Stand on the shoulders of fat-tailed giants riding cycles.

Basile Grassi

Paris School of Economics and Université Paris 1 Panthéon Sorbonne

May 15, 2012

Abstract

This small model experiences aggregate fluctuations from idiosyncratic shocks. An endogeneous fat-tailed distribution of firms size arises because of selection of the best ideas *à la* Alvarez *et al.* (2007). An entrepreneur update its idea only if it improves its technology. This leads to growth and a fat-tailed distribution of firms sizes. From this, idiosyncratic shocks on ability to manage this ideas leads to aggregate fluctuations as in Gabaix (2011).

1 Selection of ideas

Following Alvarez *et al.* (2007) and Lucas (2008), the technology frontier evolves according a differential equation. This differential equation deponds upon the way the flow of ideas is modeled. I will considered two cases.

Let us denote X the cost level (the technology average productivity level will be $1/X$) and $G(x,t)$ the counter cumulative distribution function, i.e.

$$\mathbb{P}_t\{X \geq x\} = G(x,t)$$

We will denote X by idea or by cost level.

1.1 Poisson arrivals with internal source of ideas

Each entrepreneurs at t has inherited cost level and receive a new idea drawn from the current distribution $G(x,t)$ at a Poisson arrrival rate α. She keeps the new idea only if it is a better idea that is to say that the new cost is lower the current one.

For fixed x, I follow Alvarez *et al.* (2007) to motivate an ordinary differential equation:

$$G(x,t+h) = G(x,t)\mathbb{P}\{\text{no lower cost arrives in } (t,t+h)\}$$

Then

$$\begin{aligned}
\mathbb{P}\{\text{no lower cost arrives in } (t,t+h)\} &= \mathbb{P}\{\text{no ideas arrives in } (t,t+h)\} \\
&+ \mathbb{P}\{\text{one idea } > x \text{ arrives from } G \text{ in } (t,t+h)\} \\
&+ \mathbb{P}\{\text{more than on idea } > x \text{ arrives in in } (t,t+h)\} \\
&= 1 - \alpha h + \alpha h G(x,t) + o(h)
\end{aligned}$$

which yields after some computation (rearranging and dividing through by h, and letting $h \to 0$):

$$\frac{\partial \log(G(x,t))}{\partial t} = -\alpha[1 - G(x,t)]$$

The solution of this equation is

$$G(x,t) = \frac{G(x,0)}{G(x,0) + e^{\alpha t}(1 - G(x,0))}$$

We are a focusing on a "balanced growth path" which is to say whether there is a function φ and a number ν such that

$$G(x,t) = \varphi(e^{\nu t}x)$$

which after choosing $\nu = \alpha$ and some computations leads to

$$\varphi'(x) = -\frac{1}{x}\varphi(x)(1 - \varphi(x))$$

which has a solution $\varphi : \mathbb{R}_+ \to [0,1]$

$$\varphi(x) = \frac{1}{1+\phi x} = \mathbb{P}\{X \geq x\}$$

where ϕ is a parameter.

Alavarez *et al.* (2007) shows that given an initial distribution $G(x,0)$, $\lim_{t\to 0} G(e^{-\alpha t}x,t) = \varphi(x)$ and that $\phi = -G_x(0,0)$.

Thus the distribution of cost X is characterized by the counter cumulative distribution function (CCDF) $\varphi(.)$ and then the average TFP is defined by $Y = \frac{1}{X}$. The distribution of Y is characterized by a CCDF

$$
\begin{aligned}
\mathbb{P}(Y \geq y) = \mathbb{P}(\frac{1}{X} \geq y) &= \mathbb{P}(\frac{1}{y} \geq X) \\
&= 1 - \mathbb{P}(X > \frac{1}{y}) \\
&= 1 - \frac{1}{1+\phi\frac{1}{y}} \\
&= \frac{\phi\frac{1}{y}}{1+\phi\frac{1}{y}} \\
\mathbb{P}(Y \geq y) &= \frac{1}{1+\frac{y}{\phi}} := g(y)
\end{aligned}
$$

1.2 Poisson arrivals with external source of ideas and growth in the arrival rate

Leads to an asymptotic exponential distribution with parameter $-\beta H'(0)$ of $G(x,t)$.

2 A simple model of aggregate fluctuation from idiosyncratic shock

2.1 Environment

This economy is populated by N entrepreneurs who draw a ideas φ from the stationnary distribution of ideas characterized by $g(.)$, they keep this ideas during the two periods of this economy.

Their are endowned with a tehcnology from which they produce zn^α unit of final good with n unit of input at cost w per unit. The TFP term z is at the first period $z_1 = \varphi^{1-\alpha}$ and at the second period $z_2 = f(\varphi, x)^{1-\alpha}$ where x is drawn from a given distribution characterized by a probability distribution $h(.)$[1] and $f(.,.)$ will be defined later.

For a given productivity level z, each entrepreneur maximize its profit. Thus the program of the entrepreneur is:

$$\pi(z, w) = \max_{n \geq 0}\{zn^\alpha - wn\}$$

which leads to

$$\begin{aligned}
\text{Input demand: } n^*(z,w) &= \Big(\frac{z\alpha}{w}\Big)^{\frac{1}{1-\alpha}} \\
\text{Supply of final good: } y^*(z,w) &= z^{\frac{1}{1-\alpha}}\big(\frac{\alpha}{w}\big)^{\frac{\alpha}{1-\alpha}} \\
\text{Profit: } \pi^*(z,w) &= z^{\frac{1}{1-\alpha}}\big(\frac{1}{w}\big)^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}\big(\frac{1}{\alpha} - 1\big)
\end{aligned}$$

For a given ideas y and a given ability x the output at the first period is $y_1 = \varphi\big(\frac{1}{w}\big)^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}(\frac{1}{\alpha} - 1)$ and $y_2 = f(\varphi, x)\big(\frac{1}{w}\big)^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}(\frac{1}{\alpha} - 1)$.

At this point, an entrepreneur $i \in \{1..N\}$ is characterized by (φ_i, x_i) where φ_i stand for its idea and x_i the ability of managing this idea in the second period. She produce y_1^i and y_2^i in the first and the second period respectively.

2.2 Aggregate volatility

The total GDP of this economy is just the sum over the entrepreneurs of each production y_t^i:

$$Y_t = \sum_{i=1}^{N} y_t^i$$

[1] x could be interpreted as the ability of a given entrepreneur to manage the ideas y. Here we suppose that the draw of y and the draw of x are independant.

The GDP growth is defined by

$$\frac{\Delta Y_{t+1}}{Y_t} = \frac{1}{Y_t}\sum_{i=1}^{N} \Delta y^i_{t+1}$$

The absolute growth of a firm own by the entrepreneur i is

$$\Delta y^i_{t+1} = (\frac{\alpha}{w})^{\frac{\alpha}{1-\alpha}}\Big(f(\varphi_i, x_i) - \varphi_i\Big)$$

Let us assume that $f(\varphi_i, x_i) = \varphi_i x_i$

$$\begin{aligned}\Delta y^i_{t+1} &= \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}\varphi_i(x_i - 1)\\ &= y^i_{idea}(x_i - 1)\end{aligned}$$

where $y^i_{idea} := \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}\varphi_i$[2]. From this equation, one can see that the growth of firm follow a Gibrat's law - the growth of firm does not depend on its size[3].

Finally, the GDP is thus

$$\frac{\Delta Y_{t+1}}{Y_t} = \frac{1}{Y_t}\sum_{i=1}^{N} y^i_{idea}(x_i - 1)$$

from which, I can derive the GDP volatility

$$\sigma_{GDP} = \sqrt{\mathbb{V}ar_t(\frac{\Delta Y_{t+1}}{Y_t})} = \Big(\sum_{i=1}^{N}\sigma^2(\frac{y^i_{idea}}{Y_t})^2\Big)^{1/2}$$

where σ^2 is the variance of the ability shock $h(.)$.

[2]Here $y^i_{idea} = y^i_t$

[3]This will be false as soon as in the first period a ability shock is drawn by the entrepreneur. In this last case $\Delta y^i_{t+1} = y^i_{idea}(x^i_{t+1} - x^i_t)$. in this last case all the following result are still true.

Let us compute the distribution from which y^i_{idea} are drawn.

$$\begin{aligned}
\mathbb{P}\{y_{idea} > y\} &= \mathbb{P}\{\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}\varphi > y\} \\
&= \mathbb{P}\left\{\varphi > \frac{y}{\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}}\right\} \\
&= g\left(\frac{y}{\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}}\right) \\
&= \frac{1}{1+\frac{y}{\phi(\frac{\alpha}{w})^{\frac{\alpha}{1-\alpha}}}}
\end{aligned}$$

this distribution does not have a variance.

From this one can apply Proposition 2 from Gabaux (2011) and conclude that $\sigma_{GDP} \sim \frac{v}{\ln N}\sigma$, where v is a random variable followinf a distribution which does not depend on N or σ[4].

[4] v follow the square root of a stable lévy distribution with exponent 1/2.