Mean-field limit of non-exchangeable systems

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1 Introduction

In this paper we will analyse the eponymous article [JPS22] by Pierre-Emmanuel Jabin, David Poyato and Juan Soler. This article uses ideas coming from graph theory, stochastic analysis and partial differential equations in order to obtain new results regarding the derivation of mean-field limits of certain non-exchangeable sparse graphs. The paper you're about to read is the written assignment for class MAP512 from Ecole Polytechnique.

So far in classic mean-field limit theory, researchers were mainly focused on exchangeable systems in dense graphs. This corresponds to a plethora of physical systems where all particles interact with each other in the same way. This setting was very satisfactory to study problems in statistical physics where all the physical particles are identical. However, as other scientific fields in Biology or Social Sciences became interested in using the tools from this theory, there was a growing need for more robust results where we dropped some of our previous hypothesis about the particle system. To this end, recent articles on this subject obtained new results with sparse graphs, or non-exchangeable systems. However, [JPS22] can be viewed as a breakthrough in this area of research since it obtains a convergence result despite working under sparse and non-exchangeable hypothesis. Moreover, it only requires the graph to follow some specific sparsity conditions, but no convergence conditions.

In section 2, we'll review the graph convergence theory in order to present [JPS22]'s new definition of a graph limit and the results one may obtain with it. In section 3, we will introduce the classical notions and methods of mean-field limits and propagation of chaos in the exchangeable case. This section is mostly based on the foundational work of [Szn91]. Understanding the classical methods presented there is essential to address the non-exchangeable case. Section 4 is where we will show subtleties introduced by Jabin et al. to extend these classical arguments to the non-exchangeable case. The main result of this section is the propagation of independence proposition 4.1, that we will need to prove the main theorem of [JPS22] stated in Section 5. In Section 5, we will outline the main ideas of the proof of the main theorem to establish the link between the other important results of [JPS22] and see how they are useful to prove the main theorem. Lastly, in section 6, we overview new directions we could have taken to further understand the ideas presented in [JPS22].

2 Graphons and extended graphons

An innovative aspect of the article we studied pertains to graph theory. The dynamics of an N agents system depends on the graph containing all the information about their interactions. In order to obtain a mean-field limit of our system, one will also obtain some limit object for the interaction graph. However, previous research on this topic was mainly concerned with graphons, who behave well as limits for dense graphs. In [JPS22], there was a need for new theory describing limits of sparse graphs. Indeed, the authors invented a new type of graph limit in order to better capture the nature of a sparse graph limit. We shall then introduce some classical elements of graph theory, and present the new extended graphon.

2.1 The graphon

The graphon is a limit object for graphs that occurs when one thinks of limits using homomorphism densities. The following development can also be found on [Lov13].

Definition 2.1. *A homomorphism between the simple graphs F and G is an application that preserves all the edges. More explicitly, $\varphi : V(F) \to V(G)$ is a homomorphism if and only if $\forall (i,j) \in E(F), (\varphi(i), \varphi(j)) \in E(G)$.*

Let's say we have two simple graphs F and G, the homomorphism density of (F, G) is exactly the proportion of homomorphisms from F to G over all the applications from F to G.

Definition 2.2. *The homomorphism density from graph F to graph G is given by:*

$$t(F,G) = \frac{\hom(F,G)}{|V(G)|^{|V(F)|}}.$$

Where $\hom(F,G)$ *is the number of homomorphisms from F to G.*

With this definition, one can already define convergence for a sequence of graphs $(G_n)_n$.

Definition 2.3. *We say that the sequence of graphs $(G_n)_n$ converges in the t sense if for any simple graph F, we have that $t(F, G_n)$ converges in $\mathbb{R}$.*

Now, we will seek to generalize this notion for graphs with weighted edges.

Definition 2.4. *We define the quantity of homomorphisms from the simple graph F to the weighted graph G as:*

$$\hom(F, G) = \sum_{\varphi: V(F) \to V(G)} \prod_{(i,j) \in E(F)} w_{\varphi(i)\varphi(j)}(G).$$

Remark 2.1. *Note that we impose F to remain a simple graph. If, also, G is simple, then:*

$$\prod_{(i,j) \in E(F)} w_{\varphi(i)\varphi(j)}(G) = \mathbb{1}_{\{\varphi \text{ is a homomorphism}\}}.$$

Therefore, the latter definition is coherent with definition 2.2 when G is a simple graph.

We may now generalize $t(F, G)$ to the case where G is no longer a weighted graph, but a graphon. This object is introduced because the space of graphs is not complete for the convergence of homomorphism densities. The graphs are converging to something else entirely, and graphons are one nice and intuitive way of capturing the nature of this limit object, although its limitations will eventually be strong in certain contexts. A graphon is a symmetric bounded measurable function from $[0, 1]^2$ to $\mathbb{R}$. If F is a simple graph and W is a graphon, then we have the following formula:

$$t(F, W) = \int_{[0,1]^{|V(F)|}} \prod_{ij \in E(F)} W(x_i, x_j) dx.$$

A way of understanding that graphs approximate graphons is to consider that any given graph defines a graphon, and that a convergent sequence of graphs defines a convergent sequence of graphons, whose limit object is a graphon. Say you have a graph G, with $|V(G)| = n$, then one may split the $[0, 1]^2$ square into a grid with n^2 small squares. We then define the graphon W_G to be the step function with constant values on each small square:

$$W_G(x, y) = \sum_{1 \le i,j \le |V(G)|} \mathbb{1}_{]\frac{i-1}{|V(G)|}, \frac{i}{|V(G)|}]}(x) \mathbb{1}_{]\frac{j-1}{|V(G)|}, \frac{j}{|V(G)|}]}(y) w_{ij}(G) = w_{\lceil nx \rceil \lceil ny \rceil}(G). \tag{1}$$

This definition is visually direct when one considers the adjacency matrix of G being transformed into an application from $[0, 1]^2$ to $\mathbb{R}$. It now comes immediately that for any simple graph F and weighted graph G:

$$t(F, G) = t(F, W_G).$$

Moreover, the following theorem from [Bor+06] nicely wraps up the theory of convergent dense graphs:

Theorem 2.1. *If $(G_n)_n$ is a convergent weighted graph sequence with uniformly bounded weights, then there exists a graphon W such that for any simple graph F, we have that $t(F, G_n) \to t(F, W)$.*

The above theory of graph convergence gives us a coherent framework to think about dense graph limits. The theorem above also implies that this type of convergence is equivalent to convergence according to the cut metric, applied to either graphs or graphons.

2.2 The case of sparse graphs

However, this theory behaves badly when one considers sparse graphs. Indeed, all sparse graphs converge to an almost everywhere null graphon. We obtain a limit objet which has lost much information about the graph sequence.

Definition 2.5. *One considers a graph sequence $(G_n)_n$ to be sparse when:*

$$\frac{|E(G_n)|}{|V(G_n)|^2} = \frac{\sum_{ij \in E(G_n)} |w_{ij}|}{|V(G_n)|^2} = o(1).$$

Now, let $(G_n)_n$ be a convergent sparse graph sequence and K_2 be the simple graph with 2 connected vertices. We may obtain straightforwardly that:

$$t(K_2, G_n) = \frac{|E(G_n)|}{|V(G_n)|^2} \to 0.$$

Therefore we know that the limit graphon W verifies the equation $t(K_2, W) = 0$, which means that:

$$\int_{[0,1]^2} W(x_1, x_2) dx = 0.$$

And given the fact that W is positive, this means that W is equal to zero almost everywhere.

These observations have a big impact on [JPS22], where we make two fundamental assumptions on the graphs. First, the total interaction received of transmitted by an agent i, which is $\sum_{j=1}^{N} w_{ij}$ or $\sum_{i=1}^{N} w_{ij}$ is of order 1. Secondly, no single agent is allowed to have a dominant role:

$$\max_{1\leq i\leq N} \sum_{j=1}^{N} |w_{ij}| = O(1) \quad \max_{1\leq j\leq N} \sum_{i=1}^{N} |w_{ij}| = O(1) \quad \text{as} \quad N \to \infty \tag{2}$$

$$\max_{1\leq i,j\leq N} |w_{ij}| = o(1) \quad \text{as} \quad N \to \infty \tag{3}$$

Proposition 2.1. *If a graph sequence $(G_N)_{N\in\mathbb{N}}$ satisfies condition 2 then it is a sparse graph sequence.*

Proof. Condition 2 gives us:

$$\max_{1\leq i\leq N} \sum_{1\leq j\leq N} |w_{ij}| = O(1).$$

Therefore, if we consider a sequence of graphs $(G_n)_n$ satisfying this condition, with also $|V(G_n)| = n$, we can easily prove that $(G_n)_n$ is a sparse graph sequence.

$$\frac{|E(G_N)|}{|V(G_N)|^2} = \frac{\sum_{1\leq i,j\leq N} |w_{ij}|}{N^2} \leq \frac{N \max_{1\leq i\leq N} \sum_{1\leq j\leq N} |w_{ij}|}{N^2} = \frac{1}{N} O(1) = o(1).$$

□

This reasoning remains true for any graph sequence with the number of nodes tending towards infinity. Thus, the authors of the article had to work with a different theory of graph limits. One that extends the concepts we've just explored.

In the case of dense graphs, the definition of the empirical graphon is given by eq. (1). Yet, if the graph is sparse, the right notion of empirical graphon is the same as previously, but with a rescaling to avoid convergence to zero of the graphon.

Definition 2.6. *The empirical graphon of a sparse graph G with N vertices is defined as :*

$$W_G(x, y) = \sum_{1\leq i,j\leq N} N \mathbb{1}_{]\frac{i-1}{N},\frac{i}{N}]}(x) \mathbb{1}_{]\frac{j-1}{N},\frac{j}{N}]}(y) w_{ij}(G) = N w_{\lceil nx\rceil\lceil ny\rceil}(G). \tag{4}$$

2.3 The extended graphon

To circumvent all the previous issues, [JPS22] creates a new limit object for graph sequences, the extended graphon.

Definition 2.7. *We call $L^\infty_\xi M_\zeta \cap L^\infty_\zeta M_\xi$ the space of extended graphons. The Bochner space $L^\infty_\xi M_\zeta$ corresponds to the essentially bounded maps $\xi \in [0,1] \to w(\xi, d\zeta) \in \mathcal{M}_\zeta([0,1])$. This means that the following seminorm is finite:*

$$||w||_{L^\infty_\xi M_\zeta} = \sup_{||\varphi||_{C([0,1])}\leq 1} \operatorname*{ess\,sup}_{\xi\in[0,1]} |\int_0^1 \varphi(\zeta) w(\xi, d\zeta)| < +\infty.$$

Hence, $w \in L^\infty_\xi M_\zeta \cap L^\infty_\zeta M_\xi \iff w(\xi, d\zeta) \in L^\infty([0,1], \mathcal{M}([0,1]))$ and $w(d\xi, \zeta) \in L^\infty([0,1], \mathcal{M}([0,1]))$.

Given this new type of graphons, we define a new generalization of the homomorphism density given by definition 2.2. This new definition will take into account the averaged information of the system, or equivalently a probability law of the particles:

Definition 2.8. *Consider any $w \in L^\infty_\xi M_\zeta \cap L^\infty_\zeta M_\xi$ and any $f \in L^\infty_\xi L^1_x$. Then we define for every tree $T \in \mathbf{T}$:*

$$\tau(T, w, f)(x_1, ..., x_{|T|}) := \int_{[0,1]^{|T|}} \prod_{(k,l)\in T} w(\xi_k, \xi_l) \prod_{m=1}^{|T|} f(x_m, \xi_m) d\xi_1...d\xi_{|T|}.$$

The restriction to trees, simple graphs without cycles, is useful since it yields the following inequality:

$$||\tau(T, w, f)||_{W^{k,p}(\mathbb{R}^{d|T|})} \leq ||w||^{|T|-1}_{L^\infty_\xi M_\zeta \cap L^\infty_\zeta M_\xi} ||f||^{|T|}_{L^\infty_\xi W^{k,p}_x}. \tag{5}$$

This result is obtained in [JPS22] by a straightforward induction argument followed by a generalization to the new τ operator. We point out that the proof only works thanks to the fact that we restricted the domain of this operator to trees, and not all simple graphs.

Since the new τ operator, when evaluated at some T, w and f yields not a real number, but a function on $\mathbb{R}^{d|T|}$, it makes sense for us to think of its L^2 norm. This consideration enables us to consider the new norm, defined for any $\lambda > 0$:

$$||\tau(., w, f)||_\lambda = \sup_{T\in\mathbf{T}} \lambda^{|T|/2} ||\tau(T, w, f)||_{L^2(\mathbb{R}^{d|T|})}.$$

Having made this definition, [JPS22] obtains a non trivial estimate on the difference of two probability distributions satisfying the same partial differential equation:

Theorem 2.2. *Let $f, \tilde{f} \in L^\infty([0,t_*], L^\infty_\xi(L^1_x \cap L^\infty_x \cap H^1_x))$ associated with $w, \tilde{w} \in L^\infty_\xi M_\zeta \cap L^\infty_\zeta M_\xi$ weak solution to the following partial differential equation:*

$$\partial_t f(t,x,\xi) + div_x \left(f(t,x,\xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) f(t,dy,\zeta) \right) = 0.$$

Where $K \in L^\infty \cap W^{1,1}$ and $divK \in L^\infty$. Then, we have for some $C, \lambda > 0$:

$$\left|\left| \int_0^1 (f-\tilde{f})(t,.,\xi) d\xi \right|\right|_{L^2_x} \leq \frac{C}{(ln|ln||\tau(.,w,f^0) - \tau(.,\tilde{w},\tilde{f}^0)||_\lambda|)_+^{\frac{1}{2}}}$$

Finally, we will now state a crucial result on extended graphons obtained by [JPS22]. We have a result that is equivalent to theorem 2.1, but for extended graphons. This result is more complex, but it follows the same logic that a graph sequence which respects a certain uniform boundary will converge to something:

Theorem 2.3. *Let $\{w_N\}_N$ and $\{f_N\}_N$ be sequences such that the following hold:*

$$(i) \sup_{N\in\mathbb{N}} \sup_{\xi\in[0,1]} \int_0^1 |w_N(\xi,\zeta)| d\zeta < \infty, \sup_{N\in\mathbb{N}} \sup_{\zeta\in[0,1]} \int_0^1 |w_N(\xi,\zeta)| d\xi < \infty,$$

$$(ii) \sup_{N\in\mathbb{N}} ||f_N||_{L^\infty_\xi(W^{1,1}_x \cap W^{1,\infty}_x)} < \infty.$$

Then, there exists $(\varphi(n))_n$ a subsequence of $\mathbb{N}$, $w \in L^\infty_\zeta M_\xi \cap L^\infty_\xi M\zeta$ and $f \in L^\infty_\xi(W^{1,1}_x, W^{1,\infty}_x)$ such that:

$$\tau(T, w_{\varphi(N)}, f_{\varphi(N)}) \to \tau(T, w, f) \text{ in } L^p_{loc}(\mathbb{R}^{d|T|}),$$

For any $T \in \mathbf{T}$ and $1 \leq P < \infty$.

3 Classical mean-field limits in the exchangeable case and propagation of chaos

3.1 Definitions

We begin by analyzing the foundational work of Sznitman. In [Szn91], we consider $b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ to be bounded Lipschitz. We define our particle system of interaction kernel b with the following equations:

$$\begin{cases} dX_i^N(t) = \frac{1}{N}\sum_{j=1}^N b(X_i^N(t), X_j^N(t))dt + dW_i(t), \ i \in \{1,\dots,N\} \\ X_i^N(t=0) = X_i^0 \end{cases} . \tag{6}$$

where the $(W_i(t))_{t\geq 0}$ are N independent Wiener processes. In eq. (6), the particle system is exchangeable since the weighting of each term of the sum is the same, equal to $\frac{1}{N}$.

Remark 3.1. *Since b is bounded Lipschitz, we will consider the constant K so that:*

$$|b(x_1,y_1) - b(x_2,y_2)| \leq K|(x_1,y_1) - (x_2,y_2)| \wedge 1$$

Definition 3.1. *We define the empirical measure over the $(X_i^N)_{i=1}^N$ as :*

$$\mu_N(t,x) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N(t)}(x)$$

Definition 3.2. *We define the Wasserstein distance of order 1 on the set $M(C)$ of probability measures on $C = C([0,T], \mathbb{R}^d)$ for any pair (μ,ν) of probability measures on C:*

$$W_1^T(\mu,\nu) = \inf\{\mathbb{E}[\sup_{t\leq T} |X(t) - Y(t)| \wedge 1] : I\!P_X = \mu, I\!P_Y = \nu\}$$

Where $I\!P$ is the law of the pair of random variables (X,Y) and $I\!P_X$ (resp. $I\!P_Y$) is the marginal with respect to the first (resp.second) coordinate.

The Wasserstein distance of order 1 on R^d is given, for any pair (μ,ν) of probability measures on $C = \mathbb{R}^d$, by:

$$W_1(\mu,\nu) = \inf\{\mathbb{E}[|X-Y|] : I\!P_X = \mu, I\!P_Y = \nu\}$$

We will admit the following fact stated in [Szn91](under formula 1.4): The Wasserstein distance defines a complete metric on M(C) (whether C equald $\mathbb{R}^d$ or $C = C([0,T], \mathbb{R}^d)$), which gives to M(C) the topology of weak convergence.

In the definition of the Wasserstein distance on $C([0,T], \mathbb{R}^d)$, we voluntarily bound the distance by 1. In [Szn91], this is consistent with our definition of the constant K stated in remark 3.1.

Proposition 3.1. *We recall the dual representation of W_1 (in the case of probability measures on $\mathbb{R}^d$):*

$$W_1(\mu,\nu) = \sup\{\int_{\mathbb{R}^d} \phi(x) d(\mu-\nu)(x) : \phi : \mathbb{R}^d \to \mathbb{R}, Lip(\phi) \leq 1\}$$

where $Lip(\phi)$ is the Lipschitz constant of a Lipschitz continuous function ϕ.

We note that the proofs from Sznitman use the Wasserstein metric on $C([0,T],\mathbb{R}^d)$ while Jabin et al. use the one on $\mathbb{R}^d$.

Definition 3.3. *A random vector $(Z_1,\dots,Z_N)$ is symmetric (under permutation) if for all permutation $\sigma\in\mathcal{S}_N$, $(Z_{\sigma(i)})_{i\le N}$ has same probability law.*

A random process $(Z_i(t))_{t\ge0}, i\in\{1,\dots,N\}$ is symmetric if for all permutation $\sigma\in\mathcal{S}_N$, $(Z_{\sigma(i)}(t))_{t\ge0}, i\in\{1,\dots,N\}$ has same probability law.

It is essential to remark that, in the exchangeable case ($w_{ij}=1/N$), the random process $(X_i(t))_{t\ge0}, i\in\{1,\dots,N\}$ is symmetric. Indeed, for all $i\le N$, we have :

$$dX_i(t)=\frac{1}{N}\sum_{j=1}^{N}b(X_i(t),X_j(t))dt+dW_i(t)=g(X_1(t),\dots,X_N(t))+dW_i(t)$$

g is a deterministic measurable function and the brownian motions are identically distributed.

Definition 3.4 (Weak and strong solutions of SDEs)**.** *Let $(B_t)_{t\ge0}$ be a Brownian motion with admissible filtration $(\mathcal{F}_t)_{t\ge0}$. A progressively measurable process $(X_t,\mathcal{F}_t)$ is a strong solution with initial condition $x_0\in\mathbb{R}^d$ if*

$$\begin{cases}X_t-X_0=\int_0^t\sigma(s,X_s)dB_s+\int_0^t b(s,X_s)ds,\\ X_0=x_0\end{cases}\tag{7}$$

holds almost surely for all $t\ge0$.

A stochastic process $(X_t,\mathcal{F}_t)$ on some probability space $(\Omega,\mathcal{F},I\!P)$ is called a weak solution with initial distribution μ if there exists a Brownian motion $(B_t)_{t\ge0}$ on $(\Omega,\mathcal{F},I\!P)$ such that $(\mathcal{F}_t)_{t\ge0}$ is an admissible filtration, $I\!P(X_0\in\cdot)=\mu(\cdot)$ and

$$X_t-X_0=\int_0^t\sigma(s,X_s)dB_s+\int_0^t b(s,X_s)ds\tag{8}$$

holds almost surely for all $t\ge0$.

We have to consider different notions of uniqueness. For strong solutions we are looking for pathwise unique solutions, i.e. if $(X_t^{(1)})_{t\ge0}$ and $(X_t^{(2)})_{t\ge0}$ are strong solutions to eq. (7) with the same initial condition, then pathwise uniqueness means

$$I\!P(\sup_{t\ge0}|X_t^{(1)}-X_t^{(2)}|=0)=1.$$

The notion of uniqueness for weak solutions is weak uniqueness, that is uniqueness in distribution.

3.2 Classical exchangeable case: laboratory example

The goal of this laboratory example section is to show that when N goes to infinity each X_i^N has a natural limit $\bar{X}_i$. Each $\bar{X}_i$ is an independent copy of a new object: "the nonlinear process".

Definition 3.5 (Nonlinear process)**.**

$$\begin{cases}dX(t)=\int_{\mathbb{R}^d}b(X(t),y)f(t,dy)dt+dW(t)\\ X(t=0)=X_0\end{cases}.\tag{9}$$

where $f(t,\cdot)=Law(X(t))$.

The random process $(X(t))_{t\ge0}$ is nonlinear because the integral in right-hand side of the equation is with respect to the law of $X(t)$. First, one must ensure existence of a solution to this equation, hence Theorem 1.1. of [Szn91]:

Theorem 3.1. *There is existence and uniqueness, trajectorial and in law for the solutions of eq.* (9)*.*

Proof. The proof is based on fixed-point arguments. To begin, we define ϕ the map which associates to $f\in M(C([0,T],\mathbb{R}^d))$ the law, in $M(C([0,T],\mathbb{R}^d))$, of the (strong) solution of

$$\begin{cases}dX(t)=\int_{\mathbb{R}^d}b(X(t),y)f(t,dy)dt+dW(t)\\ X(t=0)=X_0\end{cases}.\tag{10}$$

We observe that $(X(t))_{t\le T}$ is a solution of eq. (9) if, and only if its law on $C([0,T],\mathbb{R}^d)$ is a fixed point of ϕ.

Lemma 1.3 of [Szn91] can be stated as:

Lemma 3.2. *For all $t\le T$ and any pair (μ,ν) of probability measures on $C([0,T],\mathbb{R}^d)$:*

$$W_1^t(\phi(\mu),\phi(\nu))\le c_T\int_0^t W_1^u(\mu,\nu)du$$

where c_T is a constant. Since $W_1^t(\mu,\nu)\le W_1^T(\mu,\nu)\forall t\le T$, we have :

$$W_1^T(\phi(\mu),\phi(\nu))\le c_T T W_1^T(\mu,\nu)$$

Proof. The lemma by showing the inequality for any coupling and then using Gronwall's lemma. Let us consider μ and ν in $M(C([0,T],\mathbb{R}^d))$ and the processes $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ defined by:

$$\begin{cases} X(t) = x_0 + W(t) + \int_0^t \int_C b(X(s), w(s))d\mu(w)ds \\ Y(t) = x_0 + W(t) + \int_0^t \int_C b(Y(s), w(s))d\nu(w)ds \end{cases} \tag{11}$$

First, we can write

$$\sup_{s\leq t} |X(s) - Y(s)| \leq \int_0^t ds| \int_C b(X(s), w(s))d\mu(w) - \int_C b(Y(s), w(s))d\nu(w)|$$

By triangular inequality introducing a $(b(y, w^1(s))-b(y, w^1(s)))d\mathbb{P}(w^1, w^2)$ term (where $\mathbb{P}$ is any coupling of μ, ν on $C([0,s],\mathbb{R}^d)$)in the integrand and using remark 3.1, it comes:

$$|\int_C b(X(s), w(s)d\mu(w) - \int_C b(Y(s), w(s)|d\nu(w)| \leq K[|x-y| \wedge 1 + \int_{C\times C} |w^1(s) - w^2(s)| \wedge 1 d\mathbb{P}(w^1, w^2)]$$

Then, putting the two previous inequalities together:

$$\sup_{s\leq t} |X(s) - Y(s)| \leq K \int_0^t ds|X(s) - Y(s)| \wedge 1 + K \int_0^t W_1^s(\mu, \nu)ds$$

Finally, using Gronwall's lemma:

$$\sup_{s\leq t} |X(s) - Y(s)| \leq Ke^{KT} \int_0^t W_1^s(\mu, \nu)ds$$

The lemma follows by taking the expectation of this expression with respect to any coupling of μ, ν. □

We now can achieve the proof of the theorem. From the lemma, we have weak and strong uniqueness of solutions of eq. (9).

Indeed, let μ and ν be two laws on $C([0,T],\mathbb{R}^d)$ of two weak solutions of eq. (9). Then, by the observation made above in the proof of the theorem, μ and ν are fixed points of ϕ. Thus, by lemma 3.2, we have, for all $t \leq T$:

$$W_1^T(\phi(\mu), \phi(\nu)) = W_1^T(\mu, \nu) \leq \int_0^t W_1^u(\mu, \nu)du$$

Therefore, by Gronwall's lemma :

$$W_1^T(\mu, \nu) \leq 0 \cdot \exp(\int_0^t c_T ds) = 0$$

Hence $\mu = \nu$ and the weak uniqueness part.

For the strong uniqueness part, let $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ be strong solutions of eq. (9) (therefore with same initial condition). Let μ and ν be respective laws on $C([0,T],\mathbb{R}^d)$. Thus, we have by definition

$$\begin{cases} X(t) = x_0 + W(t) + \int_0^t \int_C b(X(s), w(s))d\mu(w)ds \\ Y(t) = x_0 + W(t) + \int_0^t \int_C b(Y(s), w(s))d\nu(w)ds \end{cases} \tag{12}$$

where $C = C([0,T],\mathbb{R}^d)$. We can therefore use the last inequality in the proof of lemma 3.2 and the fact that, by weak uniqueness, we have $\mu = \nu$:

$$\sup_{s\leq t} |X(s) - Y(s)| \wedge 1 \leq Ke^{KT} \int_0^t W_1^s(\mu, \nu)ds = 0$$

Hence the strong uniqueness part.

For the existence part of the theorem, let $T > 0$, $k \in \mathbb{N}$ and $f \in M(C([0,T],\mathbb{R}^d))$. We have, iterating lemma 3.2:

$$W_1^T(\phi^{k+1}(f), \phi^k(f)) \leq c_T^k \frac{T^k}{k!} W_1^T(\phi(f), f)$$

Therefore, $(\phi^k(f))$ is a Cauchy sequence and, by completeness, converges to a limit, which we denote f_T. By using that ϕ and W_1^T are continuous, it comes: $W_1^T(\phi(f_T), f_T) = \lim_{k\to\infty} W_1^T(\phi^{k+1}(f_T), \phi^k(f_T)) = W_1^T(f_T, f_T) = 0$. Thus, f_T is a fixed point of ϕ.

Let $T' < T$, the image of f_T on $C([0,T'],\mathbb{R}^d)$ is still a fixed point, so the f_T are a consistent family, yielding a P on $C([0,T'],\mathbb{R}^d)$. Thus, P is a required solution to eq. (9), which proves the theorem.

□

In [Szn91], the classical arguments of propagation of chaos are presented. First, he defines the following McKean-Vlasov SDE:

Definition 3.6. *By theorem 3.1, we introduce the random processes $(\bar{X}_i(t))_{t\geq 0}$ pour $i \in \{1, \ldots, N\}$:*

$$\begin{cases} d\bar{X}_i(t) = \int_{\mathbb{R}^d} b(\bar{X}_i(t), y)\bar{f}(t, dy)dt + dW_i(t) \\ \bar{X}_i(0) = X_i^0 \end{cases} . \tag{13}$$

where $\bar{f}(t, \cdot) = Law(\bar{X}_i(t))$.

First, we note that the initial values of this nonlinear system of stochastic differential equations $\bar{X}_i(0)$ are taken equal to the ones of the particle system X_i^0. Moreover, the brownian motion of each $\bar{X}_i$ is the same as the corresponding X_i. One may note that, in this system, for all $1 \leq i \leq N$, the law of $\bar{X}_i$ is the same. We have N copies of the non-linear process.

Remark 3.2. *Moreover, we can show that they are independent (thus i.i.d.) as soon as the $(X_i^0)_{i\geq 1}$ are. Indeed, since the initial conditions and the Brownian motions of each $\bar{X}_i$ are independent, the solutions of the SDEs must be independent. The proof can be done by defining a deterministic functional, which associates an initial condition and a brownian trajectory to the solution of the SDE with this initial condition and this Brownian trajectory.*

Theorem 1.4 of [Szn91] gives the propagation of chaos property of particle system described by eq. (6), and is stated here:

Theorem 3.3 (Propagation of chaos)**.** *For all $i \geq 1, T > 0$ we have, for $(X_i(t))_{t\geq 0}$ defined in eq.* (9) *and $(\bar{X}_i(t))_{t\geq 0}$ defined in eq.* (13)*:*

$$\sup_N \sqrt{N}\mathbb{E}[\sup_{t\leq T} |X_i^N(t) - \bar{X}_i^N(t)|] < \infty$$

Thus, Sznitman proves a convergence of the process at a speed $N^{-1/2}$, which is consistant with Jabin et al.'s convergence at speed $\sup_{1\leq i,j\leq N} |w_{ij}|^{1/2}$ since the maximum weight in the exchangeable case is $\frac{1}{N}$.

Proof. We will drop the N superscript for notational simplicity. First, since the initial condition and brownian motion of X_i and $\bar{X}_i$ are the same, we have:

$$\begin{aligned}
X_i(t) - \bar{X}_i(t) &= \int_0^t [\frac{1}{N}\sum_{j=1}^N b(X_i(s), X_j(s)) - \int_{\mathbb{R}^d} b(\bar{X}_i(s), y)\bar{f}(s, dy)]ds \\
&= \int_0^t ds \frac{1}{N}\sum_{j=1}^N \Bigg[b(X_i(s), X_j(s)) - b(\bar{X}_i(s), X_j(s)) + b(\bar{X}_i(s), X_j(s)) - b(\bar{X}_i(s), \bar{X}_j(s)) \\
&\qquad + b(\bar{X}_i(s), \bar{X}_j(s)) - \int_{\mathbb{R}^d} b(\bar{X}_i(s), y)\bar{f}(s, dy)\Bigg] \qquad (14)
\end{aligned}$$

Now, let us denote $b_s(x, x') = b(x, x') - \int_{\mathbb{R}^d} b(x, y)\bar{f}(s, dy)$. Using remark 3.1 with triangular inequality we get:

$$|X_i - \bar{X}_i|(t) \leq K \int_0^t ds \Big(|X_i(s) - \bar{X}_i(s)| + \frac{1}{N}\sum_{j=1}^N |X_j(s) - \bar{X}_j(s)| + |\frac{1}{N}\sum_{j=1}^N b_s(\bar{X}_i(s), \bar{X}_j(s))| \Big)$$

Thus, taking the supremum for $t \leq T$ on both sides in the previous inequality, taking the expectation and summing over i:

$$\sum_{i=1}^N \mathbb{E}[\sup_{t\leq T} |X_i - \bar{X}_i|(t)] \leq K \int_0^T ds \sum_{i=1}^N \Big(\mathbb{E}[|X_i(s) - \bar{X}_i(s)|] + \frac{1}{N}\sum_{j=1}^N \mathbb{E}|X_j(s) - \bar{X}_j(s)| + \mathbb{E}|\frac{1}{N}\sum_{j=1}^N b_s(\bar{X}_i(s), \bar{X}_j(s))| \Big)$$

Now, we shall denote $|X_i - \bar{X}_i|_T^* := \sup_{t\leq T} |X_i - \bar{X}_i|(t)$. Using the symmetry of the processes $(X_i(t))_{t\geq 0}, i \in \{1, \dots, N\}$ and $(\bar{X}_i(t))_{t\geq 0}, i \in \{1, \dots, N\}$, we get:

$$\begin{aligned}
N\mathbb{E}[|X_1 - \bar{X}_1|(T)] &\leq K \int_0^T ds \sum_{i=1}^N \Big(2\mathbb{E}[|X_i(s) - \bar{X}_i(s)|] + \mathbb{E}|\frac{1}{N}\sum_{j=1}^N b_s(\bar{X}_i(s), \bar{X}_j(s))| \Big) \\
&\leq K' \int_0^T ds \sum_{i=1}^N \Big(\mathbb{E}[|X_i(s) - \bar{X}_i(s)|] + \mathbb{E}|\frac{1}{N}\sum_{j=1}^N b_s(\bar{X}_i(s), \bar{X}_j(s))| \Big) \qquad (15)
\end{aligned}$$

Thus, applying Gronwall's lemma and symmetry, we get:

$$\mathbb{E}[|X_i - \bar{X}_i|_T^*] \leq K(T) \int_0^T ds \mathbb{E}|\frac{1}{N}\sum_{j=1}^N b_s(\bar{X}_i(s), \bar{X}_j(s))|$$

where $K(T)$ is a constant only depending on T. Now, the theorem will follow if we prove that

$$\mathbb{E}|\frac{1}{N}\sum_{j=1}^N b_s(\bar{X}_i(s), \bar{X}_j(s))| \leq \frac{C(T)}{\sqrt{N}}$$

We have:

$$\mathbb{E}|\frac{1}{N}\sum_{j=1}^N b_s(\bar{X}_i(s), \bar{X}_j(s))|^2 = \frac{1}{N^2}\mathbb{E}[\sum_{j,k=1}^N b_s(\bar{X}_i(s), \bar{X}_j(s)) b_s(\bar{X}_i(s), \bar{X}_k(s))] \qquad (16)$$

If $Law(X'(s)) = \bar{f}(s,\cdot)$ and x is fixed, then $\mathbb{E}[b_s(x, X'(s))] = \mathbb{E}[b(x, X'(s))] - \int_{\mathbb{R}^d} b(x,y)\bar{f}(s,dy) = 0$. Thus, for all $j \neq k$:

$$\mathbb{E}[b_s(\bar{X}_i(s), \bar{X}_j(s)) b_s(\bar{X}_i(s), \bar{X}_k(s))] = 0$$

Therefore, there only remains N terms in eq. (16) and, using Jensen inequality (or Cauchy-Schwarz), we obtain the upper bound $\frac{C(T)}{\sqrt{N}}$ we were seeking. □

Remark 3.3. *We can check that the time marginals of the nonlinear process of eq.* (9) *satisfy in a weak sens the following equation:*

$$\partial_t f(t,x) = \frac{1}{2}\Delta_x f(t,x) - div_x(f(t,x)\int b(x,y)f(t,dy)) \tag{17}$$

Indeed, let g be a $C^2(\mathbb{R}^d)$ bounded function. By Ito's formula we have:

$$g(X(t)) = g(X(0)) + \int_0^t g'(X(s))dB(s) + \int_0^t (\frac{1}{2}\Delta g + \int_{\mathbb{R}^d} b(X(s),y)f(s,dy)\nabla f(X(s)))ds$$

Integrating this expression, using the transfer lemma and integrating by parts, like done in Step 2 of the proof of proposition 4.1 we indeed get a weak version of eq. (17). *The introduction of a "nonlinear process", and not only of a nonlinear equation like eq.* (17) *is natural. For each t and k, the joint distribution of $(X_1^N(t), \dots, X_k^N(t))$ is converging to $f(t,\cdot)^{\otimes k}$ but we have proven something stronger: we have convergence at the level of processes.*

3.3 Exchangeable case: general result on propagation of chaos

After the laboratory example, Sznitman gives a broader setting for the propagation of chaos.

Definition 3.7. *Let E be a separable metric space, f_N a sequence of symmetric probabilities on E^N. We say that f_N is f-chaotic, f being a probability on E, if for all $k \geq 1$, $\phi_1, \dots, \phi_k \in C_b(E)$,*

$$\lim_{N\to\infty} \langle f_N, \phi_1 \otimes \dots \phi_k \otimes 1 \dots \otimes 1\rangle = \lim_{N\to\infty} \int_{E^N} \phi(x_1)\dots\phi_k(x_k) f_N(dx_1, \dots, dx_N) \tag{18}$$

$$= \prod_{i=1}^k \langle f, \phi_i\rangle \tag{19}$$

$$= \prod_{i=1}^k \int_E \phi_i(x) f(dx) \tag{20}$$

In the above definition, the laws of the particles 1 to k, which correspond to k canonical coordinates of E^N are need not be independent as long as N is finite. Yet, we say that f_N is f-chaotic if, when N goes to infinity, every finite number k of canonical coordinates are "independent under the limit of f_N.

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = (E^N, \mathcal{B}(E^N), f_N)$. On this space, we define the canonical coordinates $X_k(w) = w_k$ for all $w \in E^N$. Now, we can state the general proposition of [Szn91] on propagation of chaos:

Proposition 3.2. *A probability measure f_N on E^N is f-chaotic if, and only if $\bar{X}_N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i}$ (which is a $M(E)$-valued random variable on E^N) converges in law to the constant random variable f.*

Thus, proposition 3.2 states that convergence in law of the empirical measure $\bar{X}_N$ to the deterministic limit distribution f is equivalent to f_N being f-chaotic.

4 Non-exchangeable case and propagation of independence

We define a system of particles of interacting kernel K with the following system of equations :

$$\begin{cases} dX_i^N(t) = \sum_{j=1}^N w_{ij} K(X_i^N(t) - X_j^N(t))dt \\ X_i^N(0) = X_{i,0}^N \end{cases} . \tag{21}$$

This defintion is a bit less general than eq. (6) since the interaction kernel b of Sznitman does not only depend on the difference $X_i - \bar{X}_i$ and Sznitman deals with the stochastic case by adding a $dW_i(t)$ term. Yet, a preliminary and essential remark in the introduction of [JPS22] is that while they consider for simplicity deterministic systems like eq. (21), their analysis would extend in a straightforward manner to stochastic multi-agent systems with additive noise.

We have seen in the previous section what an exchangeable particle system is. Now, since the weights of the particles are not the same anymore, we are in the non-exchangeable case. In that case, we lose the symmetry property of the random vector $(X_1, \dots, X_N)$.

The weight of the interaction between each pair of agents can be represented by a weighted, a priori non-symmetric graph G_N where each vertex represents an agent. For notational simplicity, we may omit the N superscript. Essential assumptions we make on this interaction graph are eq. (2) and eq. (3).

In the non-exchangeable case, we cannot show a propagation of chaos property for our system eq. (21). Jabin et al. prove a propagation of independence property, which is the analogue of propagation of chaos. The propagation of independence leads to weak convergence of the empirical measure results similar to those obtained with propagation of chaos. The proof requires a careful extension of the usual arguments that we presented in the previous section. First, let us state Lemma 2.1 of Jabin et al., where we propose an extension of the classical [Szn91] McKean SDE associated with the particles system to the case of non-uniform weights.

Lemma 4.1. *Consider the nonlinear system of SDE for $(\bar{X}_1, \ldots, \bar{X}_N)$ given by*

$$\begin{cases} d\bar{X}_i = \sum_{j=1}^N w_{i,j} \int_{\mathbb{R}^d} K(\bar{X}_i - y) \bar{f}_j(t, dy) dt \\ \bar{X}_i(0) = \bar{X}_i^0 \end{cases} . \tag{22}$$

where $K \in W^{1,\infty}$ and we denote $\bar{f}_i(t, \cdot) = Law(\bar{X}_i(t))$. Then, for any random initial data $(\bar{X}_1^0, \ldots, \bar{X}_N^0)$ such that $\mathbb{E}[|\bar{X}_i^0|] < \infty$ there is existence and uniqueness, trajectorial and in law of solutions of eq. (22). In addition, if $\bar{X}_i^0$ are independent then $\bar{X}_i(t)$ are also independent for each $t \in \mathbb{R}_+$.

Proof. The proof is a straightforward extension of theorem 3.1, with nonlinear processes adapted to the non-exchangeable case, that is each process $\bar{X}_i$ has its own law $\bar{f}_i$. The proof of the conservation of independence works just like in remark 3.2. □

The $\bar{X}_i^0$ are not identically distributed but it is essential to assume that they are independent, to ensure that the $\bar{X}_i(t)$ are independent. Since the weights are non-uniform and we do not have symmetry anymore, we must use different laws $\bar{f}_i$, while in the exchangeable case eq. (13), we had a unique $\bar{f}$ because symmetry was propagated.
Moreover, the propagation of independence theorem below (Proposition 2.2 in [JPS22]) extends theorem 3.3. For exchangeable systems, propagation of chaos often directly implies that the limit as $N \to +\infty$ of the 1-particle distribution of the system $f(t, x)$ solves the mean-field equation :

$$\partial_t f(t, x) + \operatorname{div}_x \Big(\bar{w} f(t, x) \int_{\mathbb{R}^d} K(x - y) f(t, dy) \Big) \tag{23}$$

We see in the following proposition an analogous equation for the non-exchangeable case.

Proposition 4.1 (Propagation of independence)**.** *Let $(X_1, \ldots, X_N)$ be a solution to eq. (21) with a convolution kernel $K \in W^{1,\infty}$ and consider the associated laws $f_i(t, \cdot) = Law(X_i(t))$. Assume that the initial positions X_i^0 are independent random variables such that $\mathbb{E}[|X_i^0|^2] < \infty$ and the following uniform estimates (in N) hold*

$$\sup_{1 \le i \le N} \sqrt{\mathbb{E}[|X_i^0|^2]} \le M, \qquad \sup_{1 \le i \le N} \sum_{j=1}^N |w_{ij}| \le C, \tag{24}$$

for every $N \in \mathbb{N}$ and appropriate $M, C \in \mathbb{R}_+$. Consider the solution to the coupled PDE system

$$\begin{cases} \partial_t \bar{f}_i + div_x \Big(\bar{f}_i(t, x) \sum_{j=1}^N w_{i,j} \int_{\mathbb{R}^d} K(x - y) \bar{f}_j(t, dy) \Big) = 0 \\ \bar{f}_i(0, x) = f_i(0, x) \end{cases} . \tag{25}$$

Then, the following estimate holds

$$\sup_{1 \le i \le N} W_1(f_i(t, \cdot), \bar{f}_i(t, \cdot)) \le C_1(t) \sup_{1 \le i,j \le N} |w_{ij}|^{1/2}, \tag{26}$$

for every $t \in \mathbb{R}_+$. We also have a direct control on the empirical measure of the system, namely

$$\mathbb{E} W_1 \Big(\frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}, \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, \cdot) \Big) \le \frac{\tilde{C} C_2(t)}{N^\theta} + C_1(t) \sup_{1 \le i,j \le N} |w_{ij}|^{1/2}, \tag{27}$$

for every $t \in R_+$ and appropriate constants $\tilde{C}, \theta > 0$ depending only on the dimension d. In addition, $C_1(t)$ and $C_2(t)$ depend on $M, C, \|K\|_{W^{1,\infty}}$ and t, and they can be made explicit by

$$C_1(t) = \sqrt{\frac{2}{C}} (e^{2Ct\|K\|_{W^{1,\infty}}} - 1), \quad C_2(t) = (2M^2 + 2C^2 \|K\|_{L^\infty} t^2)^{1/2}$$

We should remark that the initial values of the $\bar{f}_i$ functions, which are the laws to which the laws of the particles f_i converge, are taken equal to the initial values of the f_i.

We should interpret this proposition in the following way. Because of eq. (3), eq. (26) entails that the laws of the particles f_i converge to the laws f_i which satisfy eq. (25) with respect to the Wasserstein distance. Similarly, the empirical measure of the system, which is a random measure, converges to the mean of the limit laws $\bar{f}_i$. We will, like in [Szn91], prove in reality a convergence at the level of processesX_i and $\bar{X}_i$, which will imply a convergence of the laws f_i and $\bar{f}_i$.

Eventually, we note that the assumptions on the kernel K are the same as [Szn91] on kernel b: they are Lipschitz bounded.

Proof. The $X_i(t)$ are the positions of the particles over time. Let $\bar{X}_i(t)$, $i \in \{1, \ldots, N\}$ be the unique solutions of eq. (22) with initial conditions $\bar{X}_i(0) = X_i^0$. The proof is based on a coupling of the X_i to the $\bar{X}_i$. Let us denote the laws $\bar{f}_i(t, \cdot) = Law(\bar{X}_i(t))$ and consider the sub-σ-algebras $\mathcal{F}_i(t) = \sigma(\bar{X}_i(t))$, for each $t \in \mathbb{R}_+$ and $i \in \{1, \ldots, N\}$. The proof is based on the introduction of the process $\bar{X}_i(t)$, although it does not appear in the statement of the proposition, and showing in the proof that this random process has its laws $\bar{f}_i(t,\cdot)$ which verify eq. (25).

Step 1: First, we show that

$$\sup_{1\leq i\leq N} \mathbb{E}|X_i - \bar{X}_i| \leq C_1(t) \sup_{1\leq i,j\leq N} |w_{ij}|^{1/2} \tag{28}$$

Showing such a claim is the exact analogue of theorem 3.3 Since the variables $\bar{X}_i$ and $\bar{X}_j$ are independent by lemma 4.1, for any $j \in \{1,\dots,N\}$ with $j \neq i$, we have:

$$\int_{\mathbb{R}^d} K(\bar{X}_i - y)\bar{f}_j(t,dy) = \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)] \tag{29}$$

where we denote $\mathbb{E}_i = \mathbb{E}[\cdot|\mathcal{F}_i(t)]$. Taking the difference between eq. (21) and eq. (22), we get:

$$\begin{aligned}\frac{d}{dt}(X_i - \bar{X}_i) &= \sum_{j=1}^N w_{ij}(K(X_i - X_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)])\\ &= \sum_{j=1}^N w_{ij}(K(X_i - X_j) - K(\bar{X}_i - \bar{X}_j)) + \sum_{j=1}^N w_{ij}(K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)])\end{aligned} \tag{30}$$

Taking the total expectation we get:

$$\frac{d}{dt}\mathbb{E}|X_i - \bar{X}_i| \leq 2C[K]_{Lip} \sup_{j\leq N} \mathbb{E}|X_j - \bar{X}_j| + \mathbb{E}\left[\left|\sum_{j=1}^N w_{ij}(K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)])\right|\right] \tag{31}$$

In theorem 3.3, we controlled the term $\mathbb{E}\left[\left|\sum_{j=1}^N w_{ij}(K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)])\right|\right]$, which was denoted $\mathbb{E}|\frac{1}{N}\sum_{j=1}^N b_s(\bar{X}_i(s),\bar{X}_j(s))|$, with its second moment. We do just the same here. We develop the square in the expectation and make most terms of the double sum disappear. For that, now that all $\bar{X}_i$s do not have the same law (which was the case in theorem 3.3), we must use the fact that $K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]$ and $K(\bar{X}_i - \bar{X}_k) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_k)]$ are conditionally independent given $\mathcal{F}_i$. That way, it comes:

$$\mathbb{E}\left[\left|\sum_{j=1}^N w_{ij}(K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)])\right|\right] \leq 8C\|K\|_{L^\infty}^2 \sup_{j\leq N} |w_{ij}|$$

Then, just like in theorem 3.3, we can use Jensen's (or Cauchy-Schwarz) inequality to upper bound the first moment and finally apply Gronwall's lemma. This gives:

$$\sup_{1\leq i\leq N} \mathbb{E}|X_i - \bar{X}_i| \leq C_1(t) \sup_{1\leq i,j\leq N} |w_{ij}|^{1/2} \tag{32}$$

Step 2: We now must show that the laws $\bar{f}_i(t,\cdot)$ verify eq. (25). Let $\phi \in C_c^1(\mathbb{R}^d)$ be a test function. We have, by derivation under the $\int$ sign, and then using eq. (22) and eq. (29):

$$\frac{d}{dt}\int_{\mathbb{R}^d} \phi(x)\bar{f}_i(t,x)dx = \frac{d}{dt}\mathbb{E}[\phi(\bar{X}_i)] = \mathbb{E}[\nabla\phi(\bar{X}_i)\cdot\frac{d\bar{X}_i}{dt}] = \sum_{j=1}^N w_{ij}\mathbb{E}[\nabla\phi(\bar{X}_i)\cdot\mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]] \tag{33}$$

In the above expression, since $\mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)] \in \mathbb{R}$, we can replace $\nabla\phi(\bar{X}_i)$ by $div\phi(\bar{X}_i)$.

By definition of the conditional expectation, $\mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)] = \mathbb{E}[K(\bar{X}_i - \bar{X}_j)|\bar{X}_i] \circ \bar{X}_i$. Here, time is fixed so we can introduce a new notation to highlight the randomness in $\bar{X}_i$: we denote the random variable $\bar{X}_i : v \in \Omega \to \bar{X}_i(v) \in \mathbb{R}^d$. Here, $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space we do not want to give more details about since it is not the center of our argument. We can now write:

$$\begin{aligned}\mathbb{E}[\nabla\phi(\bar{X}_i)\cdot\mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]] &= \int_\Omega div\phi(\bar{X}_i(v))\mathbb{E}[K(\bar{X}_i(v) - \bar{X}_j)|\bar{X}_i = \bar{X}_i(v)]\mathbb{P}(dv) && (34)\\ &= \int_\Omega div\phi(\bar{X}_i(v))\mathbb{E}[K(\bar{X}_i(v) - \bar{X}_j)]\mathbb{P}(dv) && (35)\\ &= \int_\Omega div\phi(\bar{X}_i(v))\Big(\int_{\mathbb{R}^d} K(\bar{X}_i(v) - y)\bar{f}_j(t,dy)\Big)\mathbb{P}(dv) && (36)\\ &= div\phi(x)K(x-y)\bar{f}_i(t,dx)\bar{f}_j(t,dy) && (37)\end{aligned}$$

The second line comes from the independence between $\bar{X}_i$ and $\bar{X}_j$. The last line is key and obtained with the transfer lemma since $\forall B \in \mathcal{B}(\mathbb{R}^d)$, $((\bar{X}_i)_*\mathbb{P})(B) = \mathbb{P}(\bar{X}_i \in B) = \bar{f}_i(t,B)$. Putting this last expression in eq. (33) and integrating by parts gives a weak version of eq. (25).

Step 3: We now must show that eq. (28) implies bounds eq. (26) and eq. (27).

In [JPS22], this step of the proof is made using the dual representation of the Wasserstein distance of order 1. Here, we voluntarily propose a version of the proof using the primal representation of the distance to highlight the chosen couplings necessary to derive the bounds.

Since $Law(X_i(t)) = f_i(t,\cdot)$ and $Law(\bar{X}_i(t)) = \bar{f}_i(t,\cdot)$, we have by definition of the Wasserstein distance:

$$W_1(f_i(t,.), \bar{f}_i(t,.)) = \inf_{\mathbb{P}_X = f_i(t,.), \mathbb{P}_Y = \bar{f}_i(t,.)} \mathbb{E}[|X - Y|] \leq \mathbb{E}[|X_i(t) - \bar{X}_i(t)|]$$

where $\mathbb{E}$ in the right-hand side is with respect to the joint probability distribution of $(X_i(t), \bar{X}_i(t))$. Using eq. (32), we directly obtain the first bound we want in step 3, which is eq. (26).

In the same way, we denote the respective empirical measures on $\mathbb{R}^d$ of the X_is and the $\bar{X}_i$s:

$$\mu_N(t, dx) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i(t)}(dx), \bar{\mu}_N(t, dx) = \frac{1}{N}\sum_{i=1}^N \delta_{\bar{X}_i(t)}(dx)$$

These random variables take their values in the space of measures on $\mathbb{R}^d$, denoted $M(\mathbb{R}^d)$. We can take the Wasserstein distance between any pair of outcomes of the two random variables. Then, we can take the expectation of this distance, which necessarily is with respect to the randomness of the X_is and the $\bar{X}_i$s:

$$\mathbb{E}_{(X_i(t),\bar{X}_i(t))_{i\geq 1}}[W_1(\mu_N(t,\cdot), \bar{\mu}_N(\cdot))] = \mathbb{E}_{(X_i(t),\bar{X}_i(t))_{i\geq 1}}[\inf_{\mathbb{P}_X = \mu_N(t,.), \mathbb{P}_Y = \bar{\mu}_N(t,.)} \mathbb{E}[|X - Y|]] \tag{38}$$

$$\leq \mathbb{E}_{(X_i(t),\bar{X}_i(t))_{i\geq 1}}[\mathbb{E}[|X - Y|]] \tag{39}$$

where the coupling of (X, Y) in the right-hand side of the second line is the following: $X \sim U\{X_1(t), \dots, X_N(t)\}$, $Y \sim U\{\bar{X}_1(t), \dots, \bar{X}_N(t)\}$ and $\{X = X_i(t)\} = \{Y = \bar{X}_i(t)\}$. It is easy to check that the marginal laws of follow respectively $\mu_N(t,.)$ and $\bar{\mu}_N(t,.)$. Thus: $\mathbb{E}[|X - Y|] = \frac{1}{N}\sum_{i=1}^N |X_i(t) - \bar{X}_i(t)|$. Putting it into eq. (38), we get:

$$\mathbb{E}_{(X_i(t),\bar{X}_i(t))_{i\geq 1}}[W_1(\mu_N(t,\cdot), \bar{\mu}_N(t,\cdot))] \leq \mathbb{E}_{(X_i(t),\bar{X}_i(t))_{i\geq 1}} \frac{1}{N}\sum_{i=1}^N |X_i(t) - \bar{X}_i(t)| \leq \sup_{i\geq 1} \mathbb{E}|X_i(t) - \bar{X}_i(t)| \tag{40}$$

Now, by the independence of the $\bar{X}_i$s, we have, through a straightforward extension of the proofs in Theorem 1 in [FG13] which consider the exchangeable case, that, for some $\tilde{C}, \theta > 0$:

$$\mathbb{E}W_1\left(\bar{\mu}_N(\cdot)), \frac{1}{N}\sum_{i=1}^N \bar{f}_i(t,\cdot)\right) \leq \frac{\tilde{C}}{N^\theta} \sup_{i\leq N} \mathbb{E}[|X_i(t)|^2]^{1/2} \leq \frac{\tilde{C}C_2(t)}{N^\theta} \tag{41}$$

Indeed, the second inequality comes from the fact that, integrating eq. (21) with respect to time and using the fact that K is bounded :

$$\mathbb{E}[|X_i|^2] \leq \mathbb{E}[(|X_i^0|^2 + C\|K\|_\infty t)^2] \leq C_2(t)^2$$

Applying the triangular inequality to eq. (40) and eq. (41), we deduce eq. (27). □

5 Main theorem

5.1 Stating the theorem

Consider a sequence $(X_i(t))_{1\leq i\leq N}$ solving the following system:

$$\begin{cases} \frac{dX_i}{dt} = \sum_{j=1}^N w_{ij} K(X_i - X_j), \\ X_i(0) = X_i^0. \end{cases}$$

Where $K \in W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d)$ and the weights $(\{w_{ij}\}_{ij})_N$ satisfy 2 and 3. Also, assume that the X_i^0 are independent random variables with laws f_i^0. Finally, assume that:

$$\sup_{N\in\mathbb{N}} \sup_{1\leq i\leq N} \mathbb{E}[|X_i^0|^2] < \infty, \text{ and } \sup_{N\in\mathbb{N}} \sup_{1\leq i\leq N} ||f_i^0||_{W^{1,1}\cap W^{1,\infty}(\mathbb{R}^d)} < \infty.$$

Then, we obtain the two following results.

(1) There exists $w \in L_\xi^\infty M_\zeta \cap L_\zeta^\infty M_\xi$ and $f \in L^\infty([0, t_*] \times [0,1], W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d))$, for any $t_* > 0$, such that (w, f) solves:

$$\partial_t f(t,x,\xi) + div_x \left(f(t,x,\xi)\int_0^1 w(\xi, d\zeta)\int_{\mathbb{R}^d} K(x-y) f(t,dy,\zeta)\right) = 0. \tag{42}$$

(2) Up to the extraction of a subsequence,

$$\sup_{0\leq t\leq t_*} \mathbb{E}W_1\left(\int_0^1 f(t,.,\xi)d\xi, \mu_N(t,.)\right) \to 0. \tag{43}$$

Where W_1 is the usual Wasserstein distance on $\mathbb{R}^d$ and $\mu_N(t,x) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i(t)}(x)$ is the empirical measure associated to $(X_i(t))_{1\leq i\leq N}$.

5.2 Quick overview of the proof

Using everything we have introduced in the previous parts, the proof is not too complicated. Indeed, obtaining eq. (42) is a straightforward consequence of theorem 2.3, along with some well-known results from the theory of partial differential equations.

Proving the second result requires a more involved reasoning. The way to tackle this part of the proof is starting by defining $(\bar{f}_i)_{1\le i\le N}$, for any $N \in \mathbb{N}$, which solve eq. (42), but for the interaction graph w associated with the system of N particles. Doing this allows us to define $\frac{1}{N}\sum_{i=1}^N \bar{f}_i(t,.)$ as a new probability measure in $\mathbb{R}^d$. Applying the triangular inequality with this new measure to eq. (43), we'll obtain two terms that should converge to zero. For one of them, we'll simply use proposition 4.1 in a straightforward manner. For the other term, we'll rely on more elaborate arguments and theorem 2.2 to finish the proof.

5.3 An attempt of a proof

We start our proof by defining the empirical graphon and the empirical law at time zero:

$$w_N(\xi,\zeta) = \sum_{1\le i,j\le N} N w_{ij} \mathbb{1}_{]\frac{i-1}{N},\frac{i}{N}]}(\xi)\mathbb{1}_{]\frac{j-1}{N},\frac{j}{N}]}(\zeta),$$

$$f_N^0(x,\xi) = \sum_{i=1}^N f_i^0(x)\mathbb{1}_{[\frac{i-1}{N},\frac{i}{N}[}(\xi).$$

We now wish to apply theorem 2.3. The necessary hypothesis to use it are satisfied, indeed:

$$\int_0^1 |w_N(\xi,\zeta)|d\zeta = \int_0^1 \left|\sum_{1\le i,j\le N} N w_{ij} \mathbb{1}_{]\frac{i-1}{N},\frac{i}{N}]}(\xi)\mathbb{1}_{]\frac{j-1}{N},\frac{j}{N}]}(\zeta)\right| d\zeta =$$

$$\sum_{i,j=1}^N N w_{ij} \mathbb{1}_{]\frac{i-1}{N},\frac{i}{N}]}(\xi)\int_0^1 \mathbb{1}_{]\frac{j-1}{N},\frac{j}{N}]}(\zeta)d\zeta = \sum_{i,j=1}^N w_{ij}\mathbb{1}_{]\frac{i-1}{N},\frac{i}{N}]}(\xi).$$

Thus, we obtain that:

$$\sup_{\xi\in[0,1]}\int_0^1 |w_N(\xi,\zeta)|d\zeta = \sup_{1\le i\le N}\sum_{j=1}^N w_{ij} = O(1) \text{ by eq. (2).}$$

The same reasoning works for $\int_0^1 |w_N(\xi,\zeta)|d\xi$, so condition (i) of theorem 2.3 is fulfilled. For condition (ii), we simply notice that by hypothesis of the main theorem:

$$\sup_{N\in\mathbb{N}} ||f_N^0||_{L^\infty_\xi(W_x^{1,1}\cap W_x^{1,\infty})} = \sup_{N\in\mathbb{N}}\sup_{\xi\in[0,1]} ||f_N^0(.,\xi)||_{W_x^{1,1}\cap W_x^{1,\infty}} =$$

$$\sup_{N\in\mathbb{N}}\sup_{\xi\in[0,1]} ||\sum_{i=1}^N f_i^0(.)\mathbb{1}_{[\frac{i-1}{N},\frac{i}{N}[}(\xi)||_{W_x^{1,1}\cap W_x^{1,\infty}} = \sup_{N\in\mathbb{N}}\max_{1\le i\le N} ||f_i^0||_{W_x^{1,1}\cap W_x^{1,\infty}} < +\infty.$$

Therefore, theorem 2.3 applies and it gives us a subsequence $(\varphi(n))_n$, an extended graphon $w \in L^\infty_\zeta M_\xi \cap L^\infty_\xi M_\zeta$ and $f^0 \in L^\infty_\xi(W_x^{1,1}\cap W_x^{1,\infty})$ such that for any tree $T\in\mathbf{T}$ and $1\le p<\infty$ we have that:

$$\tau(T, w_{\varphi(N)}, f^0_{\varphi(N)}) \to \tau(T,w,f^0) \text{ in } L^p_{loc}(\mathbb{R}^{d|T|}),$$

And we achieve eq. (42) using a classic fixed point argument to obtain the existence of $f\in L^\infty([0,t_*]\times[0,1], W^{1,1}\cap W^{1,\infty}(\mathbb{R}^d))$ such that (w,f) is a weak solution of eq. (42).

For the second result of the theorem, we must define, for all $N\in\mathbb{N}$, $\bar{f}_i$ that solves the system:

$$\begin{cases} \partial_t \bar{f}_i + div_x\left(\bar{f}_i(t,x)\sum_{j=1}^N w_{ij}\int_{\mathbb{R}^d} K(x-y)\bar{f}_j(t,dy)\right) \\ \bar{f}_i(0,x) = f_i^0(x). \end{cases}$$

Thus, applying the triangular inequality to W_1:

$$\mathbb{E}\left[W_1\left(\int_0^1 f(t,.,\xi)d\xi, \mu_N(t,.)\right)\right] \le W_1\left(\int_0^1 f(t,.,\xi)d\xi, \frac{1}{N}\sum_{i=1}^N \bar{f}_i(t,.)\right) + \mathbb{E}\left[W_1\left(\frac{1}{N}\sum_{i=1}^N \bar{f}_i(t,.), \mu_N(t,.)\right)\right].$$

Starting with the second term above, we'll use proposition 4.1 to argue that it converges to zero as N tends to infinity. Indeed, it is simple to prove that all the estimates necessary to use proposition 4.1 hold, so we obtain the following result:

$$\mathbb{E}W_1\Big(\frac{1}{N}\sum_{i=1}^N \delta_{X_i(t)}, \frac{1}{N}\sum_{i=1}^N \bar{f}_i(t,\cdot)\Big) \le \frac{\tilde{C}C_2(t)}{N^\theta} + C_1(t)\sup_{1\le i,j\le N}|w_{ij}|^{1/2}.$$

Where $\tilde{C}, \theta > 0$ and C_1, C_2 are bounded in $[0, t_*]$. Then, eq. (3): $\max_{1\leq i,j\leq N} |w_{ij}| = o(1)$ allows us to state that this term tend to zero, even when taking the supremum over t.

For the other term we may start by defining:

$$f_N(t,x,\xi) = \sum_{i=1}^{N} \bar{f}_i(t,x)\mathbb{1}_{[\frac{i-1}{N},\frac{i}{N}[}(\xi).$$

One may observe that:

$$\int_0^1 f_N(t,x,\xi)d\xi = \frac{1}{N}\sum_{i=1}^{N}\bar{f}_i(t,x), \text{ and}$$

$$f_N(0,x,\xi) = f_N^0(x,\xi).$$

It is immediate to show that (w_N, f_N) is a weak solution of eq. (42), so we may invoke theorem 2.2 to obtain the following inequality:

$$\left|\left|\int_0^1 (f - f_N)(t,.,\xi)d\xi\right|\right|_{L_x^2} \leq \frac{C(t)}{(\log|\log||\tau(.,w_N,f_N^0) - \tau(.,w,f^0)||_\lambda|)_+^{\frac{1}{2}}}.$$

Now, using the fact that $\tau(T, w_N, f_N^0)$ converges to $\tau(T, w, f^0)$ in L^p_{loc} up to a subsequence and eq. (5), we can prove that up to a subsequence we have the following limit when $N \to +\infty$:

$$||\tau(.,w_N,f_N^0) - \tau(.,w,f^0)||_\lambda \to 0.$$

Such a result implies, by the previous inequality, that $\frac{1}{N}\sum_{i=1}^{N}\bar{f}_i(t,.) = \int_0^1 f_N(t,.,\xi)d\xi$ converges to $\int_0^1 f(t,.,\xi)d\xi$ in L_x^2. Finally, since the L^2 norm locally dominates the Wasserstein distance, we have that up to a subsequence:

$$W_1\left(\int_0^1 f(t,.,\xi)d\xi, \frac{1}{N}\sum_{i=1}^{N}\bar{f}_i(t,.)\right) \to 0.$$

Thus concluding our proof.

6 Perspectives

In this final part, we'll discuss what we think are the next steps to better understand [JPS22]. Indeed, we are aware that we could only go so far in the understanding of this article, and there remains a few unresolved questions.

The main innovation of this paper is indubitably the extended graphon. Our presentation of this object does a good job at presenting the historical framework of graph convergence theory, we are quite comfortable seeing the τ operator as a generalization of the homomorphism density. However, we still don't fully grasp why it was defined specifically this way. To this end, we think that an exploration of [KLS16] and its idea of s-graphons would be a good starting place. Consequently, the proofs for some important results on extended graphons remain a bit mysterious to us. For instance, we believe our presentation of theorem 2.3 is a bit shallow.

For the extensions of [Szn91], we believe that the reading of other works such as [FG13] would have given us a more comfortable understanding of the subject. Indeed, doing so would perhaps have enabled us to give a better explanation of proposition 4.1.

A suggestion on which we would have liked to follow up is the simulation of a N particles system using a simple kernel K satisfying the hypotheses of the main theorem. We believe that such a direct observation of the systems we are studying would have given us a better intuitive feeling of what's going on. Also, doing so would have opened the door to getting a more visual representation of our problem using images of our simulations. This would have been a great way to tell a richer story of all the discoveries made on [JPS22].

Lastly, one thing we think we would have enjoyed doing but didn't have the time for is exploring the ideas [JPS22] presents in its appendix. Indeed, working at the intersection between mathematics and neuroscience seems like a promising research area to us. Moreover, having the application in mind might have helped us ask better questions to the mathematics.

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