HABIB UNIVERSITY

EE-252 Midterm-2 Exam
Signals and Systems
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Duration: 60 min

Select one from Q1 and Q2; one from Q3 and Q4; one from Q5 and Q6.

Q1 [20 points]: Rayleigh's energy theorem states that the energy E of a signal $v(t)$ is related to its spectrum $V(f)$ by

$$E = \int_{-\infty}^{\infty} |V(f)|^2 df = \int_{-\infty}^{\infty} |v(t)|^2 dt,$$

and we know the following transform pair:

$$e^{-b|t|} \longleftrightarrow \frac{2b}{b^2 + (2\pi f)^2}$$

Using the above stated facts, prove that

$$\int_0^{\infty} \frac{1}{(a^2+x^2)^2} dx = \frac{\pi}{4a^3}$$

Q2 [20 points]: Consider a finite-duration sinusoid in Fig. 2.3-3a, sometimes referred to as an radio-frequency (RF) pulse when f_c falls in the radio-frequency band.

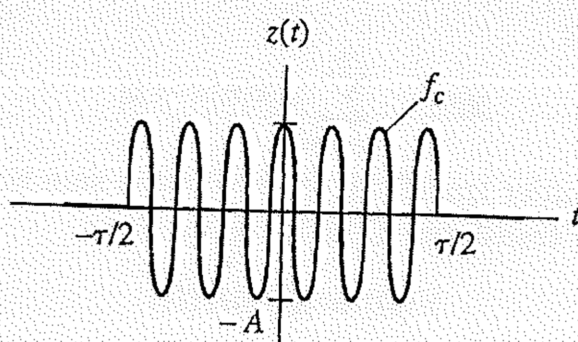


Figure 1: RF pulse.

The signal is expressed as:

$$z(t) = A\Pi\left(\frac{t}{\tau}\right)\cos\omega_c t$$

Obtain its Fourier transform and plot both the transform, $Z(f)$, and its "amplitude" spectrum, that is, $|Z(f)|$.

Q3 [50 points]: Consider a continuous-time LTI system, where the system's output $y(t)$ may be expressed as the convolution of input signal $x(t)$ and the impulse response $h(t)$:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

We know that the Fourier transform of convolution integral is equal to product of continuous-time Fourier transform (CTFT), so we have:

$$Y(f) = X(f)H(f)$$

where $x(t) \longleftrightarrow X(f), \quad h(t) \longleftrightarrow H(f), \quad y(t) \longleftrightarrow Y(f)$.

Now consider $x(t) = \text{sinc}(4t)$ and $h(t) = 2\,\text{sinc}(t/2)$; you have to obtain $y(t)$ without doing explicit convolution integration.

(a) Find $X(f)$ and $H(f)$.

(b) Obtain $Y(f)$. Obtain $y(t)$ from $Y(f)$.

- You may use some known transform pair in solving **(a)** and **(b)**.

Q4 [50 points]: Consider a discrete-time LTI system, where the system's output $y[n]$ may be expressed as the convolution of input signal $x[n]$ and the impulse response $h[n]$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

We know that the Fourier transform of convolution sum is equal to product of Discrete-time Fourier transform (DTFT), so we have:

$$Y(\Omega) = X(\Omega)H(\Omega)$$

where $x[n] \longleftrightarrow X(\Omega), \quad h[n] \longleftrightarrow H(\Omega), \quad y[n] \longleftrightarrow Y(\Omega)$.

Now consider $x[n] = \text{sinc}(4n)$ and $h[n] = 2\,\text{sinc}(n/2)$; you have to obtain $y[n]$ without doing explicit convolution sum.

(a) Find $X(\Omega)$ and $H(\Omega)$.

(b) Obtain $Y(\Omega)$. Obtain $y[n]$ from $Y(\Omega)$.

- You may use some known transform pair in solving **(a)** and **(b)**.

Q5 [50 points]: We know the transform pairs:

$$x[n] = \alpha^n u[n], \quad |\alpha| < 1 \quad X(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}}$$

$$x[n] = \begin{cases} 1, & -L \le n \le L \\ 0, & \text{otherwise} \end{cases}, \qquad X(\Omega) = \frac{\sin\left(\frac{\Omega}{2}(2L+1)\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

Also, for a transform pair

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\Omega)$$

it can be shown that

$$x[n]\cos(\Omega_0 n) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)\right]$$

and

$$x[n]\sin(\Omega_0 n) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[X(\Omega - \Omega_0)\, e^{-j\pi/2} + X(\Omega + \Omega_0)\, e^{j\pi/2}\right]$$

Using these information, find the DTFT of the following signals:

(a) $x[n] = (0.5)^n u[n] \sin(0.2\pi n)$
(b) $x[n] = (u[n] - u[n-8]) \cos(0.2\pi n)$

Sketch the DTFT spectrum.

Hint: First obtain the DTFT of the functions $(0.5)^n u[n]$ and $(u[n] - u[n-4])$; you may use known transform pairs. Next, apply the modulation property.

Q6 [50 points]: Signals listed below have odd symmetry. For each signal determine the DTFT. Graph the magnitude and the phase of the transform.

$$x[n] = \begin{cases} -1/4, & n = -2 \\ -1/3, & n = -1 \\ 1/3, & n = 1 \\ 1/4, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} n, & n = -5, \ldots, 5 \\ 0, & \text{otherwise} \end{cases}$$

Q1 [20 points]

$$e^{-b|t|} \longleftrightarrow \frac{2b}{b^2+(2\pi f)^2}$$

$$\Rightarrow v(t) = e^{-2\pi a|t|} \longleftrightarrow \frac{4\pi a}{(2\pi a)^2+(2\pi f)^2}$$

$$= \frac{a/\pi}{a^2+f^2} =: V(f)$$

$$\int_{-\infty}^{\infty} v^2(t)\,dt = \int_{-\infty}^{\infty} |V(f)|^2 df$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(e^{-2a\pi|t|}\right)^2 dt = \int_{-\infty}^{\infty} \left(\frac{a/\pi}{a^2+f^2}\right)^2 df$$

$$I_1 = I_2$$

Let us solve these integrals.

$$I_1 = \int_{-\infty}^{\infty} e^{-4\pi a|t|}\,dt$$

$$= 2\int_0^{\infty} e^{-4\pi a t}\,dt = \left.\frac{2e^{-4\pi a t}}{-4\pi a}\right|_0^{\infty} = \frac{1}{2\pi a}$$

$$I_2 = \int_{-\infty}^{\infty} \left(\frac{a}{\pi}\right)^2 \frac{1}{(a^2+f^2)^2} df$$

$$= 2\left(\frac{a}{\pi}\right)^2 \int_0^{\infty} \frac{1}{(a^2+f^2)^2} df$$

$$= I_1 = \frac{1}{2\pi a}$$

$$\Rightarrow \int_0^{\infty} \frac{1}{(a^2+f^2)^2} df = \left(\frac{\pi}{a}\right)^2 \frac{1}{2} * \frac{1}{2\pi a}.$$

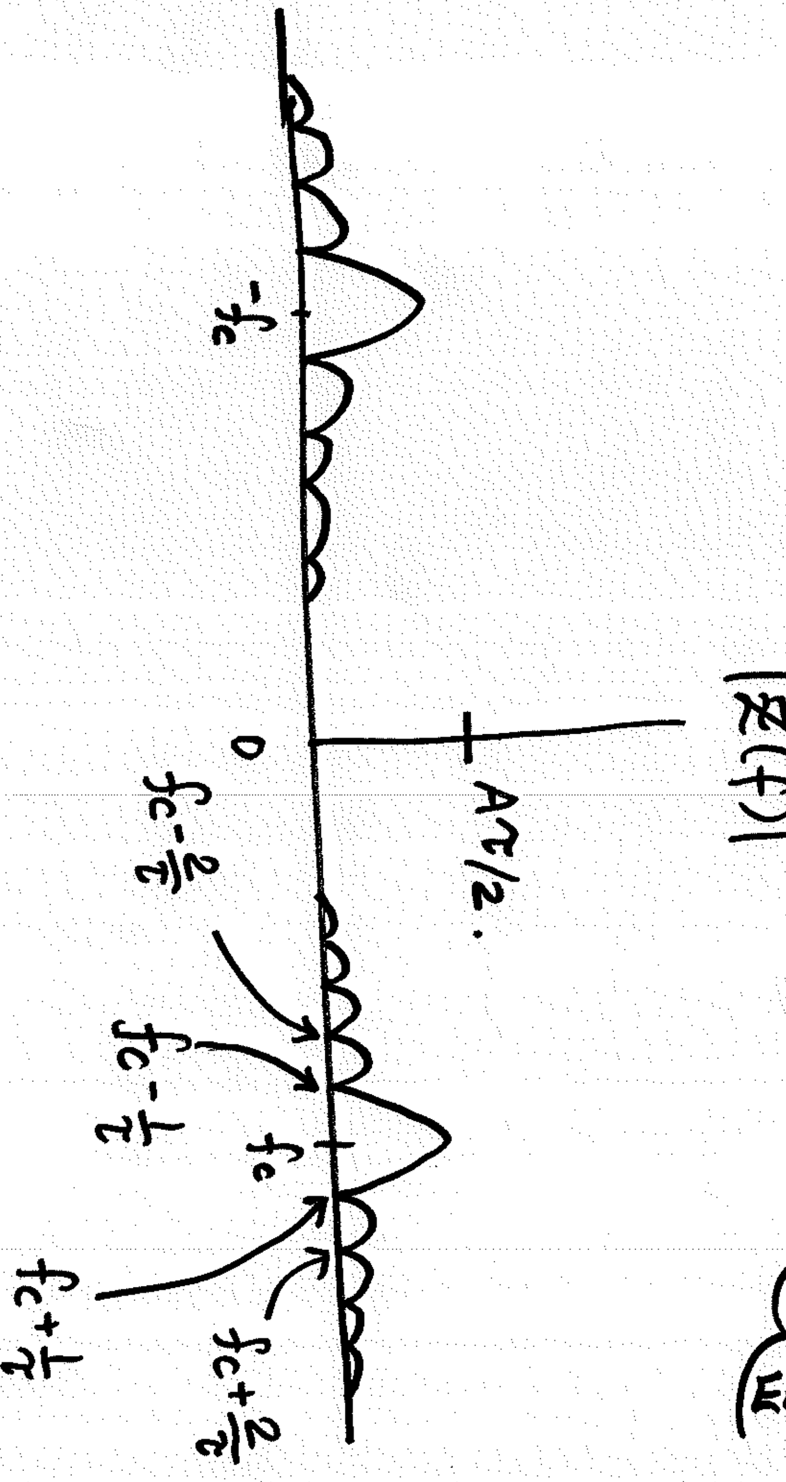
$$= \frac{\pi^2}{4\pi a^2 . a} = \frac{\pi}{4\pi a^3}$$

proved.

Q2 [20 points].

$$z(t) = A\,\Pi\left(\frac{t}{\tau}\right)\cos(2\pi f_c t)$$

$$Z(f) = \frac{A\tau}{2}\,\mathrm{sinc}\left[(f-f_c)\tau\right] + \frac{A\tau}{2}\,\mathrm{sinc}\left[(f+f_c)\tau\right].$$



Q3. $$A\Pi\left(\frac{t}{\tau}\right) \longleftrightarrow A\tau\,\mathrm{sinc}(f\tau)$$ ④

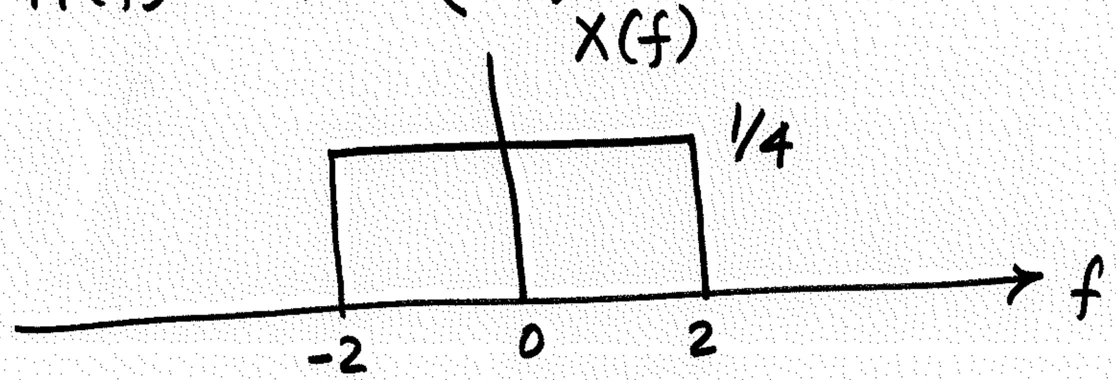
$$A\,\mathrm{sinc}(2Wt) \longleftrightarrow \frac{A}{2W}\Pi\left(\frac{f}{2W}\right)$$

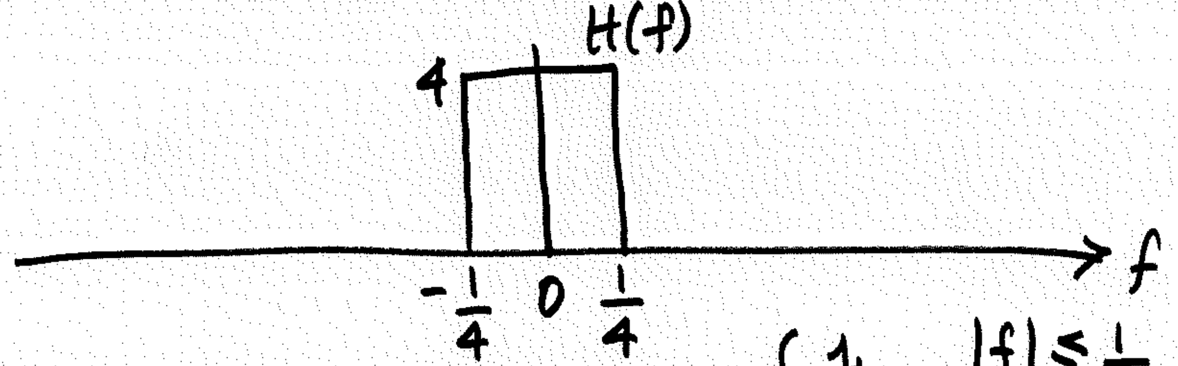
$$x(t) = \mathrm{sinc}(4t) \Rightarrow (A=1,\ 2W=4)$$

$$X(f) = \frac{1}{4}\Pi\left(\frac{f}{4}\right)$$

$$h(t) = 2\,\mathrm{sinc}(t/2) \Rightarrow \left(A=2,\ 2W=\frac{1}{2}\right)$$

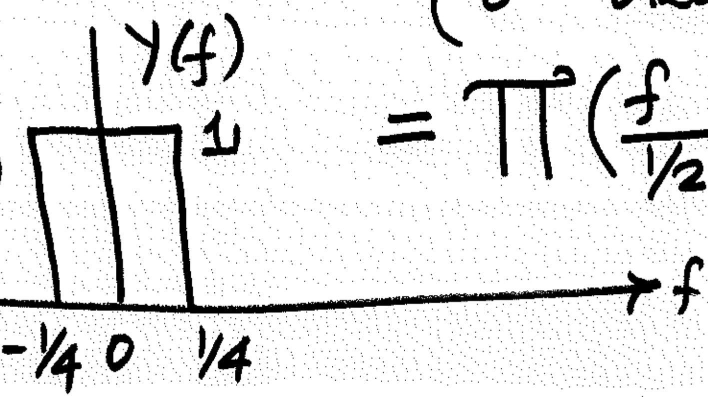
$$H(f) = 4\Pi(2f)$$





Since $$Y(f) = X(f)H(f) = \begin{cases} 1 & |f| \le \frac{1}{4} \\ 0 & \text{otherwise} \end{cases} = \Pi\left(\frac{f}{1/2}\right) = \Pi(2f)$$

$$y(t) = \mathcal{F}^{-1}\{Y(f)\} = \frac{1}{2}\mathrm{sinc}(t/2)$$



Q5.

(a)

$$(0.5)^n u[n] \longleftrightarrow \frac{1}{1-0.5e^{-j\Omega}}$$

$$(0.5)^n u[n] \sin(0.2\pi n) \longleftrightarrow \frac{1}{2}\left[\frac{e^{-j\pi/2}}{1-0.5e^{-j(\Omega-0.2\pi)}} + \frac{e^{+j\pi/2}}{1-0.5e^{-j(\Omega+0.2\pi)}}\right]$$

(b)

$$u[n] - u[n-8] \longleftrightarrow \frac{1-e^{-j8\Omega}}{1-e^{-j\Omega}}$$

$$\left(u[n]-u[n-8]\right)\cos(0.2\pi n) \longleftrightarrow \frac{1}{2}\left[\frac{1-e^{-j8(\Omega-0.2\pi)}}{1-e^{-j(\Omega-0.2\pi)}} + \frac{1-e^{-j8(\Omega+0.2\pi)}}{1-e^{-j(\Omega+0.2\pi)}}\right].$$