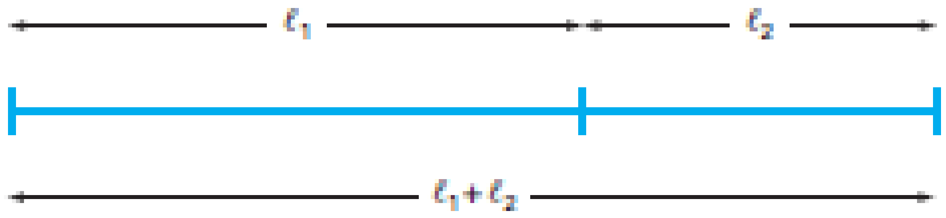
7.2.1 Golden-Section Search

The actual value of the golden ratio can be derived by expressing Euclid's definition as

FIGURE 7.5
Euclid's definition of the golden ratio is based on dividing a line into two segments so that the ratio of the whole line to the larger segment is equal to the ratio of the larger segment to the smaller segment. This ratio is called the golden ratio.



$$\frac{\ell_1 + \ell_2}{\ell_1} = \frac{\ell_1}{\ell_2}$$

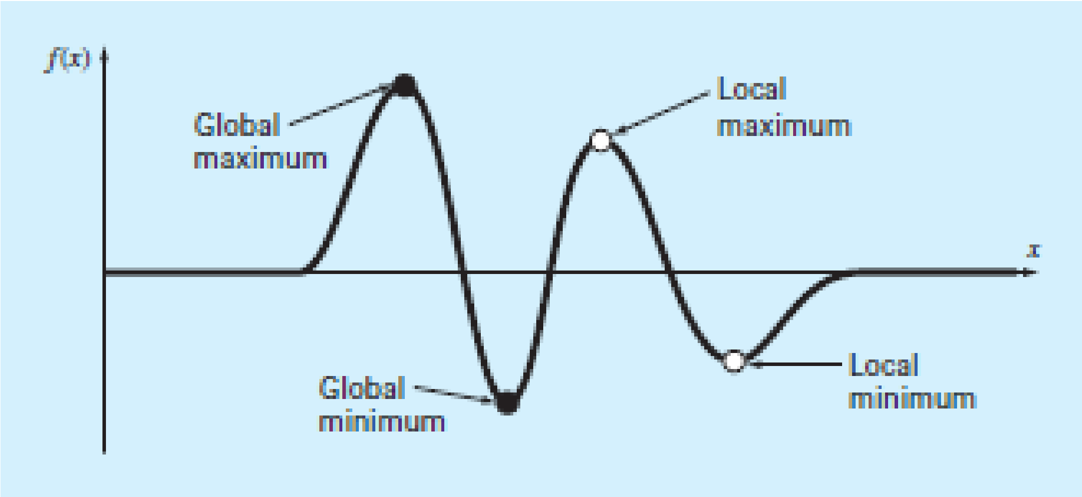
Multiplying by ℓ_1/ℓ_2 and collecting terms yields

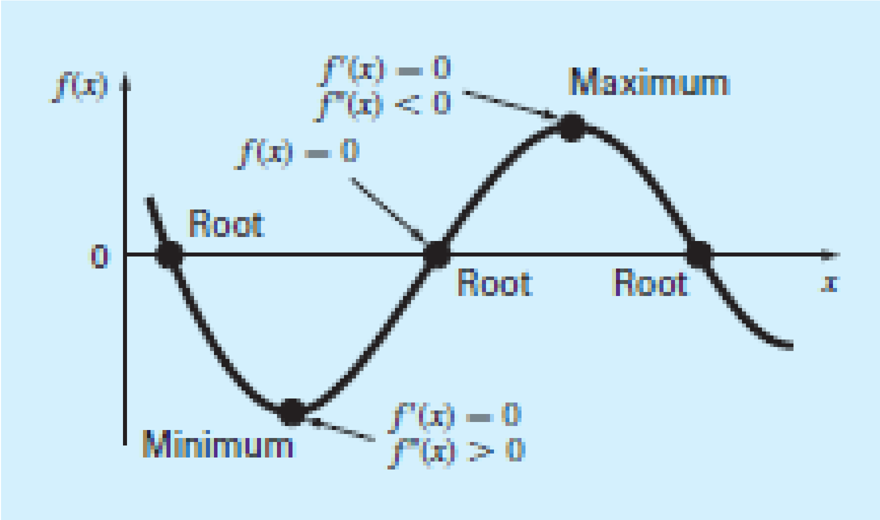
$$\phi^2 - \phi - 1 = 0$$

where $\phi = \ell_1/\ell_2$. The positive root of this equation is the golden ratio:

$$\phi = \frac{1 + \sqrt{5}}{2} = 1.61803398874989\ldots$$

INTRODUCTION AND BACKGROUND

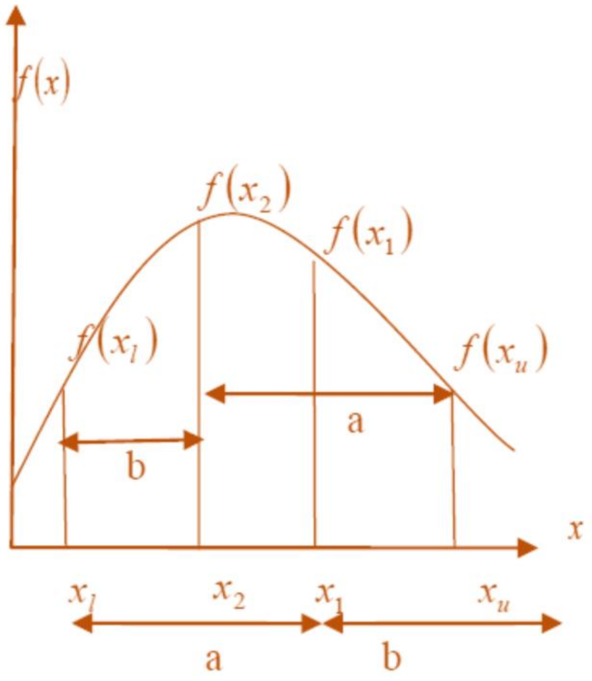




Finally, the process of finding a maximum versus finding a minimum is essentially identical because the same value x^* both minimizes $f(x)$ and maximizes $-f(x)$. This

The Golden Section Search Algorithm

The following algorithm can be used to determine the maximum of a function $f(x)$



Initialization:

Determine x_l and x_u which is known to contain the maximum of the function $f(x)$.

Step 1

Determine two intermediate points x_1 and x_2 such that

where $$x_1 = x_l + d$$ $$x_2 = x_u - d$$ $\Rightarrow$ $$d = \frac{\sqrt{5}-1}{2}(x_u - x_l)$$

Step 2

Evaluate $f(x_1)$ and $f(x_2)$.

If $f(x_1) > f(x_2)$, then determine new x_l, x_1, x_2 and x_u as shown in Equation set (5). Note that the only new calculation is done to determine the new x_1.

$$
\begin{aligned}
x_l &= x_2 \\
x_2 &= x_1 \\
x_u &= x_u \\
x_1 &= x_l + \frac{\sqrt{5}-1}{2}(x_u - x_l)
\end{aligned}
\qquad (5)
$$

If $f(x_1) < f(x_2)$, then determine new x_l, x_1, x_2 and x_u as shown in Equation set (6). Note that the only new calculation is done to determine the new x_2.

$$\begin{aligned} x_l &= x_l \\ x_u &= x_1 \\ x_1 &= x_2 \\ x_2 &= x_u - \frac{\sqrt{5}-1}{2}(x_u - x_l) \end{aligned} \tag{6}$$

<u>Step 3</u>

If $x_u - x_l < \varepsilon$ (a sufficiently small number), then the maximum occurs at $\frac{x_u + x_l}{2}$ and stop iterating, else go to Step 2.

What happens after choosing the first two intermediate points?

Next we determine a new and smaller interval where the maximum value of the function lies in. We know that the new interval is either $[x_l, x_2, x_1]$ or $[x_2, x_1, x_u]$. To determine which of these intervals will be considered in the next iteration, the function is evaluated at the intermediate points x_2 and x_1. If $f(x_2) > f(x_1)$, then the new region of interest will be $[x_l, x_2, x_1]$; else if $f(x_2) < f(x_1)$, then the new region of interest will be $[x_2, x_1, x_u]$. In Figure 6, we see that $f(x_2) > f(x_1)$, therefore our new region of interest is $[x_l, x_2, x_1]$. We should point out that the boundaries of the new smaller region are now determined by x_l and x_1, and we already have one of the intermediate points, namely x_2, conveniently located at a point where the ratio of the distance to the boundaries is the Golden Ratio. All that is left to do is to determine the location of the second intermediate point. Can you determine if the second point will be closer to x_l or x_1? This process of determining a new smaller region of interest and a new intermediate point will continue until the distance between the boundary points are sufficiently small.

EXAMPLE 7.2 Golden-Section Search

Problem Statement. Use the golden-section search to find the minimum of

$$f(x) = \frac{x^2}{10} - 2\sin x$$

within the interval from $x_l = 0$ to $x_u = 4$.

Solution. First, the golden ratio is used to create the two interior points:

$$d = 0.61803(4 - 0) = 2.4721$$
$$x_1 = 0 + 2.4721 = 2.4721$$
$$x_2 = 4 - 2.4721 = 1.5279$$

$$x_1 = x_l + d$$
$$x_2 = x_u - d$$

where

$$d = (\phi - 1)(x_u - x_l)$$

The function can be evaluated at the interior points:

$$f(x_2) = \frac{1.5279^2}{10} - 2\sin(1.5279) = -1.7647$$

$$f(x_1) = \frac{2.4721^2}{10} - 2\sin(2.4721) = -0.6300$$

$$d = 0.61803(2.4721 - 0) = 1.5279$$
$$x_2 = 2.4721 - 1.5279 = 0.9443$$

$$f(0.9943) = -1.5310.$$

i	x_l	$f(x_l)$	x_2	$f(x_2)$	x_1	$f(x_1)$	x_u	$f(x_u)$	d
1	0	0	1.5279	−1.7647	2.4721	−0.6300	4.0000	3.1136	2.4721
2	0	0	0.9443	−1.5310	1.5279	−1.7647	2.4721	−0.6300	1.5279
3	0.9443	−1.5310	1.5279	−1.7647	1.8885	−1.5432	2.4721	−0.6300	0.9443
4	0.9443	−1.5310	1.3050	−1.7595	1.5279	−1.7647	1.8885	−1.5432	0.5836
5	1.3050	−1.7595	1.5279	−1.7647	1.6656	−1.7136	1.8885	−1.5432	0.3607
6	1.3050	−1.7595	1.4427	−1.7755	1.5279	−1.7647	1.6656	−1.7136	0.2229
7	1.3050	−1.7595	1.3901	−1.7742	1.4427	−1.7755	1.5279	−1.7647	0.1378
8	1.3901	−1.7742	1.4427	−1.7755	1.4752	−1.7732	1.5279	−1.7647	0.0851

Note that the current minimum is highlighted for every iteration. After the eighth iteration, the minimum occurs at $x = 1.4427$ with a function value of -1.7755. Thus, the result is converging on the true value of -1.7757 at $x = 1.4276$.

Find the angle θ which maximizes the cross-sectional area of the gutter. Using an initial interval of $[0, \pi/2]$, find the solution after 2 iterations. Use an initial $\varepsilon = 0.05$.

The function to be maximized is

$$f(\theta) = 4\sin\theta(1+\cos\theta)$$

Iteration 1:

Given the values for the boundaries of $x_l = 0$ *and* $x_u = \pi/2$, we can calculate the initial intermediate points as follows:

$$x_1 = x_l + \frac{\sqrt{5}-1}{2}(x_u - x_l)$$

$$= 0 + \frac{\sqrt{5}-1}{2}(1.5708)$$

$$= 0.97080$$

$$x_2 = x_u - \frac{\sqrt{5}-1}{2}(x_u - x_l)$$

$$= 1.5708 - \frac{\sqrt{5}-1}{2}(1.5708)$$

$$= 0.60000$$

The function is evaluated at the intermediate points as $f(0.9708) = 5.1654$ and $f(0.60000) = 4.1227$. Since $f(x_1) > f(x_2)$, we eliminate the region to the left of x_2 and update the lower boundary point as $x_l = x_2$. The upper boundary point x_u remains unchanged. The second intermediate point x_2 is updated to assume the value of x_1 and finally the first intermediate point x_1 is re-calculated as follows:

$$
\begin{aligned}
x_1 &= x_l + \frac{\sqrt{5}-1}{2}(x_u - x_l) \\
&= 0.60000 + \frac{\sqrt{5}-1}{2}(1.5708 - 0.60000) \\
&= 1.2000
\end{aligned}
$$

To check the stopping criteria the difference between x_u and x_l is calculated to be

$$x_u - x_l = 1.5708 - 0.60000 = 0.97080$$

which is greater than $\varepsilon = 0.05$. The process is repeated in the second iteration.

To check the stopping criteria the difference between x_u and x_l is calculated to be

$$x_u - x_l = 1.5708 - 0.60000 = 0.97080$$

which is greater than $\varepsilon = 0.05$. The process is repeated in the second iteration.

Iteration 2:

The values for the boundary and intermediate points used in this iteration were calculated in the previous iteration as shown below.

$$x_l = 0.60000$$

$$x_u = 1.5708$$

$$x_1 = 1.2000$$

$$x_2 = 0.97080$$

Again the function is evaluated at the intermediate points as $f(1.20000) = 5.0791$ and $f(0.97080) = 5.1654$. Since $f(x_1) < f(x_2)$, the opposite of the case seen in the first iteration, we eliminate the region to the right of x_1 and update the upper boundary point as $x_u = x_1$. The lower boundary point x_l remains unchanged. The first intermediate point x_1 is updated to assume the value of x_2 and finally the second intermediate point x_2 is re-calculated as follows:

$$x_2 = x_u - \frac{\sqrt{5}-1}{2}(x_u - x_l)$$
$$= 1.2000 - \frac{\sqrt{5}-1}{2}(1.2000 - 0.60000)$$
$$= 0.82918$$

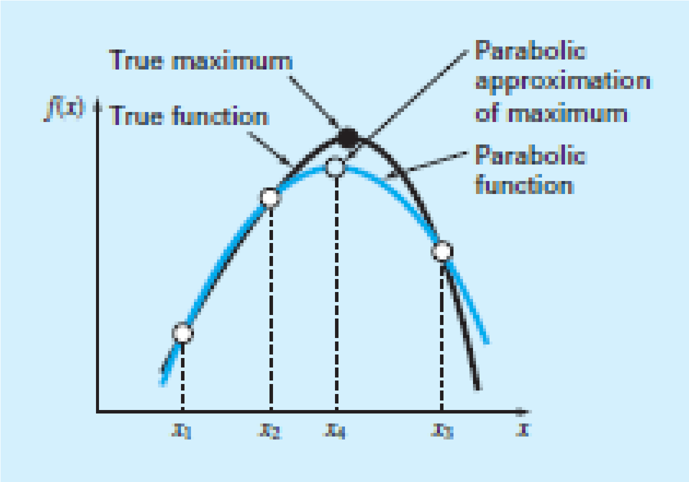
To check the stopping criteria the difference between x_u and x_l is calculated to be

$$x_u - x_l = 1.2000 - 0.60000$$
$$= 0.60000$$

Iteration	x_l	x_u	x_1	x_2	$f(x_1)$	$f(x_2)$	ε
1	0.00000	1.5708	0.97081	0.59999	5.1654	4.1226	1.5708
2	0.59999	1.5708	1.2000	0.97081	5.0791	5.1654	0.97081
3	0.59999	1.2000	0.97081	0.82917	5.1654	4.9418	0.59999
4	0.82917	1.2000	1.0583	0.97081	5.1955	5.1654	0.37081
5	0.97081	1.2000	1.1124	1.0583	5.1743	5.1955	0.22918

7.2.2 Parabolic Interpolation

$$x_4 = x_2 - \frac{1}{2}\frac{(x_2 - x_1)^2[f(x_2) - f(x_3)] - (x_2 - x_3)^2[f(x_2) - f(x_1)]}{(x_2 - x_1)[f(x_2) - f(x_3)] - (x_2 - x_3)[f(x_2) - f(x_1)]}$$



where x_1, x_2, and x_3 are the initial guesses, and x_4 is the value of x that corresponds to the optimum value of the parabolic fit to the guesses.

EXAMPLE 7.3 Parabolic Interpolation

Problem Statement. Use parabolic interpolation to approximate the minimum of

$$f(x) = \frac{x^2}{10} - 2\sin x$$

with initial guesses of $x_1 = 0$, $x_2 = 1$, and $x_3 = 4$.

Solution. The function values at the three guesses can be evaluated:

$$\begin{aligned} x_1 &= 0 & f(x_1) &= 0 \\ x_2 &= 1 & f(x_2) &= -1.5829 \\ x_3 &= 4 & f(x_3) &= 3.1136 \end{aligned}$$

$$x_4 = x_2 - \frac{1}{2}\frac{(x_2 - x_1)^2[f(x_2) - f(x_3)] - (x_2 - x_3)^2[f(x_2) - f(x_1)]}{(x_2 - x_1)[f(x_2) - f(x_3)] - (x_2 - x_3)[f(x_2) - f(x_1)]}$$

$$x_4 = 1 - \frac{1}{2}\frac{(1-0)^2[-1.5829 - 3.1136] - (1-4)^2[-1.5829 - 0]}{(1-0)[-1.5829 - 3.1136] - (1-4)[-1.5829 - 0]} = 1.5055$$

which has a function value of $f(1.5055) = -1.7691$.

2nd Iteration:

$$x_1 = 1 \qquad f(x_1) = -1.5829$$
$$x_2 = 1.5055 \qquad f(x_2) = -1.7691$$
$$x_3 = 4 \qquad f(x_3) = 3.1136$$

$$x_4 = 1.5055 - \frac{1}{2}\frac{(1.5055-1)^2[-1.7691-3.1136]-(1.5055-4)^2[-1.7691-(-1.5829)]}{(1.5055-1)[-1.7691-3.1136]-(1.5055-4)[-1.7691-(-1.5829)]}$$
$$= 1.4903$$

which has a function value of $f(1.4903) = -1.7714$. The process can be repeated, with the results tabulated here:

i	x_1	$f(x_1)$	x_2	$f(x_2)$	x_3	$f(x_3)$	x_4	$f(x_4)$
1	0.0000	0.0000	1.0000	−1.5829	4.0000	3.1136	1.5055	−1.7691
2	1.0000	−1.5829	1.5055	−1.7691	4.0000	3.1136	1.4903	−1.7714
3	1.0000	−1.5829	1.4903	−1.7714	1.5055	−1.7691	1.4256	−1.7757
4	1.0000	−1.5829	1.4256	−1.7757	1.4903	−1.7714	1.4266	−1.7757
5	1.4256	−1.7757	1.4266	−1.7757	1.4903	−1.7714	1.4275	−1.7757

Thus, within five iterations, the result is converging rapidly on the true value of −1.7757 at $x = 1.4276$.

Class Activity

7.7 Employ the following methods to find the maximum of

$$f(x) = 4x - 1.8x^2 + 1.2x^3 - 0.3x^4$$

(a) Golden-section search ($x_l = -2$, $x_u = 4$, $\varepsilon_s = 1\%$).
(b) Parabolic interpolation ($x_1 = 1.75$, $x_2 = 2$, $x_3 = 2.5$, iterations = 5).

7.10 Consider the following function:

$$f(x) = 2x + \frac{3}{x}$$

Perform 10 iterations of parabolic interpolation to locate the minimum. Comment on the convergence of your results ($x_1 = 0.1$, $x_2 = 0.5$, $x_3 = 5$)