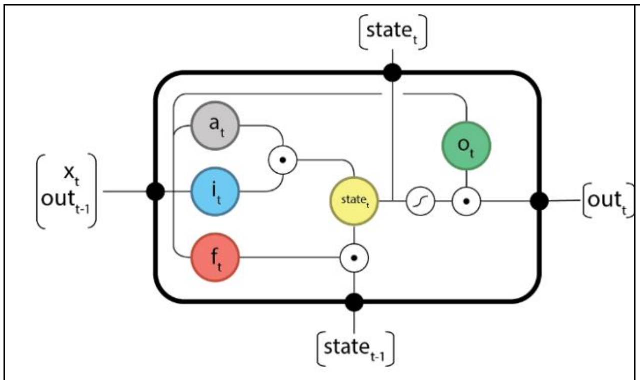


- Above $\odot$ is the element-wise product or Hadamard product.
- Inner products will be represented as $\cdot$
- Outer products will be respresented as $\otimes$
- σ represents the sigmoid function

Gates are defined as

The gates are defined as:

$$\begin{aligned}
&\text{Input activation:}\\
&\quad a_t = \tanh(W_a \cdot x_t + U_a \cdot out_{t-1} + b_a)\\
&\text{Input gate:}\\
&\quad i_t = \sigma(W_i \cdot x_t + U_i \cdot out_{t-1} + b_i)\\
&\text{Forget gate:}\\
&\quad f_t = \sigma(W_f \cdot x_t + U_f \cdot out_{t-1} + b_f)\\
&\text{Output gate:}\\
&\quad o_t = \sigma(W_o \cdot x_t + U_o \cdot out_{t-1} + b_o)
\end{aligned}$$

Which leads to:

$$\begin{aligned}
&\text{Internal state:}\\
&\quad state_t = a_t \odot i_t + f_t \odot state_{t-1}\\
&\text{Output:}\\
&\quad out_t = \tanh(state_t) \odot o_t
\end{aligned}$$

Note for simplicity we define:

$$gates_t = \begin{bmatrix} a_t \\ i_t \\ f_t \\ o_t \end{bmatrix}, \; W = \begin{bmatrix} W_a \\ W_i \\ W_f \\ W_o \end{bmatrix}, \; U = \begin{bmatrix} U_a \\ U_i \\ U_f \\ U_o \end{bmatrix}, \; b = \begin{bmatrix} b_a \\ b_i \\ b_f \\ b_o \end{bmatrix}$$

The backward components

Given:

- ΔT the output difference as computed by any subsequent layers (i.e. the rest of your network), and;
- Δout the output difference as computed by the next time-step LSTM (the equation for t-1 is below).

Find:

$$
\begin{aligned}
\delta out_t &= \Delta_t + \Delta out_t \\
\delta state_t &= \delta out_t \odot o_t \odot (1 - \tanh^2(state_t)) + \delta state_{t+1} \odot f_{t+1} \\
\delta a_t &= \delta state_t \odot i_t \odot (1 - a_t^2) \\
\delta i_t &= \delta state_t \odot a_t \odot i_t \odot (1 - i_t) \\
\delta f_t &= \delta state_t \odot state_{t-1} \odot f_t \odot (1 - f_t) \\
\delta o_t &= \delta out_t \odot \tanh(state_t) \odot o_t \odot (1 - o_t) \\
\delta x_t &= W^T \cdot \delta gates_t \\
\Delta out_{t-1} &= U^T \cdot \delta gates_t
\end{aligned}
$$

The final updates to the internal parameters is computed as:

$$
\begin{aligned}
\delta W &= \sum_{t=0}^{T} \delta gates_t \otimes x_t \\
\delta U &= \sum_{t=0}^{T-1} \delta gates_{t+1} \otimes out_t \\
\delta b &= \sum_{t=0}^{T} \delta gates_{t+1}
\end{aligned}
$$

Let us begin by defining out internal weights:

$$
\begin{aligned}
W_a &= \begin{bmatrix} 0.45 \\ 0.25 \end{bmatrix}, U_a = \begin{bmatrix} 0.15 \end{bmatrix}, b_a = \begin{bmatrix} 0.2 \end{bmatrix} \\
W_i &= \begin{bmatrix} 0.95 \\ 0.8 \end{bmatrix}, U_i = \begin{bmatrix} 0.8 \end{bmatrix}, b_i = \begin{bmatrix} 0.65 \end{bmatrix} \\
W_f &= \begin{bmatrix} 0.7 \\ 0.45 \end{bmatrix}, U_f = \begin{bmatrix} 0.1 \end{bmatrix}, b_f = \begin{bmatrix} 0.15 \end{bmatrix} \\
W_o &= \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}, U_o = \begin{bmatrix} 0.25 \end{bmatrix}, b_o = \begin{bmatrix} 0.1 \end{bmatrix}
\end{aligned}
$$

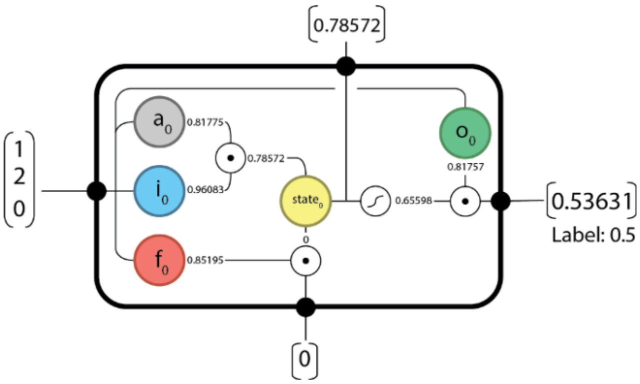
And now input data:

$$
\begin{aligned}
x_0 &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ with label: } 0.5 \\
x_1 &= \begin{bmatrix} 0.5 \\ 3 \end{bmatrix} \text{ with label: } 1.25
\end{aligned}
$$

Lets consider sequence length of two to demonstrate the unrolling over time

(a) Show that for t=0; following were the values of a_0, i_0, f_0, $state_0$, and out_0

Forward @ t=0



Sol:

$$a_0 = \tanh(W_a \cdot x_0 + U_a \cdot out_{-1} + b_a) = \tanh(\begin{bmatrix}0.45 & 0.25\end{bmatrix} \begin{bmatrix}1 \\ 2\end{bmatrix} + \begin{bmatrix}0.15\end{bmatrix} \begin{bmatrix}0\end{bmatrix} + \begin{bmatrix}0.2\end{bmatrix}) = 0.81775$$

$$i_0 = \sigma(W_i \cdot x_0 + U_i \cdot out_{-1} + b_i) = \sigma(\begin{bmatrix}0.95 & 0.8\end{bmatrix} \begin{bmatrix}1 \\ 2\end{bmatrix} + \begin{bmatrix}0.8\end{bmatrix} \begin{bmatrix}0\end{bmatrix} + \begin{bmatrix}0.65\end{bmatrix}) = 0.96083$$

$$f_0 = \sigma(W_f \cdot x_0 + U_f \cdot out_{-1} + b_f) = \sigma(\begin{bmatrix}0.7 & 0.45\end{bmatrix} \begin{bmatrix}1 \\ 2\end{bmatrix} + \begin{bmatrix}0.1\end{bmatrix} \begin{bmatrix}0\end{bmatrix} + \begin{bmatrix}0.15\end{bmatrix}) = 0.85195$$

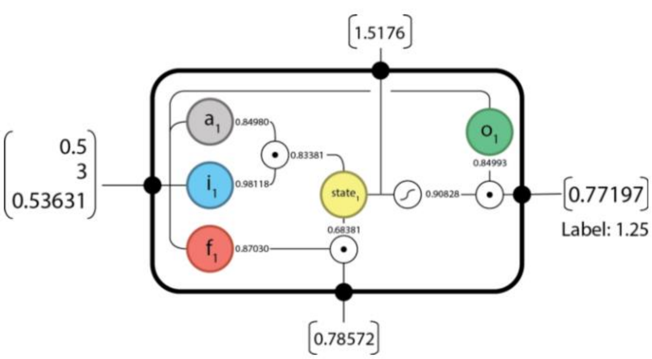
$$o_0 = \sigma(W_o \cdot x_0 + U_o \cdot out_{-1} + b_o) = \sigma(\begin{bmatrix}0.6 & 0.4\end{bmatrix} \begin{bmatrix}1 \\ 2\end{bmatrix} + \begin{bmatrix}0.25\end{bmatrix} \begin{bmatrix}0\end{bmatrix} + \begin{bmatrix}0.1\end{bmatrix}) = 0.81757$$

$$state_0 = a_0 \odot i_0 + f_0 \odot state_{-1} = 0.81775 \times 0.96083 + 0.85195 \times 0 = 0.78572$$

$$out_0 = \tanh(state_0) \odot o_0 = \tanh(0.78572) \times 0.81757 = 0.53631$$

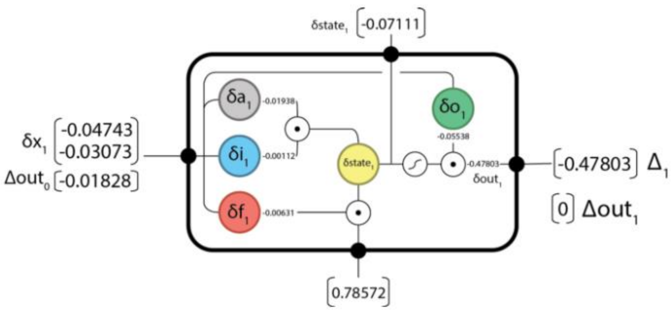
Similarly

Forward @ t=1



Then we will compute Backward for t = 1

Backward @ t=1



(b) For Backward @t=0; show the steps to get the values of various derivatives e.g. derivate of state, x,out_0.

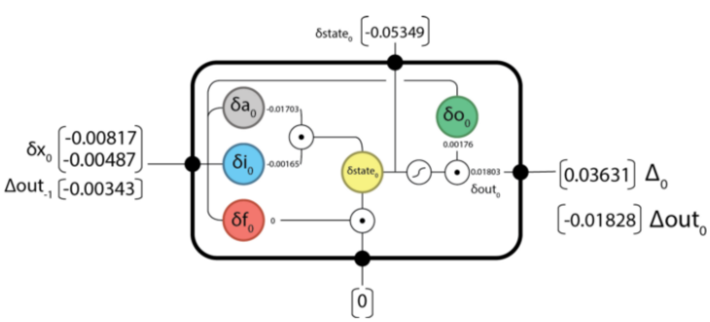
Use L2 Loss

$$E(x, \hat{x}) = \frac{(x - \hat{x})^2}{2}$$

The derivate w.r.t. x is:

$$\partial_x E(x, \hat{x}) = x - \hat{x}$$

Backward @ t=0



Sol:

$$\Delta_0 = \partial_x E = 0.53631 - 0.5 = 0.03631$$

$$\Delta out_0 = -0.01828, \text{ passed back from T=1}$$

$$\delta out_0 = \Delta_0 + \Delta out_0 = 0.03631 + -0.01828 = 0.01803$$

$$\delta state_0 = \delta out_0 \odot o_0 \odot (1 - \tanh^2(state_0)) + \delta state_1 \odot f_1 = 0.01803 \times 0.81757 \times (1 - \tanh^2(0.78572)) + -0.07111 \times 0.87030 = -0.05349$$

$$\delta a_0 = \delta state_0 \odot i_0 \odot (1 - a_0^2) = -0.05349 \times 0.96083 \times (1 - 0.81775^2) = -0.01703$$

$$\delta i_0 = \delta state_0 \odot a_0 \odot i_0 \odot (1 - i_0) = -0.05349 \times 0.81775 \times 0.96083 \times (1 - 0.96083) = -0.00165$$

$$\delta f_0 = \delta state_0 \odot state_{-1} \odot f_0 \odot (1 - f_0) = -0.05349 \times 0 \times 0.85195 \times (1 - 0.85195) = 0$$

$$\delta o_0 = \delta out_0 \odot \tanh(state_0) \odot o_0 \odot (1 - o_0) = 0.01803 \times \tanh(0.78572) \times 0.81757 \times (1 - 0.81757) = 0.00176$$

$$\delta x_0 = W^T \cdot \delta gates_0$$

$$= \begin{bmatrix} 0.45 & 0.95 & 0.70 & 0.60 \\ 0.25 & 0.80 & 0.45 & 0.40 \end{bmatrix} \begin{bmatrix} -0.01703 \\ -0.00165 \\ 0 \\ 0.00176 \end{bmatrix} = \begin{bmatrix} -0.00817 \\ -0.00487 \end{bmatrix}$$

$$\Delta out_{-1} = U^T \cdot \delta gates_1$$

$$= \begin{bmatrix} 0.15 & 0.80 & 0.10 & 0.25 \end{bmatrix} \begin{bmatrix} -0.01703 \\ -0.00165 \\ 0 \\ 0.00176 \end{bmatrix} = -0.00343$$

(c) Use Forward t=0, Backward t=0 above, Forward t=1, Backward t = 1 below, compute δW, δU and δb

Sol:

$$\delta W = \sum_{t=0}^{T} \delta gates_t \otimes x_t$$

$$= \begin{bmatrix} -0.01703 \\ -0.00165 \\ 0 \\ 0.00176 \end{bmatrix} \begin{bmatrix} 1.0 & 2.0 \end{bmatrix} + \begin{bmatrix} -0.01938 \\ -0.00112 \\ -0.00631 \\ -0.05538 \end{bmatrix} \begin{bmatrix} 0.5 & 3.0 \end{bmatrix} = \begin{bmatrix} -0.02672 & -0.0922 \\ -0.00221 & -0.00666 \\ -0.00316 & -0.01893 \\ -0.02593 & -0.16262 \end{bmatrix}$$

$$\delta U = \sum_{t=0}^{T-1} \delta gates_{t+1} \otimes out_t$$

$$= \begin{bmatrix} -0.01938 \\ -0.00112 \\ -0.00631 \\ -0.05538 \end{bmatrix} \begin{bmatrix} 0.53631 \end{bmatrix} = \begin{bmatrix} -0.01039 \\ -0.00060 \\ -0.00338 \\ -0.02970 \end{bmatrix}$$

$$\delta b = \sum_{t=0}^{T} \delta gates_{t+1}$$

$$= \begin{bmatrix} -0.01703 \\ -0.00165 \\ 0 \\ 0.00176 \end{bmatrix} + \begin{bmatrix} -0.01938 \\ -0.00112 \\ -0.00631 \\ -0.05538 \end{bmatrix} = \begin{bmatrix} -0.03641 \\ -0.00277 \\ -0.00631 \\ -0.05362 \end{bmatrix}$$

(d) Update parameters based on SGD update

$$W^{new} = W^{old} - \lambda * \delta W^{old}$$

Sol:

$$W^{new} = W^{old} - \lambda * \delta W^{old}$$

$$W_a = \begin{bmatrix} 0.45267 \\ 0.25922 \end{bmatrix}, U_a = \begin{bmatrix} 0.15104 \end{bmatrix}, b_a = \begin{bmatrix} 0.20364 \end{bmatrix}$$

$$W_i = \begin{bmatrix} 0.95022 \\ 0.80067 \end{bmatrix}, U_i = \begin{bmatrix} 0.80006 \end{bmatrix}, b_i = \begin{bmatrix} 0.65028 \end{bmatrix}$$

$$W_f = \begin{bmatrix} 0.70031 \\ 0.45189 \end{bmatrix}, U_f = \begin{bmatrix} 0.10034 \end{bmatrix}, b_f = \begin{bmatrix} 0.15063 \end{bmatrix}$$

$$W_o = \begin{bmatrix} 0.60259 \\ 0.41626 \end{bmatrix}, U_o = \begin{bmatrix} 0.25297 \end{bmatrix}, b_o = \begin{bmatrix} 0.10536 \end{bmatrix}$$