CHAPTER 4

Mathematical Expectation

Mean of a Random Variable

Variance and Covariance of Random Variables

Let X be a random variable with probability distribution $f(x)$. The **mean**, or **expected value**, of X is

$$\mu = E(X) = \sum_x x f(x)$$

if X is discrete, and

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\ dx$$

if X is continuous.

Example:

Let X represent the number of good components in the sample. The probability distribution of X is

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2, 3.$$

Find the expected value of the number of good components

Example:

Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \frac{20{,}000}{x^3}, & x > 100, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected life of this type of device.

Solution:

$$\mu = E(X) = \int_{100}^{\infty} x \frac{20,000}{x^3}\, dx = \int_{100}^{\infty} \frac{20,000}{x^2}\, dx = 200.$$

Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x)$$

if X is discrete, and

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

if X is continuous.

Example:

Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X = x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

$$E[g(X)] = E(2X - 1) = \sum_{x=4}^{9}(2x - 1)f(x)$$

Class Activity:

Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $g(X) = 4X + 3$.

Let X and Y be random variables with joint probability distribution $f(x, y)$. The mean, or expected value, of the random variable $g(X, Y)$ is

$$\mu_{g(X,Y)} = E[g(X, Y)] = \sum_x \sum_y g(x, y) f(x, y)$$

if X and Y are discrete, and

$$\mu_{g(X,Y)} = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y)\, dx\, dy$$

if X and Y are continuous.

Example:

Let X and Y be the random variables with joint probability distribution indicated in Table 3.1 on page 96. Find the expected value of $g(X,Y) = XY$. The table is reprinted here for convenience.

$f(x,y)$		x = 0	1	2	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$$E(XY) = \sum_{x=0}^{2}\sum_{y=0}^{2} xyf(x,y) = \frac{3}{14}.$$

Class Activity:

Find $E(Y/X)$ for the density function

$$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$E\left(\frac{Y}{X}\right) = \frac{5}{8}.$$

Practice:

4.17 Let X be a random variable with the following probability distribution:

x	-3	6	9
$f(x)$	$1/6$	$1/2$	$1/3$

Find $\mu_{g(X)}$, where $g(X) = (2X+1)^2$.

4.26 Let X and Y be random variables with joint density function

$$f(x,y) = \begin{cases} 4xy, & 0 < x,\ y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $Z = \sqrt{X^2+Y^2}$.

Practice:

4.23 Suppose that X and Y have the following joint probability function:

$f(x,y)$		$x = 2$	$x = 4$
y	1	0.10	0.15
	3	0.20	0.30
	5	0.10	0.15

(a) Find the expected value of $g(X,Y) = XY^2$.

(b) Find μ_X and μ_Y.

Variance of Random variable

Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is

$$\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x), \qquad \text{if } X \text{ is discrete, and}$$

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx, \qquad \text{if } X \text{ is continuous.}$$

The positive square root of the variance, σ, is called the **standard deviation** of X.

The variance of a random variable X is

$$\sigma^2 = E(X^2) - \mu^2.$$

Class Activity 1:

Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested. The following is the probability distribution of X.

x	0	1	2	3
$f(x)$	0.51	0.38	0.10	0.01

Using Theorem 4.2, calculate σ^2.

Class Activity 2:

The weekly demand for a drinking-water product, in thousands of liters, from a local chain of efficiency stores is a continuous random variable X having the probability density

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the mean and variance of X.

Let X be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$ is

$$\sigma_{g(X)}^2 = E\{[g(X) - \mu_{g(X)}]^2\} = \sum_x [g(x) - \mu_{g(X)}]^2 f(x)$$

if X is discrete, and

$$\sigma_{g(X)}^2 = E\{[g(X) - \mu_{g(X)}]^2\} = \int_{-\infty}^{\infty} [g(x) - \mu_{g(X)}]^2 f(x)\, dx$$

if X is continuous.

Class Activity:

Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

Class Activity:

Solution:

$$\mu_{2X+3} = E(2X+3) = \sum_{x=0}^{3} (2x+3) f(x) =$$

$$\sigma^2_{2X+3} = E[(2X+3-6)^2] = E(4X^2 - 12X + 9)$$

$$= \sum_{x=0}^{3} (4x^2 - 12x + 9) f(x)$$

$$= 4.$$

Covariance of Random Variables

Let X and Y be random variables with joint probability distribution $f(x, y)$. The covariance of X and Y is

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \sum_x \sum_y (x - \mu_X)(y - \mu_y) f(x, y)$$

if X and Y are discrete, and

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_y) f(x, y) \, dx \, dy$$

if X and Y are continuous.

The covariance of two random variables X and Y with means μ_X and μ_Y, respectively, is given by

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y.$$

Example:

Example 3.14 on page 95 describes a situation involving the number of blue refills X and the number of red refills Y. Two refills for a ballpoint pen are selected at random from a certain box, and the following is the joint probability distribution:

	$f(x,y)$	$x=0$	$x=1$	$x=2$	$h(y)$
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
	$g(x)$	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$$E(XY) = \sum_{x=0}^{2}\sum_{y=0}^{2} xyf(x,y)$$

$$E(XY) = 3/14.$$

Find the covariance of X and Y.

Solution:

$$\mu_X = \sum_{x=0}^{2} xg(x) = (0)\left(\frac{5}{14}\right) + (1)\left(\frac{15}{28}\right) + (2)\left(\frac{3}{28}\right) = \frac{3}{4},$$

$$\mu_Y = \sum_{y=0}^{2} yh(y) = (0)\left(\frac{15}{28}\right) + (1)\left(\frac{3}{7}\right) + (2)\left(\frac{1}{28}\right) = \frac{1}{2}.$$

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{3}{14} - \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) = -\frac{9}{56}.$$

Example:

The fraction X of male runners and the fraction Y of female runners who compete in marathon races are described by the joint density function

$$f(x,y) = \begin{cases} 8xy, & 0 \le y \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the covariance of X and Y.

Solution:

$$g(x) = \begin{cases} 4x^3, & 0 \le x \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

$$h(y) = \begin{cases} 4y(1-y^2), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The **marginal distributions** of X alone and of Y alone are

$$g(x) = \sum_y f(x,y) \quad \text{and} \quad h(y) = \sum_x f(x,y)$$

for the discrete case, and

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy \quad \text{and} \quad h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

for the continuous case.

Class Activity:

Solution contd:

$$\mu_X = E(X) = \int_0^1 4x^4 \; dx = \frac{4}{5} \text{ and } \mu_Y = \int_0^1 4y^2(1 - y^2) \; dy = \frac{8}{15}.$$

$$E(XY) = \int_0^1 \int_y^1 8x^2y^2 \; dx \; dy = \frac{4}{9}.$$

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right) = \frac{4}{225}.$$

The correlation coefficient of X and Y is

Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y, respectively. The correlation coefficient of X and Y is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

correlation coefficient satisfies the inequality $-1 \leq \rho_{XY} \leq 1$.

Example:

Find the correlation coefficient between X and Y in Example 4.13.

	$f(x,y)$	$x = 0$	$x = 1$	$x = 2$	$h(y)$
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
	$g(x)$	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Solution (Step by Step):

Mean

$$\mu_X = \sum_{x=0}^{2} x g(x)$$

$$\mu_Y = \sum_{y=0} y h(y)$$

Covar

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$

$$E(X^2) =$$

$$E(Y^2) =$$

$$\sigma^2 = E(X^2) - \mu^2.$$

Var

$$\sigma_X^2 =$$

$$\sigma_Y^2 =$$

Class Activity:

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = -\frac{1}{\sqrt{5}}.$$

Class Activity:

Find the correlation coefficient of X and Y in Example 4.14.

The fraction X of male runners and the fraction Y of female runners who compete in marathon races are described by the joint density function

$$f(x, y) = \begin{cases} 8xy, & 0 \le y \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Solution:

the marginal density functions. They are

$$g(x) = \begin{cases} 4x^3, & 0 \le x \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

$$h(y) = \begin{cases} 4y(1 - y^2), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Solution (Step by Step):

Class Activity:

Mean

$$\mu = E(X)$$

$$\mu_Y =$$

$$E(XY) = \frac{4}{9}.$$

Covar ➡

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$

$$E(X^2) =$$

$$E(Y^2) =$$

Var

$$\sigma_X^2 =$$

$$\sigma_Y^2 =$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} =$$

$$\rho_{XY} = \frac{4/225}{\sqrt{(2/75)(11/225)}} = \frac{4}{\sqrt{66}}.$$