Numerical Integration

Bound Error

Example:

Apply the Trapezoid Rule and Simpson's Rule to approximate

$$\int_1^2 \ln x \, dx,$$

and find an upper bound for the error in your approximations.

Solution:

$$\int_1^2 \ln x \, dx \approx \frac{h}{2}(y_0 + y_1) = \frac{1}{2}(\ln 1 + \ln 2) = \frac{\ln 2}{2} \approx 0.3466.$$

The error for the Trapezoid Rule is $-h^3 f''(c)/12$, where $1 < c < 2$. Since $f''(x) = -1/x^2$, the magnitude of the error is at most

$$\frac{1^3}{12c^2} \leq \frac{1}{12} \approx 0.0834.$$

$$\int_1^2 \ln x \, dx = 0.3466 \pm 0.0834.$$

Simpson's Rule yields the estimate

$$\int_1^2 \ln x\, dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2) = \frac{0.5}{3}\left(\ln 1 + 4\ln\frac{3}{2} + \ln 2\right) \approx 0.3858.$$

The error for Simpson's Rule is $-h^5 f^{(iv)}(c)/90$, where $1 < c < 2$. Since $f^{(iv)}(x) = -6/x^4$, the error is at most

$$\frac{6(0.5)^5}{90c^4} \leq \frac{6(0.5)^5}{90} = \frac{1}{480} \approx 0.0021.$$

$$\int_1^2 \ln x\, dx = 0.3858 \pm 0.0021,$$

The integral can be computed exactly by using integration by parts:

$$\int_1^2 \ln x\, dx = x\ln x\big|_1^2 - \int_1^2 dx$$
$$= 2\ln 2 - 1\ln 1 - 1 \approx 0.386294.$$

Example:

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. Note that the exact value of the integral can be determined analytically to be 1.640533.

Solution:

Trapezoidal Rule

$$I = (0.8 - 0)\frac{0.2 + 0.232}{2} = 0.1728$$

$$E_a = -\frac{1}{12}(-60)(0.8)^3 = 2.56$$

Simpson's Rule

For $n = 2$ $(h = 0.4)$:

$$f(0) = 0.2 \qquad f(0.4) = 2.456 \qquad f(0.8) = 0.232$$

$$I = 0.8\frac{0.2 + 2(2.456) + 0.232}{4} = 1.0688$$

$$E_t = 1.640533 - 1.0688 = 0.57173 \qquad \varepsilon_t = 34.9\%$$

$$E_a = -\frac{0.8^3}{12(2)^2}(-60) = 0.64$$

Example(Composite)

Carry out four-panel approximations of

Find the bound error $$\int_1^2 \ln x \, dx,$$

using the composite Trapezoid Rule and composite Simpson's Rule.

Solution:

The error is at most

$$\frac{(b-a)h^2}{12}|f''(c)| = \frac{1/16}{12}\frac{1}{c^2} \leq \frac{1}{(16)(12)(1^2)} = \frac{1}{192} \approx 0.0052.$$

the error cannot be more than

$$\frac{(b-a)h^4}{180}|f^{(iv)}(c)| = \frac{(1/8)^4}{180}\frac{6}{c^4} \leq \frac{6}{8^4 \cdot 180 \cdot 1^4} \approx 0.000008.$$

Example 2 Determine values of h that will ensure an approximation error of less than 0.00002 when approximating $\int_0^\pi \sin x \, dx$ and employing
(a) Composite Trapezoidal rule and (b) Composite Simpson's rule.

Formula

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180}h^4 f^{(4)}(\mu).$$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu).$$

Solution (a) The error form for the Composite Trapezoidal rule for $f(x) = \sin x$ on $[0, \pi]$ is

$$\left|\frac{\pi h^2}{12} f''(\mu)\right| = \left|\frac{\pi h^2}{12}(-\sin \mu)\right| = \frac{\pi h^2}{12}|\sin \mu|.$$

$$\frac{\pi h^2}{12}|\sin \mu| \le \frac{\pi h^2}{12} < 0.00002.$$

Since $h = \pi/n$ implies that $n = \pi/h$, we need

$$\frac{\pi^3}{12n^2} < 0.00002 \quad \text{which implies that} \quad n > \left(\frac{\pi^3}{12(0.00002)}\right)^{1/2} \approx 359.44.$$

and the Composite Trapezoidal rule requires $n \ge 360$.

(b) The error form for the Composite Simpson's rule for $f(x) = \sin x$ on $[0, \pi]$ is

$$\left|\frac{\pi h^4}{180} f^{(4)}(\mu)\right| = \left|\frac{\pi h^4}{180} \sin \mu\right| = \frac{\pi h^4}{180} |\sin \mu|.$$

$$\frac{\pi h^4}{180} |\sin \mu| \leq \frac{\pi h^4}{180} < 0.00002.$$

Using again the fact that $n = \pi / h$ gives

$$\frac{\pi^5}{180 n^4} < 0.00002 \quad \text{which implies that} \quad n > \left(\frac{\pi^5}{180(0.00002)}\right)^{1/4} \approx 17.07.$$

So Composite Simpson's rule requires only $n \geq 18$.

EXERCISE SET 4.4

11. Determine the values of n and h required to approximate

$$\int_0^2 e^{2x} \sin 3x \, dx$$

to within 10^{-4}. Use

a. Composite Trapezoidal rule.

b. Composite Simpson's rule.

c. Composite Midpoint rule.

13. Determine the values of n and h required to approximate

$$\int_0^2 \frac{1}{x+4} \, dx$$

to within 10^{-5} and compute the approximation. Use

a. Composite Trapezoidal rule.

b. Composite Simpson's rule.

c. Composite Midpoint rule.

Answers:

11. a. The Composite Trapezoidal rule requires $h < 0.000922295$ and $n \geq 2168$.

b. The Composite Simpson's rule requires $h < 0.037658$ and $n \geq 54$.

c. The Composite Midpoint rule requires $h < 0.00065216$ and $n \geq 3066$.

13. a. The Composite Trapezoidal rule requires $h < 0.04382$ and $n \geq 46$. The approximation is 0.405471.

b. The Composite Simpson's rule requires $h < 0.44267$ and $n \geq 6$. The approximation is 0.405466.

c. The Composite Midpoint rule requires $h < 0.03098$ and $n \geq 64$. The approximation is 0.405460.


thank you!