Best Ideas, Idiosyncratic Shocks and Aggregate Fluctuations

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Abstract

This paper derives a model of aggregate fluctuations from idiosyncratic shocks. An endogeneous fat-tailed distribution of firms size arises because of selection of the best ideas *à la* Alvarez *et al.* (2008) . An entrepreneur update its idea only if it improves its technology. This leads to growth and a fat-tailed distribution of firms sizes. From this, idiosyncratic shocks leads to aggregate fluctuations as in Gabaix (2011) .

1 Selection of ideas

Following Alvarez *et al.* (2008) and Lucas (2009), the technology frontier evolves according to a differential equation. This differential equation depends on the way the flow of ideas is modeled.

Let us denote X the cost level (the average TFP will be $1/X$) and $G(x,t)$ the counter cumulative distribution function, i.e.

$$\mathbb{P}_t\{X \geq x\} = G(x,t)$$

We will call X an idea or a cost level.

1.1 Poisson arrivals with internal source of ideas

An entrepreneur at t inherits cost level and receives a new idea drawn from the current distribution $G(x,t)$ at a Poisson arrrival rate α. She keeps the new idea only if it is a better idea that is to say that the new cost is lower the current one.

For fixed x, I follow Alvarez *et al.* (2008) to motivate an ordinary differential equation:

$$G(x,t+h) = G(x,t)\mathbb{P}\{\text{no lower cost arrives in } (t,t+h)\}$$

Then

$$\begin{aligned}\mathbb{P}\{\text{no lower cost arrives in } (t,t+h)\} &= \mathbb{P}\{\text{no ideas arrives in } (t,t+h)\}\\ &+ \mathbb{P}\{\text{one idea } > x \text{ arrives from } G \text{ in } (t,t+h)\}\\ &+ \mathbb{P}\{\text{more than on idea } > x \text{ arrives in in } (t,t+h)\}\\ &= 1-\alpha h+\alpha h G(x,t)+o(h)\end{aligned}$$

which yields after some computations (rearranging and dividing through by h, and letting $h \to 0$):

$$\frac{\partial \log(G(x,t))}{\partial t} = -\alpha[1-G(x,t)]$$

The solution of this equation is

$$G(x,t) = \frac{G(x,0)}{G(x,0)+e^{\alpha t}(1-G(x,0))}$$

We are a focusing on a "balanced growth path" which is to say whether there is a function φ and a number ν such that

$$G(x,t) = \varphi(e^{\nu t}x)$$

which after choosing $\nu = \alpha$ and some computations leads to

$$\varphi'(x) = -\frac{1}{x}\varphi(x)(1-\varphi(x))$$

which has a solution $\varphi : \mathbb{R}_+ \to [0,1]$

$$\varphi(x) = \frac{1}{1+\phi x} = \mathbb{P}\{X \geq x\}$$

where ϕ is a parameter.

Alavarez *et al.* (2008) shows that given an initial distribution $G(x, 0)$, $\lim_{t\to 0} G(e^{-\alpha t}x, t) = \varphi(x)$ and that $\phi = -G_x(0, 0)$.

Thus the distribution of cost X is characterized by the counter cumulative distribution function (CCDF) $\varphi(.)$ and then the average TFP is defined by $Y = \frac{1}{X}$. The distribution of Y is characterized by a CCDF

$$\begin{aligned}
\mathbb{P}(Y \geq y) = \mathbb{P}(\frac{1}{X} \geq y) &= \mathbb{P}(\frac{1}{y} \geq X) \\
&= 1 - \mathbb{P}(X > \frac{1}{y}) \\
&= 1 - \frac{1}{1 + \phi\frac{1}{y}} \\
&= \frac{\phi\frac{1}{y}}{1 + \phi\frac{1}{y}} \\
\mathbb{P}(Y \geq y) &= \frac{1}{1 + \frac{y}{\phi}} := g(y)
\end{aligned}$$

2 A simple model of aggregate fluctuations from idiosyncratic shocks

2.1 Environment

This economy is populated by N entrepreneurs who draw ideas φ from the stationary distribution of ideas characterized by $g(.)$, they keep this ideas during the two periods of this economy.

They are endowned with a technology from which they produce zn^α units of final good with n units of input at cost w per unit of final output. The TFP term z is at the first period $z_1 = \varphi^{1-\alpha}$ and at the second period $z_2 = f(\varphi, x)^{1-\alpha}$ where x is drawn from a given distribution characterized by a probability distribution $h(.)$[1] and $f(.,.)$ will be defined later[2].

For a given productivity level z, each entrepreneur maximizes its profit. Thus the program of the entrepreneur is:

$$\pi(z,w) = \max_{n\geq 0}\{zn^{\alpha} - wn\}$$

which leads to

$$\begin{aligned}
\text{Input demand: } n^*(z,w) &= \left(\frac{z\alpha}{w}\right)^{\frac{1}{1-\alpha}} \\
\text{Supply of final good: } y^*(z,w) &= z^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \\
\text{Profit: } \pi^*(z,w) &= z^{\frac{1}{1-\alpha}}\left(\frac{1}{w}\right)^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}\left(\frac{1}{\alpha} - 1\right)
\end{aligned}$$

For a given ideas y and a given ability x the output at the first period is $y_1 = \varphi\left(\frac{1}{w}\right)^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}\left(\frac{1}{\alpha} - 1\right)$ and $y_2 = f(\varphi,x)\left(\frac{1}{w}\right)^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}\left(\frac{1}{\alpha} - 1\right)$.

At this point, an entrepreneur $i \in \{1,2,\ldots,N\}$ is characterized by (φ_i, x_i) where φ_i stand for the idea and x_i the idiosyncratic shock in the second period. She produces y_1^i and y_2^i in the first and the second period respectively.

2.2 Aggregate volatility

The total GDP of this economy is just the sum over the entrepreneurs of each production y_t^i:

$$Y_t = \sum_{i=1}^{N} y_t^i$$

[1] x could be interpreted as the relative growth of its ideas compare to the balanced growth of each ideas. Here we suppose that the draw of y and the draw of x are independant.

[2] The hypothesis that $z_1 = \varphi^{1-\alpha}$ and $z_2 = f(\varphi,x)^{1-\alpha}$ is critical. Because this hypothesis leads to a one to one mapping between the tail parameter of the size distribution of firms and the one of the productivity distribution. Whitout this hypothesis the tail parameter will be $\frac{1}{1-\alpha} \approx 3$ which implies a thin tail distribution of firms size.

The GDP growth is defined by

$$\frac{\Delta Y_{t+1}}{Y_t} = \frac{1}{Y_t} \sum_{i=1}^{N} \Delta y_{t+1}^i \tag{1}$$

The absolute growth of a firm own by the entrepreneur i is

$$\Delta y_{t+1}^i = (\frac{\alpha}{w})^{\frac{\alpha}{1-\alpha}} \Big(f(\varphi_i, x_i) - \varphi_i \Big)$$

Let us assume that $f(\varphi_i, x_i) = \varphi_i x_i$ which yields[3].

$$\begin{aligned} \Delta y_{t+1}^i &= \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \varphi_i (x_i - 1) \\ &= y_{idea}^i (x_i - 1) \end{aligned}$$

where $y_{idea}^i := \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \varphi_i$[4]. From this equation, one can see that the growth of firm follow a Gibrat's law - the growth of firm does not depend on its size[5].

Finally, the GDP is thus

$$\frac{\Delta Y_{t+1}}{Y_t} = \frac{1}{Y_t} \sum_{i=1}^{N} y_{idea}^i (x_i - 1)$$

from which, I can derive the GDP volatility

$$\sigma_{GDP} = \sqrt{\mathbb{V}ar_t \Big(\frac{\Delta Y_{t+1}}{Y_t} \Big)} = \Big(\sum_{i=1}^{N} \sigma^2 \Big(\frac{y_{idea}^i}{Y_t} \Big)^2 \Big)^{1/2}$$

where σ^2 is the variance of the idiosyncratic shock $h(.)$.

[3]This assumption is also key, since for $f(\varphi_i, x_i) = \varphi_i + x_i$, Δy_{t+1}^i will not be proportional to y_{idea}^i and Gabaix' theorem does not applied anymore.

[4]Here $y_{idea}^i = y_t^i$

[5]This will be false as soon as in the first period a ability shock is drawn by the entrepreneur. In this last case $\Delta y_{t+1}^i = y_{idea}^i (x_{t+1}^i - x_t^i)$. in this last case all the following result are still true.

Let us compute the distribution from which y^i_{idea} are drawn.

$$\begin{aligned}\mathbb{P}\{y_{idea} > y\} &= \mathbb{P}\{\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}\varphi > y\} \\ &= \mathbb{P}\Big\{\varphi > \frac{y}{(\frac{\alpha}{w})^{\frac{\alpha}{1-\alpha}}}\Big\} \\ &= g\left(\frac{y}{(\frac{\alpha}{w})^{\frac{\alpha}{1-\alpha}}}\right) \\ &= \frac{1}{1 + \frac{y}{\phi(\frac{\alpha}{w})^{\frac{\alpha}{1-\alpha}}}}\end{aligned}$$

this distribution does not have a variance.

From this one can apply Proposition 2 from Gabaix (2011) and conclude that $\sigma_{GDP} \sim \frac{v}{\ln N}\sigma$, where v is a random variable following a distribution which does not depend on N or σ[6].

2.3 Simulation

I assume that the idiosyncratic shock x is equal to $\exp(s)$ where s follow a normal distribution with mean 0 and variance σ^2. The parameters value are $\phi = 2, \alpha = 0.6, w = 0.3$ and $\sigma = 0.01$. To compute figure 1, for each number of firms N, I draw $M = 1000$ economies. Each economy is characterized by a sequence $\{(\varphi_i, x_i)\}_{1\ldots N}$. For each economy, I compute the GDP growth using equation (1), and the empirical standard deviation over this M economies.

To see if the theoretical result of the previous section holds, I regress the corresponding empirical standard deviation on a constant and $1/\log(N)$.

[6] v follow the square root of a stable lévy distribution with exponent 1/2.

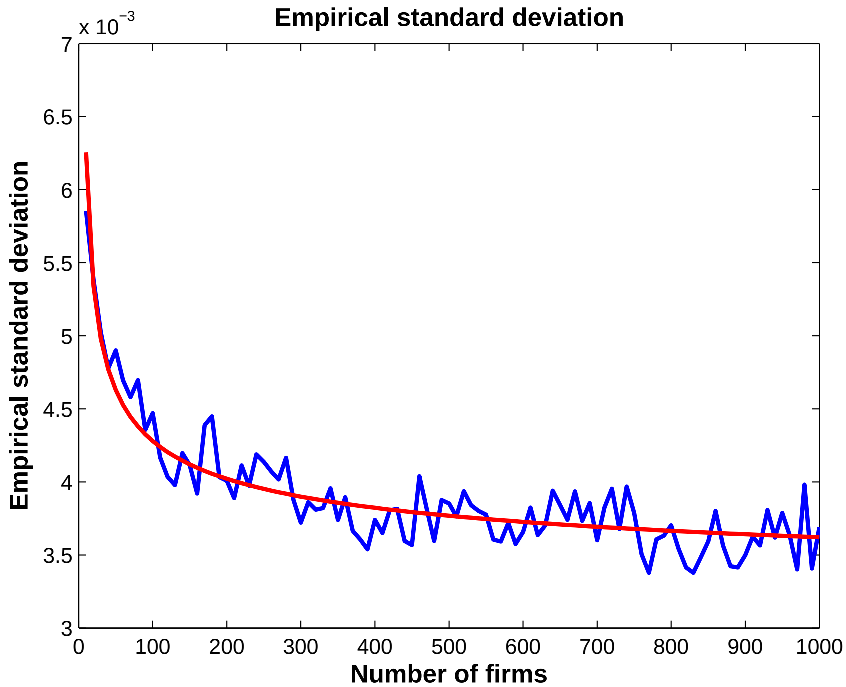

Figure 1: Empirical standard deviation as a function of N (blue) and a fit to $1/\log(N)$ (red).

References

[1] Fernando E. Alvarez, Francisco J. Buera, and Jr. Robert E. Lucas. Models of idea flows. NBER Working Papers 14135, National Bureau of Economic Research, Inc, June 2008.

[2] Xavier Gabaix. The granular origins of aggregate fluctuations. *Econometrica*, 79(3):733–772, 05 2011.

[3] Robert E. Lucas. Ideas and growth. *Economica*, 76(301):1–19, 02 2009.