CONNECTIVITY AND FLOW

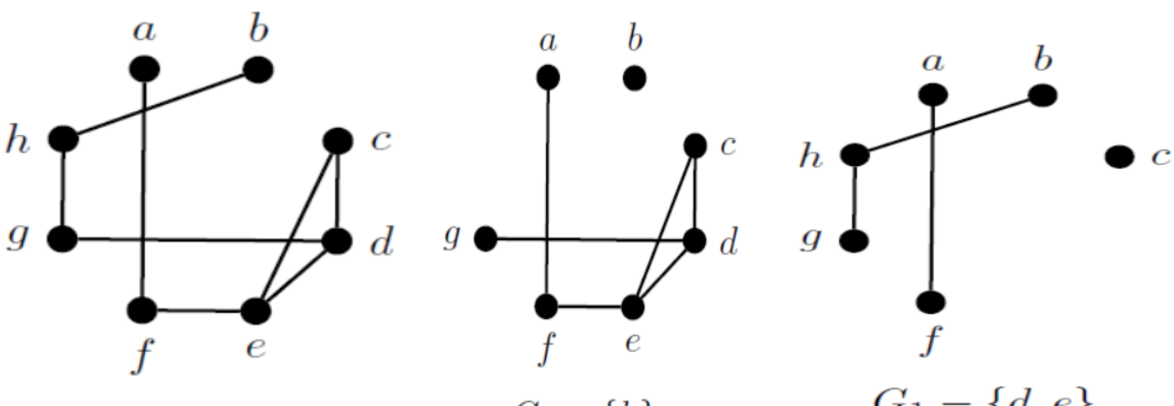
- We have used connectivity in the context of problems. For example, we needed to know if a graph was connected to determine if it has Eulerian circuit, Hamiltonian cycle, and we define trees as minimally connected graphs; since the removal of any edge would disconnect the graph.
- This chapter focuses on connectivity as its own topic, where we now consider how connected a graph is, and not just whether it is connected or not.
- One way to describe the clumping is in a connected graph, how many edges or vertices would need to be removed before the graph is no longer connected, which is one way we measure connectivity.

Connectivity Measures:

When we define a graph to be connected, we refer to the existence of a way to move between any two vertices in a graph, specifically as the existence of a path between any pair of vertices.

Definition 4.1 A ***cut-vertex*** of a graph G is a vertex v whose removal disconnects the graph, that is, G is connected but $G - v$ is not. A set S of vertices within a graph G is a ***cut-set*** if $G - S$ is disconnected.

- Note that any connected graph that is not complete has a cut-set, whereas K_n does not have a cut-set.
- Moreover, a graph can have many different cuts-sets of varying sizes.
- For example, two different cut-sets are shown below in graph G_1.



G_1 $\qquad$ $G_1 - \{h\}$ $\qquad$ $G_1 - \{d, e\}$

k-Connected:

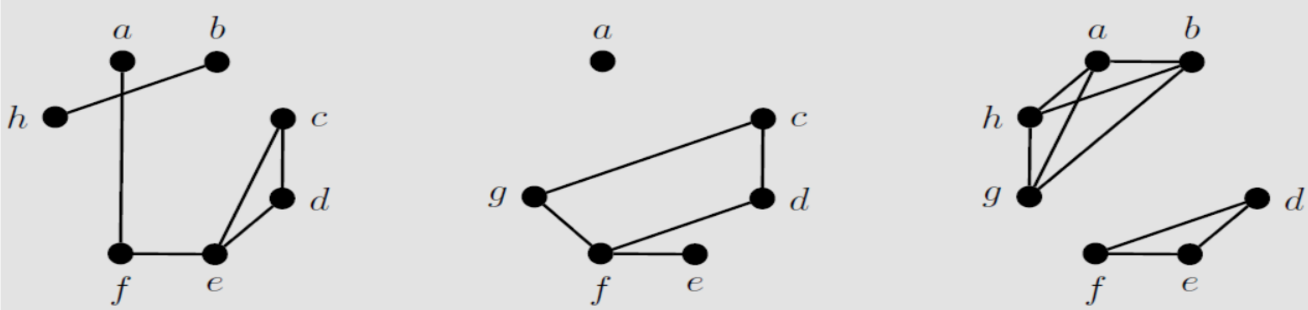
Definition 4.2 For any graph G, we say G is ***k*-connected** if the smallest cut-set is of size at least k.

Define the connectivity of G, $\kappa(G) = k$, to be the maximum k such that G is k-connected, that is there is a cut-set S of size k, yet no cut-set exists of size $k - 1$ or less. Define $\kappa(K_n) = n - 1$.

- The distinction between *k*-connected and connectivity *k* is subtle yet important.
- For example, if we say a graph is 3-connected, then we know there cannot be a cut-set of size 2 or less in the graph; however, we only know that its connectivity is at least 3 ($\kappa(G) \geq 3$).

Example 4.1 Find $\kappa(G)$ for each of the graphs shown above on page 169.

Solution: The removal of any one of d, e, f, g, or h in G_1 will disconnect the graph, so $\kappa(G_1) = 1$. Similarly, $G_3 - c$ has two components and so $\kappa(G_3) = 1$. However, $\kappa(G_2) = 2$ since the removal of any one vertex will not disconnect the graph, yet $S = \{b, h\}$ is a cut-set. Note this means G_2 is both 1-connected and 2-connected, but not 3-connected.



$G_1 - \{g\}$ $\qquad$ $G_2 - \{b, h\}$ $\qquad$ $G_3 - \{c\}$

- The example above demonstrates that more than one minimal cut-set can exist within a graph.
- Moreover, any connected graph is 1-connected.
- We are more interested in how large k can be before G fails to be k-connected.

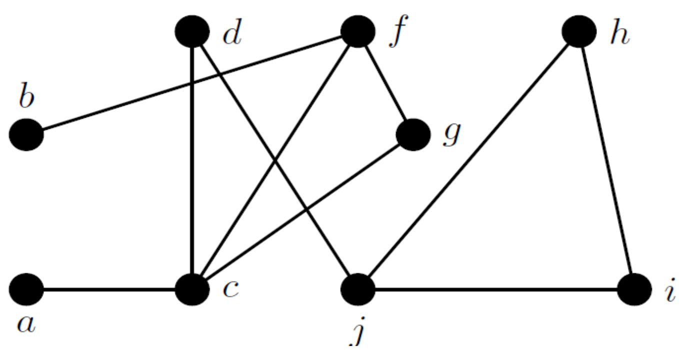
k-Edge-Connected:

- We now look at how many edges need to be removed before the graph is disconnected.
- Recall that when we remove an edge e = xy from a graph, we are not removing the endpoints x and y.

Definition 4.3 A ***bridge*** in a graph $G = (V, E)$ is an edge e whose removal disconnects the graph, that is, G is connected but $G - e$ is not. An ***edge-cut*** is a set $F \subseteq E$ so that $G - F$ is disconnected.

- ✓ Clearly every connected graph has an edge-cut since removing all the edges from a graph will result in just a collection of isolated vertices.
- ✓ As with the vertex version, we are more concerned with the smallest size of an edge-cut.

Find all bridges in the following graph:

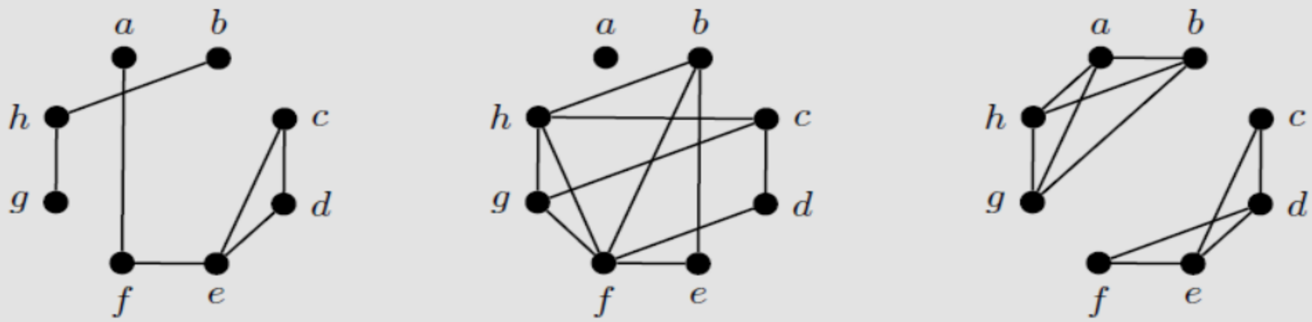


Definition 4.4 We say G is ***k*-edge-connected** if the smallest edge-cut is of size at least k.

Define $\kappa'(G) = k$ to be the maximum k such that G is k-edge-connected, that is there exists a edge-cut F of size k, yet no edge-cut exists of size $k-1$.

Example 4.2 Find $\kappa'(G)$ for each of the graphs shown on page 169.

Solution: There are many options for a single edge whose removal will disconnect G_1 (for example af or dg). Thus $\kappa'(G_1) = 1$. For G_2, no one edge can disconnect the graph with its removal, yet removing both ab and ah will isolate a and so $\kappa(G_2) = 2$. Similarly $\kappa'(G_3) = 2$, since the removal of bc and cg will create two components, one with vertices a, b, g, h and the other with c, d, e, f.



$G_1 - dg$ $\quad$ $G_2 - \{ab, ah\}$ $\quad$ $G_3 - \{bc, cg\}$

Whitney's Theorem:

- What do you think, is there any relationship between the vertex and edge connectivity measures?
- The examples above should demonstrate that these measures need not be equal, though they can be.
- How does the minimum degree of a graph play a role in these?
- Notice how in both G_2 and G_3 above we found an edge-cut by removing both edges incident to a specific vertex.

Theorem 4.5 (Whitney's Theorem) For any graph G, $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Remark:

- ✓ Whitney's Theorem provides an indication that high connectivity (or edge-connectivity) requires a large minimum degree. But is the converse true?
- ✓ Can a graph have a high minimum degree but low connectivity?

CONNECTIVITY AND PATHS:

- Now we have some familiarity with connectivity, we turn to its relationship to paths within a graph.
- We will assume the graphs are connected, as otherwise, the results are trivial.
- We begin by relating cut-vertices and bridges to paths.

Theorem 4.6 A vertex v is a cut-vertex of a graph G if and only if there exist vertices x and y such that v is on every $x - y$ path.

Proof: First suppose v is a cut-vertex in a graph G. Then $G-v$ must have at least two components. Let x and y be vertices in different components of $G - v$. Since G is connected, we know there must exist an $x - y$ path in G that does not exist in $G - v$. Thus v must lie on this path.

Conversely, let v be a vertex and suppose there exist vertices x and y such that v is on every $x - y$ path. Then none of these paths exist in $G - v$, and so x and y cannot be in the same component of $G - v$. Thus G must have at least two components and so v is a cut-vertex.

Theorem 4.7 An edge e is a bridge of G if and only if there exist vertices x and y such that e is on every $x - y$ path.

- It should be obvious that any edge along a cycle cannot be a bridge since its removal will only break the cycle, not disconnect the graph.
- More surprising is that all edges not on a cycle are in fact bridges.

Theorem 4.8 Every nontrivial connected graph contains at least two vertices that are not cut-vertices.

Theorem 4.9 An edge e is a bridge of G if and only if e lies on no cycle of G.

Definition 4.10 Let P_1 and P_2 be two paths within the same graph G. We say these paths are

- ***disjoint*** if they have no vertices or edges in common.
- ***internally disjoint*** if the only vertices in common are the starting and ending vertices of the paths.
- ***edge-disjoint*** if they have no edges in common.

✓ Two disjoint paths are automatically internally disjoint and edge-disjoint, but two edge-disjoint paths may or may not be internally disjoint.

Definition 4.11 Let x and y be two vertices in a graph G. A set S (of either vertices or edges) ***separates*** x and y if x and y are in different components of $G - S$. When this happens, we say S is a separating set for x and y.

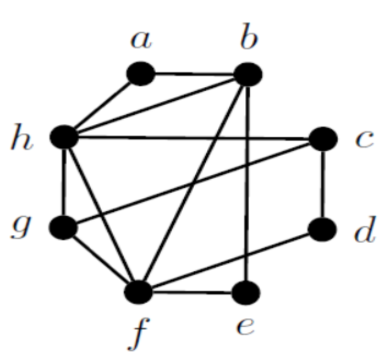
Remarks:

✓ Note that a cut-set may or may not be a separating set for a specific pair of vertices.

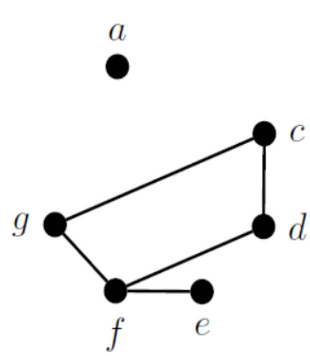
✓ Consider graph G_2 from page 169. We have already shown that {b, h} is a cut-set and {ab, ah} is an edge-cut. If we want to separate b and c then we cannot use b in the separating set and using the edges ab and ah will only isolate a, leaving b and c in the same component.

✓ We can separate b and c using the vertices {f, h} and the edges {cd, cg, ch}.

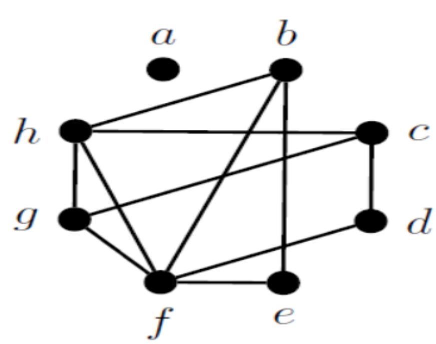
✓ Note that you cannot separate b and c with fewer or edges.



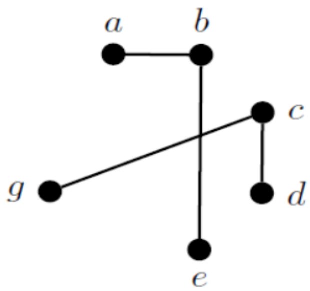
G_2



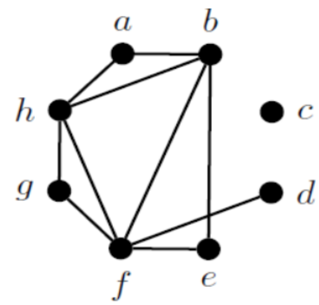
$G_2 - \{b, h\}$



$G_2 - \{ab, ah\}$



$G_2 - \{f, h\}$



$G_2 - \{cd, cg, ch\}$

Menger's Theorem:

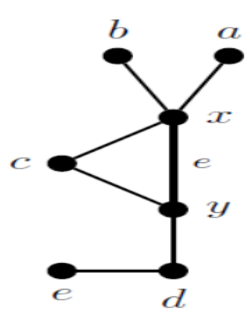
- The following theorems generalize the results relating a cut-vertex or bridge to paths in a graph.
- Menger's Theorem, and the resulting theorems, show the number of internally disjoint (or edge-disjoint) paths directly corresponds to the connectivity (or edge-connectivity) of a graph.

Intuitive Idea:

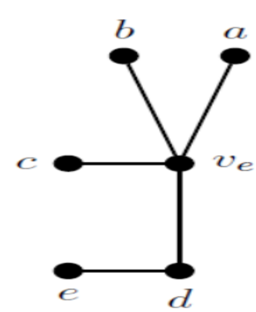
For example, in G_2 above we could separate *b* and *c* using two vertices and it should be easy to see that *b h c* and *b e f d c* are internally disjoint *b* − *c* paths.

- ✓ However, if we try to find more than two *b* − *c* paths then one of them cannot be internally disjoint from the others.

Definition 4.12 Let $e = xy$ be an edge of a graph G. The ***contraction*** of e, denoted G/e, replaces the edge e with a vertex v_e so that any vertices adjacent to either x or y are now adjacent to v_e.



G_4



G_4/e

Remarks:

- Contracting an edge creates a smaller graph, both in terms of the number of vertices and edges but keeps much of the structure of a graph intact.
- In particular, contracting an edge cannot disconnect a graph.

Menger's Theorem Statement:

Theorem 4.13 (Menger's Theorem) Let x and y be nonadjacent vertices in G. Then the minimum number of vertices that separate x and y equals the maximum number of internally disjoint $x - y$ paths in G.

- ✓ An immediate result from Menger's Theorem refers to the global condition of connectivity as opposed to the separation of two specific vertices.

Theorem 4.14 A nontrivial graph G is k-connected if and only if for each pair of distinct vertices x and y there are at least k internally disjoint $x - y$ paths.

- ✓ Now an edge version exists for the two previous theorems.

Theorem 4.15 Let x and y be distinct vertices in G. Then the minimum number of edges that separate x and y equals the maximum number of edge-disjoint $x - y$ paths in G.

Theorem 4.16 A nontrivial graph G is k-edge-connected if and only if for each pair of distinct vertices x and y there are at least k edge disjoint $x - y$ paths.

Remark:

- When we are investigating graphs that are not trees, Menger's Theroem (and the resulting theorems) allow us to conduct similar analyses (about connectivity & paths).
- Where the level to which a graph is connected is equal to the number of paths that would need to be broken in order to separate two vertices.