CHAPTER 4: GENERAL VECTOR SPACES

4.1 Real Vector Spaces

1. **(a)** $\mathbf{u}+\mathbf{v}=(-1+3,2+4)=(2,6)$; $k\mathbf{u}=(0,3\cdot 2)=(0,6)$

(b) For any $\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$ in V, $\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2)$ is an ordered pair of real numbers, therefore $\mathbf{u}+\mathbf{v}$ is in V. Consequently, V is closed under addition.

For any $\mathbf{u}=(u_1,u_2)$ in V and for any scalar k, $k\mathbf{u}=(0,ku_2)$ is an ordered pair of real numbers, therefore $k\mathbf{u}$ is in V. Consequently, V is closed under scalar multiplication.

(c) Axioms 1-5 hold for V because they are known to hold for R^2.

(d) Axiom 7: $k((u_1,u_2)+(v_1,v_2))=k(u_1+v_1,u_2+v_2)=(0,k(u_2+v_2))=(0,ku_2)+(0,kv_2)$
$=k(u_1,u_2)+k(v_1,v_2)$ for all real k, u_1, u_2, v_1, and v_2;

Axiom 8: $(k+m)(u_1,u_2)=(0,(k+m)u_2)=(0,ku_2+mu_2)=(0,ku_2)+(0,mu_2)$

$=k(u_1,u_2)+m(u_1,u_2)$ for all real k, m, u_1, and u_2;

Axiom 9: $k(m(u_1,u_2))=k(0,mu_2)=(0,kmu_2)=(km)(u_1,u_2)$ for all real k, m, u_1, and u_2;

(e) Axiom 10 fails to hold: $1(u_1,u_2)=(0,u_2)$ does not generally equal (u_1,u_2).
Consequently, V is not a vector space.

3. Let V denote the set of all real numbers.

Axiom 1: $x+y$ is in V for all real x and y;

Axiom 2: $x+y=y+x$ for all real x and y;

Axiom 3: $x+(y+z)=(x+y)+z$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0}=0$, we have $0+x=x+0=x$ for all real x;

Axiom 5: for each $\mathbf{u}=x$, let $-\mathbf{u}=-x$; then $x+(-x)=(-x)+x=0$

Axiom 6: kx is in V for all real k and x;

Axiom 7: $k(x+y)=kx+ky$ for all real k, x, and y;

Axiom 8: $(k+m)x=kx+mx$ for all real k, m, and x;

Axiom 9: $k(mx) = (km)x$ for all real k, m, and x;

Axiom 10: $1x = x$ for all real x.

This is a vector space – all axioms hold.

5. Axiom 5 fails whenever $x \neq 0$ since it is then impossible to find (x', y') satisfying $x' \geq 0$ for which $(x, y) + (x', y') = (0, 0)$. (The zero vector from axiom 4 must be $\mathbf{0} = (0, 0)$.)

Axiom 6 fails whenever $k < 0$ and $x \neq 0$.

This is not a vector space.

7. Axiom 8 fails to hold:

$$(k+m)\mathbf{u} = \left((k+m)^2 x, (k+m)^2 y, (k+m)^2 z\right)$$

$$k\mathbf{u} + m\mathbf{u} = \left(k^2x, k^2y, k^2z\right) + \left(m^2x, m^2y, m^2z\right) = \left(\left(k^2+m^2\right)x, \left(k^2+m^2\right)y, \left(k^2+m^2\right)z\right)$$

therefore in general $(k+m)\mathbf{u} \neq k\mathbf{u} + m\mathbf{u}$.

This is not a vector space.

9. Let V be the set of all 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ (i.e., all diagonal 2×2 matrices)

Axiom 1: the sum of two diagonal 2×2 matrices is also a diagonal 2×2 matrix.

Axiom 2: follows from part (a) of Theorem 1.4.1.

Axiom 3: follows from part (b) of Theorem 1.4.1.

Axiom 4: taking $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; follows from part (a) of Theorem 1.4.2.

Axiom 5: let the negative of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ be $\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$;

follows from part (c) of Theorem 1.4.2 and Axiom 2.

Axiom 6: the scalar multiple of a diagonal 2×2 matrix is also a diagonal 2×2 matrix.

Axiom 7: follows from part (h) of Theorem 1.4.1.

Axiom 8: follows from part (j) of Theorem 1.4.1.

Axiom 9: follows from part (l) of Theorem 1.4.1.

Axiom 10: $1\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for all real a and b.

This is a vector space – all axioms hold.

11. Let V denote the set of all pairs of real numbers of the form $(1,x)$.

Axiom 1: $(1,y)+(1,y')=(1,y+y')$ is in V for all real y and y';

Axiom 2: $(1,y)+(1,y')=(1,y+y')=(1,y'+y)=(1,y')+(1,y)$ for all real y and y';

Axiom 3: $(1,y)+((1,y')+(1,y''))=(1,y)+(1,y'+y'')=(1,y+y'+y'')=(1,y+y')+(1,y'')$

$=((1,y)+(1,y'))+(1,y'')$ for all real y, y', and y'';

Axiom 4: taking $\mathbf{0}=(1,0)$, we have $(1,0)+(1,y)=(1,y)$ and $(1,y)+(1,0)=(1,y)$ for all real y;

Axiom 5: for each $\mathbf{u}=(1,y)$, let $-\mathbf{u}=(1,-y)$;

then $(1,y)+(1,-y)=(1,0)$ and $(1,-y)+(1,y)=(1,0)$;

Axiom 6: $k(1,y)=(1,ky)$ is in V for all real k and y;

Axiom 7: $k((1,y)+(1,y'))=k(1,y+y')=(1,ky+ky')=(1,ky)+(1,ky')=k(1,y)+k(1,y')$

for all real k, y, and y';

Axiom 8: $(k+m)(1,y)=(1,(k+m)y)=(1,ky+my)=(1,ky)+(1,my)=k(1,y)+m(1,y)$

for all real k, m, and y;

Axiom 9: $k(m(1,y))=k(1,my)=(1,kmy)=(km)(1,y)$ for all real k, m, and y;

Axiom 10: $1(1,y)=(1,y)$ for all real y.

This is a vector space – all axioms hold.

13. Axiom 3: follows from part (b) of Theorem 1.4.1 since

$$\mathbf{u}+(\mathbf{v}+\mathbf{w})=\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\left(\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}+\begin{bmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{bmatrix}\right)$$

$$=\left(\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}\right)+\begin{bmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{bmatrix}=(\mathbf{u}+\mathbf{v})+\mathbf{w}$$

Axiom 7: follows from part (h) of Theorem 1.4.1 since

$$k(\mathbf{u}+\mathbf{v})=k\left(\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}\right)=k\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+k\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}=k\mathbf{u}+k\mathbf{v}$$

Axiom 8: follows from part (j) of Theorem 1.4.1 since

$$(k+m)\mathbf{u}=(k+m)\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=k\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+m\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=k\mathbf{u}+m\mathbf{u}$$

Axiom 9: follows from part (l) of Theorem 1.4.1 since

$$k(m\mathbf{u})=k\left(m\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}\right)=(km)\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=(km)\mathbf{u}$$

15. Axiom 1: $(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$ is in V

Axiom 2: $(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)=(v_1+u_1,v_2+u_2)=(v_1,v_2)+(u_1,u_2)$

Axiom 3: $(u_1,u_2)+((v_1,v_2)+(w_1,w_2))=(u_1,u_2)+(v_1+w_1,v_2+w_2)$

$=(u_1+v_1+w_1,u_2+v_2+w_2)=(u_1+v_1,u_2+v_2)+(w_1,w_2)$

$=((u_1,u_2)+(v_1,v_2))+(w_1,w_2)$

Axiom 4: taking $\mathbf{0}=(0,0)$, we have $(0,0)+(u_1,u_2)=(u_1,u_2)$ and $(u_1,u_2)+(0,0)=(u_1,u_2)$

Axiom 5: for each $\mathbf{u}=(u_1,u_2)$, let $-\mathbf{u}=(-u_1,-u_2)$;
then $(u_1,u_2)+(-u_1,-u_2)=(0,0)$ and $(-u_1,-u_2)+(u_1,u_2)=(0,0)$

Axiom 6: $k(u_1,u_2)=(ku_1,0)$ is in V

Axiom 7: $k((u_1,u_2)+(v_1,v_2))=k(u_1+v_1,u_2+v_2)=(ku_1+kv_1,0)=(ku_1,0)+(kv_1,0)=k(u_1,u_2)+k(v_1,v_2)$

Axiom 8: $(k+m)(u_1,u_2)=((k+m)u_1,0)=(ku_1+mu_1,0)=(ku_1,0)+(mu_1,0)$
$=k(u_1,u_2)+m(u_1,u_2)$

Axiom 9: $k(m(u_1,u_2))=k(mu_1,0)=(kmu_1,0)=(km)(u_1,u_2)$

19. $\frac{1}{u}=u^{-1}$

21.

$\mathbf{u}+\mathbf{w}=\mathbf{v}+\mathbf{w}$	Hypothesis
$(\mathbf{u}+\mathbf{w})+(-\mathbf{w})=(\mathbf{v}+\mathbf{w})+(-\mathbf{w})$	Add $-\mathbf{w}$ to both sides
$\mathbf{u}+[\mathbf{w}+(-\mathbf{w})]=\mathbf{v}+[\mathbf{w}+(-\mathbf{w})]$	Axiom 3
$\mathbf{u}+\mathbf{0}=\mathbf{v}+\mathbf{0}$	Axiom 5
$\mathbf{u}=\mathbf{v}$	Axiom 4

True-False Exercises

(a) True. This is a part of Definition 1.

(b) False. Example 1 discusses a vector space containing only one vector.

(c) False. By part (d) of Theorem 4.1.1, if $k\mathbf{u} = \mathbf{0}$ then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

(d) False. Axiom 6 fails to hold if $k < 0$. (Also, Axiom 4 fails to hold.)

(e) True. This follows from part (c) of Theorem 4.1.1.

(f) False. This function must have a value of zero at *every* point in $(-\infty,\infty)$.

4.2 Subspaces

1. **(a)** Let W be the set of all vectors of the form $(a,0,0)$, i.e. all vectors in R^3 with last two components equal to zero.

This set contains at least one vector, e.g. $(0,0,0)$.

Adding two vectors in W results in another vector in W: $(a,0,0)+(b,0,0)=(a+b,0,0)$ since the result has zeros as the last two components.

Likewise, a scalar multiple of a vector in W is also in W: $k(a,0,0)=(ka,0,0)$ - the result also has zeros as the last two components.

According to Theorem 4.2.1, W is a subspace of R^3.

(b) Let W be the set of all vectors of the form $(a,1,1)$, i.e. all vectors in R^3 with last two components equal to one. The set W is not closed under the operation of vector addition since $(a,1,1)+(b,1,1)=(a+b,2,2)$ does not have ones as its last two components thus it is outside W.

According to Theorem 4.2.1, W is not a subspace of R^3.

(c) Let W be the set of all vectors of the form (a,b,c), where $b = a + c$.

This set contains at least one vector, e.g. $(0,0,0)$. (The condition $b = a + c$ is satisfied when $a = b = c = 0$.)

Adding two vectors in W results in another vector in W
$(a,a+c,c)+(a',a'+c',c')=(a+a',\ a+c+a'+c',\ c+c')$ since in this result, the second component is the sum of the first and the third: $a+c+a'+c'=(a+a')+(c+c')$.

Likewise, a scalar multiple of a vector in W is also in W: $k(a,a+c,c)=(ka,k(a+c),kc)$ since in this result, the second component is once again the sum of the first and the third:
$k(a+c)=ka+kc$.

According to Theorem 4.2.1, W is a subspace of R^3.

3. **(a)** Let W be the set of all $n \times n$ diagonal matrices.
This set contains at least one matrix, e.g. the zero $n \times n$ matrix.
Adding two matrices in W results in another $n \times n$ diagonal matrix, i.e. a matrix in W :

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}+b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}+b_{nn} \end{bmatrix}$$

Likewise, a scalar multiple of a matrix in W is also in W :

$$k\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} ka_{11} & 0 & \cdots & 0 \\ 0 & ka_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ka_{nn} \end{bmatrix}$$

According to Theorem 4.2.1, W is a subspace of M_{nn}.

(b) Let W be the set of all $n \times n$ matrices such whose determinant is zero. We shall show that W is not closed under the operation of matrix addition. For instance, consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ - both have determinant equal 0, therefore both matrices are in W. However, $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has nonzero determinant, thus it is outside W.
According to Theorem 4.2.1, W is not a subspace of M_{nn}.

(c) Let W be the set of all $n \times n$ matrices with zero trace.
This set contains at least one matrix, e.g., the zero $n \times n$ matrix is in W.
Let us assume $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are both in W, i.e. $\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = 0$ and $\text{tr}(B) = b_{11} + b_{22} + \cdots + b_{nn} = 0$.
Since $\text{tr}(A+B) = (a_{11}+b_{11}) + (a_{22}+b_{22}) + \cdots + (a_{nn}+b_{nn})$
$= a_{11} + a_{22} + \cdots + a_{nn} + b_{11} + b_{22} + \cdots + b_{nn} = 0 + 0 = 0$, it follows that $A + B$ is in W.
A scalar multiple of the same matrix A with a scalar k has
$tr(kA) = ka_{11} + ka_{22} + \cdots + ka_{nn} = k(a_{11} + a_{22} + \cdots + a_{nn}) = 0$ therefore kA is in W as well.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

(d) Let W be the set of all symmetric $n \times n$ matrices (i.e., $n \times n$ matrices such that $A^T = A$).
This set contains at least one matrix, e.g., I_n is in W.
Let us assume A and B are both in W, i.e. $A^T = A$ and $B^T = B$. By Theorem 1.4.8(b), their sum satisfies $(A+B)^T = A^T + B^T = A + B$ therefore W is closed under addition.

From Theorem 1.4.8(d), a scalar multiple of a symmetric matrix is also symmetric: $(kA)^T = kA^T = kA$ which makes W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

5. **(a)** Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

This set contains at least one polynomial, $0 + 0x + 0x^2 + 0x^3 = 0$.
Adding two polynomials in W results in another polynomial in W:
$(0 + a_1x + a_2x^2 + a_3x^3) + (0 + b_1x + b_2x^2 + b_3x^3)$
$= 0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$.
Likewise, a scalar multiple of a polynomial in W is also in W:
$k(0 + a_1x + a_2x^2 + a_3x^3) = 0 + (ka_1)x + (ka_2)x^2 + (ka_3)x^3$.
According to Theorem 4.2.1, W is a subspace of P_3.

(b) Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$, i.e. all polynomials that can be expressed in the form $-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3$.
Adding two polynomials in W results in another polynomial in W
$(-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3) + (-b_1 - b_2 - b_3 + b_1x + b_2x^2 + b_3x^3)$
$= (-a_1 - a_2 - a_3 - b_1 - b_2 - b_3) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$
since we have $(-a_1 - a_2 - a_3 - b_1 - b_2 - b_3) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = 0$.
Likewise, a scalar multiple of a polynomial in W is also in W
$k(-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3) = -ka_1 - ka_2 - ka_3 + ka_1x + ka_2x^2 + ka_3x^3$
since it meets the condition $(-ka_1 - ka_2 - ka_3) + (ka_1) + (ka_2) + (ka_3) = 0$.
According to Theorem 4.2.1, W is a subspace of P_3.

7. **(a)** Let W be the set of all functions f in $F(-\infty,\infty)$ for which $f(0) = 0$.

This set contains at least one function, e.g., the constant function $f(x) = 0$.

Assume we have two functions f and g in W, i.e., $f(0) = g(0) = 0$. Their sum $f + g$ is also a function in $F(-\infty,\infty)$ and satisfies $(f + g)(0) = f(0) + g(0) = 0 + 0 = 0$ therefore W is closed under addition.
A scalar multiple of a function f in W, kf, is also a function in $F(-\infty,\infty)$ for which $(kf)(0) = k(f(0)) = 0$ making W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of $F(-\infty,\infty)$.

(b) Let W be the set of all functions f in $F(-\infty,\infty)$ for which $f(0) = 1$.

We will show that W is not closed under addition. For instance, let $f(x) = 1$ and $g(x) = \cos x$ be two functions in W. Their sum, $f+g$, is not in W since $(f+g)(0)=f(0)+g(0)=1+1=2$. We conclude that W is not a subspace of $F(-\infty,\infty)$.

9. **(a)** Let W be the set of all sequences in R^∞ of the form $(v,0,v,0,v,0,\ldots)$.
This set contains at least one sequence, e.g. $(0,0,0,\ldots)$.
Adding two sequences in W results in another sequence in W:
$(v,0,v,0,v,0,\ldots)+(w,0,w,0,w,0,\ldots)=(v+w,0,v+w,0,v+w,0,\ldots)$.
Likewise, a scalar multiple of a vector in W is also in W: $k(v,0,v,0,v,0,\ldots)=(kv,0,kv,0,kv,0,\ldots)$.
According to Theorem 4.2.1, W is a subspace of R^∞.

(b) Let W be the set of all sequences in R^∞ of the form $(v,1,v,1,v,1,\ldots)$.
This set is not closed under addition since
$(v,1,v,1,v,1,\ldots)+(w,1,w,1,w,1,\ldots)=(v+w,2,v+w,2,v+w,2,\ldots)$ is not in W.
We conclude that W is not a subspace of R^∞.

11. **(a)** Let W be the set of all matrices of form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$. This set contains at least one matrix, e.g. the zero matrix. Adding two matrices in W results in another matrix in W:

$$\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}+\begin{bmatrix} a' & 0 \\ b' & 0 \end{bmatrix}=\begin{bmatrix} a+a' & 0 \\ b+b' & 0 \end{bmatrix}.$$

Likewise, a scalar multiple of a matrix in W is also in W:

$k\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}=\begin{bmatrix} ka & 0 \\ kb & 0 \end{bmatrix}$. According to Theorem 4.2.1, W is a subspace of M_{22}.

(b) Let W be the set of all matrices of form $\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}$. This set is not closed under scalar multiplication when the scalar is 0. Consequently, W is not a subspace of M_{22}.

(c) Let W be the set of all 2×2 matrices A such that $A\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. This set is not closed under addition since if A and B are matrices in W then

$(A+B)\begin{bmatrix} 1 \\ -1 \end{bmatrix}=A\begin{bmatrix} 1 \\ -1 \end{bmatrix}+B\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 2 \\ 0 \end{bmatrix}+\begin{bmatrix} 2 \\ 0 \end{bmatrix}=\begin{bmatrix} 4 \\ 0 \end{bmatrix}$. Consequently, the matrix $A+B$ is not contained in W. According to Theorem 4.2.1, W is not a subspace of M_{22}.

13. **(a)** Let W be the set of all vectors in R^4 of form (a,a^2,a^3,a^4). This set is not closed under addition. For example, the vector $(1,1,1,1)$ is in W but $(1,1,1,1)+(1,1,1,1)=(2,2,2,2)$ is not. According to Theorem 4.2.1, W is not a subspace of R^4.

(b) Let W be the set of all vectors in R^4 of form $(a,0,b,0)$. This set contains at least one vector, e.g. the zero vector. Adding two vectors in W results in another vector in W:

$(a,0,b,0)+(a',0,b',0)=(a+a',0,b+b',0)$.

Likewise, a scalar multiple of a vector in W is also in W: $k(a,0,b,0)=(ka,0,kb,0)$. According to Theorem 4.2.1, W is a subspace of R^4.

15. (a) Let W be the set of all polynomials of degree less than or equal to six. This set is not empty. For example, $p(x)=x$ is contained in W. Adding two polynomials in W results in another polynomial in W because the sum of two polynomials of degree at most six is another polynomial of degree at least six. Likewise, a scalar multiple of a polynomial of degree at most six is another polynomial of degree at most six. According to Theorem 4.2.1, W is a subspace of P_∞.

(b) Let W be the set of all polynomials of degree equal to six. This set is not closed under addition. For example, $p(x)=x^6+x$ and $q(x)=-x^6$ are both polynomials in W but

$p(x)+q(x)=x^6+x-x^6=x$ has degree 1 so it is not contained in W. According to Theorem 4.2.1, W is not a subspace of P_∞.

(c) Let W be the set of all polynomials of degree greater than or equal to six. This set is not closed under addition. For example, $p(x)=x^6+x$ and $q(x)=-x^6$ are both polynomials in W but

$p(x)+q(x)=x^6+x-x^6=x$ has degree 1 so it is not contained in W. According to Theorem 4.2.1, W is not a subspace of P_∞.

17. (a) Let W be the set of all sequences of the form $(v_1,v_2,v_3,\ldots)$ such that $\lim_{n\to\infty} v_n=0$. This set is nonempty (e.g. it contains the zero sequence $(0,0,0,\cdots)$). Adding two sequences $(v_1,v_2,v_3,\ldots)$ and $(w_1,w_2,w_3,\ldots)$ in W results in the sequence $(v_1+w_1,v_2+w_2,v_3+w_3,\ldots)$ which is also in W since $\lim_{n\to\infty} v_n+\lim_{n\to\infty} w_n=\lim_{n\to\infty}(v_n+w_n)=0$. Likewise, a scalar multiple of a sequence $(v_1,v_2,v_3,\ldots)$ in W is also in W because $k(\lim_{n\to\infty} v_n)=\lim_{n\to\infty} kv_n=0$. (These results both follow because sums and constant multiples of convergent sequences are also convergent.). According to Theorem 4.2.1, W is a subspace of R^∞.

(b) Let W be the set of all sequences of the form $(v_1,v_2,v_3,\ldots)$ such that $\lim_{n\to\infty} v_n$ exists and is finite. This set is nonempty (e.g. it contains the zero sequence $(0,0,0,\cdots)$). Adding two sequences $(v_1,v_2,v_3,\ldots)$ and $(w_1,w_2,w_3,\ldots)$ in W results in the sequence $(v_1+w_1,v_2+w_2,v_3+w_3,\ldots)$ which is also in W. This follows because both $\lim\limits_{n\to\infty} v_n$ and $\lim\limits_{n\to\infty} w_n$ exist and are finite so that

$\lim\limits_{n\to\infty} v_n+\lim\limits_{n\to\infty} w_n=\lim\limits_{n\to\infty}(v_n+w_n)$ also exists and is finite. Likewise, a scalar multiple of a sequence

$(v_1, v_2, v_3, \ldots)$ in W is also in W because $k\left(\lim_{n\to\infty} v_n\right) = \lim_{n\to\infty} kv_n$. According to Theorem 4.2.1, W is a subspace of R^∞.

(c) Let W be the set of all sequences of the form $(v_1, v_2, v_3, \ldots)$ such that $\sum_{n=1}^{\infty} v_n = 0$. This set is nonempty (e.g. it contains the zero sequence $(0,0,0,\cdots)$). Adding two sequences $(v_1, v_2, v_3, \ldots)$ and $(w_1, w_2, w_3, \ldots)$ in W results in the sequence $(v_1 + w_1, v_2 + w_2, v_3 + w_3, \ldots)$ which is also in W. This follows because both $\sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} w_n$ converge to zero so

that $\sum_{n=1}^{\infty} v_n + \sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty} v_n + w_n = 0$. Likewise, a scalar multiple of a sequence $(v_1, v_2, v_3, \ldots)$ in W is also in W because $k\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} kv_n = 0$. According to Theorem 4.2.1, W is a subspace of R^∞.

(d) Let W be the set of all sequences of the form $(v_1, v_2, v_3, \ldots)$ such that $\sum_{n=1}^{\infty} v_n$ converges. This set is nonempty (e.g. it contains the zero sequence $(0,0,0,\cdots)$). Adding two sequences $(v_1, v_2, v_3, \ldots)$ and $(w_1, w_2, w_3, \ldots)$ in W results in the sequence $(v_1 + w_1, v_2 + w_2, v_3 + w_3, \ldots)$ which is also in W. This follows because both $\sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} w_n$ converge so $\sum_{n=1}^{\infty} v_n + \sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty} (v_n + w_n)$ also converges. Likewise, a scalar multiple of a sequence $(v_1, v_2, v_3, \ldots)$ in W is also in W because $k\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} kv_n$. According to Theorem 4.2.1, W is a subspace of R^∞.

19. (a) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$ therefore the solution are $x = -\frac{1}{2}t$, $y = -\frac{3}{2}t$, $z = t$. These are parametric equations of a line through the origin.

(b) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ therefore the only solution is $x = y = z = 0$ - the origin.

(c) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which corresponds to an equation of a plane through the origin $x - 3y + z = 0$.

(d) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ therefore the solutions are $x = -3t$, $y = -2t$, $z = t$. These are parametric equations of a line through the origin.