1-Error Analysis

2-Solution of non-linear equation

Example 2 Show that $x^5 - 2x^3 + 3x^2 - 1 = 0$ has a solution in the interval $[0, 1]$.

Solution Consider the function defined by $f(x) = x^5 - 2x^3 + 3x^2 - 1$. The function f is continuous on $[0, 1]$. In addition,

$$f(0) = -1 < 0 \quad \text{and} \quad 0 < 1 = f(1).$$

The Intermediate Value Theorem implies that a number x exists, with $0 < x < 1$, for which $x^5 - 2x^3 + 3x^2 - 1 = 0$. ■

Decimal Machine Numbers

normalized *decimal* floating-point form

$$\pm 0.d_1d_2\ldots d_k \times 10^n, \quad 1 \le d_1 \le 9, \quad \text{and} \quad 0 \le d_i \le 9,$$

for each $i = 2, \ldots, k$. Numbers of this form are called k-digit *decimal machine numbers*.

$$y = 0.d_1d_2\ldots d_kd_{k+1}d_{k+2}\ldots \times 10^n.$$

$$fl(y) = 0.d_1d_2\ldots d_k \times 10^n.$$ Chopping

Rounding

For rounding, when $d_{k+1} \ge 5$, we add 1 to d_k to obtain $fl(y)$;

$d_{k+1} < 5$, we simply chop off all but the first k digits; so we *round down*.

Significant digits are those digits that can be used with confidence.

- **Non zero numbers are always significant**

  1.23 45.6 6,7263

- **In between zeros are always significant**

  1.005 70206

- **Leading zeros are never significant**

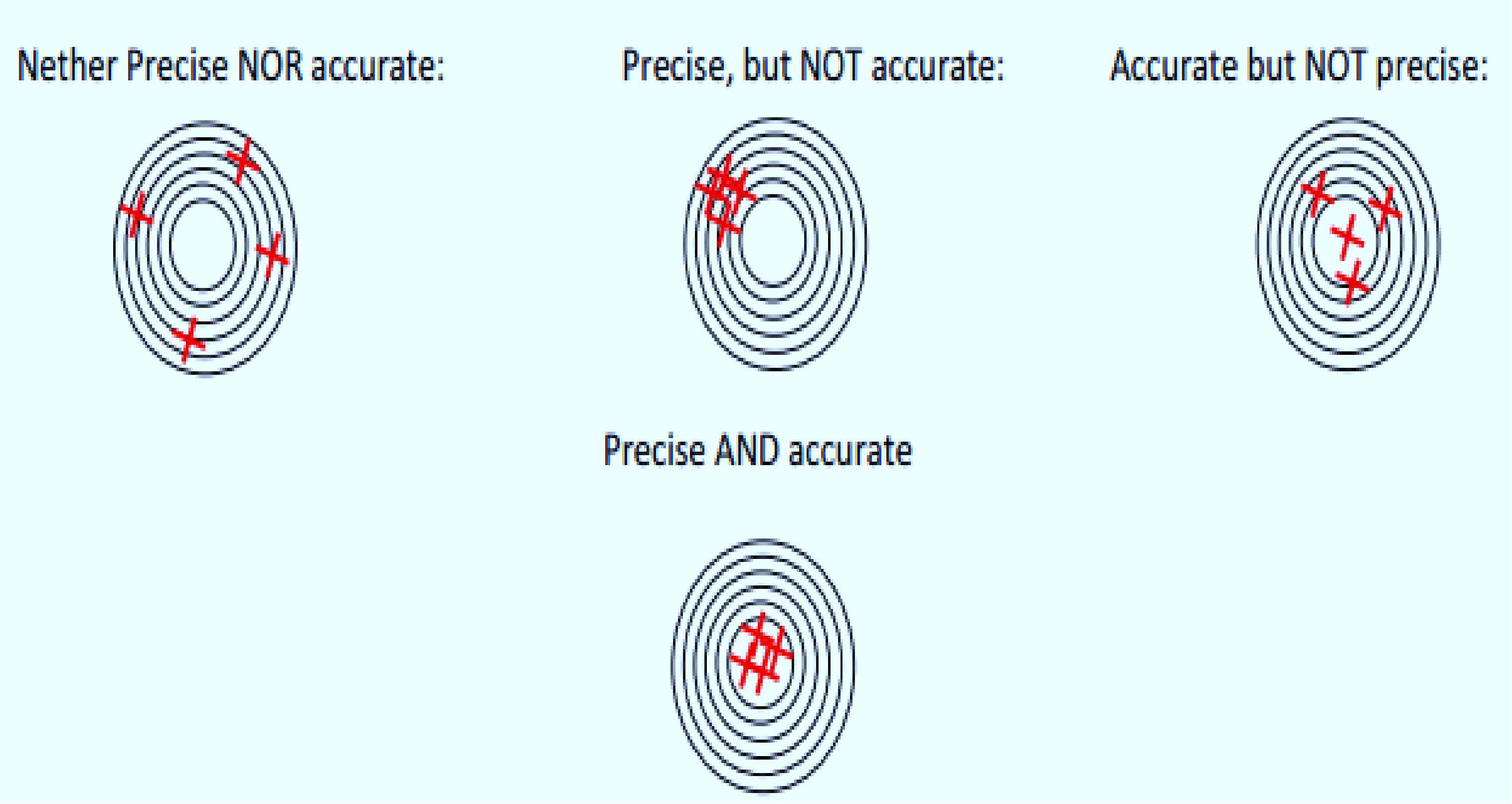
  0.0055 0.0302

- **Trailing zeros are some time significant**

  70,000 70,000. 1,030 1030.0000

Accuracy and Precision

- Accuracy is related to the closeness to the true value.
- Precision is related to the closeness to other estimated values.



Rounding and Chopping

Rounding: Replace the number by the nearest

machine number. OR

its impossible to represent all real numbers exactly on machine with finite

Chopping: Throw all or drop the extra digits.

Error: is difference between an approximation of number used in computation

and its exact value

OR Error = True value – approximate value

- √2 =1.414213562373095048801168872
- ∏ = 3.141592653589793238462643383

∏round =3.1416

∏chop =3.1415

ERROR Analysis:

- Truncation Error:

  are when an iterative method is terminated

  OR mathematical procedure is approximated and

  approximate solution differs from exact solution

- Discretization Error :

  are committed when a solution

  of discrete problem does not coincide with solution

  of continuous problem

Error in CM – True Error

Can be computed if the true value is known:

$$\text{Absolute Error}:$$

$$AE = \left|\text{true value} - \text{approximation}\right|$$

$$\text{Absolute Relative Error}:$$

$$ARE = \left|\frac{\text{true value} - \text{approximation}}{\text{true value}}\right|$$

Error in CM – Estimated Error

When the true value is not known:

$$\text{Estimated Absolute Error}$$

$$AE = \left|\text{current estimate} - \text{previous estimate}\right|$$

$$\text{Estimated Absolute Relative Error}$$

$$ARE = \left|\frac{\text{current estimate} - \text{previous estimate}}{\text{current estimate}}\right|$$

Example 1 Determine the five-digit (a) chopping and (b) rounding values of the irrational number π.

Definition 1.15 Suppose that p^* is an approximation to p. The **absolute error** is $|p - p^*|$, and the **relative error** is $\dfrac{|p - p^*|}{|p|}$, provided that $p \neq 0$. ■

Example 2 Determine the absolute and relative errors when approximating p by p^* when

(a) $p = 0.3000 \times 10^1$ and $p^* = 0.3100 \times 10^1$;

(b) $p = 0.3000 \times 10^{-3}$ and $p^* = 0.3100 \times 10^{-3}$;

(c) $p = 0.3000 \times 10^4$ and $p^* = 0.3100 \times 10^4$.

Example 3 Suppose that $x = \frac{5}{7}$ and $y = \frac{1}{3}$. Use five-digit chopping for calculating $x + y$, $x - y$, $x \times y$, and $x \div y$.

Solution Note that

$$x = \frac{5}{7} = 0.\overline{714285} \quad \text{and} \quad y = \frac{1}{3} = 0.\overline{3}$$

implies that the five-digit chopping values of x and y are

$$fl(x) = 0.71428 \times 10^0 \quad \text{and} \quad fl(y) = 0.33333 \times 10^0.$$

Thus

$$\begin{aligned} x \oplus y &= fl(fl(x) + fl(y)) = fl\left(0.71428 \times 10^0 + 0.33333 \times 10^0\right) \\ &= fl\left(1.04761 \times 10^0\right) = 0.10476 \times 10^1. \end{aligned}$$

The true value is $x + y = \frac{5}{7} + \frac{1}{3} = \frac{22}{21}$, so we have

$$\text{Absolute Error} = \left| \frac{22}{21} - 0.10476 \times 10^1 \right| = 0.190 \times 10^{-4}$$

and

$$\text{Relative Error} = \left| \frac{0.190 \times 10^{-4}}{22/21} \right| = 0.182 \times 10^{-4}.$$

$$x \oplus y = fl(fl(x) + fl(y)),$$

Table 1.2

$$x \ominus y = fl(fl(x) - fl(y)),$$

Operation	Result	Actual value	Absolute error	Relative error
$x \oplus y$	0.10476×10^1	$22/21$	0.190×10^{-4}	0.182×10^{-4}
$x \ominus y$	0.38095×10^0	$8/21$	0.238×10^{-5}	0.625×10^{-5}
$x \otimes y$	0.23809×10^0	$5/21$	0.524×10^{-5}	0.220×10^{-4}
$x \oslash y$	0.21428×10^1	$15/7$	0.571×10^{-4}	0.267×10^{-4}

$\boxed{\text{The maximum relative error for the operations in Example 3 is } 0.267 \times 10^{-4},}$

Example 5 Let $p = 0.54617$ and $q = 0.54601$. Use four-digit arithmetic to approximate $p - q$ and determine the absolute and relative errors using **(a)** rounding and **(b)** chopping.

Loss of significance:

occurs in numerical calculations when too many significant digits cancel

The quadratic formula states that the roots of $ax^2 + bx + c = 0$, when $a \neq 0$, are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{1.1}$$

which simplifies to an alternate quadratic formula

$$x_1 = \frac{-2c}{b + \sqrt{b^2 - 4ac}}. \tag{1.2}$$

The rationalization technique can also be applied to give the following alternative quadratic formula for x_2:

$$x_2 = \frac{-2c}{b - \sqrt{b^2 - 4ac}}. \tag{1.3}$$

Activity

(Taylor's Theorem)

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

7. Let $f(x) = x^3$.

a. Find the second Taylor polynomial $P_2(x)$ about $x_0 = 0$.

b. Find $R_2(0.5)$ and the actual error in using $P_2(0.5)$ to approximate $f(0.5)$.

c. Repeat part (a) using $x_0 = 1$.

d. Repeat part (b) using the polynomial from part (c).

8. Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = \sqrt{x+1}$ about $x_0 = 0$. Approximate $\sqrt{0.5}$, $\sqrt{0.75}$, $\sqrt{1.25}$, and $\sqrt{1.5}$ using $P_3(x)$, and find the actual errors.

9. Find the second Taylor polynomial $P_2(x)$ for the function $f(x) = e^x \cos x$ about $x_0 = 0$.

a. Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5) - P_2(0.5)|$ using the error formula, and compare it to the actual error.

Activity

1. Compute the absolute error and relative error in approximations of p by p^*.

a. $p = \pi, p^* = 22/7$

b. $p = \pi, p^* = 3.1416$

c. $p = e, p^* = 2.718$

d. $p = \sqrt{2}, p^* = 1.414$

e. $p = e^{10}, p^* = 22000$

f. $p = 10^{\pi}, p^* = 1400$

5. Use three-digit rounding arithmetic to perform the following calculations. Compute the absolute error and relative error with the exact value determined to at least five digits.

a. $133 + 0.921$

b. $133 - 0.499$

c. $(121 - 0.327) - 119$

d. $(121 - 119) - 0.327$

e. $\dfrac{\frac{13}{14} - \frac{6}{7}}{2e - 5.4}$

f. $-10\pi + 6e - \dfrac{3}{62}$

13. Use four-digit rounding arithmetic and the formulas (1.1), (1.2), and (1.3) to find the most accurate approximations to the roots of the following quadratic equations. Compute the absolute errors and relative errors.

a. $\frac{1}{3}x^2 - \frac{123}{4}x + \frac{1}{6} = 0$

b. $\frac{1}{3}x^2 + \frac{123}{4}x - \frac{1}{6} = 0$

c. $1.002x^2 - 11.01x + 0.01265 = 0$

d. $1.002x^2 + 11.01x + 0.01265 = 0$

14. Repeat Exercise 13 using four-digit chopping arithmetic.

Nested Arithmetic Accuracy loss due to round-off error can also be reduced by rearranging calculations, as shown in the next example.

Example 6 Evaluate $f(x) = x^3 - 6.1x^2 + 3.2x + 1.5$ at $x = 4.71$ using three-digit arithmetic.

$$\text{Three-digit (chopping):} \quad f(4.71) = ((104. - 134.) + 15.0) + 1.5 = -13.5,$$

$$\text{Three-digit (rounding):} \quad f(4.71) = ((105. - 135.) + 15.1) + 1.5 = -13.4.$$

$$\text{Chopping:} \left| \frac{-14.263899 + 13.5}{-14.263899} \right| \approx 0.05, \quad \text{and Rounding:} \left| \frac{-14.263899 + 13.4}{-14.263899} \right| \approx 0.06.$$

Nested Arithmetic

As an alternative approach, the polynomial $f(x)$ in Example 6 can be written in a nested manner as

$$f(x) = x^3 - 6.1x^2 + 3.2x + 1.5 = ((x - 6.1)x + 3.2)x + 1.5.$$

Using three-digit chopping arithmetic now produces

$$\begin{aligned} f(4.71) &= ((4.71 - 6.1)4.71 + 3.2)4.71 + 1.5 = ((-1.39)(4.71) + 3.2)4.71 + 1.5 \\ &= (-6.54 + 3.2)4.71 + 1.5 = (-3.34)4.71 + 1.5 = -15.7 + 1.5 = -14.2. \end{aligned}$$

$$\text{Three-digit (chopping):} \quad \left| \frac{-14.263899 + 14.2}{-14.263899} \right| \approx 0.0045;$$

$$\text{Three-digit (rounding):} \quad \left| \frac{-14.263899 + 14.3}{-14.263899} \right| \approx 0.0025.$$

Nesting has reduced the relative error for the chopping approximation to less than 10% of that obtained initially. For the rounding approximation the improvement has been even more dramatic; the error in this case has been reduced by more than 95%. □

2-Solution of non linear equation in one variable

1-Bracketing Methods
2-Open Methods

Bracketing Methods:

- In bracketing methods, the method starts with an interval that contains the root and a procedure is used to obtain a smaller interval containing the root.

- Examples of bracketing methods:

  - Bisection method
  - False position method

Open Methods:

- In the open methods, the method starts with one or more initial guess points. In each iteration, a new guess of the root is obtained.

- Open methods are usually more efficient than bracketing methods.
- They may not converge to a root.
  - Fixed point ,
  - Newton and
  - Secant are examples of open method

Bisection

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs:

INPUT endpoints a, b; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$;
$FA = f(a)$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = a + (b - a)/2$; *(Compute p_i.)*
$FP = f(p)$.

Step 4 If $FP = 0$ or $(b - a)/2 < TOL$ then
OUTPUT (p); *(Procedure completed successfully.)*
STOP.

Step 5 Set $i = i + 1$.

Step 6 If $FA \cdot FP > 0$ then set $a = p$; *(Compute a_i, b_i.)*
$FA = FP$
else set $b = p$. *(FA is unchanged.)*

Step 7 OUTPUT ('Method failed after N_0 iterations, $N_0 =$', N_0);
(The procedure was unsuccessful.)
STOP.

Algorithm

To determine a root of $f(x) = 0$ that is accurate within a specified tolerance value, given values x_1 and x_2 such that $f(x_1) * f(x_2) < 0$,

Repeat
Set $x_3 = (x_1 + x_2)/2$.

If $f(x_3) * f(x_1) < 0$ Then
Set $x_2 = x_3$
Else Set $x_1 = x_3$ End If.
Until ($|x_1 - x_2|$) $< 2 *$ tolerance value).

The final value of x_3 approximates the root, and it is in error by not more than $|x_1 - x_2|/2$.
Note: The method may produce a false root if $f(x)$ is discontinuous on $[x_1, x_2]$.

Stopping criteria:

select a tolerance $\varepsilon > 0$, and construct $p_1, \ldots p_N$ until

1) $$|p_N - p_{N-1}| < \varepsilon,$$

2) $$\frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon, \quad p_N \neq 0,$$

3) $$|f(p_N)| < \varepsilon.$$

The number of iteration required for given tolerance :

$$|p - p_n| \leq \frac{b-a}{2^n} \leq \epsilon,$$

then taking the logarithms of both sides yields

$$\boxed{n \geq \frac{\log\left(\frac{b-a}{\epsilon}\right)}{\log(2)}}.$$

Determine the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} using $a_1 = 1$ and $b_1 = 2$.

Example 1 Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$, and use the Bisection method to determine an approximation to the root that is accurate to at least within 10^{-4}.

Table 2.1

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	−0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	−0.01605
11	1.364257813	1.365234375	1.364746094	−0.00799
12	1.364746094	1.365234375	1.364990235	−0.00396
13	1.364990235	1.365234375	1.365112305	−0.00194

Pros

1. The method has the important property that it **always converges** to a solution.

Cons

1. It is **relatively slow to converge**.
2. A **good intermediate approximation might be inadvertently discarded**.
3. If a function $f(x)$ is such that it just **touches the *x*-axis** such as $f(x) = x^2 = 0$, it will be unable to find the lower guess a and the upper guess b such that $f(a) \cdot f(b) < 0$.

Fixed-Point Iteration

Definition 2.2 The number p is a **fixed point** for a given function g if $g(p) = p$.

Example 1 Determine any fixed points of the function $g(x) = x^2 - 2$.

Solution A fixed point p for g has the property that

$$p = g(p) = p^2 - 2 \quad \text{which implies that} \quad 0 = p^2 - p - 2 = (p+1)(p-2).$$

so g has two fixed points, one at $p = -1$ and the other at $p = 2$.

Example 2 Show that $g(x) = (x^2 - 1)/3$ has a unique fixed point on the interval $[-1, 1]$.

Example

- Fixed Point Iteration

$$f(x) = x^2 - 2x - 3 = 0$$

(ans: $x = 3$ or -1)

Case a:

$$\boxed{\begin{aligned} & x^2 - 2x - 3 = 0 \\ & \Rightarrow x^2 = 2x + 3 \\ & \Rightarrow x = \sqrt{2x+3} \\ & \Rightarrow g(x) = \sqrt{2x+3} \end{aligned}}$$

Case b:

$$\boxed{\begin{aligned} & x^2 - 2x - 3 = 0 \\ & \Rightarrow x(x-2) - 3 = 0 \\ & \Rightarrow x = \frac{3}{x-2} \\ & \Rightarrow g(x) = \frac{3}{x-2} \end{aligned}}$$

Case c:

$$\boxed{\begin{aligned} & x^2 - 2x - 3 = 0 \\ & \Rightarrow 2x = x^2 - 3 \\ & \Rightarrow x = \frac{x^2 - 3}{2} \\ & \Rightarrow g(x) = \frac{x^2 - 3}{2} \end{aligned}}$$

So which one is better?

Case a	Case b	Case c
$x_{i+1} = \sqrt{2x_i + 3}$	$x_{i+1} = \dfrac{3}{x_i - 2}$	$x_{i+1} = \dfrac{x_i^2 - 3}{2}$
1. $x_0 = 4$	1. $x_0 = 4$	1. $x_0 = 4$
2. $x_1 = 3.31662$	2. $x_1 = 1.5$	2. $x_1 = 6.5$
3. $x_2 = 3.10375$	3. $x_2 = -6$	3. $x_2 = 19.625$
4. $x_3 = 3.03439$	4. $x_3 = -0.375$	4. $x_3 = 191.070$
5. $x_4 = 3.01144$	5. $x_4 = -1.263158$	
6. $x_5 = 3.00381$	6. $x_5 = -0.919355$	
	7. $x_6 = -1.02762$	
	8. $x_7 = -0.990876$	
	9. $x_8 = -1.00305$	
Converge!	Converge, but slower	Diverge!

Iteration Algorithm with the Form $x = g(x)$

To determine a root of $f(x) = 0$, given a value x_1 reasonably close to the root,

Rearrange the equation to an equivalent form $x = g(x)$.

Repeat
Set $x_2 = x_1$.
Set $x_1 = g(x_1)$
Until $|x_1 - x_2| <$ tolerance value

Note: The method may converge to a root different from the expected one, or it may diverge. Different rearrangements will converge at different rates.

Fixed-Point Iteration

To find a solution to $p = g(p)$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = g(p_0)$. *(Compute p_i.)*

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); *(The procedure was successful.)*
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. *(Update p_0.)*

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(The procedure was unsuccessful.)
STOP.

Theorem 2.4 (Fixed-Point Theorem)

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all x in $[a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \le k, \quad \text{for all } x \in (a,b).$$

Then for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \ge 1,$$

converges to the unique fixed point p in $[a,b]$. ■

Newton's

To find a solution to $f(x) = 0$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = p_0 - f(p_0)/f'(p_0)$. *(Compute p_i.)*

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); *(The procedure was successful.)*
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. *(Update p_0.)*

Step 7 OUTPUT ('The method failed after N_0 iterations, N_0 =', N_0);
(The procedure was unsuccessful.)
STOP.

Newton's Method

To determine a root of $f(x) = 0$, given x_0 reasonably close to the root,

Compute $f(x_0), f'(x_0)$.
If ($f(x_0) \neq 0$) And ($f'(x_0) \neq 0$) Then
Repeat
    Set $x_1 = x_0$.
    Set $x_0 = x_0 - f(x_0)/f'(x_0)$.
    Until ($|x_1 - x_0| <$ tolerance value 1) Or
    $|f(x_0)| <$ tolerance value 2).
    End If.

Note: The method may converge to a root different from the expected one or diverge if the starting value is not close enough to the root.

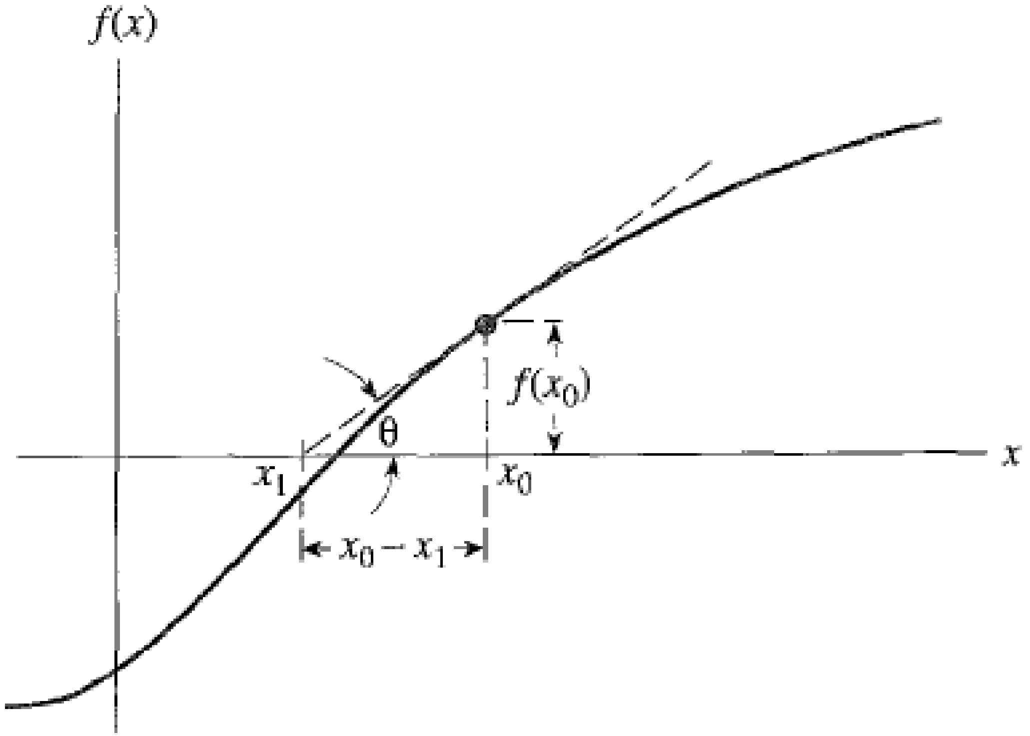
$$\tan\theta = f'(x_0) = \frac{f(x_0)}{x_0 - x_1},$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots.$$

$$x_{n+1} = x_n - \frac{f(x_n)}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$



Example 1 Consider the function $f(x) = \cos x - x = 0$. Approximate a root of f using (a) a fixed-point method, and (b) Newton's Method

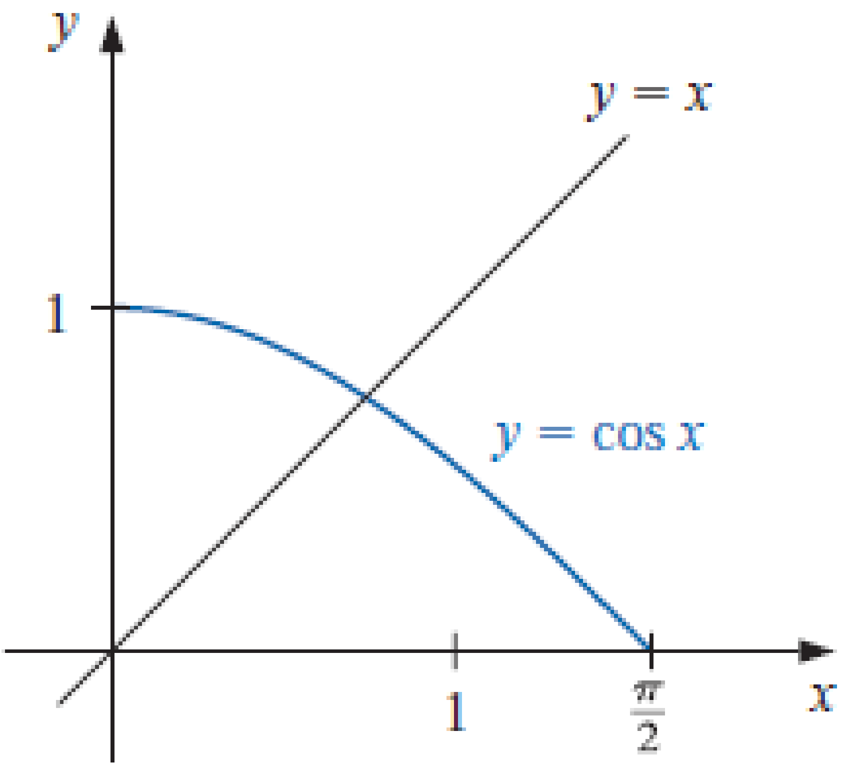
Table 2.3

n	p_n
0	0.7853981635
1	0.7071067810
2	0.7602445972
3	0.7246674808
4	0.7487198858
5	0.7325608446
6	0.7434642113
7	0.7361282565

Table 2.4

Newton's Method

n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332



Comparison b/w Secant and Newton

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1.$$

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$

An Algorithm for the Secant Method

To determine a root of $f(x) = 0$, given two values, x_0 and x_1, that are near the root,

If $|f(x_0)| < |f(x_1)|$ Then
  Swap x_0 with x_1.
Repeat
  Set $x_2 = x_1 - f(x_1) * \dfrac{x_0 - x_1}{f(x_0) - f(x_1)}$
  Set $x_0 = x_1$.
  Set $x_1 = x_2$.
Until $|f(x_2)| <$ tolerance value.

Note: If $f(x)$ is not continuous, the method may fail.

The Secant Method

Example 2 Use the Secant method to find a solution to $x = \cos x$, and compare the approximations with those given in Example 1 which applied Newton's method.

	Newton
n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

Table 2.5

	Secant
n	p_n
0	0.5
1	0.7853981635
2	0.7363841388
3	0.7390581392
4	0.7390851493
5	0.7390851332

An Algorithm for the Method of False Position *(regula falsi)*

To determine a root of $f(x) = 0$, given two values of x_0 and x_1 that bracket a root: that is, $f(x_0)$ and $f(x_1)$ are of opposite sign,

Repeat

Set $x_2 = x_1 - f(x_1) * \dfrac{x_0 - x_1}{f(x_0) - f(x_1)}$

If $f(x_2)$ is of opposite sign to $f(x_0)$ Then

Set $x_1 = x_2$

Else

Set $x_0 = x_2$

End If.

Until $|f(x_2)| < $ tolerance value.

The Method of False Position

Example 3 Use the method of False Position to find a solution to $x = \cos x$, and compare the approximations with those given in Example 1 which applied fixed-point iteration and Newton's method, and to those found in Example 2 which applied the Secant method.

Table 2.6

	False Position	Secant	Newton
n	p_n	p_n	p_n
0	0.5	0.5	0.7853981635
1	0.7853981635	0.7853981635	0.7395361337
2	0.7363841388	0.7363841388	0.7390851781
3	0.7390581392	0.7390581392	0.7390851332
4	0.7390848638	0.7390851493	0.7390851332
5	0.7390851305	0.7390851332	
6	0.7390851332		

Comparison of methods, $f(x) = 3x + \sin(x) - e^x = 0$, $x_0 = 0$, $x_1 = 1$

	Interval halving		False position		Secant method	
Iteration	x	$f(x)$	x	$f(x)$	x	$f(x)$
1	0.5	0.330704	0.470990	0.265160	0.470990	0.265160
2	0.25	−0.286621	0.372277	0.029533	0.372277	0.029533
3	0.375	0.036281	0.361598	$2.94 * 10^{-3}$	0.359904	$-1.29 * 10^{-3}$
4	0.3125	−0.121899	0.360538	$2.90 * 10^{-4}$	0.360424	$5.55 * 10^{-6}$
5	0.34375	−0.041956	0.360433	$2.93 * 10^{-5}$	0.360422	$3.55 * 10^{-7}$
Error after 5 iterations	0.01667		$-1.17 * 10^{-5}$		$< -1 * 10^{-7}$	

(Exact value of root is 0.360421703.)

Features of Bisection:

- ✓Type – closed bracket
- ✓No. of initial guesses – 2
- ✓Convergence – linear
- ✓Rate of convergence – slow but steady

- ✓Accuracy – good
- ✓Programming effort – easy
- ✓Approach – middle point

Features of Reguli false:

- ✓No. of initial guesses – 2
- ✓Type – closed bracket
- ✓Convergence – linear
- ✓Rate of convergence – slow
- ✓Accuracy – good
- ✓Approach – interpolation
- ✓Programming effort – easy

Features of Newton Raphson:

- ✓Type – open bracket
- ✓No. of initial guesses – 1
- ✓Convergence – quadratic
- ✓Rate of convergence – faster
- ✓Accuracy – good
- ✓Programming effort – easy
- ✓Approach – Taylor’s series

Futures of Secant Method:

- ✓No. of initial guesses – 2
- ✓Type – open bracket
- ✓Rate of convergence – faster
- ✓Convergence – super linear
- ✓Accuracy – good
- ✓Approach – interpolation
- ✓Programming effort – tedious

Summary

Method	Pros	Cons
Bisection	- Easy, Reliable, Convergent - One function evaluation per iteration - No knowledge of derivative is needed	- Slow - Needs an interval [a,b] containing the root, i.e., $f(a)f(b)<0$
Newton	- Fast (if near the root) - Two function evaluations per iteration	- May diverge - Needs derivative and an initial guess x_0 such that $f'(x_0)$ is nonzero
Secant	- Fast (slower than Newton) - One function evaluation per iteration - No knowledge of derivative is needed	- May diverge - Needs two initial points guess x_0, x_1 such that $f(x_0) - f(x_1)$ is nonzero