Elements of Numerical Integration

EXERCISE SET 4.3

EXERCISE SET 4.4

Elements of Numerical Integration

The methods of quadrature in this section are based on the interpolation polynomials

The range of integration $(b - a)$ is divided into a *finite* number of intervals in numerical integration. The integration techniques consisting of equal intervals are based on formulas known as *Newton-Cotes closed quadrature formulas*.

In this chapter, we present the following methods of integration with illustrative examples:

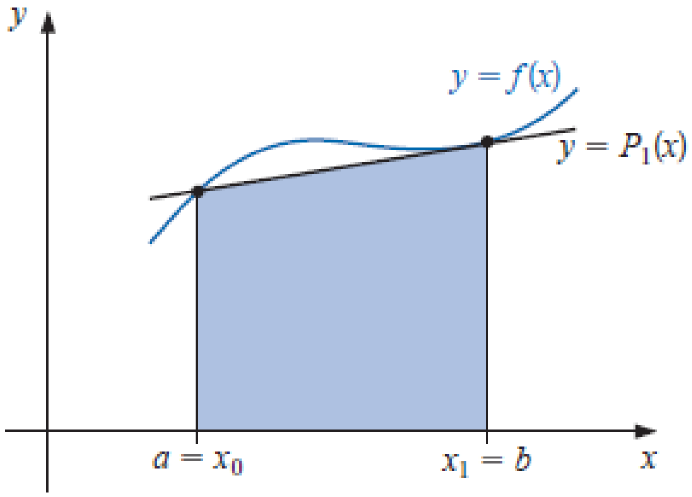
1. Trapezoidal rule.
2. Simpson's 1/3 rule.
3. Simpson's 3/8 rule.
4. Boole's and Weddle's rules.

a-Trapezoidal and Simpson's rule (4.3)
b- Closed and open Newton-cotes formula
c-Composite Numerical Integration (4.4)

The Trapezoidal Rule

To derive the Trapezoidal rule for approximating $\int_a^b f(x)\,dx$, let $x_0 = a, x_1 = b, h = b - a$ and use the linear Lagrange polynomial:

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1).$$



Then

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} \left[\frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1) \right] dx$$

$$\int_a^b f(x)\,dx = \left[\frac{(x - x_1)^2}{2(x_0 - x_1)} f(x_0) + \frac{(x - x_0)^2}{2(x_1 - x_0)} f(x_1) \right]_{x_0}^{x_1} = \frac{(x_1 - x_0)}{2}[f(x_0) + f(x_1)] = \frac{h}{2}[f(x_0) + f(x_1)]$$

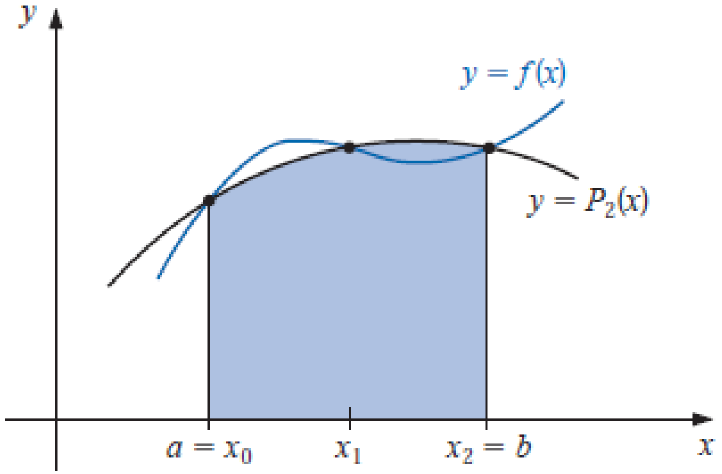
Similarly $\int_{x1}^{x2} f(x)dx = \frac{h}{2}[f(x1) + f(x2)]$, *and* $\int_{x(n-1)}^{xn} f(x)dx = \frac{h}{2}[f(x_{n-1}) + f(xn)]$,

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) + E_n$$

Is called trapezoidal rule

Simpson's Rule

Simpson's rule results from integrating over $[a, b]$ the second Lagrange polynomial with equally-spaced nodes $x_0 = a$, $x_2 = b$, and $x_1 = a + h$, where $h = (b - a)/2$. (See Figure 4.4.)



Therefore

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} \left[\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \right] dx$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

similarly $\int_{x2}^{x4} f(x)dx = \frac{h}{3}[f(x2) + 4f(x3) + f(x4)]$ and

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \cdots + y_{2N-1}) + 2(y_2 + y_4 + \cdots + y_{2N-2}) + y_{2N}] + \text{Error term}$$

Is called Simpson's 1/3 rule

QUADRATURE FORMULAS:

TRAPEZOIDAL RULE

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) + E_n$$

SIMPSON'S 1/3 RULE

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \cdots + y_{2N-1}) + 2(y_2 + y_4 + \cdots + y_{2N-2}) + y_{2N}] + \text{Error term}$$

Simpson's 3/8 rule is

$$\int_a^b f(x)dx = \frac{3}{8}h[y(a) + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y(b)]$$

Closed-Newton-Cotes (Quadrature formulas)

Theorem 4.2 Suppose that $\sum_{i=0}^{n} a_i f(x_i)$ denotes the $(n+1)$-point closed Newton-Cotes formula with $x_0 = a$, $x_n = b$, and $h = (b-a)/n$. There exists $\xi \in (a, b)$ for which

Some of the common **closed Newton-Cotes formulas** with their error terms are listed. Note that in each case the unknown value ξ lies in (a, b).

$n = 1$: Trapezoidal rule

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi), \quad \text{where} \quad x_0 < \xi < x_1. \tag{4.25}$$

$n = 2$: Simpson's rule

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi), \quad \text{where} \quad x_0 < \xi < x_2. \tag{4.26}$$

$n = 3$: Simpson's Three-Eighths rule

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] - \frac{3h^5}{80} f^{(4)}(\xi), \qquad (4.27)$$

where $x_0 < \xi < x_3$.

$n = 4$:

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] - \frac{8h^7}{945} f^{(6)}(\xi),$$

where $x_0 < \xi < x_4$. $\qquad (4.28)$

Open-Newton-Cotes (Quadrature formulas)

Theorem 4.3 Suppose that $\sum_{i=0}^{n} a_i f(x_i)$ denotes the $(n+1)$-point open Newton-Cotes formula with $x_{-1} = a$, $x_{n+1} = b$, and $h = (b-a)/(n+2)$. There exists $\xi \in (a,b)$ for which

$n = 0$: Midpoint rule

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2h f(x_0).$$

$n = 1$:

$$\int_{x_{-1}}^{x_2} f(x)\,dx = \frac{3h}{2}[f(x_0) + f(x_1)]$$

$n = 2$:

$$\int_{x_{-1}}^{x_3} f(x)\,dx = \frac{4h}{3}[2f(x_0) - f(x_1) + 2f(x_2)].$$

$n = 3$:

$$\int_{x_{-1}}^{x_4} f(x)\,dx = \frac{5h}{24}[11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)].$$

Example 2 Compare the results of the closed and open Newton-Cotes formulas listed as (4.25)–(4.28) and (4.29)–(4.32) when approximating

$$\int_0^{\pi/4} \sin x \, dx = 1 - \sqrt{2}/2 \approx 0.29289322.$$

Solution For the closed formulas we have

$$n = 1: \quad \frac{(\pi/4)}{2}\left[\sin 0 + \sin \frac{\pi}{4}\right] \approx 0.27768018$$

$$n = 2: \quad \frac{(\pi/8)}{3}\left[\sin 0 + 4\sin \frac{\pi}{8} + \sin \frac{\pi}{4}\right] \approx 0.29293264$$

$$n = 3: \quad \frac{3(\pi/12)}{8}\left[\sin 0 + 3\sin \frac{\pi}{12} + 3\sin \frac{\pi}{6} + \sin \frac{\pi}{4}\right] \approx 0.29291070$$

$$n = 4: \quad \frac{2(\pi/16)}{45}\left[7\sin 0 + 32\sin \frac{\pi}{16} + 12\sin \frac{\pi}{8} + 32\sin \frac{3\pi}{16} + 7\sin \frac{\pi}{4}\right] \approx 0.29289318$$

and for the open formulas we have

$$n = 0: \quad 2(\pi/8)\left[\sin\frac{\pi}{8}\right] \approx 0.30055887$$

$$n = 1: \quad \frac{3(\pi/12)}{2}\left[\sin\frac{\pi}{12} + \sin\frac{\pi}{6}\right] \approx 0.29798754$$

$$n = 2: \quad \frac{4(\pi/16)}{3}\left[2\sin\frac{\pi}{16} - \sin\frac{\pi}{8} + 2\sin\frac{3\pi}{16}\right] \approx 0.29285866$$

$$n = 3: \quad \frac{5(\pi/20)}{24}\left[11\sin\frac{\pi}{20} + \sin\frac{\pi}{10} + \sin\frac{3\pi}{20} + 11\sin\frac{\pi}{5}\right] \approx 0.29286923$$

Class Activity

Example: Compute the integral $I = \sqrt{\frac{2}{\pi}}\int_0^1 e^{-x^2/2}dx$ using Simpson's 1/3 rule,

Taking h = 0.125.

EXERCISE SET 4.3

1. Approximate the following integrals using the Trapezoidal rule.

a. $\int_{0.5}^{1} x^4\,dx$

b. $\int_{0}^{0.5} \frac{2}{x-4}\,dx$

c. $\int_{1}^{1.5} x^2 \ln x\,dx$

d. $\int_{0}^{1} x^2 e^{-x}\,dx$

e. $\int_{1}^{1.6} \frac{2x}{x^2-4}\,dx$

f. $\int_{0}^{0.35} \frac{2}{x^2-4}\,dx$

g. $\int_{0}^{\pi/4} x \sin x\,dx$

h. $\int_{0}^{\pi/4} e^{3x} \sin 2x\,dx$

2. Approximate the following integrals using the Trapezoidal rule.

a. $\int_{-0.25}^{0.25} (\cos x)^2\,dx$

b. $\int_{-0.5}^{0} x \ln(x+1)\,dx$

c. $\int_{0.75}^{1.3} \left((\sin x)^2 - 2x \sin x + 1\right)\,dx$

d. $\int_{e}^{e+1} \frac{1}{x \ln x}\,dx$

3. Find a bound for the error in Exercise 1 using the error formula, and compare this to the actual error.

4. Find a bound for the error in Exercise 2 using the error formula, and compare this to the actual error.

EXERCISE SET 4.3

5. Repeat Exercise 1 using Simpson's rule.
6. Repeat Exercise 2 using Simpson's rule.
7. Repeat Exercise 3 using Simpson's rule and the results of Exercise 5.
8. Repeat Exercise 4 using Simpson's rule and the results of Exercise 6.
9. Repeat Exercise 1 using the Midpoint rule.
10. Repeat Exercise 2 using the Midpoint rule.

22. Given the function f at the following values,

x	1.8	2.0	2.2	2.4	2.6
$f(x)$	3.12014	4.42569	6.04241	8.03014	10.46675

approximate $\int_{1.8}^{2.6} f(x)\,dx$ using all the appropriate quadrature formulas of this section.

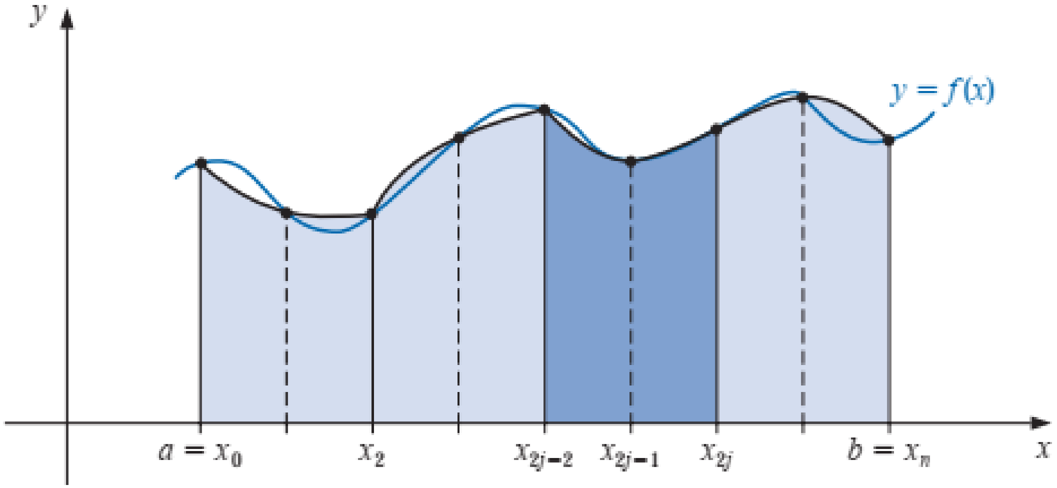
4.4 Composite Numerical Integration

Theorem 4.4 Let $f \in C^4[a,b]$, n be even, $h = (b-a)/n$, and $x_j = a + jh$, for each $j = 0, 1, \ldots, n$. There exists a $\mu \in (a,b)$ for which the **Composite Simpson's rule** for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180}h^4 f^{(4)}(\mu).$$

Derivation:

To generalize this procedure for an arbitrary integral $\int_a^b f(x)\,dx$, choose an even integer n. Subdivide the interval $[a,b]$ into n subintervals, and apply Simpson's rule on each consecutive pair of subintervals. (See Figure 4.7.)



Derivation:

Simpson's Rules (Composite Forms)

Simpson's 1/3 rule

$$h = \frac{b-a}{n}$$

The total integral can be represented as

$$I = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

Substituting Simpson's 1/3 rule for the individual integral yields

$$I \cong 2h\frac{f(x_0) + 4\,f(x_1) + f(x_2)}{6} + 2h\frac{f(x_2) + 4\,f(x_3) + f(x_4)}{6}$$

$$+ \cdots + 2h\frac{f(x_{n-2}) + 4\,f(x_{n-1}) + f(x_n)}{6}$$

$$I = \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{2N-2} + 4y_{2N-1} + y_{2N})]$$

$$\boxed{\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \cdots + y_{2N-1}) + 2(y_2 + y_4 + \cdots + y_{2N-2}) + y_{2N}] + \text{Error term}}$$

Derivation:

Simpson's 3/8 rule

Similarly in deriving composite Simpson's 3/8 rule, we divide the interval of integration into n sub-intervals, where n is divisible by 3, and applying the integration formula

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \cdots + \int_{x_{n-3}}^{x_n} f(x)dx$$

$$\int_{x_0}^{x_3} f(x)dx = \frac{3}{8}h(y_0 + 3y_1 + 3y_2 + y_3)$$

We obtain the composite form of Simpson's 3/8 rule as

$$\int_a^b f(x)dx = \frac{3}{8}h[y(a) + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots$$
$$+ 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y(b)]$$

Theorem 4.5 Let $f \in C^2[a,b]$, $h = (b-a)/n$, and $x_j = a + jh$, for each $j = 0, 1, \ldots, n$. There exists a $\mu \in (a,b)$ for which the **Composite Trapezoidal rule** for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu). \qquad \blacksquare$$

The Trapezoidal Rule (Composite Form)

The Newton-Cotes formula is based on approximating $y = f(x)$ between (x_0, y_0) and $(x_1, y_1)$by a straight line, thus forming a trapezium, is called trapezoidal rule. In order to evaluate the definite integral

$$I = \int_a^b f(x)dx$$

Derivation:

we divide the interval [a, b] into n sub-intervals, each of size $h = (b - a)/n$ and denote the sub-intervals by $[x_0, x_1]$, $[x_1, x_2]$, …, $[x_{n-1}, x_n]$, such that $x_0 = a$ and $x_n = b$ and $x_k = x_0 + k_h$, $k = 1, 2, \ldots, n - 1$.

Thus, we can write the above definite integral as a sum. Therefore,

$$I = \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

Substituting the trapezoidal rule for each integral yields

$$I = h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + \cdots + h\frac{f(x_{n-1}) + f(x_n)}{2}$$

or, grouping terms,

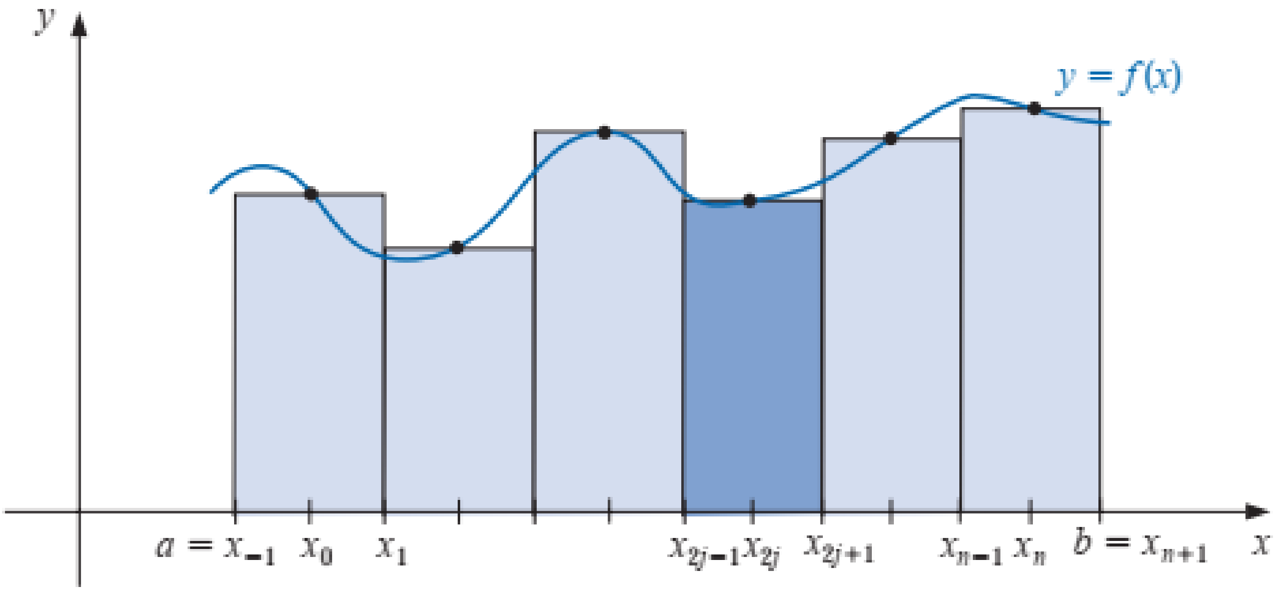
$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

OR $\int_a^b f(x)dx = \frac{h}{2}\sum_{i=0}^{n-1}(f_i + f_{i+1})$

$$\boxed{\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) + E_n}$$

Theorem 4.6 Let $f \in C^2[a,b]$, n be even, $h = (b-a)/(n+2)$, and $x_j = a + (j+1)h$ for each $j = -1, 0, \ldots, n+1$. There exists a $\mu \in (a,b)$ for which the **Composite Midpoint rule** for $n+2$ subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = 2h\sum_{j=0}^{n/2} f(x_{2j}) + \frac{b-a}{6}h^2 f''(\mu). \quad \blacksquare$$



Composite Trapezoidal rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu).$$

Composite Simpson's Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180}h^4 f^{(4)}(\mu).$$

Composite Midpoint rule

$$\int_a^b f(x)\,dx = 2h\sum_{j=0}^{n/2} f(x_{2j}) + \frac{b-a}{6}h^2 f''(\mu).$$

Example 1 Use Simpson's rule to approximate $\int_0^4 e^x\,dx$ and compare this to the results obtained by adding the Simpson's rule approximations for $\int_0^2 e^x\,dx$ and $\int_2^4 e^x\,dx$. Compare these approximations to the sum of Simpson's rule for $\int_0^1 e^x\,dx$, $\int_1^2 e^x\,dx$, $\int_2^3 e^x\,dx$, and $\int_3^4 e^x\,dx$.

$$\boxed{\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]}$$

Solution Simpson's rule on $[0, 4]$ uses $h = 2$ and gives

$$\int_0^4 e^x\,dx \approx \frac{2}{3}(e^0 + 4e^2 + e^4) = 56.76958.$$

$$\boxed{\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]}$$

Applying Simpson's rule on each of the intervals $[0, 2]$ and $[2, 4]$ uses $h = 1$ and gives

$$\begin{aligned}
\int_0^4 e^x\,dx &= \int_0^2 e^x\,dx + \int_2^4 e^x\,dx \\
&\approx \frac{1}{3}\left(e^0 + 4e + e^2\right) + \frac{1}{3}\left(e^2 + 4e^3 + e^4\right) \\
&= \frac{1}{3}\left(e^0 + 4e + 2e^2 + 4e^3 + e^4\right) \\
&= 53.86385.
\end{aligned}$$

$$\boxed{\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]}$$

For the integrals on [0, 1],[1, 2],[3, 4], and [3, 4] we use Simpson's rule four times with $h = \frac{1}{2}$ giving

$$\begin{aligned}
\int_0^4 e^x\,dx &= \int_0^1 e^x\,dx + \int_1^2 e^x\,dx + \int_2^3 e^x\,dx + \int_3^4 e^x\,dx \\
&\approx \frac{1}{6}\left(e_0 + 4e^{1/2} + e\right) + \frac{1}{6}\left(e + 4e^{3/2} + e^2\right) \\
&\quad + \frac{1}{6}\left(e^2 + 4e^{5/2} + e^3\right) + \frac{1}{6}\left(e^3 + 4e^{7/2} + e^4\right) \\
&= \frac{1}{6}\left(e^0 + 4e^{1/2} + 2e + 4e^{3/2} + 2e^2 + 4e^{5/2} + 2e^3 + 4e^{7/2} + e^4\right) \\
&= 53.61622.
\end{aligned}$$

Example 2 Determine values of h that will ensure an approximation error of less than 0.00002 when approximating $\int_0^\pi \sin x \, dx$ and employing
(a) Composite Trapezoidal rule and **(b)** Composite Simpson's rule.

How many subinterval of [0,pi] are required

Solution **(a)** The error form for the Composite Trapezoidal rule for $f(x) = \sin x$ on $[0, \pi]$ is

$$\left|\frac{\pi h^2}{12} f''(\mu)\right| = \left|\frac{\pi h^2}{12}(-\sin \mu)\right| = \frac{\pi h^2}{12}|\sin \mu|.$$

$$\frac{\pi h^2}{12}|\sin \mu| \le \frac{\pi h^2}{12} < 0.00002.$$

h = 0.00874

Since $h = \pi/n$ implies that $n = \pi/h$, we need

$$\frac{\pi^3}{12n^2} < 0.00002 \quad \Longrightarrow \quad n > \left(\frac{\pi^3}{12(0.00002)}\right)^{1/2} \approx 359.44.$$

$\boxed{\text{Composite Trapezoidal rule requires } n \ge 360.}$

(b) The error form for the Composite Simpson's rule for $f(x) = \sin x$ on $[0, \pi]$ is

$$\left|\frac{\pi h^4}{180} f^{(4)}(\mu)\right| = \left|\frac{\pi h^4}{180} \sin \mu\right| = \frac{\pi h^4}{180} |\sin \mu|.$$

$$\frac{\pi h^4}{180} |\sin \mu| \le \frac{\pi h^4}{180} < 0.00002.$$

Using again the fact that $n = \pi / h$ gives

$$\frac{\pi^5}{180 n^4} < 0.00002 \quad \Longrightarrow \quad n > \left(\frac{\pi^5}{180(0.00002)}\right)^{1/4} \approx 17.07.$$

So Composite Simpson's rule requires only $n \ge 18$.

EXERCISE SET 4.4

HOME ACTIVITY

1. Use the Composite Trapezoidal rule with the indicated values of n to approximate the following integrals.

   a. $\int_1^2 x \ln x \, dx, \quad n = 4$

   b. $\int_{-2}^2 x^3 e^x \, dx, \quad n = 4$

   c. $\int_0^2 \frac{2}{x^2 + 4} \, dx, \quad n = 6$

   d. $\int_0^\pi x^2 \cos x \, dx, \quad n = 6$

   e. $\int_0^2 e^{2x} \sin 3x \, dx, \quad n = 8$

   f. $\int_1^3 \frac{x}{x^2 + 4} \, dx, \quad n = 8$

   g. $\int_3^5 \frac{1}{\sqrt{x^2 - 4}} \, dx, \quad n = 8$

   h. $\int_0^{3\pi/8} \tan x \, dx, \quad n = 8$

2. Use the Composite Trapezoidal rule with the indicated values of n to approximate the following integrals.

   a. $\int_{-0.5}^{0.5} \cos^2 x \, dx, \quad n = 4$

   b. $\int_{-0.5}^{0.5} x \ln(x + 1) \, dx, \quad n = 6$

   c. $\int_{.75}^{1.75} (\sin^2 x - 2x \sin x + 1) \, dx, \quad n = 8$

   d. $\int_e^{e+2} \frac{1}{x \ln x} \, dx, \quad n = 8$

3. Use the Composite Simpson's rule to approximate the integrals in Exercise 1.

4. Use the Composite Simpson's rule to approximate the integrals in Exercise 2.

CLASS ACTIVITY

7. Approximate $\int_0^2 x^2 \ln(x^2+1)\,dx$ using $h = 0.25$. Use

- **a.** Composite Trapezoidal rule.
- **b.** Composite Simpson's rule.
- **c.** Composite Midpoint rule.

11. Determine the values of n and h required to approximate

$$\int_0^2 e^{2x} \sin 3x \, dx$$

to within 10^{-4}. Use

- **a.** Composite Trapezoidal rule.
- **b.** Composite Simpson's rule.
- **c.** Composite Midpoint rule.

13. Determine the values of n and h required to approximate

$$\int_0^2 \frac{1}{x+4}\,dx$$

to within 10^{-5} and compute the approximation. Use

- **a.** Composite Trapezoidal rule.
- **b.** Composite Simpson's rule.
- **c.** Composite Midpoint rule.

QUIZ

2. Use the Composite Trapezoidal rule with the indicated values of n to approximate the following integrals.

a. $\int_{-0.5}^{0.5} \cos^2 x \, dx, \quad n = 4$

b. $\int_{-0.5}^{0.5} x \ln(x+1) \, dx, \quad n = 6$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right]$$

4. Use the Composite Simpson's rule to approximate the integrals in Exercise 2.

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right]$$

9. Suppose that $f(0) = 1$, $f(0.5) = 2.5$, $f(1) = 2$, and $f(0.25) = f(0.75) = \alpha$. Find α if the Composite Trapezoidal rule with $n = 4$ gives the value 1.75 for $\int_0^1 f(x)\,dx$.

Solution:

The Composite Trapezoidal Rule

The composite Trapezoidal rule is given by

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f\left(x_j\right) + f(b)\right]$$

Given Data

$$f(0) = 1,$$
$$f(0.5) = 2.5,$$
$$f(1) = 2,$$
$$f(0.25) = f(0.75) = \alpha$$

Also given that

$$\int_0^1 f(x)dx = 1.75$$

Apply the formula

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f\left(x_j\right) + f(b)\right]$$

$$\int_0^1 f(x)dx = \frac{h}{2}\left[f\left(0\right) + 2\left(f\left(0.25\right) + f\left(0.5\right) + f\left(0.75\right)\right) + f\left(1\right)\right]$$

$$1.75 = \frac{0.25}{2}\left[1 + 2\left(2.5 + \alpha + \alpha\right) + 2\right]$$

$$\frac{3.5}{0.25} = \left[3 + 5 + 4\alpha\right]$$

$$8 + 4\alpha = 14$$

$$\boxed{\alpha = 1.5}$$

More Example
(Tabular form)

Example

Find the approximate value of $y = \int_0^{\pi} \sin x dx$ using

(i) Trapezoidal rule

n = 6

Solution

We shall at first divide the range of integration $(0, \pi)$ into six equal parts so that each part is of width $\pi/6$ and write down the table of values:

X	0	π/6	π/3	π/2	2π/3	5π/6	π
Y=sinx	0.0	0.5	0.8660	1.0	0.8660	0.5	0.0

Applying trapezoidal rule, we have

$$\int_0^{\pi} \sin x dx = \frac{h}{2}[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$y = \int_0^{\pi} \sin x dx = \frac{\pi}{12}[0 + 0 + 2(3.732)] = \frac{3.1415}{6} \times 3.732 = 1.9540$$

Example :

From the following data, estimate the value of $\int_1^5 \log x dx$ using Simpson's 1/3

X	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Y=logx	0.0000	0.4055	0.6931	0.9163	1.0986	1.2528	1.3863	1.5041	1.6094

$$\int_1^5 \log x dx = \frac{h}{3}[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{0.5}{3}[(0 + 1.6094) + 4(4.0787) + 2(3.178)]$$

$$= \frac{0.5}{3}(1.6094 + 16.3148 + 6.356) = 4.0467$$

Example: Compute the integral $I=\sqrt{\frac{2}{\pi}}\int_0^1 e^{-x^2/2}dx$ using Simpson's 1/3 rule,

Taking h = 0.125.

Solution At the outset, we shall construct the table of the function as required.

X	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1.0
$y=\sqrt{\frac{2}{\pi}}\exp(-x^2/2)$	0.7979	0.7917	0.7733	0.7437	0.7041	0.6563	0.6023	0.5441	0.4839

Using Simpson's 1.3 rule, we have

$$=\frac{h}{3}[y_0+y_8+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)]$$

$$=\frac{0.125}{3}[0.7979+0.4839+4(0.7917+0.7437+0.6563+0.5441)$$
$$+2(0.7733+0.7041+0.6023)]$$

$$=\frac{0.125}{3}(1.2812+10.9432+4.1594) \quad =0.6827$$

Solve using all composite quadrature formula

Evaluate $\int_2^6 \log_{10} x \, dx$ by using trapezoidal rule, taking $n = 8$, correct to five decimal places.

$$f(x) = \log_{10} x$$

$$a = 2,\ b = 6,\ n = 8$$

$$h = \frac{b-a}{n} = \frac{6-2}{8} = \frac{1}{2} = 0.5$$

x	2	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
f(x)	0.30103 y_0	0.39794 y_1	0.47712 y_2	0.54407 y_3	0.60206 y_4	0.65321 y_5	0.69897 y_6	0.74036 y_7	0.77815 y_8

$$I = \frac{h}{2}[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$I = \frac{0.5}{2}[(0.30103 + 0.77815) + 2(0.39794 + 0.47712 + 0.54407 + 0.60206 + 0.65321$$

$$+ 0.69897 + 0.74036 + 0.77815)]$$

$$I = 2.32666$$

Evaluate $\int_2^6 \log_{10} x \, dx$ by using Simpson's 1/3 rule, taking $n = 6$.

$$f(x) = \log_{10} x$$

$$a = 2,\ b = 6,\ n = 6$$

$$h = \frac{b-a}{n} = \frac{6-2}{6} = \frac{2}{3}$$

x	2 = 6/3	8/3	10/3	12/3 = 4	14/3	16/3	18/3 = 6
y = f(x)	0.30103 y_0	0.42597 y_1	0.52288 y_2	0.60206 y_3	0.66901 y_4	0.72700 y_5	0.77815 y_6

The Simpson's 1/3 rule is

$$I = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$I = \frac{2/3}{3}[(0.30103 + 0.77815) + 4(0.42597 + 0.60206 + 0.72700) + 2(0.52288 + 0.66901)]$$

$$I = 2.32957$$

using Simpson's 3/8 rule, taking $n = 6$,

$$f(x) = \log_{10} x$$

$$a = 2,\ b = 6,\ n = 6$$

$$h = \frac{b-a}{n} = \frac{6-2}{6} = \frac{2}{3}$$

x	2 = 6/3	8/3	10/3	12/3 = 4	14/3	16/3	18/3 = 6
y = f(x)	0.30103 y_0	0.42597 y_1	0.52288 y_2	0.60206 y_3	0.66901 y_4	0.72700 y_5	0.77815 y_6

The Simpson's three-eighth's rule is

$$I = \frac{3 \cdot h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$I = \frac{3(2/3)}{8}[(0.30103 + 0.77815) + 3(0.42597 + 0.52288 + 0.66901 + 0.72700) + 2(0.60206)]$$

$$I = 2.32947$$

ANY Questions?