\times \end{bmatrix}$$

### 2.2.3 Electromagnetic field and electromagnetic field equations

The pair of Maxwell equations

$$\begin{cases} \rho_f = \nabla \cdot \vec{d} \\ \vec{j}_f = -\partial_t \vec{d} + \nabla \times \vec{h} \end{cases}$$

can be re-written in 4-formalism, using 4-gradient in Cartesian coordinates

$$\boldsymbol{\nabla} = \mathbf{E}^\alpha \frac{\partial}{\partial X^\alpha} = \mathbf{E}_0 \frac{\partial}{c\partial t} + \mathbf{E}_i \frac{\partial}{\partial x^i} = \mathbf{E}_0 \frac{\partial}{c\partial t} + \nabla \,,$$

and the definition of the 4-current density vector

$$\mathbf{J} = J^\alpha \mathbf{E}_\alpha = c\rho\, \mathbf{E}_0 + \vec{j}$$

so that

$$c\rho \mathbf{E}_0 + \vec{j} = \boldsymbol{\nabla} \cdot \mathbf{F} = \boldsymbol{\nabla} \cdot [(0\,\mathbf{E}_0 + c\vec{d}) \otimes \mathbf{E}_0 + (-\mathbf{E}_0 c\vec{d} + \vec{h}_\times)] \,,$$

with the displacement field tensor,

$$\mathbf{D} = D^{\alpha\beta}\, \mathbf{E}_\alpha\, \mathbf{E}_\beta \,,$$

with components (rows for the first index, columns for the second index)

$$D^{\alpha\beta} = \begin{bmatrix} 0 & -cd_x & -cd_y & -cd_z \\ cd_x & 0 & -h_z & h_y \\ cd_y & h_z & 0 & -h_x \\ cd_z & -h_y & h_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -c\underline{d}^T \\ c\underline{d} & \underline{h}_\times \,. \end{bmatrix}$$

The pair of Maxwell equations

$$\begin{cases}\nabla \cdot \vec{b} = 0 \\ \partial_t \vec{b} + \nabla \times \vec{e} = \vec{0}\end{cases}$$

can be re-written in 4-formalism as

$$0 = \partial_\mu F_{\eta\xi} + \partial_\eta F_{\xi\mu} + \partial_\xi F_{\mu\eta}$$

Among these $64 = 4^3$ equations, there are only 4 independent equations.

- If 2 indices are the same, the corresponding equation is the identity $0 = 0$. As an example, if $\mu = \eta$

  $$0 = \partial_\mu F_{\mu\xi} + \partial_\mu \underbrace{F_{\xi\mu}}_{-F_{\mu xi}} + \partial_\xi \underbrace{F_{\mu\mu}}_{=0} = 0 \,,$$

  thus only combinations with different indices may provide some information.

- Given the ordered set of indices $(\mu, \eta, \xi)$, switching a pair of indices provides the same equation. As an example, switching $\mu$ and $\eta$

  $$\begin{aligned} 0 &= \partial_\eta F_{\mu\xi} + \partial_\mu F_{\xi\eta} + \partial_\xi F_{\eta\mu} = \\ &= \partial_\eta(-F_{\xi\mu}) + \partial_\mu(-F_{\eta\xi}) + \partial_\xi(-F_{\mu\eta}) \,. \end{aligned}$$

- Thus, only 4 combination of different indices, without taking order into account, provide independent information

  $$\begin{aligned} (1,2,3): &\quad 0 = \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \partial_x(-b_x) + \partial_y(-b_y) + \partial_z(-b_z) \\ (2,3,0): &\quad 0 = \partial_2 F_{30} + \partial_3 F_{02} + \partial_0 F_{23} = \partial_y\left(-\frac{e_z}{c}\right) + \partial_z\left(\frac{e_y}{c}\right) + \partial_{ct}(-b_x) \\ (3,0,1): &\quad 0 = \partial_3 F_{01} + \partial_0 F_{13} + \partial_1 F_{30} = \partial_z\left(\frac{e_x}{c}\right) + \partial_{ct}(-b_y) + \partial_x\left(-\frac{e_z}{c}\right) \\ (0,1,2): &\quad 0 = \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = \partial_{ct}(-b_z) + \partial_x\left(-\frac{e_y}{c}\right) + \partial_y\left(\frac{e_x}{c}\right) \end{aligned}$$

  i.e.

  $$\begin{aligned} (1,2,3): &\quad 0 = -\nabla \cdot \vec{b} \\ (2,3,0): &\quad 0 = -\frac{1}{c}\left[\partial_t b_x + (\partial_y e_z - \partial_z e_y)\right] \\ (3,0,1): &\quad 0 = -\frac{1}{c}\left[\partial_t b_y + (\partial_z e_x - \partial_x e_z)\right] \\ (0,1,2): &\quad 0 = -\frac{1}{c}\left[\partial_t b_z + (\partial_x e_y - \partial_y e_x)\right] \end{aligned}$$

  i.e.

  $$\begin{cases} 0 = \nabla \cdot \vec{b} \\ \vec{0} = \partial_t \vec{b} + \nabla \times \vec{e} \end{cases}$$

### 2.2.4 Point particle in electromagnetic field

Lorentz 4-force acting on a point charge of electric charge charge $q$ reads

$$\mathbf{f} = \mathbf{F} \cdot \mathbf{J} = q \, \mathbf{F} \cdot \mathbf{U} \,.$$

so that the dynamical equation reads

$$m \, \mathbf{X}'' = q \, \mathbf{F} \cdot \mathbf{X}'$$

### 2.2.5 Energy balance

$$\frac{\partial u}{\partial t} =$$
$$\frac{\partial \vec{s}}{\partial t} =$$

…

$$\boldsymbol{\nabla} \cdot \mathbf{T} = -\mathbf{F} \cdot \mathbf{J}$$

# Part II

# General Relativity

CHAPTER

THREE

# GENERAL RELATIVITY

CHAPTER

# FOUR

# GENERAL RELATIVITY - NOTES

# Part III

# Statistical Mechanics

CHAPTER

# FIVE

# STATISTICAL PHYSICS

CHAPTER

# SIX

# STATISTICAL PHYSICS - NOTES

# Part IV

# Quantum Mechanics

CHAPTER

SEVEN

# QUANTUM MECHANICS

- Principles and postulates
  - statistics and measurements outcomes (Heisenberg built its matrix mechanics only on observables…)
  - CCR
- angluar momentum, spin, and atom

## 7.1 Mathematical tools for quantum mechanics

**Definition 1 (Operator)**

**Definition 2 (Adjoint operator)**

Given an operator $\hat{A} : U \rightarrow V$, its self-adjoint $\hat{A}^* : V \rightarrow U$ is the operator s.t.

$$(\mathbf{v},\ \hat{A}\mathbf{u})_V = (\mathbf{u}, \hat{A}^*\mathbf{v})_U$$

holds for $\forall \mathbf{u} \in U,\ \mathbf{v} \in V$.

**Definition 3 (Hermitian (self-adjoint) operator)**

If $\hat{A} : U \rightarrow U$, it is a self-adjoint operator if $\hat{A}^* = \hat{A}$.

Self-adjoint operators have real eigenvalues, and orthogonal eigenvectors (at least those associated to different eigenvalues; those associated with the same eigenvalues can be used to build an orthogonal set of vectors with orthogonalization process).

## 7.2 Postulates of Quantum Mechanics

- …
- Canonical Commutation Relation (CCR) *and Canonical Anti-Commutation Relation…*
- …

## 7.3 Non-relativistic Mechanics

### 7.3.1 Statistical Interpretation and Measurement

#### Wave function

The state of a system is described by a wave function $|\Psi\rangle$

**todo**

- properties: domain, image,…
- unitary $1 = \langle\Psi|\Psi\rangle = |\Psi|^2$, for statistical interpretation of $|\Psi|^2$ as a density probability function

#### Operators and Observables

Physical **observable** quantities are represented by *Hermitian operators*. Possible outcomes of measurement are the eigenvalues of the operator

Given $\hat{A}$ and the set of its eigenvectors $\{|A_i\rangle\}_i$ (**todo** *continuous or discrete spectrum…, need to treat this difference quite in details*), with associated eigenvalues $\{a_i\}_i$

$$\hat{A}|A_i\rangle = a_i|A_i\rangle$$

$$|\Psi\rangle = |A_i\rangle\langle A_i|\Psi\rangle = |A_i\rangle\Psi_i^A$$
$$\langle A_j|\Psi\rangle = \langle A_j|A_i\rangle\langle A_i|\Psi\rangle = \Psi_j^A$$

and thus

$$\Psi_j^A = \langle A_j|\Psi\rangle$$
$$\Psi_j^{A*} = \langle\Psi|A_j\rangle$$

- identity operator $\sum_i |A_i\rangle\langle A_i| = \mathbb{1}$, since

$$\sum_i |A_i\rangle\langle A_i|\Psi\rangle = \sum_i |A_i\rangle\langle A_i|\Psi_j^A A_j\rangle = \sum_i |A_i\rangle\delta_{ij}\Psi_j^A = \sum_i |A_i\rangle\Psi_i^A = |\Psi\rangle$$

- Normalization:

$$1 = \langle\Psi|\Psi\rangle = \Psi_j^{A*}\underbrace{\langle A_j|A_i\rangle}_{\delta_{ij}}\Psi_i^A = \sum_i \left|\Psi_i^A\right|^2$$

with $|\Psi_i^A|^2$ that can be interpreted as the probability of finding the system in state $|\Psi_i^a\rangle$

- Expected value of the physical quantity in the a state $|\Psi\rangle$, with possible values $a_i$ with probability $|\Psi_i^A|^2$

$$\begin{aligned}\bar{A}_\Psi &= \sum_i a_i |\Psi_i^A|^2 = \\ &= \sum_i a_i \Psi_i^{A*}\Psi_i^A = \\ &= \sum_i a_i \langle\Psi|A_i\rangle\langle A_i|\Psi\rangle = \\ &= \langle\Psi|\left(\sum_i a_i |A_i\rangle\langle A_i|\right)|\Psi\rangle = \\ &= \langle\Psi|\hat{A}|\Psi\rangle =\end{aligned}$$

since an operator $\hat{A}$ can be written as a function of its eigenvalues and eigenvectors

$$\begin{aligned}\left(\sum_i a_i |A_i\rangle\langle A_i|\right)\Psi\rangle = \left(\sum_i a_i |A_i\rangle\langle A_i|\right) c_k|A_k\rangle &= \\ = \sum_i a_i |A_i\rangle c_i &= \\ = \sum_i \hat{A}|A_i\rangle c_i &= \\ = \hat{A}\sum_i |A_i\rangle c_i &= \hat{A}|\Psi\rangle\ .\end{aligned}$$

### Space Representation

**Position operator** $\hat{\mathbf{r}}$ has eigenvalues $\mathbf{r}$ identifying the possible measurements of the position

$$\hat{\mathbf{r}}|\mathbf{r}\rangle = \mathbf{r}|\mathbf{r}\rangle\ ,$$

being $\mathbf{r}$ the result of the measurement (position in space, mathematically it could be a vector), and $|\mathbf{r}\rangle$ the state function corresponding to the measurement $\mathbf{r}$ of the position.

- Result of measurement, $\mathbf{r}$, is a position in space. As an example, it could be a point in an Euclidean space $P \in E^n$. It could be written using properties of Dirac's delta "function"

$$\mathbf{r} = \int_{\mathbf{r}'} \delta(\mathbf{r}' - \mathbf{r})\,\mathbf{r}'\,d\mathbf{r}'$$

- Projection of wave function over eigenstates of position operator

$$\begin{aligned}\langle\mathbf{r}|\Psi\rangle(t) = \Psi(\mathbf{r},t) &= \int_{\mathbf{r}'} \delta(\mathbf{r}-\mathbf{r}')\Psi(\mathbf{r}',t)d\mathbf{r}' = \\ &= \int_{\mathbf{r}'} \langle\mathbf{r}|\mathbf{r}'\rangle\Psi(\mathbf{r}',t)d\mathbf{r}' = \\ &= \int_{\mathbf{r}'} \langle\mathbf{r}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle(t)d\mathbf{r}' = \\ &= \langle\mathbf{r}|\underbrace{\left(\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'|d\mathbf{r}'\right)}_{=\hat{\mathbb{1}}}|\Psi\rangle(t)\ .\end{aligned}$$

- having used orthogonality (**todo** *why? provide definition and examples of operators with continuous spectrum*)

$$\langle\mathbf{r}'|\mathbf{r}\rangle = \delta(\mathbf{r}' - \mathbf{r})$$

- Expansion of a state function $|\Psi\rangle(t)$ over the basis of the position operator

$$|\Psi\rangle(t) = \hat{\mathbf{1}}|\Psi\rangle(t) = \left(\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'d\mathbf{r}'\right)|\Psi\rangle(t) = \int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle(t)\,d\mathbf{r}' \ .$$

- Unitariety and probability density

$$\begin{aligned}
1 = \langle\Psi|\Psi\rangle(t) = \langle\Psi| &\left(\int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'d\mathbf{r}'\right)|\Psi\rangle \\
&= \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle\,d\mathbf{r}' \\
&= \int_{\mathbf{r}'} \Psi^*(\mathbf{r}',t)\Psi(\mathbf{r}',t)\,d\mathbf{r}' \\
&= \int_{\mathbf{r}'} |\Psi(\mathbf{r}',t)|^2\,d\mathbf{r}'
\end{aligned}$$

and thus $|\Psi(\mathbf{r},t)|^2$ can be interpreted as the **probability density function** of measuring position of the system equal to $\mathbf{r}'$.

- Average value of the operator

$$\begin{aligned}
\bar{\mathbf{r}} &= \langle\Psi|\hat{\mathbf{r}}|\Psi\rangle = \\
&= \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|d\mathbf{r}'\ |\hat{\mathbf{r}}| \int_{\mathbf{r}''} |\mathbf{r}''\rangle\langle\mathbf{r}''|\Psi\rangle\,d\mathbf{r}'' \\
&= \int_{\mathbf{r}'}\int_{\mathbf{r}''} \langle\Psi|\mathbf{r}'\rangle\langle\mathbf{r}'|\hat{\mathbf{r}}|\mathbf{r}''\rangle\langle\mathbf{r}''|\Psi\rangle\,d\mathbf{r}'d\mathbf{r}'' = \\
&= \int_{\mathbf{r}'}\int_{\mathbf{r}''} \langle\Psi|\mathbf{r}'\rangle\ \underbrace{\langle\mathbf{r}'|\mathbf{r}''\rangle}_{=\delta(\mathbf{r}'-\mathbf{r}'')}\ \mathbf{r}''\langle\mathbf{r}''|\Psi\rangle\,d\mathbf{r}'d\mathbf{r}'' = \\
&= \int_{\mathbf{r}'} \langle\Psi|\mathbf{r}'\rangle\mathbf{r}'\langle\mathbf{r}'|\Psi\rangle\,d\mathbf{r}' = \\
&= \int_{\mathbf{r}'} \Psi^*(\mathbf{r}',t)\ \mathbf{r}'\ \Psi(\mathbf{r}',t)\,d\mathbf{r}' = \\
&= \int_{\mathbf{r}'} |\Psi(\mathbf{r}',t)|^2\ \mathbf{r}'\,d\mathbf{r}' \ .
\end{aligned}$$

### Momentum Representation

**Momentum operator** as the limit of…**todo** *prove the expression of the momentum operator as the limit of the generator of translation*

$$\langle\mathbf{r}|\hat{\mathbf{p}} = -i\hbar\nabla\langle\mathbf{r}|$$

- Spectrum

$$\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$$

$$\langle\mathbf{r}|\hat{\mathbf{p}}|\mathbf{p}\rangle = -i\hbar\nabla\langle\mathbf{r}|\mathbf{p}\rangle = \mathbf{p}\langle\mathbf{r}|\mathbf{p}\rangle$$

and thus the eigenvectors in space base $\mathbf{p}(\mathbf{r}) = \langle\mathbf{r}|\mathbf{p}\rangle$ are the solution of the differential equation

$$-i\hbar\nabla\mathbf{p}(\mathbf{r}) = \mathbf{p}\mathbf{p}(\mathbf{r}) \ ,$$

that in Cartesian coordinates reads

$$-i\hbar\partial_j p_k(\mathbf{r}) = p_j p_k(\mathbf{r})$$

$$p_k(\mathbf{r}) = p_{k,0} \exp\left[i\frac{p_j}{\hbar}r_j\right]$$

or

$$\langle\mathbf{r}|\mathbf{p}\rangle = \mathbf{p}(\mathbf{r}) = \mathbf{p}_0 \exp\left[i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}\right]$$

**todo**

- normalization factor $\frac{1}{(2\pi)^{\frac{3}{2}}}$

$$\mathcal{F}\{\delta(x)\}(k) = \int_{-\infty}^{\infty} \delta(x)e^{-ikx}\,dx = 1$$

- Fourier transform and inverse Fourier transform: definitions and proofs (link to a math section)
- representation in basis of wave vector operator $\hat{\mathbf{k}}$, $\hat{\mathbf{p}} = \hbar\hat{\mathbf{k}}$

### From position to momentum representation

Momentum and wave vector, $\mathbf{p} = \hbar\mathbf{k}$

$$\begin{aligned}\langle\mathbf{p}|\Psi\rangle &= \langle\mathbf{p}|\int_{\mathbf{r}'}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \int_{\mathbf{r}'}\langle\mathbf{p}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'}\exp\left[i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}\right]\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' =\end{aligned}$$

Relation between position and wave-number representation can be represented with a Fourier transform

$$\begin{aligned}\langle\mathbf{k}|\Psi\rangle &= \langle\mathbf{k}|\int_{\mathbf{r}'}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \int_{\mathbf{r}'}\langle\mathbf{k}|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'}\exp\left[i\mathbf{k}\cdot\mathbf{r}'\right]\langle\mathbf{r}'|\Psi\rangle d\mathbf{r}' = \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbf{r}'}\exp\left[i\mathbf{k}\cdot\mathbf{r}'\right]\Psi(\mathbf{r}')d\mathbf{r}' = \\ &= \mathcal{F}\{\Psi(\mathbf{r})\}(\mathbf{k})\end{aligned}$$

## 7.3.2 Schrodinger Equation

$$i\hbar\frac{d}{dt}|\Psi\rangle = \hat{H}|\Psi\rangle$$

being $\hat{H}$ the Hamiltonian operator and $|\Psi\rangle$ the wave function, as a function of time $t$ as an independent variable.

### Stationary States

Eigenspace of the Hamiltonian operator

$$\hat{H}|\Psi_k\rangle = E_k|\Psi_k\rangle \ ,$$

with $E_k$ possible values of energy measurements. *If no eigenstates with the same eigenvalue exists, then…otherwise… Without external influence* **todo** *be more detailed!*, energy values and eigenstates of the systems are constant in time.

Thus, exapnding the state of the system $|\Psi\rangle$ over the stationary states gives $|\Psi_k\rangle$, $|\Psi\rangle = |\Psi_k\rangle c_k(t)$, and inserting in Schrodinger equation

$$i\hbar\dot{c}_k|\Psi_k\rangle = c_k E_k|\Psi_k\rangle$$

and exploiting orthogonality of eigenstates, a diagonal system for the amplitudes of stationary states ariese,

$$i\hbar\dot{c}_k = c_k E_k \ .$$

whose solution reads

$$c_k(t) = c_{k,0}\exp\left[-i\frac{E_k}{\hbar}t\right]$$

Thus the state of the system evolves like a superposition of monochromatic waves with frequencies $\omega_k = \frac{E_k}{\hbar}$,

$$|\Psi\rangle = |\Psi_k\rangle c_k(t) = |\Psi_k\rangle c_{k,0}\exp\left[-i\frac{E_k}{\hbar}t\right] \ .$$

$$\begin{aligned}
\frac{d}{dt}\bar{A} = \frac{d}{dt}\left(\langle\Psi|\hat{A}|\Psi\rangle\right) = \\
&= \frac{d}{dt}\langle\Psi|\hat{A}|\Psi\rangle + \langle\Psi|\frac{d\hat{A}}{dt}|\Psi\rangle + \langle\Psi|\hat{A}\frac{d}{dt}|\Psi\rangle = \\
&= \langle\Psi|\frac{d\hat{A}}{dt}|\Psi\rangle + \frac{i}{\hbar}\langle\Psi|\hat{H}\hat{A}|\Psi\rangle - \frac{i}{\hbar}\langle\Psi|\hat{A}\hat{H}|\Psi\rangle = \\
&= \langle\Psi|\left(\frac{i}{\hbar}[\hat{H},\hat{A}] + \frac{d\hat{A}}{dt}\right)|\Psi\rangle \ .
\end{aligned}$$

### Pictures

- Schrodinger
- Heisenberg
- Interaction

### Schrodinger

If $\hat{H}$ not function of time

$$|\Psi\rangle(t) = \exp\left[-i\frac{\hat{H}}{\hbar}(t-t_0)\right]|\Psi\rangle(t_0) = \hat{U}(t,t_0)|\Psi\rangle(t_0)$$

$$\bar{A} = \langle\Psi|\hat{A}|\Psi\rangle = \langle\Psi_0|\hat{U}^*(t,t_0)\hat{A}\hat{U}(t,t_0)|\Psi_0\rangle$$

### Heisenberg

…

for $\hat{H}$ independent from time $t$,

$$\frac{d}{dt}\bar{\mathbf{r}} = \overline{\frac{i}{\hbar}\left[\hat{H}, \hat{\mathbf{r}}\right]}$$
$$\frac{d}{dt}\bar{\mathbf{p}} = \overline{\frac{i}{\hbar}\left[\hat{H}, \hat{\mathbf{p}}\right]}$$

### Hamiltonian Mechanics

From Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q_q$$

$q$ generalized coordinates, $p := \frac{\partial L}{\partial \dot{q}}$ generalized momenta.

Hamiltonian

$$H(p,q,t) = p\dot{q} - L(\dot{q},q,t)$$

Increment of the Hamiltonian

$$dH = \partial_p H dp + \partial_q H dq + \partial_t H dt$$

$$\begin{aligned} dH &= \dot{q}dp + pd\dot{q} - \partial_{\dot{q}}Ld\dot{q} - \partial_q Ldq - \partial_t Ldt = \\ &= \dot{q}dp - \partial_q Ldq - \partial_t Ldt = \\ &= \dot{q}dp - \left(\dot{p} + Q_q\right)dq - \partial_t Ldt = \end{aligned}$$

$$\begin{cases} \frac{\partial H}{\partial p} = \dot{q} \\ \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -\dot{p} + Q_q \\ \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \end{cases}$$

Physical quantity $f(p(t), q(t), t)$. Its time derivative reads

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t} = \\ &= \frac{\partial f}{\partial p}\left[-\frac{\partial H}{\partial q} + Q_q\right] + \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial f}{\partial t} = \\ &= \{H, f\} + \partial_t f + Q_q \partial_p f \end{aligned}$$

If $Q_q = 0$, the correspondence between quantum mechanics and classical mechanics

$$\frac{df}{dt} = \{H,f\} + \partial_t f \qquad \leftrightarrow \qquad \frac{d}{dt}\bar{\hat{f}} = \overline{\frac{i}{\hbar}[\hat{H},\hat{f}]} + \overline{\frac{\partial \hat{f}}{\partial t}}$$

$$\{H,f\} \qquad \leftrightarrow \qquad \frac{i}{\hbar}[\hat{H},\hat{f}]$$

### Interaction

## 7.3.3 Matrix Mechanics

### Attualization of 1925 papers

…to find the canonical commutation relation,

$$[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = i\hbar \mathbb{I}\,\hat{\mathbf{1}}\ .$$

$$\begin{aligned}
[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = \hat{\mathbf{r}}\hat{\mathbf{p}} - \hat{\mathbf{p}}\hat{\mathbf{r}} = & \\
& = \hat{\mathbf{r}} \int_{\mathbf{r}} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r} \hat{\mathbf{p}} - \hat{\mathbf{p}} \int_{\mathbf{r}} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r}\, \hat{\mathbf{r}} \int_{\mathbf{r}'} |\mathbf{r}'\rangle\langle\mathbf{r}'| d\mathbf{r}' = \\
& = - \int_{\mathbf{r}} \mathbf{r}|\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\, d\mathbf{r} - \hat{\mathbf{p}} \int_{\mathbf{r}}\int_{\mathbf{r}'} |\mathbf{r}\rangle\mathbf{r}' \underbrace{\langle\mathbf{r}|\mathbf{r}'\rangle}_{\delta(\mathbf{r}-\mathbf{r}')} \langle\mathbf{r}'| d\mathbf{r}' = \\
& = - \int_{\mathbf{r}} \mathbf{r}|\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\, d\mathbf{r} - \hat{\mathbf{p}} \int_{\mathbf{r}} \mathbf{r}|\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r} = \\
& = - \int_{\mathbf{r}} \mathbf{r}|\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\, d\mathbf{r} - \int_{\mathbf{r}'} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r}\, \hat{\mathbf{p}} \int_{\mathbf{r}'} \mathbf{r}'|\mathbf{r}'\rangle\langle\mathbf{r}'| d\mathbf{r}' = \\
& = - \int_{\mathbf{r}} \mathbf{r}|\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}|\, d\mathbf{r} + \int_{\mathbf{r}} |\mathbf{r}\rangle i\hbar\nabla\langle\mathbf{r}| d\mathbf{r} \int_{\mathbf{r}'} \mathbf{r}'|\mathbf{r}'\rangle\langle\mathbf{r}'| d\mathbf{r}' = \ldots
\end{aligned}$$

$$\begin{aligned}
[\hat{\mathbf{r}}, \hat{\mathbf{p}}]\,|\Psi\rangle = & - \int_{\mathbf{r}} \mathbf{r}|\mathbf{r}\rangle i\hbar\nabla\Psi(\mathbf{r},t) + \int_{\mathbf{r}} |\mathbf{r}\rangle i\hbar\nabla\left(\mathbf{r}\Psi(\mathbf{r},t)\right) = \\
& = - \int_{\mathbf{r}} |\mathbf{r}\rangle i\hbar \left[\mathbf{r}\nabla\Psi(\mathbf{r},t) + \mathbb{I}\Psi(\mathbf{r},t) + \mathbf{r}\nabla\Psi(\mathbf{r},t)\right] = \\
& = i\hbar \underbrace{\int_{\mathbf{r}} |\mathbf{r}\rangle\langle\mathbf{r}| d\mathbf{r}}_{\hat{\mathbf{1}}}\, |\Psi\rangle\ ,
\end{aligned}$$

and since $|\Psi\rangle$ is arbitrary

$$[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = i\hbar\mathbb{I}\hat{\mathbf{1}}\ .$$

$$[\hat{r}_a, \hat{p}_b] = i\hbar\delta_{ab}\ .$$

## 7.3.4 Heisenberg Uncertainty “principle”

- Heisenberg uncertainty “principle” is a relation between product of variance of two physical quantities and their commutation,
- **todo** *relation with measurement process and outcomes. Commutation as a measurement process: first measure $B$ and then $A$, or first measure $A$ and then $B$*

$$\sigma_A\sigma_B \geq \frac{1}{2}\left|\overline{[\hat{A}, \hat{B}]}\right|\ .$$

### Proof of Heisenberg uncertainty "principle"

$$
\begin{aligned}
\sigma_A^2\sigma_B^2 &= \langle\Psi|\left(\hat{A}-\bar{A}\right)^2|\Psi\rangle\langle\Psi|\left(\hat{B}-\bar{B}\right)^2|\Psi\rangle = \\
&= \langle(\hat{A}-\bar{A})\Psi|(\hat{A}-\bar{A})\Psi\rangle\langle(\hat{B}-\bar{B})\Psi|(\hat{B}-\bar{B})\Psi\rangle = \\
&= \|(\hat{A}-\bar{A})\Psi\|^2\|(\hat{B}-\bar{B})\Psi\|^2 = \\
&\geq \left|\langle(\hat{A}-\bar{A})\Psi|(\hat{B}-\bar{B})\Psi\rangle\right|^2 = \\
&= \left|\langle\Psi|(\hat{A}-\bar{A})(\hat{B}-\bar{B})\Psi\rangle\right|^2 = \\
&= \left|\langle\Psi|\hat{A}\hat{B}-\hat{A}\bar{B}-\bar{A}\hat{B}+\bar{A}\bar{B}|\Psi\rangle\right|^2 = \\
&= \left|\langle\Psi|\hat{A}\hat{B}-\bar{A}\bar{B}|\Psi\rangle\right|^2 = \\
&= \left|\frac{\langle\Psi|\hat{A}\hat{B}-\hat{B}\hat{A}|\Psi\rangle}{2i}\right|^2 = \\
&= \frac{\left|\langle\Psi|[\hat{A},\hat{B}]|\Psi\rangle\right|^2}{4} = \frac{1}{4}\left|\overline{[\hat{A},\hat{B}]}\right|^2
\end{aligned}
$$

having used Cauchy triangle inequality and

$$
|z| = \frac{\mathrm{re}\{z\}+\mathrm{re}\{z^*\}}{2} = \frac{\mathrm{im}\{z\}-\mathrm{im}\{z^*\}}{2i}
$$

Hesienberg uncertainty principles applied to position and momentum reads

$$
\sigma_{r_a}\sigma_{p_b} \geq \frac{1}{2}\left|\overline{[\hat{r}_a,\hat{p}_b]}\right| = \frac{\hbar}{2}\delta_{ab}\ .
$$

## 7.4 Many-body problem

Wave function with symmetries: Fermions and Bosons

CHAPTER

# EIGHT

# QUANTUM MECHANICS - NOTES

# PROOF INDEX


## Adjoint Operator

`Adjoint Operator` (*ch/quantum-mechanics/intro*), 29

## Operator

`Operator` (*ch/quantum-mechanics/intro*), 29

## Self-Adjoint Operator

`Self-Adjoint Operator` (*ch/quantum-mechanics/intro*), 29