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Linear Algebra (MT-1004)

Lecture # 27

Theorem 4.9.1

The row space and the column space of a matrix A have the same dimension.

Definition 1

The common dimension of the row space and column space of a matrix A is called the ***rank*** of A and is denoted by rank(A); the dimension of the null space of A is called the ***nullity*** of A and is denoted by nullity(A).

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Theorem 4.9.2

Dimension Theorem for Matrices

If A is a matrix with n columns, then

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n \tag{4}$$

EXAMPLE 3 | The Sum of Rank and Nullity

The matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

has 6 columns, so

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = 6$$

This is consistent with Example 1, where we showed that

$$\operatorname{rank}(A) = 2 \quad \text{and} \quad \operatorname{nullity}(A) = 4$$

Theorem 4.9.3

If A is an $m \times n$ matrix, then

(a) $\operatorname{rank}(A)$ = the number of leading variables in the general solution of $A\mathbf{x} = \mathbf{0}$.

(b) $\operatorname{nullity}(A)$ = the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

Theorem 4.9.4

If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system of m equations in n unknowns, and if A has rank r, then the general solution of the system contains $n - r$ parameters.

The Fundamental Spaces of a Matrix

There are six important vector spaces associated with an $m \times n$ matrix A and its transpose A^T:

row space of A	row space of A^T
column space of A	column space of A^T
null space of A	null space of A^T

However, transposing a matrix converts row vectors into column vectors and conversely, so except for a difference in notation, the row space of A^T is the same as the column space of A, and the column space of A^T is the same as the row space of A. Thus, of the six spaces listed above, only the following four are distinct:

row space of A	column space of A
null space of A	null space of A^T

These are called the ***fundamental spaces*** of the matrix A. The row space and null space of A are subspaces of R^n, whereas the column space of A and the null space of A^T are subspaces of R^m. The null space of A^T is also called the ***left null space of A*** because transposing both sides of the equation $A^T\mathbf{x} = \mathbf{0}$ produces the equation $\mathbf{x}^T A = \mathbf{0}^T$ in which the unknown is on the left. The dimension of the left null space of A is called the ***left nullity of A***. We will now consider how the four fundamental spaces are related.

Let us focus for a moment on the matrix A^T. Since the row space and column space of a matrix have the same dimension, and since transposing a matrix converts its columns to rows and its rows to columns, the following result should not be surprising.

Theorem 4.9.5

If A is any matrix, then $\text{rank}(A) = \text{rank}(A^T)$.

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Bases for the Fundamental Spaces

An efficient way to obtain bases for the four fundamental spaces of an $m \times n$ matrix A is to adjoin the $m \times m$ identity matrix to A to obtain an augmented matrix $[A \mid I]$ and apply elementary row operations to this matrix to put A in reduced row echelon form R, thereby putting the augmented matrix in the form $[R \mid E]$. In the case where A is invertible the matrix E will be A^{-1}, but in general it will not. The rank r of A can then be obtained by counting the number of pivots (leading l's) in R, and the nullity of A^T can be obtained from the relationship

$$\text{nullity}(A^T) = m - r \tag{7}$$

that follows from Formula (5). Bases for three of the fundamental spaces can be obtained directly from $[R \mid E]$ as follows:

- A basis for row(A) will be the r rows of R that contain the leading 1's (the pivot rows).
- A basis for col(A) will be the r columns of A that contain the leading 1's of R (the pivot columns).
- A basis for null(A^T) will be the bottom $m - r$ rows of E (see the proof at the end of this section)

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EXAMPLE 5 | Bases for the Fundamental Spaces

In Example 1 we found a basis for the null space of the 4×6 matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

$$\left[\begin{array}{cccccc|cccc} -1 & 2 & 0 & 4 & 5 & -3 & 1 & 0 & 0 & 0 \\ 3 & -7 & 2 & 0 & 1 & 4 & 0 & 1 & 0 & 0 \\ 2 & -5 & 2 & 4 & 6 & 1 & 0 & 0 & 1 & 0 \\ 4 & -9 & 2 & -4 & -4 & 7 & 0 & 0 & 0 & 1 \end{array}\right]$$

$A \qquad\qquad I$

$$\left[\begin{array}{cccccc|cccc} 1 & 0 & -4 & -28 & -37 & 13 & 0 & 0 & -\frac{9}{2} & \frac{5}{2} \\ 0 & 1 & -2 & -12 & -16 & 5 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$R \qquad\qquad E$

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$$\text{row space basis: } \left\{ \begin{bmatrix} -1 \\ 0 \\ -4 \\ -28 \\ -37 \\ 13 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ -12 \\ -16 \\ 5 \end{bmatrix} \right\}, \text{ column space basis: } \left\{ \begin{bmatrix} -1 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ -5 \\ -9 \end{bmatrix} \right\}$$

$$\text{left null space basis: } \left\{ \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right\}$$

Definition 2

If W is a subspace of R^n, then the set of all vectors in R^n that are orthogonal to every vector in W is called the ***orthogonal complement*** of W and is denoted by the symbol $W^\perp$.

Theorem 4.9.6

If W is a subspace of R^n, then:

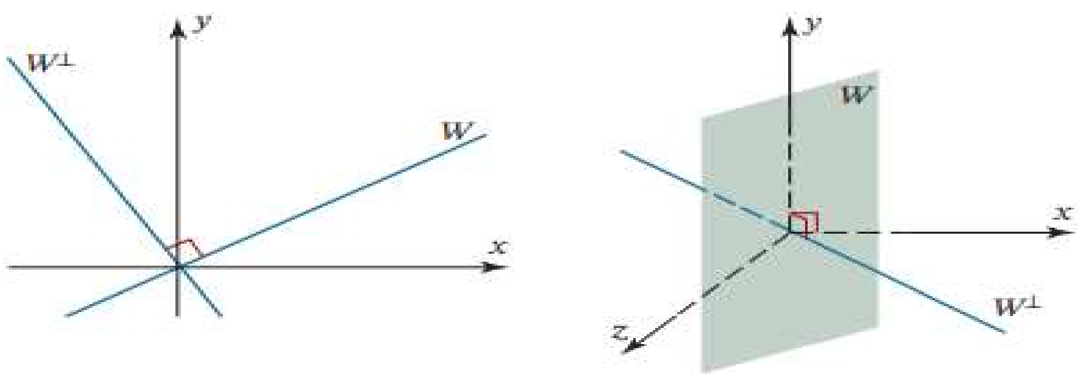
(a) $W^\perp$ is a subspace of R^n.

(b) The only vector common to W and $W^\perp$ is $\mathbf{0}$.

(c) The orthogonal complement of $W^\perp$ is W.

EXAMPLE 6 | Orthogonal Complements

In R^2 the orthogonal complement of a line W through the origin is the line through the origin that is perpendicular to W (Figure 4.9.1a); and in R^3 the orthogonal complement of a plane W through the origin is the line through the origin that is perpendicular to that plane (Figure 4.9.1b).



(a) (b)

FIGURE 4.9.1

Theorem 4.9.7

If A is an $m \times n$ matrix, then:

(a) The null space of A and the row space of A are orthogonal complements in R^n.

(b) The null space of A^T and the column space of A are orthogonal complements in R^m.

Theorem 4.9.8

Equivalent Statements

If A is an $n \times n$ matrix in which there are no duplicate rows and no duplicate columns, then the following statements are equivalent.

(*a*) A is invertible.

(*b*) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*c*) The reduced row echelon form of A is I_n.

(*d*) A is expressible as a product of elementary matrices.

(*e*) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(*f*) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

(*g*) $\det(A) \neq 0$.

(*h*) The column vectors of A are linearly independent.

(*i*) The row vectors of A are linearly independent.

(*j*) The column vectors of A span R^n.

(*k*) The row vectors of A span R^n.

(*l*) The column vectors of A form a basis for R^n.

(*m*) The row vectors of A form a basis for R^n.

(*n*) A has rank n.

(*o*) A has nullity 0.

(*p*) The orthogonal complement of the null space of A is R^n.

(*q*) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.

Theorem 4.9.9

Let A be an $m \times n$ matrix.

(*a*) (***Overdetermined Case***). If $m > n$, then the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent for at least one vector $\mathbf{b}$ in R^n.

(*b*) (***Underdetermined Case***). If $m < n$, then for each vector $\mathbf{b}$ in R^m the linear system $A\mathbf{x} = \mathbf{b}$ is either inconsistent or has infinitely many solutions.

In engineering and physics, the occurrence of an overdetermined or underdetermined linear system often signals that one or more variables were omitted in formulating the problem or that extraneous variables were included. This often leads to some kind of complication.

Applications of Rank

The advent of the Internet has stimulated research on finding efficient methods for transmitting large amounts of digital data over communications lines with limited bandwidths. Digital data are commonly stored in matrix form, and many techniques for improving transmission speed use the rank of a matrix in some way. Rank plays a role because it measures the "redundancy" in a matrix in the sense that if A is an $m \times n$ matrix of rank k, then $n - k$ of the column vectors and $m - k$ of the row vectors can be expressed in terms of k linearly independent column or row vectors. The essential idea in many data compression schemes is to approximate the original data set by a data set with smaller rank that conveys nearly the same information, then eliminate redundant vectors in the approximating set to speed up the transmission time.