Department of Mathematics

Mathematics for Business Informatics II

Linear Algebra

# Lecture 3

# The Gaussian & Gauss-Jordan Elimination methods

By

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# Contents

# 1. Direct Methods for Solving Linear Systems

## 1.1. Augmented Matrix & Rank theorem

We know that the System of Linear equations can take the following form:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1m}x_m &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2m}x_m &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nm}x_m &= b_n
\end{aligned}
$$

From this system of equations, we get the matrix form ***(A* I *b*)** :

$$
(A|b) = \left(\begin{array}{ccccc|c}
a_{11} & a_{12} & a_{13} & \cdots & a_{1m} & b_1 \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2m} & b_2 \\
\vdots & & & & \vdots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} & b_n
\end{array}\right)
$$

This form is called the Augmented matrix (or extended coefficient matrix of the system of linear equations).

Note that: The vertical bar serves to remind us of the equal signs in equations.

**Definition 1.** The Rank ( $r$ ) of a matrix is the number of nonzero rows in its Row Echelon or in its Reduced Echelon forms.

**Theorem 1. (About the Rank of the matrix)**

If we have $A$ is the coefficient matrix of a system of linear equations with $n$ variables. If the system is consistent (has a solution(s)); then: number of free variables (parameters) $= n - r(A)$; where $r(A)$ is the rank of the matrix $A$.

**Conclusion:** Based on Echelon and Reduced Echelon Forms and by using elementary row operations; we can perform methods for solving systems of linear equations as follows:

1. Gaussian Elimination.
2. Gauss-Jordan Elimination method.

We shall discuss both methods in the following sections.

# 1.2. The Gaussian Elimination Method.

This method depends on finding echelon form using elementary row operations(ERO's) as the following steps:

**Step 1.** Write the **augmented matrix** of the systems of linear equations.

**Step 2.** Use ERO's to reduce Augmented matrix to **Row Echelon** .

**Step 3.** Using back substitution, solve the equivalent system that corresponds to the Row Echelon Form.

To achieve such Echelon form we do the following guidelines:

(a) Locate the leftmost column that is not all zeros.

(b) Create a leading entry at the top of this column (preferably =1).

(c) Use the leading entry to create zeros below it.

(d) Cover up the row containing the leading entry, and go back to (a) to repeat the procedure on the remaining of the matrix.

**Stop:** ***when the entire matrix in row Echelon form.***

**Example 1.** Solve the system of linear equations by using the Gaussian Elimination method.

$$\begin{aligned} 2y + 3z &= 8 \\ 2x + 3y + z &= 5 \\ x - y - 2z &= -5 \end{aligned}$$

**Solution.** Use Elementary Row Operations(ERO's) as follows:

Step 1 (augmented matrix) & Step 2 (echelon form) as follows:

$$\left(\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{array}\right)$$

$$\xrightarrow{\frac{1}{5}R_2} \left(\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{array}\right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Step 3: The augmented matrix is now in row echelon form, then, we construct the corresponding system of linear equations:

$$\begin{aligned} x - y - 2z &= -5 \\ y + z &= 3 \\ z &= 2 \end{aligned}$$

We write the Solution Vector of the systems as a Column Vector: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

# 1.3. The Gauss-Jordan elimination method

We changes the matrix using ERO's into Reduced Row Echelon form as follows:

**Step 1.** Write Augmented matrix of the systems of linear equations

**Step 2.** Use ERO's to reduce Augmented matrix to Reduced Echelon

**Step 3.** If the resulting system is consistent, solve for the leading variables in term of any remaining free variables.

## Important Remarks:

1. In Gauss-Jordan elimination we proceed as Gauss elimination but reduce the Augmented matrix to Reduced row Echelon form.

2. The rank *r* of different Echelon forms of certain matrix is the same and this *r* is the same for its corresponding Reduced Echelon form .

**Theorem 2.** The solution of a system of linear equations with real coefficients has either of the following 3 Cases:

(a) a unique solution (a consistent system) or

(b) infinitely many solutions ( a consistent system) or

(c) no solution (an inconsistent system).

Example 2. Solve the following system of equations

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ -x_1 + x_2 &= 2 \end{aligned}$$

using **Gauss-Jordan** elimination method.

**Solution:** Write this system in augmented matrix form & use *ERO's* to reduce the augmented matrix to Reduced row Echelon Form:

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & 1 & 2 \end{array}\right) \xrightarrow[R_2 \to R_2 + R_1]{} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 3 & 3 \end{array}\right) \xrightarrow[R_2 \to \frac{1}{3} R_2]{}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow[R_1 \to R_1 - 2R_2]{} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

**So, we have one (unique) solution *(the <u>1st case</u> of* Theorem 2*):***

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

**(*Verify the result?*)**

# 2. Solution of Homogenous System of Equations

Let us have the homogeneous system of linear equations:

$$a_{11}\ x_1 + \ a_{12}\ x_2\ + \ldots\ldots + \ a_{1n}\ x_n \ = \ 0$$

$$a_{21}\ x_1 + \ a_{22}\ x_2 + \ldots\ldots + \ a_{2n}\ x_n \ = \ 0$$

$$\ldots$$

$$a_{m1}\ x_1 + \ a_{m2}\ x_2\ + \ldots\ldots + \ a_{mn}\ x_n = 0\ .$$

Now, we give the following Theorems for solutions to this system:

**Theorem 3.** If we have the 2 matrices: $A$ & $\underline{x}$ of dimensions $m$ x $n$ & $n$ x 1 respectively; then the homogenous system $A\ .\ \underline{x} = \underline{0}$ always has at least one of the solutions:

(a) either the solution is: $x_1 = x_2 = \ldots = x_n = 0$ for the homogenous system which is called the trivial solution.

(b) or, if the solution is $x_1, x_2, \ldots, x_n$ to the system in which not all vectors (unknowns) are zero is called a nontrivial solution.

## Theorem 4.

Let $A$ be the reduced coefficient matrix of homogenous system of $m$ linear equation in $n$ unknowns.

If $A$ has exactly $k$ non zero rows, the $k \leq n$ & has 2 cases:

**Case 1.** If $k < n$, the system has infinitely many solution (Non-trivial solution) &

**Case 2.** If $k = n$, the system has a unique solution (Trivial solution)

**We can summarize the Results of Theorems 3&4 as follows:**

(1) The homogeneous system cannot have the case: no solution.

(2)In the case when $k \leq n$ , there can be either a unique solution (trivial solution) or infinitely many solutions (nontrivial solution).

(3) If we have number of equation $m$ less than number of unknowns $n$, then the system has a non- trivial solution.

**Example 3.** Use <u>Gauss-Jordan method</u> to determine whether the following homogenous system have a unique solution or infinitely many solutions; then solve the system:

$$x - 2y + z = 0$$
$$2x - y + 5z = 0$$
$$x + y + 4z = 0$$

**Solution.** By reducing the coefficient matrix, we have:

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & -1 & 5 \\ 1 & 1 & 4 \end{pmatrix} \to \dots \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We have number of nonzero-rows = 2, which in the reduced coefficient matrix is less than the number of unknowns 3.

i.e. rank = 2 & $n$ = 3.

Hence by Theorem 2, there are infinitely many solutions.

Since the reduced coefficient matrix corresponds to: $x + 3z = 0$

$$y + z = 0$$

Now, let $z = r$, then $y = -r$ & $x = -3r$ ; where $r$ is any real number.

Then, the solution is: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$.

# 3. Solved Examples

**Example 4.** Determine whether the following homogenous system have a unique solution or many infinitely solutions; then solve the system:

$$\begin{aligned} 3x + 4y &= 0 \\ x - 2y &= 0 \\ 2x + y &= 0 \\ 2x + 3y &= 0 \end{aligned}$$

**Solution:** By reducing the coefficient matrix, we have:

$$\begin{pmatrix} 3 & 4 \\ 1 & -2 \\ 2 & 1 \\ 2 & 3 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

From the Reduced coefficient matrix; we get that: the number of nonzero-rows = 2 equals the number of unknowns = 2.
Then by applying Theorem 1, we conclude that the above system must have a unique solution; which is the trivial solution $x = 0$, $y = 0$.

**Example 5.** Find the **Rank** $\boldsymbol{r}$ *of the following matrices:*

1. *Let* $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$. *We have* $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \xrightarrow[R_2 \to R_2 + R_1]{} \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}$

$$\xrightarrow[R_2 \to \frac{1}{5} R_2]{} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \xrightarrow[R_1 \to R_1 - 2R_2]{} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\boxed{\textit{so } A \textit{ has rank } 2.}$

2. *Let* $A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$. *Here we find:*

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \xrightarrow[R_1 \to \frac{1}{2} R_1]{} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} \left(\begin{array}{c|c} 1 & 2 \\ \hline 0 & 0 \end{array}\right).$$

We see that this matrix $\boldsymbol{A}$ **has rank = 1**.

Example 6. By using Gauss-Jordan elimination method, solve the following system of 3 linear equations:

$$\begin{aligned} x_1 \quad\quad + \quad x_3 &= -1 \\ 2x_1 - x_2 + 3x_3 &= -6 \\ -2x_1 \quad\quad - 2x_3 &= 2 \end{aligned}$$

**Solution:**

**By applying Gauss- Jordan elimination method we get:**

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 2 & -1 & 3 & -6 \\ -2 & 0 & -2 & 2 \end{array}\right) \xrightarrow[R_2 \to R_2 - 2R_1]{} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -4 \\ -2 & 0 & -2 & 2 \end{array}\right) \xrightarrow[R_3 \to R_3 + 2R_1]{}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_2 \to (-1) \cdot R_2]{} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 4 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

***Now, we repeat last step of*** example 6:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \to (-1)\cdot R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 4 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

Since we have 2 non-zero rows (which is the rank $r$ of the matrix) & 3 variables; then we **have infinitely many solutions**.

Since we have $n = 3$ & $r = 2$;

then the number of **free variables (parameters) = 3 – 2 = 1.**

Now suppose the 3rd (free) variable $= s$.

Hence, the solution of the system will be given by:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 - s \\ 4 + s \\ s \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}; \quad with\, s \in \mathbb{R}.$$

This is the 2nd <u>case</u> of solutions of theorem 2 (@ slide 7);

i.e. we have infinitely many solutions.

(*Verify the result?*)

**Example 7.** By using **Gauss - Jordan** method;

find the solution of the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= 4 \end{aligned}$$

**Solution:**

We write the system of linear equations in **Augmented matrix** as follows:

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 4 \end{array}\right) \underset{R_2 \to R_2 - 2R_1}{\rightsquigarrow} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ \hline 0 & 0 & 2 \end{array}\right)$$

**So, there is No Solution (the 3rd case of solutions of theorem 2).**

**(*Verify the result?*)**

**Example 8.** Using Gaussian-Elimination method find the solution of the homogeneous system

$$\begin{aligned} x + y - z &= 0 \\ 2x - 3y + z &= 0 \\ x - 4y + 2z &= 0 \end{aligned}$$

**Solution.** We use Gaussian elimination method & we get:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & -4 & 2 & 0 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & 0 \\ 0 & -5 & 3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -3/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

By using the last reduced matrix; we get:

$$\begin{aligned} x + y - z &= 0 \\ y - 3/5\, z &= 0 \\ 0 &= 0 \end{aligned}$$

Since the rank $r = 2$ less than number of unknowns = 3, then we have infinitely many solutions or Non-trivial solution.

*e.g.* If we choose $z = w$, then $y = 3/5\, w$ & $x = 2/5\, w$.

**<u>Example 9</u>** Find the solution of the system:

$$\begin{aligned} x + y - z &= 0 \\ 2x + 4y - z &= 0 \\ 3x + 2y + 2z &= 0 \end{aligned}$$

## Solution

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 4 & -1 & 0 \\ 3 & 2 & 2 & 0 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) \xrightarrow{....} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad Verify?$$

**Finally, we have $r = 3$ & $n = 3$ , then we have unique solution $(0, 0, 0)$ which is the trivial solution.**

**Example 10**. Find the type of the solution of the homogeneous system of linear equations:

$$\begin{aligned} x + 2y - 7z - 3w &= 0 \\ x - y + 9z - 2w &= 0 \\ 2x + y - 5z - w &= 0 \end{aligned}$$

**Solution.**

By using the results of Theorem 1:

Since we have 3 equations; $m = 3$ which less than the number of unknowns; $n = 4$,

then by using Theorem 1; the system has a non-trivial solution.

*i.e*. not all the values of $x$, $y$, $z$ & $w$ are zeros.

# THANK YOU

Prof. Helail