CHAPTER

3

Interpolation and Polynomial Approximation

Fitting a Polynomial to Data

Suppose that we have the following data pairs—x-values and $f(x)$-values—where $f(x)$ is some unknown function:

x	$f(x)$
3.2	22.0
2.7	17.8
1.0	14.2
4.8	38.3
5.6	51.7

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

$$\text{when } x = 3.2:\ a(3.2)^3 + b(3.2)^2 + c(3.2) + d = 22.0,$$

$$\text{if } x = 2.7:\ a(2.7)^3 + b(2.7)^2 + c(2.7) + d = 17.8,$$

$$\text{if } x = 1.0:\ a(1.0)^3 + b(1.0)^2 + c(1.0) + d = 14.2,$$

$$\text{if } x = 4.8:\ a(4.8)^3 + b(4.8)^2 + c(4.8) + d = 38.3.$$

$$
\begin{aligned}
a &= -0.5275, \\
b &= 6.4952, \\
c &= -16.1177, \\
d &= 24.3499,
\end{aligned}
$$

polynomial is

$$-0.5275x^3 + 6.4952x^2 - 16.1177x + 24.3499.$$

At $x = 3.0$, the estimated value is 20.212.

Complexity Why do we use polynomials? Polynomials are very often used for interpolation because of their straightforward mathematical properties. There is a simple theory about when an interpolating polynomial of a given degree exists for a given set of points. More important, in a real sense, polynomials are the most fundamental of functions for digital computers. Central processing units usually have fast methods in hardware for adding and multiplying floating point numbers, which are the only operations needed to evaluate a polynomial. Complicated functions can be approximated by interpolating polynomials in order to make them computable with these two hardware operations.

Interpolation and the Lagrange Polynomial

Lagrange Interpolating Polynomials

the linear Lagrange polynomial

$$p_1(x) = L_0(x) f_0 + L_1(x) f_1 = \frac{x - x_1}{x_0 - x_1} \cdot f_0 + \frac{x - x_0}{x_1 - x_0} \cdot f_1 .$$

Quadratic interpolation is interpolation of given (x_0, f_0), (x_1, f_1), (x_2, f_2) by a second-degree polynomial $p_2(x)$, which by Lagrange's idea is

$$p_2(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2$$

with $L_0(x_0) = 1, L_1(x_1) = 1, L_2(x_2) = 1$, and $L_0(x_1) = L_0(x_2) = 0$, etc. We claim that

Example 2 (a) Use the numbers (called *nodes*) $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = 1/x$.

(b) Use this polynomial to approximate $f(3) = 1/3$.

$$L_0(x) = \frac{l_0(x)}{l_0(x_0)} = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$P(x) = \sum_{k=0}^{2} f(x_k)L_k(x) \qquad L_1(x) = \frac{l_1(x)}{l_1(x_1)} = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$L_2(x) = \frac{l_2(x)}{l_2(x_2)} = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

$$= \frac{1}{3}(x - 2.75)(x - 4) - \frac{64}{165}(x - 2)(x - 4) + \frac{1}{10}(x - 2)(x - 2.75)$$

$$= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}.$$

$$f(3) \approx P(3) = \frac{9}{22} - \frac{105}{88} + \frac{49}{44} = \frac{29}{88} \approx 0.32955.$$

$$P_3(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f_1$$
$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f_3$$

Example:

Fit a cubic through the first four points of the preceding table and use it to find the interpolated value for $x = 3.0$.

x	$f(x)$
3.2	22.0
2.7	17.8
1.0	14.2
4.8	38.3

$$P_3(3.0) = \frac{(3.0 - 2.7)(3.0 - 1.0)(3.0 - 4.8)}{(3.2 - 2.7)(3.2 - 1.0)(3.2 - 4.8)}(22.0)$$
$$+ \frac{(3.0 - 3.2)(3.0 - 1.0)(3.0 - 4.8)}{(2.7 - 3.2)(2.7 - 1.0)(2.7 - 4.8)}(17.8)$$
$$+ \frac{(3.0 - 3.2)(3.0 - 2.7)(3.0 - 4.8)}{(1.0 - 3.2)(1.0 - 2.7)(1.0 - 4.8)}(14.2)$$
$$+ \frac{(3.0 - 3.2)(3.0 - 2.7)(3.0 - 1.0)}{(4.8 - 3.2)(4.8 - 2.7)(4.8 - 1.0)}(38.3).$$

Carrying out the arithmetic, $P_3(3.0) = 20.21$.

EXERCISE SET 3.1

1. For the given functions $f(x)$, let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(0.45)$, and find the absolute error.

   a. $f(x) = \cos x$

   b. $f(x) = \sqrt{1+x}$

   c. $f(x) = \ln(x+1)$

   d. $f(x) = \tan x$

5. Use appropriate Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:

   a. $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$

   b. $f\left(-\frac{1}{3}\right)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$

7. The data for Exercise 5 were generated using the following functions. Use the error formula to find a bound for the error, and compare the bound to the actual error for the cases $n = 1$ and $n = 2$.

   a. $f(x) = x \ln x$

   b. $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$

   c. $f(x) = x \cos x - 2x^2 + 3x - 1$

   d. $f(x) = \sin(e^x - 2)$

3.3 Divided Differences

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0)\cdots(x - x_{k-1}).$$

$$\boxed{\begin{aligned} P_n(x) = f[x_0] &+ (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0 \ldots x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2)f[x_0 \ldots x_3] + \cdots \\ &+ (x - x_0)(x - x_1)\ldots(x - x_{n-1})f[x_0 \ldots x_n]. \end{aligned}}$$

Algorithm 3.2

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1}),$$

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
x_0 & y_0 = f[x_0] \arrow[dr, "a0"] & & & \\
& & f[x_0, x_1] \arrow[dr, "a1"] & & \\
x_1 & y_1 = f[x_1] \arrow[ur] \arrow[dr] & & f[x_0, x_1, x_2] \arrow[dr, "a2"] & \\
& & f[x_1, x_2] \arrow[ur] \arrow[dr] & & f[x_0, x_1, x_2, x_3] \\
x_2 & y_2 = f[x_2] \arrow[ur] \arrow[dr] & & f[x_1, x_2, x_3] \arrow[ur] & \\
& & f[x_2, x_3] \arrow[ur] & & \\
x_3 & y_3 = f[x_3] \arrow[ur] & & &
\end{tikzcd}
$$

$$
\boxed{f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}}
$$

$$
f[x_k] = f(x_k)
$$

$$
f[x_k \ \ x_{k+1}] = \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k}
$$

$$
f[x_k \ \ x_{k+1} \ \ x_{k+2}] = \frac{f[x_{k+1} \ \ x_{k+2}] - f[x_k \ \ x_{k+1}]}{x_{k+2} - x_k}
$$

$$
f[x_k \ \ x_{k+1} \ \ x_{k+2} \ \ x_{k+3}] = \frac{f[x_{k+1} \ \ x_{k+2} \ \ x_{k+3}] - f[x_k \ \ x_{k+1} \ \ x_{k+2}]}{x_{k+3} - x_k},
$$

Table 3.9

x	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
		$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
		$f[x_1, x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \dfrac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
		$f[x_2, x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2}$		$f[x_1, x_2, x_3, x_4] = \dfrac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$
x_3	$f[x_3]$		$f[x_2, x_3, x_4] = \dfrac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	
		$f[x_3, x_4] = \dfrac{f[x_4] - f[x_3]}{x_4 - x_3}$		$f[x_2, x_3, x_4, x_5] = \dfrac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}$
x_4	$f[x_4]$		$f[x_3, x_4, x_5] = \dfrac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$	
		$f[x_4, x_5] = \dfrac{f[x_5] - f[x_4]}{x_5 - x_4}$		
x_5	$f[x_5]$			

Example 1 Complete the divided difference table for the data used in Example 1 of Section 3.2, and reproduced in Table 3.10, and construct the interpolating polynomial that uses all this data.

i	x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, \ldots, x_i]$	$f[x_{i-4}, \ldots, x_i]$
0	1.0	0.7651977				
			−0.4837057			
1	1.3	0.6200860		−0.1087339		
			−0.5489460		0.0658784	
2	1.6	0.4554022		−0.0494433		0.0018251
			−0.5786120		0.0680685	
3	1.9	0.2818186		0.0118183		
			−0.5715210			
4	2.2	0.1103623				

Table 3.10

x	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

$$
\begin{aligned}
P_4(x) = {} & 0.7651977 - 0.4837057(x - 1.0) - 0.1087339(x - 1.0)(x - 1.3) \\
& + 0.0658784(x - 1.0)(x - 1.3)(x - 1.6) \\
& + 0.0018251(x - 1.0)(x - 1.3)(x - 1.6)(x - 1.9).
\end{aligned}
$$

$$P_4(1.5) = 0.5118200$$

NDDT:

Example 1 Complete the divided difference table for the data used in Example 1 of Section 3.2, and reproduced in Table 3.10, and construct the interpolating polynomial that uses all this data.

i	x_i	$f[x_i]$	$f[x_{i-1},x_i]$	$f[x_{i-2},x_{i-1},x_i]$	$f[x_{i-3},\ldots,x_i]$	$f[x_{i-4},\ldots,x_i]$
0	1.0	0.7651977				
			−0.4837057			
1	1.3	0.6200860		−0.1087339		
			−0.5489460		0.0658784	
2	1.6	0.4554022		−0.0494433		0.0018251
			−0.5786120		0.0680685	
3	1.9	0.2818186		0.0118183		
			−0.5715210			
4	2.2	0.1103623				

a) Approximate f(1.1) use Newton forward difference
b) Approximate f(2.0) use Newton backward difference

$$
\begin{aligned}
P_4(1.1) &= P_4(1.0 + \tfrac{1}{3}(0.3)) \\
&= 0.7651977 + \frac{1}{3}(0.3)(-0.4837057) + \frac{1}{3}\left(-\frac{2}{3}\right)(0.3)^2(-0.1087339) \\
&\quad + \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(0.3)^3(0.0658784) \\
&\quad + \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)(0.3)^4(0.0018251) \\
&= 0.7196460.
\end{aligned}
$$

$$
\begin{aligned}
P_4(2.0) &= P_4\left(2.2 - \frac{2}{3}(0.3)\right) \\
&= 0.1103623 - \frac{2}{3}(0.3)(-0.5715210) - \frac{2}{3}\left(\frac{1}{3}\right)(0.3)^2(0.0118183) \\
&\quad - \frac{2}{3}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)(0.3)^3(0.0680685) - \frac{2}{3}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)(0.3)^4(0.0018251) \\
&= 0.2238754.
\end{aligned}
$$

Class Activity:

$$f(x) \approx f_0 + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})f[x_0, \cdots, x_n].$$

Compute $f(9.2)$ from the values shown in the first two columns of the following table.

x_j	$f_j = f(x_j)$	$f[x_j, x_{j+1}]$	$f[x_j, x_{j+1}, x_{j+2}]$	$f[x_j, \cdots, x_{j+3}]$
8.0	2.079442			
		0.117783		
9.0	2.197225		−0.006433	
		0.108134		0.000411
9.5	2.251292		−0.005200	
		0.097735		
11.0	2.397895			

$$f(x) \approx p_3(x) = 2.079442 + 0.117783(x - 8.0) - 0.006433(x - 8.0)(x - 9.0) + 0.000411(x - 8.0)(x - 9.0)(x - 9.5).$$

$$f(9.2) \approx 2.079442 + 0.141340 - 0.001544 - 0.000030 = 2.219208.$$

EXERCISE SET 3.3

1. Use Eq. (3.10) or Algorithm 3.2 to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
   a. $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$
   b. $f(0.9)$ if $f(0.6) = -0.17694460$, $f(0.7) = 0.01375227$, $f(0.8) = 0.22363362$, $f(1.0) = 0.65809197$
2. Use Eq. (3.10) or Algorithm 3.2 to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
   a. $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$
   b. $f(0)$ if $f(-0.5) = 1.93750$, $f(-0.25) = 1.33203$, $f(0.25) = 0.800781$, $f(0.5) = 0.687500$
3. Use Newton the forward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
   a. $f\left(-\frac{1}{3}\right)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$
   b. $f(0.25)$ if $f(0.1) = -0.62049958$, $f(0.2) = -0.28398668$, $f(0.3) = 0.00660095$, $f(0.4) = 0.24842440$

EQUAL SPACING:

Newton Forward-Difference Formula

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0)$$

$h = x_{i+1} - x_i$, for each $i = 0, 1, \ldots, n-1$ and let $x = x_0 + sh$.

Newton Backward–Difference Formula

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!},$$

$$P_n(x) = f[x_n] + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_n)$$

$$P_n(x) = f[x_n] + (-1)^1 \binom{-s}{1} \nabla f(x_n) + (-1)^2 \binom{-s}{2} \nabla^2 f(x_n) + \cdots + (-1)^n \binom{-s}{n} \nabla^n f(x_n).$$

If, in addition, the nodes are equally spaced with $x = x_n + sh$

EQUAL SPACING:

$$x_0, \quad x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad \cdots, \quad x_n = x_0 + nh.$$

forward difference interpolation formula

$$f(x) \approx p_n(x) = \sum_{s=0}^{n} \binom{r}{s} \Delta^s f_0 \qquad (x = x_0 + rh, \quad r = (x - x_0)/h)$$

$$= f_0 + r\Delta f_0 + \frac{r(r-1)}{2!} \Delta^2 f_0 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!} \Delta^n f_0$$

backward difference interpolation formula

$$f(x) \approx p_n(x) = \sum_{s=0}^{n} \binom{r+s-1}{s} \nabla^s f_0 \qquad (x = x_0 + rh, r = (x - x_0)/h)$$

$$= f_0 + r\nabla f_0 + \frac{r(r+1)}{2!} \nabla^2 f_0 + \cdots + \frac{r(r+1)\cdots(r+n-1)}{n!} \nabla^n f_0.$$

Newton's Forward and Backward Interpolations

Class Activity:

Compute a 7D-value of the Bessel function $J_0(x)$ for $x = 1.72$ from the four values in the following table, using (a) Newton's forward formula (14), (b) Newton's backward formula (18).

j_{for}	j_{back}	x_j	$J_0(x_j)$	1st Diff.	2nd Diff.	3rd Diff.
0	−3	1.7	0.3979849			
				−0.0579985		
1	−2	1.8	0.3399864		−0.0001693	
				−0.0581678		0.0004093
2	−1	1.9	0.2818186		0.0002400	
				−0.0579278		
3	0	2.0	0.2238908			

Solution. The computation of the differences is the same in both cases. Only their notation differs.

(a) **Forward.** In (14) we have $r = (1.72 - 1.70)/0.1 = 0.2$, and j goes from 0 to 3 (see first column). In each column we need the first given number, and (14) thus gives

$$J_0(1.72) \approx 0.3979849 + 0.2\,(-0.0579985) + \frac{0.2(-0.8)}{2}\,(-0.0001693) + \frac{0.2(-0.8)(-1.8)}{6} \cdot 0.0004093$$

$$= 0.3979849 - 0.0115997 + 0.0000135 + 0.0000196 = 0.3864183,$$

which is exact to 6D, the exact 7D-value being 0.3864185.

(b) **Backward.** For (18) we use j shown in the second column, and in each column the last number. Since $r = (1.72 - 2.00)/0.1 = -2.8$, we thus get from (18)

$$J_0(1.72) \approx 0.2238908 - 2.8\,(-0.0579278) + \frac{-2.8\,(-1.8)}{2} \cdot 0.0002400 + \frac{-2.8(-1.8)(-0.8)}{6} \cdot 0.0004093$$

$$= 0.2238908 + 0.1621978 + 0.0006048 - 0.0002750$$

$$= 0.3864184.$$

Forward Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

Finite Differences

Backward Difference Table:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_4	y_4		$\nabla^2 y_5$			
		∇y_5				
x_5	y_5					

We have Newton's forward difference interpolation formula

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$p = \frac{x - x_0}{h}$$

The Newton-Gregory Backward Interpolation Formula

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

$$+ \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n y_n + \text{Error}$$

$$p = \frac{x - x_n}{h}$$

Centered Differences

The Newton forward- and backward-difference formulas are not appropriate for approximat ing $f(x)$ when x lies near the center of the table We will consider only one centered-

difference formula, Stirling's method.

For the centered-difference formulas, we choose x_0 near the point being approximated and label the nodes directly below x_0 as $x_1, x_2, \ldots$ and those directly above as $x_{-1}, x_{-2}, \ldots$. With this convention, **Stirling's formula** is given by

$$\begin{aligned}
P_n(x) = P_{2m+1}(x) = {} & f[x_0] + \frac{sh}{2}(f[x_{-1}, x_0] + f[x_0, x_1]) + s^2h^2 f[x_{-1}, x_0, x_1] \\
& + \frac{s(s^2-1)h^3}{2} f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2]) \\
& + \cdots + s^2(s^2-1)(s^2-4)\cdots(s^2-(m-1)^2)h^{2m} f[x_{-m}, \ldots, x_m] \\
& + \frac{s(s^2-1)\cdots(s^2-m^2)h^{2m+1}}{2}(f[x_{-m-1}, \ldots, x_m] + f[x_{-m}, \ldots, x_{m+1}]),
\end{aligned}$$

Table 3.13

x	$f(x)$	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
x_{-2}	$f[x_{-2}]$				
		$f[x_{-2}, x_{-1}]$			
x_{-1}	$f[x_{-1}]$		$f[x_{-2}, x_{-1}, x_0]$		
		$\underline{f[x_{-1}, x_0]}$		$\underline{f[x_{-2}, x_{-1}, x_0, x_1]}$	
x_0	$\underline{f[x_0]}$		$\underline{f[x_{-1}, x_0, x_1]}$		$\underline{f[x_{-2}, x_{-1}, x_0, x_1, x_2]}$
		$\underline{f[x_0, x_1]}$		$\underline{f[x_{-1}, x_0, x_1, x_2]}$	
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$			
x_2	$f[x_2]$				

Example 2 Consider the table of data given in the previous examples. Use Stirling's formula to approximate $f(1.5)$ with $x_0 = 1.6$.

Table 3.14

x	$f(x)$	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
1.0	0.7651977				
		−0.4837057			
1.3	0.6200860		−0.1087339		
		−0.5489460		0.0658784	
1.6	0.4554022		−0.0494433		0.0018251
		−0.5786120		0.0680685	
1.9	0.2818186		0.0118183		
		−0.5715210			
2.2	0.1103623				

The formula, with $h = 0.3$, $x_0 = 1.6$, and $s = -\frac{1}{3}$, becomes

$$\boxed{\begin{aligned} P_n(x) = P_{2m+1}(x) &= f[x_0] + \frac{sh}{2}(f[x_{-1}, x_0] + f[x_0, x_1]) + s^2h^2 f[x_{-1}, x_0, x_1] \\ &+ \frac{s(s^2-1)h^3}{2} f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2]) \\ &+ \cdots + s^2(s^2-1)(s^2-4)\cdots(s^2-(m-1)^2)h^{2m} f[x_{-m}, \ldots, x_m] \end{aligned}}$$

$$\begin{aligned} f(1.5) &\approx P_4\left(1.6 + \left(-\frac{1}{3}\right)(0.3)\right) \\ &= 0.4554022 + \left(-\frac{1}{3}\right)\left(\frac{0.3}{2}\right)((-0.5489460) + (-0.5786120)) \\ &+ \left(-\frac{1}{3}\right)^2 (0.3)^2(-0.0494433) \\ &+ \frac{1}{2}\left(-\frac{1}{3}\right)\left(\left(-\frac{1}{3}\right)^2 - 1\right)(0.3)^3(0.0658784 + 0.0680685) \\ &+ \left(-\frac{1}{3}\right)^2\left(\left(-\frac{1}{3}\right)^2 - 1\right)(0.3)^4(0.0018251) = 0.5118200. \end{aligned}$$

Activity

6. Use the Newton backward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.

   a. $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$

   b. $f(0.25)$ if $f(-1) = 0.86199480$, $f(-0.5) = 0.95802009$, $f(0) = 1.0986123$, $f(0.5) = 1.2943767$

9. **a.** Approximate $f(0.05)$ using the following data and the Newton forward-difference formula:

x	0.0	0.2	0.4	0.6	0.8
$f(x)$	1.00000	1.22140	1.49182	1.82212	2.22554

   b. Use the Newton backward-difference formula to approximate $f(0.65)$.

   c. Use Stirling's formula to approximate $f(0.43)$.