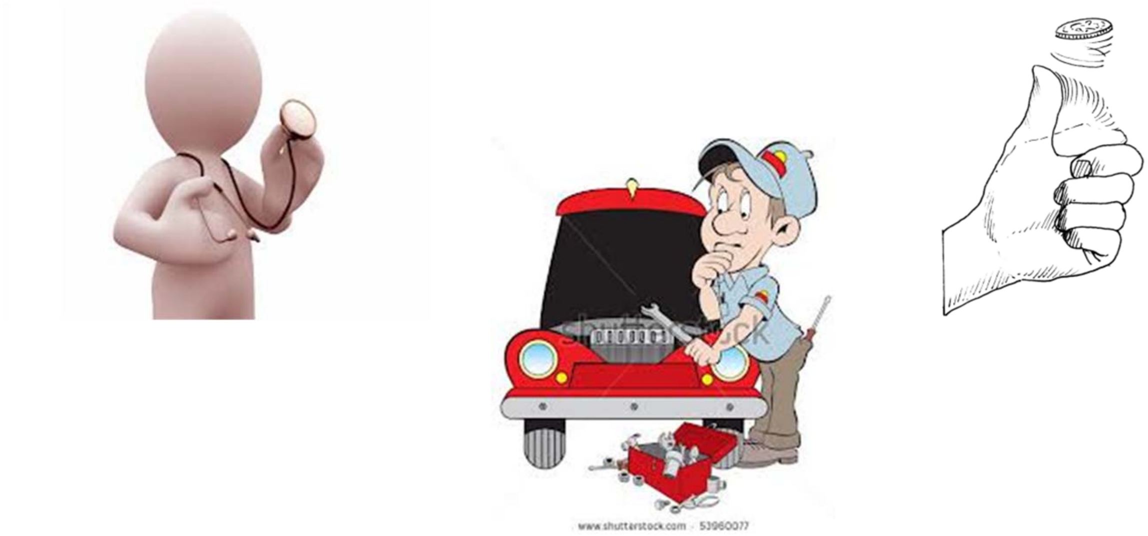
PROBABILISTIC REASONING

Induction inferences

Everyday inferences may be extremely different in content, but less so in structure: most of them are **inductive** (i.e., invalid) because they venture beyond the information given to draw conclusions that are probable given the available evidence, but are not logically implied by it



Induction inferences

Sherlock Holmes (*The Sign of the Four*)

«Three qualities are necessary

for the ideal detective:

knowledge, *observation*, ~~*deduction*~~»

Sherlock Holmes (*The Sign of the Four*)

«Three qualities are necessary

for the ideal detective:

knowledge, *observation*, *induction*»

Probability

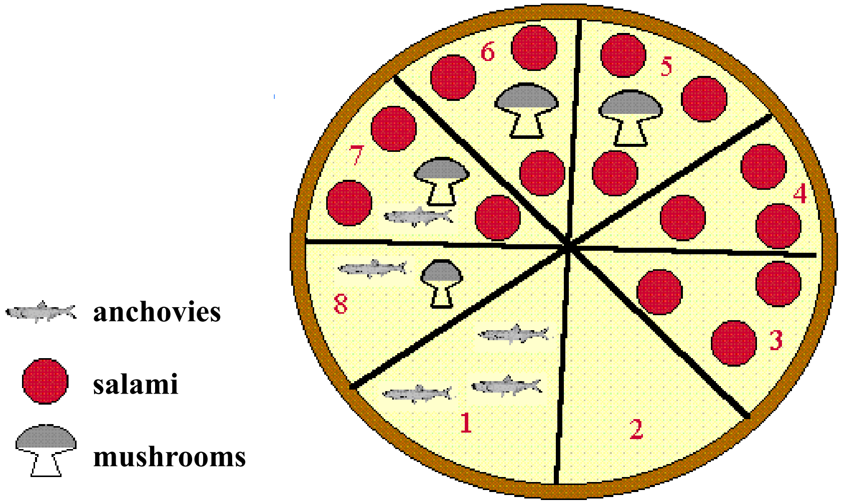
A little step back:

Probability of *h*, pr(*h*), is a value between “0” (=*h* is certainly false) and “1” (= *h* is certainly true), .5 indicates that *h* has the same probability of being true or false

Conditional probability of *h* in light of *e* is the probability that *h* will occur given the knowledge /assumption that *e* has already occurred

$$p(h \mid e) = \frac{p(h\&e)}{p(e)}$$

Example



A slice of pizza is taken at random. There is salami on it.

What's the probability that on the same slide of pizza there are also mushrooms?

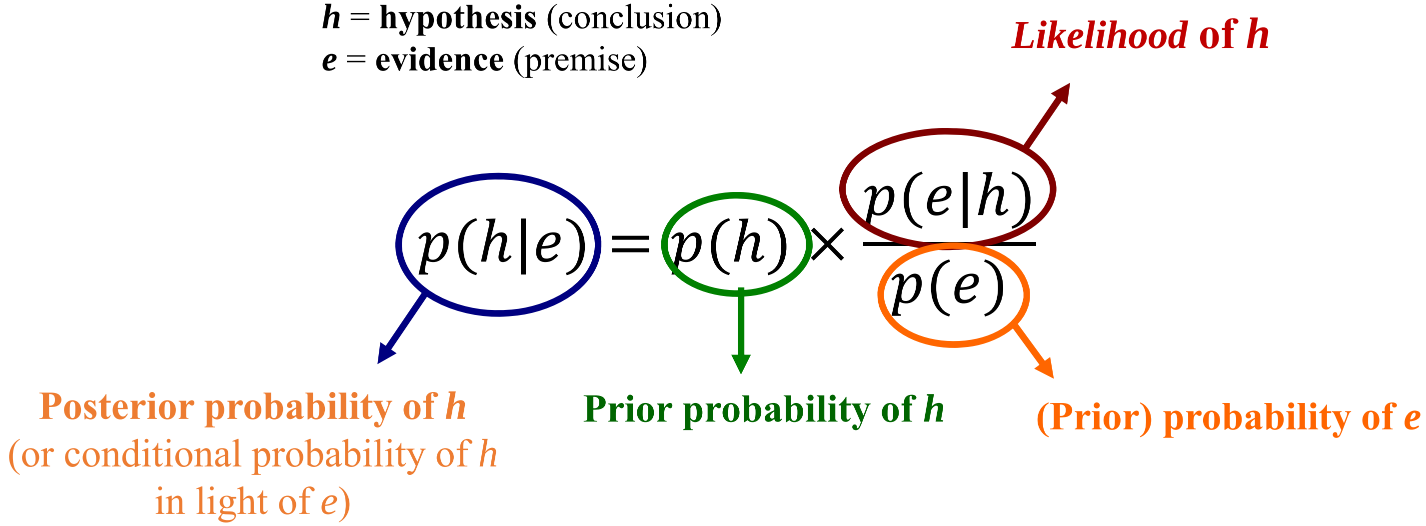
$$p(m \mid s) = \frac{p(m\&s)}{p(s)} = \frac{3/8}{5/8} = \frac{3}{5}$$

BAYES THEOREM

With regards to the **degree of belief** in a hypothesis *h* in light of evidence *e*, there is one normative model of reference: the **Bayes' theorem**

Bayes' theorem expresses how a subjective degree of belief in *h* should rationally change to account for evidence *e*

h = **hypothesis** (conclusion)
e = **evidence** (premise)



$$p(h|e) = p(h) \times \frac{p(e|h)}{p(e)}$$

Likelihood* of *h

Posterior probability of *h*
(or conditional probability of *h*
in light of *e*)

Prior probability of *h*

(Prior) probability of *e*

Bayes Theorem

Proof is very simple

$$p(h \mid e) = \frac{p(h\&e)}{p(e)}$$ conditional probability def.

Rearranged becomes:

$$p(h \,\&\, e) = p(e)\, p(h \mid e)$$
$$p(e \,\&\, h) = p(h)\, p(e \mid h)$$

Therefore:

$$p(e)\, p(h \mid e) = p(h)\, p(e \mid h)$$

$$p(h|e) = p(h) \times \frac{p\,(e|h)}{p(e)}$$

Sometimes $p(e)$ is not explicitly given, so we need to compute it:

$$p(\mathrm{e}) = p(e|h) \text{ x } p(h) + p(e|\text{non-}h) \text{ x } p(\text{non-}h)$$

Bayes theorem can therefore be written as:

$$p(h|e) = \frac{p(e|h) \times p(h)}{p(e|h) \times p(h) + p(e|\neg h) \times p(\neg h)}$$

(See also the odds version of Bayes)

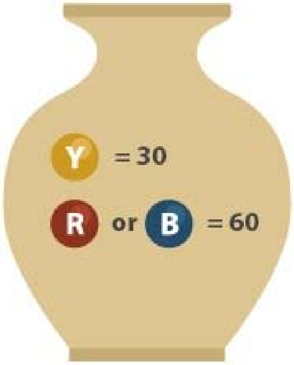
How are people’s probability judgements?

Ellsberg paradox (Ellsberg, 1961)

In the urn, there are:

- 30 yellow balls
- X red balls
- Y blue balls

X + Y = 60



Situation A

Bet Y or R

Situation B

Bet R / B or Y / B

You receive €100 if you draw a ball of the color that you chose

Do you choose yellow / red?

Do you choose red / blue or yellow / blue?

Ellsberg paradox (Ellsberg, 1961)

In the urn, there are:

- 30 yellow balls
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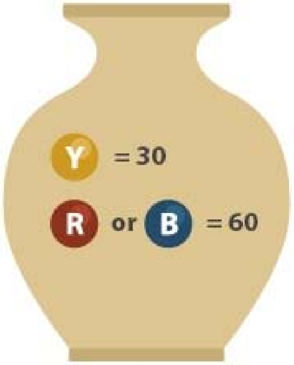
- Y blue balls

X + Y = 60



You receive €100 if you draw a ball of the color that you chose

Situation A

Bet Y or R

Most bet

Y

Do you choose yellow / red?

Situation B

Bet R / B or Y / B

R / B

Do you choose red / blue or yellow / blue?

Ellsberg paradox

To sum up

	30 balls	60 balls	
	yellow	red	blue
Option 1: yellow ball	€100	0	0
Option 2: red ball	0	€100	0

	30 balls	60 balls	
	yellow	red	blue
Option 1: yellow or blue ball	€100	0	€100
Option 2: red or blue ball	0	€100	€100

Most people prefer option 1 in the first bet and option 2 in the second

Why is this problematic?

Ellsberg paradox

To sum up

To prefer 1 over 2 in the first bet seems to suggest that there are more blue than red balls (B > R)

	30 balls	60 balls	
	yellow	red	blue
Option 1: yellow ball	€100	0	0
Option 2: red ball	0	€100	0

	30 balls	60 balls	
	yellow	red	blue
Option 1: yellow or blue ball	€100	0	€100
Option 2: red or blue ball	0	€100	€100

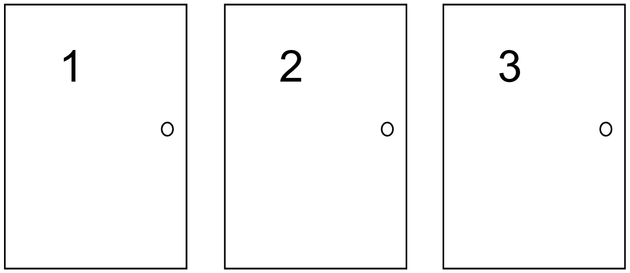
Ellsberg paradox is generally taken to be evidence for **ambiguity aversion** (i.e, preference for known risks over unknown risks)

To prefer 1 over 2 in the second bet seems to suggest that there are less blue than red balls (B < R)

Monty hall

Let's Make a Deal was a game show (hosted by Monty Hall and Carol Merril) originally ran from 1963 to 1977 on network TV

The highlight of the show was the "Big Deal"



Three doors:

- behind one there is **a shiny new car**
- behind the other **two** there are **goats**

Our contestant will select a door (without opening it)

Monty Hall (**who knows what's behind each of the doors**) then opens one of the other two doors (always a door with a goat)

Finally, he gives the contestant the option of switching doors or sticking with his/her original choice

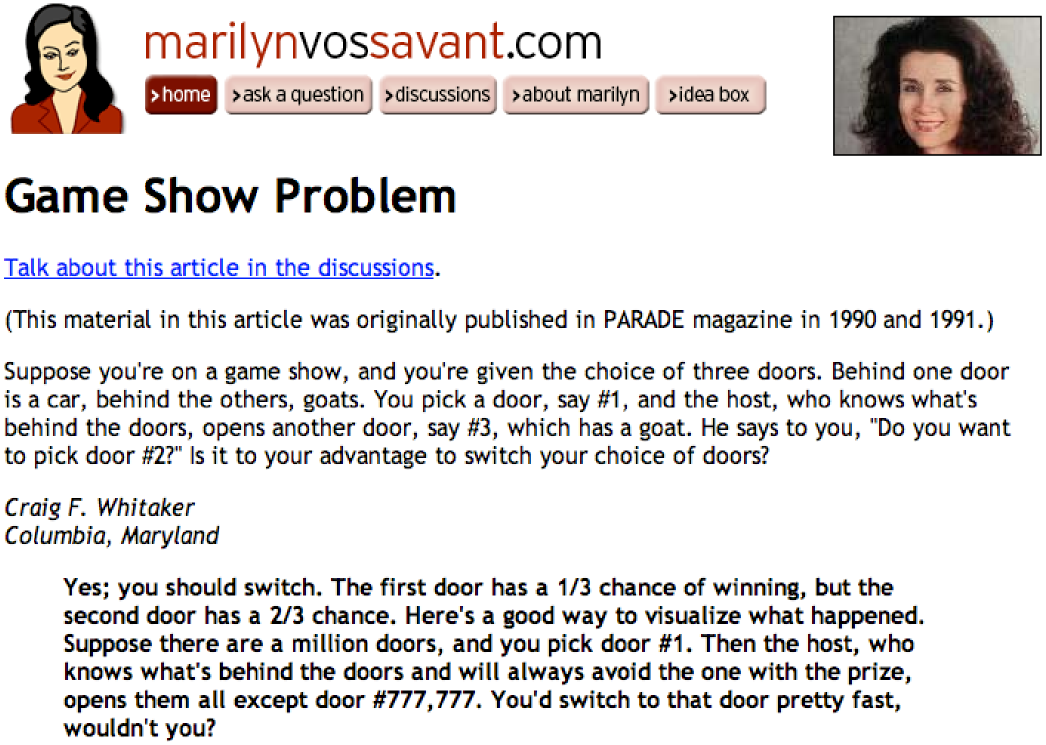
The question is: **should the contestant switch?**

Monty Hall

The answer is **yes**, s/he should switch!

The contestant would **double his/her odds of winning by switching doors**

Let’s Make a Deal inspired a probability problem that can **confuse and anger the best mathematicians**…



Game Show Problem

Talk about this article in the discussions.

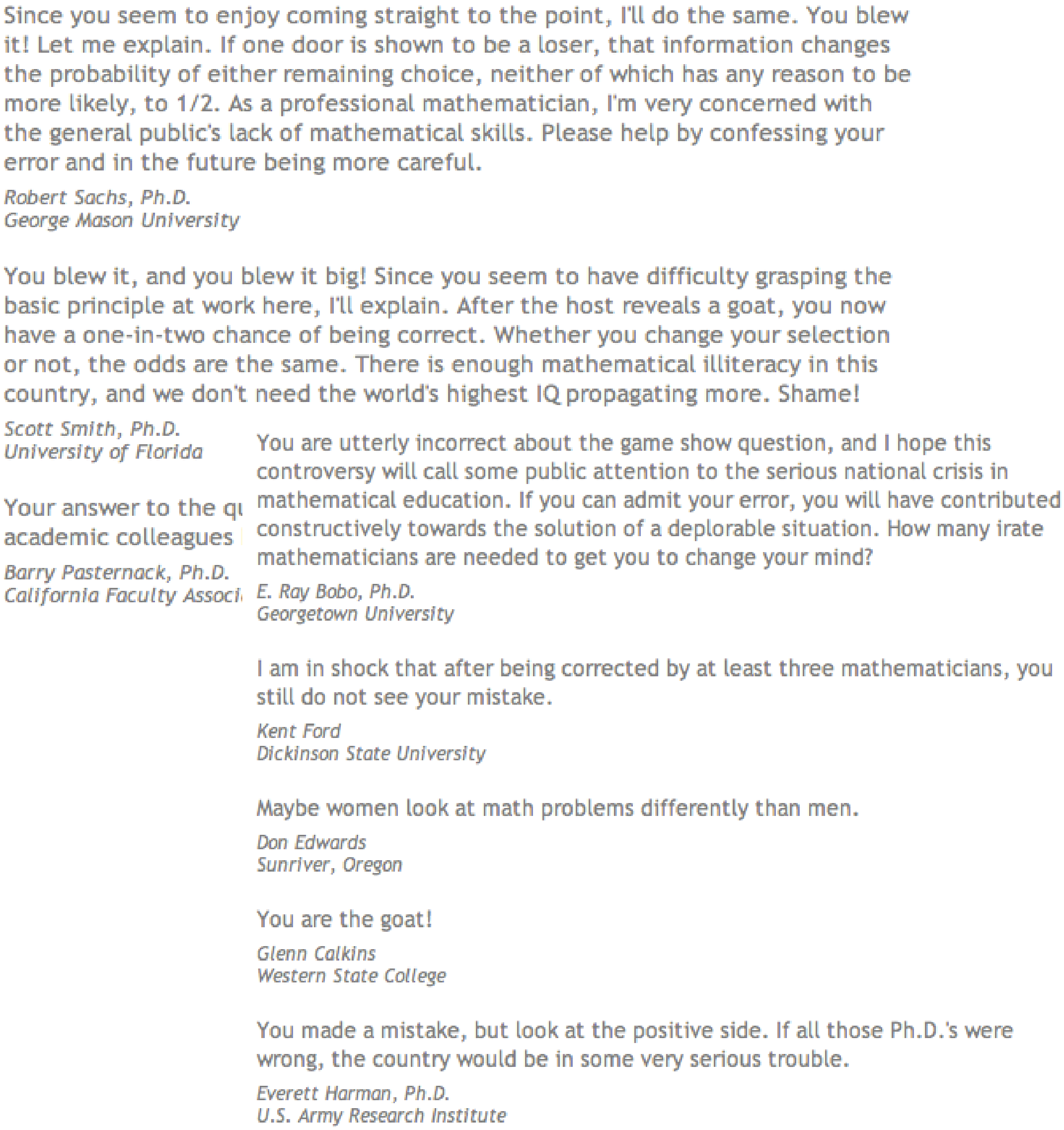
(This material in this article was originally published in PARADE magazine in 1990 and 1991.)

Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say #1, and the host, who knows what's behind the doors, opens another door, say #3, which has a goat. He says to you, "Do you want to pick door #2?" Is it to your advantage to switch your choice of doors?

Craig F. Whitaker
Columbia, Maryland

> **Yes; you should switch. The first door has a 1/3 chance of winning, but the second door has a 2/3 chance. Here's a good way to visualize what happened. Suppose there are a million doors, and you pick door #1. Then the host, who knows what's behind the doors and will always avoid the one with the prize, opens them all except door #777,777. You'd switch to that door pretty fast, wouldn't you?**

Monty Hall



Since you seem to enjoy coming straight to the point, I'll do the same. You blew it! Let me explain. If one door is shown to be a loser, that information changes the probability of either remaining choice, neither of which has any reason to be more likely, to 1/2. As a professional mathematician, I'm very concerned with the general public's lack of mathematical skills. Please help by confessing your error and in the future being more careful.

Robert Sachs, Ph.D.
George Mason University

You blew it, and you blew it big! Since you seem to have difficulty grasping the basic principle at work here, I'll explain. After the host reveals a goat, you now have a one-in-two chance of being correct. Whether you change your selection or not, the odds are the same. There is enough mathematical illiteracy in this country, and we don't need the world's highest IQ propagating more. Shame!

Scott Smith, Ph.D.
University of Florida

Your answer to the qu
academic colleagues

Barry Pasternack, Ph.D.
California Faculty Associ

You are utterly incorrect about the game show question, and I hope this controversy will call some public attention to the serious national crisis in mathematical education. If you can admit your error, you will have contributed constructively towards the solution of a deplorable situation. How many irate mathematicians are needed to get you to change your mind?

E. Ray Bobo, Ph.D.
Georgetown University

I am in shock that after being corrected by at least three mathematicians, you still do not see your mistake.

Kent Ford
Dickinson State University

Maybe women look at math problems differently than men.

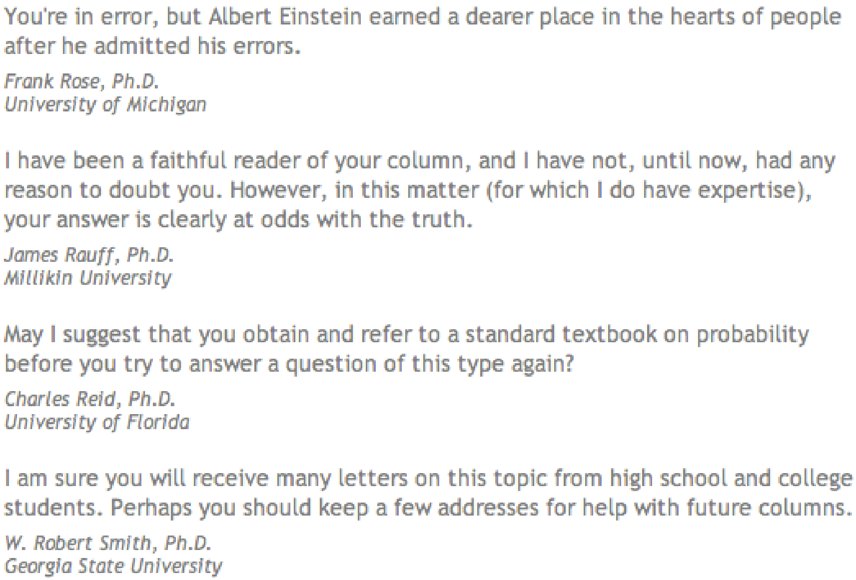
Don Edwards
Sunriver, Oregon

You are the goat!

Glenn Calkins
Western State College

You made a mistake, but look at the positive side. If all those Ph.D.'s were wrong, the country would be in some very serious trouble.

Everett Harman, Ph.D.
U.S. Army Research Institute



You're in error, but Albert Einstein earned a dearer place in the hearts of people after he admitted his errors.

Frank Rose, Ph.D.
University of Michigan

I have been a faithful reader of your column, and I have not, until now, had any reason to doubt you. However, in this matter (for which I do have expertise), your answer is clearly at odds with the truth.

James Rauff, Ph.D.
Millikin University

May I suggest that you obtain and refer to a standard textbook on probability before you try to answer a question of this type again?

Charles Reid, Ph.D.
University of Florida

I am sure you will receive many letters on this topic from high school and college students. Perhaps you should keep a few addresses for help with future columns.

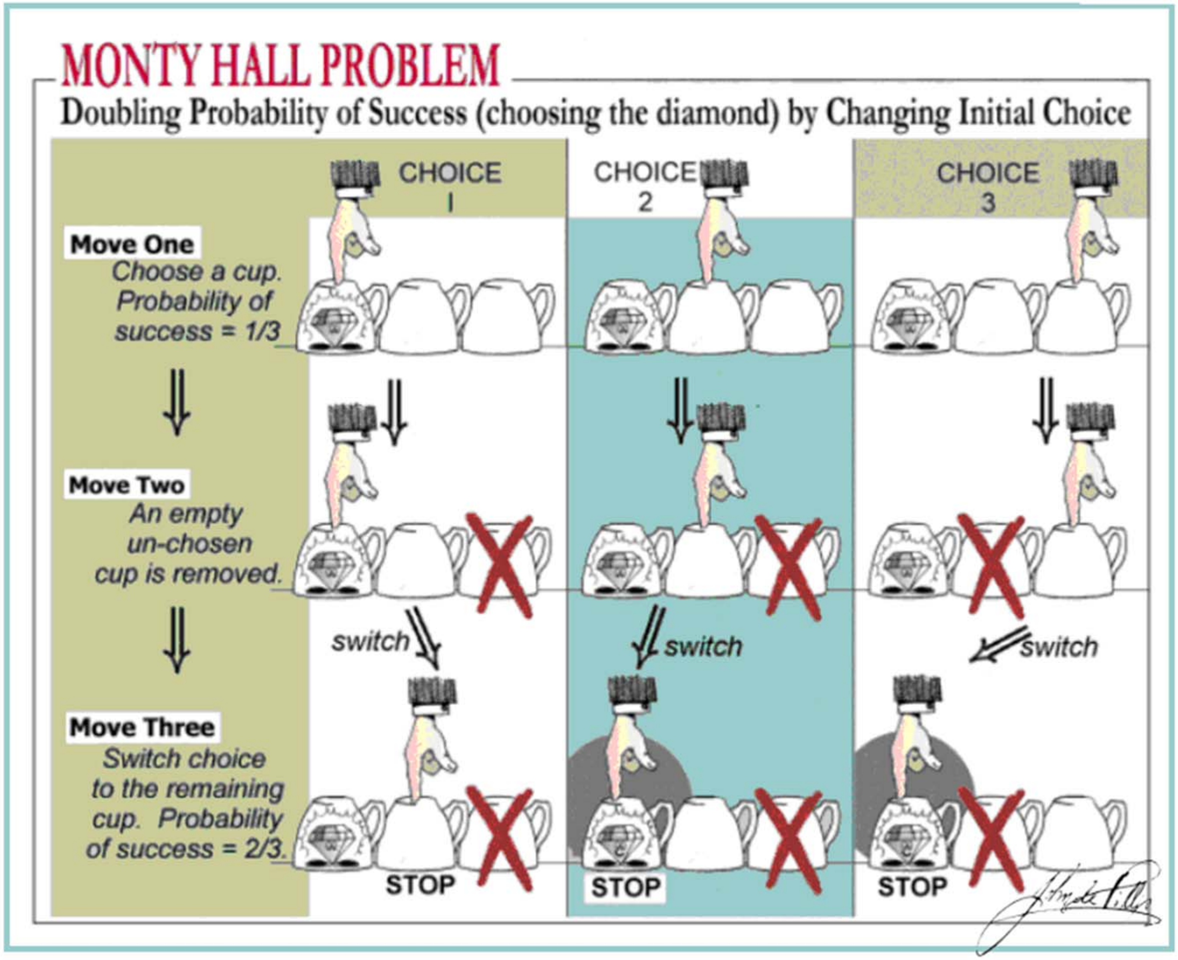
W. Robert Smith, Ph.D.
Georgia State University

If the contestant switches…

Monty Hall

	A B C	A **B** C	A B **C**	
 B C	s/he **loses**	s/he **wins**	s/he **wins**	2 / 3
A  C	s/he **wins**	s/he **loses**	s/he **wins**	2 / 3
A B 	s/he **wins**	s/he **wins**	s/he **loses**	2 / 3

Monty Hall


MONTY HALL PROBLEM
Doubling Probability of Success (choosing the diamond) by Changing Initial Choice
CHOICE 1
CHOICE 2
CHOICE 3
Move One
Choose a cup. Probability of success = 1/3
Move Two
An empty un-chosen cup is removed.
Move Three
Switch choice to the remaining cup. Probability of success = 2/3.
switch
switch
switch
STOP
STOP
STOP

Conservatism

Edwards (1968)

There are two opaque urns, one containing

A	**B**
7 blue chips	3 blue chips
3 red chips	7 red chips

One of the two urns is randomly selected

Then, some chips are randomly drawn (one at a time), with replacement, from the selected urn

Overall, you get **8 blue** and **4 red** chips

What is the probability that the selected urn is A?

Correct answer*: 0.97

Most common answer: around 0.7 (= people updated beliefs **conservatively**, that is **more slowly** than Bayes' theorem)

Conservatism

*

$P(E|A)= 0.7 \times 0.7 \times 0.7 \times 0.7 \times 0.7 \times 0.7 \times 0.7 \times 0.7 \times 0.3 \times 0.3 \times 0.3 \times 0.3 = 0.000466949$

$P(A) = 0.5$

$P(E|B)= 0.3 \times 0.3 \times 0.3 \times 0.3 \times 0.3 \times 0.3 \times 0.3 \times 0.3 \times 0.7 \times 0.7 \times 0.7 \times 0.7 = 0.000015753$

$P(B) = 0.5$

$$P(H|E) = 0.967365$$

Base rate fallacy

“**Cab** scenario” (Tversky & Kahneman, 1980)

A cab was involved in a hit and run accident at night.

Two cab companies, the Green and the Blue, operate in the city: 85% of the cabs in the city are Green and 15% are Blue.

A witness identified the cab as Blue. The court tested the reliability of the witness under the same circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colors 80% of the time and failed 20% of the time.

What is the probability that the cab involved in the accident was Blue rather than Green knowing that this witness identified it as Blue?

Most participants say > 50%, many even 80%

But, this would be correct only if there were equal number of blue and green cars...

Base rate fallacy

Correct answer:

H = the cab involved in the hit and run accident is blue

E = the witness identified the cab as blue

$$P(H \mid E) = \frac{P(E \mid H) \times P(H)}{P(E \mid H) \times P(H) + P(E \mid \neg H) \times P(\neg H)}$$

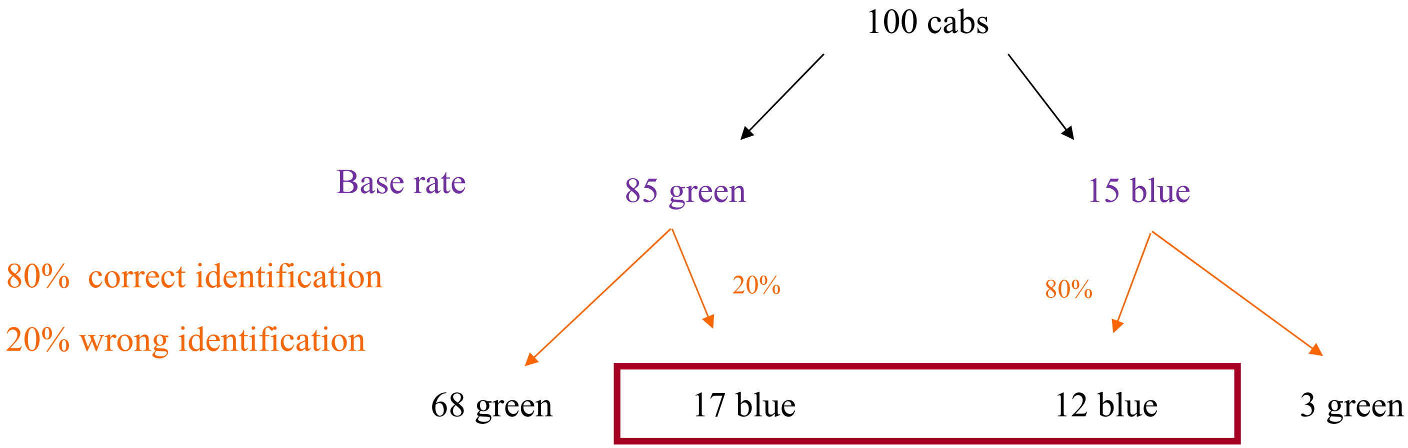
$$P(H \mid E) = \frac{0.80 \times 0.15}{0.80 \times 0.15 + .20 \times .85} = 0.41$$

Participants often say P(H|E) > 0.50, or even 0.80

↓

They seem to eglect the base rate

Base rate fallacy



100 cabs

Base rate

85 green

15 blue

80% correct identification

20% wrong identification

20%

80%

68 green

17 blue

12 blue

3 green

Total of 29 answers “blue”

Correct answers: 12/29 = 0.41

Watch out for **false positives**!!!

Conjunction fallacy

The **conjunction rule:**

$p(h_1 \wedge h_2) \leq p(h_1)$ because $h_1 \wedge h_2 \models h_1$

… and of course: $p(h_1 \wedge h_2|e) \leq p(h_1|e)$

The **conjunction fallacy:**

Judgments of the kind $\boldsymbol{p(h_1 \wedge h_2) > p(h_1)}$

… and of course: $\boldsymbol{p(h_1 \wedge h_2|e) > p(h_1|e)}$

Linda scenario

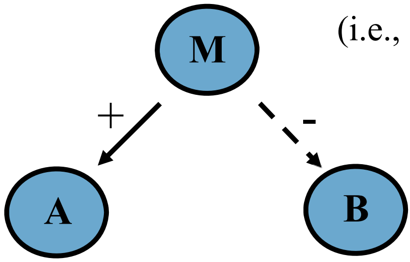
Linda scenario (Tversky & Kahneman, 1983)

Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations. (e)

Linda is a teacher in elementary school.
Linda works in a bookstore and takes Yoga classes.
Linda is active in the feminist movement. (h_2)
Linda is a psychiatric social worker.
Linda is a member of the League of Women Voters.
Linda is a bank teller. (h_1)
Linda is an insurance salesperson.
Linda is a bank teller and is active in the feminist movement. ($h_1 \wedge h_2$)

The great majority of participants ranked the conjunction ($h_1 \wedge h_2$) as more probable than its less representative constituent (h_1)

The “M $\longrightarrow$ A paradigm” (Tversky & Kahneman, 1983)



A “causal model” **M** (evidence ***e***)
(i.e., Linda’s personality)

An added event **A** (hypothesis $\boldsymbol{h_2}$)
which is highly “representative” of model M
(i.e., feminist)

A basic event **B** (hypothesis $\boldsymbol{h_1}$)
which is “unrepresentative” of model M
(i.e., bank teller)

Mr. F scenario (Tversky & Kahneman, 1983)

A health survey was conducted in a representative sample of adult males in British Columbia of all ages and occupations. Mr. F. was included in the sample. He was selected by chance from the list of participants.

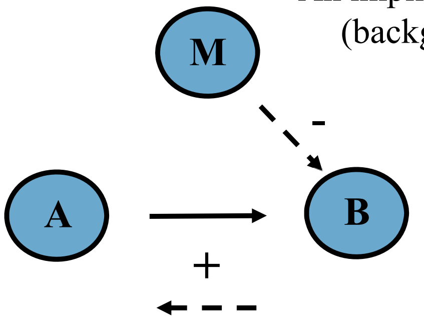
What is more probable?

Mr. F. has had one or more heart attacks. (h_1)

✗ Mr. F. has had one or more heart attacks, and he is over 55. ($h_1 \wedge h_2$)

The AB paradigm

The **"A ⟶ B paradigm"** (Tversky & Kahneman, 1983)



An implicit "causal model" **M**
(background knowledge)

An added event **A** (hypothesis h_2)
(i.e., over 55)

A basic event **B** (hypothesis h_1)
which is highly "representative" of event A
(i.e., one or more heart attacks)

Controversy on the nature of CF

From the very beginning the CF phenomenon has been described as a violation of “the simplest and the most basic qualitative law of probability” (Tversky & Kahneman, 1983)

Gigerenzer (1994; et al. 1988; 1999): Lack of ecological validity in the CF paradigm (i.e. the task and response format used to explore the CF are not representative of those typically encountered in daily life)

Is the CF a genuine reasoning error or is it an artifact?
Do participants interpret the experimental stimuli in a way that could justify their answers?

Three possible sources of misunderstanding…

A real fallacy?

➡ **the single conjunct h_1 could be interpreted as “$h_1 \wedge \textit{not-}h_2$”**

Many techniques have been developed to prevent the misinterpretation of h_1

- Rephrasing the conjunct “h_1”

  E.g. “Linda is a bank teller *whether or not she is active in the feminist movement*”
  “Linda is a bank teller *regardless of whether or not she is also active in the feminist movement*”
  “Linda is a bank teller *who may or may not be active in the feminist movement”*

- Including the conjunction “$h_1 \wedge \textit{not-}h_2$” in the set of options

⇨ Moderate / high CF

➡ **the term *probable* could be interpreted in a non-mathematical way (as *believable*, *plausible*, etc.)**

The most common technique to prevent the misinterpretation of the term *probable*

- Using a betting paradigm (on future events)

Bonini, Tentori & Osherson (2004, exp. 2)

In order to reduce traffic fatalities, the government will…
- launch a publicity campaign
- launch a publicity campaign and penalize more harshly dangerous traffic violations (both events must happen for you to win the money placed on this bet)
- launch a publicity campaign and not penalize more harshly dangerous traffic violations (both events must happen for you to win the money placed on this bet)

⇨ High CF (in the example, average bet on $h_1 \wedge h_2$: €3.29, on h_1: €2.46)

A real fallacy?

⇨ **the connective *and* could be interpreted as *or***

Many techniques have been developed to prevent the misinterpretation of *and*

- Avoiding the word *and* (i.e., using implicit conjunctions)
  E.g. “Linda is a *feminist bank teller*”

- Explicitly pointing out the conjunctive meaning of *and*
  E.g. by adding “both events must happen for you to win the money placed on this bet” (see the previous slide)

- Controlling for the interpretation of *and* after the CF task
  E.g. by means of Venn diagrams (“Shade with the pen the area corresponding to women who are “bank tellers and feminists”)
  E.g. by means of Implication questions

⇨ High CF

A real fallacy?

Scandinavia Scenario

The Scandinavian peninsula is the European area with the greatest percentage of people with blond hair and blue eyes. This is the case even though (as in Italy) every possible combination of hair and eye color occurs. Suppose we choose at random an individual from the Scandinavian population. (e)

The individual has blond hair (h_1)
The individual has blond hair and blue eyes $(h_1 \wedge h_2)$
The individual has blond hair and does not have blue eyes $(h_1 \wedge \text{not-}h_2)$

Implication question for the Scandinavia problem

Luke is in his last year of high school. One morning he met Mika, a new student from Finland. Mika has blond hair and blue eyes. Speaking together, Luca learned that Mika likes to play the piano and is in Italy because his father was transferred to the Milan branch of a large foreign bank. At home Luca tells his sister about Mika, and makes several claims about him. From among the statements shown below, please indicate which are true, which are false, and which might be either.

(a) Mika was born in Helsinki
(b) Mika hates to play the piano
TRUE (c) Mika has blue eyes
(d) Mika likes living in Milan
TRUE (e) Mika has blond hair
(f) Mika says that his family moves often because of his father's work

CF rate: 69%

A real fallacy?

➡ Last resort? The CF is a real fallacy, but it is limited to **single-case probability judgments**

Tentori, Bonini & Osherson (2004)

Scandinavia

The Scandinavian peninsula is the European area with the greatest percentage of people with blond hair and blue eyes. This is the case even though (as in Italy) every possible combination of hair and eye color occurs. Suppose we choose at random 100 individuals from the Scandinavian population. Which group do you think is the most numerous?

Individuals who have blond hair
Individuals who have blond hair and blue eyes
Individuals who have blond hair and do not have blue eyes

⇨ High CF (No sig. difference in CF rates between frequency (61%) and probability (69%) formats

A real fallacy?

To sum up…

The results shown point to a **genuine** (and elementary) **error** in reasoning about chance (not limited to the "single-case" probability)

Under a careful control of stimuli, the **CF rate** seems to be around **50-60%**

CF occurs in a wide variety of contexts and also with **highly educated participants,** including when the participants are given a monetary reward for logically correct answers

What can the conjunction fallacy tell us about human reasoning?

I. Human probability judgments exhibit systematic departures from relevant standards of rationality even when simple reasoning tasks are at issue (and proving it hasn't been a walk in the park…)

Explaining the CF

But WHY?

Theories are needed to account for people’s actual reasoning behavior and its departures from rationality

Various explanations of CF, including:

- Costello (2008): systematic effect of **random error**
- Nilsson (2008): non-normative **averaging rule** as applied to the probability of conjuncts
- …

all share the assumption that **CF rates should rise as the probability of the added conjunct h_2 rises**

Explaining the CF

In classical CF scenarios…

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in antinuclear demonstrations. (e)

Linda is a bank teller (h_1)
Linda is active in the feminist movement (h_2)

$p(h_1|e)$ is very low but also $c(h_1,e)$ is negative

$p(h_2|e \wedge h_1)$ is relatively high but also $c(h_2,e|h_1)$ is positive

Explaining the CF

Disentangling impact and posterior probability in CF scenarios

$$p(h_2|e \wedge h_1) < p(h_3|e \wedge h_1)$$
$$c(h_2, e|h_1) > c(h_3, e|h_1)$$

From Tentori, Crupi, & Russo (2013)

L. is a swiss man (e)
Which of the following hypotheses do you think is more probable?

	27%	☐ L. knows the tiramisù recipe (h_1)			
p<.01 CF	63%	☐ L. knows the tiramisù recipe and can ski $(h_1 \wedge h_2)$	68%	+ 4.7
	10%	☐ L. knows the tiramisù recipe and has a driver's license $(h_1 \wedge h_3)$	83%	- 0.6
			p<.01	p<.01

O. received a degree in violin studies from the Conservatory (e)
Which of the following hypotheses do you think is more probable?

	20%	☐ O. is an expert mountaineer (h_1)		
p<.01 CF	67%	☐ O. is an expert mountaineer and gives music lessons $(h_1 \wedge h_2)$	35%	+ 5.6
	13%	☐ O. is an expert mountaineer and owns an umbrella $(h_1 \wedge h_3)$	67%	- 0.1
			p<.01	p<.01