Basile TERVER

TP4 Computational Statistics

Exercice 1

(1) Let us show that

$$P((x,y), dx'dy') = \mathbb{1}_{B=0} P_1((x,y),dx') + \mathbb{1}_{B=1} P_2((x,y)dy')$$

with $B \sim Ber(\frac{1}{2})$. Let $X_1 \sim \mathbb{1}_{B=0} P_1((x,y)dx') + \mathbb{1}_{B=1} P_2((x,y),dy')$ and h a continuous bounded function. Then, we have :

$$E[h(X_1)|X_0] = E[h(X_1)(\mathbb{1}_{B=0} + \mathbb{1}_{B=1})|X_0]$$
$$= E[h(X_1)|X_0, B=0]P(B=0) + E[h(X_1)|X_0, B=1]P(B=1)$$
$$= \frac{1}{2}P_1(X_0,h(X_1)) + \frac{1}{2}P_2(X_0,h(X_1))$$
$$= P(X_0,h(X_1))$$

$E[X|A] = E[X\mathbb{1}_A]/P(A)$

This concludes the proof.

Exercice 3

(1) $$p(X,\mu,\sigma^2,\tau^2|Y) = \frac{p(X,\mu,\sigma^2,\tau^2,Y)}{p(Y)}$$

$$= p(X,\mu,\sigma^2,\tau^2)\, p(Y|X,\mu,\sigma^2,\tau^2) \times C$$

constant independent of (X,μ,σ^2,τ^2)

$$= \pi_{prior}(\mu,\sigma^2,\tau^2)\, p(X|\mu,\sigma^2,\tau^2)\, p(Y|X,\mu,\sigma^2,\tau^2) \times C$$

Having, for all $i \in [\![1,N]\!]$, $j \in [\![1,k_i]\!]$:

$$p(X|\mu,\sigma^2,\tau^2) = \prod_{i=1}^{N} \mathcal{N}(X_i|\mu,\sigma^2) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(X_i-\mu)^2}{2\sigma^2}\right)$$
$$\propto \prod_{i=1}^{N} \frac{1}{\sigma} \exp\left(-\frac{(X_i-\mu)^2}{2\sigma^2}\right)$$

$$p(Y|X,\mu,\sigma^2,\tau^2) = \prod_{i=1}^{N}\prod_{j=1}^{k_i} \mathcal{N}(Y_{ij}|X_i,\tau^2) \propto \prod_{i=1}^{N}\prod_{j=1}^{k_i} \frac{1}{\tau} \exp\left(-\frac{(Y_{ij}-X_i)^2}{2\tau^2}\right)$$

$$\pi_{prior}(\mu,\sigma^2,\tau^2) \propto \frac{1}{\sigma^{2(1+\alpha)}} \exp\left(-\frac{\beta}{\sigma^2}\right) \frac{1}{\tau^{2(1+\gamma)}} \exp\left(-\frac{\beta}{\tau^2}\right)$$

So :

$$p(X,\mu,\sigma^2,\tau^2|Y) \propto \tau^{-K-2(1+\gamma)} \sigma^{-N-2(1+\alpha)} \exp\left(-\frac{1}{2\tau^2}\sum_{i=1}^{N}\sum_{j=1}^{k_i}(y_{ij}-X_i)^2\right)$$
$$\times \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(X_i-\mu)^2\right)$$
$$\times \exp\left(-\beta\left(\frac{1}{\tau^2}+\frac{1}{\sigma^2}\right)\right)$$

Therefore the log posterior is given by:

$$\boxed{\begin{aligned}\log p(X, \mu, \sigma^2, \tau^2 \mid Y) = &-[K+2(1+\gamma)]\log(\tau) - [N+2(1+\alpha)]\log(\sigma) \\ &- \frac{1}{2\tau^2}\sum_{i=1}^{N}\sum_{j=1}^{K_i}(y_{ij}-X_i)^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(X_i-\mu)^2 - \beta\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right) + C\end{aligned}}$$

with C independent of $(X, \mu, \sigma^2, \tau^2)$.

② $$p(X \mid \mu, \sigma^2, \tau^2, Y) = \frac{p(X, \mu, \sigma^2, \tau^2, Y)}{p(\mu, \sigma^2, \tau^2, Y)} = \frac{p(X, \mu, \sigma^2, \tau^2 \mid Y)\, p(Y)}{p(\mu, \sigma^2, \tau^2, Y)}$$
$$\propto p(X, \mu, \sigma^2, \tau^2 \mid Y)$$

up to a constant independent of X. Thus:

- $$\log p(X_i \mid \mu, \sigma^2, \tau^2, \{Y_{ij}\}_{j\in[1,K_i]}) = -\frac{1}{2\tau^2}\sum_{j=K_i}(X_i^2 - 2y_{ij}X_i) - \frac{1}{2\sigma^2}\times(X_i^2 - 2X_i\mu) + \underbrace{C}_{\text{independent of } X}$$
$$= -\frac{1}{2}\left[X_i^2\left(\frac{K_i}{\tau^2}+\frac{1}{\sigma^2}\right) - 2X_i\left(\frac{\sum_{j=1}^{K_i} y_{ij}}{\tau^2} + \frac{\mu}{\sigma^2}\right)\right] + \underbrace{C}_{\text{independent of } X}$$
$$= -\frac{1}{2\frac{\sigma^2\tau^2}{K_i\sigma^2+\tau^2}}\left[X_i^2 - 2X_i\frac{\frac{\sum_{j=1}^{K_i} y_{ij}}{\tau^2}+\frac{\mu}{\sigma^2}}{\frac{K_i\sigma^2+\tau^2}{\sigma^2\tau^2}}\right] + cst$$

We hereby identify the above expression to the general expression of the log density of a $\mathcal{N}(\mu', (\sigma')^2)$ distribution, up to an additive constant independent of x, which is $\log p(x \mid \mu', (\sigma')^2) = -\frac{1}{2(\sigma')^2}(x^2 - 2\mu' x)$.

$$\boxed{\begin{aligned}&\text{Therefore: } (X_i \mid \mu, \sigma^2, \tau^2, \{Y_{ij}\}_{j\in[1,K_i]}) \sim \mathcal{N}(\mu', (\sigma')^2) \\ &\text{with } \sigma' = \frac{\sigma\tau}{\sqrt{K_i\sigma^2+\tau^2}}, \quad \mu' = \frac{\sigma^2\sum y_{ij} + \tau^2\mu}{K_i\sigma^2+\tau^2}\end{aligned}}$$

- Similarly, we have:

$$p(\mu \mid X, \sigma^2, \tau^2, Y) = -\frac{1}{2\sigma^2}\sum_{i=1}^{N}(\mu^2 - 2X_i\mu) = -\frac{N}{2\sigma^2}\left(\mu^2 - 2\mu\frac{\sum_{i=1}^{N} X_i}{N}\right) + cst$$

and therefore $\boxed{(\mu \mid X, \sigma^2, \tau^2, Y) \sim \mathcal{N}\left(\frac{\sum_1^N X_i}{N}, \frac{\sigma}{\sqrt{N}}\right)}$

• Similarly we have:

$$\log p(\tau^2 | X, \mu, \sigma^2, Y) = -[K+2(1+\gamma)] \log(\tau) - \frac{1}{\tau^2}\left[\sum_{i=1}^{N}\sum_{j=1}^{K_i} \frac{(y_{ij}-X_i)^2}{2} + \beta\right] + \underset{\text{independent of } \tau^2}{cst}$$

And by identification we have

$$\boxed{(\tau^2 | X, \mu, \sigma^2, Y) \sim I\Gamma\left(\frac{K}{2}+\gamma ; \beta + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{K_i} \frac{(y_{ij}-X_i)^2}{2}\right)}$$

• Eventually, we also have:

$$\log p(\sigma^2 | X, \mu, \tau^2, Y) = -\left[\frac{N}{2}+1+\gamma\right] \log(\sigma^2) - \frac{1}{\sigma^2}\left[\frac{1}{2}\sum_{i=1}^{N}(X_i-\mu)^2 + \beta\right] + \underset{\text{independent of } \sigma^2}{cst}$$

Thus, by identification:

$$\boxed{(\sigma^2 | X, \mu, \sigma^2, Y) \sim I\Gamma\left(\frac{N}{2}+\alpha ; \beta + \frac{1}{2}\sum_{i=1}^{N}(X_i-\mu)^2\right)}$$

Thus, we do not need to use Hastings-Metropolis to sample from the marginal posteriors since we now they follow usual distributions (normal and inverse gamma), we use a pure Gibbs-sampler.

③ Now, let us look at the posterior of (X, μ):

$$\log p(X, \mu | Y, \sigma^2, \tau^2) = -\frac{1}{2\tau^2}\sum_{i=1}^{N}\sum_{j=1}^{k_i}(X_i^2 - 2X_i y_{ij}) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(X_i-\mu)^2 + \underset{\text{indep of } (X,\mu)}{Cst}$$

$$= -\frac{1}{2}\sum_{i=1}^{N}\left[X_i^2\left(\frac{k_i}{\tau^2}+\frac{1}{\sigma^2}\right) - \frac{2X_i\mu}{\sigma^2}\right] + X_i \frac{\sum_{j=1}^{k_i} y_{ij}}{\tau^2} - \frac{1}{2}\frac{N\mu^2}{\sigma^2} + cst \qquad (E)$$

Denote $Z = (X, \mu) \in \mathbb{R}^{N+1}$. Let us identify the parameters of $p(Z | Y, \sigma^2, \tau^2)$, which, we know, follows $\mathcal{N}(\lambda, \Sigma)$. with $\Sigma \in S_{N+1}^{++}$ and $\lambda \in \mathbb{R}^{N+1}$.

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We have the general formula of the log posterior of a multivariate normal distribution:

$$\log \mathcal{N}(Z \mid \lambda, \Sigma) = -\frac{1}{2}\log\det(\Sigma) - \frac{1}{2}(Z-\lambda)^T \Sigma^{-1} (Z-\lambda) + cst$$

$$= -\frac{1}{2}\log\det(\Sigma) - \frac{1}{2} Z^T \Sigma^{-1} Z + Z^T \Sigma^{-1}\lambda - \frac{1}{2}\lambda^T \Sigma^{-1}\lambda$$

We have: $\begin{cases} Z^T \Sigma^{-1} Z = \sum_{i=1}^{N+1} Z_i^2 \Sigma_{ii} + 2\sum_{i<j} Z_i Z_j \Sigma_{ij}^{-1} \\ Z^T \Sigma^{-1}\lambda = \sum_{i=1}^{N+1} Z_i (\Sigma^{-1}\lambda)_i \end{cases}$

Thus, by identification with (E):

$$Z^T \Sigma^{-1} Z = \sum_{i=1}^{N} Z_i^2 \left(\frac{\sigma^2 k_i + \tau^2}{\sigma^2 \tau^2}\right) + Z_{N+1}^2 \frac{N}{\sigma^2} + 2\sum_{i=1}^{N} Z_i Z_{N+1}\left(-\frac{1}{\sigma^2}\right)$$

$$\boxed{\text{Hence: } (Z \mid Y, \sigma^2, \tau^2) \sim \mathcal{N}(\lambda, \Sigma) \text{ with: } \quad \Sigma^{-1} = \begin{pmatrix} \frac{\sigma^2 k_1 + \tau^2}{\sigma^2\tau^2} & 0 & \cdots & 0 & -\frac{1}{\sigma^2} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \frac{\sigma^2 k_n + \tau^2}{\sigma^2\tau^2} & -\frac{1}{\sigma^2} \\ -\frac{1}{\sigma^2} & \cdots & \cdots & -\frac{1}{\sigma^2} & \frac{N}{\sigma^2} \end{pmatrix}}$$

Likewise, by identification with (E) we get:

$$Z^T \Sigma^{-1}\lambda = \sum_{i=1}^{N} Z_i \left(\frac{1}{\tau^2}\sum_{j=1}^{k_i} y_{ij}\right)$$

Hence: $\Sigma^{-1}\lambda = \begin{pmatrix} \frac{1}{\tau^2}\sum_{j=1}^{k_1} y_{1j} \\ \vdots \\ \frac{1}{\tau^2}\sum_{j=1}^{k_n} y_{nj} \\ 0 \end{pmatrix}$

Therefore: $\boxed{\lambda = \Sigma \begin{pmatrix} \frac{1}{\tau^2}\sum_{j=1}^{k_1} y_{1j} \\ \vdots \\ \frac{1}{\tau^2}\sum_{j=1}^{k_n} y_{nj} \\ 0 \end{pmatrix}}$