Homework I
Advanced Topics in Stochastic Modelling

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2018 March 25

1 Question 1

Let X_T be the fraction of time that it rains given T days have elapsed. Where,

$$
\begin{aligned}
X_T &= \frac{r_1 + r_2 + \cdots + r_n}{T} \\
\sum_i^n r_i + \sum_i^m d_i &= T \\
r_i &\text{ is the length of } i\text{th raining spell} \\
d_i &\text{ is the length of } i\text{th dry spell}
\end{aligned}
$$

The long-run fraction of time that it rain can be expressed by:

$$
\begin{aligned}
\lim_{T\to\infty} X_T &= \lim_{T\to\infty} \frac{\sum_i^n r_i}{T} \\
&= \frac{E[r_i]}{E[T_i]} \\
&= \frac{E[r_i]}{E[r_i] + E[d_i]} \qquad \# \; r_i \perp d_i \\
&= \frac{2}{2+7} \\
&= \frac{2}{9}
\end{aligned}
$$

where $T_i = r_i + d_i$, the period of the ith renewal cycle of the system.

2 Question 2

2.1 Part a

Let X_T be the fraction of people who go to Hilton Hotel given T hours have elapsed.
Where,

$$X_T = \frac{y_1 + y_2 + \cdots + y_n}{P_T}$$
$$\sum_{i=1}^{n} y_i + \sum_{i=1}^{m} n_i = P_T \quad \# \; y_i = 7 \; \forall i$$

y_i is the number of people who do go to Hilton Hotel during the ith renewal cycle
d_i is the number of people who go elsewhere during the ith renewal cycle

The long-run fraction of people who go to Hilton Hotel can be expressed by:

$$\lim_{T\to\infty} X_T = \lim_{T\to\infty} \frac{\sum_i^n y_i}{P_T}$$
$$\text{as } T \to \infty \;, \; P_T \to \infty$$
$$= \lim_{P_T\to\infty} \frac{\sum_i^n y_i}{P_T}$$
$$= \frac{E[y_i]}{E[T_i]}$$
$$= \frac{7}{7 + \frac{36}{60} \cdot 10} \quad \# \text{ proportionality property}$$
$$= \frac{7}{13}$$

where $T_i = y_i + n_i$, the number of people in the ith renewal cycle.

2.2 Part b

Let X_i be amount of time spent by the ith person to reach Hilton Hotel at each renewal cycle.
Let Y_i be the inter-arrival time of the ith person and $i-1$th person
Where,

$$Y_i \sim \text{Exp}(10) \quad \forall i \quad \text{\# independent increment property}$$

$$\begin{aligned}
X_i &= \frac{36}{60} + \sum_{j=i+1}^{7} Y_j \quad \forall i \setminus \{7\} \\
X_7 &= \frac{36}{60}
\end{aligned}$$

The expected time spent by people going to Hilton Hotel can be expressed by:

$$\begin{aligned}
E[X] &= \sum_{i=1}^{7} \left(\frac{1}{7} \cdot \sum_{j=i+1}^{7} Y_j \right) \qquad \text{\# proportionality property} \\
&= \frac{1}{7} E\left[\sum_{i=2}^{7} Y_i\right] + \frac{1}{7} E\left[\sum_{i=3}^{7} Y_i\right] + \frac{1}{7} E\left[\sum_{i=4}^{7} Y_i\right] \\
&+ \frac{1}{7} E\left[\sum_{i=5}^{7} Y_i\right] + \frac{1}{7} E\left[\sum_{i=6}^{7} Y_i\right] + \frac{1}{7} E\left[\sum_{i=7}^{7} Y_i\right] + \frac{1}{7} E[36] \\
&= \frac{6}{7} E[Y_7] + \frac{5}{7} E[Y_6] + \frac{4}{7} E[Y_5] + \frac{3}{7} E[Y_4] + \frac{2}{7} E[Y_3] + \frac{1}{7} E[Y_2] \\
&= \frac{36}{60} + \frac{6}{7} \cdot \frac{1}{10} + \frac{5}{7} \cdot \frac{1}{10} + \frac{4}{7} \cdot \frac{1}{10} + \frac{3}{7} \cdot \frac{1}{10} + \frac{2}{7} \cdot \frac{1}{10} + \frac{1}{7} \cdot \frac{1}{10} \\
&= \frac{9}{10} \quad (hours)
\end{aligned}$$

3 Question 3

3.1 Part a

Let X_T be the fraction of shots thrown by any child given T is the total number of shots thrown.
Where,

$$X_T = \frac{a_1 + a_2 + \cdots + a_n}{T}$$
$$\sum_{i=1}^{n} a_i + \sum_{i=1}^{m} b_i + \sum_{i=1}^{k} c_i = T$$
$$a_i \sim Geo(p_1)$$
$$b_i \sim Geo(p_2)$$
$$c_i \sim Geo(p_3)$$

a_i is the number of shots thrown by child a at the ith renewal cycle
b_i & c_i are the number of shots not thrown by child a at the ith renewal cycle

The long-run fraction of shots thrown by any child can be expressed by:

$$\begin{aligned}
\lim_{T\to\infty} X_T &= \lim_{T\to\infty} \frac{\sum_i^n a_i}{T} \\
&= \frac{E[a_i]}{E[T_i]} \\
&= \frac{E[a_i]}{E[a_i] + E[b_i] + E[c_i]} \qquad \# \ a_i \perp b_i \perp c_i \\
&= \frac{\frac{1}{p_1}}{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}} \\
&= \frac{\frac{1}{p_1}}{\frac{p_2p_3 + p_1p_3 + p_1p_2}{p_1p_2p_3}} \\
&= \frac{p_2p_3}{p_2p_3 + p_1p_3 + p_1p_2}
\end{aligned}$$

where $T_i = a_i + b_i + c_i$, the period of the ith renewal cycle of the system.

This solution can be applied to any of the three children by reassigning the values of p_1, p_2 & p_3

3.2 Part b

$p_1 = \frac{2}{3}$, $p_2 = \frac{3}{4}$, $p_3 = \frac{4}{5}$

For Child A

$$\begin{aligned}
\lim_{T\to\infty} X_T &= \lim_{T\to\infty} \frac{\sum_i^n a_i}{T} \\
&= \frac{p_2p_3}{p_2p_3 + p_1p_3 + p_2p_3} \\
&= \frac{\frac{12}{20}}{\frac{12}{20} + \frac{8}{15} + \frac{6}{12}} \\
&= \frac{18}{49}
\end{aligned}$$

For Child B

$$\begin{aligned}
\lim_{T\to\infty} X_T &= \lim_{T\to\infty} \frac{\sum_i^n b_i}{T} \\
&= \frac{p_1p_3}{p_2p_3 + p_1p_3 + p_2p_3} \\
&= \frac{\frac{8}{15}}{\frac{12}{20} + \frac{8}{15} + \frac{6}{12}} \\
&= \frac{16}{49}
\end{aligned}$$

For Child C

$$\begin{aligned}
\lim_{T\to\infty} X_T &= \lim_{T\to\infty} \frac{\sum_i^n c_i}{T} \\
&= \frac{p_1p_2}{p_2p_3 + p_1p_3 + p_2p_3} \\
&= \frac{\frac{6}{12}}{\frac{12}{20} + \frac{8}{15} + \frac{6}{12}} \\
&= \frac{15}{49}
\end{aligned}$$

4 Question 4

Let X_T be the fraction of fully repaired machines given T number of machines. Where,

$$X_T = \frac{r_1 + r_2 + \cdots + r_T}{T}$$
$$\sum_{i=1}^{T} r_i + \sum_{i=1}^{T} b_i = T$$
$$r_i = \begin{cases} 1 & i\text{th machine is fully repaired} \\ 0 & \text{otherwise} \end{cases}$$

Let Y be the time to complete repairs on any given machine
Let Z be the time since the start of repair for mistake to occur.

$$Y \sim \text{Exp}(\mu)$$
$$Z \sim \text{Exp}(\frac{1}{\lambda})$$
$$Z \sim \text{Exp}(\lambda^{-1})$$
$$\Pr(Y < Z) = \frac{\mu}{\mu + \lambda^{-1}} \quad \text{\# difference of two exp. distributions}$$

$$r_i = \begin{cases} 1 & Y < Z \\ 0 & Y > Z \end{cases}$$
$$E[r_i] = \frac{\mu}{\mu + \lambda^{-1}}$$
$$b_i = \begin{cases} 1 & i\text{th machine is broken} \\ 0 & \text{otherwise} \end{cases}$$

The long-run fraction of fully repaired machines can be expressed by:

$$\begin{aligned}
\lim_{T\to\infty} X_T &= \lim_{T\to\infty} \frac{\sum_i^T r_i}{T} \\
&= \frac{E[r_i]}{E[T_i]} \\
&= \frac{E[r_i]}{1} \quad \text{\# machine will always be checked} \\
&= \frac{\mu}{\mu + \lambda^{-1}}
\end{aligned}$$