Midterm II Solution

Q1. SOLUTION.

$v(t) = \delta(t-t_1)$

$v(t) \longleftrightarrow V(f)$

$V(f) = \mathcal{F}\{v(t)\} = \mathcal{F}\{\delta(t-t_1)\} = e^{-j\omega t_1}$

Note $\mathcal{F}\{\delta(t)\} = 1.\ \forall f.$

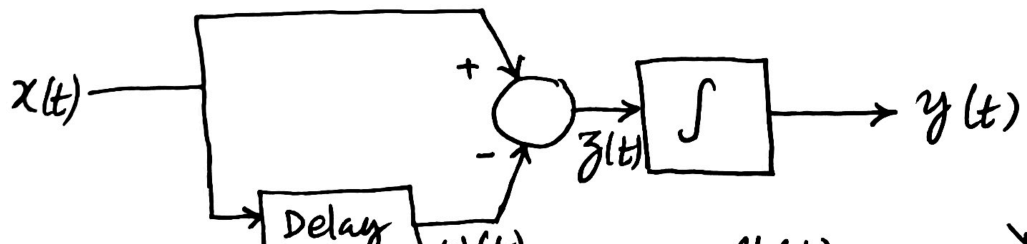
Similarly $W(f) = \mathcal{F}\{w(t)\} = \mathcal{F}\{\delta(t-t_2)\} = e^{-j\omega t_2}.$

$$V(f)W(f) = e^{-j\omega(t_1+t_2)}$$

$$\begin{aligned} v(t) * w(t) &= \mathcal{F}^{-1}\{V(f)W(f)\} \\ &= \mathcal{F}^{-1}\{e^{-j\omega(t_1+t_2)}\} \\ &= \delta(t-t_1-t_2). \end{aligned}$$

Q2. SOLUTION.

(a)



$y(t) \longleftrightarrow Y(f)$ \
$z(t) \longleftrightarrow Z(f)$ } given.

$\int z(t)dt \longleftrightarrow \dfrac{Z(f)}{j2\pi f}$

(b) $y(t) = \int z(t)\,dt$

$$Y(f) = \frac{Z(f)}{j2\pi f}$$

$$\Rightarrow \frac{Y(f)}{Z(f)} = \frac{1}{j2\pi f}$$

Q2 (c) $\omega(t) = x(t-T)$

$$x(t) \leftrightarrow X(f), \quad \omega(t) \leftrightarrow W(f) \quad \text{given.}$$

$$x(t-T) \leftrightarrow X(f)e^{-j\omega T}$$

$$\mathcal{F}\{\omega(t)\} = \mathcal{F}\{x(t-T)\}$$

$$W(f) = X(f)e^{-j\omega T}$$

$$\Rightarrow \frac{W(f)}{X(f)} = e^{-j\omega T}$$

Q2 (d) $$\frac{Y(f)}{X(f)} = \frac{Y(f)}{Z(f)}\,\frac{Z(f)}{X(f)}$$

We know $z(t) = x(t) - \omega(t)$

$$Z(f) = X(f) - W(f)$$

$$Z(f) = X(f)\left(1 - e^{-j\omega T}\right)$$

$$\frac{Z(f)}{X(f)} = 1 - e^{-j\omega T}$$

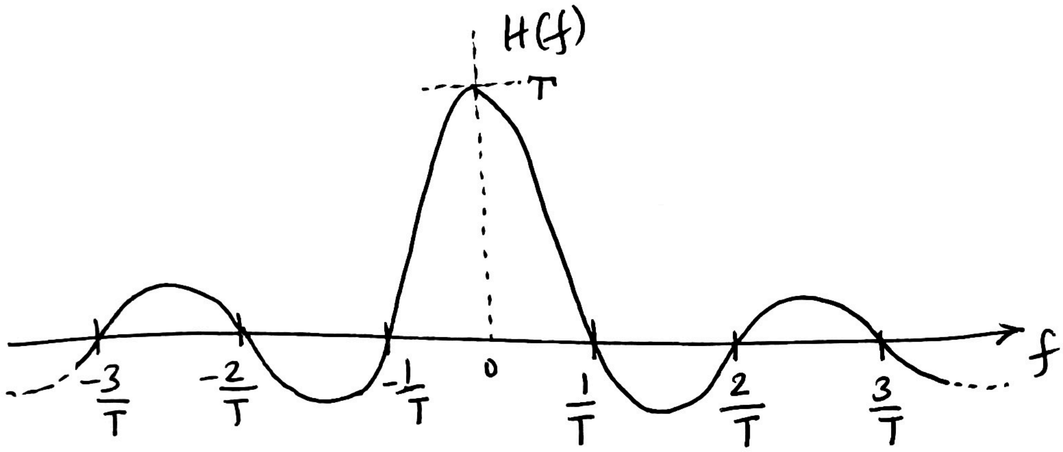
$$\Rightarrow \frac{Y(f)}{X(f)} = \frac{1}{j2\pi f}\left(1 - e^{-j\omega T}\right) = T\,\mathrm{sinc}(fT)\,e^{-j\pi f T}$$

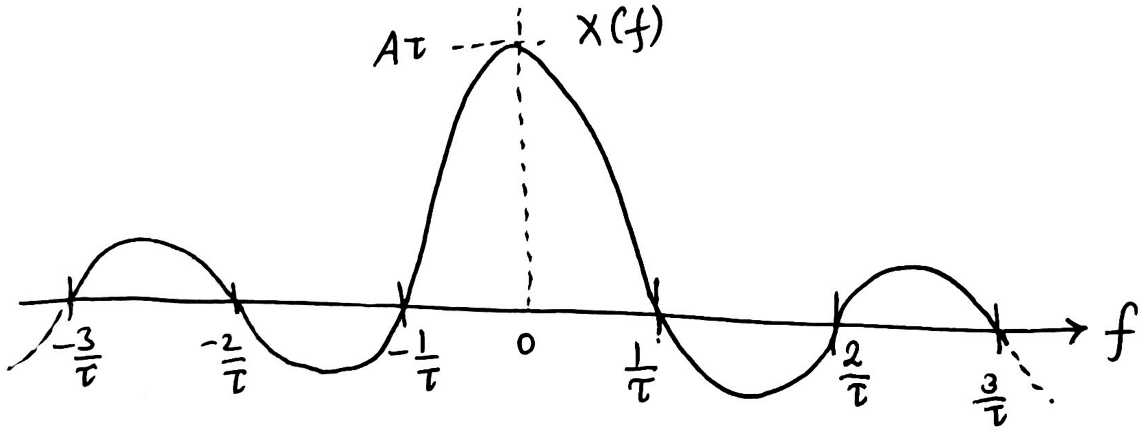
You can prove this yourself.

Q3. Given. $y(t) = x(t) * h(t) = \int x(\lambda)h(t-\lambda)\,d\lambda$ ③

$$\left.\begin{aligned} h(t) &\longleftrightarrow H(f) \\ x(t) &\longleftrightarrow X(f) \\ H(f) &= T\,\mathrm{sinc}(fT) \\ x(t) &= A\Pi(t/\tau) \\ \Rightarrow X(f) &= A\tau\,\mathrm{sinc}(f\tau) \end{aligned}\right\} \quad Y(f) = TA\tau\,\mathrm{sinc}(f\tau) \times \mathrm{sinc}(fT)$$

$Y(f)$ is the product of two sinc functions





When $\tau = T$

$$Y(f) = AT^2\,\mathrm{sinc}^2(fT)$$

Recall that $A\Lambda(t/T) \longleftrightarrow AT\,\mathrm{sinc}^2(fT)$

$$\Rightarrow y(t) = TA\Lambda\left(\frac{t}{T}\right) = \begin{cases} TA\left(1 - \frac{|t|}{T}\right) & |t| < T \\ 0 & |t| > T \end{cases}$$

$y(t)$ is triangular pulse.

When $\tau \ll T$ or in otherwords T is very large compared to τ.

if τ is very small Then $\frac{1}{\tau}$ must be very large.
This means that

$$X(f) \approx A\tau \text{ (Constant)} \Rightarrow \boxed{\text{sinc}(f\tau) \approx 1}$$

for the range of frequencies where $H(f)$ is significant.

Or you may notice that

$$\lim_{\tau \to 0} \text{sinc}(f\tau) \to 1$$

Therefore $$Y(f) \cong TA\tau\,\text{sinc}(fT)$$

$$\Rightarrow y(t) \cong TA\tau\,\Pi\left(\frac{t}{T}\right)$$

When $\tau \gg T$

Similarly when $T \ll \tau$

$$\lim_{T \to 0} \text{sinc}(fT) \to 1$$

$$Y(f) \cong TA\tau\,\text{sinc}(f\tau)$$

$$\Rightarrow y(t) \cong TA\tau\,\Pi\left(\frac{t}{\tau}\right).$$

Q5.

$$x(t) = \cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t)$$

where $\omega_0 = 2\pi \Rightarrow f_0 = 1$ Hz.

$$H(f) = e^{-j\omega/3} =: e^{-j\omega t_d} \quad \text{(delay)}.$$

$$\Rightarrow t_d = \frac{1}{3} \text{ sec.}$$

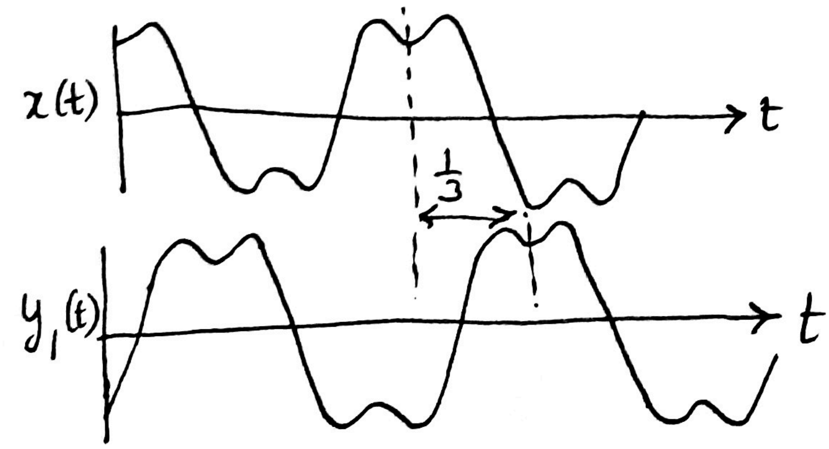
$$\begin{aligned} y_1(t) &= x(t) * h(t) \\ &= \cos\left(\omega_0\left(t-\frac{1}{3}\right)\right) - \frac{1}{3}\cos\left(3\omega_0\left(t-\frac{1}{3}\right)\right) \\ &= \cos\left(\omega_0 t - \frac{2\pi}{3}\right) - \frac{1}{3}\cos\left(3\omega_0 t - 2\pi\right) \end{aligned}$$

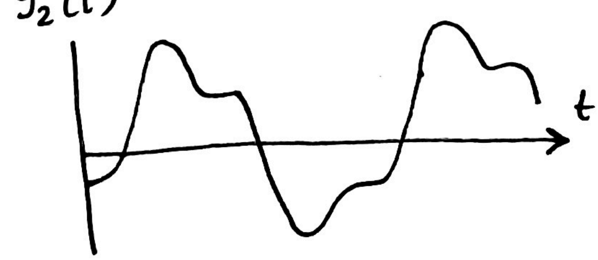
Whatever is the plot of signal $x(t)$ $y_1(t)$ is just a delayed version of $x(t)$ where the delay is $\frac{1}{3}$ seconds. Do MATLAB plots yourself.

In contrast

$$y_2(t) = \cos\left(\omega_0 t - \frac{2\pi}{3}\right) - \frac{1}{3}\cos\left(3\omega_0 t - \frac{2\pi}{3}\right)$$

is not a delayed version. It is rather a very distorted version of $x(t)$. No this is not possible to find transfer function for this case.





Question 6.

(6)

(a) $$y(t) = x(t) + \frac{1}{2}x^2(t) + \frac{1}{3}x^3(t)$$

$$\Rightarrow Y(f) = X(f) + \frac{1}{2}X(f) * X(f) + \frac{1}{3}X(f) * X(f) * X(f)$$

"$*$" denotes convolution.

(b) $$x(t) = \cos(\omega_0 t)$$

$$y(t) = \frac{1}{12}\left(3 + 15\cos(\omega_0 t) + 3\cos(2\omega_0 t) + \cos(3\omega_0 t)\right)$$

prove it yourself

(c) One sided spectrum of $y(t)$ is.