Proving an enhanced Sanov-type Large Deviation Principle on multidimensional (α, β) rough paths

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September 4th, 2023

Plan

Plan

Stochastic and rough volatility

Let S be an asset's price, and σ be its volatility process. The Stochastic Differential Equation (SDE) of stochastic volatility is

$$dS_t = S_t \sigma_t dX_t,$$

where X is a standard BM, constant correlation $\rho := \frac{d\langle X, W\rangle_t}{dt}$ is modelled by defining a 2D standard BM $(W, \bar{W})$ and setting $X := \rho W + \bar{\rho}\bar{W}$, with $\bar{\rho} := \sqrt{1-\rho^2}$.

Rough volatility: volatility of the asset price has a lower Hölder regularity than the asset price process.
Simple rough volatility: $\sigma_t := f(\hat{X}_t)$, $\hat{X}_t := \int_0^t k_H(t-s)dW_s$, with $k_H(r) := \frac{1}{\Gamma(H+1/2)} r^{H-\frac{1}{2}}$.

Fukasawa and Takano's contribution

- Forde and Zhang [3] were able to derive small-time Large Deviations Principles (LDPs) on rough volatility but only for σ at most linear, whereas the case we want to treat is f exponential.
- Regularity structures approach [1] allows to deal with such f.
- Fukasawa and Takano [5] wanted a direct rough path version for stochastic integral $\int f(\hat{X})dX$ appearing in rough volatility, because regularity structures approach is quite complex.
- Using classical rough path theory was impossible because $\hat{X}$ not controlled by X in Gubinelli's [6] sense since $\hat{X}$ less Hölder regular. Also, lifting an fBM with $H \leq 1/4$ (empirically the case we want to treat) is impossible in classical rough path theory for Gaussian processes.
- Hence the introduction of (α, β) rough paths.

The enhanced Sanov theorem

Let B^i, B^j be two independent d-dimensional BMs with initial measure λ and assume some exponential integrability condition of λ.
Denote B^{ij} the path (B^i, B^j) and $\mathbf{B}^{ij} = (B^{ij}, \mathbb{B}^{ij})$ the corresponding rough path lift of B^{ij}.
The enhanced empirical measure $\mathbf{L}_n^{\mathbf{B}}$ is defined as

$$\mathbf{L}_n^{\mathbf{B}} := \frac{1}{n^2} \sum_{i,j=1}^{n} \delta_{(B^{ij}, \mathbb{B}^{ij})}.$$

Theorem (Theorem 3.6 of [2])

The family $\{\boldsymbol{L}_n^{\boldsymbol{B}}{}_*\mathbb{P} : n \in \mathbb{N}\}$ satisfies a LDP on $P_1(\mathcal{C}_g^{0,\alpha}([0, T], \mathbb{R}^{2d}))$ endowed with the 1-Wasserstein metric, with scale (or rate or speed) n and good rate function $\boldsymbol{I}$ (see report).

Our aim: an enhanced Sanov LDP for (α, β) rough paths.

Plan

Plan

1. Introduction
2. Definition of the N-dimensional (α, β) lifting
   - The N-dimensional (α, β) space
   - Definition of the N-dimensional (α, β) lifting
   - Proof that the lifting yields an N-dimensional (α, β) rough path
3. Proving a strong approximation lemma of N-dimensional (α, β) rough paths
   - Desired property
   - Proof of small interval case
   - Problem with case of endpoints belonging to dissection
4. Following the classical lifting of Gaussian processes to classical rough paths
   - The difference with our previous attempt
   - The problem: using the 2D Young estimate requires a 2D increment
5. Conclusion

The (α, β) space I

Set $\alpha \in (0, 1/2), \beta \in (0, 1/2)$ and $I := I(\alpha, \beta)$ and $J := J(\alpha, \beta)$ as in Fukasawa [5].

An N-dimensional (α, β) rough path $\tilde{\mathbb{X}} = (\hat{X}, \tilde{X}^{(i)}, \tilde{X}^{(jk)})_{i \in I, (j,k) \in J}$ is defined on Δ_T and satisfies the following conditions; for any $i \in I, (j, k) \in J$ and $s \leq u \leq t$,

(i) X is $\mathbb{R}^N$-valued, $\hat{X}$ is $\mathbb{R}^N$-valued, $X^{(i)}$ is $(\mathbb{R}^N)^{\otimes 2}$-valued, and $X^{(jk)}$ is $(\mathbb{R}^N)^{\otimes 4}$-valued.

(ii) *Modified Chen's relation*: $X_{st} = X_{su} + X_{ut}, \hat{X}_{st} = \hat{X}_{su} + \hat{X}_{ut}$, and for all $a, b \in [\![1, N]\!]$

$$\tilde{X}^{(i),ab}_{st} = \tilde{X}^{(i),ab}_{su} + \sum_{p=0}^{i} \frac{1}{(i-p)!} (\hat{X}^{a}_{su})^{i-p} \tilde{X}^{(p),ab}_{ut}.$$

The (α, β) space II

and for all $a, b, c, d \in [\![1, N]\!]$

$$\tilde{X}_{st}^{(jk),ab,cd} - \tilde{X}_{su}^{(jk),ab,cd} = \tilde{X}_{su}^{(j),cd} \sum_{q=0}^{k} \frac{(\hat{X}_{su}^{a})^{k-q}}{(k-q)!} \tilde{X}_{ut}^{(q),ab} + \sum_{p=0}^{j} \frac{(\hat{X}_{su}^{c})^{j-p}}{(j-p)!} \sum_{q=0}^{k} \frac{(\hat{X}_{su}^{a})^{k-q}}{(k-q)!} \tilde{X}_{ut}^{(pq),ab,cd}.$$

(iii) Hölder regularity: there exists a constant C such that, for all $(s, t) \in \Delta_T$,

$$|\hat{X}_{s,t}| \leq C(t-s)^{\beta}, \quad |X_{s,t}^{(i)}| \leq C(t-s)^{i\beta+\alpha}, \quad |X_{s,t}^{(jk)}| \leq C(t-s)^{(j+k)\beta+2\alpha}.$$

Let $\Omega_{(\alpha,\beta)-\mathrm{Hol}}^{N}$ denote the set of the N-dimensional (α, β) rough paths.

Plan

Definition of the N-dimensional (α, β) driving noise

Denote:

$$X = (X^1, \dots, X^N), \quad \hat{X} = (\hat{X}^1, \dots, \hat{X}^N)$$

with $(X^i, \hat{X}^i)_{i \leq N}$ i.i.d. and, for each $i \in [\![1, N]\!]$, $\hat{X}^i$ a 1-dimensional fractional Brownian motion possibly correlated to X^i, a 1-dimensional standard Brownian motion. We denote $X^i = \rho W^i + \bar{\rho}\bar{W}^i$ with $(W^i, \bar{W}^i)$ being a 2-dimensional Brownian motion, $\bar{\rho} = \sqrt{1 - \rho^2}$ and $\hat{X}^i_t := \int_0^t k_H(t-s) dW^i_s$, $H \in (0, 1)$ similarly to the simple rough volatility classical framework.

Definition of the N-dimensional (α, β) lifting

Define, for all $(s, t) \in \Delta_T$, $i \in I, (j, k) \in J$

$$\tilde{X}_{st}^{(i)} = \begin{pmatrix} \tilde{X}_{st}^{(i),11} & \dots & \tilde{X}_{st}^{(i),1N} \\ \vdots & & \vdots \\ \tilde{X}_{st}^{(i),N1} & \dots & \tilde{X}_{st}^{(i),NN} \end{pmatrix} = \begin{pmatrix} \frac{1}{i!} \int_s^t (\hat{X}_{sr}^1)^i dX_r^1 & \dots & \frac{1}{i!} \int_s^t (\hat{X}_{sr}^1)^i dX_r^N \\ \vdots & & \vdots \\ \frac{1}{i!} \int_s^t (\hat{X}_{sr}^N)^i dX_r^1 & \dots & \frac{1}{i!} \int_s^t (\hat{X}_{sr}^N)^i dX_r^N \end{pmatrix}$$

and, similarly, define

$$\tilde{X}_{st}^{(jk)} := \frac{1}{k!} \int_s^t (\hat{X}_{sr})^k \otimes \tilde{X}_{sr}^{(j)} \otimes dX_r \in (\mathbb{R}^N)^{\otimes 4},$$

that is, for all $a, b, c, d \in [\![1, N]\!]$

$$\tilde{X}_{st}^{(jk),ab,cd} = \frac{1}{k!} \int_s^t (\hat{X}_{sr}^a)^k \tilde{X}_{sr}^{(j),cd} dX_r^b.$$

Proof that this lifting yields an (α, β) rough path

Proposition

Then we have that, for almost every $\omega \in \Omega$, $\tilde{\mathbb{X}}(\omega) := (\hat{X}(\omega), \tilde{X}^{(i)}(\omega), \tilde{X}^{(jk)}(\omega))_{i \in I(\alpha,\beta),(j,k) \in J(\alpha,\beta)}$ is an N-dimensional (α, β) rough path for any $\alpha \in (0, 1/2), \beta \in (0, H)$.
This allows us, for all $\alpha \in (0, 1/2), \beta \in (0, H)$ to define the lift of $(X, \hat{X})$ to a N-layer (α, β) rough path as:

$$\tilde{S}^N : C^{\alpha-Hol}([0,T], \mathbb{R}^N) \times C^{\beta-Hol}([0,T], \mathbb{R}^N) \to \Omega^N_{(\alpha,\beta)-Hol}$$
$$(X, \hat{X}) = (X^1, \dots, X^N, \hat{X}^1, \dots, \hat{X}^N) \mapsto (\hat{X}, \tilde{X}^{(i)}_{i \in I}, \tilde{X}^{(jk)}_{(j,k) \in J})$$

where $\Omega^N_{(\alpha,\beta)-Hol}$ is the set of the N-layer (α, β)-rough paths defined above.

Plan

Proof

(i) Modified Chen's relation: linearity of integral and binomial theorem
(ii) Hölder regularity: As in Fukasawa, for $i \in I \setminus \{0\}$, $\pi_i(\tilde{S}^N(X, \hat{X}))$ belongs to the $i+1$-Wiener chaos, for $(jk) \in J$, $\pi_{jk}(\tilde{S}^N(X, \hat{X}))$ belongs to the $k+j+2$-Wiener chaos. Thus, apply hypercontractivity, Itô isometry, and Gaussian integrability condition for the fBM.

Plan

Plan

Desired property

Fix $\alpha \in (0, \frac{1}{2})$ and $\beta \in (0, H)$. Then fix $\alpha' \in (\alpha, \frac{1}{2})$. For all $i \in I, (j,k) \in J, s < t \in [0,1]$ and $p \in [1, \infty)$,

$$\|X_{st}^{(i)} - X_{st}^{D,(i)}\|_{L^p} \leq p^{\frac{i+1}{2}} C(\alpha,\beta,H)|D|^{\frac{1}{2}-\alpha'}(t-s)^{i\beta+\alpha'}$$
$$\|X_{st}^{(jk)} - X_{st}^{D,(jk)}\|_{L^p} \leq p^{\frac{j+k+2}{2}} C(\alpha,\beta,H)|D|^{\frac{1}{2}-\alpha'}(t-s)^{(j+k)\beta+2\alpha'}.$$

A similar property with any strictly positive power on $|D|$ would be sufficient for our final objective.
We omit the $\tilde{\cdot}$ over the levels of the (α, β) rough path for notational simplicity.
Using Kolmogorov's continuity theorem and definition of homogenous (α, β) norm (long proof in Appendix), this would allow to show

$$\|d_{(\alpha,\beta)-Hol}(\mathbf{X}, \tilde{S}^N(X^D, \hat{X}^D))\|_{L^p} \leq C'(\alpha,\beta,H)|D|^{\frac{1}{2}-\alpha'}p^{\frac{1}{2}}.$$

Plan

Proof of small interval case $s_D \leq s < t \leq s^D$ I

Reminder: For all $D \ni r_D \leq r \leq r^D \in D$ (uniform dissection of mesh $1/m$): $dX^D_r = \frac{dr}{r^D - r_D} X_{r_D, r^D} = m X_{r_D, r^D} dr$. Thus:

$$\pi_i(\tilde{S}^N(X^D, \hat{X}^D)_{st})^{a,b} = \frac{1}{i!} \int_s^t \left(\hat{X}^{D,a}_{sr}\right)^i dX^{D,b}_r$$

$$= \frac{1}{i!} m X^b_{s_D, s^D} \int_s^t \left(m(r-s)\hat{X}^a_{s_D,s^D}\right)^i dr = \frac{1}{i!} m^{i+1} X^b_{s_D,s^D} (\hat{X}^a_{s_D,s^D})^i \int_s^t (r-s)^i dr$$

$$= \frac{1}{i!} m^{i+1} X^b_{s_D,s^D} (\hat{X}^a_{s_D,s^D})^i \frac{(t-s)^{i+1}}{i+1}.$$

By Cauchy-Schwarz and Gaussian square root growth of moments:

$$\|X^b_{s_D,s^D} (\hat{X}^a_{s_D,s^D})^i\|_p \leq \|X^b_{s_D,s^D}\|_{2p} \|(\hat{X}^a_{s_D,s^D})^i\|_{2p}$$

$$\leq C(H,\alpha,\beta) p^{i/2} m^{-iH} p^{1/2} m^{-1/2} = C(H,\alpha,\beta) p^{(i+1)/2} \left(\frac{1}{m}\right)^{iH+1/2}.$$

Proof of small interval case $s_D \leq s < t \leq s^D$ II

Idea: Interpolation = We want to upper bound $\|\tilde{X}_{st}^{(i),ab} - \tilde{X}_{st}^{D,(i),ab}\|_2$ by $C(H,\alpha,\beta)\left(\frac{1}{m}\right)^{\theta}(t-s)^{\epsilon}$ with ϵ,θ big enough to "trade" some positive power of $\min(|D|, t-s)$ for some positive power of $\max(|D|, t-s)$ to end up with the exponents on $|D|, t-s$ of the desired strong approximation lemma.

In the current small interval case, it gives

$$\|\tilde{X}_{st}^{(i),ab} - \tilde{X}_{st}^{D,(i),ab}\|_p \leq C(H,\alpha,\beta)p^{(i+1)/2}\left(\frac{1}{m}\right)^{\frac{1}{2}-\alpha'}(t-s)^{i\beta+\alpha'},$$

with $\alpha' \in \left(0, \frac{1}{2}\right)$.

Plan

1. Introduction
2. Definition of the N-dimensional (α, β) lifting
   - The N-dimensional (α, β) space
   - Definition of the N-dimensional (α, β) lifting
   - Proof that the lifting yields an N-dimensional (α, β) rough path
3. Proving a strong approximation lemma of N-dimensional (α, β) rough paths
   - Desired property
   - Proof of small interval case
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   - The problem: using the 2D Young estimate requires a 2D increment
5. Conclusion

Case $s, t \in D$ but arbitrarily far. Let $s = t_c$ and $t = t_d$ for some $c < d$. We have $(d - c) = m(t - s)$. Recall m is destined to $\to \infty$.

We prove a generalisation of the modified Chen's relation to any number of intervals:

$$\tilde{X}_{st}^{(i),ab} = \sum_{l=c}^{d-1} \sum_{j=0}^{i} \frac{\tilde{X}_{t_l,t_{l+1}}^{(j),ab} \left(\hat{X}_{t_c,t_l}^{a}\right)^{i-j}}{(i-j)!}, \ \tilde{X}_{st}^{D,(i),ab} = \sum_{l=c}^{d-1} \sum_{j=0}^{i} \frac{\tilde{X}_{t_l,t_{l+1}}^{D,(j),ab} \left(\hat{X}_{t_c,t_l}^{a}\right)^{i-j}}{(i-j)!}$$

The reason why it failed is that we started with triangular inequality on $\sum_{l=c}^{d-1}$:

$$\|\tilde{X}_{st}^{(i),ab} - \tilde{X}_{st}^{D,(i),ab}\|_p \leq \sum_{l=c}^{d-1} \sum_{k=0}^{i} \frac{1}{(i-k)!} \| \left(\hat{X}_{t_c,t_l}^{a}\right)^{i-k} \left(\tilde{X}_{t_l,t_{l+1}}^{(k),ab} - \tilde{X}_{t_l,t_{l+1}}^{D,(k),ab}\right) \|_p.$$

Problem with first attempt II

Indeed, using Cauchy-Schwarz, the small interval case we treated previsouly and Gaussian integrability (square root growth of moments) of the fBM, this yields:

$$\|\tilde{X}_{st}^{(i),ab} - \tilde{X}_{st}^{D,(i),ab}\|_p \leq C(H,\alpha,\beta) p^{\frac{i+1}{2}} \sum_{l=c}^{d-1} \sum_{k=0}^{i} (t_l - t_c)^{H(i-k)} \frac{1}{m^{kH+\frac{1}{2}}}.$$

Problem, for $(d-c) = m(t-s)$:

$$\sum_{l=c}^{d-1} \sum_{k=0}^{i} (t_l - t_c)^{H(i-k)} \frac{1}{m^{kH+\frac{1}{2}}} \sim_{m\to\infty} m^{1/2},$$

while the desired property was, for $\alpha' \in (\alpha, 1/2)$,
$\|X_{st}^{(i)} - X_{st}^{D,(i)}\|_{L^p} \leq p^{\frac{i+1}{2}} C(\alpha,\beta,H)|D|^{\frac{1}{2}-\alpha'}(t-s)^{i\beta+\alpha'}$.

Plan

Plan

1. Introduction
2. Definition of the N-dimensional (α, β) lifting
   - The N-dimensional (α, β) space
   - Definition of the N-dimensional (α, β) lifting
   - Proof that the lifting yields an N-dimensional (α, β) rough path
3. Proving a strong approximation lemma of N-dimensional (α, β) rough paths
   - Desired property
   - Proof of small interval case
   - Problem with case of endpoints belonging to dissection
4. Following the classical lifting of Gaussian processes to classical rough paths
   - The difference with our previous attempt
   - The problem: using the 2D Young estimate requires a 2D increment
5. Conclusion

The difference with our previous attempt I

Previously, we wanted to use an explicit expression (generalisation of modified Chen's relation) of $\tilde{X}^{(i)} - \tilde{X}^{D,(i)}$ as a function of increments "adapted to the dissection" (like) to control $\|\tilde{X}^{(i)} - \tilde{X}^{D,(i)}\|_{L^2}$.

Now, control $\|\tilde{X}^{(i)} - \tilde{X}^{D,(i)}\|_{L^2}$ by ω the control on the covariance of $(X, \hat{X}, X^D, \hat{X}^D)$. Motivation for this: from Friz-Victoir, we can Hölder dominate the covariances R_{X,X^D} and $R_{\hat{X},\hat{X}^D}$ by ω_D and $\hat{\omega}_D$.

What follows is the core of the method of Friz-Victoir: Let $\hat{X}$ be a bounded variation Gaussian process (that will be a piecewise linear approximation to the fBM $\hat{X}$) and X, a bounded variation Gaussian process (that will be an approximation to the BM X).

Plan

1. Introduction
2. Definition of the N-dimensional (α, β) lifting
   - The N-dimensional (α, β) space
   - Definition of the N-dimensional (α, β) lifting
   - Proof that the lifting yields an N-dimensional (α, β) rough path
3. Proving a strong approximation lemma of N-dimensional (α, β) rough paths
   - Desired property
   - Proof of small interval case
   - Problem with case of endpoints belonging to dissection
4. Following the classical lifting of Gaussian processes to classical rough paths
   - The difference with our previous attempt
   - The problem: using the 2D Young estimate requires a 2D increment
5. Conclusion

The problem: using the 2D Young estimate requires a 2D increment

Simpler case first: suppose $\hat{X}$ independent of X.

$$\mathbb{E}\left[\int_s^t (\hat{X}_{sr})^i dX_r \int_{s'}^{t'} (\hat{X}_{s'r})^i dX_r\right] = \mathbb{E}\left[\int_s^t \int_{s'}^{t'} (\hat{X}_{su})^i (\hat{X}_{s'v})^i dX_u d(X)_v\right]$$

$$= \int_s^t \int_{s'}^{t'} \mathbb{E}\left[(\hat{X}_{su})^i (\hat{X}_{s'v})^i dX_u dX_v\right] = \int_s^t \int_{s'}^{t'} \mathbb{E}[(\hat{X}_{su})^i (\hat{X}_{s'v})^i] d\mathbb{E}[X_u X_v]$$

We want to use the 2D Young estimate on this integral. We need the integrand to be a 2D increment, defined as

$$f\begin{pmatrix} s, t \\ u, v \end{pmatrix} := f\begin{pmatrix} s \\ u \end{pmatrix} + f\begin{pmatrix} t \\ v \end{pmatrix} - f\begin{pmatrix} s \\ v \end{pmatrix} - f\begin{pmatrix} t \\ u \end{pmatrix}$$

for any $f : [0, T]^2 \to \mathbb{R}^d$. But is seems impossible.

Plan

Conclusion

- Established contribution: generalisation of the (α, β) lifting to N dimensions. Conceptually non-trivial because it allows to generalise the SDEs considered in rough volatility to i.i.d. sequences of driving noises.
- Two main attempts to prove the strong approximation of (α, β) rough paths by piecewise linear approximations. The first one, based on the Brownian case of Friz-Victoir [4], allowed to show small interval case. The case of arbitrarily far endpoints is part of ongoing work with Fukasawa.
- Second main attempt was to follow stochastic integration of Gaussian processes depicted in Section 15 of [4]. Not sufficient because the powers $(\hat{X}_{s,r})^i$ in $\tilde{X}^{(i)}$ do not seem naturally associated to the control on covariance.

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