CHAPTER 5: EIGENVALUES AND EIGENVECTORS

5.1 Eigenvalues and Eigenvectors

1. $A\mathrm{x} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1\mathrm{x}$ therefore x is an eigenvector of A corresponding to the eigenvalue -1.

2. $A\mathrm{x} = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\mathrm{x}$ therefore x is an eigenvector of A corresponding to the eigenvalue 4.

3. $A\mathrm{x} = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 5\mathrm{x}$ therefore x is an eigenvector of A corresponding to the eigenvalue 5.

4. $A\mathrm{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\mathrm{x}$ therefore x is an eigenvector of A corresponding to the eigenvalue 0.

5. **(a)** $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 3) - (-4)(-2) = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$.

The characteristic equation is $(\lambda - 5)(\lambda + 1) = 0$. The eigenvalues are $\lambda = 5$ and $\lambda = -1$.

The reduced row echelon form of $5I - A = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. The general solution of $(5I - A)\mathrm{x} = 0$ is $x_1 = t$, $x_2 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = 5$ is $\{(1,1)\}$.

The reduced row echelon form of $-1I - A = \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix}$ is $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. The general solution of $(-1I - A)\mathrm{x} = 0$ is $x_1 = -2t$, $x_2 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = -1$ is $\{(-2,1)\}$.

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda + 2 & 7 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda + 2)(\lambda - 2) - (7)(-1) = \lambda^2 + 3$.

The characteristic equation is $\lambda^2 + 3 = 0$. There are no real eigenvalues.

(c) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2$.

The characteristic equation is $(\lambda - 1)^2 = 0$. The eigenvalue is $\lambda = 1$.

The matrix $I - A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is already in reduced row echelon form. The general solution of $(I - A)\mathrm{x} = 0$ is $x_1 = s$, $x_2 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = 1$ is $\{(1,0),(0,1)\}$.

(d) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 2 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2$.

The characteristic equation is $(\lambda - 1)^2 = 0$. The eigenvalue is $\lambda = 1$.

The reduced row echelon form of $I - A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The general solution of $(I - A)\mathrm{x} = 0$ is $x_1 = t$, $x_2 = 0$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = 1$ is $\{(1,0)\}$.

6. **(a)** $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 - (-1)^2 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$.

The characteristic equation is $(\lambda - 1)(\lambda - 3) = 0$. The eigenvalues are $\lambda = 1$ and $\lambda = 3$.

The reduced row echelon form of $I - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. The general solution of $(I - A)\mathrm{x} = 0$ is $x_1 = -t$, $x_2 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = 1$ is $\{(-1,1)\}$.

The reduced row echelon form of $3I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. The general solution of $(3I - A)\mathrm{x} = 0$ is $x_1 = t$, $x_2 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = 3$ is $\{(1,1)\}$.

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -3 \\ 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2$.

The characteristic equation is $(\lambda - 2)^2 = 0$. The eigenvalue is $\lambda = 2$.

The reduced row echelon form of $2I - A = \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The general solution of

$(2I - A)\mathrm{x} = 0$ is $x_1 = t$, $x_2 = 0$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = 2$ is $\{(1,0)\}$.

(c) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2$.

The characteristic equation is $(\lambda - 2)^2 = 0$. The eigenvalue is $\lambda = 2$.

The matrix $2I - A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is already in reduced row echelon form.

The general solution of $(2I - A)\mathrm{x} = 0$ is $x_1 = s$, $x_2 = t$. In vector form,

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda = 2$ is $\{(1,0),(0,1)\}$.

(d) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 \\ 2 & \lambda + 1 \end{vmatrix} = (\lambda - 1)(\lambda + 1) - (-2)(2) = \lambda^2 + 3$.

The characteristic equation is $\lambda^2 + 3 = 0$. There are no real eigenvalues.

7. Cofactor expansion along the second column yields $\det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & 0 & -1 \\ 2 & \lambda - 1 & 0 \\ 2 & 0 & \lambda - 1 \end{vmatrix}$

$= (\lambda - 1)\begin{vmatrix} \lambda - 4 & -1 \\ 2 & \lambda - 1 \end{vmatrix} = (\lambda - 1)\left[(\lambda - 4)(\lambda - 1) - (-1)(2)\right] = (\lambda - 1)(\lambda^2 - 5\lambda + 6)$

$= (\lambda - 1)(\lambda - 2)(\lambda - 3)$. The characteristic equation is $(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$.

The eigenvalues are $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$.

The reduced row echelon form of $I - A = \begin{bmatrix} -3 & 0 & -1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of

$(I - A)\mathrm{x} = 0$ is $x_1 = 0$, $x_2 = t$, $x_3 = 0$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. A basis for the eigenspace

corresponding to $\lambda = 1$ is $\{(0,1,0)\}$.

The reduced row echelon form of $2I-A=\begin{bmatrix}-2 & 0 & -1\\ 2 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$. The general solution of

$(2I-A)\mathrm{x}=0$ is $x_1=-\frac{1}{2}t$, $x_2=t$, $x_3=t$. In vector form, $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}t\\ t\\ t\end{bmatrix}=t\begin{bmatrix}-\frac{1}{2}\\ 1\\ 1\end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda=2$ is $\{(-1,2,2)\}$ (scaled by a factor of 2 for convenience).

The reduced row echelon form of $3I-A=\begin{bmatrix}-1 & 0 & -1\\ 2 & 2 & 0\\ 2 & 0 & 2\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$. The general solution of

$(3I-A)\mathrm{x}=0$ is $x_1=-t$, $x_2=t$, $x_3=t$. In vector form, $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-t\\ t\\ t\end{bmatrix}=t\begin{bmatrix}-1\\ 1\\ 1\end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda=3$ is $\{(-1,1,1)\}$.

8. Cofactor expansion along the second column yields $\det(\lambda I-A)=\begin{vmatrix}\lambda-1 & 0 & 2\\ 0 & \lambda & 0\\ 2 & 0 & \lambda-4\end{vmatrix}$

$=\lambda\begin{vmatrix}\lambda-1 & 2\\ 2 & \lambda-4\end{vmatrix}=\lambda\left[(\lambda-1)(\lambda-4)-(2)(2)\right]=\lambda\left(\lambda^2-5\lambda\right)=\lambda^2(\lambda-5)$.

The characteristic equation is $\lambda^2(\lambda-5)=0$. The eigenvalues are $\lambda=0$ and $\lambda=5$.

The reduced row echelon form of $0I-A=\begin{bmatrix}-1 & 0 & 2\\ 0 & 0 & 0\\ 2 & 0 & -4\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & -2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$. The general solution of

$(0I-A)\mathrm{x}=0$ is $x_1=2t$, $x_2=s$, $x_3=t$. In vector form, $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}2t\\ s\\ t\end{bmatrix}=s\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}2\\ 0\\ 1\end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda=0$ is $\{(0,1,0),(2,0,1)\}$.

The reduced row echelon form of $5I-A=\begin{bmatrix}4 & 0 & 2\\ 0 & 5 & 0\\ 2 & 0 & 1\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$. The general solution of

$(5I-A)\mathrm{x}=0$ is $x_1=-\frac{1}{2}t$, $x_2=0$, $x_3=t$. In vector form, $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}t\\ 0\\ t\end{bmatrix}=t\begin{bmatrix}-\frac{1}{2}\\ 0\\ 1\end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda=5$ is $\{(-1,0,2)\}$ (scaled by a factor of 2 for convenience).

9. Cofactor expansion along the second row yields $\det(\lambda I - A) = \begin{vmatrix} \lambda-6 & -3 & 8 \\ 0 & \lambda+2 & 0 \\ -1 & 0 & \lambda+3 \end{vmatrix}$

$= (\lambda+2)\begin{vmatrix} \lambda-6 & 8 \\ -1 & \lambda+3 \end{vmatrix} = (\lambda+2)\left[(\lambda-6)(\lambda+3)-(8)(-1)\right] = (\lambda+2)(\lambda^2-3\lambda-10)$

$= (\lambda+2)(\lambda+2)(\lambda-5)$. The characteristic equation is $(\lambda+2)^2(\lambda-5)=0$.

The eigenvalues are $\lambda=-2$ and $\lambda=5$.

The reduced row echelon form of $-2I-A = \begin{bmatrix} -8 & -3 & 8 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of

$(-2I-A)\mathrm{x}=0$ is $x_1=t$, $x_2=0$, $x_3=t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace

corresponding to $\lambda=-2$ is $\{(1,0,1)\}$.

The reduced row echelon form of $5I-A = \begin{bmatrix} -1 & -3 & 8 \\ 0 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of

$(5I-A)\mathrm{x}=0$ is $x_1=8t$, $x_2=0$, $x_3=t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace

corresponding to $\lambda=5$ is $\{(8,0,1)\}$.

10. Cofactor expansion along the first row yields $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix}$

$= \lambda\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & \lambda \end{vmatrix} + (-1)\begin{vmatrix} -1 & \lambda \\ -1 & -1 \end{vmatrix} = \lambda(\lambda^2-1)+(-\lambda-1)-(1+\lambda)$

$= \lambda(\lambda-1)(\lambda+1)-(\lambda+1)-(\lambda+1) = (\lambda+1)\left[\lambda(\lambda-1)-1-1\right] = (\lambda+1)(\lambda^2-\lambda-2) = (\lambda+1)(\lambda+1)(\lambda-2)$.

We conclude that the eigenvalues are -1 and 2.

The reduced row echelon form of $-1I-A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of

$(-1I-A)\mathrm{x}=0$ is $x_1=-s-t$, $x_2=s$, $x_3=t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda=-1$ is $\{(-1,1,0),(-1,0,1)\}$.

The reduced row echelon form of $2I - A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

The general solution of $(2I - A)\mathrm{x} = 0$ is $x_1 = t$, $x_2 = t$, $x_3 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = 2$ is $\{(1,1,1)\}$.

11. Cofactor expansion along the second column yields $\det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & 0 & 1 \\ 0 & \lambda - 3 & 0 \\ -1 & 0 & \lambda - 2 \end{vmatrix}$

$= (\lambda - 3)\begin{vmatrix} \lambda - 4 & 1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 3)\left[(\lambda - 4)(\lambda - 2) - (1)(-1)\right] = (\lambda - 3)(\lambda^2 - 6\lambda + 9) = (\lambda - 3)^3$.

The characteristic equation is $(\lambda - 3)^3 = 0$. The eigenvalue is $\lambda = 3$.

The reduced row echelon form of $3I - A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The general solution of $(3I - A)\mathrm{x} = 0$ is $x_1 = t$, $x_2 = s$, $x_3 = t$. In vector form,

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda = 3$ is $\{(0,1,0),(1,0,1)\}$.

12. We use the arrow technique to evaluate the determinant: $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 3 & -3 \\ -3 & \lambda + 5 & -3 \\ -6 & 6 & \lambda - 4 \end{vmatrix}$

$= \left[(\lambda - 1)(\lambda + 5)(\lambda - 4) + 54 + 54\right] - \left[18(\lambda + 5) - 18(\lambda - 1) - 9(\lambda - 4)\right] = \lambda^3 - 12\lambda - 16$.

The characteristic equation is $\lambda^3 - 12\lambda - 16 = 0$.

Following the procedure described in Example 3, we determine that the only integer solutions of the characteristic equation are $\pm 1, \pm 2, \pm 4, \pm 8,$ and ± 16. Successively substituting these into the characteristic polynomial, we find $\det(-2I - A) = 0$ so that $\lambda + 2$ must be a factor of the polynomial. Dividing $\lambda + 2$ into $\lambda^3 - 12\lambda - 16$ we obtain

$\det(\lambda I - A) = (\lambda + 2)(\lambda^2 - 2\lambda - 8) = (\lambda + 2)(\lambda + 2)(\lambda - 4)$.

We conclude that the eigenvalues are -2 and 4.

The reduced row echelon form of $-2I-A=\begin{bmatrix}-3 & 3 & -3\\ -3 & 3 & -3\\ -6 & 6 & -6\end{bmatrix}$ is $\begin{bmatrix}1 & -1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$. The general solution of

$(-2I-A)\mathrm{x}=0$ is $x_1=s-t$, $x_2=s$, $x_3=t$. In vector form, $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}s-t\\ s\\ t\end{bmatrix}=s\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda=-2$ is $\{(1,1,0),(-1,0,1)\}$.

The reduced row echelon form of $4I-A=\begin{bmatrix}3 & 3 & -3\\ -3 & 9 & -3\\ -6 & 6 & 0\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & -\frac{1}{2}\\ 0 & 1 & -\frac{1}{2}\\ 0 & 0 & 0\end{bmatrix}$. The general solution of

$(4I-A)\mathrm{x}=0$ is $x_1=\frac{1}{2}t$, $x_2=\frac{1}{2}t$, $x_3=t$. In vector form, $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}\frac{1}{2}t\\ \frac{1}{2}t\\ t\end{bmatrix}=t\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ 1\end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda=4$ is $\{(1,1,2)\}$ (scaled by a factor of 2 for convenience).

13. The matrix $\lambda I-A$ is lower triangular, therefore by Theorem 2.1.2 its determinant is the product of the entries on the main diagonal. Therefore the characteristic equation is $(\lambda-3)(\lambda-7)(\lambda-1)=0$.

14. The matrix $\lambda I-A$ is upper triangular, therefore by Theorem 2.1.2 its determinant is the product of the entries on the main diagonal. Therefore the characteristic equation is $(\lambda-9)(\lambda+1)(\lambda-3)(\lambda-7)=0$.

15. $T(x,y)=\begin{bmatrix}x+4y\\ 2x+3y\end{bmatrix}=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$; the standard matrix for the operator T is $A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$.

The following results were obtained in the solution of Exercise 5(a). These statements apply to the matrix A, therefore they also apply to the associated operator T:

- the eigenvalues are $\lambda=5$ and $\lambda=-1$,
- a basis for the eigenspace corresponding to $\lambda=5$ is $\{(1,1)\}$,
- a basis for the eigenspace corresponding to $\lambda=-1$ is $\{(-2,1)\}$.

16. $T(x,y,z)=\begin{bmatrix}2x-y-z\\ x-z\\ -x+y+2z\end{bmatrix}=\begin{bmatrix}2 & -1 & -1\\ 1 & 0 & -1\\ -1 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$; the standard matrix for T is $A=\begin{bmatrix}2 & -1 & -1\\ 1 & 0 & -1\\ -1 & 1 & 2\end{bmatrix}$.

We use the arrow technique to evaluate the determinant: $\det(\lambda I-A)=\begin{vmatrix}\lambda-2 & 1 & 1\\ -1 & \lambda & 1\\ 1 & -1 & \lambda-2\end{vmatrix}$

$=\left[(\lambda-2)\lambda(\lambda-2)+1+1\right]-\left[\lambda-(\lambda-2)-(\lambda-2)\right]=\lambda^3-4\lambda^2+5\lambda-2$.

The characteristic equation is $\lambda^3-4\lambda^2+5\lambda-2=0$.

Following the procedure described in Example 3, we determine that the only possible integer solutions of the characteristic equation are ±1 and ±2.

Since $\det(I-A)=0$, $\lambda-1$ must be a factor of the characteristic polynomial. Dividing $\lambda-1$ into $\lambda^3-4\lambda^2+5\lambda-2$ leads to $\det(\lambda I-A)=(\lambda-1)(\lambda^2-3\lambda+2)=(\lambda-1)(\lambda-1)(\lambda-2)$.

We conclude that the eigenvalues are 1 and 2.

The reduced row echelon form of $1I-A=\begin{bmatrix}-1 & 1 & 1\\ -1 & 1 & 1\\ 1 & -1 & -1\end{bmatrix}$ is $\begin{bmatrix}1 & -1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$. The general solution of $(1I-A)\mathrm{x}=0$ is $x_1=s+t$, $x_2=s$, $x_3=t$. In vector form, $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}s+t\\ s\\ t\end{bmatrix}=s\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda=1$ is $\{(1,1,0),(1,0,1)\}$.

The reduced row echelon form of $2I-A=\begin{bmatrix}0 & 1 & 1\\ -1 & 2 & 1\\ 1 & -1 & 0\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$. The general solution of $(2I-A)\mathrm{x}=0$ is $x_1=-t$, $x_2=-t$, $x_3=t$. In vector form, $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-t\\ -t\\ t\end{bmatrix}=t\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda=2$ is $\{(-1,-1,1)\}$.

17. **(a)** The transformation D^2 maps any function $\mathbf{f}=f(x)$ in $C^\infty(-\infty,\infty)$ into its second derivative, i.e. $D^2(\mathbf{f})=f''(x)$. From calculus, we have

$$D^2(\mathbf{f}+\mathbf{g})=\frac{d}{dx}\frac{d}{dx}\big(f(x)+g(x)\big)=\frac{d}{dx}\big(f'(x)+g'(x)\big)=f''(x)+g''(x)=D^2(\mathbf{f})+D^2(\mathbf{g}) \text{ and}$$

$$D^2(k\mathbf{f})=\frac{d}{dx}\frac{d}{dx}\big(kf(x)\big)=\frac{d}{dx}\big(kf'(x)\big)=kf''(x)=kD^2(\mathbf{f}).$$ We conclude that D^2 is linear.

(b) Denote $\mathbf{f}=\sin\sqrt{\omega}x$ and $\mathbf{g}=\cos\sqrt{\omega}x$. We have

$$D^2(\mathbf{f})=\frac{d}{dx}\frac{d}{dx}\big(\sin\sqrt{\omega}x\big)=\frac{d}{dx}\big(\sqrt{\omega}\cos\sqrt{\omega}x\big)=\big(\sqrt{\omega}\big)\big(-\sqrt{\omega}\sin\sqrt{\omega}x\big)=-\omega\sin\sqrt{\omega}x=-\omega\mathbf{f} \text{ and}$$

$$D^2(\mathbf{g})=\frac{d}{dx}\frac{d}{dx}\big(\cos\sqrt{\omega}x\big)=\frac{d}{dx}\big(-\sqrt{\omega}\sin\sqrt{\omega}x\big)=\big(-\sqrt{\omega}\big)\big(\sqrt{\omega}\cos\sqrt{\omega}x\big)=-\omega\cos\sqrt{\omega}x=-\omega\mathbf{g}.$$

It follows that $\mathbf{f}=\sin\sqrt{\omega}x$ and $\mathbf{g}=\cos\sqrt{\omega}x$ are eigenvectors of D^2; $\lambda=-\omega$ is the eigenvalue associated with both of these eigenvectors.

18. Denote $\mathbf{f}=\sinh\sqrt{\omega}x$ and $\mathbf{g}=\cosh\sqrt{\omega}x$. We have

$$D^2(\mathbf{f})=\frac{d}{dx}\frac{d}{dx}\big(\sinh\sqrt{\omega}x\big)=\frac{d}{dx}\big(\sqrt{\omega}\cosh\sqrt{\omega}x\big)=\big(\sqrt{\omega}\big)\big(\sqrt{\omega}\sinh\sqrt{\omega}x\big)=\omega\sinh\sqrt{\omega}x=\omega\mathbf{f} \text{ and}$$

$$D^2(\mathbf{g})=\frac{d}{dx}\frac{d}{dx}\big(\cosh\sqrt{\omega}x\big)=\frac{d}{dx}\big(\sqrt{\omega}\sinh\sqrt{\omega}x\big)=\big(\sqrt{\omega}\big)\big(\sqrt{\omega}\cosh\sqrt{\omega}x\big)=\omega\cosh\sqrt{\omega}x=\omega\mathbf{g}.$$

It follows that $\mathbf{f} = \sinh\sqrt{\omega}x$ and $\mathbf{g} = \cosh\sqrt{\omega}x$ are eigenvectors of D^2; $\lambda = \omega$ is the eigenvalue associated with both of these eigenvectors.

19. **(a)** The reflection of any vector on the line $y = x$ is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,1)\}$.
The reflection of any vector perpendicular to the line $y = x$ (i.e., on the line $y = -x$) is the negative of the original vector: an eigenvalue $\lambda = -1$ corresponds to the eigenspace $\text{span}\{(-1,1)\}$.

(b) The projection of any vector on the x-axis is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,0)\}$.
The projection of any vector perpendicular to the x-axis (i.e., on the y-axis) is the zero vector: an eigenvalue $\lambda = 0$ corresponds to the eigenspace $\text{span}\{(0,1)\}$.

(c) The result of the rotation through $90°$ of a nonzero vector is never a scalar multiple of the original vector. Consequently, this operator has no real eigenvalues.

(d) The result of the contraction of any vector $\mathbf{v}$ is a scalar multiple $k\mathbf{v}$ therefore the only eigenvalue is $\lambda = k$ and the corresponding eigenspace is the entire space R^2.

(e) The result of the shear applied to any vector on the x-axis is the same vector whereas the result of the shear applied to a nonzero vector in any other direction is not a scalar multiple of the original vector. The only eigenvalue is $\lambda = 1$ and the corresponding eigenspace is $\text{span}\{(1,0)\}$.

20. **(a)** The reflection of any vector on the y-axis is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(0,1)\}$.
The reflection of any vector perpendicular to the y-axis (i.e., on the x-axis) is the negative of the original vector: an eigenvalue $\lambda = -1$ corresponds to the eigenspace $\text{span}\{(1,0)\}$.

(b) The result of the rotation through $180°$ of any vector is the negative of the original vector: the only eigenvalue is $\lambda = -1$ and the corresponding eigenspace is the entire space R^2.

(c) The result of the dilation of any vector $\mathbf{v}$ is a scalar multiple $k\mathbf{v}$ therefore the only eigenvalue is $\lambda = k$ and the corresponding eigenspace is the entire space R^2.

(d) The result of the expansion applied to any vector on the x-axis is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,0)\}$.
The result of the expansion applied to any vector $\mathbf{v}$ on the y-axis is the scalar multiple $k\mathbf{v}$: an eigenvalue $\lambda = k$ corresponds to the eigenspace $\text{span}\{(0,1)\}$.

(e) The result of the shear applied to any vector on the y-axis is the same vector whereas the result of the shear applied to a nonzero vector in any other direction is not a scalar multiple of the original vector. The only eigenvalue is $\lambda = 1$ and the corresponding eigenspace is $\text{span}\{(0,1)\}$.

21. **(a)** The reflection of any vector on the xy-plane is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,0,0),(0,1,0)\}$.

The reflection of any vector perpendicular to the xy-plane (i.e., on the z-axis) is the negative of the original vector: an eigenvalue $\lambda = -1$ corresponds to the eigenspace $\text{span}\{(0,0,1)\}$.

(b) The projection of any vector on the xz-plane is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,0,0),(0,0,1)\}$.
The projection of any vector perpendicular to the xz-plane (i.e., on the y-axis) is the zero vector: an eigenvalue $\lambda = 0$ corresponds to the eigenspace $\text{span}\{(0,1,0)\}$.

(c) The result of the rotation applied to any vector on the x-axis is the same vector whereas the result of the rotation applied to a nonzero vector in any other direction is not a scalar multiple of the original vector. The only eigenvalue is $\lambda = 1$ and the corresponding eigenspace is $\text{span}\{(1,0,0)\}$.

(d) The result of the contraction of any vector $\mathbf{v}$ is a scalar multiple $k\mathbf{v}$ therefore the only eigenvalue is $\lambda = k$ and the corresponding eigenspace is the entire space R^3.

22. **(a)** The reflection of any vector on the xz-plane is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,0,0),(0,0,1)\}$.
The reflection of any vector perpendicular to the xz-plane (i.e., on the y-axis) is the negative of the original vector: an eigenvalue $\lambda = -1$ corresponds to the eigenspace $\text{span}\{(0,1,0)\}$.

(b) The projection of any vector on the yz-plane is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(0,1,0),(0,0,1)\}$.
The projection of any vector perpendicular to the yz-plane (i.e., on the x-axis) is the zero vector: an eigenvalue $\lambda = 0$ corresponds to the eigenspace $\text{span}\{(1,0,0)\}$.

(c) The result of the rotation applied to any vector on the y-axis is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(0,1,0)\}$.
The result of the rotation applied to any vector perpendicular to the y-axis (i.e., on the xz-plane) is the negative of the original vector: an eigenvalue $\lambda = -1$ corresponds to the eigenspace $\text{span}\{(1,0,0),(0,0,1)\}$.

(d) The result of the dilation of any vector $\mathbf{v}$ is a scalar multiple $k\mathbf{v}$ therefore the only eigenvalue is $\lambda = k$ and the corresponding eigenspace is the entire space R^3.

23. A line through the origin in the direction of $\mathbf{x} \neq 0$ is invariant under A if and only if $\mathbf{x}$ is an eigenvector of A.

(a) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & 1 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda - 4)(\lambda - 1) - (1)(-2) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$.

The characteristic equation is $(\lambda - 2)(\lambda - 3) = 0$. The eigenvalues are $\lambda = 2$ and $\lambda = 3$.

The reduced row echelon form of $2I - A = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$. The general solution of $(2I - A)\mathrm{x} = 0$ is $x = \frac{1}{2}t$, $y = t$. Therefore $y = 2x$ is an equation of the corresponding invariant line.

The reduced row echelon form of $3I - A = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. The general solution of $(3I - A)\mathrm{x} = 0$ is $x_1 = t$, $x_2 = t$. Therefore $y = x$ is an equation of the corresponding invariant line.

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - (-1)(1) = \lambda^2 + 1$.

There are no real eigenvalues and no invariant lines.

24. Since $p(\lambda) = \det(\lambda I - A)$, it follows that $p(0) = \det(-A) = (-1)^n \det(A)$.

(a) $n = 3$ and $p(0) = 5$ therefore $\det(A) = -5$.

(b) $n = 4$ and $p(0) = 7$ therefore $\det(A) = 7$.

25. **(a)** Since the degree of $p(\lambda)$ is 6, A is a 6×6 matrix (see Exercise 37).

(b) $p(0) \neq 0$, therefore 0 is not an eigenvalue of A. From parts (a) and (r) of Theorem 5.1.5, A is invertible.

(c) A has three eigenspaces since it has three distinct eigenvalues, each corresponding to an eigenspace.

27. Substituting the given eigenvectors $\mathbf{x}$ and the corresponding eigenvalues λ into $A\mathbf{x} = \lambda\mathbf{x}$ yields

$$A\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},\ A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \text{ and } A\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can combine these three equations into a single equation $A\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Since the matrix $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ is invertible, we can multiply both sides on the right by its inverse,

$\begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & -1 \end{bmatrix}$, resulting in $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

(Note that this exercise could also be solved by assigning nine unknown values to the elements of $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then solving the system of nine equations in nine unknowns resulting from the equation $A\mathbf{x} = \lambda\mathbf{x}$.)

28. Denoting $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we have $\det(\lambda I - A)=\begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix}$

$=(\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} = \lambda^2 - \underbrace{(a_{11}+a_{22})}_{\mathrm{tr}(A)}\lambda + \underbrace{a_{11}a_{22} - a_{12}a_{22}}_{\det(A)}$.

31. It follows from Exercise 28 that if the characteristic polynomial $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is $p(\lambda)=\lambda^2 + c_1\lambda + c_2$ then $c_1 = -\mathrm{tr}(A) = -a_{11} - a_{22}$ and $c_2 = \det(A) = a_{11}a_{22} - a_{12}a_{21}$. Therefore

$p(A) = A^2 + c_1 A + c_2 I$

$$=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + (-a_{11}-a_{22})\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + (a_{11}a_{22} - a_{12}a_{21})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{21}a_{12} + a_{22}^2 \end{bmatrix} + \begin{bmatrix} -a_{11}^2 - a_{22}a_{11} & -a_{11}a_{12} - a_{22}a_{12} \\ -a_{11}a_{21} - a_{22}a_{21} & -a_{11}a_{22} - a_{22}^2 \end{bmatrix}$$

$$+\begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

33. By Theorem 5.1.5(r), it follows from A being invertible that A cannot have a zero eigenvalue. Multiplying both sides of the equation $A\mathbf{x} = \lambda\mathbf{x}$ by A^{-1} on the left and applying Theorem 1.4.1 yields $A^{-1}A\mathbf{x} = \lambda A^{-1}\mathbf{x}$. Since $A^{-1}A = I$, dividing both sides of the equation by λ we obtain $\frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}$. This shows that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} associated with an eigenvector $\mathbf{x}$.

34. Subtracting $sI\mathbf{x} = s\mathbf{x}$ from both sides of the equation $A\mathbf{x} = \lambda\mathbf{x}$ and applying Theorem 1.4.1 yields $(A - sI)\mathbf{x} = (\lambda - s)\mathbf{x}$. This shows that $\lambda - s$ is an eigenvalue of $A - sI$ associated with the eigenvector $\mathbf{x}$.

35. Multiplying both sides of the equation $A\mathbf{x} = \lambda\mathbf{x}$ by the scalar s yields $s(A\mathbf{x}) = s(\lambda\mathbf{x})$. By Theorem 1.4.1, the equation can be rewritten as $(sA)\mathbf{x} = (s\lambda)\mathbf{x}$. This shows that $s\lambda$ is an eigenvalue of sA associated with the eigenvector $\mathbf{x}$.

36. $\det(\lambda I - A) = \begin{vmatrix} \lambda+2 & -2 & -3 \\ 2 & \lambda-3 & -2 \\ 4 & -2 & \lambda-5 \end{vmatrix} = \lambda^3 - 6\lambda^2 + 11\lambda - 6$;

The only possibilities for integer solutions of the characteristic equation are $\pm 1, \pm 2, \pm 3, \pm 6$. Since $\det(1I - A) = 0$, we divide $\lambda - 1$ into $\lambda^3 - 6\lambda^2 + 11\lambda - 6$ to obtain

$\det(\lambda I - A) = (\lambda - 1)(\lambda^2 - 5\lambda + 6) = (\lambda - 1)(\lambda - 2)(\lambda - 3)$.

The reduced row echelon form of $I-A=\begin{bmatrix}3 & -2 & -3\\ 2 & -2 & -2\\ 4 & -2 & -4\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$.

The general solution of $(I-A)\mathrm{x}=0$ is $x_1=t$, $x_2=0$, $x_3=t$.

A basis for the eigenspace of the matrix A corresponding to $\lambda=1$ is $\{(1,0,1)\}$.

The reduced row echelon form of $2I-A=\begin{bmatrix}4 & -2 & -3\\ 2 & -1 & -2\\ 4 & -2 & -3\end{bmatrix}$ is $\begin{bmatrix}1 & -\frac{1}{2} & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$.

The general solution of $(2I-A)\mathrm{x}=0$ is $x_1=\dfrac{1}{2}t$, $x_2=t$, $x_3=0$.

A basis for the eigenspace of the matrix A corresponding to $\lambda=2$ is $\{(1,2,0)\}$.

The reduced row echelon form of $3I-A=\begin{bmatrix}5 & -2 & -3\\ 2 & 0 & -2\\ 4 & -2 & -2\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$.

The general solution of $(3I-A)\mathrm{x}=0$ is $x_1=t$, $x_2=t$, $x_3=t$.

A basis for the eigenspace of the matrix A corresponding to $\lambda=3$ is $\{(1,1,1)\}$.

(a) From the result of Exercise 33, the matrix A^{-1} has

- eigenvalue 1 with a basis for the corresponding eigenspace $\{(1,0,1)\}$,
- eigenvalue $\frac{1}{2}$ with a basis for the corresponding eigenspace $\{(1,2,0)\}$,
- eigenvalue $\frac{1}{3}$ with a basis for the corresponding eigenspace $\{(1,1,1)\}$.

(b) From the result of Exercise 34, the matrix $A-3I$ has

- eigenvalue -2 with a basis for the corresponding eigenspace $\{(1,0,1)\}$,
- eigenvalue -1 with a basis for the corresponding eigenspace $\{(1,2,0)\}$,
- eigenvalue 0 with a basis for the corresponding eigenspace $\{(1,1,1)\}$.

(c) From the result of Exercise 34, the matrix $A+2I$ has

- eigenvalue 3 with a basis for the corresponding eigenspace $\{(1,0,1)\}$,
- eigenvalue 4 with a basis for the corresponding eigenspace $\{(1,2,0)\}$,
- eigenvalue 5 with a basis for the corresponding eigenspace $\{(1,1,1)\}$.

True-False Exercises

(a) False. The vector $\mathbf{x}$ must be nonzero (without that requirement, $A\mathbf{x}=\lambda\mathbf{x}$ holds true for <u>all</u> $n\times n$ matrices A and <u>all</u> values λ by taking $\mathbf{x}=\mathbf{0}$).

(b) False. If λ is an eigenvalue of A then $(\lambda I-A)\mathbf{x}=\mathbf{0}$ must have nontrivial solutions.

(c) True. Since $p(0)=1\neq 0$, zero is not an eigenvalue of A. By Theorem 5.1.5(r), we conclude that A is invertible.

(d) False. Every eigenspace must include the zero vector, which is not an eigenvector.

(e) False. E.g., the only eigenvalue of $A=2I$ is 2. However, the reduced row echelon form of A is I, whose only eigenvalue is 1.

(f) False. By Theorem 5.1.5(h), the set of columns of A must be linearly dependent.

5.2 Diagonalization

1. $\begin{vmatrix}1 & 1\\ 3 & 2\end{vmatrix}=-1$ does not equal $\begin{vmatrix}1 & 0\\ 3 & -2\end{vmatrix}=-2$ therefore, by Table 1 in Section 5.2, A and B are not similar matrices.

2. $\begin{vmatrix}4 & -1\\ 2 & 4\end{vmatrix}=18$ does not equal $\begin{vmatrix}4 & 1\\ 2 & 4\end{vmatrix}=14$ therefore, by Table 1 in Section 5.2, A and B are not similar matrices.

3. $\begin{vmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{vmatrix}=(1)(1)(1)=1$ does not equal $\begin{vmatrix}1 & 2 & 0\\ \frac{1}{2} & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=0-0+1\begin{vmatrix}1 & 2\\ \frac{1}{2} & 1\end{vmatrix}=0$ therefore, by Table 1 in Section 5.2, A and B are not similar matrices.

4. The reduced row echelon forms of the matrices A and B are $\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$, respectively. Since $\operatorname{rank}(A)=1$ does not equal $\operatorname{rank}(B)=2$, by Table 1 in Section 5.2, A and B are not similar matrices.

5. $\det(\lambda I-A)=\begin{vmatrix}\lambda-1 & 0\\ -6 & \lambda+1\end{vmatrix}=(\lambda-1)(\lambda+1)$ therefore A has eigenvalues 1 and -1.

The reduced row echelon form of $1I-A$ is $\begin{bmatrix}1 & -\frac{1}{3}\\ 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1=1$ consists of vectors $\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ where $x_1=\frac{1}{3}t$, $x_2=t$. A vector $\mathrm{p}_1=\begin{bmatrix}1\\ 3\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-1I-A$ is $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_2 = -1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = 0$, $x_2 = t$. A vector $\mathrm{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix P using the column vectors p_1 and p_2: $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the two columns can be interchanged.)

Calculating $P^{-1} = \frac{1}{(1)(1)-(0)(3)} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ and performing matrix multiplications we check that $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

6. $\det(\lambda I - A) = \begin{vmatrix} \lambda + 14 & -12 \\ 20 & \lambda - 17 \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$ therefore A has eigenvalues 1 and 2.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & -\frac{4}{5} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{4}{5}t$, $x_2 = t$. A vector $\mathrm{p}_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{3}{4}t$, $x_2 = t$. A vector $\mathrm{p}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix P using the column vectors p_1 and p_2: $P = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the two columns can be interchanged.)

Calculating $P^{-1} = \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}$ and performing matrix multiplications we check that

$$P^{-1}AP = \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

7. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 0 & 2 \\ 0 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 3)^2$ thus A has eigenvalues 2 and 3 (with algebraic multiplicity 2).

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = 0$, $x_3 = 0$. A vector $\mathrm{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = \lambda_3 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -2t$, $x_2 = s$, $x_3 = t$. We can write

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ therefore vectors $\mathrm{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathrm{p}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace. Note that the geometric multiplicity of this eigenvalue matches its algebraic multiplicity.

We form a matrix P using the column vectors p_1, p_2, and p_3: $P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the columns can be interchanged.)

To invert the matrix P, we can employ the procedure introduced in Section 1.5: since the reduced row echelon form of the matrix $\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$, we have

$$P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We check that $P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.

8. Cofactor expansion along the second column yields $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & -1 & \lambda-1 \end{vmatrix} =$

$(\lambda-1)\begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix} = (\lambda-1)\left[(\lambda-1)^2-1\right] = (\lambda-1)(\lambda-2)\lambda$ thus A has eigenvalues 0, 1 and 2.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 0$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0$, $x_2 = -t$, $x_3 = t$. A vector $\mathrm{p}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = 0$, $x_3 = 0$. A vector $\mathrm{p}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0$, $x_2 = t$, $x_3 = t$. A vector $\mathrm{p}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix P using the column vectors p_1, p_2, and p_3: $P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the columns can be interchanged.)

Calculating $P^{-1}=\begin{bmatrix}0 & -\frac{1}{2} & \frac{1}{2}\\ 1 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$ and performing matrix multiplications we check that

$$P^{-1}AP=\begin{bmatrix}0 & -\frac{1}{2} & \frac{1}{2}\\ 1 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 1 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}.$$

9. **(a)** Cofactor expansion along the second column yields

$$\det(\lambda I-A)=\begin{vmatrix}\lambda-4 & 0 & -1\\ -2 & \lambda-3 & -2\\ -1 & 0 & \lambda-4\end{vmatrix}=(\lambda-3)\begin{vmatrix}\lambda-4 & -1\\ -1 & \lambda-4\end{vmatrix}=(\lambda-3)\left[(\lambda-4)^2-1\right]=(\lambda-3)^2(\lambda-5)$$

therefore A has eigenvalues 3 (with algebraic multiplicity 2) and 5.

(b) The reduced row echelon form of $3I-A$ is $\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$, consequently $\operatorname{rank}(3I-A)=1$.

The reduced row echelon form of $5I-A$ is $\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -2\\ 0 & 0 & 0\end{bmatrix}$, consequently $\operatorname{rank}(5I-A)=2$.

(c) Based on part (b), the geometric multiplicities of the eigenvalues $\lambda=3$ and $\lambda=5$ are $3-1=2$ and $3-2=1$, respectively. Since these are equal to the corresponding algebraic multiplicities, by Theorem 5.2.4(b) A is diagonalizable.

10. **(a)** $\det(\lambda I-A)=\begin{vmatrix}\lambda-3 & 0 & 0\\ 0 & \lambda-2 & 0\\ 0 & -1 & \lambda-2\end{vmatrix}=(\lambda-3)(\lambda-2)^2$ therefore A has eigenvalues 2 (with algebraic multiplicity 2) and 3.

(b) The reduced row echelon form of $2I-A$ is $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$, consequently $\operatorname{rank}(2I-A)=2$.

The reduced row echelon form of $3I-A$ is $\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$, consequently $\operatorname{rank}(3I-A)=2$.

(c) Based on part (b), the geometric multiplicity of the eigenvalue $\lambda=3$ is $3-2=1$, which is equal to the algebraic multiplicity of this eigenvalue. However, the geometric multiplicity of the eigenvalue $\lambda=2$ is $3-2=1$, which is less than the corresponding algebraic multiplicity (2), therefore by Theorem 5.2.4(b) A is not diagonalizable.

11. Cofactor expansion along the second row yields

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & -4 & 2 \\ 3 & \lambda-4 & 0 \\ 3 & -1 & \lambda-3 \end{vmatrix} = -3\begin{vmatrix} -4 & 2 \\ -1 & \lambda-3 \end{vmatrix} + (\lambda-4)\begin{vmatrix} \lambda+1 & 2 \\ 3 & \lambda-3 \end{vmatrix}$$

$$= (-3)\left[(-4)(\lambda-3)-(2)(-1)\right] + (\lambda-4)\left[(\lambda+1)(\lambda-3)-(2)(3)\right] = \lambda^3 - 6\lambda^2 + 11\lambda - 6.$$

Following the procedure described in Example 3 of Section 5.1, we determine that the only possibilities for integer solutions of the characteristic equation are ± 1, ± 2, ± 3, and ± 6. Since $\det(1I - A) = 0$, $\lambda - 1$ must be a factor of the characteristic polynomial. Dividing $\lambda - 1$ into $\lambda^3 - 6\lambda^2 + 11\lambda - 6$ leads to $\det(\lambda I - A) = (\lambda-1)(\lambda^2 - 5\lambda + 6) = (\lambda-1)(\lambda-2)(\lambda-3)$.

We conclude that the eigenvalues are 1, 2, and 3 - each of them has the algebraic multiplicity 1.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$, $x_3 = t$. A vector $\mathrm{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace. This eigenvalue has geometric multiplicity 1.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{2}{3}t$, $x_2 = t$, $x_3 = t$. A vector $\mathrm{p}_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace. This eigenvalue has geometric multiplicity 1.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{1}{4}t$, $x_2 = \frac{3}{4}t$, $x_3 = t$. A vector $\mathrm{p}_3 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace. This eigenvalue has geometric multiplicity 1.

Since for each eigenvalue the geometric multiplicity matches the algebraic multiplicity, by Theorem 5.2.4(b) A is diagonalizable.

We form a matrix P using the column vectors p_1, p_2, and p_3: $P=\begin{bmatrix}1&2&1\\1&3&3\\1&3&4\end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the columns can be interchanged.)

$$P^{-1}AP=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}.$$

12. Since $\det(\lambda I-A)=\begin{vmatrix}\lambda-19&-9&6\\-25&\lambda+11&9\\-17&9&\lambda+4\end{vmatrix}=\lambda^3-4\lambda^2+5\lambda-2=(\lambda-1)^2(\lambda-2)$, A has eigenvalues $\lambda=1$ with the algebraic multiplicity 2 and $\lambda=2$ with the algebraic multiplicity 1.

The reduced row echelon form of $1I-A$ is $\begin{bmatrix}1&0&-1\\0&1&-\frac{4}{3}\\0&0&0\end{bmatrix}$ so that $\lambda=1$ has the geometric multiplicity 1.

The reduced row echelon form of $2I-A$ is $\begin{bmatrix}1&0&-\frac{3}{4}\\0&1&-\frac{3}{4}\\0&0&0\end{bmatrix}$ so that $\lambda=2$ has the geometric multiplicity 1.

Since for $\lambda=1$ the geometric multiplicity and the algebraic multiplicity are not equal, we conclude from Theorem 5.2.4(b) that A is not diagonalizable.

13. $\det(\lambda I-A)=\begin{vmatrix}\lambda&0&0\\0&\lambda&0\\-3&0&\lambda-1\end{vmatrix}=\lambda^2(\lambda-1)$ so the eigenvalues are $\lambda=0$ with the algebraic multiplicity 2 and $\lambda=1$ with the algebraic multiplicity 1.

The reduced row echelon form of $0I-A$ is $\begin{bmatrix}1&0&\frac{1}{3}\\0&0&0\\0&0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1=\lambda_2=0$ contains vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ where $x_1=-\frac{1}{3}t$, $x_2=s$, $x_3=t$. We can write

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-\frac{1}{3}t\\s\\t\end{bmatrix}=s\begin{bmatrix}0\\1\\0\end{bmatrix}+t\begin{bmatrix}-\frac{1}{3}\\0\\1\end{bmatrix}$ therefore vectors $\mathrm{p}_1=\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\mathrm{p}_2=\begin{bmatrix}-1\\0\\3\end{bmatrix}$ form a basis for this eigenspace. This eigenvalue has the geometric multiplicity 2.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0$, $x_2 = 0$, $x_3 = t$. A vector $\mathrm{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace. This eigenvalue has the geometric multiplicity 1.

Since for each eigenvalue the geometric multiplicity matches the algebraic multiplicity, by Theorem 5.2.4(b) A is diagonalizable.

We form a matrix P using the column vectors p_1, p_2, and p_3: $P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$. (Note that this answer is not unique.)

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

14. From Theorem 5.1.2, A has only one eigenvalue $\lambda = 5$ with the algebraic multiplicity 3.

The reduced row echelon form of $5I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that $\lambda = 5$ has the geometric multiplicity 1.

Since the geometric multiplicity and the algebraic multiplicity are not equal, we conclude from Theorem 5.2.4(b) that A is not diagonalizable.

15. **(a)** The degree of the characteristic polynomial of A is 3 therefore A is a 3×3 matrix. All three eigenspaces (for $\lambda = 1$, $\lambda = -3$, and $\lambda = 5$) must have dimension 1.

(b) The degree of the characteristic polynomial of A is 6 therefore A is a 6×6 matrix.
The possible dimensions of the eigenspace corresponding to $\lambda = 0$ are 1 or 2.
The dimension of the eigenspace corresponding to $\lambda = 1$ must be 1.
The possible dimensions of the eigenspace corresponding to $\lambda = 2$ are 1, 2, or 3.

16. **(a)** The degree of the characteristic polynomial of A is 5 therefore A is a 5×5 matrix.
In factored form, $\lambda^3\left(\lambda^2 - 5\lambda - 6\right) = \lambda^3(\lambda - 6)(\lambda + 1)$.

The possible dimensions of the eigenspace corresponding to $\lambda = 0$ are 1, 2, or 3.
The dimension of the eigenspace corresponding to $\lambda = 6$ must be 1.
The dimension of the eigenspace corresponding to $\lambda = -1$ must be 1.

(b) The degree of the characteristic polynomial of A is 3 therefore A is a 3×3 matrix.
In factored form, $\lambda^3-3\lambda^2+3\lambda-1=(\lambda-1)^3$.
The possible dimensions of the eigenspace corresponding to $\lambda=1$ are 1, 2, or 3.

17. $\det(\lambda I-A)=\begin{vmatrix}\lambda & -3\\ -2 & \lambda+1\end{vmatrix}=\lambda(\lambda+1)-(-3)(-2)=\lambda^2+\lambda-6=(\lambda-2)(\lambda+3)$ therefore A has eigenvalues 2 and -3, each with the algebraic multiplicity 1.

The reduced row echelon form of $2I-A$ is $\begin{bmatrix}1 & -\frac{3}{2}\\ 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda=2$ consists of vectors $\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ where $x_1=\frac{3}{2}t$, $x_2=t$. A vector $\mathrm{p}_1=\begin{bmatrix}3\\ 2\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-3I-A$ is $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda=-3$ consists of vectors $\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ where $x_1=-t$, $x_2=t$. A vector $\mathrm{p}_2=\begin{bmatrix}-1\\ 1\end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix $P=\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$ and calculate $P^{-1}=\frac{1}{(3)(1)-(-1)(2)}\begin{bmatrix}1 & 1\\ -2 & 3\end{bmatrix}=\frac{1}{5}\begin{bmatrix}1 & 1\\ -2 & 3\end{bmatrix}=\begin{bmatrix}\frac{1}{5} & \frac{1}{5}\\ -\frac{2}{5} & \frac{3}{5}\end{bmatrix}$ so that $P^{-1}AP=\begin{bmatrix}2 & 0\\ 0 & -3\end{bmatrix}=D$.

Therefore $A^{10}=PD^{10}P^{-1}=\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}2^{10} & 0\\ 0 & (-3)^{10}\end{bmatrix}\begin{bmatrix}\frac{1}{5} & \frac{1}{5}\\ -\frac{2}{5} & \frac{3}{5}\end{bmatrix}=\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}1{,}024 & 0\\ 0 & 59{,}049\end{bmatrix}\begin{bmatrix}\frac{1}{5} & \frac{1}{5}\\ -\frac{2}{5} & \frac{3}{5}\end{bmatrix}$

$=\begin{bmatrix}24{,}234 & -34{,}815\\ -23{,}210 & 35{,}839\end{bmatrix}$.

18. From Theorem 5.1.2, A has eigenvalues 1 and 2, each with the algebraic multiplicity 1.

The reduced row echelon form of $1I-A$ is $\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda=1$ consists of vectors $\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ where $x_1=t$, $x_2=t$. A vector $\mathrm{p}_1=\begin{bmatrix}1\\ 1\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $2I-A$ is $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda=2$ consists of vectors $\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ where $x_1=0$, $x_2=t$. A vector $\mathrm{p}_2=\begin{bmatrix}0\\ 1\end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix $P=\begin{bmatrix}1&0\\1&1\end{bmatrix}$ and calculate $P^{-1}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}$ so that $P^{-1}AP=\begin{bmatrix}1&0\\0&2\end{bmatrix}=D$.

Therefore $A^{10}=PD^{10}P^{-1}=\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}1^{10}&0\\0&2^{10}\end{bmatrix}\begin{bmatrix}1&0\\-1&1\end{bmatrix}=\begin{bmatrix}1&0\\-1023&1024\end{bmatrix}$.

19. To invert the matrix P, we can employ the procedure introduced in Section 1.5: since the reduced row echelon form of the matrix $\begin{bmatrix}1&1&1&1&0&0\\0&0&1&0&1&0\\1&0&5&0&0&1\end{bmatrix}$ is $\begin{bmatrix}1&0&0&0&-5&1\\0&1&0&1&4&-1\\0&0&1&0&1&0\end{bmatrix}$, we have

$$P^{-1}=\begin{bmatrix}0&-5&1\\1&4&-1\\0&1&0\end{bmatrix}.$$

We verify that $P^{-1}AP=\begin{bmatrix}0&-5&1\\1&4&-1\\0&1&0\end{bmatrix}\begin{bmatrix}-1&7&-1\\0&1&0\\0&15&-2\end{bmatrix}\begin{bmatrix}1&1&1\\0&0&1\\1&0&5\end{bmatrix}=\begin{bmatrix}-2&0&0\\0&-1&0\\0&0&1\end{bmatrix}=D$ is a diagonal matrix therefore P diagonalizes A.

$$A^{11}=PD^{11}P^{-1}=\begin{bmatrix}1&1&1\\0&0&1\\1&0&5\end{bmatrix}\begin{bmatrix}(-2)^{11}&0&0\\0&(-1)^{11}&0\\0&0&1^{11}\end{bmatrix}\begin{bmatrix}0&-5&1\\1&4&-1\\0&1&0\end{bmatrix}$$

$$=\begin{bmatrix}1&1&1\\0&0&1\\1&0&5\end{bmatrix}\begin{bmatrix}-2{,}048&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&-5&1\\1&4&-1\\0&1&0\end{bmatrix}=\begin{bmatrix}-1&10{,}237&-2{,}047\\0&1&0\\0&10{,}245&-2{,}048\end{bmatrix}$$

20. After calculating $P^{-1}=\begin{bmatrix}0&1&0\\0&0&1\\1&-1&4\end{bmatrix}$, we verify that

$P^{-1}AP=\begin{bmatrix}0&1&0\\0&0&1\\1&-1&4\end{bmatrix}\begin{bmatrix}1&-2&8\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&-4&1\\1&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}=D$ is a diagonal matrix therefore P diagonalizes A.

(a) $$A^{1000}=PD^{1000}P^{-1}=\begin{bmatrix}1&-4&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}(-1)^{1000}&0&0\\0&(-1)^{1000}&0\\0&0&1^{1000}\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&-1&4\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

(b) $A^{-1000} = PD^{-1000}P^{-1} = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} (-1)^{-1000} & 0 & 0 \\ 0 & (-1)^{-1000} & 0 \\ 0 & 0 & 1^{-1000} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $A^{2301} = PD^{2301}P^{-1} = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} (-1)^{2301} & 0 & 0 \\ 0 & (-1)^{2301} & 0 \\ 0 & 0 & 1^{2301} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(d) $A^{-2301} = PD^{-2301}P^{-1} = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} (-1)^{-2301} & 0 & 0 \\ 0 & (-1)^{-2301} & 0 \\ 0 & 0 & 1^{-2301} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

21. Cofactor expansion along the first row yields $\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-3 \end{vmatrix}$

$= (\lambda-3)\begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & \lambda-3 \end{vmatrix} = (\lambda-3)\left[(\lambda-2)(\lambda-3)-1\right] - (\lambda-3)$

$= (\lambda-3)\left[\lambda^2 - 5\lambda + 6 - 1 - 1\right] = (\lambda-3)(\lambda^2 - 5\lambda + 4) = (\lambda-1)(\lambda-3)(\lambda-4),$

therefore A has eigenvalues 1, 3, and 4, each with the algebraic multiplicity 1.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = 2t$, $x_3 = t$. A vector $\mathrm{p}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -t$, $x_2 = 0$, $x_3 = t$. A vector $\mathrm{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_3 = 4$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = -t$, $x_3 = t$. A vector $\mathrm{p}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix $P = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ and find its inverse using the procedure introduced in Section 1.5. Since the reduced row echelon form of the matrix $\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ is

$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$, we have $P^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

We conclude that $A^n = PD^nP^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

22. Since $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & -1 & \lambda - 1 \end{vmatrix} = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda - 3)$, A has eigenvalues $\lambda = 3$ with the algebraic multiplicity 1 and $\lambda = 0$ with the algebraic multiplicity 2.

The geometric multiplicity of $\lambda_1 = 3$ must be 1.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the geometric multiplicity of $\lambda_2 = \lambda_3 = 0$ is 2.

Since for each eigenvalue the geometric multiplicity matches the algebraic multiplicity, by Theorem 5.2.4(b) A is diagonalizable, i.e., there exists an invertible matrix P such that

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B.$$

Consequently, the matrices A and B are similar.

23. By inspection, both A and B have rank 1 (both matrices are in reduced row echelon form).

Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $AP = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ and $PB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$.

Setting $AP = PB$ requires that $a = 0$, $b = a$, and $c = 0$.

For any value d, the matrix $P=\begin{bmatrix}0 & 0\\ 0 & d\end{bmatrix}$ satisfies the equality $AP=PB$. However, $P=\begin{bmatrix}0 & 0\\ 0 & d\end{bmatrix}$ has zero determinant therefore it is not invertible so that the similarity condition $B=P^{-1}AP$ cannot be met.

24. By inspection, both A and B have only one eigenvalue, $\lambda=1$.

Let $P=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. Then $AP=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}a+c & b+d\\ c & d\end{bmatrix}$ and $PB=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$.

Setting $AP=PB$ requires that $a+c=a$ and $b+d=b$, consequently $c=d=0$.

For any a and b, the matrix $P=\begin{bmatrix}a & b\\ 0 & 0\end{bmatrix}$ satisfies the equality $AP=PB$. However, $P=\begin{bmatrix}a & b\\ 0 & 0\end{bmatrix}$ has zero determinant therefore it is not invertible so that the similarity condition $B=P^{-1}AP$ cannot be met.

25. Since there exist matrices P and Q such that $B=P^{-1}AP$ and $C=Q^{-1}BQ$, we can write $C=Q^{-1}\left(P^{-1}AP\right)Q=\left(PQ\right)^{-1}A\left(PQ\right)$. Consequently, A is similar to C.

26. **(a)** Any square matrix A is similar to itself since $A=I^{-1}AI$.

(b) If $O_{n\times n}=P^{-1}AP$ then $A=PO_{n\times n}P^{-1}=O_{n\times n}$.

(c) Table 1 lists invertibility as one of the similarity invariants. Consequently, a nonsingular matrix cannot be similar to a singular matrix.

27. **(a)** The dimension of the eigenspace must be at least 1, but cannot exceed the algebraic multiplicity of the corresponding eigenvalue. Since the algebraic multiplicities of the eigenvalues $1,3$, and 4 are 1, 2, and 3, respectively, we conclude that

- The dimension of the eigenspace corresponding to $\lambda=1$ must be 1.
- The possible dimensions of the eigenspace corresponding to $\lambda=3$ are 1 or 2.
- The possible dimensions of the eigenspace corresponding to $\lambda=4$ are 1, 2, or 3.

(b) If A is diagonalizable then by Theorem 5.2.4(b) for each eigenvalue the dimension of the eigenspace must be equal to the algebraic multiplicity. Therefore

- The dimension of the eigenspace corresponding to $\lambda=1$ must be 1.
- The dimension of the eigenspace corresponding to $\lambda=3$ must be 2.
- The dimension of the eigenspace corresponding to $\lambda=4$ must be 3.

(c) If the dimension of the eigenspace were smaller than 3 then by Theorem 4.6.2(a), a set of three vectors from that eigenspace would have to be linearly dependent. Consequently, for the set of the three vectors to be linearly independent, the eigenspace containing the set must be

of dimension at least 3. This is only possible for the eigenspace corresponding to the eigenvalue $\lambda = 4$.

29. Using the result obtained in Exercise 30 of Section 5.1, we can take $P = \begin{bmatrix} -b & -b \\ a-\lambda_1 & a-\lambda_2 \end{bmatrix}$ where $\lambda_1 = \frac{1}{2}\left[(a+d)+\sqrt{(a-d)^2+4bc}\right]$ and $\lambda_2 = \frac{1}{2}\left[(a+d)-\sqrt{(a-d)^2+4bc}\right]$.

30. $T(x_1,x_2) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix for the operator T is $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.

$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 1 \\ -1 & \lambda-1 \end{vmatrix} = (\lambda-2)(\lambda-1)-(1)(-1) = \lambda^2 - 3\lambda + 3$

The discriminant of this quadratic polynomial, $b^2 - 4ac = (-3)^2 - 4(1)(3) = -3$ is negative, therefore the characteristic polynomial has no real zeros. Consequently, A has no real eigenvalues and it cannot be diagonalized.

31. $T(x_1,x_2) = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix for the operator T is $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

$\det(\lambda I - A) = \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$; thus, A has eigenvalues 1 and -1, both with algebraic multiplicities 1.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t$, $x_2 = t$. A vector $\mathrm{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = -1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$. A vector $\mathrm{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix P using the column vectors p_1 and p_2: $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the two columns can be interchanged.)

32. $T(x_1,x_2,x_3)=\begin{bmatrix}8x_1+3x_2-4x_3\\-3x_1+x_2+3x_3\\4x_1+3x_2\end{bmatrix}=\begin{bmatrix}8&3&-4\\-3&1&3\\4&3&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$;

the standard matrix for T is $A=\begin{bmatrix}8&3&-4\\-3&1&3\\4&3&0\end{bmatrix}$. Since $\det(\lambda I-A)=\begin{vmatrix}\lambda-8&-3&4\\3&\lambda-1&-3\\-4&-3&\lambda\end{vmatrix}=$

$(\lambda-4)^2(\lambda-1)$, thus A has eigenvalues $\lambda=1$ with algebraic multiplicity 1 and $\lambda=4$ with algebraic multiplicity 2.

The reduced row echelon form of $4I-A$ is $\begin{bmatrix}1&0&-1\\0&1&0\\0&0&0\end{bmatrix}$ so that $\lambda=4$ has geometric multiplicity 1.

Since the geometric multiplicity and the algebraic multiplicity are not equal, we conclude from Theorem 5.2.4(b) that A is not diagonalizable.

33. $T(x_1,x_2,x_3)=\begin{bmatrix}3x_1\\x_2\\x_1-x_2\end{bmatrix}=\begin{bmatrix}3&0&0\\0&1&0\\1&-1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$; the standard matrix for T is $A=\begin{bmatrix}3&0&0\\0&1&0\\1&-1&0\end{bmatrix}$.

Since $\det(\lambda I-A)=\begin{vmatrix}\lambda-3&0&0\\0&\lambda-1&0\\-1&1&\lambda\end{vmatrix}=(\lambda-3)(\lambda-1)\lambda$, thus A has eigenvalues 0, 1, and 3, each with algebraic multiplicity 1.

The reduced row echelon form of $0I-A$ is $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_1=0$ contains vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ where $x_1=0$, $x_2=0$, $x_3=t$. A vector $\mathrm{p}_1=\begin{bmatrix}0\\0\\1\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $1I-A$ is $\begin{bmatrix}1&0&0\\0&1&1\\0&0&0\end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_2=1$ contains vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ where $x_1=0$, $x_2=-t$, $x_3=t$. A vector $\mathrm{p}_2=\begin{bmatrix}0\\-1\\1\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 3t$, $x_2 = 0$, $x_3 = t$. A vector $\mathrm{p}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix P using the column vectors p_1, p_2, and p_3: $P = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the columns can be interchanged.)

True-False Exercises

(a) False. E.g., $A = I_2$ has only one eigenvalue $\lambda = 1$, but it is diagonalizable with $P = I_2$.

(b) True. This follows from Theorem 5.2.1.

(c) True. Multiplying $A = P^{-1}BP$ on the left by P yields $PA = BP$.

(d) False. The matrix P is not unique. For instance, interchanging two columns of P results in a different matrix which also diagonalizes A.

(e) True. Since A is invertible, we can take the inverse on both sides of the equality

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \text{ obtaining } P^{-1}A^{-1}P = D^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\lambda_n \end{bmatrix}.$$

Consequently, P diagonalizes both A and A^{-1}.

(f) True. We can transpose both sides of the equality $P^{-1}AP = D$ obtaining $P^T A^T \left(P^T\right)^{-1} = D^T = D$, i.e., $\left(\left(P^T\right)^{-1}\right)^{-1} A^T \left(P^T\right)^{-1} = D$. Consequently, $\left(P^T\right)^{-1}$ diagonalizes A^T.

(g) True. A basis for R^n must be a linearly independent set of n vectors, so by Theorem 5.2.1 A is diagonalizable.

(h) True. This follows from Theorem 5.2.2(b).

(i) True. From Theorem 5.1.5 we have $\det(A) = 0$. Since $\det\left(A^2\right) = \left(\det(A)\right)^2 = 0^2 = 0$, it follows from the same theorem that A^2 is singular.