Abraham Wald's Work on Aircraft Survivability

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While he was a member of the Statistical Research Group (SRG), Abraham Wald worked on the problem of estimating the vulnerability of aircraft, using data obtained from survivors. This work was published as a series of SRG memoranda and was used in World War II and in the wars in Korea and Vietnam. The memoranda were recently reissued by the Center for Naval Analyses. This article is a condensation and exposition of Wald's work, in which his ideas and methods are described. In the final section, his main results are reexamined in the light of classical statistical theory and more recent work.

KEY WORDS: Survivability; Missing data; Approximate methods; Maximum likelihood.

1. INTRODUCTION

December 7, 1981, was the 40th anniversary of the attack on Pearl Harbor, the subsequent entry of the United States into World War II, and also the birth of the Statistical Research Group (SRG) and the Antisubmarine Warfare Operations Research Group (ASWORG, later renamed the Operations Evaluation Group (OEG) and now part of the Center for Naval Analyses). The early histories of SRG and ASWORG/OEG were described recently by their original leaders, W.A. Wallis (1980) and P.M. Morse (1977), respectively. While in the SRG, Abraham Wald developed techniques for estimating the survivability of aircraft encountering enemy ground fire. Wald's methods were used in World War II and by the Navy and Air Force during the wars in Korea and Vietnam. Although this work was declassified many years ago, it has not appeared in the open literature. At the end of his historical paper, Wallis (1980) mentions that the Wald work will soon appear in print. The papers Wald wrote describing the methods were recently reprinted by the Center for Naval Analyses (Wald 1980); there are eight memoranda, totaling over 100 pages.

The primary goal of this article is to present an expository survey of Wald's work. Wald's work is interesting from several perspectives. It is of historical interest, since the questions Wald addressed were most urgent at the time but are substantively different from questions of interest to the defense establishment today. Second, Wald's work is interesting in the light of more recent developments (e.g., isotonic regression and numerical methods in missing data problems). It is interesting in a third way, too, for it gives us another example of a great mind in action.

In writing this exposition, we have tried to stay as close to Wald's work as possible. We have followed the logical order of the arguments in the order in which he wrote the memoranda. The work is quite complicated, and many of the details are quite technical. For ease of exposition, we have eliminated as many details as possible while attempting to retain cohesiveness and clarity. The reader interested in full details can obtain copies of the original memoranda from the Center for Naval Analyses.

In the next section, the operational and statistical problems are formulated, some sample data are given, and an overview of the SRG memoranda is given. Section 3 is a survey of Wald's work, beginning with the derivation of his basic equation. Various bounds and approximations for the survivability are then derived. The section concludes with a discussion of the effects of sampling errors. In the last section, we reexamine Wald's work in light of classical statistical theory as well as more recent work. This reexamination leads us to the general conclusion that Wald's treatment of these problems was definitive.

2. THE PROBLEMS AND AN OVERVIEW OF WALD'S WORK

2.1 The Operational and Statistical Problems

The *operational* problem can be stated as follows. Aircraft returning from missions have hits by enemy weapons distributed over various parts of the plane (e.g., wings, tail, fuselage, etc.). The operational commander must decide (a) what tactics would improve survivability, and (b) how to reinforce various parts of the aircraft to improve survivability. Reinforcement means, of course, that the aircraft is heavier, and this impairs its mission. The operational commander does not know the distribution of hits on an aircraft that did not return. This is the basic difficulty in making a decision.

The statistical treatment of the problems that Wald studied is complicated by the fact that data on downed aircraft are unobservable. If these missing data were available, survival probabilities could be estimated by the methods of isotonic regression. Without such data, Wald

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set to work on the problem of estimating the probability that an aircraft that has sustained a fixed number of hits will survive an additional hit. He also attempted to estimate the survival probability of an aircraft sustaining a hit to one of various portions of the body, with different failure rates (e.g., a hit to the nose is more lethal than a hit to the middle of the fuselage). Wald's problems were compounded by a lack of modern computing equipment, a present-day recourse when one is faced with hard problems that resist analytical solution.

2.2 A Hypothetical Set of Data

Throughout the memoranda, Wald used data to illustrate his methods. Although Wald used different data values to illustrate the analysis, we have redone the calculations for one set of data. This helps one see the usefulness of the more complicated analyses.

The set of data is divided into two subsets. The first subset pertains only to hits on the aircraft, ignoring location of the hit. Assume that 400 aircraft were sent on a mission and that the numbers of aircraft returning with i hits anywhere, A_i, are $A_0 = 320$, $A_1 = 32$, $A_2 = 20$, $A_3 = 4$, $A_4 = 2$, and $A_5 = 2$. The second subset assumes that the location of the hits is known. Subdivide the aircraft into 4 main parts: engines (part 1), fuselage (part 2), fuel system (part 3), everything else (part 4), and let $\gamma(i)$ be the fraction of the area of the aircraft occupied by part i. The total number of hits to all returning aircraft in this case is $\sum_{i=1}^{5} iA_i = 102$. Assume that the hits are distributed according to the following observations:

Part number	$\gamma(i)$	Number of hits (N_i) observed on part
1	.269	19
2	.346	39
3	.154	18
4	.231	26

In anticipation of what follows, let $\delta(i)$ be the fraction of hits observed on part i. Then $\delta(1) = .186$, $\delta(2) = .382$, $\delta(3) = .176$, $\delta(4) = .255$.

These are the kinds of data that the operational commander would obtain and pass on to the statistician working for him. We suggest that the reader now reread the operational problems described in Section 2.1, consider the data again, and then decide how one might attack the problem.

2.3 An Outline of Wald's SRG Memoranda

The basic observational variables are the number N of aircraft participating in the combat, the number A_i of aircraft returning with i hits, and $a_i = A_i/N$. From these data, one wants to find P_i, the probability that an aircraft is downed by the ith hit, and p_i, the conditional probability that an aircraft is downed by the ith hit, given that it received at least $i - 1$ hits and was not downed.

Wald then introduced distributions of hits over the aircraft and found analogous quantities for each subregion of the aircraft. Figure 1 is a flowchart of Wald's work on this problem.

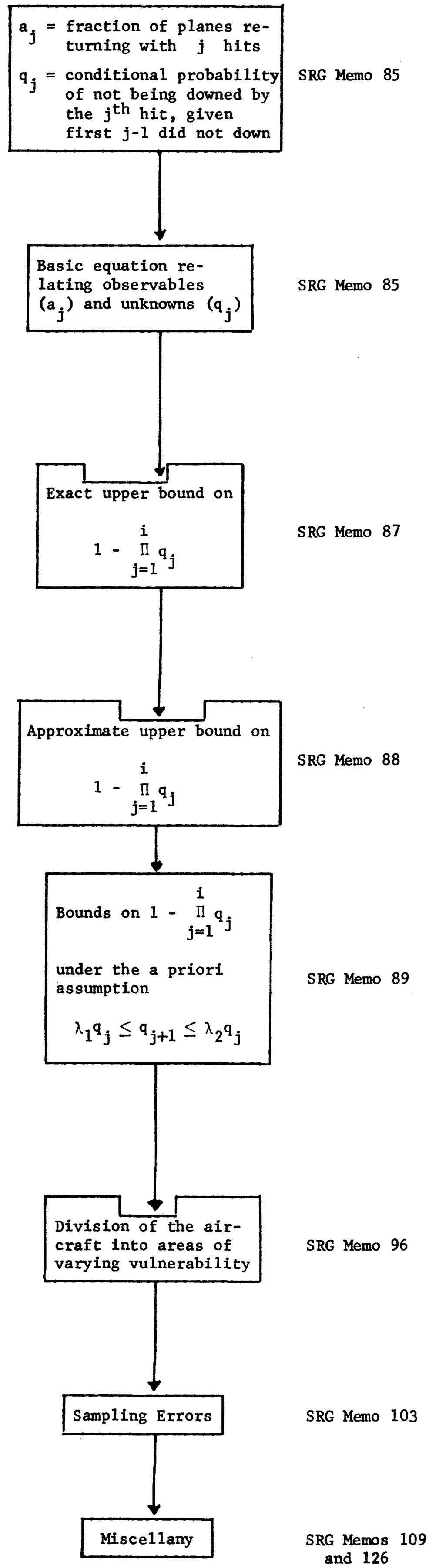

Figure 1. Schematic Outline of Wald's Memoranda.

3. SURVEY OF WALD'S MEMORANDA

This section is a survey of the memoranda. Until Section 3.6, it is assumed that sampling errors are negligible.

3.1 Wald's Basic Equation

In this section, we derive the basic equation satisfied by the probabilities P_i (or $q_i \equiv 1 - p_i$). Let $a_i \equiv A_i/N$ be the fraction of aircraft returning with i hits. Wald assumed that $a_i = 0$ if $i > n$, for some n. Thus, the fraction of aircraft lost is $L = 1 - \sum_{i=0}^{n} a_i$. Wald also assumed that an unhit aircraft always returns and that there is a value m such that the probability of receiving more than m hits is zero. He argued that $m = n + 1$.

Let x_i be the fraction of aircraft downed by the ith hit. (Thus $x_0 \equiv 0$.) Then $\sum_{i=0}^{n} x_i = L$. The x_i's then satisfy the recursion relationship

$$x_i = p_i\left(1 - \sum_{j=0}^{i-1} a_j - \sum_{j=0}^{i-1} x_j\right), \quad i = 1, \ldots, n. \qquad (3.1)$$

The term in brackets in (3.1) is the proportion of aircraft receiving at least i hits. If c_i is defined by $c_i = 1 - \sum_{j=0}^{i-1} a_j$, then (3.1) becomes

$$x_i + p_i \sum_{j=0}^{i-1} x_j = p_i c_i, \quad i = 1, 2, \ldots, n. \qquad (3.2)$$

For some of the analysis, Wald found (3.2) more useful than (3.1). The goal now is to somehow relate the observables (a_j) to the probabilities. In SRG 85, Wald derives the following equation, which is basic to all of his work.

$$\sum_{j=1}^{n} \frac{a_j}{q_1 \cdots q_j} = 1 - a_0. \qquad (3.3)$$

Equation (3.3) relates the observables a_j, the fractions of aircraft returning with j hits, and the unknowns q_j, the conditional probability of not being downed by the jth hit given that the first $j - 1$ hits did not down the aircraft. It is the fundamental equation of the analysis. In the next section, we compare Wald's work with other approaches to this problem. For this reason, it helps to review Wald's derivation of (3.3).

Let b_i be the hypothetical proportion of aircraft hit i times if dummy bullets were used. Then $b_i \geq a_i$; set $y_i = b_i - a_i$. In addition, $y_i = P_i b_i = P_i(a_i + y_i)$. Thus $y_i = (P_i/Q_i)\, a_i$, where as before, $P_i = 1 - q_1 q_2 \cdots q_i$ and $Q_i = q_1 \cdots q_i$. Hence we obtain $y_i = (a_i/q_1 \cdots q_i) - a_i$. Summing up to n and noting that $\sum_{i=1}^{n} y_i = L$ gives (3.3).

Equation (3.3) is easily solved with the simplifying assumption of constant $q_j \equiv q$. For example, for the data, (3.3) becomes the fifth-order equation

$$\frac{.08}{q} + \frac{.05}{q^2} + \frac{.01}{q^3} + \frac{.005}{q^4} + \frac{.005}{q^5} = .20, \qquad (3.4)$$

which yields $q = .851$. Hence p_i, the probability of the ith hit downing the aircraft given that the first $i - 1$ hits did not down it, is $p_i = .149$ (for all i).

Once we know p_i, we can compute x_i, the ratio of the number of aircraft downed by the ith to the total number of aircraft participating, recursively from Equations (3.1) or (3.2). We find that $x_1 = .02980$, $x_2 = .01344$, $x_3 = .00399$, $x_4 = .00190$, and $x_5 = .00087$.

These results are easily obtained, but are based on the assumption of $q_1 = q_2 = \cdots = q_n$. This severely limits their usefulness. The rest of Wald's memoranda studies ways of relaxing this assumption.

3.2 A Least Upper Bound for the Probability of i Hits Downing an Aircraft

Wald's next step was to find a bound on $P_i = 1 - \prod_{j=1}^{i} q_i$, which is the probability of an aircraft being downed by i hits. The bound he found is sharp and its attainment corresponds to the worst case in terms of survivability.

The problem of interest may be written as follows:

$$\text{minimize} \quad \prod_{j=1}^{i} q_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} \frac{a_j}{q_1 \cdots q_j} = 1 - a_0. \qquad (3.5)$$

Equation (3.5) is a nonlinear optimization problem (Avriel 1976). Wald obtained an iterative solution as follows. First he showed that if a set $\{q_1^*, \ldots, q_n^*\}$ solves (3.5), then $q_i^* = q_{i+1}^* = \cdots = q_n^*$; that is, that the q_j are all equal for $j \geq i$.

Applying this result when $i = 1$ means that q_1 is minimized if it satisfies

$$\sum_{j=1}^{n} \frac{a_j}{q_1^j} = 1 - a_0. \qquad (3.6)$$

Assume now that q_1 is known by solving the algebraic equation (3.6). Next one needs to find the value of q_2 that minimizes $q_1 q_2$. Using the result given above, problem (3.5) becomes

$$\text{minimize} \quad q_1 q_2$$

$$\text{subject to} \quad \frac{1}{q_1} \sum_{j=1}^{n} \frac{a_j}{q_2^{j-1}} = 1 - a_0. \qquad (3.7)$$

Straightforward solution via the Lagrange multiplier method gives

$$q_1 = \frac{1}{1 - a_0} \sum_{j=2}^{n} \frac{(j-1)a_j}{q_2^{j-1}}$$

and

$$\sum_{j=2}^{n-1} \frac{(j-1)a_{j+1}}{q_2^j} = a_1. \qquad (3.8)$$

Elementary arguments show that these equations have exactly one root in (q_1, q_2).

Wald then generalized this argument to determine the minimum of $\prod_{j=1}^{i} q_j$. He followed the same kind of reasoning, starting with the assumption that $q_j = q_2$, $i \geq j \geq 2$; then one wants to minimize $q_1 q_2^{i-1}$. The Lagrange multiplier method is used again; only the details change.

It is clear that even with present-day computing abilities this approach quickly becomes complicated and time-consuming. In 1943, the task of exact computations was hopeless for any problems of operational interest; thus Wald considered various approximation schemes.

3.3 Approximate Bounds on P_i

Wald's next step was to obtain approximate upper and lower bounds on P_i. Let P_i^* be the maximum value of P_i and let $Q_i^* = 1 - P_i^*$. The first step is to find the lower bound z_i of Q_i^*, that is, to find a bound on the minimum of Q_i. Wald used an interesting kind of hypothetical reasoning: Let y_j be the fraction of the returning aircraft that would be downed if they were to receive $i - j$ additional hits. Then one obtains

$$P_i = \sum_{j=0}^{i-1} y_j + \sum_{j=1}^{i} x_j, \quad i = 1, 2, \ldots, n. \tag{3.9}$$

After some algebraic manipulations, Wald obtained the bounds

$$1 - \frac{\sum_{j=1}^{i} x_j}{\left(1 - \sum_{j=0}^{i-1} a_j\right)} < Q_i < 1 - \sum_{j=1}^{i} x_j. \tag{3.10}$$

Equation (3.10) provides a lower bound on Q_i, once an upper bound on $\sum_{j=1}^{i} x_j$ is known. Wald showed that the maximum value of $X_i \equiv \sum_{j=1}^{i} x_j$ occurs when $p_1 = p_2 = \cdots = p_n = p$. In such a case, the solution of (3.6) gives $q_1 = 1 - p$, and then the x_i are obtained from (3.1). We will let z_i be the lower bound on Q_i obtained in this manner.

Next Wald turned to the problem of estimating an upper bound on the value of Q_i. He showed that such an upper bound is given by

$$t_i = \min[\tilde{u}_1^{\,i}, \tilde{u}_2^{\,i-1}, \ldots, \tilde{u}_{i-1}^{\,2}, \tilde{u}_i], \tag{3.11}$$

where $\tilde{u}_r$ is the positive root of the equation

$$\sum_{j=r}^{n} \frac{a_j}{u^{j-r+1}} = 1 - \sum_{j=0}^{r-1} a_0. \tag{3.12}$$

He obtained (3.11) and (3.12) by a sequence of manipulations on equations analogous to (3.5) and (3.6).

Let us now apply these results to the data. First we will find the lower bound z_i. The first step is to find q_0, the solution of (3.3) when $q_1 = q_2 = \cdots = q_n$. In this case, we have found q_0 as the solution of (3.4); that is, $q_0 = .851$. We have also found the x_i and thus obtain the upper bounds $X_i = \sum_{j=1}^{i} x_j$. For the data, we obtain $X_1 = .02980$, $X_2 = .04324$, $X_3 = .04723$, $X_4 = .04913$, $X_5 = .05000$. According to (3.10), our lower bound is $z_i = 1 - (X_i/(1 - \sum_{j=0}^{i-1} a_j))$. Hence we obtain $z_1 = .85100$, $z_2 = .63967$, $z_3 = .32529$, and $z_4 = .18117$. It is not necessary to calculate z_5, since Q_5 can be obtained directly. In this case, $z_5 = .090909$.

Now consider the upper bounds t_i. Let us write out some of the Equations (3.12), to see what they look like. For $r = 1$, we obtain (3.4), so that $\tilde{u}_1 = .851$. For $r = 2$, (3.12) becomes

$$\frac{a_2}{u} + \frac{a_3}{u^2} + \frac{a_4}{u^3} + \frac{a_5}{u^4} = 1 - a_0 - a_1, \tag{3.13}$$

which has solution $\tilde{u}_2 = .722$. In a similar way, one finds $\tilde{u}_3 = .531$, $\tilde{u}_4 = .333$. The t_i are given by (3.11); namely

$$\begin{aligned} t_1 &= .851 \\ t_2 &= \min(\tilde{u}_1^2, \tilde{u}_2) = .722 \\ t_3 &= \min(\tilde{u}_1^3, \tilde{u}_2^2, \tilde{u}_3) = .521 \\ t_4 &= \min(\tilde{u}_1^4, \tilde{u}_2^3, \tilde{u}_3^2, \tilde{u}_4) = .282. \end{aligned} \tag{3.14}$$

Note that t_5 is not calculated since the exact value of Q_5 can be found.

In Table 1, we compare the exact result obtained by the method of the previous section with lower bound (z_i), upper bound (t_i), and the value obtained assuming all hits are equally lethal ($q_0^{\,i}$).

3.4 Bounds on P_i Under Additional Assumptions

The results of the previous section are, from a computational viewpoint, less cumbersome than the exact results. They are still complicated to use, however, so Wald studied the bounds on survival probability under additional assumptions. These assumptions are that

$$\lambda_1 q_j \leq q_{j+1} \leq \lambda_2 q_j, \quad j = 1, 2, \ldots, n - 1 \tag{3.15}$$

for fixed known λ_1 and λ_2, and that

$$\sum_{j=1}^{n} a_j \lambda_1^{-j(j-1)/2} < 1 - a_0. \tag{3.16}$$

Note that (3.16) need not be true if λ_1 is too small; but if λ_1 is close enough to 1, then (3.16) will be true. The basic Equations (3.3) and (3.16) imply that $q_1 < 1$.

Wald first calculated the values of $q_1, \ldots, q_n$, which make Q_i ($i < n$) a minimum. Denote these by $q_1^*, \ldots, q_n^*$. By using a straightforward proof by contradiction, he proved the following: (a) for $j = i, i + 1, \ldots, n - 1$, $q_{j+1}^* = \lambda_2 q_j^*$; and (b) if j is the smallest integer such

Table 1. Exact and Approximate Values of Q_i

	Value			
i	Exact Value	Lower Bound	Upper Bound	Equal Lethality of Hits
1	.851	.851	.851	.851
2	.721	.640	.722	.724
3	.517	.325	.521	.616
4	.282	.181	.282	.525

that $q_{k+1}^* = \lambda_2 q_k^*$ for all $k \geq j$, then $q_r^* = \lambda_1 q_{r-1}^*$ for $r = 2, 3, \ldots, j - 1$. These results can be viewed as analogs of the results in Section 3.3.

Let E_{ir}, $r = 1, \ldots, i - 1$, be the minimum value of Q_i under the restriction that $q_{j+1} = \lambda_1 q_j$, $j = 1, \ldots, r - 1$, and $q_{j+1} = \lambda_2 q_j$ for $j = r + 1, \ldots, n - 1$. The above results show that $Q_i = \min\{E_{i1}, E_{i2}, \ldots, E_{i,i-1}\}$. The results in Sections 3.2 and 3.3 show how the E_{ir} can be calculated. In particular, Wald showed that if g_r is the positive root (in q) of the equation (for $r = 0, 1, 2, \ldots, i - 1$)

$$\sum_{j=1}^{r+1} a_j \lambda_1^{-j(j-1)/2} q^{-j} + \sum_{j=1}^{n-r-1} \{a_{r+1+j} \lambda_1^{-r(r+1)/2 - rj}\} \times \{\lambda_2^{-j(j+1)/2} q^{-(r+1+j)}\} = 1 - a_0 \quad (3.17)$$

then an approximation to E_{ir} is

$$E_{ir} \simeq \lambda_1^{r(r+1)/2 + r(i-r-1)} \lambda_2^{(i-r)(i-r-1)/2} q_r^{\,i}. \quad (3.18)$$

Similar arguments show that if $q_1^*, \ldots, q_n^*$ are values of q_j minimizing $Q_n = \prod_{j=1}^{n} q_j$, then $q_{j+1}^* = \lambda_1 q_j^*$, $j = 1, \ldots, n - 1$. This means that if q is the root of the equation

$$\sum_{j=1}^{n} a_j \lambda_1^{-j(j-1)/2} q^{-j} = 1 - a_0, \quad (3.19)$$

then the minimum value of Q_n is $\lambda_1^{n(n-1)/2} q^n$.

Wald proceeded in the same fashion to show that the maximum of Q_n is $\lambda_2^{n(n-1)/2} q^n$, where q is a solution of the (3.19) with λ_1 replaced by λ_2.

There is a quantity analogous to E_{ir}. Namely, if D_{ir} is the maximum of Q_i under the restriction that $q_{j+1} = \lambda_1 q_j$ for $j = r + 1, \ldots, n - 1$ and $q_{j+1} = \lambda_2 q_j$ for $j = 1, \ldots, r - 1$, then Wald showed that the maximum of Q_i is $\max\{D_{i1}, \ldots, D_{i,i-1}\}$. He showed that a good approximation to D_{ir} is obtained from (3.17) and (3.18) with the λ_1 and λ_2 interchanged.

We apply these results to the data with $\lambda_1 = .85$, $\lambda_2 = .95$. It is easy to check that (3.16) is satisfied.

To find the lower limit of Q_i, the four equations (for $r = 0, 1, 2, 3$) (3.17) must be solved. For example, for $r = 0$ this equation is

$$\frac{a_1}{q} + \frac{a_2}{\lambda_2 q^2} + \frac{a_3}{\lambda_2^3 q^3} + \frac{a_4}{\lambda_2^6 q^4} + \frac{a_5}{\lambda_2^{10} q^5} = 1 - a_0. \quad (3.20)$$

The roots of (3.17) for the values $r = 0, 1, 2, 3$ are $g_0 = .887$, $g_1 = .938$, $g_2 = .964$, and $g_3 = .979$. Next, the E_{ir} are found approximately from (3.18), and then Q_i is the minimum of the E_{ir}. Table 2 shows the results of such calculations. The lower limit of Q_5 is found by using (3.19). In this case, the root of (3.19) is $q = .986$ and the lower limit of $Q_5 = \lambda_1^{10} q^5$ is .183.

To find the maximum value of Q_i, the same procedure is followed. Since the details are the same, only the final results will be given. Table 3 shows both bounds.

Table 2. Estimating the Minimum of the Survival Probability

i	r	g_r	E_{ir} Approximately	min Q_i Approximately
1	0	.887	.887	.887
2	0	.887	.747	
	1	.938	.747	.747
3	0	.887	.598	
	1	.938	.567	
	2	.964	.550	.550
4	0	.887	.455	
	1	.938	.408	
	2	.964	.364	
	3	.979	.347	.347

3.5 Analysis of Vulnerability Areas of the Aircraft

Wald considered next the problem of determining the vulnerability of different parts of the aircraft. The idea here is that the location of the hits on returning aircraft provides useful information on the vulnerability of various parts of the aircraft. Wald began with the premise that one knows the conditional probability $\gamma_i(i_1, \ldots, i_k)$ that area m will receive i_m hits given a total of $i = \sum_{m=1}^{k} i_m$ hits. He argued that $\gamma_i(i_1, \ldots, i_k)$ can be experimentally determined by firing dummy bullets at real aircraft. The quantity of interest here is $Q_i(i_1, \ldots, i_k)$, the probability that an aircraft is not downed given i_m hits to area m, with $\sum_{m=1}^{k} i_m = i$. Wald first formulated the problem in a very general setting, where it is essentially intractable.

To make any progress, he needed to introduce an assumption of independence. Thus, he assumed that if $q(i)$ is the probability that one hit on area i will not down the aircraft and if $\gamma(i)$ is the conditional probability that area i is hit given that one hit occurred, then

$$Q_i(i_1, \ldots, i_k) = \prod_{m=1}^{k} [q(m)]^{i_m}, \quad (3.21)$$

$$\gamma_i(i_1, \ldots, i_k) = \frac{i!}{\prod_{m=1}^{k} i_m!} \prod_{m=1}^{k} [\gamma(m)]^{i_m}. \quad (3.22)$$

In (3.21) and (3.22), it is understood that $\sum_{m=1}^{k} i_m = i$. Let $\delta(i)$ be the probability that area i is hit, given that the aircraft received exactly one hit that did not down it. Then

Table 3. Lower and Upper Bounds on Q_i

i	Lower Bound on Q_i	Upper Bound on Q_i
1	.887	.986
2	.747	.826
3	.550	.631
4	.347	.463
5	.183	.329

by its definition

$$\delta(i) = \frac{\gamma(i)q(i)}{\sum_{i=1}^{k} \gamma(i)q(i)}. \tag{3.23}$$

In (3.23), recognize the summation as the probability q that a single shot did not down the aircraft. Under the assumption of independence, q will satisfy (3.3) with $q_j \equiv q$ and may be replaced by the solution to that equation. Equation (3.23) is rewritten as

$$q(i) = \frac{\delta(i)q}{\gamma(i)}, \tag{3.24}$$

where $\gamma(i)$ is assumed to be known from auxiliary tests or equated with the proportion of surface area associated with part i, and $\delta(i)$ may be estimated from the data as

$$\delta(i) = \frac{\sum_{j_k} \cdots \sum_{j_1} j_i a(j_1, \ldots, j_k)}{\sum_{j_k} \cdots \sum_{j_1} (j_1 + \cdots + j_k) a(j_1, \ldots, j_k)}. \tag{3.25}$$

The interpretation of $\delta(i)$ is that it is the ratio of the total number of hits in area i of the returning aircraft to the total number of hits on the returning aircraft. Thus, $\delta(i)$ is empirically determined and $q(i)$ is computed by applying (3.23) to the data. Such analyses have actually been performed on real data, with success.

We apply this approach to the data. We have already seen that the positive root of (3.3) with equal q_j is $q_0 = .851$. Thus q_0 is the overall probability of surviving a hit. The probability of surviving a hit to part i is given by (3.24). The q in (3.4) is q_0; $\gamma(i)$ (the fraction of area occupied by part i) and $\delta(i)$ (the fraction of hits to part i) were given along with the data. The results of the calculations are shown in Table 4. For these data, the most vulnerable portion of the aircraft is the engine area.

3.6 Effects of Sampling Errors

Wald considered sampling errors in the special case of equal (but unknown) q_j, and he derived confidence limits for q.

In the absence of sampling errors, the x_i are recursively defined by (3.1) with equal p_i. When there are sampling errors, (3.1) is replaced by

$$x_i = \bar{p}_i\left(1 - \sum_{j=0}^{i-1} a_j - \sum_{j=1}^{i-1} x_j\right), \tag{3.26}$$

Table 4. Probability of Surviving a Single Hit to a Given Part

Part	Probability of Surviving a Single Hit
Entire Aircraft	.851
Engine	.588
Fuselage	.940
Fuel System	.973
Others	.939

where $\bar{p}_i$ has the distribution of the success ratio in a sequence of $N_i = N(1 - \sum_{j=0}^{i-1} a_j - \sum_{j=1}^{i-1} x_j)$ independent trials. Still assuming that $x_i = 0$ for $i > n$ (which is not really true for the case with sampling errors), the basic equation (3.3) becomes

$$\sum_{j=1}^{n} \frac{a_j}{\bar{q}_i \cdots \bar{q}_j} = 1 - a_0. \tag{3.27}$$

Here $\bar{q}_j = 1 - \bar{p}_j$ is an estimate for q; but the $\bar{q}_j$'s are unknown.

Wald derived confidence bounds in the following manner. Consider a hypothetical experiment in which b_i is the fraction of aircraft that would be hit exactly i times if dummy bullets were used. The distribution of Na_i is the same as the distribution of the number of successes in a sequence of Nb_i independent trials, each trial having a probability of success q^i. This gives

$$E(a_i/q^i) = b_i, \quad \operatorname{var}(a_i/q^i) = \frac{b_i(1 - q^i)}{Nq^i}. \tag{3.28}$$

Summing (3.28) gives

$$E\left(\sum_{i=1}^{n} a_i/q^i\right) = \sum_{i=1}^{n} b_i = 1 - a_0,$$

$$\operatorname{var}\left(\sum_{i=1}^{n} \frac{a_i}{q^i}\right) = \sum_{i=1}^{n} \frac{b_i(1 - q^i)}{Nq^i}. \tag{3.29}$$

For moderate to large N, appeal to the central limit theorem and conclude that if

$$\int_{-\lambda_\alpha}^{\lambda_\alpha} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt = \alpha,$$

then an α confidence interval for q is

$$1 - a_0 - \lambda_\alpha\left(\sum_{i=1}^{n} \frac{b_i(1 - q^i)}{Nq^i}\right)^{1/2} \leq \sum_{i=1}^{n} \frac{a_i}{q^i} \leq 1 - a_0 + \lambda_\alpha\left(\sum_{i=1}^{n} \frac{b_i(1 - q^i)}{Nq^i}\right)^{1/2}. \tag{3.30}$$

The only trouble with (3.30) is that the b_i are not known. Again appealing to limit theorems, Wald replaced b_i by a_i/q^i (this replacement is accurate to $O(1/\sqrt{n})$). Hence we obtain a confidence interval of the form

$$1 - a_0 - \lambda_\alpha\left(\sum_{i=1}^{n} \frac{a_i(1 - q^i)}{Nq^{2i}}\right)^{1/2} \leq \sum_{i=1}^{n} \frac{a_i}{q^i} \leq 1 - a_0 + \lambda_\alpha\left(\sum_{i=1}^{n} \frac{a_i(1 - q^i)}{Nq^{2i}}\right)^{1/2}. \tag{3.31}$$

A final simplification is achieved by another appeal to a limit theorem. If q_0 is the root of (3.3) with equal q_i, then as $N \to \infty$, $q \to q_0$, so Wald replaced q^{2i} by q_0^{2i} in (3.31), and the resulting confidence limit is now very simple.

These results can be summarized in the following elegant fashion. If a_i are subject to sampling error and q is the true parameter, then $\sum_{i=1}^{n} a_i/q^i$ is normally distributed with mean $1 - a_0$ and variance given by (3.29).

To show how this works, we will derive the 95% and 99% confidence intervals for the data. The first step is to find the positive solution, q_0, of (3.3) with equal q_j. In this case, $q_0 = .851$. The second step is to find the approximate variance of $\sum_{i=1}^{n} a_i/q^i$. This variance is

$$\sigma^2 = \sum_{i=1}^{n} a_i(1 - q_0^i)/Nq_0^{2i}, \tag{3.32}$$

and in this case we find $\sigma = .01373$. According to (3.31), the confidence limits are found by solving

$$\sum_{i=1}^{n} \frac{a_i}{q^i} = 1 - a_0 \pm \lambda_\alpha \sigma, \tag{3.33}$$

where $\lambda_\alpha = 1.960, 2.576$ for the 95% and 99% limits, respectively. For the 95% confidence limit on q_0, the solution of (3.33) gives [.797, .921] and for the 99% confidence limit, [.782, .947].

3.7 Miscellany

SRG Memoranda 109 and 126 deal, very briefly, with these topics: (a) factors that are nonconstant in combat, (b) nonprobabilistic interpretation of the results, (c) the situation when $\gamma(i)$ are unknown, and (d) vulnerability to different kinds of guns. The most interesting of these topics is the last one, in which Wald generalizes the previous work to include different kinds of weapons. Namely, instead of working with $q(i)$, the probability that an aircraft survives a hit to part i, he works with $q(i, j)$, the probability that an aircraft survives a hit to part i by weapon type j. The generalization is conceptually straightforward, although the details are complicated.

4. DISCUSSION

In this section, we propose to reexamine Wald's work on aircraft survivability, relating his results to classical statistical theory as well as to more recent statistical thought. We believe that such a development makes Wald's recommendations more easily understood. It also allows us to support the general conclusion that Wald's treatment of this problem was definitive, since, through this reexamination, we are able to identify the optimal character of Wald's estimators and to explain why treatment of more general problems is impossible with the data Wald had available to him.

Let us consider the first data set. Wald does not explicitly discuss a model for the data he seeks to fit. It is clear, however, that the appropriate model is multinomial. It is also clear that there are missing data. It is useful to picture the data as embedded in the following scheme.

$$\begin{array}{cccccc} X_{01} & X_{11} & X_{21} & X_{31} & X_{41} & X_{51} \\ & X_{12} & X_{22} & X_{32} & X_{42} & X_{52} \end{array} \tag{4.1}$$

where X_{i1} = the number of aircraft returning with i hits, and X_{i2} = the number of aircraft downed with i hits. Data Set 1 amounts to X_{i1}, $i = 0, \ldots, 5$, while X_{i2}, $i = 1, \ldots, 5$ are unobservable. The multinomial distribution based on 400 observations classified into 11 cells represents the full model for the collection $\{X_{ij}\}$. Let the parameters of the full model be denoted by $\{p_{ij}\}$. Wald prefers to use the parameterization:

(1) $p_{01}, \ldots, p_{51}$ (for which $a_0, \ldots, a_5$ are the corresponding sample proportions in Wald's notation)
(2) $Q_1, \ldots, Q_5$, where

$$Q_i = \frac{p_{i1}}{p_{i1} + p_{i2}}. \tag{4.2}$$

Whatever the parameterization, the critical fact vis-a-vis the estimation problem of interest is that the full model is determined by 10 parameters while the available data have only six degrees of freedom. Put another way, the 10-parameter model for the available data is not identifiable; indeed, the likelihood depends on $\{p_{12}, \ldots, p_{52}\}$ only through the value of $\sum_{i=1}^{5} p_{i2}$. The nonidentifiability of the model for X_{i1}, $i = 0, \ldots, 5$ explains the role of the assumption

$$Q_i = q^i \quad \text{for all } i. \tag{4.3}$$

This restriction renders the estimation problem well defined. The necessity of identifiability also dictates the assumption (for the purpose of analyzing the data set) that the probability of sustaining more than five hits is zero.

We now turn to the derivation of the maximum likelihood estimators for the parameters of the multinomial distribution with missing data under the restriction (4.3). Initially, we write the likelihood as

$$\mathcal{L} \propto \left(\prod_{i=0}^{5} p_{i1}^{x_{i1}}\right)\left(1 - \sum_{i=0}^{5} p_{i1}\right)^{400 - \sum_{i=0}^{5} x_{i1}}.$$

The likelihood equations

$$\left\{\frac{\partial}{\partial p_{i1}} \mathcal{L} = 0\right\}_{i=0}^{5}$$

are equivalent to

$$\hat{p}_{i1} = \frac{x_{i1}}{N} \qquad i = 0, \ldots, 5.$$

Now, the parametric analog of Wald's fundamental equation (3.3) is

$$\sum_{j=1}^{n} \frac{p_{j1}}{\prod_{i=1}^{j} q_i} = 1 - p_{01}. \tag{4.4}$$

The latter equation can be shown to be algebraically equivalent to

$$\sum_{j=1}^{n} (p_{j1} + p_{j2}) = 1 - p_{01}, \tag{4.5}$$

which simply specifies that all cell probabilities sum to one. Under restriction (4.3), Equation (4.4) becomes

$$\sum_{j=1}^{n} \frac{p_{j1}}{q^j} = 1 - p_{01}, \tag{4.6}$$

specifying q implicitly as a function of $\{p_{i1}, i = 0, \ldots, n\}$. Now, let $\hat{q}$ be the solution of (3.3), which, for the first data set, can be written as

$$\sum_{j=1}^{5} \frac{\hat{p}_{j1}}{\hat{q}^j} = 1 - \hat{p}_{01}. \tag{4.7}$$

From the invariance property of the MLE's, it is clear that $\hat{q}$ is the MLE of the parameter q.

The regularity of the multinomial model implies the asymptotic optimality of Wald's estimators of the parameters $\{p_{i1}\}$ and p. Wald's confidence interval for the survival probability q can be obtained via MLE theory and thus, its optimality in large samples can be asserted. Since interesting larger models cannot be treated with the data available, Wald's estimation results are, with a sufficiently large sample size, the best possible. For larger models, Wald appropriately turns to the development of bounds on survival probabilities.

Two important areas of statistical analysis having some bearing on Wald's work have been developed since Wald's time. The first is the area of isotonic regression, a subject treated in depth in the recent book by Barlow et al. (1972). The second is the treatment of problems with missing data via the EM algorithm (see Dempster, Laird, and Rubin 1977). Isotonic regression would appear to be an appropriate methodology in Wald's problem, since aircraft vulnerability undoubtedly increases with the number of hits sustained; that is, it is reasonable to expect that $p_1 \le p_2 \le \cdots \le p_n$. In spite of its intuitive appeal, the isotonic version of Wald's problem suffers from nonidentifiability, since ordering of parameters does not reduce the dimension of the parameter space. Thus, given Wald's data, estimation via the methods of isotonic regression proves impossible without additional assumptions. If complete data were available, the unrestricted MLE's for the q_i's are given by

$$\prod_{j=1}^{i} \hat{q}_j = \frac{x_{i1}}{x_{i1} + x_{i2}} \quad i = 1, \ldots, 5. \tag{4.8}$$

The problem of "isotonizing" these estimates is formally equivalent to the problem of estimating ordered binomial parameters treated by Barlow et al. (1972, p. 102).

The EM algorithm does not help for similar reasons. When the model is not identifiable, a starting value $\mathbf{p}^{(0)}$ for the parameter produces expected $\mathbf{X}$ values, which in turn produce $\mathbf{p}^{(1)} = \mathbf{p}^{(0)}$. In the reduced model, subject to (4.3), one can treat maximum likelihood estimation analytically, and there is no need to employ the EM algorithm.

Let us now examine Wald's estimators for the survival probabilities of various aircraft sections. The portion of the data set classifying hits by part can be viewed as embedded in the array

$$\begin{array}{ccccc} Y_{11} & Y_{21} & Y_{31} & Y_{41} & N_1 \\ Y_{12} & Y_{22} & Y_{32} & Y_{42} & N_2 \end{array} \tag{4.9}$$

where Y_{i1} = # of hits to part i on returning aircraft; Y_{i2} = # of hits to part i on downed aircraft; $N_1 = \sum_{i=1}^4 Y_{i1}$; $N_2 = \sum_{i=1}^4 Y_{i2}$. The data consist of $Y_{i1}, i = 1, \ldots, 4$ and N_1, while $Y_{i2}, i = 1, \ldots, 4$ and N_2 are unobservable. Define the following events:

$$A_i = \{\text{the } i\text{th section is hit}\}$$

$$A = \{\text{the aircraft is hit}\}$$

$$B = \{\text{the aircraft is not downed}\}.$$

Wald's parameters may be identified as

$$q = P(B \mid A),\ q(i) = P(B \mid A_i)$$

$$\delta(i) = P(A_i \mid A \cap B),\ \gamma(i) = P(A_i \mid A). \tag{4.10}$$

With complete data as pictured in (4.9), the MLE's of $q(i)$ are simply

$$\hat{q}(i) = \frac{Y_{i1}}{Y_{i1} + Y_{i2}} \quad i = 1, \ldots, 4. \tag{4.11}$$

With the incomplete data available to Wald, one must make use of the structural relationship (3.23) (which is immediate from the definitions in (4.10)) and the assumption that $\gamma(i), i = 1, \ldots, 4$ are known. Wald explicitly remarks on the impossibility of estimating $\gamma(i)$ and $q(i)$ simultaneously from his data. However, MLE's for $\{\delta(i)\}$ and q may be obtained from the data, and the estimates

$$\hat{q}(i) = \frac{\hat{\delta}(i)}{\gamma(i)} \cdot \hat{q} \quad i = 1, \ldots, 4 \tag{4.12}$$

are maximum likelihood estimates by invariance, provided these estimates lie in the unit interval. Wald does not deal with estimation problems in which one or more of the estimates $\hat{q}(i)$ exceed one. In such cases, the MLE of the vector $(q(1), \ldots, q(4))$ lies on the boundary of the parameter space, and its identification is tedious but straightforward.

In our discussion of Wald's formulation and solution of a variety of problems dealing with aircraft survivability, we have mentioned a number of assumptions he imposed to obtain closed-form solutions or efficient bounds. These assumptions deserve scrutiny. Among the assumptions one encounters are (a) constant vulnerability, that is, $q_i \equiv q$, which is an independence assumption; (b) known bounds on rate of growth of vulnerability, that is, $\lambda_1 q_j \le q_{j+1} \le \lambda_2 q_j$; and (c) independence of survival among and within areas of different vulnerability. The main cause for concern regarding these assumptions is that the data available do not provide a means for investigating their validity. Consider assumption (a), for example. With complete data (corresponding to $\{x_{ij}\}$ in (4.1)) one could investigate statistically, via a likelihood ratio test or otherwise, the validity of the assumption $q_i \equiv q$. With the type of data available to Wald, such an option is not open because of the lack of identifiability of larger models. Wald cautioned his readers that the solution he provides should be used only "if it is known *a priori* that $q_1 = q_2 = \cdots = q_n$." How and whether such a priori knowledge could be garnered is open to debate. Wald does provide an option for those who are more conservative. The lower bounds for Q_i may be considered conservative estimates of survival probabilities, although they might often be too small to be useful. The dilemma one encounters with the foregoing three assumptions mentioned is similar to that faced in competing risks methodology, where considerable recent work has focused on identifiability and bounds for survival probabilities (see Tsiatis 1975 and Peterson 1976).

Viewing Wald's work on aircraft survivability in light of the state of the art at the time it was done, it seems to us to be a remarkable piece of work. While the field of statistics has grown considerably since the early 1940's, Wald's work on this problem is difficult to improve upon. Much of the work appears to be ad hoc—there are few allusions to modeling and no reference to classical statistical approaches or results. By the sheer power of his intuition, Wald was led to subtle structural relationships (e.g., Equations (3.3) and (3.24)), and was able to deal with both structural and inferential questions in a definitive way.

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Comment

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The authors are to be congratulated on a fine paper. They have distilled the key ideas in Wald's work on aircraft survivability, and have successfully related the ideas to standard statistical methods. The bulk of this discussion will be concerned with this relationship of the work to standard statistical methods, particularly the use of statistical models to describe the situation. Some attention will also be given to decision-theoretic issues.

1. STATISTICAL MODELING

As indicated in the paper, the primary quantities studied can be considered

$$P_{i1} = P\,(i \text{ hits and survival}) = Q_i \cdot \lambda_i,$$

where

$$Q_i = P\,(\text{survival} \mid i \text{ hits}),$$

$$\lambda_i = P\,(i \text{ hits}),$$

and

$$P_0{}^* = P\,(\text{not surviving}) = 1 - \sum_{i=0}^{\infty} P_{i1}.$$

If the observations can be assumed to be independent, and out of a total of n missions the data are

X_{i1} = the number of aircraft that receive i hits and survive,

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