Chapter 7 : Iterative methods in Matrix Algebra

7.3 The Jacobi and Gauss-Siedel Iterative Techniques

1-Diagonal Dominant form

2-Iterative methods for system of linear equation A x = b

a-Jacobi methods

b-Gauss Seidal Techniques

Diagonally dominant: The coefficient on the diagonal must be at least equal to the sum of the other coefficients in that row and at least one row with a diagonal coefficient greater than the sum of the other coefficients in that row.

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \text{for all 'i'}$$

Which coefficient matrix is diagonally dominant?

$$[A] = \begin{bmatrix} 2 & 5.81 & 34 \\ 45 & 43 & 1 \\ 123 & 16 & 1 \end{bmatrix} \qquad [B] = \begin{bmatrix} 124 & 34 & 56 \\ 23 & 53 & 5 \\ 96 & 34 & 129 \end{bmatrix}$$

Most physical systems do result in simultaneous linear equations that have diagonally dominant coefficient matrices.

Checking if the coefficient matrix is diagonally dominant

$$[A]=\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

The inequalities are all true and at least one row is *strictly* greater than:

Therefore: The solution should converge

An *iterative* method.

Basic Procedure:

i. Algebraically solve each system of linear equation for x_i

ii. Assume an initial guess solution array

iii. Check for diagonally dominant ?

iv. Arrange for each x_i and repeat

v. Use absolute relative approximate error after each iteration to check if error is within a pre-specified tolerance.

Stopping criteria:

Calculate the Absolute Relative estimated Error

$$\left|\in_x\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right|$$

$$OR$$

Absolute error

$$\left|\in_x\right|_i = \| x_i^{new} - x_i^{old} \|$$

The iterations are stopped when the absolute relative approximate error is less than a pre specified tolerance for all unknowns.

Jacobi Method

The Jacobi Method is a form of fixed-point iteration for a system of equations. In FPI the first step is to rewrite the equations, solving for the unknown. The first step of the Jacobi Method is to do this in the following standardized way: Solve the ith equation for the ith unknown. Then, iterate as in Fixed-Point Iteration, starting with an initial guess.

► **EXAMPLE** Apply the Jacobi Method to the system $3u + v = 5, u + 2v = 5$.

Begin by solving the first equation for u and the second equation for v. We will use the initial guess $(u_0, v_0) = (0, 0)$. We have

$$u = \frac{5 - v}{3}$$

$$v = \frac{5 - u}{2}.$$

The two equations are iterated:

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/2}{3} \\ \frac{5-5/3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{5}{3} \end{bmatrix}$$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{5-5/3}{3} \\ \frac{5-5/6}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{25}{12} \end{bmatrix}.$$

Further steps of Jacobi show convergence toward the solution, which is $[1, 2]$.

Apply the Jacobi Method to the system $3u + v = 5, u + 2v = 5$.

Now suppose that the equations are given in the reverse order.

Solve the first equation for the first variable u and the second equation for v. We begin with

$$u = 5 - 2v$$
$$v = 5 - 3u.$$

The two equations are iterated as before, but the results are quite different:

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 5 - 2v_0 \\ 5 - 3u_0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$
$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 5 - 2v_1 \\ 5 - 3u_1 \end{bmatrix} = \begin{bmatrix} -5 \\ -10 \end{bmatrix}$$
$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 5 - 2(-10) \\ 5 - 3(-5) \end{bmatrix} = \begin{bmatrix} 25 \\ 20 \end{bmatrix}.$$

In this case the Jacobi Method fails, as the iteration diverges.

Theorem

If the $n \times n$ matrix A is strictly diagonally dominant, then (1) A is a nonsingular matrix, and (2) for every vector b and every starting guess, the Jacobi Method applied to $Ax = b$ converges to the (unique) solution.

Gauss–Seidel Method

Apply the Jacobi Method to the system $3u + v = 5, u + 2v = 5$.

Closely related to the Jacobi Method is an iteration called the **Gauss–Seidel** Method. The only difference between Gauss–Seidel and Jacobi is that in the former, the most recently updated values of the unknowns are used at each step, even if the updating occurs in the current step.

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-5/3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{3} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/3}{3} \\ \frac{5-10/9}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{35}{18} \end{bmatrix}$$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{5-v_2}{3} \\ \frac{5-u_3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-35/18}{3} \\ \frac{5-55/54}{2} \end{bmatrix} = \begin{bmatrix} \frac{55}{54} \\ \frac{215}{108} \end{bmatrix}.$$

$$u = \frac{5-v}{3}$$
$$v = \frac{5-u}{2}.$$

$$U_{K+1} = \frac{1}{3}[5 - V_k]$$
$$V_{K+1} = \frac{1}{2}[5 - U_{k+1}]$$

Note the difference between Gauss–Seidel and Jacobi: The definition of v_1 uses u_1, not u_0. We see the approach to the solution $[1, 2]$ as with the Jacobi Method, but somewhat more accurately at the same number of steps. Gauss–Seidel often converges faster than Jacobi if the method is convergent. Theorem verifies that the Gauss–Seidel Method, like Jacobi, converges to the solution as long as the coefficient matrix is strictly diagonally dominant.

► **EXAMPLE**

Apply the Gauss–Seidel Method to the system

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 4 & 1 \\ -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

Starting with $x_0 = [u_0, v_0, w_0] = [0, 0, 0]$,

The Gauss–Seidel iteration is

$$u_{k+1} = \frac{4 - v_k + w_k}{3}$$

$$v_{k+1} = \frac{1 - 2u_{k+1} - w_k}{4}$$

$$w_{k+1} = \frac{1 + u_{k+1} - 2v_{k+1}}{5}.$$

Generalized form

First Iteration:

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \frac{4-0-0}{3} = \frac{4}{3} \\ \frac{1-8/3-0}{4} = -\frac{5}{12} \\ \frac{1+4/3+5/6}{5} = \frac{19}{30} \end{bmatrix} \approx \begin{bmatrix} 1.3333 \\ -0.4167 \\ 0.6333 \end{bmatrix}$$

2nd Iteration:

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{101}{60} \\ -\frac{3}{4} \\ \frac{251}{300} \end{bmatrix} \approx \begin{bmatrix} 1.6833 \\ -0.7500 \\ 0.8367 \end{bmatrix}.$$

The system is strictly diagonally dominant, and therefore the iteration will converge to the solution $[2, -1, 1]$. ◀

Example

Solve the system by jacobi's iterative method

$$8x-3y+2z=20$$

$$4x+11y-z=33$$

$$6x+3y+12z=35$$

(Perform only four iterations)

the system is diagonally dominant

$$x=\frac{1}{8}[20+3y-2z]$$

$$y=\frac{1}{11}[33-4x+z]$$

$$z=\frac{1}{12}[35-6x-3y]$$

Generalized form

$$x_{k+1}=\frac{1}{8}[20+3y_k-2z_k]$$

$$y_{k+1}=\frac{1}{11}[33-4x_k+z_k]$$

$$z_{k+1}=\frac{1}{12}[35-6x_k-3y_k]$$

initial aproximation $x_0=y_0=z_0=0$

first iteration

$$x_1 = \frac{1}{8}\left[20 + 3(0) - 2(0)\right] = 2.5$$

$$y_1 = \frac{1}{11}\left[33 - 4(0) + 0\right] = 3$$

$$z_1 = \frac{1}{12}\left[35 - 6(0) - 3(0)\right] = 2.916667$$

Second iteration

$$x_2 = \frac{1}{8}\left[20 + 3(3) - 2(2.9166667)\right] = 2.895833$$

$$y_2 = \frac{1}{11}\left[33 - 4(2.5) + 2.9166667\right] = 2.3560606$$

$$z_2 = \frac{1}{12}\left[35 - 6(2.5) - 3(3)\right] = 0.9166666$$

third iteration

$$x_3 = \frac{1}{8}\left[20 + 3(2.3560606) - 2(0.9166666)\right] = 3.1543561$$

$$y_3 = \frac{1}{11}\left[33 - 4(2.8958333) + 0.9166666\right] = 2.030303$$

$$z_3 = \frac{1}{12}\left[35 - 6(2.8958333) - 3(2.3560606)\right] = 0.8797348$$

fourth iteration

$$x_4 = \frac{1}{8}\left[20 + 3(2.030303) - 2(0.8797348)\right] = 3.0419299$$

$$y_4 = \frac{1}{11}\left[33 - 4(3.1543561) + 0.8797348\right] = 1.9329373$$

$$z_4 = \frac{1}{12}\left[35 - 6(3.1543561) - 3(2.030303)\right] = 0.8319128$$

Example: If not diagonally dominant

Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$12x_1 + 3x_2 - 5x_3 = 1$$

Solution:

$$12x_1 + 3x_2 - 5x_3 = 1$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$3x_1 + 7x_2 + 13x_3 = 76$$

$\Rightarrow$

$$x_1 = \frac{1 + 5x_3 - 3x_2}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

Home Activity:

Example 1 The linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned}
E_1: &\quad 10x_1 - x_2 + 2x_3 = 6,\\
E_2: &\quad -x_1 + 11x_2 - x_3 + 3x_4 = 25,\\
E_3: &\quad 2x_1 - x_2 + 10x_3 - x_4 = -11,\\
E_4: &\quad 3x_2 - x_3 + 8x_4 = 15
\end{aligned}$$

has the unique solution $\mathbf{x} = (1, 2, -1, 1)^t$. Use Jacobi's iterative technique to find approximations $\mathbf{x}^{(k)}$ to $\mathbf{x}$ starting with $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Example 3 Use the Gauss-Seidel iterative technique to find approximate solutions to

$$\begin{aligned}
10x_1 - x_2 + 2x_3 &= 6, \\
-x_1 + 11x_2 - x_3 + 3x_4 &= 25, \\
2x_1 - x_2 + 10x_3 - x_4 &= -11, \\
3x_2 - x_3 + 8x_4 &= 15
\end{aligned}$$

starting with $\mathbf{x} = (0, 0, 0, 0)^t$ and iterating until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Solution The solution $\mathbf{x} = (1, 2, -1, 1)^t$ was approximated by Jacobi's method in Example 1. For the Gauss-Seidel method we write the system, for each $k = 1, 2, \ldots$ as

Generalized form

$$\begin{aligned}
x_1^{(k)} &= \qquad\qquad\quad \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} \qquad\qquad\quad + \frac{3}{5},\\
x_2^{(k)} &= \frac{1}{11}x_1^{(k)} \qquad\qquad\quad + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11},\\
x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} \qquad\qquad\quad + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10},\\
x_4^{(k)} &= \qquad\qquad - \frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} \qquad\qquad\quad + \frac{15}{8}.
\end{aligned}$$

When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have $\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$. Subsequent iterations give the values in Table 7.2.

Table 7.2

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	−0.9873	−1.014	−1.0025	−1.0003	−1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

Because

$$\frac{\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\|_\infty}{\|\mathbf{x}^{(5)}\|_\infty} = \frac{0.0008}{2.000} = 4 \times 10^{-4},$$

Example

Solve the system by Gauss-Seidel iterative method

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

$$6x + 3y + 12z = 35$$

(Perform only four iterations)

the system is diagonally dominant

$$x = \frac{1}{8}[20 + 3y - 2z]$$

$$y = \frac{1}{11}[33 - 4x + z]$$

$$z = \frac{1}{12}[35 - 6x - 3y]$$

Generalized form

$$x_{k+1} = \frac{1}{8}[20 + 3y_k - 2z_k]$$

$$y_{k+1} = \frac{1}{11}[33 - 4x_{k+1} + z_k]$$

$$z_{k+1} = \frac{1}{12}[35 - 6x_{k+1} - 3y_{k+1}]$$

we start with an initial aproximation $x_0 = y_0 = z_0 = 0$

first iteration

$$x_1 = \frac{1}{8}[20 + 3(0) - 2(0)] = 2.5$$

$$y_1 = \frac{1}{11}[33 - 4(2.5) + 0] = 2.0909091$$

$$z_1 = \frac{1}{12}[35 - 6(2.5) - 3(2.0909091)] = 1.1439394$$

Second iteration

$$x_2 = \frac{1}{8}[20 + 3y_1 - z_1] = \frac{1}{8}[20 + 3(2.0909091) - 2(1.1439394)] = 2.9981061$$

$$y_2 = \frac{1}{11}[33 - 4x_2 + z_1] = \frac{1}{11}[33 - 4(2.9981061) + 1.1439394] = 2.0137741$$

$$z_2 = \frac{1}{12}[35 - 6x_2 - 3y_2] = \frac{1}{12}[35 - 6(2.9981061) - 3(2.0137741)] = 0.9141701$$

third iteration

$$x_3 = \frac{1}{8}\left[20 + 3(2.0137741) - 2(0.9141701)\right] = 3.0266228$$

$$y_3 = \frac{1}{11}\left[33 - 4(3.0266228) + 0.9141701\right] = 1.9825163$$

$$z_3 = \frac{1}{12}\left[35 - 6(3.0266228) - 3(1.9825163)\right] = 0.9077262$$

fourth iteration

$$x_4 = \frac{1}{8}\left[20 + 3(1.9825163) - 2(0.9077262)\right] = 3.0165121$$

$$y_4 = \frac{1}{11}\left[33 - 4(3.0165121) + 0.9077262\right] = 1.9856071$$

$$z_4 = \frac{1}{12}\left[35 - 6(3.0165121) - 3(1.9856071)\right] = 0.8319128$$

<u>EXAMPLE</u> Use Gauss-Seidel Method

$$\begin{aligned} 3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\ 0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\ 0.3x_1 - 0.2x_2 + 10x_3 &= 71.4 \end{aligned}$$

Note that the exact solution is $[x]^T = \lfloor 3 \quad -2.5 \quad 7 \rfloor$

Solution. First, solve each of the equations for its unknown on the diagonal:

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3} \qquad \text{(E1)}$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7} \qquad \text{(E2)}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10} \qquad \text{(E3)}$$

By assuming that x_2 and x_3 are zero

$$x_1 = \frac{7.85 + 0.1(0) + 0.2(0)}{3} = 2.616667$$

This value, along with the assumed value of x_3 =0, can be substituted into Eq.(E2) to calculate

$$x_2 = \frac{-19.3 - 0.1(2.616667) + 0.3(0)}{7} = -2.794524$$

The first iteration is completed by substituting the calculated values for x_1 and x_2 into Eq.(E3) to yield

$$x_3 = \frac{71.4 - 0.3(2.616667) + 0.2(-2.794524)}{10} = 7.005610$$

For the second iteration, the same process is repeated to compute

$$x_1 = \frac{7.85 + 0.1(-2.794524) + 0.2(7.005610)}{3} = 2.990557$$

$$x_2 = \frac{-19.3 - 0.1(2.990557) + 0.3(7.005610)}{7} = -2.499625$$

$$x_3 = \frac{71.4 - 0.3(2.990557) + 0.2(-2.499625)}{10} = 7.000291$$

The method is, therefore, converging on the true solution. Additional iterations could be applied to improve the answers. Consequently, we can estimate the error. For example , for x_1

$$\varepsilon_{x,1} = \left| \frac{2.990557 - 2.616667}{2.990557} \right| \times 100\% = 12.5\%$$

For x_2 and x_3 , the error estimates are

$$\varepsilon_{x,2} = 11.8\%$$

$$\varepsilon_{x,3} = 0.076\%$$

Repeat to it again until the result is known to at least the tolerance specified by ε_s.

Example -Gauss-Seidel Method:

Given the system of equations

$$
\begin{aligned}
12x_1 + 3x_2 - 5x_3 &= 1 \\
x_1 + 5x_2 + 3x_3 &= 28 \\
3x_1 + 7x_2 + 13x_3 &= 76
\end{aligned}
\quad \Rightarrow \quad
\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 28 \\ 76 \end{bmatrix}
$$

With an initial guess of

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
$$

Will the solution converge using the Gauss-Siedel method?

$$12x_1 + 3x_2 - 5x_3 = 1$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$3x_1 + 7x_2 + 13x_3 = 76$$

Rewriting each equation

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = \frac{1 - 3(0) + 5(1)}{12} = 0.50000$$

$$x_2 = \frac{28 - (0.5) - 3(1)}{5} = 4.9000$$

$$x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

The absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{0.50000-1.0000}{0.50000}\right| \times 100 = 100.00\%$$

$$\left|\in_a\right|_2 = \left|\frac{4.9000-0}{4.9000}\right| \times 100 = 100.00\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.0923-1.0000}{3.0923}\right| \times 100 = 67.662\%$$

After Iteration #1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 4.9000 \\ 3.0923 \end{bmatrix}$$

The maximum absolute relative error after the first iteration is 100%

Substituting the x values into the arranged equations

$$x_1 = \frac{1 - 3(4.9000) + 5(3.0923)}{12} = 0.14679$$

$$x_2 = \frac{28 - (0.14679) - 3(3.0923)}{5} = 3.7153$$

$$x_3 = \frac{76 - 3(0.14679) - 7(4.900)}{13} = 3.8118$$

After Iteration #2

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.14679 \\ 3.7153 \\ 3.8118 \end{bmatrix}$$

Iteration #2 absolute relative approximate error

$$\left|\in_x\right|_1 = \left|\frac{0.14679 - 0.50000}{0.14679}\right| \times 100 = 240.61\%$$

$$\left|\in_x\right|_2 = \left|\frac{3.7153 - 4.9000}{3.7153}\right| \times 100 = 31.889\%$$

$$\left|\in_x\right|_3 = \left|\frac{3.8118 - 3.0923}{3.8118}\right| \times 100 = 18.874\%$$

The maximum absolute relative error after the first iteration is 240.61%

This is much larger than the maximum absolute relative error obtained in iteration #1. Is this a problem?

Repeating more iterations, the following values are obtained

Iteration	x_1	$\lvert\in_x\rvert_1 \%$	x_2	$\lvert\in_x\rvert_2 \%$	x_3	$\lvert\in_x\rvert_3 \%$
1	0.50000	100.00	4.9000	100.00	3.0923	67.662
2	0.14679	240.61	3.7153	31.889	3.8118	18.876
3	0.74275	80.236	3.1644	17.408	3.9708	4.0042
4	0.94675	21.546	3.0281	4.4996	3.9971	0.65772
5	0.99177	4.5391	3.0034	0.82499	4.0001	0.074383
6	0.99919	0.74307	3.0001	0.10856	4.0001	0.00101

The solution obtained $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.99919 \\ 3.0001 \\ 4.0001 \end{bmatrix}$ is close to the exact solution of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

Example: If not diagonally dominant

Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$12x_1 + 3x_2 - 5x_3 = 1$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Rewriting the equations

$$x_1 = \frac{76 - 7x_2 - 13x_3}{3}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{1 - 12x_1 - 3x_2}{-5}$$

Will the solution converge using the Gauss-Siedel method?

Using Gauss-Seidel Method:

Conducting six iterations, the following values are obtained

Iteration	x_1	$\left\|\in_x\right\|_1 \%$	x_2	$\left\|\in_x\right\|_2 \%$	x_3	$\left\|\in_x\right\|_3 \%$
1	21.000	95.238	0.80000	100.00	50.680	98.027
2	-196.15	110.71	14.421	94.453	-462.30	110.96
3	-1995.0	109.83	-116.02	112.43	4718.1	109.80
4	-20149	109.90	1204.6	109.63	-47636	109.90
5	2.0364×10^5	109.89	-12140	109.92	4.8144×10^5	109.89
6	-2.0579×10^5	109.89	1.2272×10^5	109.89	-4.8653×10^6	109.89

The values are not converging.

this mean that the Gauss-Seidel method cannot be used.

EXERCISE SET 7.3

1. Find the first two iterations of the Jacobi method for the following linear systems, using $\mathbf{x}^{(0)} = \mathbf{0}$:

a.
$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1, \\ 3x_1 + 6x_2 + 2x_3 &= 0, \\ 3x_1 + 3x_2 + 7x_3 &= 4. \end{aligned}$$

b.
$$\begin{aligned} 10x_1 - x_2 \qquad\quad &= 9, \\ -x_1 + 10x_2 - 2x_3 &= 7, \\ -2x_2 + 10x_3 &= 6. \end{aligned}$$

c.
$$\begin{aligned} 10x_1 + 5x_2 \qquad\qquad\quad &= 6, \\ 5x_1 + 10x_2 - 4x_3 \qquad &= 25, \\ -4x_2 + 8x_3 - x_4 &= -11, \\ -x_3 + 5x_4 &= -11. \end{aligned}$$

d.
$$\begin{aligned} 4x_1 + x_2 + x_3 + x_5 &= 6, \\ -x_1 - 3x_2 + x_3 + x_4 &= 6, \\ 2x_1 + x_2 + 5x_3 - x_4 - x_5 &= 6, \\ -x_1 - x_2 - x_3 + 4x_4 &= 6, \\ 2x_2 - x_3 + x_4 + 4x_5 &= 6. \end{aligned}$$

3. Repeat Exercise 1 using the Gauss-Seidel method.

8. Use the Gauss-Seidel method to solve the linear systems in Exercise 2, with $TOL = 10^{-3}$ in the l_∞ norm.


ANY
QUESTIONS