(f) False. The set $\left\{\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$ is linearly dependent since $\begin{bmatrix}1&1\\0&0\end{bmatrix}=(-1)\begin{bmatrix}0&0\\1&1\end{bmatrix}+\begin{bmatrix}1&0\\1&0\end{bmatrix}+\begin{bmatrix}0&1\\0&1\end{bmatrix}$.

(g) True. Requiring that for all x values $a(x-1)(x+2)+bx(x+2)+cx(x-1)=0$ holds true implies that the equality must be true for any specific x value. Setting $x=0$ yields $a=0$. Likewise, $x=1$ implies $b=0$, and $x=-2$ implies $c=0$. Since $a=b=c=0$ is required, we conclude that the three given polynomials are linearly independent.

(h) False. The functions f_1 and f_2 are linearly dependent if there exist scalars k_1 and k_2, <u>not both equal 0</u>, such that $k_1f_1(x)+k_2f_2(x)=0$ for <u>all</u> real numbers x.

4.5 Coordinates and Basis

1. Vectors $(2,1)$ and $(3,0)$ are linearly independent if the vector equation

$$c_1(2,1)+c_2(3,0)=(0,0)$$

has only the trivial solution. For these vectors to span R^2, it must be possible to express every vector $\mathbf{b}=(b_1,b_2)$ in R^2 as

$$c_1(2,1)+c_2(3,0)=(b_1,b_2)$$

These two equations can be rewritten as linear systems

$$\begin{aligned}2c_1+3c_2&=0\\c_1&=0\end{aligned}\qquad\text{and}\qquad\begin{aligned}2c_1+3c_2&=b_1\\c_1&=b_2\end{aligned}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix}2&3\\1&0\end{vmatrix}=-3\neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values b_1 and b_2. Therefore the vectors $(2,1)$ and $(3,0)$ are linearly independen and span R^2 so that they form a basis for R^2.

2. Vectors $(3,1,-4)$, $(2,5,6)$, and $(1,4,8)$ are linearly independent if the vector equation

$$c_1(3,1,-4)+c_2(2,5,6)+c_3(1,4,8)=(0,0,0)$$

has only the trivial solution. For these vectors to span R^3, it must be possible to express every vector $\mathbf{b}=(b_1,b_2,b_3)$ in R^3 as.

$$c_1(3,1,-4)+c_2(2,5,6)+c_3(1,4,8)=(b_1,b_2,b_3)$$

These two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} 3c_1 & + & 2c_2 & + & 1c_3 & = & 0 \\ 1c_1 & + & 5c_2 & + & 4c_3 & = & 0 \\ -4c_1 & + & 6c_2 & + & 8c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} 3c_1 & + & 2c_2 & + & 1c_3 & = & b_1 \\ 1c_1 & + & 5c_2 & + & 4c_3 & = & b_2 \\ -4c_1 & + & 6c_2 & + & 8c_3 & = & b_3 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} = 26 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values b_1, b_2, and b_3. Therefore the vectors $(3,1,-4)$, $(2,5,6)$, and $(1,4,8)$ are linearly independent and span R^3 so that they form a basis for R^3.

3. Polynomials x^2+1, x^2-1, and $2x-1$ are linearly independent if the equation

$$c_1\left(x^2+1\right)+c_2\left(x^2-1\right)+c_3\left(2x-1\right)=0$$

has only the trivial solution. For these polynomials to span P_2, it must be possible to express every polynomial $a_0+a_1x+a_2x^2$. as

$$c_1\left(x^2+1\right)+c_2\left(x^2-1\right)+c_3\left(2x-1\right)=a_0+a_1x+a_2x^2$$

Grouping the terms on the left hand side of both equations as $\left(c_1-c_2-c_3\right)+\left(2c_3\right)x+\left(c_1+c_2\right)x^2$ these equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} 1c_1 & - & 1c_2 & - & 1c_3 & = & 0 \\ 0c_1 & + & 0c_2 & + & 2c_3 & = & 0 \\ 1c_1 & + & 1c_2 & + & 0c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} 1c_1 & - & 1c_2 & - & 1c_3 & = & a_0 \\ 0c_1 & + & 0c_2 & + & 2c_3 & = & a_1 \\ 1c_1 & + & 1c_2 & + & 0c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix} = -4 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the polynomials x^2+1, x^2-1, and $2x-1$ are linearly independent and span P_2 so that they form a basis for P_2.

4. Polynomials $1+x$, $1-x$, $1-x^2$, and $1-x^3$ are linearly independent if the equation

$$c_1\left(1+x\right)+c_2\left(1-x\right)+c_3\left(1-x^2\right)+c_4\left(1-x^3\right)=0$$

has only the trivial solution. For these polynomials to span P_3, it must be possible to express every polynomial $a_0+a_1x+a_2x^2+a_3x^3$ as

$$c_1(1+x)+c_2(1-x)+c_3(1-x^2)+c_4(1-x^3)=a_0+a_1x+a_2x^2+a_3x^3$$

Grouping the terms on the left hand side of both equations as $(c_1+c_2+c_3+c_4)+(c_1-c_2)x-c_3x^2-c_4x^3$ these equations can be rewritten as linear systems

$$\begin{array}{rcrcrcrcl} 1c_1 & + & 1c_2 & + & 1c_3 & + & 1c_4 & = & 0 \\ 1c_1 & - & 1c_2 & + & 0c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 0c_2 & - & 1c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 0c_2 & + & 0c_3 & - & 1c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 1c_1 & + & 1c_2 & + & 1c_3 & + & 1c_4 & = & a_0 \\ 1c_1 & - & 1c_2 & + & 0c_3 & + & 0c_4 & = & a_1 \\ 0c_1 & + & 0c_2 & - & 1c_3 & + & 0c_4 & = & a_2 \\ 0c_1 & + & 0c_2 & + & 0c_3 & - & 1c_4 & = & a_3 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -2 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, a_2, and a_3. Therefore the polynomials $1+x$, $1-x$, $1-x^2$ and $1-x^3$ are linearly independent and span P_3 so that they form a basis for P_3.

5. Matrices $\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ are linearly independent if the equation

$$c_1\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}+c_2\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}+c_4\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ as

$$c_1\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}+c_2\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}+c_4\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Equating corresponding entries on both sides yields linear systems

$$\begin{array}{rcrcrcrcl} 3c_1 & + & 0c_2 & + & 0c_3 & + & 1c_4 & = & 0 \\ 6c_1 & - & 1c_2 & - & 8c_3 & + & 0c_4 & = & 0 \\ 3c_1 & - & 1c_2 & - & 12c_3 & - & 1c_4 & = & 0 \\ -6c_1 & + & 0c_2 & - & 4c_3 & + & 2c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 3c_1 & + & 0c_2 & + & 0c_3 & + & 1c_4 & = & a_{11} \\ 6c_1 & - & 1c_2 & - & 8c_3 & + & 0c_4 & = & a_{12} \\ 3c_1 & - & 1c_2 & - & 12c_3 & - & 1c_4 & = & a_{21} \\ -6c_1 & + & 0c_2 & - & 4c_3 & + & 2c_4 & = & a_{22} \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{vmatrix} = 48 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the

nonhomogeneous system is consistent for all real values a_{11}, a_{12}, a_{21}, and a_{22}. Therefore the matrices $\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ are linearly independent and span M_{22} so that they form a basis for M_{22}.

6. Matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are linearly independent if the equation

$$c_1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ as

$$c_1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Equating corresponding entries on both sides in each equation yields linear systems

$$\begin{array}{rcrcrcrcl} 1c_1 & + & 1c_2 & + & 0c_3 & + & 1c_4 & = & 0 \\ 1c_1 & - & 1c_2 & - & 1c_3 & + & 0c_4 & = & 0 \\ 1c_1 & + & 0c_2 & + & 1c_3 & + & 0c_4 & = & 0 \\ 1c_1 & + & 0c_2 & + & 0c_3 & + & 0c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 1c_1 & + & 1c_2 & + & 0c_3 & + & 1c_4 & = & a_{11} \\ 1c_1 & - & 1c_2 & - & 1c_3 & + & 0c_4 & = & a_{12} \\ 1c_1 & + & 0c_2 & + & 1c_3 & + & 0c_4 & = & a_{21} \\ 1c_1 & + & 0c_2 & + & 0c_3 & + & 0c_4 & = & a_{22} \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_{11}, a_{12}, a_{21}, and a_{22}. Therefore the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are linearly independent and span M_{22} so that they form a basis for M_{22}.

7. **(a)** Vectors $(2,-3,1)$, $(4,1,1)$, and $(0,-7,1)$ are linearly independent if the vector equation

$$c_1(2,-3,1) + c_2(4,1,1) + c_3(0,-7,1) = (0,0,0)$$

has only the trivial solution. This equation can be rewritten as a linear system

$$\begin{array}{rcrcrcl} 2c_1 & + & 4c_2 & + & 0c_3 & = & 0 \\ -3c_1 & + & 1c_2 & - & 7c_3 & = & 0 \\ 1c_1 & + & 1c_2 & + & 1c_3 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} = 0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $(2,-3,1)$, $(4,1,1)$, and $(0,-7,1)$ are linearly dependent, they do not form a basis for R^3.

(b) Vectors $(1,6,4)$, $(2,4,-1)$, and $(-1,2,5)$ are linearly independent if the vector equation

$$c_1(1,6,4)+c_2(2,4,-1)+c_3(-1,2,5)=(0,0,0)$$

has only the trivial solution. This equation can be rewritten as a linear system

$$\begin{array}{rcrcrcl} 1c_1 & + & 2c_2 & - & 1c_3 & = & 0 \\ 6c_1 & + & 4c_2 & + & 2c_3 & = & 0 \\ 4c_1 & - & 1c_2 & + & 5c_3 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix} = 0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $(1,6,4)$, $(2,4,-1)$, and $(-1,2,5)$ are linearly dependent, they do not form a basis for R^3.

8. Vectors $\mathbf{p}_1 = 1-3x+2x^2$, $\mathbf{p}_2 = 1+x+4x^2$, and $\mathbf{p}_3 = 1-7x$ are linearly independent if the vector equation $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$ has only the trivial solution.

By grouping the terms on the left hand side as $c_1(1-3x+2x^2)+c_2(1+x+4x^2)+c_3(1-7x)=$ $(c_1+c_2+c_3)+(-3c_1+c_2-7c_3)x+(2c_1+4c_2)x^2$ this equation can be rewritten as the linear system

$$\begin{array}{rcrcrcl} c_1 & + & c_2 & + & c_3 & = & 0 \\ -3c_1 & + & c_2 & - & 7c_3 & = & 0 \\ 2c_1 & + & 4c_2 & & & = & 0 \end{array}$$

The coefficient matrix of this system has determinant $\begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{vmatrix} = 0$, thus it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly dependent, we conclude that they do not form a basis for P_2.

9. Matrices $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ are linearly independent if the equation

$$c_1\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}+c_2\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}+c_3\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}+c_4\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has only the trivial solution. Equating corresponding entries on both sides yields a linear system

$$\begin{array}{rcrcrcrcl} 1c_1 & + & 2c_2 & + & 1c_3 & + & 0c_4 & = & 0 \\ 0c_1 & - & 2c_2 & - & 1c_3 & - & 1c_4 & = & 0 \\ 1c_1 & + & 3c_2 & + & 1c_3 & + & 1c_4 & = & 0 \\ 1c_1 & + & 2c_2 & + & 0c_3 & + & 1c_4 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -1 & -1 \\ 1 & 3 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{vmatrix} = 0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the matrices $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ are linearly dependent, we conclude that they do not form a basis for M_{22}.

10. **(a)** The identity $\cos^2 x - \sin^2 x = \cos 2x$ implies that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, therefore it is not a basis for V.

(b) For the equation $c_1 \cos^2 x + c_2 \sin^2 x = 0$ to hold for all real x values, we must have $c_1 = 0$ (required when $x = 0$) and $c_2 = 0$ (required when $x = \frac{\pi}{2}$). Therefore the vectors $\mathbf{v}_1 = \cos^2 x$ and $\mathbf{v}_2 = \sin^2 x$ are linearly independent.

Any vector $\mathbf{v}$ in V can be expressed as $\mathbf{v} = k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x$. However, from the identity $\cos^2 x - \sin^2 x = \cos 2x$ it follows that we can express $\mathbf{v}$ as a linear combination of $\cos^2 x$ and $\sin^2 x$ alone: $\mathbf{v} = k_1 \cos^2 x + k_2 \sin^2 x + k_3 \left(\cos^2 x - \sin^2 x\right) = \left(k_1 + k_3\right)\cos^2 x + \left(k_2 - k_3\right)\sin^2 x$. This proves that the vectors $\mathbf{v}_1 = \cos^2 x$ and $\mathbf{v}_2 = \sin^2 x$ span V.

We conclude that $\mathbf{v}_1 = \cos^2 x$ and $\mathbf{v}_2 = \sin^2 x$ form a basis for V.

(Note that $\{\mathbf{v}_1, \mathbf{v}_3\}$ and $\{\mathbf{v}_2, \mathbf{v}_3\}$ are also bases for V.)

11. **(a)** Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,1) = c_1(2,-4) + c_2(3,8)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & 1 \\ -4c_1 & + & 8c_2 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{bmatrix}$. The solution of the linear system is $c_1 = \frac{5}{28}$, $c_2 = \frac{3}{14}$, therefore the coordinate vector is $(\mathbf{w})_S = \left(\frac{5}{28}, \frac{3}{14}\right)$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(a,\ b) = c_1(1,1) + c_2(0,2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} 1c_1 & + & 0c_2 & = & a \\ 1c_1 & + & 2c_2 & = & b \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = a$, $c_2 = \frac{b-a}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = \left(a, \frac{b-a}{2}\right)$.

12. **(a)** Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,0) = c_1(1,-1) + c_2(1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} c_1 & + & c_2 & = & 1 \\ -c_1 & + & c_2 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = \left(\frac{1}{2}, \frac{1}{2}\right)$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(0,1) = c_1(1,-1) + c_2(1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ -c_1 & + & c_2 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = -\frac{1}{2}$, $c_2 = \frac{1}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

13. **(a)** Expressing $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$(2,-1,3) = c_1(1,0,0) + c_2(2,2,0) + c_3(3,3,3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} c_1 & + & 2c_2 & + & 3c_3 & = & 2 \\ & & 2c_2 & + & 3c_3 & = & -1 \\ & & & & 3c_3 & = & 3 \end{array}$$

which can be solved by back-substitution to obtain $c_3 = 1$, $c_2 = -2$, and $c_1 = 3$. The coordinate vector is $(\mathbf{v})_S = (3,-2,1)$.

(b) Expressing $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$(5,-12,3) = c_1(1,2,3) + c_2(-4,5,6) + c_3(7,-8,9)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 1c_1 & - & 4c_2 & + & 7c_3 & = & 5 \\ 2c_1 & + & 5c_2 & - & 8c_3 & = & -12 \\ 3c_1 & + & 6c_2 & + & 9c_3 & = & 3 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution of the linear system is $c_1 = -2$, $c_2 = 0$, and $c_3 = 1$. The coordinate vector is $(\mathbf{v})_S = (-2,0,1)$.

14. **(a)** Since $\mathbf{p} = 4\mathbf{p}_1 + (-3)\mathbf{p}_2 + 1\mathbf{p}_3$ we conclude that the coordinate vector is $(\mathbf{p})_S = (4,-3,1)$.

(b) Expressing $\mathbf{p}$ as a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ we obtain

$$2 - x + x^2 = c_1(1+x) + c_2(1+x^2) + c_3(x+x^2)$$

Grouping the terms on the right hand side according to powers of x yields

$$2 - x + x^2 = (c_1 + c_2) + (c_1 + c_3)x + (c_2 + c_3)x^2$$

For this equality to hold for all real x, the coefficients associated with the same power of x on both sides must match. This leads to the linear system

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & & & = & 2 \\ c_1 & & & + & c_3 & = & -1 \\ & & c_2 & + & c_3 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. The solution is $c_1 = 0$, $c_2 = 2$, $c_3 = -1$, therefore the coordinate vector is $(\mathbf{p})_S = (0,2,-1)$.

15. Matrices (vectors in M_{22}) A_1, A_2, A_3, and A_4 are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = B$$

The left hand side of each of these equations is the matrix $\begin{bmatrix} k_1 & k_1 + k_2 \\ k_1 + k_2 + k_3 & k_1 + k_2 + k_3 + k_4 \end{bmatrix}$. Equating corresponding entries, these two equations can be rewritten as linear systems

$$\begin{array}{rcl} k_1 & = & 0 \\ k_1 + k_2 & = & 0 \\ k_1 + k_2 + k_3 & = & 0 \\ k_1 + k_2 + k_3 + k_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcl} k_1 & = & a \\ k_1 + k_2 & = & b \\ k_1 + k_2 + k_3 & = & c \\ k_1 + k_2 + k_3 + k_4 & = & d \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a, b, c and d. Therefore the matrices A_1, A_2, A_3, and A_4 are linearly independent and span M_{22} so that they form a basis for M_{22}.

To express $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ as a linear combination of the matrices A_1, A_2, A_3, and A_4, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{rcl} k_1 & = & 1 \\ k_1 + k_2 & = & 0 \\ k_1 + k_2 + k_3 & = & 1 \\ k_1 + k_2 + k_3 + k_4 & = & 0 \end{array}$$

which can be solved by forward-substitution to obtain $k_1 = 1$, $k_2 = -1$, $k_3 = 1$, $k_4 = -1$.
This allows us to express $A = 1A_1 - 1A_2 + 1A_3 - 1A_4$.
The coordinate vector is $(A)_S = (1,-1,1,-1)$.

16. Matrices (vectors in M_{22}) A_1, A_2, A_3, and A_4 are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $B=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$k_1A_1+k_2A_2+k_3A_3+k_4A_4=B$$

The left hand side of each of these equations is the matrix $\begin{bmatrix} k_1+k_2+k_3 & k_2 \\ k_1+k_4 & k_3 \end{bmatrix}$. Equating corresponding entries, these two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcrcl} k_1 & + & k_2 & + & k_3 & & & = & 0 \\ & & k_2 & & & & & = & 0 \\ k_1 & & & & & + & k_4 & = & 0 \\ & & & & k_3 & & & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} k_1 & + & k_2 & + & k_3 & & & = & a \\ & & k_2 & & & & & = & b \\ k_1 & & & & & + & k_4 & = & c \\ & & & & k_3 & & & = & d \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}=-1\neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a, b, c and d. Therefore the matrices A_1, A_2, A_3, and A_4 are linearly independent and span M_{22} so that they form a basis for M_{22}.

To express $A=\begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$ as a linear combination of the matrices A_1, A_2, A_3, and A_4, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{rcrcrcrcl} k_1 & + & k_2 & + & k_3 & & & = & 6 \\ & & k_2 & & & & & = & 2 \\ k_1 & & & & & + & k_4 & = & 5 \\ & & & & k_3 & & & = & 3 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ therefore the solution is $k_1=1$, $k_2=2$, $k_3=3$, $k_4=4$.

This allows us to express $A=1A_1+2A_2+3A_3+4A_4$. The coordinate vector is $(A)_S=(1,2,3,4)$.

17. Vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent if the vector equation

$$c_1\mathbf{p}_1+c_2\mathbf{p}_2+c_3\mathbf{p}_3=\mathbf{0}$$

has only the trivial solution. For these vectors to span P_2, it must be possible to express every vector $\mathbf{p} = a_0 + a_1x + a_2x^2$ in P_2 as

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$$

Grouping the terms on the left hand sides as $c_1\left(1+x+x^2\right)+c_2\left(x+x^2\right)+c_3x^2 = c_1+\left(c_1+c_2\right)x+\left(c_1+c_2+c_3\right)x^2$ these two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} c_1 & & & & & = & 0 \\ c_1 & + & c_2 & & & = & 0 \\ c_1 & + & c_2 & + & c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} c_1 & & & & & = & a_0 \\ c_1 & + & c_2 & & & = & a_1 \\ c_1 & + & c_2 & + & c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent and span P_2 so that they form a basis for P_2.

To express $\mathbf{p} = 7 - x + 2x^2$ as a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{rcrcrcr} c_1 & & & & & = & 7 \\ c_1 & + & c_2 & & & = & -1 \\ c_1 & + & c_2 & + & c_3 & = & 2 \end{array}$$

which can be solved by forward-substitution to obtain $c_1 = 7$, $c_2 = -8$, $c_3 = 3$.

This allows us to express $\mathbf{p} = 7\mathbf{p}_1 - 8\mathbf{p}_2 + 3\mathbf{p}_3$. The coordinate vector is $(\mathbf{p})_S = (7,-8,3)$.

18. Vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent if the vector equation

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$$

has only the trivial solution. For these vectors to span P_2, it must be possible to express every vector $\mathbf{p} = a_0 + a_1x + a_2x^2$ in P_2 as

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$$

Grouping the terms on the left hand sides as $c_1\left(1+2x+x^2\right)+c_2\left(2+9x\right)+c_3\left(3+3x+4x^2\right) = \left(c_1+2c_2+3c_3\right)+\left(2c_1+9c_2+3c_3\right)x+\left(c_1+4c_3\right)x^2$ these two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} c_1 & + & 2c_2 & + & 3c_3 & = & 0 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & 0 \\ c_1 & & & + & 4c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} c_1 & + & 2c_2 & + & 3c_3 & = & a_0 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & a_1 \\ c_1 & & & + & 4c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = -1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent and span P_2 so that they form a basis for P_2.

To express $\mathbf{p} = 2 + 17x - 3x^2$ as a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{rcrcrcr} c_1 & + & 2c_2 & + & 3c_3 & = & 2 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & 17 \\ c_1 & & & + & 4c_3 & = & -3 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ therefore the solution is $c_1 = 1$, $c_2 = 2$, $c_3 = -1$. This allows us to express $\mathbf{p} = 1\mathbf{p}_1 + 2\mathbf{p}_2 + (-1)\mathbf{p}_3$.

The coordinate vector is $(\mathbf{p})_S = (1,2,-1)$.

19. **(a)** The third vector is a sum of the first two. This makes the set linearly dependent, hence it cannot be a basis for R^2.

(b) The two vectors generate a plane in R^3, but they do not span all of R^3. Consequently, the set is not a basis for R^3.

(c) For instance, the polynomial $\mathbf{p} = 1$ cannot be expressed as a linear combination of the given two polynomials. This means these two polynomials do not span P_2, hence they do not form a basis for P_2.

(d) For instance, the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ cannot be expressed as a linear combination of the given four matrices. This means these four matrices do not span M_{22}, hence they do not form a basis for M_{22}.

20. If the set contains at least two vectors, then the zero vector can be expressed as a scalar product of any other vector in the set and zero scalar. According to Definition 1 in Section 4.3, this makes the set linearly dependent.

A set with only one vector is linearly dependent if and only if the vector is a zero vector (see the margin note next to Definition 1 in Section 4.3).

21. **(a)** We have $T_A(1,0,0)=(1,0,-1)$, $T_A(0,1,0)=(1,1,2)$, and $T_A(0,0,1)=(1,-3,0)$. The vector equation

$$k_1(1,0,-1)+k_2(1,1,2)+k_3(1,-3,0)=(0,0,0)$$

can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcl} 1k_1 & + & 1k_2 & + & 1k_3 & = & 0 \\ 0k_1 & + & 1k_2 & - & 3k_3 & = & 0 \\ -1k_1 & + & 2k_2 & + & 0k_3 & = & 0 \end{array}$$

The determinant of the coefficient matrix of this system is $\det(A)=10\neq 0$, therefore by Theorem 2.3.8, the system has only the trivial solution. We conclude that the set $\{T_A(\mathbf{e}_1),T_A(\mathbf{e}_2),T_A(\mathbf{e}_3)\}$ is linearly independent.

(b) We have $T_A(1,0,0)=(1,0,-1)$, $T_A(0,1,0)=(1,1,2)$, and $T_A(0,0,1)=(2,1,1)$. By inspection,

$$(2,1,1)=(1,0,-1)+(1,1,2)$$

We conclude that the set $\{T_A(\mathbf{e}_1),T_A(\mathbf{e}_2),T_A(\mathbf{e}_3)\}$ is linearly dependent.

22. **(a)** Expressing $T_A(\mathbf{u})=(4,-2,0)$ as a linear combination of the vectors in S we obtain

$$(4,-2,0)=c_1(1,1,0)+c_2(0,1,1)+c_3(1,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 1c_1 & + & 0c_2 & + & 1c_3 & = & 4 \\ 1c_1 & + & 1c_2 & + & 1c_3 & = & -2 \\ 0c_1 & + & 1c_2 & + & 1c_3 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 6 \end{bmatrix}$.

The solution of the linear system is $c_1=-2$, $c_2=-6$, and $c_3=6$.

The coordinate vector is $(T_A(\mathbf{u}))_S=(-2,-6,6)$.

(b) Expressing $T_A(\mathbf{u})=(-2,0,-1)$ as a linear combination of the vectors in S we obtain

$$(-2,0,-1)=c_1(1,1,0)+c_2(0,1,1)+c_3(1,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 1c_1 & + & 0c_2 & + & 1c_3 & = & -2 \\ 1c_1 & + & 1c_2 & + & 1c_3 & = & 0 \\ 0c_1 & + & 1c_2 & + & 1c_3 & = & -1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$.

The solution of the linear system is $c_1 = 1$, $c_2 = 2$, and $c_3 = -3$.

The coordinate vector is $\left(T_A(\mathbf{u})\right)_S = (1,2,-3)$.

23. We have $\mathbf{u}_1 = (\cos 30°, \sin 30°) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\mathbf{u}_2 = (0,1)$.

(a) By inspection, we can express $\mathbf{w} = \left(\sqrt{3},1\right)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$

$$\left(\sqrt{3},1\right) = 2\left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right) + 0(0,1)$$

therefore the coordinate vector is $(\mathbf{w})_S = (2,0)$.

(b) Expressing $\mathbf{w} = \left(\sqrt{3},1\right)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,0) = c_1\left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right) + c_2(0,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} \tfrac{\sqrt{3}}{2}c_1 \qquad\quad &= 1 \\ \tfrac{1}{2}c_1 + c_2 &= 0 \end{aligned}$$

The first equation yields $c_1 = \frac{2}{\sqrt{3}}$, then the second equation can be solved to obtain $c_2 = -\frac{1}{\sqrt{3}}$. The coordinate vector is $(\mathbf{w})_S = \left(\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

(c) By inspection, we can express $\mathbf{w} = (0,1)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$

$$(0,1) = 0\left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right) + 1(0,1)$$

therefore the coordinate vector is $(\mathbf{w})_S = (0,1)$.

(d) Expressing $\mathbf{w} = (a, b)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(a, b) = c_1\left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right) + c_2(0,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} \tfrac{\sqrt{3}}{2}c_1 \qquad\quad &= a \\ \tfrac{1}{2}c_1 + c_2 &= b \end{aligned}$$

The first equation yields $c_1 = \frac{2a}{\sqrt{3}}$, then the second equation can be solved to obtain $c_2 = b - \frac{a}{\sqrt{3}}$. The coordinate vector is $(\mathbf{w})_S = \left(\frac{2a}{\sqrt{3}}, b - \frac{a}{\sqrt{3}}\right)$.

24. **(a)** $\left(0, \sqrt{2}\right)$; **(b)** $(1,0)$; **(c)** $\left(-1, \sqrt{2}\right)$; **(d)** $\left(a-b, \sqrt{2}b\right)$

25. **(a)** Polynomials 1, $2t$, $-2+4t^2$, and $-12t+8t^3$ are linearly independent if the equation

$$c_1(1) + c_2(2t) + c_3\left(-2+4t^2\right) + c_4\left(-12t+8t^3\right) = 0$$

has only the trivial solution. For these polynomials to span P_3, it must be possible to express every polynomial $a_0 + a_1 t + a_2 t^2 + a_3 t^3$ as

$$c_1(1) + c_2(2t) + c_3\left(-2+4t^2\right) + c_4\left(-12t+8t^3\right) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Grouping the terms on the left hand side of both equations as
$(c_1 - 2c_3) + (2c_2 - 12c_4)t + 4c_3 t^2 + 8c_4 t^3$ these equations can be rewritten as linear systems

$$\begin{array}{rcrcrcrcr} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & 0 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcr} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & a_0 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & a_1 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & a_2 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & a_3 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{vmatrix} = 64 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, a_2, and a_3. Therefore the polynomials 1, $2t$, $-2+4t^2$, and $-12t+8t^3$ are linearly independent and span P_3 so that they form a basis for P_3.

(b) To express $\mathbf{p} = -1 - 4t + 8t^2 + 8t^3$ as a linear combination of the four vectors in B, we form the nonhomogeneous system as was done in part (a), with the appropriate right hand side values

$$\begin{array}{rcrcrcrcr} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & -1 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & -4 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & 8 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & 8 \end{array}$$

Back-substitution yields $c_4 = 1$, $c_3 = 2$, $c_2 = 4$, and $c_1 = 3$.
The coordinate vector is $(\mathbf{p})_B = (3,4,2,1)$.

26. **(b)** $(\mathbf{p})_B = (2,-8,0,1)$

27. **(a)** $\mathbf{w} = 6(3,1,-4) - 1(2,5,6) + 4(1,4,8) = (20,17,2)$

(b) $\mathbf{q} = 3(x^2+1) + 0(x^2-1) + 4(2x-1) = 3x^2 + 8x - 1$

(c) $B = -8\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + 7\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -21 & -103 \\ -106 & 30 \end{bmatrix}$

True-False Exercises

(a) False. The set must also be linearly independent.

(b) False. The subset must also span V.

(c) True. This follows from Theorem 4.5.1.

(d) True. For any vector $\mathbf{v} = (a_1,\ldots,a_n)$ in R^n, we have $\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$ therefore the coordinate vector of $\mathbf{v}$ with respect to the standard basis $S = \{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ is $(\mathbf{v})_S = (a_1,\ldots,a_n) = \mathbf{v}$.

(e) False. For instance, $\{1+t^4, t+t^4, t^2+t^4, t^3+t^4, t^4\}$ is a basis for P_4.

4.6 Dimension

1. The augmented matrix of the linear system $\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1 = t$, $x_2 = 0$, $x_3 = t$. In vector form

$$(x_1, x_2, x_3) = (t, 0, t) = t(1, 0, 1)$$

therefore the solution space is spanned by a vector $\mathbf{v}_1 = (1,0,1)$. This vector is nonzero, therefore it forms a linearly independent set (Theorem 4.4.2(b)). We conclude that $\mathbf{v}_1$ forms a basis for the solution space and that the dimension of the solution space is 1.

2. The augmented matrix of the linear system $\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}$. The general solution is $x_1 = -\frac{1}{4}s$, $x_2 = -\frac{1}{4}s - t$, $x_3 = s$, $x_4 = t$. In vector form

$$(x_1,x_2,x_3,x_4)=\left(-\tfrac{1}{4}s,-\tfrac{1}{4}s-t,s,t\right)=s\left(-\tfrac{1}{4},-\tfrac{1}{4},1,0\right)+t(0,-1,0,1)$$

therefore the solution space is spanned by vectors $\mathrm{v}_1=\left(-\tfrac{1}{4},-\tfrac{1}{4},1,0\right)$ and $\mathrm{v}_2=(0,-1,0,1)$. These vectors are linearly independent since neither of them is a scalar multiple of the other (Theorem 4.4.2(c)). We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is 2.

3. The augmented matrix of the linear system $\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. The only solution is $x_1=x_2=x_3=0$.

The solution space has no basis - its dimension is 0.

4. The augmented matrix of the linear system $\begin{bmatrix} 1 & -4 & 3 & -1 & 0 \\ 2 & -8 & 6 & -2 & 0 \end{bmatrix}$ has the reduced row echelon form

$\begin{bmatrix} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1=4r-3s+t$, $x_2=r$, $x_3=s$, $x_4=t$. In vector form

$$(x_1,x_2,x_3,x_4)=(4r-3s+t,r,s,t)=r(4,1,0,0)+s(-3,0,1,0)+t(1,0,0,1)$$

therefore the solution space is spanned by vectors $\mathbf{v}_1=(4,1,0,0)$, $\mathbf{v}_2=(-3,0,1,0)$, and $\mathbf{v}_3=(1,0,0,1)$. By inspection, these vectors are linearly independent since $r\mathbf{v}_1+s\mathbf{v}_2+t\mathbf{v}_3=\mathbf{0}$ implies $r=s=t=0$. We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for the solution space and that the dimension of the solution space is 3.

5. The augmented matrix of the linear system $\begin{bmatrix} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{bmatrix}$ has the reduced row echelon form

$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1=3s-t$, $x_2=s$, $x_3=t$. In vector form

$$(x_1,x_2,x_3)=(3s-t,s,t)=s(3,1,0)+t(-1,0,1)$$

therefore the solution space is spanned by vectors $\mathbf{v}_1=(3,1,0)$ and $\mathbf{v}_2=(-1,0,1)$. These vectors are linearly independent since neither of them is a scalar multiple of the other (Theorem 4.4.2(c)). We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is 2.

6. The augmented matrix of the linear system $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form

$\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x = 4t$, $y = -5t$, $z = t$. In vector form

$$(x,y,z) = (4t,-5t,t) = t(4,-5,1)$$

therefore the solution space is spanned by vector $\mathbf{v}_1 = (4,-5,1)$. By Theorem 4.4.2(b), this vector forms a linearly independent set since it is not the zero vector. We conclude that $\mathbf{v}_1$ forms a basis for the solution space and that the dimension of the solution space is 1.

7. **(a)** If we let $y = s$ and $z = t$ be arbitrary values, we can solve the plane equation for x: $x = \frac{2}{3}s - \frac{5}{3}t$. Expressing the solution in vector form $(x,y,z) = \left(\frac{2}{3}s - \frac{5}{3}t, s, t\right) = s\left(\frac{2}{3},1,0\right) + t\left(-\frac{5}{3},0,1\right)$. By Theorem 4.4.2(c), $\left\{\left(\frac{2}{3},1,0\right),\left(-\frac{5}{3},0,1\right)\right\}$ is linearly independent since neither vector in the set is a scalar multiple of the other. A basis for the subspace is $\left\{\left(\frac{2}{3},1,0\right),\left(-\frac{5}{3},0,1\right)\right\}$. The dimension of the subspace is 2.

(b) If we let $y = s$ and $z = t$ be arbitrary values, we can solve the plane equation for x: $x = s$. Expressing the solution in vector form $(x,y,z) = (s,s,t) = s(1,1,0) + t(0,0,1)$. By Theorem 4.4.2(c), $\{(1,1,0),(0,0,1)\}$ is linearly independent since neither vector in the set is a scalar multiple of the other. A basis for the subspace is $\{(1,1,0),(0,0,1)\}$. The dimension of the subspace is 2.

(c) In vector form, $(x,y,z) = (2t,-t,4t) = t(2,-1,4)$. By Theorem 4.4.2(b), the vector $(2,-1,4)$ forms a linearly independent set since it is not the zero vector. A basis for the subspace is $\{(2,-1,4)\}$. The dimension of the subspace is 1.

(d) The subspace contains all vectors $(a,a+c,c) = a(1,1,0) + c(0,1,1)$ thus we can express it as as $\text{span}(S)$ where $S = \{(1,1,0),(0,1,1)\}$. By Theorem 4.4.2(c), S is linearly independent since neither vector in the set is a scalar multiple of the other. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 2.

8. **(a)** The given subspace can be expressed as $\text{span}(S)$ where $S = \{(1,0,0,0),(0,1,0,0),(0,0,1,0)\}$ is a set of linearly independent vectors. Therefore S forms a basis for the subspace, so its dimension is 3.

(b) The subspace contains all vectors $(a,b,a+b,a-b) = a(1,0,1,1) + b(0,1,1,-1)$ thus we can express it as $\text{span}(S)$ where $S = \{(1,0,1,1),(0,1,1,-1)\}$. By Theorem 4.4.2(c), S is linearly independent since

neither vector in the set is a scalar multiple of the other. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 2.

(c) The subspace contains all vectors $(a,a,a,a)=a(1,1,1,1)$ thus we can express it as as $\text{span}(S)$ where $S=\{(1,1,1,1)\}$. By Theorem 4.4.2(b), S is linearly independent since it contains a single nonzero vector. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 1.

9. **(a)** Let W be the space of all diagonal $n\times n$ matrices. We can write

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = d_1\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{A_1} + d_2\underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{A_2} + \cdots + d_n\underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{A_n}$$

The matrices $A_1,\ldots,A_n$ are linearly independent and they span W; hence, $A_1,\ldots,A_n$ form a basis for W. Consequently, the dimension of W is n.

(b) A basis for this space can be constructed by including the n matrices $A_1,\ldots,A_n$ from part (a), as well as $(n-1)+(n-2)+\cdots+3+2+1=\frac{n(n-1)}{2}$ matrices B_{ij} (for all $i<j$) where all entries are 0 except for the (i,j) and (j,i) entries, which are both 1.

For instance, for $n=3$, such a basis would be:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_1},\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_2},\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_3},\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{B_{12}},\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{B_{13}},\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{B_{23}}$$

The dimension is $n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}$.

(c) A basis for this space can be constructed by including the n matrices $A_1,\ldots,A_n$ from part (a), as well as $(n-1)+(n-2)+\cdots+3+2+1=\frac{n(n-1)}{2}$ matrices C_{ij} (for all $i<j$) where all entries are 0 except for the (i,j) entry, which is 1.

For instance, for $n=3$, such a basis would be:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_1},\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_2},\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_3},\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{C_{12}},\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{C_{13}},\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{C_{23}}$$

The dimension is $n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}$.

10. The given subspace can be expressed as $\text{span}(S)$ where $S=\{x,x^2,x^3\}$ is a set of linearly independent vectors in P_3. Therefore S forms a basis for the subspace. The dimension of the subspace is 3.

11. **(a)** W is the set of all polynomials $a_0+a_1x+a_2x^2$ for which $a_0+a_1+a_2=0$, i.e. all polynomials that can be expressed in the form $-a_1-a_2+a_1x+a_2x^2$.

Adding two polynomials in W results in another polynomial in W

$$\left(-a_1-a_2+a_1x+a_2x^2\right)+\left(-b_1-b_2+b_1x+b_2x^2\right)$$
$$=\left(-a_1-a_2-b_1-b_2\right)+\left(a_1+b_1\right)x+\left(a_2+b_2\right)x^2$$

since we have $\left(-a_1-a_2-b_1-b_2\right)+\left(a_1+b_1\right)+\left(a_2+b_2\right)=0$.

Likewise, a scalar multiple of a polynomial in W is also in W

$$k\left(-a_1-a_2+a_1x+a_2x^2\right)=-ka_1-ka_2+ka_1x+ka_2x^2$$

since it meets the condition $\left(-ka_1-ka_2\right)+\left(ka_1\right)+\left(ka_2\right)=0$.

According to Theorem 4.2.1, W is a subspace of P_2.

(c) From part (a), an arbitrary polynomial in W can be expressed in the form

$$-a_1-a_2+a_1x+a_2x^2=a_1\left(-1+x\right)+a_2\left(-1+x^2\right)$$

therefore, the polynomials $-1+x$ and $-1+x^2$ span W. Also, $a_1\left(-1+x\right)+a_2\left(-1+x^2\right)=0$ implies $a_1=a_2=0$, so $-1+x$ and $-1+x^2$ are linearly independent, hence they form a basis for W. The dimension of W is 2.

12. **(a)** Either $(1,0,0)$ or $(0,1,0)$ can be used since neither is in $\text{span}\{\mathbf{v}_1,\mathbf{v}_2\}$

(e.g., with $(1,0,0)$, linear independence can be easily shown calculating $\begin{vmatrix}-1 & 1 & 1\\ 2 & -2 & 0\\ 3 & -2 & 0\end{vmatrix}=2\neq 0$ then using parts (b) and (g) of Theorem 2.3.8; the set forms a basis by Theorem 4.6.4)

(b) Any of the three standard basis vector for R^3 can be used since none of them is in $\text{span}\{\mathbf{v}_1,\mathbf{v}_2\}$

(e.g., with $(1,0,0)$, linear independence can be easily shown calculating $\begin{vmatrix}1 & 3 & 1\\ -1 & 1 & 0\\ 0 & -2 & 0\end{vmatrix}=2\neq 0$ then using parts (b) and (g) of Theorem 2.3.8; the set forms a basis by Theorem 4.6.4)

13. The equation $k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{e}_1+k_4\mathbf{e}_2+k_5\mathbf{e}_3+k_6\mathbf{e}_4=\mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcrcrcrcrcr}
k_1 & - & 3k_2 & + & k_3 & & & & & & & = & 0\\
-4k_1 & + & 8k_2 & & & + & k_4 & & & & & = & 0\\
2k_1 & - & 4k_2 & & & & & + & k_5 & & & = & 0\\
-3k_1 & + & 6k_2 & & & & & & & + & k_6 & = & 0
\end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -2 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 \end{bmatrix}$.

Based on the leading entries in the first, second, fourth, and fifth columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are linearly independent. Since $\dim(R^4) = 4$, it follows by Theorem 4.6.4 that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form a basis for R^4. (The answer is not unique.)

14. The equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = 0$ implies $c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_2) + c_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = 0$, i.e., $(c_1 + c_2 + c_3)\mathbf{v}_1 + (c_2 + c_3)\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$, which by linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ requires that

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}$$

Solving this system by back-substitution yields $c_1 = c_2 = c_3 = 0$ therefore $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent. Since the dimension of V is 3 (as its basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ contains three vectors), by Theorem 4.6.4 $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ must also be a basis for V.

15. The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 = \mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcrcrcrcr} k_1 & & & + & k_3 & & & & & = & 0 \\ -2k_1 & + & 5k_2 & & & + & k_4 & & & = & 0 \\ 3k_1 & - & 3k_2 & & & & & + & k_5 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{5}{9} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{9} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{5}{9} & 0 \end{bmatrix}$.

Based on the leading entries in the first three columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$).

Therefore the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{e}_1$ are linearly independent. Since $\dim\left(R^3\right)=3$, it follows by Theorem 4.6.4 that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{e}_1$ form a basis for R^3. (The answer is not unique.)

16. One of the infinitely many ways to enlarge the given set to a basis for R^4 is by adding the vectors $(0,0,1,0)$ and $(0,0,0,1)$ to the set. Since the resulting set contains $\dim\left(R^4\right)=4$ vectors, by Theorem 4.6.4 we only need to establish the linear independence of the set to be able to conclude that it forms a basis for R^4. The homogeneous equation $k_1(1,0,0,0)+k_2(1,1,0,0)+k_3(0,0,1,0)+k_4(0,0,0,1)=(0,0,0,0)$ can be rewritten as a linear system whose coefficient matrix $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has determinant 1. Using parts (b) and (g) of Theorem 2.3.8, we conclude that there is only the trivial solution, therefore the enlarged set of four vectors is linearly independent (and, consequently, forms a basis for R^4).

17. The equation $k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3+k_4\mathbf{v}_4=\mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcrcrcl} 1k_1 & + & 1k_2 & + & 2k_3 & + & 0k_4 & = & 0 \\ 0k_1 & + & 0k_2 & + & 0k_3 & + & 0k_4 & = & 0 \\ 0k_1 & + & 1k_2 & + & 1k_3 & - & 1k_4 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

For arbitrary values of s and t, we have $k_1=-s-t$, $k_2=-s+t$, $x_3=s$, $k_4=t$.
Letting $s=1$ and $t=0$ allows us to express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_3=\mathbf{v}_1+\mathbf{v}_2$.
Letting $s=0$ and $t=1$ allows us to express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_4=\mathbf{v}_1-\mathbf{v}_2$.
By part (b) of Theorem 4.6.3, $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2\}=\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$.

Based on the leading entries in the first two columns, the vector equation $k_1\mathbf{v}_1+k_2\mathbf{v}_2=\mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. We conclude that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$. (The answer is not unique.)

18. The equation $k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3+k_4\mathbf{v}_4=\mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{ccccccccc} 1k_1 & + & 2k_2 & + & 0k_3 & + & 3k_4 & = & 0 \\ 1k_1 & + & 2k_2 & + & 0k_3 & + & 3k_4 & = & 0 \\ 1k_1 & + & 2k_2 & + & 0k_3 & + & 3k_4 & = & 0 \\ 1k_1 & + & 0k_2 & + & 3k_3 & + & 4k_4 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 3 & 4 & 0 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

For arbitrary values of s and t, we have $k_1 = -3s - 4t$, $k_2 = \frac{3}{2}s + \frac{1}{2}t$, $k_3 = s$, $k_4 = t$.

Letting $s = 1$ and $t = 0$ allows us to express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_3 = 3\mathbf{v}_1 - \frac{3}{2}\mathbf{v}_2$.

Letting $s = 0$ and $t = 1$ allows us to express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_4 = 4\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2$.

By part (b) of Theorem 4.6.3, $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Based on the leading entries in the first two columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. We conclude that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. (The answer is not unique.)

19. The space of all vectors $\mathbf{x} = (x_1, x_2, x_3)$ for which $T_A(\mathbf{x}) = \mathbf{0}$ is the solution space of $A\mathbf{x} = \mathbf{0}$.

(a) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = -t$, $x_2 = t$, $x_3 = t$. In vector form, $(x_1, x_2, x_3) = (-t, t, t) = t(-1,1,1)$. Since $\{(-1,1,1)\}$ is a basis for the space, the dimension is 1.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = -2s$, $x_2 = s$, $x_3 = t$. In vector form, $(x_1, x_2, x_3) = (-2s, s, t) = s(-2,1,0) + t(0,0,1)$. Since $\{(-2,1,0), (0,0,1)\}$ is a basis for the space, the dimension is 2.

(c) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = 0$, $x_2 = -t$, $x_3 = t$. In vector form, $(x_1, x_2, x_3) = (0, -t, t) = t(0,-1,1)$. Since $\{(0,-1,1)\}$ is a basis for the space, the dimension is 1.

20. The space of all vectors $\mathbf{x}=(x_1,x_2,x_3,x_4)$ for which $T_A(\mathbf{x})=\mathbf{0}$ is the solution space of $A\mathbf{x}=\mathbf{0}$.

(a) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$ so $x_1=-2s+t$, $x_2=-\frac{1}{2}s+\frac{1}{4}t$, $x_3=s$, $x_4=t$. In vector form,

$(x_1,x_2,x_3,x_4)=(-2s+t,-\frac{1}{2}s+\frac{1}{4}t,s,t)=s(-2,-\frac{1}{2},1,0)+t(1,\frac{1}{4},0,1)$.

Since $\{(-2,-\frac{1}{2},1,0),(1,\frac{1}{4},0,1)\}$ is a basis for the space, the dimension is 2.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ so $x_1=x_2=x_3=-t$, $x_4=t$. In vector form,

$(x_1,x_2,x_3,x_4)=(-t,-t,-t,t)=t(-1,-1,-1,1)$.

Since $\{(-1,-1,-1,1)\}$ is a basis for the space, the dimension is 1.

27. In parts (a) and (b), we will use the results of Exercises 18 and 19 by working with coordinate vectors with respect to the standard basis for P_2, $S=\{1,x,x^2\}$.

(a) Denote $\mathbf{v}_1=-1+x-2x^2$, $\mathbf{v}_2=3+3x+6x^2$, $\mathbf{v}_3=9$.

Then $(\mathbf{v}_1)_S=(-1,1,-2)$, $(\mathbf{v}_2)_S=(3,3,6)$, $(\mathbf{v}_3)_S=(9,0,0)$.

Setting $k_1(\mathbf{v}_1)_S+k_2(\mathbf{v}_2)_S+k_3(\mathbf{v}_3)_S=\mathbf{0}$ we obtain a linear system with augmented matrix

$\begin{bmatrix} -1 & 3 & 9 & 0 \\ 1 & 3 & 0 & 0 \\ -2 & 6 & 0 & 0 \end{bmatrix}$ whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Since there is only the trivial solution, it follows that the three coordinate vectors are linearly independent, and, by the result of Exercise 22, so are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Because the number of these vector matches $\dim(P_2)=3$, from Theorem 4.6.4 the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for P_2.

(b) Denote $\mathbf{v}_1=1+x$, $\mathbf{v}_2=x^2$, $\mathbf{v}_3=2+2x+3x^2$.

Then $(\mathbf{v}_1)_S=(1,1,0)$, $(\mathbf{v}_2)_S=(0,0,1)$, $(\mathbf{v}_3)_S=(2,2,3)$.

Setting $k_1(\mathbf{v}_1)_S+k_2(\mathbf{v}_2)_S+k_3(\mathbf{v}_3)_S=\mathbf{0}$ we obtain a linear system with augmented matrix

$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$ whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

This yields solutions $k_1=-2t$, $k_2=-3t$, $k_3=t$. Taking $t=1$, we can express $(\mathbf{v}_3)_S$ as a linear combination of $(\mathbf{v}_1)_S$ and $(\mathbf{v}_2)_S$: $(\mathbf{v}_3)_S=2(\mathbf{v}_1)_S+3(\mathbf{v}_2)_S$ - the same relationship holds true for the vectors themselves: $\mathbf{v}_3=2\mathbf{v}_1+3\mathbf{v}_2$. By part (b) of Theorem 4.6.3,

$\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2\}=\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

Based on the leading entries in the first two columns, the vector equation $k_1(\mathbf{v}_1)_S + k_2(\mathbf{v}_2)_S = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$). Therefore the coordinate vectors $(\mathbf{v}_1)_S$ and $(\mathbf{v}_2)_S$ are linearly independent and, by the result of Exercise 18, so are the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

We conclude that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(c) Clearly, $1 + x - 3x^2 = \frac{1}{2}(2 + 2x - 6x^2) = \frac{1}{3}(3 + 3x - 9x^2)$ therefore from Theorem 4.6.3(b), the subspace is spanned by $1 + x - 3x^2$. By Theorem 4.4.2(b), a set containing a single nonzero vector is linearly independent.

We conclude that $1 + x - 3x^2$ forms a basis for this subspace of P_2.

True-False Exercises

(a) True.

(b) True. For instance, $\mathbf{e}_1, \ldots, \mathbf{e}_{17}$.

(c) False. This follows from Theorem 4.6.2(b).

(d) True. This follows from Theorem 4.6.4.

(e) True. This follows from Theorem 4.6.4.

(f) True. This follows from Theorem 4.6.5(a).

(g) True. This follows from Theorem 4.6.5(b).

(h) True. For instance, invertible matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ form a basis for M_{22}.

(i) True. The set has $n^2 + 1$ matrices, which exceeds $\dim(M_{nn}) = n^2$.

(j) False. This follows from Theorem 4.6.6(c).

(k) False. For instance, for any constant c, $\operatorname{span}\{x - c, x^2 - c^2\}$ is a two-dimensional subspace of P_2 consisting of all polynomials in P_2 for which $p(c) = 0$. Clearly, there are infinitely many different subspaces of this type.

4.7 Change of Basis