Bisection method (Bolzano)

Suppose you have to locate the root of the equation f(x)=0 in an interval say (x_0, x_1) ,let $f(x_0)$ and $f(x_1)$ are of opposite signs such that $f(x_0)f(x_1) < 0$

Then the graph of the function crossed the x-axis between x_0 *and* x_1 which exists the existence of at least one root in the interval (x_0, x_1).

The desired root is approximately defined by the mid point $x_2 = \frac{x_0 + x_1}{2}$ if $f(x_2) = 0$ then x_2 is the root of the equation otherwise the root lies either between x_0 *and* x_2 or x_1 *and* x_2

Example:

Carry out the five iterations for the function $f(x) = 2x\cos(2x) - (x+1)^2$

Note: All the calculations should be done in the radians.
Solution:

$f(x) = 2x\cos(2x) - (x+1)^2$

$f(-1) = 2(-1)\cos(-2) - (-1+1)^2 = -2(-0.4161) = +0.8322 > 0$

$f(0) = 2(0)\cos(0) - (0+1)^2 = -1 = -1 < 0$

so the root lies between 0 *and* -1 *as* $f(0)f(-1) < 0$

$$x_2 = \frac{0-1}{2} = -0.5$$

$f(-0.5) = 2(-0.5)\cos(-1) - (-0.5+1)^2 = -0.5403 - 0.25 = -0.7903 < 0$

so root lies between -1 *and* -0.5 *as* $f(-1)f(-0.5)$

$$x_3 = \frac{-0.5-1}{2} = -0.75$$

$f(-0.75) = 2(-0.75)\cos(-1.5) - (-0.75+1)^2 = -0.106 - 0.0625 = -0.1686 < 0$

so root lies between -1 *and* -0.75 *as* $f(-1)f(-0.75)$

$$x_4 = \frac{-0.75-1}{2} = -0.875$$

$f(-0.875) = 2(-0.875)\cos(-1.75) - (-0.875+1)^2 = 0.3119 - 0.015625 = 0.296275 > 0$

so root lies between -0.875 *and* -0.75 *as* $f(-0.75)f(-0.875)$

$$x_5 = \frac{-0.75-0.875}{2} = -0.8125$$

$f(-0.8125) = 2(-0.8125)\cos(-1.625) - (-0.8125+1)^2 = 0.0880 - 0.0351 = 0.052970 > 0$

so root lies between -0.8125 *and* -0.75 *as* $f(-0.75)f(-0.8125)$

$$x_5 = \frac{-0.75-0.8125}{2} = -0.78125$$

Regula-Falsi method (Method of false position)

Here we choose two points x_n and x_{n-1} such that $f(x_n)$ *and* $f(x_{n-1})$ have opposite signs. Intermediate value property suggests that the graph of the y=f(x) crosses the x-axis between these two points and therefore, a root lies between these two points.

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

Example

Use the Regula-Falsi method to compute a real root of the equation $x3 - 9x + 1 = 0$,
(i) if the root lies between 2 and 4
(ii) if the root lies between 2 and 3.
Comment on the results.

Solution

Let
f (x) = x3 - 9x + 1
$f(2) = 2^3 - 9(2) + 1 = 8 - 18 + 1 = -9$ and $f(4) = 4^3 - 9(4) + 1 = 64 - 36 + 1 = 29$.
Since $f(2)$ and $f(4)$ are of opposite signs, the root of $f(x) = 0$ lies between 2 and 4.
Taking $x1 = 2$, $x2 = 4$ and using Regula-Falsi method, the first approximation is given by

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 4 - \frac{4-2}{29-(-9)}(29) = 4 - \frac{2(29)}{38} = 4 - 1.5263 = 2.4736$$

Now
$f(x3) = 2.4736^3 - 9(2.4736) + 1 = 15.13520 - 22.2624 + 1 = -6.12644$.
Since $f(x2)$ and $f(x3)$ are of opposite signs, the root lies between $x2$ and $x3$.
The second approximation to the root is given as

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.4736 - \frac{2.4736 - 4}{-6.12644 - 29}(-6.12644)$$

$$= 2.4736 - \frac{-1.5264}{-35.12644}(-6.12644) = 2.4736 - (0.04345)(-6.12644)$$

$$= 2.4736 + 0.26619 = 2.73989$$

Therefore
$f(x4) = 2.73989^3 - 9(2.73989) + 1 = 20.5683 - 24.65901 + 1 = = -3.090707$.
Now, since $f(x2)$ and $f(x4)$ are of opposite signs, the third approximation is obtained from

$$x_5 = x_4 - \frac{x_4 - x_2}{f(x_4) - f(x_2)} f(x_4) = 2.73989 - \frac{2.73989 - 4}{-3.090707 - 29}(-3.090707) = 2.86125$$

$$= 2.73989 - \frac{-1.26011}{-32.090707}(-3.090707) = 2.73989 + 0.039267(3.090707) = 2.73989 + 0.121363 = 2.86125$$

Now

$f(x5) = 2.86125^3 - 9(2.86125) + 1 = 23.42434 - 25.75125 + 1 = -1.326868.$

(ii)

Here

$f(x) = x3 - 9x + 1$

$f(2) = 2^3 - 9(2) + 1 = 8 - 18 + 1 = -9$ and $f(3) = 3^3 - 9(3) + 1 = 27 - 27 + 1 = 1.$

Since $f(2)$ and $f(3)$ are of opposite signs, the root of $f(x) = 0$ lies between 2 and 3.
Taking $x1 = 2$, $x2 = 3$ and using Regula-Falsi method, the first approximation is given by

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 3 - \frac{3-2}{1+9}(1)$$

$$= 3 - \frac{1}{10} = 2.9$$

$f(x_3) = 2.9^3 - 9(2.9) + 1 = 24.389 - 26.1 + 1 = -0.711$

Since $f(x2)$ and $f(x3)$ are of opposite signs, the root lies between $x2$ and $x3$.
The second approximation to the root is given as

$$x_4 = 2.9 - \frac{2.9-3}{-0.711-1}(-0.711) = 2.9 - \frac{-0.1}{-1.711}(-0.711) = 2.9 - \frac{-0.1}{-1.711}(-0.711)$$

$$= 2.9 - (0.05844)(-0.711) = 2.9\_0.04156 = 2.94156$$

$f(x_4) = -0.0207$

$f(x_4) = 2.94156^3 - 9(2.94156) + 1 = 25.45265 - 26.47404 + 1 = -0.0207$

Now, we observe that $f(x2)$ and $f(x4)$ are of opposite signs; the third approximation is obtained from

$$x_5 = 2.94156 - \frac{2.94156 - 3}{-0.0207 - 1}(-0.0207) = 2.94156 - \frac{-0.05844}{-1.0207}(-0.0207)$$

$$= 2.94156 - (-0.05725)(-0.0207) = 2.94275$$

$f(x_5) = 2.94275^3 - 9(2.94275) + 1 = 25.48356 - 26.48475 + 1 = -0.0011896$

We observe that the value of the root as a third approximation is evidently different in both the cases, while the value of $x5$, when the interval considered is (2, 3), is closer to the root.

Example:

Use method of false position to solve $e^{-x}+2^{-x}+2\cos x-6=0 \quad 1\le x\le 2$

Solution:

$$f(x)=e^{x}+2^{-x}+2\cos x-6$$
$$x_0=1 \;,\quad x_1=2$$
$$now$$
$$x_{n+1}=\frac{x_n-x_{n-1}}{f(x_n)-f(x_{n-1})}f(x_n)$$
$$f(1)=e^{1}+2^{-1}+2\cos 1-6=2.7182+0.5+2(0.5403)-6=-1.7011$$
$$f(2)=e^{2}+2^{-2}+2\cos 2-6=7.3886+0.25+2(-0.4161)-6=0.8068$$
$$now\ for\ n=1$$
$$x_2=x_1-\frac{x_1-x_0}{f(x_1)-f(x_0)}f(x_1)=2-\frac{2-1}{0.8068+1.7011}(0.8068)$$
$$x_2=2-\frac{1}{2.5079}(0.8068)=1.6783$$
$$f(1.6783)=e^{1.6783}+2^{-1.6783}+2\cos(1.6783)-6=-0.5457$$
$$now\ for\ n=2$$
$$x_3=x_2-\frac{x_2-x_1}{f(x_2)-f(x_1)}f(x_2)=1.6783-\frac{1.6783-2}{(-0.5457)-0.8068}(-0.5457)$$
$$x_3=1.6783-\frac{(-0.3217)}{(-1.3525)}(-0.5457)=1.6783+0.12979=1.8081$$

$$f(1.8081)=e^{1.6783}+2^{-1.8081}+2\cos(1.8081)-6=-0.8575$$

$$now\ for\ n=3$$

$$x_4=x_3-\frac{x_3-x_2}{f(x_3)-f(x_2)}f(x_3)=1.8081-\frac{1.8081-1.6783}{(-0.08575)+0.5457}(-0.08575)$$

$$x_3=1.8081-\frac{0.1298}{0.45995}(-0.08575)=1.6783+0.12979=1.8323$$

$$f(1.8323)=e^{1.8323}+2^{-1.8323}+2\cos(1.8323)-6=0.1199$$

$$now\ for\ n=4$$

$$x_5=x_4-\frac{x_4-x_3}{f(x_4)-f(x_3)}f(x_4)=1.8323-\frac{1.8323-1.8081}{0.01199+0.08575}(0.01199)$$

$$x_5=1.8323-\frac{0.0242}{0.09774}(0.01199)=1.8323-0.00296=1.8293$$

$$f(1.8293)=e^{1.8293}+2^{-1.8293}+2\cos(1.8293)-6=-0.000343$$

$$now\ for\ n=5$$

$$x_6=x_5-\frac{x_5-x_4}{f(x_5)-f(x_4)}f(x_5)=1.8293-\frac{1.8293-1.8323}{-0.000343-0.01199}(-0.000343)$$

$$x_6=1.8293-\frac{(-0.003)}{-0.012333}(-0.000343)=1.8293$$

Newton -Raphson Method

This method is one of the most powerful method and well known methods, used for finding a root of f(x)=0 the formula many be derived in many ways the simplest way to derive this formula is by using the first two terms in Taylor's series expansion of the form,

$$f(x_{n+1})=f(x_n)+(x_{n+1}-x_n)f'(x_n)$$

$$setting\ f(x_{n+1})=0\ gives,$$

$$f(x_n)+(x_{n+1}-x_n)f'(x_n)=0$$

$$thus\ on\ simplification, we\ get\ ,$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\ for\ n=0,1,2...$$

Example

Find the first three iteration of the equation $f(x) = x - 0.8 - 0.2\sin x$ in the interval $[0, \pi/2]$.

Solution

$$f(0) = 0 - 0.8 - 0.2\sin(0) = 0 - 0.8 - 0.2(0) = -0.8$$

$$\begin{aligned} f(1.57) &= 1.57 - 0.8 - 0.2\sin(1.75) \\ &= 1.57 - 0.8 - 0.2(0.99999) \\ &= 1.57 - 0.8 - 0.199998 = 0.570002 \end{aligned}$$

$$f'(x) = 1 - 0.2\cos x$$

$$f'(0) = 1 - 0.2\cos(0) = 1 - 0.2 = 0.8$$

$$here\,|f(0)|\,is\ greater\ then\ x_0 = 0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-0.8}{0.8} = 1$$

$$now$$

$$\begin{aligned} f(1) &= 1 - 0.8 - 0.2\sin(1) \\ &= 1 - 0.8 - 0.1683 \\ &= 0.0317 \end{aligned}$$

$$f'(x) = 1 - 0.2\cos x$$

$$f'(1) = 1 - 0.2\cos(1) = 1 - 0.1081 = 0.8919$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{0.0317}{0.8919} = 1 - 0.0355 = 0.9645$$

$$\begin{aligned} f(0.9645) &= 0.9645 - 0.8 - 0.2\sin(0.9645) \\ &= 0.9645 - 0.8 - 0.1645 \\ &= 0.0002 \end{aligned}$$

$$f'(x) = 1 - 0.2\cos x$$

$$f'(0.9645) = 1 - 0.2\cos(0.9645) = 1 - 0.11396 = 0.88604$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.9645 - \frac{0.0002}{0.88604} = 0.9645 - 0.00022 = 0.9643$$

Example

Perform three iteration of the equation $\ln(x-1)+\cos(x-1)=0 \; when \; 1.2 \le x \le 2$ .Use Newton Raphson method to calculate the root.

Solution

Here

$\ln(x-1)+\cos(x-1)=0 \; when \; 1.2 \le x \le 2$

$f(x)=\ln(x-1)+\cos(x-1)$

$$f(x)=\ln(x-1)+\cos(x-1)$$
$$f(1.2)=\ln(1.2-1)+\cos(1.2-1)$$
$$=-1.6094+0.9801=-0.6293$$
$$f(2)=\ln(2-1)+\cos(2-1)$$
$$=0+0.5403=0.5403$$
$$now \; f(x)=\ln(x-1)+\cos(x-1)$$
$$f'(x)=\frac{1}{x-1}-\sin(x-1)$$
$$f'(1.2)=\frac{1}{1.2-1}-\sin(1.2-1)$$
$$=5-0.1986=4.8014$$
$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=1.2-\frac{-0.6293}{4.8014}=1.2+0.1311=1.3311$$
$$f(1.311)=\ln(1.3311-1)+\cos(1.3311-1)$$
$$=-1.1053+0.9457=-0.1596$$
$$f'(1.3311)=\frac{1}{1.3311-1}-\sin(1.3311-1)$$
$$=3.0202-0.3251=2.6951$$
$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=1.3311-\frac{-0.1596}{2.6951}=1.3311+0.0592=1.3903$$
$$f(1.3903)=\ln(1.3903-1)+\cos(1.3903-1)$$
$$=-0.9408+0.9248=-0.016$$
$$f'(1.3903)=\frac{1}{1.3903-1}-\sin(1.3903-1)$$
$$=2.5621-0.3805=2.1816$$
$$x_3=x_2-\frac{f(x_2)}{f'(x_2)}=1.3903-\frac{-0.016}{2.1816}=1.3903+0.0073=1.3976$$

Secant Method

The secant method is modified form of Newton-Raphson method .If in Newton-Raphson method; we replace the derivative $f'(x_n)$ by the difference ratio, i.e,

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Where x_n and x_{n-1} are two approximations of the root we get

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$= \frac{x_n f(x_n) - x_n f(x_{n-1}) - f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$= \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Provided $f(x_n) \neq f(x_{n-1})$

Example

Do three iterations of secant method to find the root of

$f(x) = x^3 - 3x + 1 = 0,$

Taking

$x_0 = 1,\ x_1 = 0.5$

$n = 1,$

$f(x_0) = f(1) = 1^3 - 3(1) + 1 = -1$

$f(x_1) = f(0.5) = 0.5^3 - 3(0.5) + 1 = 0.125 - 1.5 + 1 = -0.375$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{(1)(-0.375) - (0.5)(-1)}{-0.375 - (-1)} = 0.2$$

$n = 2,$

$f(x_2) = f(0.2) = 0.2^3 - 3(0.2) + 1 = 0.008 - 0.6 + 1 = 0.408$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{(0.5)(0.408) - 0.2(-0.375)}{0.408 - (-0.375)} = 0.3563$$

$n = 3,$

$f(x_3) = f(0.3563) = 0.3563^3 - 3(0.3563) + 1 = 0.04523 - 1.0689 + 1 = -0.02367$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{(0.2) f(0.3563) - 0.3563 f(0.2)}{f(0.3563) - f(0.2)}$$

$x_5 = 0.3473,\ f(x_5) = -0.0000096$

and $|x_5 - x_4| = 0.0004.$