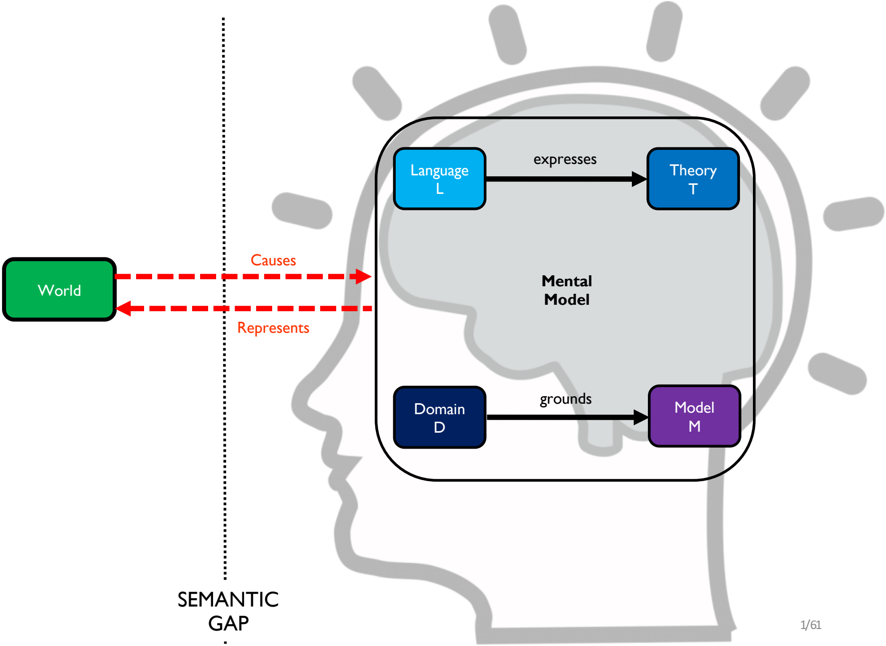
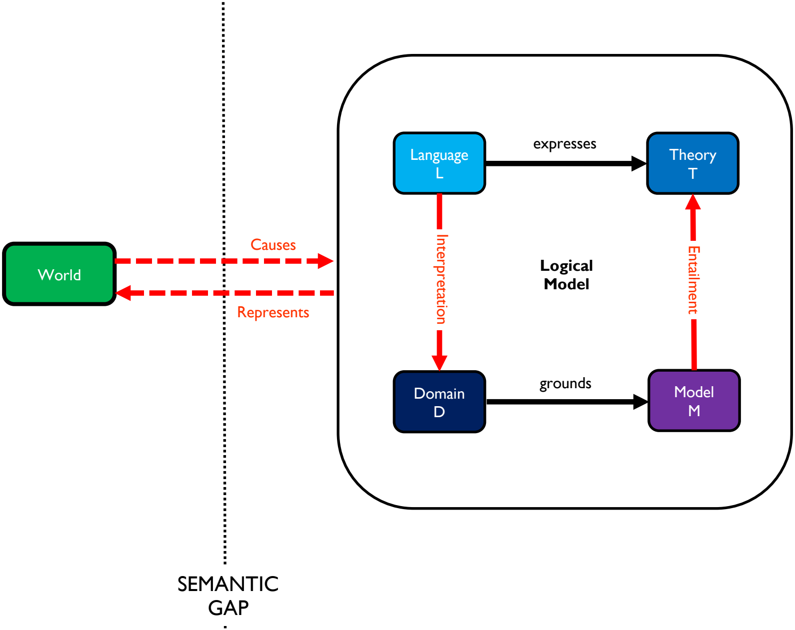
Mental Model

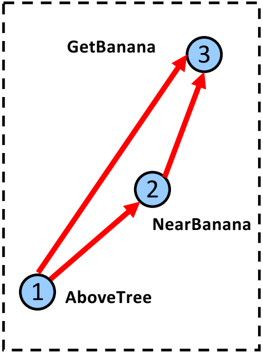


Logical Model



Logical Model

World | Logical Model | Language L | Domain D | Theory T | Model M



L = "*AboveTree, NearBanana, GetBanana 1, 2,* $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\Box$, $\Diamond$, …"

T = "{<1, 2>, <1, 3>, <2, 3>}"

D: {1, 2, 3}

I: "I(AboveTree) = 1, I(NearBanana) = 2, I(GetBanana) = 3"

M,1 $\models$ *AboveTree*, **M,2** $\models$ $\Box$*GetBanana*, …

SEMANTIC GAP

What is Modality?

- A modality is an expression that is used to *qualify* the truth of a judgement (or, in other words, an operator that expresses a "mode" in which a proposition is true)
- It can be seen as an operator that takes a proposition and returns a more complex proposition.

Proposition	Modal Expression
John drives a Ferrari	John *is able to* drive a Ferrari
Everybody pays taxes	It is *obligatory* that everybody pays taxes

- Modalities are expressed in natural language through **modal verbs** such as *can/could, may/might, must, will/would,* and *shall/should.*

What is Modality?

- In logic modalities are formalized using an operator such as $\Box$ ($\Diamond$) that can be applied to a formula φ to obtain another formula $\Box\varphi$ ($\Diamond\varphi$).
- The truth value of $\Box\varphi$ is not a function of the truth value of φ.

Example

- The fact that John is able to drive a Ferrari may be true independently from the fact that John is actually driving a Ferrari.
- The fact that it is *obligatory* that everybody pays taxes is typically true, and this is independent from the fact that everybody actually pays taxes.

Note: $\neg$ is not a modal operator since the truth value of $\neg\varphi$ is a function of the truth value of φ.

Modalities

- A modality is an expression that is used to *qualify* the truth of a judgement.
- Historically, the first modalities formalized with modal logic were the so called alethic modalities i.e.,
  1. it is possible that a certain proposition holds, usually denoted with $\Diamond\varphi$
  2. it is necessary that a certain proposition holds, usually denoted with $\Box\varphi$
- Afterwards a number of modal logics for different "*qualifications*" have been studied. The most common are. . .

Modalities

Modality	**Symbol**	**Expression Symbolised**
Alethic	$\Box\varphi$ $\Diamond\varphi$	it is *necessary* that φ it is *possible* that φ
Deontic	$O\varphi$ $P\varphi$ $F\varphi$	it is *obligatory* that φ it is *permitted* that φ it is *forbidden* that φ
Temporal	$G\varphi$ $F\varphi$	it will *always* be the case that φ it will *eventually* be the case that φ
Epistemic	$B_a\varphi$ $K_a\varphi$	agent *a* *believes* that φ agent *a* *knows* that φ
Contextual	$ist(c, \varphi)$	φ is *true in the context c*
Dynamic	$[\alpha]\varphi$ $(\alpha)\varphi$	φ must be true after the execution of program α φ can be true after the execution of program α
Computational	$AX\varphi$ $AG\varphi$ $AF\varphi$ $A\varphi U\vartheta$ $EX\varphi$	φ is true for every immediate successor state φ is true for every successor state φ will eventually be true in all the possible evolutions φ is true until ϑ becomes true φ is true in at least one immediate successor state

Relational structures in FOL

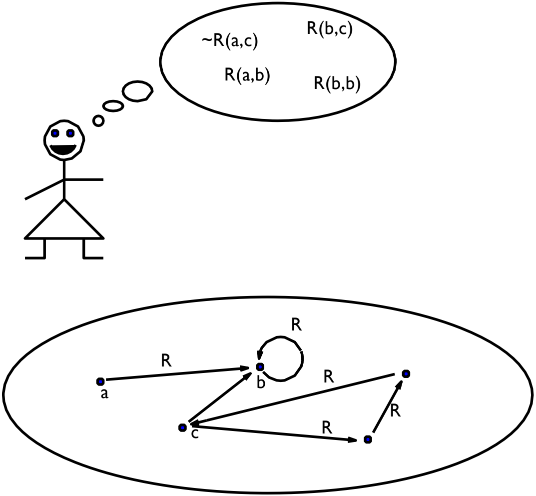
- Relational structures can be investigated in FOL;
- The language must contain at least a binary relation R, and we can formalize the properties of a relational structure using formulae such as
  - $\forall x R(x, x)$ (R is reflexive)
  - $\forall x \exists y R(x, y)$ (R is serial)
  - $\forall xy\, (R(x, y) \supset R(y, x))$ (R is symmetric)
  - . . .
- So, why do we need modal logics?

Relational structures in first order and modal logic

- In First Order Logic we describes a relational structure from an external point of view, (and our description is not relative to a particular point).
- Modal logics describe relational structures from an internal point of view, rather than from the top perspective
- A formula has a meaning in a point $w \in W$ of a structure

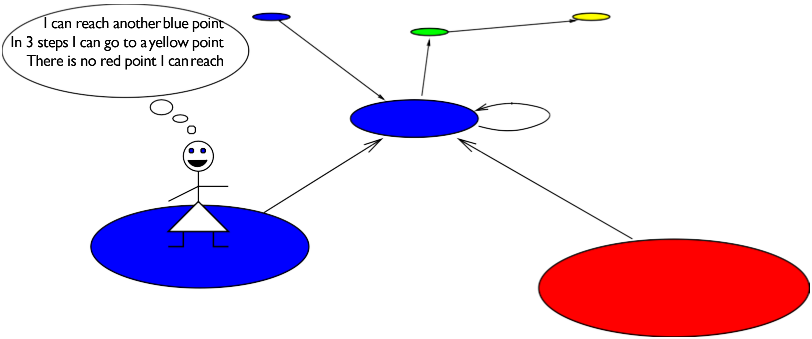
Relational structures in first order and modal logic

In first order logic, re-lational structures are described from the top point of view. each point of W and the re-lation R can benamed.



Relational structures in first order and modal logic

In modal logics, relational structures are described from an internal perspective there is no way to mention points of W and the relation R.



The Language of a basic modal logic

If P is a set of primitive proposition, the set of formulas of the basic modal logic is defined as follows:

- each $p \in$ P is a formula (atomic formula);
- if A and B are formulas then $\neg A$, $A \wedge B$, $A \vee B$, $A \supset B$ and $A \equiv B$ are formulas
- if A is a formula $\Box A$ and $\Diamond A$ are formulas.

Intuitive interpretation of the basic modal logic

The formula $\Box\varphi$ can be intuitively interpreted in many ways

- φ is necessarily true (classical modal logic)
- φ is known/believed to be true (epistemic logic)
- φ is provable in a theory (provability logic)
- φ will be always true (temporal logic)
- . . .

In all these cases $\Diamond\varphi$ is interpreted as $\neg\Box\neg\varphi$.

In other words, $\Diamond\varphi$, stands for $\neg\varphi$ is not necessarily true, that is, φ is possibly true.

Semantics for the basic modal logic

A basic frame (or simply a frame) is an algebraic structure

$$F = \langle W , R \rangle$$

where $R \subseteq W \times W$.
An interpretation I (or assignment) of a modal language in a frame F, is a function

$$I : P \to 2^W$$

Intuitively $w \in I(p)$ means that p is true in w, or that w is of type p.
A model M is a pair (*frame, interpretation*). I.e.:

$$M = \langle F, I \rangle$$

Satisfiability of modal formulas

Truth is relative to a world, so we define that relation of $\vDash$ between a world in a model and a formula

$$M, w \vDash p \text{ iff } w \in I(p)$$

$$M, w \vDash \varphi \wedge \psi \text{ iff } M, w \vDash \varphi \text{ and } M, w \vDash \psi$$

$$M, w \vDash \varphi \vee \psi \text{ iff } M, w \vDash \varphi \text{ or } M, w \vDash \psi$$

$$M, w \vDash \varphi \supset \psi \text{ iff } M, w \vDash \varphi \implies \textit{implies } M, w \vDash \psi$$

$$M, w \vDash \varphi \equiv \psi \text{ iff } M, w \vDash \varphi \text{ iff } M, w \vDash \psi$$

$$M, w \vDash \neg\varphi \text{ iff not } M, w \vDash \varphi$$

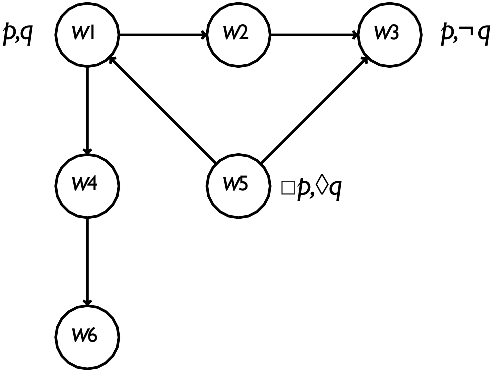
$$M, w \vDash \Box\varphi \text{ iff for all } w' \text{ s.t. } wRw', M, w' \vDash \varphi$$

$$M, w \vDash \Diamond\varphi \text{ iff there is a } w' \text{ s.t. } wRw' \text{ and } M, w' \vDash \varphi$$

φ is globally satisfied in a model M, in symbols, $M \vDash \varphi$ if

$$M, w \vDash \varphi \quad \text{for all } w \in W$$

Satisfiability example



Validity relation on frames

A formula φ is valid in a world w of a frame F, in symbols $F, w \vDash \varphi$ iff

$$M, w \vDash \varphi \text{ for all } I \text{ with } M = \langle F, I \rangle$$

A formula φ is valid in a frame F, in symbols $F \vDash \varphi$ iff

$$F, w \vDash \varphi \text{ for all } w \in W$$

If C is a class of frames, then a formula φ is valid in the class of frames C, in symbols $\vDash_C \varphi$ iff

$$F \vDash \varphi \text{ for all } F \in C$$

A formula φ is valid, in symbols $\vDash \varphi$ iff

$$F \vDash \varphi \text{ for all models frames } F$$

Logical consequence

- φ is a local logical consequence of Γ, in symbols $\Gamma \vDash \varphi$, if for every model $M = \langle F, I \rangle$ and every point $w \in W$,

$$M, w \vDash \Gamma \text{ implies that } M, w \vDash \varphi$$

- φ is a local logical consequence of Γ in a class of frames C, in symbols $\Gamma \vDash_C \varphi$ if for avery model $M = \langle F, I \rangle$ with $F \in C$ and every point $w \in W$,

$$M, w \vDash \Gamma \text{ implies that } M, w \vDash \varphi$$

Hilbert axioms for normal modal logic

A1 $\varphi \supset (\psi \supset \varphi)$

A2 $(\varphi \supset (\psi \supset \vartheta)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \vartheta))$

A3 $(\neg\psi \supset \neg\varphi) \supset ((\neg\psi \supset \varphi) \supset \varphi)$

MP $\dfrac{\varphi \quad \varphi \supset \psi}{\psi}$

K $\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$

Nec $\dfrac{\varphi}{\Box\varphi}$ the necessitation rule

The above set of axioms and rules is called **K**, and every modal logic with a validity relation closed under the rules of **K** is a Normal Modal Logic.

Remark on Nec

Notice that **Nec** rule is not the same as

$$\varphi \supset \Box\varphi \tag{3}$$

indeed formula (3) is not valid.

Assignment Find a model in which (3) is false

Satisfiability – exercises

Exercise

Show that each of the following formulas is not valid by constructing a frame $F = (W, R)$ that contains a world that does not satisfy them.

1. $\Box\bot$
2. $\Diamond\varphi \supset \Box\varphi$
3. $\Diamond\Box\varphi \supset \Box\Diamond\varphi$

Multi-Modal Logics

All the definitions given for basic modal logic can be generalized in the case in which we have n □-operators $\Box_1, \ldots, \Box_n$ (and also $\Diamond_1, \ldots, \Diamond_n$), which are interpreted in the frame

$$F = (W, R_1, \ldots R_n)$$

Every $\Box_i$ and $\Diamond_i$ is interpreted w.r.t. the relation R_i .

A logic with n modal operators is called Multi-Modal.

Multi-Modal logics are often used to model Multi-Agent systems where modality $\Box_i$ is used to express the fact that "agent i knows (believes) that φ".

Exercises

Exercise

Let $F = (W, R_1, \ldots, R_n)$ be a frame for the modal language with n modal operator $\Box_1, \ldots, \Box_n$. Show that the following properties holds:

1. $F \models \mathbf{K}_i$ (where $\mathbf{K}_i$ is obtained by replacing $\Box$ with $\Box_i$ in the axiom $\mathbf{K}$)
2. If $R_i \subseteq R_j$ then $F \models \Diamond_i \varphi \supset \Diamond_j \varphi$
3. If $R_i \subseteq R_j$ then $F \models \Box_j \varphi \supset \Box_i \varphi$
4. $F \not\models \Box_i p \supset \Box_j p$ for any primitive proposition p
5. If $R_i \subseteq R_j \circ R_k$, then[a] $F \models \Diamond_i \varphi \supset \Diamond_j \Diamond_k \varphi$

---

[a]Given two binary relations R and S on the set W, $R \circ S = \{(v, u) | (v, w) \in R \text{ and } (w, u) \in S\}$

Other exercises

Exercise

Prove that the following formulae are valid:

- $\models \Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi$
- $\models \Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi$
- $\models \neg\Diamond\varphi \equiv \Box\neg\varphi$
- $\neg\Box\Diamond\Diamond\Box\Box\Diamond\Box\varphi \equiv \Diamond\Box\Box\Diamond\Diamond\Box\Diamond\neg\varphi$ (i.e., pushing in $\neg$ changes $\Box$ into $\Diamond$ and $\Diamond$ into $\Box$)

Suggestion: keep in mind the analogy $\Box/\forall$ and $\Diamond/\exists$.

Exercise

Exercise

Consider the frame $F = (W, R)$ with

- $W = \{0, 1, \ldots n-1\}$
- $R = \{(0,1), (1,2), \ldots, (n-1, 0)\}$

Show that the following formulas are valid in F

1. $\Box\varphi \equiv \Diamond\varphi$
2. $\varphi \equiv \underbrace{\Box \ldots \Box}_{n}\varphi$

Answers also the following questions:

3. can you explain which property of the frame R is formalized by formula 1 and 2?
4. Can you imagine another frame F', different from F that satisfies formulas 1 and 2?

Expressing properties on structures

formula true at w	property of w
$\Diamond \top$	w has a successor point
$\Diamond\Diamond \top$	w has a successor point with a successor point
$\underbrace{\Diamond \ldots \Diamond}_{n} \top$	there is a path of length n starting at w
$\Box \bot$	w does not have any successor point
$\Box\Box \bot$	every successor of w does not have a successor point
$\underbrace{\Box \ldots \Box}_{n} \bot$	every path starting form w has length less then n

Expressing properties on structures

formula true at w	property of w
$\Diamond p$	w has a successor point which is p
$\Diamond\Diamond p$	w has a successor point with a successor point which is p
$\underbrace{\Diamond \ldots \Diamond}_{n} p$	there is a path of length n starting at w and ending at a point which is p
$\Box p$	every successor of w are p
$\Box\Box p$	all the successors of the successors of w are p
$\underbrace{\Box \ldots \Box}_{n} p$	all the paths of length n starting form w ends in a point which is p

Properties of accessibility relation

- Formulas can be used to shape the "form" of the structure, as in the examples expressed before or to impose properties on the accessibility relation R .
- Temporal logic: if the accessibility relation is supposed to represent a temporal relation, and wRw^{l} means that w^{l} is a future world w.r.t. w, then R must be a transitive relation. That is if w^{l} is a future world of w, then any future world of w^{l} is also a future world of w .
- Logic of knowledge: if the accessibility relation is used to represent the knowledge of an agent A, and wRw^{l} represents the fact that w^{l} is a possible situation coherent with its actual situation w , then R must be reflexive, since w is always coherent with itself.

Typical Properties of R

The following table summarizes the most relevant properties of the accessibility relation, which have been studied in modal logic, and for which it has been provided a sound and complete axiomatization

Properties of R

R is reflexive	$\forall w.R(w, w)$
R is transitive	$\forall w\,v\,u.(R(w, v) \wedge R(v, u) \supset R(w, u))$
R is symmetric	$\forall w\,v.(R(w, v) \supset R(v, w))$
R is Euclidean	$\forall w\,v\,u.(R(w, v) \wedge R(w, u) \supset R(v, u))$
R is serial	$\forall w.\exists v R(w, v)$
R is weakly dense	$\forall w\,v.R(w, v) \supset \exists u.(R(w, u) \wedge R(u, v))$
R is partly functional	$\forall w\,v\,u.(R(w, v) \wedge R(v, u) \supset v = u)$
R is functional	$\forall w\,\exists! v.R(w, v)$
R is weakly connected	$\forall u\,v\,w.(R(u, v) \wedge R(u, w) \supset R(v, w) \vee v = w \vee R(w, v))$
R is weakly directed	$\forall u\,v\,w.(R(u, v) \wedge R(u, w) \supset \exists t(R(v, t) \wedge R(w, t)))$

We will investigate only the ones in red color.

R is reflexive

The axiom **T**

If a frame is reflexive (we say that a frame has a property, when the relation R has such a property) then the formulas

$$\mathbf{T} \quad \Box\varphi \supset \varphi$$

holds. (Or alternatively $\varphi \supset \Diamond\varphi$.)

R is reflexive - soundness

Let M be a model on a reflexive frame $F = (W, R)$ and w any world in W. We prove that $M, w \models \Box\varphi \supset \varphi$.

1. Since R is reflexive then wRw
2. Suppose that $M, w \models \Box\varphi$ (Hypothesis)
3. From the satisfiability condition of $\Box$, $M, w \models \Box\varphi$, and wRw imply that
4. $M, w \models \varphi$ (Thesis)

   Since from (Hypothesis) we have derived (Thesis), we can conclude that

   $M, w \models \Box\varphi \supset \varphi$.

R is reflexive - completeness

Suppose that a frame $F = (W, R)$ is not reflexive.

1. If R is not reflexive then there is a $w \in W$ which does not access to itself. I.e., for some $w \in W$ it does not hold that wRw.
2. Let M be any model on F, and let φ be the propositional formula p. Let V the set p true in all the worlds of W but w where p is set to be false.
3. From the fact that w does not access to itself, we have that in all the worlds w
   accessible from w, p is true, i.e, $\forall w', wRw', M, w' \vDash p$.
4. Form the satisfiability condition of $\Box$ we have that M,

   $w \vDash \Box p$.
5. since $M, w \vDash p$, we have that $M, w \vDash \Box p \supset p$.

R is symmetric

The axiom B

If a frame is symmetric then the formula

$$\mathbf{B} \quad \varphi \supset \Box\Diamond\varphi$$

holds.

R is symmetric - soundness

Let M be a model on a symmetric frame $F = (W, R)$ and w any world in W. We prove that $M, w \models \varphi \supset \Box\Diamond\varphi$.

1. Suppose that $M, w \models \varphi$ (Hypothesis)
2. we want to show that $M, w \models \Box\Diamond\varphi$ (Thesis)
3. Form the satisfiability conditions of $\Box$, we need to prove that for every world w^l

   accessible from w, $M, w^l \models \Diamond\varphi$.
4. Let w^l, be any world accessible from w, i.e., wRw^l
5. from the fact that R is symmetric, we have that w^lRw
6. From the satisfiability condition of $\Diamond$, from the fact that w^lRw and that $M, w \models \varphi$, we have that $M, w^l \models \Diamond\varphi$.
7. so for every world w^l accessible from w, we have that $M, w^l \models \Diamond\varphi$.
8. From the satisfiability condition of $\Box$, $M, w \models \Box\Diamond\varphi$ (Thesis)
9. Since from (Hypothesis) we have derived (Thesis), we can conclude that $M, w \models \varphi \supset \Box\Diamond\varphi$.

R is symmetric - completeness

Suppose that a frame $F = (W, R)$ is not Symmetric.

1. If R is not symmetric then there are two worlds $w, w' \in W$ such that wRw' and not $w'Rw$
2. Let M be any model on F, and let φ be the propositional formula p. Let V the set p false in all the worlds of W but w where p is set to be true.
3. From the fact that w' does not access to w, it means that in all the worlds accessible from w', p is false,
4. i.e. there is no world w'' accessible from w' wuch that $M, w'' \vDash p$.
5. by the satisfiability conditions of $\Diamond$, we have that $M, w' \nvDash \Diamond p$.
6. Since there is a world w' accessible from w, with $M, w \nvDash \Diamond p$, form the satisfiability condition of $\Box$ we have that $M, w \nvDash \Box\Diamond p$.
7. since $M, w \vDash p$, and $M, w \nvDash \Box\Diamond p$. we have that $M, w \nvDash p \supset \Box\Diamond p$.

R is serial

The axiom D

If a frame is serial then the formula

$$\mathbf{D} \quad \Box\varphi \supset \Diamond\varphi$$

holds.

R is serial - soundness

Let M be a model on a serial frame $F = (W, R)$ and w any world in W. We prove that M, $w \vDash \Box\varphi \supset \Diamond\varphi$.

1. Since R is serial there is a world $w^l \in W$ with wRw^l
2. Suppose that $M, w \vDash \Box\varphi$ (Hypothesis)
3. From the satisfiability condition of $\Box$, $M, w \vDash \Box\varphi$ implies that M, $w^l \vDash \varphi$
4. Since there is a world w^l accessible from w that satisfies φ, from the satisfiability conditions of $\Diamond$ we have that $M, w \vDash \Diamond\varphi$ (Thesis) .
5. Since from (Hypothesis) we have derived (Thesis), we can conclude that

   $M, w \vDash \Box\varphi \supset \Diamond\varphi$.

R is serial - completeness

Suppose that a frame $F = (W, R)$ is not Serial.

1. If R is not serial then there is a $w \in W$ which does not have any accessible world. I.e., for all w' it does not hold that wRw'.
2. Let M be any model on F.
3. Form the satisfiability condition of $\Box$ and from the fact that w does not have any accessible world, we have that $M, w \vDash \Box\varphi$.
4. Form the satisfiability condition of $\Diamond$ and from the fact that w does not have any accessible world, we have that $M, w \vDash \Diamond\varphi$.
5. this implies that $M, w \vDash \Box\varphi \supset \Diamond\varphi$

R is transitive

The axiom 4

If a frame is transitive then the formula

$$\mathbf{4} \quad \Box\varphi \supset \Box\Box\varphi$$

holds.

R is transitive - soundness

Let M be a model on a transitive frame $F = (W, R)$ and w any world in W. We prove that $M, w \models \Box\varphi \supset \Box\Box\varphi$.

1. Suppose that $M, w \models \Box\varphi$ (Hypothesis).
2. We have to prove that $M, w \models \Box\Box\varphi$ (Thesis)
3. From the satisfiability condition of $\Box$, this is equivalent to prove that for all world w^I accessible from w $M, w^I \models \Box\varphi$.
4. Let w^I be any world accessible from w. To prove that $M, w^I \models \Box\varphi$ we have to prove that for all the world w^{II} accessible from w^I, $M, w^{II} \models \varphi$.
5. Let w^{II} be a world accessible from w^I, i.e., $w^I R w^{II}$.
6. From the facts wRw^I and $w^I R w^{II}$ and the fact that R is transitive, we have that

   wRw^{II}.
7. Since $M, w \models \Box\varphi$, from the satisfiability conditions of $\Box$ we have that

   $M, w^{II} \models \varphi$.
8. Since $M, w^{II} \models \varphi$ for every world w^{II} accessible from w^I, then $M, w^I \models \Box\varphi$. and
9. therefore $M, w \models \Box\Box\varphi$. (Thesis)
   Since from (Hypothesis) we have derived (Thesis), we can conclude that
10. $M, w \models \Box\varphi \supset \Box\Box\varphi$.

R is transitive - completeness

Suppose that a frame F $= (W, R)$ is not transitive.

1. If R is not transitive then there are three worlds $w, w^I, w^{II} \in W$, such that wRw^I, w^IRw^{II} but not wRw^{II}.
2. Let M be any model on F, and let φ be the propositional formula p. Let V the set p true in all the worlds of W but w^{II} where p is set to be false.
3. From the fact that w does not access to w^{II}, and that w^{II} is the only world where p is false, we have that in all the worlds accessible from w, p is true.
4. This implies that $M, w \vDash \Box p$.
5. On the other hand, we have that w^IRw^{II}, and $w^{II} \vDash p$ implies that $M, w^I \vDash \Box\varphi$.
6. and since wRw^I, we have that $M, w \vDash \Box\Box p$.
7. In summary: $M, w \vDash \Box\Box p$, and $M, w \vDash \Box P$; from which we have that $M, w \vDash \Box p \supset \Box\Box p$.

R is euclidean

The axiom 5

If a frame is euclidean then the formula

$$\mathbf{5} \quad \Diamond\varphi \supset \Box\Diamond\varphi$$

holds.

R is euclidean - soundness

Let M be a model on a euclidean frame $F = (W, R)$ and w any world in W. We prove that $M, w \vDash \Diamond\varphi \supset \Box\Diamond\varphi$.

1. Suppose that $M, w \vDash \Diamond\varphi$ (Hypothesis).
2. The satisfiability condition of $\Diamond$ implies that there is a world w^{I} accessible from w such that $M, w^{I} \vDash \varphi$.
3. We have to prove that $M, w \vDash \Box\Diamond\varphi$ (Thesis)
4. From the satisfiability condition of $\Box$, this is equivalent to prove that for all world w^{II} accessible from w $M, w^{II} \vDash \Diamond\varphi$,
5. let w^{II} be any world accessible from w. The fact that R is euclidean, the fact that wRw^{I} implies that $w^{II}Rw^{I}$.
6. Since $M, w^{I} \vDash \varphi$, the satisfiability condition of $\Diamond$ implies that $M, w^{II} \vDash \Diamond\varphi$.
7. and therefore $M, w \vDash \Box\Diamond\varphi$. (Thesis)
8. Since from (Hypothesis) we have derived (Thesis), we can conclude that $M, w \vDash \Box\varphi \supset \Box\Diamond\varphi$.

R is euclidean - completeness

Suppose that a frame F = (W, R) is not euclidean.

1. If R is not euclidean then there are three worlds $w, w', w'' \in W$, such that wRw', wRw'' but not $w'Rw''$.
2. Let M be any model on F, and let φ be the propositional formula p. Let V the set p false in all the worlds of W but w' where p is set to be true.
3. From the fact that w'' does not access to w', and in all the other worlds p is false, we have that $w'' \not\models \Diamond p$
4. this implies that $M, w \not\models \Box\Diamond p$.
5. On the other hand, we have that wRw', and $w' \models p$, and therefore $M, w \models \Diamond p$. $M, w \not\models \Box p \supset \Box\Box p$.
6. In summary: $M, w \not\models \Box\Diamond p$, and $M, w \models \Diamond P$; from which we have that M, $w \not\models \Diamond p \supset \Box\Diamond p$.

Soundness and completeness

K		the class of all frames
K4	**4**	the class of transitive frames
KT	**T**	the class of reflexive frames
KB	**B**	the class of symmetric frames
KD		the class of serial frames
KT4	**S4**	the class of reflexive and transitive frames
KT4B	**S5**	the class of frames with an equivalence relation
KT5	**S5**	the class of frames with an equivalence relation

Multi-Modal Logics

All the definitions given for basic modal logic can be generalized in the case in which we have n $\Box$-operators $\Box_1, \ldots, \Box_n$ (and also $\Diamond_1, \ldots, \Diamond_n$), which are interpreted in the frame

$$F = (W, R_1, \ldots R_n)$$

Every $\Box_i$ and $\Diamond_i$ is interpreted w.r.t. the relation R_i.

A logic with n modal operators is called Multi-Modal. Multi-Modal logics are often used to model Multi-Agent systems where modality $\Box_i$ is used to express the fact that "agent i knows (believes) …".

Exercises

Exercise

Let $F = (W, R_1, \ldots, R_n)$ be a frame for the modal language with n modal operator $\Box_1, \ldots, \Box_n$. Show that the following properties holds:

1. $F \models \mathbf{K}_i$ (where $\mathbf{K}_i$ is obtained by replacing $\Box$ with $\Box_i$ in the axiom $\mathbf{K}$)
2. If $R_i \subseteq R_j$ then $F \models \Diamond_i\varphi \supset \Diamond_j\varphi$
3. If $R_i \subseteq R_j$ then $F \models \Box_j\varphi \supset \Box_i\varphi$
4. $F \not\models \Box_i p \supset \Box_j p$ for any primitive proposition p
5. If $R_i \subseteq R_j \circ R_k$, then[a] $F \models \Diamond_i\varphi \supset \Diamond_j\Diamond_k\varphi$

---

[a] Given two binary relations R and S on the set W, $R \circ S = \{(v, u) | (v, w) \in R \text{ and } (w, u) \in S\}$

Modal logics and agents. What is an agent?

Definition

In artificial intelligence, an intelligent agent (IA) is an autonomous entity which observes and acts upon an environment (i.e. it is an agent) and directs its activity towards achieving goals (i.e. it is rational). Intelligent agents may also learn or use knowledge to achieve their goals. [Russell, Stuart J.; Norvig, Peter (2003), Artificial Intelligence: A Modern Approach (2nd ed.)]

Modal logics and agents. What is an agent?

Definition

In artificial intelligence, an intelligent agent (IA) is an autonomous entity which observes and acts upon an environment (i.e. it is an agent) and directs its activity towards achieving goals (i.e. it is rational). Intelligent agents may also learn or use knowledge to achieve their goals. [Russell, Stuart J.; Norvig, Peter (2003), Artificial Intelligence: A Modern Approach (2nd ed.)]

What is an agent?

Definition

An agent is a computer system capable of autonomous action in some environment, in order to achieve its delegated goals.[Wooldridge, Mike (2009), An Introduction to MultiAgent Systems (2nd ed.)]

What is an agent?

Definition

An agent is a computer system capable of autonomous action in some environment, in order to achieve its delegated goals.[Wooldridge, Mike (2009), An Introduction to MultiAgent Systems (2nd ed.)]

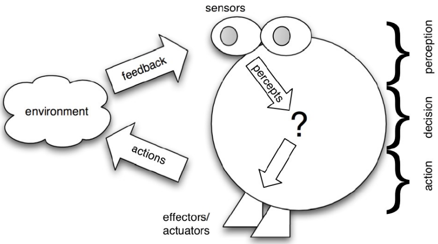
Main building blocks

- Agents act;
- Agents are able to achieve goals (often complex).

$\Downarrow$

Agents are in a close-coupled, continual interaction with their environment:

sense - decide - act - sense - decide - . . .



Simple (Uninteresting) Agents

- Thermostat
  - delegated goal is maintain room temperature
  - actions are heat on/off
- UNIX biff program
  - delegated goal is monitor for incoming email and flag it
  - actions are GUI actions.
- They are trivial because the decision making they do is trivial.

Intelligent Agents as Intentional systems

- When explaining human activity, we use statements like the following:

  Janine took her umbrella because she believed it was raining and she wanted to stay dry.

- These statements make use of a folk psychology, by which human behaviour is predicted and explained by attributing attitudes such as believing, wanting, hoping, fearing, ...

Mental attitudes

(Intelligent) agents are usually described in terms of:

- Informational attitudes:
  - Knowledge
  - Belief
- Motivational-attitudes:
  - Desire
  - Intention
  - Obligation
  - Commitment
  - Choice
  - …

Logical agent theories:

(Intelligent) agents are usually described in terms of:

- Informational attitudes (modal logic):
- Motivational-attitudes (modal logic):
- Dynamic component (temporal or dynamic logic).

Informational attitudes via Epistemic Logic

- Logic to reason about knowledge (and belief).
- Seminal book: Jaakko Hintikka, "Knowledge and Belief - An Introduction to the Logic of the Two Notions" (1962).
- $\Box\varphi$ is used to express "an agent knows that φ" ($K\varphi$) or "an agent believes that φ" ($B\varphi$).
- The multi-modal version used to represent knowledge (beliefs) of several agents

Example: "Alice does not know that Bob knows its her Birthday":

$$\neg K_{Alice} K_{Bob} AlicesBirthday$$

Examples

- "Ann knows that P implies Q"
  $K_{Ann}(P \supset Q)$
- "either Ann does or does not know P"
  $K_{Ann}P \vee K_{Ann}\neg P$
- "P is possible for Ann"
  $L_{Ann}P$ (where L is a shorthand for $\neg K \neg$)
- "Ann knows that she thinks P is possible"
  $K_{Ann}(L_{Ann}P)$

A characterization of knowledge

- Axioms for modal **K**;
- **T**: $K\varphi \supset \varphi$ (axiom of Necessity)
  "If an agent knows that φ, then φ must be true". Or, . . . an agent cannot have wrong knowledge.
- **4**: $K\varphi \supset KK\varphi$ (axiom of Positive Introspection)
  "If an agent knows that φ, then (s)he knows that s(he) knows that φ". Or, . . . an agent knows that s(he) knows.

The logic **KT4** (better known as **S4**), provides a minimal characterization of knowledge, and corresponds to the set of reflexive and transitive frames.

But, what about ignorance? We also know what we do not know!

A characterization of knowledge

- **5**: $\neg K\varphi \supset K\neg K\varphi$ (axiom of Negative Introspection)
  "If an agent does not know that φ, then (s)he knows that s(he) does not know knows that φ". Or, . . . an agent knows that s(he) does not know.

The logic **KT45** (better known as **S5**), provides the standard characterization of knowledge, and corresponds to the set of reflexive, symmetric and transitive relations (that is, all the equivalence relations).

A characterization of belief

- Axioms for modal **K**;
- Agents can have false beliefs. Therefore **T** does not hold.
- $B\varphi \supset BB\varphi$ (axiom of Positive Introspection)
  "If an agent believes that φ, then (s)he believes that s(he) believes that φ".
- **5**: $\neg B\varphi \supset B\neg B\varphi$ (axiom of Negative Introspection)
  "If an agent does not believe that φ, then (s)he believes that s(he) does not know knows that φ". Or, . . . an agent believes that s(he) does not believe.

The logic **K45** provides a minimal characterization of belief, and corresponds to the set of transitive and euclidean.

A characterization of belief

- Are beliefs mutually consistent? If yes then $\neg B(\varphi \wedge \neg\varphi)$ holds. (Axiom of Consistency)
  "an agent does not believe that" φ and $\neg\varphi$.
- An alternative formulation of this property is via the axiom **D**:
  $\Box\varphi \supset \Diamond\varphi$. (that is, $B\varphi \supset \neg B\neg\varphi$)
  "If an agent believes that φ then s(he) does not believe that not φ".

The logic **KD45** provides an alternative characterization of belief, and corresponds to the set of transitive, euclidean and serial relations

Note: the axiom **D** is a typical axiom of *Deontic logic.*
Prove that $\neg B(\varphi \wedge \neg\varphi)$ is equivalent to $\Box\varphi \supset \Diamond\varphi$.