Chap 5 Initial-Value Problems for Ordinary Differential Equations

1. Euler's method → 5.2
2. Mid Point formula
3. Modify Euler method
4. Heun's method
5. 4-RK method

$\left.\begin{array}{l}2\text{–}5\end{array}\right\}$ 5.4

A differential equation is an equation involving derivatives. In the form

$$y'(t) = f(t, y(t)),$$

a first-order differential equation expresses the rate of change of a quantity y in terms of the present time and the current value of the quantity. Differential equations are used to model, understand, and predict systems that change with time.

In general, a linear or non-linear ordinary differential equation can be written as

$$\frac{d^n y}{dt^n} = f\left(t, y, \frac{dy}{dt}, \cdots, \frac{d^{n-1}y}{dt^{n-1}}\right)$$

Here we shall focus on a system of first order differential equations of the form $\frac{dy}{dt} = f(t, y)$ with the initial condition $y(t_0) = y_0$, which is called an initial value problem (IVP).

a second order differential equation of the form

$$y'' = f(t, y, y')$$

We also examine the relationship of a system of this type to the general nth-order initial-value problem of the form

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)}),$$

for $a \le t \le b$, subject to the initial conditions

$$y(a) = \alpha_1, \quad y'(a) = \alpha_2, \quad \ldots, \quad y^{n-1}(a) = \alpha_n.$$

$$\boxed{\frac{dy}{dx} = f(x, y) \quad \text{and} \quad \frac{d^2y}{dx^2} = f(x, y, y')}$$

$$\boxed{\frac{d^ny}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}),}$$

CLASSIFICATION BY TYPE If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an **ordinary differential equation (ODE).** For example,

Examples:

$$\frac{dy}{dx} + 5y = e^x,$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0,$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

partial differential equation (PDE). For example,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t},$$

FIRST- AND SECOND-ORDER IVPS

Solve: $$\frac{dy}{dx} = f(x, y)$$

Subject to: $$y(x_0) = y_0$$

Solve: $$\frac{d^2y}{dx^2} = f(x, y, y')$$

Subject to: $$y(x_0) = y_0, y'(x_0) = y_1$$

Example:

Solve $(1 + x)\,dy - y\,dx = 0$.

Dividing by $(1 + x)y$, we can write $dy/y = dx/(1 + x)$,

$$\int \frac{dy}{y} = \int \frac{dx}{1 + x}$$

$$\ln|y| = \ln|1 + x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1}$$

$$y = c(1 + x).$$

Example:

Solve the initial-value problem $\dfrac{dy}{dx} = -\dfrac{x}{y}, \quad y(4) = -3.$

Rewriting the equation as $y\,dy = -x\,dx$,

$$\int y\,dy = -\int x\,dx \qquad \text{and} \qquad \frac{y^2}{2} = -\frac{x^2}{2} + c_1.$$

$$x^2 + y^2 = c^2$$

Now when $x = 4$, $y = -3$, so $16 + 9 = 25 = c^2$.

$$x^2 + y^2 = 25$$

5.2 Euler's Method

Derivation

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

$$\left(\frac{dy}{dt}\right)_{(t_0, y_0)} = \frac{y - y_0}{t - t_0} = f(t_0, y_0)$$

$$y = y_0 + (t - t_0) f(t_0, y_0)$$

Hence, the value of y corresponding to $t = t_1$ is given by

$$y_1 = y_0 + (t_1 - t_0) f(t_0, y_0)$$

$$y_2 = y_1 + hf(t_1, y_1)$$

recurrence relation

$$y_{m+1} = y_m + hf(t_m, y_m)$$

5.2 Euler's Method

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

The common distance between the points $h = (b - a)/N = t_{i+1} - t_i$ is called the step size.

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + h f(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N - 1.$$

ALGORITHM 5.1

Euler's

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

at $(N+1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b; integer N; initial condition α.

OUTPUT approximation w to y at the $(N+1)$ values of t.

Step 1 Set $h = (b-a)/N$;
$t = a$;
$w = \alpha$;
OUTPUT (t, w).

Step 2 For $i = 1, 2, \ldots, N$ do Steps 3, 4.

Step 3 Set $w = w + hf(t, w)$; (*Compute* w_i.)
$t = a + ih$. (*Compute* t_i.)

Step 4 OUTPUT (t, w).

Step 5 STOP.

Illustration In Example 1 we will use an algorithm for Euler's method to approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5,$$

at $t = 2$. Here we will simply illustrate the steps in the technique when we have $h = 0.5$.

$$\boxed{w_{i+1} = w_i + h f(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1.}$$

For this problem $f(t, y) = y - t^2 + 1$,

$$w_0 = y(0) = 0.5;$$

$$w_1 = w_0 + 0.5\left(w_0 - (0.0)^2 + 1\right) = 0.5 + 0.5(1.5) = 1.25;$$

$$w_2 = w_1 + 0.5\left(w_1 - (0.5)^2 + 1\right) = 1.25 + 0.5(2.0) = 2.25;$$

$$w_3 = w_2 + 0.5\left(w_2 - (1.0)^2 + 1\right) = 2.25 + 0.5(2.25) = 3.375;$$

$$y(2) \approx w_4 = w_3 + 0.5\left(w_3 - (1.5)^2 + 1\right) = 3.375 + 0.5(2.125) = 4.4375.$$

exact values given by $y(t) = (t+1)^2 - 0.5e^t$.

h=0.5

Complete the following table correct up to five decimal place:

n	t(n)	w(n)=App.value	Y(t)=Exact value	Abs.Error= Y(t)-w(t)

Example 1

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Use Algorithm 5.1 with $N = 10$ to determine approximations, and compare these with the exact values given by $y(t) = (t+1)^2 - 0.5e^t$.

Solution:

The common distance between the points $h = (b-a)/N = t_{i+1} - t_i$ is called the **step size**.

h = 0.2

$$\boxed{w_{i+1} = w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1.}$$

$$w_1 = 1.2(0.5) - 0.008(0)^2 + 0.2 = 0.8;$$

$$w_2 = 1.2(0.8) - 0.008(1)^2 + 0.2 = 1.152;$$

exact values given by $y(t) = (t+1)^2 - 0.5e^t$.

Table 5.1

	Approximate	Exact	Error
t_i	w_i	$y_i = y(t_i)$	$\lvert y_i - w_i \rvert$
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8000000	0.8292986	0.0292986
0.4	1.1520000	1.2140877	0.0620877
0.6	1.5504000	1.6489406	0.0985406
0.8	1.9884800	2.1272295	0.1387495
1.0	2.4581760	2.6408591	0.1826831
1.2	2.9498112	3.1799415	0.2301303
1.4	3.4517734	3.7324000	0.2806266
1.6	3.9501281	4.2834838	0.3333557
1.8	4.4281538	4.8151763	0.3870225
2.0	4.8657845	5.3054720	0.4396874

h = 0.2

Example: Solve IVP and find error table for h=0.2 and h=0.1

$$y' = ty + t^3 \quad \text{initial condition } y(0) = 1.$$

The exact solution $y(t) = 3e^{t^2/2} - t^2 - 2$

Solution:

$$f(t, y) = ty + t^3.$$

Euler's Method will be the iteration

$$w_0 = 1$$
$$w_{i+1} = w_i + h(t_i w_i + t_i^3).$$

step	t_i	w_i	y_i	e_i
0	0.0	1.0000	1.0000	0.0000
1	0.2	1.0000	1.0206	0.0206
2	0.4	1.0416	1.0899	0.0483
3	0.6	1.1377	1.2317	0.0939
4	0.8	1.3175	1.4914	0.1739
5	1.0	1.6306	1.9462	0.3155

step	t_i	w_i	y_i	e_i
0	0.0	1.0000	1.0000	0.0000
1	0.1	1.0000	1.0050	0.0050
2	0.2	1.0101	1.0206	0.0105
3	0.3	1.0311	1.0481	0.0170
4	0.4	1.0647	1.0899	0.0251
5	0.5	1.1137	1.1494	0.0357
6	0.6	1.1819	1.2317	0.0497
7	0.7	1.2744	1.3429	0.0684
8	0.8	1.3979	1.4914	0.0934
9	0.9	1.5610	1.6879	0.1269
10	1.0	1.7744	1.9462	0.1718

EXAMPLE 2 Comparison of Approximate and Actual Values

Consider the initial-value problem $y' = 0.2xy$, $y(1) = 1$. Use Euler's method to obtain an approximation of $y(1.5)$ using first $h = 0.1$ and then $h = 0.05$.

$$f(x, y) = 0.2xy,$$

$$y_{n+1} = y_n + h(0.2x_n y_n) \quad \text{where } x_0 = 1 \text{ and } y_0 = 1.$$

the true or actual values were calculated from the known solution $y = e^{0.1(x^2-1)}$. (Verify.) The **absolute error** is defined to be

$$|\textit{actual value} - \textit{approximation}|.$$

The **relative error** and **percentage relative error** are, in turn,

$$\frac{\textit{absolute error}}{|\textit{actual value}|} \quad \text{and} \quad \frac{\textit{absolute error}}{|\textit{actual value}|} \times 100.$$

$h = 0.1$

x_n	y_n	Actual value	Abs. error	% Rel. error
1.00	1.0000	1.0000	0.0000	0.00
1.10	1.0200	1.0212	0.0012	0.12
1.20	1.0424	1.0450	0.0025	0.24
1.30	1.0675	1.0714	0.0040	0.37
1.40	1.0952	1.1008	0.0055	0.50
1.50	1.1259	1.1331	0.0073	0.64

$h = 0.05$

x_n	y_n	Actual value	Abs. error	% Rel. error
1.00	1.0000	1.0000	0.0000	0.00
1.05	1.0100	1.0103	0.0003	0.03
1.10	1.0206	1.0212	0.0006	0.06
1.15	1.0318	1.0328	0.0009	0.09
1.20	1.0437	1.0450	0.0013	0.12
1.25	1.0562	1.0579	0.0016	0.16
1.30	1.0694	1.0714	0.0020	0.19
1.35	1.0833	1.0857	0.0024	0.22
1.40	1.0980	1.1008	0.0028	0.25
1.45	1.1133	1.1166	0.0032	0.29
1.50	1.1295	1.1331	0.0037	0.32

EXERCISE SET 5.2

1. Use Euler's method to approximate the solutions for each of the following initial-value problems.
   a. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0$, with $h = 0.5$
   b. $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1$, with $h = 0.5$
   c. $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2$, with $h = 0.25$
   d. $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1$, with $h = 0.25$

3. The actual solutions to the initial-value problems in Exercise 1 are given here. Compare the actual error at each step to the error bound.
   a. $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$
   b. $y(t) = t + \frac{1}{1-t}$
   c. $y(t) = t \ln t + 2t$
   d. $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$

Solution:

1. Euler's method gives the approximations in the following table.

a.

i	t_i	w_i	$y(t_i)$
1	0.500	0.0000000	0.2836165
2	1.000	1.1204223	3.2190993

3. a.

t	Actual Error	Error bound
0.5	0.2836165	11.3938
1.0	2.0986771	42.3654

Error Bounds for Euler's Method

Theorem 5.9

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Let $w_0, w_1, \ldots, w_N$ be the approximations generated by Euler's method for some integer N. Then, for each $i = 0, 1, 2, \ldots, N$,

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right].$$

Example 2 The solution to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5,$$

was approximated in Example 1 using Euler's method with $h = 0.2$. Use the inequality in Theorem 5.9 to find a bounds for the approximation errors and compare these to the actual errors.

Solution Because $f(t, y) = y - t^2 + 1$, we have $\partial f(t, y)/\partial y = 1$ for all y, so $L = 1$. For this problem, the exact solution is $y(t) = (t + 1)^2 - 0.5e^t$, so $y''(t) = 2 - 0.5e^t$ and

$$|y''(t)| \le 0.5e^2 - 2, \quad \text{for all } t \in [0, 2].$$

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right]..$$

Using the inequality in the error bound for Euler's method with $h = 0.2$, $L = 1$, and $M = 0.5e^2 - 2$ gives

$$|y_i - w_i| \le 0.1(0.5e^2 - 2)(e^{t_i} - 1).$$

$$|y(0.2) - w_1| \le 0.1(0.5e^2 - 2)(e^{0.2} - 1) = 0.03752;$$

$$|y(0.4) - w_2| \le 0.1(0.5e^2 - 2)(e^{0.4} - 1) = 0.08334;$$

Table 5.2

t_i	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
Actual Error	0.02930	0.06209	0.09854	0.13875	0.18268	0.23013	0.28063	0.33336	0.38702	0.43969
Error Bound	0.03752	0.08334	0.13931	0.20767	0.29117	0.39315	0.51771	0.66985	0.85568	1.08264

Solution: (Class Activity)

1. Euler's method gives the approximations in the following tables.

c. $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2$, with $h = 0.25$; actual solution $y(t) = t \ln t + 2t$.

(c)

i	t_i	w_i	$y(t_i)$
1	1.250	2.7500000	2.7789294
2	1.500	3.5500000	3.6081977
3	1.750	4.3916667	4.4793276
4	2.000	5.2690476	5.3862944

(c)

t	Actual error	Error bound
1.25	0.0289294	0.0355032
1.50	0.0581977	0.0810902
1.75	0.0876610	0.139625
2.00	0.117247	0.214785

Q-3 c)

Solution

$$f(t,y) = 1 + \frac{y}{t} \Rightarrow \left|\frac{\partial f}{\partial y}\right| = \left|\frac{1}{t}\right| \leq 1 = L$$

(Class Activity)

Then, f satisfies a Lischitz condition with $L = 1$

The exact solution is $y = t\ln t + 2t$

$$\Rightarrow |y''| = \left|\frac{1}{t}\right| \leq 1 = M, \quad \forall t \in [0,1]$$

Finding the error bound by using the inequality (Theorem 5.9)

$$|y(t_i) - w_i| \leq \frac{hM}{2L}(e^{L(t_i-a)} - 1)$$

with $L = 1, \;\; M = 1, \;\; h = 0.25, \; a = 1$

The error bound

(Home Activity)

$$|y(1.25) - w_1| \le \frac{0.5(1)}{2}(e^{(1.25-1)} - 1) = \boxed{0.0355032}$$

$$|y(1.5) - w_2| \le \frac{0.5(1)}{2}(e^{(1.5-1)} - 1) = \boxed{0.0810902}$$

$$|y(1.75) - w_3| \le \frac{0.5(1)}{2}(e^{(1.75-1)} - 1) = \boxed{0.139625}$$

$$|y(2) - w_4| \le \frac{0.5(1)}{2}(e^{(2-1)} - 1) = \boxed{0.214785}$$

(c)

t	Actual error	Error bound
1.25	0.0289294	0.0355032
1.50	0.0581977	0.0810902
1.75	0.0876610	0.139625
2.00	0.117247	0.214785

The actual error is

$$|y(1.25) - w_1| = |2.7789294 - 2.7500000| = \boxed{0.0289294}$$

$$|y(1.5) - w_2| = |3.6081977 - 3.550000| = \boxed{0.0581977}$$

$$|y(1.75) - w_3| = |4.4793276 - 4.3916667| = \boxed{0.0876609}$$

$$|y(2) - w_4| = |5.3862944 - 5.2690476| = \boxed{0.117247}$$

EXERCISE SET 5.2

Activity

2. Use Euler's method to approximate the solutions for each of the following initial-value problems.

   a. $y' = e^{t-y}, \quad 0 \le t \le 1, \quad y(0) = 1,$ with $h = 0.5$

   b. $y' = \dfrac{1+t}{1+y}, \quad 1 \le t \le 2, \quad y(1) = 2,$ with $h = 0.5$

4. The actual solutions to the initial-value problems in Exercise 2 are given here. Compute the actual error and compare this to the error bound if Theorem 5.9 can be applied.

   a. $y(t) = \ln(e^t + e - 1)$

   b. $y(t) = \sqrt{t^2 + 2t + 6} - 1$

ANY Questions?