(h) True. This set contains at least one matrix (e.g., I_n). A sum of two upper triangular matrices is also upper triangular, therefore the set is closed under addition. A scalar multiple of an upper triangular matrix is also upper triangular, hence the set is closed under scalar multiplication.

4.3 Spanning Sets

1. **(a)** For $(2,2,2)$ to be a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$, there must exist scalars a and b such that

$$a(0,-2,2)+b(1,3,-1)=(2,2,2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 0a & + & 1b & = & 2 \\ -2a & + & 3b & = & 2 \\ 2a & - & 1b & = & 2 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. The linear system is consistent, therefore $(2,2,2)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(b) For $(0,4,5)$ to be a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$, there must exist scalars a and b such that

$$a(0,-2,2)+b(1,3,-1)=(0,4,5)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 0a & + & 1b & = & 0 \\ -2a & + & 3b & = & 4 \\ 2a & - & 1b & = & 5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The last row corresponds to the equation $0=1$ which is contradictory. We conclude that $(0,4,5)$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) By inspection, the zero vector $(0,0,0)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ since

$$0(0,-2,2)+0(1,3,-1)=(0,0,0)$$

3. **(a)** For $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ to be a linear combination of A, B, and C, there must exist scalars a, b, and c such that

$$a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$$

Equating corresponding entries on both sides yields the linear system

$$\begin{array}{rcrcrcr} 4a & + & 1b & + & 0c & = & 6 \\ 0a & - & 1b & + & 2c & = & -8 \\ -2a & + & 2b & + & 1c & = & -1 \\ -2a & + & 3b & + & 4c & = & -8 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The linear system is consistent, therefore $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is a linear combination of A, B, and C.

(b) The zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a linear combination of A, B, and C since $0A + 0B + 0C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(c) For $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$ to be a linear combination of A, B, and C, there must exist scalars a, b, and c such that

$$a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$$

Equating corresponding entries on both sides yields the linear system

$$\begin{array}{rcrcrcr} 4a & + & 1b & + & 0c & = & -1 \\ 0a & - & 1b & + & 2c & = & 5 \\ -2a & + & 2b & + & 1c & = & 7 \\ -2a & + & 3b & + & 4c & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The last row corresponds to the equation $0 = 1$ which is contradictory. We conclude that $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$ is not a linear combination of A, B, and C.

5. **(a)** We need to solve the equation $k_1\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + k_3\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + k_4\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ to express the vector $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ as the desired linear combination. We can rewrite this as $\begin{bmatrix} k_1 + 2k_4 & -k_1 + k_2 + k_3 \\ k_4 & 2k_1 + k_2 - k_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Equating coefficients produces a linear system whose augmented matrix is $\begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ -1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & -1 & 4 \end{bmatrix}$. This matrix has reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & -13 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ hence $-3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + 12\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - 13\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

(b) Following part (a) we obtain a linear system whose augmented matrix is $\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ -1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & -1 & 2 \end{bmatrix}$. This matrix has reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

hence $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$.

7. **(a)** The given vectors span R^3 if an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination

$$(b_1, b_2, b_3) = k_1(2, 2, 2) + k_2(0, 0, 3) + k_3(0, 1, 1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcl} 2k_1 & + & 0k_2 & + & 0k_3 & = & b_1 \\ 2k_1 & + & 0k_2 & + & 1k_3 & = & b_2 \\ 2k_1 & + & 3k_2 & + & 1k_3 & = & b_3 \end{array}$$

By inspection, regardless of the right hand side values b_1, b_2, b_3, the first equation can be solved for k_1, then the second equation can be used to obtain k_3, and the third would yield k_2.

We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span R^3.

(b) The given vectors span R^3 if an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination

$$(b_1, b_2, b_3) = k_1(2, -1, 3) + k_2(4, 1, 2) + k_3(8, -1, 8)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcl} 2k_1 & + & 4k_2 & + & 8k_3 & = & b_1 \\ -1k_1 & + & 1k_2 & - & 1k_3 & = & b_2 \\ 3k_1 & + & 2k_2 & + & 8k_3 & = & b_3 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 0$, therefore by Theorem 2.3.8, the system cannot be consistent for all right hand side vectors $\mathbf{b}$.

We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span R^3.

9. The given polynomials span P_2 if an arbitrary polynomial in P_2, $\mathbf{p} = a_0 + a_1x + a_2x^2$ can be expressed as a linear combination

$$a_0 + a_1x + a_2x^2 = k_1(1 - x + 2x^2) + k_2(3 + x) + k_3(5 - x + 4x^2) + k_4(-2 - 2x + 2x^2)$$

Grouping the terms according to the powers of x yields

$$a_0 + a_1x + a_2x^2 = (k_1 + 3k_2 + 5k_3 - 2k_4) + (-k_1 + k_2 - k_3 - 2k_4)x + (2k_1 + 4k_3 + 2k_4)x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcrcl} 1k_1 & + & 3k_2 & + & 5k_3 & - & 2k_4 & = & a_0 \\ -1k_1 & + & 1k_2 & - & 1k_3 & - & 2k_4 & = & a_1 \\ 2k_1 & + & 0k_2 & + & 4k_3 & + & 2k_4 & = & a_2 \end{array}$$

whose augmented matrix $\begin{bmatrix} 1 & 3 & 5 & -2 & a_0 \\ -1 & 1 & -1 & -2 & a_1 \\ 2 & 0 & 4 & 2 & a_2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 2 & 1 & \frac{1}{4}a_0 - \frac{3}{4}a_1 \\ 0 & 1 & 1 & -1 & \frac{1}{4}a_0 + \frac{1}{4}a_1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \end{bmatrix}$ therefore the system has no solution if $-\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \neq 0$.

Since polynomials $\mathbf{p} = a_0 + a_1x + a_2x^2$ for which $-\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \neq 0$ cannot be expressed as a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$, we conclude that the polynomials $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ do not span P_2.

11. **(a)** The given matrices span M_{22} if an arbitrary matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as a linear combination $k_1\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + k_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We can rewrite this as $\begin{bmatrix} k_1 + k_2 & k_2 + k_3 \\ k_1 + k_4 & k_3 + k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Equating coefficients produces a linear system whose augmented matrix is $\begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 1 & 0 & 0 & 1 & c \\ 0 & 0 & 1 & 1 & d \end{bmatrix}$. The coefficient matrix has $\det\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\right) = 0$ which means the system is not consistent. We conclude that the given matrices do not span M_{22}.

(b) The given matrices span M_{22} if an arbitrary matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as a linear combination $k_1\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We can rewrite this as $\begin{bmatrix} k_1 + k_3 + k_4 & -k_1 + k_2 + k_3 \\ k_3 & k_1 + k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Equating coefficients produces a linear system whose augmented matrix is $\begin{bmatrix} 1 & 0 & 1 & 1 & a \\ -1 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 1 & 0 & 0 & 1 & d \end{bmatrix}$. The coefficient matrix has $\det\left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\right) = 1$ which means the system is consistent. We conclude that the given matrices span M_{22}.

(c) The given matrices span M_{22} if an arbitrary matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as a linear combination $k_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We can rewrite this as $\begin{bmatrix} k_1 + k_2 + k_3 + k_4 & k_2 + k_3 + k_4 \\ k_3 + k_4 & k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Equating coefficients produces a linear system whose

augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 1 & a \\ 0 & 1 & 1 & 1 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & 1 & d \end{bmatrix}$. The coefficient matrix has $\det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 1$ which means the system is consistent. We conclude that the given matrices span M_{22}.

13. **(a)** The vector $\mathbf{u} = (1,1,1)$ is in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$ if it is a linear combination of the columns of $A = \begin{bmatrix} 0 & 2 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}$: $T_A(\mathbf{e}_1) = \begin{bmatrix} 0 & 2 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $T_A(\mathbf{e}_2) = \begin{bmatrix} 0 & 2 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$. So there must exist scalars a and b such that $a(0,1,1) + b(2,-2,0) = (1,1,1)$.

Equating corresponding components on both sides leads to the linear system $\begin{array}{rcrcl} 0 & + & 2b & = & 1 \\ a & - & 2b & = & 1 \\ a & + & 0 & = & 1 \end{array}$ which is inconsistent since subtracting the last equation from the second yields $-2b = 0$ while the first equation is $2b = 1$. We conclude that the vector $\mathbf{u} = (1,1,1)$ is not in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$.

(b) The vector $\mathbf{u} = (1,1,1)$ is in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$ if it is a linear combination of the columns of $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$: $T_A(\mathbf{e}_1) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $T_A(\mathbf{e}_2) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. . So there must exist scalars a and b such that $a(0,1,2) + b(2,1,0) = (1,1,1)$. Observe that $\frac{1}{2}(0,1,2) + \frac{1}{2}(2,1,0) = (1,1,1)$. We conclude that the vector $\mathbf{u} = (1,1,1)$ is not in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$.

15. **(a)** The solution space W to the homogenous system $A\mathbf{x} = 0$ where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ is obtained from the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution in vector form is $(x,y,z,w) = (s,t,-s,-t) = s(1,0,-1,0) + t(0,1,0,-1)$ therefore the solution space is spanned by the vectors $\mathbf{v}_1 = (1,0,-1,0)$ and $\mathbf{v}_2 = (0,1,0,-1)$. We conclude that the vectors $\mathbf{u} = (1,0,-1,0)$ and $\mathbf{v} = (0,1,0,-1)$ span the solution space W.

(b) From part (a) and Theorem 4.3.2 we need to show that the vectors $\mathbf{u} = (1,0,-1,0)$ and $\mathbf{v} = (1,1,-1,-1)$ are contained in the span of the vectors $\mathbf{v}_1 = (1,0,-1,0)$ and $\mathbf{v}_2 = (0,1,0,-1)$. Observe that $\mathbf{u} = \mathbf{v}_1$ and

$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. We conclude that the vectors $\mathbf{u} = (1,0,-1,0)$ and $\mathbf{v} = (1,1,-1,-1)$ span the solution space W.

17. **(a)** The vectors $T_A(1,2) = (-1,4)$ and $T_A(-1,1) = (-2,2)$ span R^2 if an arbitrary vector $\mathbf{b} = (b_1, b_2)$ can be expressed as a linear combination

$$(b_1, b_2) = k_1(-1, 4) + k_2(-2, 2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} -1k_1 & - & 2k_2 & = & b_1 \\ 4k_1 & + & 2k_2 & = & b_2 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} -1 & -2 \\ 4 & 2 \end{vmatrix} = 6 \neq 0$, therefore by Theorem 2.3.8, the system is consistent for all right hand side vectors $\mathbf{b}$.
We conclude that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ span R^2.

(b) The vectors $T_A(1,2) = (-1,2)$ and $T_A(-1,1) = (-2,4)$ span R^2 if an arbitrary vector $\mathbf{b} = (b_1, b_2)$ can be expressed as a linear combination

$$(b_1, b_2) = k_1(-1, 2) + k_2(-2, 4)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} -1k_1 & - & 2k_2 & = & b_1 \\ 2k_1 & + & 4k_2 & = & b_2 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix} = 0$, therefore by Theorem 2.3.8, the system cannot be consistent for all right hand side vectors $\mathbf{b}$.
We conclude that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ do not span R^2.

19. Using Theorem 4.3.2, we need to show that the each of the polynomials $\mathbf{q}_1 = 2x$ and $\mathbf{q}_2 = 1 + x^2$ is in the span of the polynomials $\mathbf{p}_1 = 1 + x^2$ and $\mathbf{p}_2 = 1 + x + x^2$. Clearly, $\mathbf{q}_2 = \mathbf{p}_1$.

Observe that $2x = (-2) \quad (1 + x^2) + 2(1 + x + x^2)$ so that $\mathbf{q}_1 = (-2)\mathbf{p}_1 + 2\mathbf{p}_2$.

21. For the vector $(3,5)$ to be expressed as $\mathbf{v} + \mathbf{w}$ where $\mathbf{v}$ is in the subspace spanned by $(3,1)$ and $\mathbf{w}$ is in the subspace spanned by $(2,1)$, we must produce scalars a and b such that $a(3,1) + b(2,1) = (3,5)$.

Equating corresponding components yields a linear system with augmented matrix $\begin{bmatrix} 3 & 2 & 3 \\ 1 & 1 & 5 \end{bmatrix}$ which has

reduced row echelon form $\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 12 \end{bmatrix}$.

Therefore $\mathbf{v} = -7(3,1) = (-21,-7)$ and $\mathbf{w} = 12(2,1) = (24,12)$.

True False Exercises

(a) True.

(b) False. The span of the zero vector is just the zero vector.

(c) False. For example the vectors $(1,1,1)$ and $(2,2,2)$ span a line.

(d) True.

(e) True. This follows from part (a) of Theorem 4.2.1.

(f) False. For any nonzero vector $\mathbf{v}$ in a vector space V, both $\{\mathbf{v}\}$ and $\{2\mathbf{v}\}$ span the same subspace of V.

(g) False. The constant polynomial $p(x) = 1$ cannot be represented as a linear combination of these, since at $x = 1$ all three are zero, whereas $p(1) = 1$.

4.4 Linear Independence

1. **(a)** Since $\mathbf{u}_2 = -5\mathbf{u}_1$, linear dependence follows from Definition 1.

(b) A set of 3 vectors in R^2 must be linearly dependent by Theorem 4.4.3.

(c) Since $\mathbf{p}_2 = 2\mathbf{p}_1$, linear dependence follows from Definition 1.

(d) Since $A = (-1)B$, linear dependence follows from Definition 1.

3. **(a)** The vector equation $a(3,8,7,-3) + b(1,5,3,-1) + c(2,-1,2,6) + d(4,2,6,4) = (0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcrcl} 3a & + & 1b & + & 2c & + & 4d & = & 0 \\ 8a & + & 5b & - & 1c & + & 2d & = & 0 \\ 7a & + & 3b & + & 2c & + & 6d & = & 0 \\ -3a & - & 1b & + & 6c & + & 4d & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is $a = -t$, $b = t$, $c = -t$, $d = t$.

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

(b) The vector equation $a(3,0,-3,6)+b(0,2,3,1)+c(0,-2,-2,0)+d(-2,1,2,1)=(0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcrcl} 3a & + & 0b & + & 0c & - & 2d & = & 0 \\ 0a & + & 2b & - & 2c & + & 1d & = & 0 \\ -3a & + & 3b & - & 2c & + & 2d & = & 0 \\ 6a & + & 1b & + & 0c & + & 1d & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a=b=c=d=0$. We conclude that the given set of vectors is linearly independent.

5. **(a)** The matrix equation $a\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}+b\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}+c\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcl} 1a & + & 1b & + & 0c & = & 0 \\ 0a & + & 2b & + & 1c & = & 0 \\ 1a & + & 2b & + & 2c & = & 0 \\ 2a & + & 1b & + & 1c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a=b=c=0$. We conclude that the given matrices are linearly independent.

(b) By inspection, the matrix equation $a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}+b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}+c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has only the trivial solution $a=b=c=0$. We conclude that the given matrices are linearly independent.

7. Three vectors in R^3 lie in a plane if and only if they are linearly dependent when they have their initial points at the origin. (See the discussion following Example 6.)

(a) The vector equation $a(2,-2,0)+b(6,1,4)+c(2,0,-4)=(0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} 2a & + & 6b & + & 2c & = & 0 \\ -2a & + & 1b & + & 0c & = & 0 \\ 0a & + & 4b & - & 4c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a = b = c = 0$. We conclude that the given vectors are linearly independent, hence they do not lie in a plane.

(b) The vector equation $a(-6,7,2) + b(3,2,4) + c(4,-1,2) = (0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} -6a & + & 3b & + & 4c & = & 0 \\ 7a & + & 2b & - & 1c & = & 0 \\ 2a & + & 4b & + & 2c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is $a = \frac{1}{3}t$, $b = -\frac{2}{3}t$, $c = t$.

Since the system has nontrivial solutions, the given vectors are linearly dependent, hence they lie in a plane.

9. **(a)** The vector equation $a(0,3,1,-1) + b(6,0,5,1) + c(4,-7,1,3) = (0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} 0a & + & 6b & + & 4c & = & 0 \\ 3a & + & 0b & - & 7c & = & 0 \\ 1a & + & 5b & + & 1c & = & 0 \\ -1a & + & 1b & + & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{7}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is $a = \frac{7}{3}t$, $b = -\frac{2}{3}t$, $c = t$.

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

(b) From part (a), we have $\frac{7}{3}t\mathbf{v}_1 - \frac{2}{3}t\mathbf{v}_2 + t\mathbf{v}_3 = 0$.

Letting $t = \frac{3}{7}$, we obtain $\mathbf{v}_1 = \frac{2}{7}\mathbf{v}_2 - \frac{3}{7}\mathbf{v}_3$.

Letting $t=-\frac{3}{2}$, we obtain $\mathbf{v}_2=\frac{7}{2}\mathbf{v}_1+\frac{3}{2}\mathbf{v}_3$.

Letting $t=1$, we obtain $\mathbf{v}_3=-\frac{7}{3}\mathbf{v}_1+\frac{2}{3}\mathbf{v}_2$.

11. By inspection, when $\lambda=-\frac{1}{2}$, the vectors become linearly dependent (since they all become equal).

We proceed to find the remaining values of λ.

The vector equation $a\left(\lambda,-\frac{1}{2},-\frac{1}{2}\right)+b\left(-\frac{1}{2},\lambda,-\frac{1}{2}\right)+c\left(-\frac{1}{2},-\frac{1}{2},\lambda\right)=(0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} \lambda a & - & \frac{1}{2}b & - & \frac{1}{2}c & = & 0 \\ -\frac{1}{2}a & + & \lambda b & - & \frac{1}{2}c & = & 0 \\ -\frac{1}{2}a & - & \frac{1}{2}b & + & \lambda c & = & 0 \end{array}$$

The determinant of the coefficient matrix is $\begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix}=\lambda^3-\frac{3}{4}\lambda-\frac{1}{4}$. This determinant equals zero for all λ values for which the vectors are linearly dependent. Since we already know that $\lambda=-\frac{1}{2}$ is one of those values, we can divide $\lambda+\frac{1}{2}$ into $\lambda^3-\frac{3}{4}\lambda-\frac{1}{4}$ to obtain

$$\lambda^3-\tfrac{3}{4}\lambda-\tfrac{1}{4}=\left(\lambda+\tfrac{1}{2}\right)\left(\lambda^2-\tfrac{1}{2}\lambda-\tfrac{1}{2}\right)=\left(\lambda+\tfrac{1}{2}\right)\left(\lambda+\tfrac{1}{2}\right)(\lambda-1).$$

We conclude that the vectors form a linearly dependent set for $\lambda=-\frac{1}{2}$ and for $\lambda=1$.

13. **(a)** We calculate $T_A(1,2)=(-1,4)$ and $T_A(-1,1)=(-2,2)$. The vector equation

$$k_1(-1,4)+k_2(-2,2)=(0,0)$$

can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcl} -1k_1 & - & 2k_2 & = & 0 \\ 4k_1 & + & 2k_2 & = & 0 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} -1 & -1 \\ 4 & 1 \end{vmatrix}=6\neq 0$, therefore by Theorem 2.3.8, the system has only the trivial solution. We conclude that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ form a linearly independent set.

(b) We calculate $T_A(1,2)=(-1,2)$ and $T_A(-1,1)=(-2,4)$. Since $(-2,4)=2(-1,2)$, it follows by Definition 1 that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ form a linearly dependent set.

15. Three vectors in R^3 lie in a plane if and only if they are linearly dependent when they have their initial points at the origin. (See the discussion following Example 6.)

(a) After the three vectors are moved so that their initial points are at the origin, the resulting vectors do not lie on the same plane. Hence these vectors are linearly independent.

(b) After the three vectors are moved so that their initial points are at the origin, the resulting vectors lie on the same plane. Hence these vectors are linearly dependent.

17. The Wronskian is $W(x)=\begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix}=-x\sin x-\cos x$. Since $W(x)$ is not identically 0 on $(-\infty,\infty)$ (e.g., $W(0)=-1\neq 0$), the functions x and $\cos x$ are linearly independent.

19. **(a)** The Wronskian is $W(x)=\begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix}=e^x$. Since $W(x)$ is not identically 0 on $(-\infty,\infty)$ (e.g., $W(0)=1\neq 0$), the functions 1, x and e^x are linearly independent.

(b) The Wronskian is $W(x)=\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}=2$. Since $W(x)$ is not identically 0 on $(-\infty,\infty)$, the functions 1, x and x^2 are linearly independent.

21.
$$W(x)=\begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x-x\sin x \\ -\sin x & -\cos x & -2\sin x-x\cos x \end{vmatrix} \quad \longleftarrow \text{The Wronskian}$$

$$=\begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x-x\sin x \\ 0 & 0 & -2\sin x \end{vmatrix} \quad \longleftarrow \text{The first row was added to the third.}$$

$$=-2\sin x\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} \quad \longleftarrow \text{Cofactor expansion along the third row}$$

$$=-2\sin x\left(-\sin^2 x-\cos^2 x\right)$$

$$=(-2\sin x)(-1)=2\sin x$$

Since $W(x)$ is not identically 0 on $(-\infty,\infty)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$ are linearly independent.

True-False Exercises

(a) False. By part (b) of Theorem 4.4.2, a set containing a single *nonzero* vector is linearly independent.

(b) True. This follows directly from Definition 1.

(c) False. For instance $\{(1,1),(2,2)\}$ is a linearly dependent set that does not contain $(0,0)$.

(d) True. If $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0}$ has only one solution $a = b = c = 0$ then $a(k\mathbf{v}_1) + b(k\mathbf{v}_2) + c(k\mathbf{v}_3) = k(a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3)$ can only equal $\mathbf{0}$ when $a = b = c = 0$ as well.

(e) True. Since the vectors must be nonzero, $\{\mathbf{v}_1\}$ must be linearly independent.

Let us begin adding vectors to the set until the set $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ becomes linearly dependent, therefore, by construction, $\{\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}\}$ is linearly independent. The equation $c_1\mathbf{v}_1 + \cdots + c_{k-1}\mathbf{v}_{k-1} + c_k\mathbf{v}_k = \mathbf{0}$ must have a solution with $c_k \neq 0$, therefore $\mathbf{v}_k = -\frac{c_1}{c_k}\mathbf{v}_1 - \cdots - \frac{c_{k-1}}{c_k}\mathbf{v}_{k-1}$. Let us assume there exists another representation $\mathbf{v}_k = d_1\mathbf{v}_1 + \cdots + d_{k-1}\mathbf{v}_{k-1}$. Subtracting both sides yields $\mathbf{0} = \left(d_1 + \frac{c_1}{c_k}\right)\mathbf{v}_1 + \cdots + \left(d_{k-1} + \frac{c_{k-1}}{c_k}\right)\mathbf{v}_{k-1}$. By linear independence of $\{\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}\}$, we must have $d_1 = -\frac{c_1}{c_k}$, …, $d_{k-1} = -\frac{c_{k-1}}{c_k}$, which shows that $\mathbf{v}_k$ is a *unique* linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}$.

(f) False. The set $\left\{\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$ is linearly dependent since $\begin{bmatrix}1&1\\0&0\end{bmatrix} = (-1)\begin{bmatrix}0&0\\1&1\end{bmatrix} + \begin{bmatrix}1&0\\1&0\end{bmatrix} + \begin{bmatrix}0&1\\0&1\end{bmatrix}$.

(g) True. Requiring that for all x values $a(x-1)(x+2) + bx(x+2) + cx(x-1) = 0$ holds true implies that the equality must be true for any specific x value. Setting $x = 0$ yields $a = 0$. Likewise, $x = 1$ implies $b = 0$, and $x = -2$ implies $c = 0$. Since $a = b = c = 0$ is required, we conclude that the three given polynomials are linearly independent.

(h) False. The functions f_1 and f_2 are linearly dependent if there exist scalars k_1 and k_2, <u>not both equal 0</u>, such that $k_1 f_1(x) + k_2 f_2(x) = 0$ for <u>all</u> real numbers x.

4.5 Coordinates and Basis

1. Vectors $(2,1)$ and $(3,0)$ are linearly independent if the vector equation

$$c_1(2,1) + c_2(3,0) = (0,0)$$

has only the trivial solution. For these vectors to span R^2, it must be possible to express every vector $\mathbf{b} = (b_1, b_2)$ in R^2 as

$$c_1(2,1)+c_2(3,0)=(b_1,b_2)$$

These two equations can be rewritten as linear systems

$$\begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & 0 \\ c_1 & & & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & b_1 \\ c_1 & & & = & b_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -3 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values b_1 and b_2. Therefore the vectors $(2,1)$ and $(3,0)$ are linearly independent and span R^2 so that they form a basis for R^2.

3. Polynomials x^2+1, x^2-1, and $2x-1$ are linearly independent if the equation

$$c_1(x^2+1)+c_2(x^2-1)+c_3(2x-1)=0$$

has only the trivial solution. For these polynomials to span P_2, it must be possible to express every polynomial $a_0+a_1x+a_2x^2$ as

$$c_1(x^2+1)+c_2(x^2-1)+c_3(2x-1)=a_0+a_1x+a_2x^2$$

Grouping the terms on the left hand side of both equations as $(c_1-c_2-c_3)+(2c_3)x+(c_1+c_2)x^2$ these equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} 1c_1 & - & 1c_2 & - & 1c_3 & = & 0 \\ 0c_1 & + & 0c_2 & + & 2c_3 & = & 0 \\ 1c_1 & + & 1c_2 & + & 0c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} 1c_1 & - & 1c_2 & - & 1c_3 & = & a_0 \\ 0c_1 & + & 0c_2 & + & 2c_3 & = & a_1 \\ 1c_1 & + & 1c_2 & + & 0c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix} = -4 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the polynomials x^2+1, x^2-1, and $2x-1$ are linearly independent and span P_2 so that they form a basis for P_2.

5. Matrices $\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ are linearly independent if the equation

$$c_1\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}+c_2\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}+c_4\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ as

$$c_1\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + c_2\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Equating corresponding entries on both sides yields linear systems

$$\begin{array}{rcrcrcrcl} 3c_1 & + & 0c_2 & + & 0c_3 & + & 1c_4 & = & 0 \\ 6c_1 & - & 1c_2 & - & 8c_3 & + & 0c_4 & = & 0 \\ 3c_1 & - & 1c_2 & - & 12c_3 & - & 1c_4 & = & 0 \\ -6c_1 & + & 0c_2 & - & 4c_3 & + & 2c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 3c_1 & + & 0c_2 & + & 0c_3 & + & 1c_4 & = & a_{11} \\ 6c_1 & - & 1c_2 & - & 8c_3 & + & 0c_4 & = & a_{12} \\ 3c_1 & - & 1c_2 & - & 12c_3 & - & 1c_4 & = & a_{21} \\ -6c_1 & + & 0c_2 & - & 4c_3 & + & 2c_4 & = & a_{22} \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{vmatrix} = 48 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_{11}, a_{12}, a_{21}, and a_{22}. Therefore the matrices $\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ are linearly independent and span M_{22} so that they form a basis for M_{22}.

7. **(a)** Vectors $(2,-3,1)$, $(4,1,1)$, and $(0,-7,1)$ are linearly independent if the vector equation

$$c_1(2,-3,1) + c_2(4,1,1) + c_3(0,-7,1) = (0,0,0)$$

has only the trivial solution. This equation can be rewritten as a linear system

$$\begin{array}{rcrcrcl} 2c_1 & + & 4c_2 & + & 0c_3 & = & 0 \\ -3c_1 & + & 1c_2 & - & 7c_3 & = & 0 \\ 1c_1 & + & 1c_2 & + & 1c_3 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} = 0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $(2,-3,1)$, $(4,1,1)$, and $(0,-7,1)$ are linearly dependent, they do not form a basis for R^3.

(b) Vectors $(1,6,4)$, $(2,4,-1)$, and $(-1,2,5)$ are linearly independent if the vector equation

$$c_1(1,6,4)+c_2(2,4,-1)+c_3(-1,2,5)=(0,0,0)$$

has only the trivial solution. This equation can be rewritten as a linear system

$$\begin{array}{rcrcrcl} 1c_1 & + & 2c_2 & - & 1c_3 & = & 0 \\ 6c_1 & + & 4c_2 & + & 2c_3 & = & 0 \\ 4c_1 & - & 1c_2 & + & 5c_3 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix}=0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $(1,6,4)$, $(2,4,-1)$, and $(-1,2,5)$ are linearly dependent, they do not form a basis for R^3.

9. Matrices $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ are linearly independent if the equation

$$c_1\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}+c_2\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}+c_3\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}+c_4\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has only the trivial solution. Equating corresponding entries on both sides yields a linear system

$$\begin{array}{rcrcrcrcl} 1c_1 & + & 2c_2 & + & 1c_3 & + & 0c_4 & = & 0 \\ 0c_1 & - & 2c_2 & - & 1c_3 & - & 1c_4 & = & 0 \\ 1c_1 & + & 3c_2 & + & 1c_3 & + & 1c_4 & = & 0 \\ 1c_1 & + & 2c_2 & + & 0c_3 & + & 1c_4 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -1 & -1 \\ 1 & 3 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{vmatrix}=0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the matrices $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ are linearly dependent, we conclude that they do not form a basis for M_{22}.

11. **(a)** Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,1)=c_1(2,-4)+c_2(3,8)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & 1 \\ -4c_1 & + & 8c_2 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{bmatrix}$. The solution of the linear system is $c_1 = \frac{5}{28}$, $c_2 = \frac{3}{14}$, therefore the coordinate vector is $(\mathbf{w})_S = \left(\frac{5}{28}, \frac{3}{14}\right)$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(a,\ b) = c_1(1,1) + c_2(0,2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} 1c_1 & + & 0c_2 & = & a \\ 1c_1 & + & 2c_2 & = & b \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = a$, $c_2 = \frac{b-a}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = \left(a, \frac{b-a}{2}\right)$.

13. (a) Expressing $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$(2,-1,3) = c_1(1,0,0) + c_2(2,2,0) + c_3(3,3,3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} c_1 & + & 2c_2 & + & 3c_3 & = & 2 \\ & & 2c_2 & + & 3c_3 & = & -1 \\ & & & & 3c_3 & = & 3 \end{array}$$

which can be solved by back-substitution to obtain $c_3 = 1$, $c_2 = -2$, and $c_1 = 3$. The coordinate vector is $(\mathbf{v})_S = (3,-2,1)$.

(b) Expressing $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$(5,-12,3) = c_1(1,2,3) + c_2(-4,5,6) + c_3(7,-8,9)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 1c_1 & - & 4c_2 & + & 7c_3 & = & 5 \\ 2c_1 & + & 5c_2 & - & 8c_3 & = & -12 \\ 3c_1 & + & 6c_2 & + & 9c_3 & = & 3 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution of the linear system is $c_1 = -2$, $c_2 = 0$, and $c_3 = 1$. The coordinate vector is $(\mathbf{v})_S = (-2,0,1)$.

15. Matrices (vectors in M_{22}) A_1, A_2, A_3, and A_4 are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = B$$

The left hand side of each of these equations is the matrix $\begin{bmatrix} k_1 & k_1 + k_2 \\ k_1 + k_2 + k_3 & k_1 + k_2 + k_3 + k_4 \end{bmatrix}$.

Equating corresponding entries, these two equations can be rewritten as linear systems

$$\begin{array}{rcl} k_1 & = & 0 \\ k_1 + k_2 & = & 0 \\ k_1 + k_2 + k_3 & = & 0 \\ k_1 + k_2 + k_3 + k_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcl} k_1 & = & a \\ k_1 + k_2 & = & b \\ k_1 + k_2 + k_3 & = & c \\ k_1 + k_2 + k_3 + k_4 & = & d \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a, b, c and d. Therefore the matrices A_1, A_2, A_3, and A_4 are linearly independent and span M_{22} so that they form a basis for M_{22}.

To express $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ as a linear combination of the matrices A_1, A_2, A_3, and A_4, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{rcl} k_1 & = & 1 \\ k_1 + k_2 & = & 0 \\ k_1 + k_2 + k_3 & = & 1 \\ k_1 + k_2 + k_3 + k_4 & = & 0 \end{array}$$

which can be solved by forward-substitution to obtain $k_1 = 1$, $k_2 = -1$, $k_3 = 1$, $k_4 = -1$.

This allows us to express $A = 1A_1 - 1A_2 + 1A_3 - 1A_4$.

The coordinate vector is $(A)_S = (1,-1,1,-1)$.

17. Vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent if the vector equation

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$$

has only the trivial solution. For these vectors to span P_2, it must be possible to express every vector $\mathbf{p} = a_0 + a_1x + a_2x^2$ in P_2 as

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$$

Grouping the terms on the left hand sides as $c_1(1+x+x^2) + c_2(x+x^2) + c_3x^2 = c_1 + (c_1+c_2)x + (c_1+c_2+c_3)x^2$ these two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} c_1 & & & & & = & 0 \\ c_1 & + & c_2 & & & = & 0 \\ c_1 & + & c_2 & + & c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} c_1 & & & & & = & a_0 \\ c_1 & + & c_2 & & & = & a_1 \\ c_1 & + & c_2 & + & c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent and span P_2 so that they form a basis for P_2.

To express $\mathbf{p} = 7 - x + 2x^2$ as a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{rcrcrcr} c_1 & & & & & = & 7 \\ c_1 & + & c_2 & & & = & -1 \\ c_1 & + & c_2 & + & c_3 & = & 2 \end{array}$$

which can be solved by forward-substitution to obtain $c_1 = 7$, $c_2 = -8$, $c_3 = 3$.

This allows us to express $\mathbf{p} = 7\mathbf{p}_1 - 8\mathbf{p}_2 + 3\mathbf{p}_3$. The coordinate vector is $(\mathbf{p})_S = (7,-8,3)$.

19. **(a)** The third vector is a sum of the first two. This makes the set linearly dependent, hence it cannot be a basis for R^2.

(b) The two vectors generate a plane in R^3, but they do not span all of R^3. Consequently, the set is not a basis for R^3.

(c) For instance, the polynomial $\mathbf{p}=1$ cannot be expressed as a linear combination of the given two polynomials. This means these two polynomials do not span P_2, hence they do not form a basis for P_2.

(d) For instance, the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ cannot be expressed as a linear combination of the given four matrices. This means these four matrices do not span M_{22}, hence they do not form a basis for M_{22}.

21. **(a)** We have $T_A(1,0,0)=(1,0,-1)$, $T_A(0,1,0)=(1,1,2)$, and $T_A(0,0,1)=(1,-3,0)$. The vector equation

$$k_1(1,0,-1)+k_2(1,1,2)+k_3(1,-3,0)=(0,0,0)$$

can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcl} 1k_1 & + & 1k_2 & + & 1k_3 & = & 0 \\ 0k_1 & + & 1k_2 & - & 3k_3 & = & 0 \\ -1k_1 & + & 2k_2 & + & 0k_3 & = & 0 \end{array}$$

The determinant of the coefficient matrix of this system is $\det(A)=10\neq 0$, therefore by Theorem 2.3.8, the system has only the trivial solution. We conclude that the set $\{T_A(\mathbf{e}_1),T_A(\mathbf{e}_2),T_A(\mathbf{e}_3)\}$ is linearly independent.

(b) We have $T_A(1,0,0)=(1,0,-1)$, $T_A(0,1,0)=(1,1,2)$, and $T_A(0,0,1)=(2,1,1)$. By inspection,

$$(2,1,1)=(1,0,-1)+(1,1,2)$$

We conclude that the set $\{T_A(\mathbf{e}_1),T_A(\mathbf{e}_2),T_A(\mathbf{e}_3)\}$ is linearly dependent.

23. We have $\mathbf{u}_1=(\cos 30°,\sin 30°)=\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ and $\mathbf{u}_2=(0,1)$.

(a) By inspection, we can express $\mathbf{w}=(\sqrt{3},1)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$

$$(\sqrt{3},1)=2\left(\tfrac{\sqrt{3}}{2},\tfrac{1}{2}\right)+0(0,1)$$

therefore the coordinate vector is $(\mathbf{w})_S=(2,0)$.

(b) Expressing $\mathbf{w}=(\sqrt{3},1)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,0)=c_1\left(\tfrac{\sqrt{3}}{2},\tfrac{1}{2}\right)+c_2(0,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} \frac{\sqrt{3}}{2}c_1 & & & = & 1 \\ \frac{1}{2}c_1 & + & c_2 & = & 0 \end{array}$$

The first equation yields $c_1 = \frac{2}{\sqrt{3}}$, then the second equation can be solved to obtain $c_2 = -\frac{1}{\sqrt{3}}$. The coordinate vector is $(\mathbf{w})_S = \left(\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

(c) By inspection, we can express $\mathbf{w} = (0,1)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$

$$(0,1) = 0\left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right) + 1(0,1)$$

therefore the coordinate vector is $(\mathbf{w})_S = (0,1)$.

(d) Expressing $\mathbf{w} = (a,\ b)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(a,\ b) = c_1\left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right) + c_2(0,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} \frac{\sqrt{3}}{2}c_1 & & & = & a \\ \frac{1}{2}c_1 & + & c_2 & = & b \end{array}$$

The first equation yields $c_1 = \frac{2a}{\sqrt{3}}$, then the second equation can be solved to obtain $c_2 = b - \frac{a}{\sqrt{3}}$. The coordinate vector is $(\mathbf{w})_S = \left(\frac{2a}{\sqrt{3}}, b - \frac{a}{\sqrt{3}}\right)$.

25. (a) Polynomials 1, $2t$, $-2+4t^2$, and $-12t+8t^3$ are linearly independent if the equation

$$c_1(1) + c_2(2t) + c_3\left(-2+4t^2\right) + c_4\left(-12t+8t^3\right) = 0$$

has only the trivial solution. For these polynomials to span P_3, it must be possible to express every polynomial $a_0 + a_1t + a_2t^2 + a_3t^3$ as

$$c_1(1) + c_2(2t) + c_3\left(-2+4t^2\right) + c_4\left(-12t+8t^3\right) = a_0 + a_1t + a_2t^2 + a_3t^3$$

Grouping the terms on the left hand side of both equations as
$(c_1 - 2c_3) + (2c_2 - 12c_4)t + 4c_3t^2 + 8c_4t^3$ these equations can be rewritten as linear systems

$$\begin{array}{rcrcrcrcl} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & 0 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & a_0 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & a_1 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & a_2 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & a_3 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{vmatrix} = 64 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, a_2, and a_3. Therefore the polynomials 1, $2t$, $-2+4t^2$, and $-12t+8t^3$ are linearly independent and span P_3 so that they form a basis for P_3.

(b) To express $\mathbf{p} = -1-4t+8t^2+8t^3$ as a linear combination of the four vectors in B, we form the nonhomogeneous system as was done in part (a), with the appropriate right hand side values

$$\begin{array}{rcrcrcrcr} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & -1 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & -4 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & 8 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & 8 \end{array}$$

Back-substitution yields $c_4 = 1$, $c_3 = 2$, $c_2 = 4$, and $c_1 = 3$.

The coordinate vector is $(\mathbf{p})_B = (3,4,2,1)$.

27. **(a)** $\mathbf{w} = 6(3,1,-4) - 1(2,5,6) + 4(1,4,8) = (20,17,2)$

(b) $\mathbf{q} = 3(x^2+1) + 0(x^2-1) + 4(2x-1) = 3x^2+8x-1$

(c) $B = -8\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + 7\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -21 & -103 \\ -106 & 30 \end{bmatrix}$

True-False Exercises

(a) False. The set must also be linearly independent.

(b) False. The subset must also span V.

(c) True. This follows from Theorem 4.5.1.

(d) True. For any vector $\mathbf{v} = (a_1,\ldots,a_n)$ in R^n, we have $\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$ therefore the coordinate vector of $\mathbf{v}$ with respect to the standard basis $S = \{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ is $(\mathbf{v})_S = (a_1,\ldots,a_n) = \mathbf{v}$.

(e) False. For instance, $\{1+t^4, t+t^4, t^2+t^4, t^3+t^4, t^4\}$ is a basis for P_4.

4.6 Dimension