solution: Sns-Midterm Exam Date 30th March 2020

Q1 a) T_0 for $x(t)$ is given by 2

$x(t)$ as defined for one time period = $\begin{cases} -t-1 & -1<t<0 \\ t & 0<t<1 \end{cases}$

Compact fourier series = $d_0 + \sum_{k=1}^{\infty} d_k \cos(k\omega_0 t + \phi_k)$

$d_0 = 0$ as signal is symetric on the x-axis.

$$d_k = \sqrt{a_k^2 + b_k^2} \qquad \phi_k = \tan^{-1}\left(\frac{b_k}{a_k}\right)$$

$$a_k = \frac{2}{T_0}\int_{T_0} x(t)\cos(k\omega_0 t)\,dt$$

$$a_k = \frac{2}{2}\left[\int_{-1}^{0}(-t-1)\cos(k\omega_0 t)\,dt + \int_0^1 t\cos(k\omega_0 t)\,dt\right]$$

$$a_k = \int_{-1}^{0} -t\cos\left(k\frac{2\pi}{2}t\right)dt + \int_{-1}^{0} -\cos\left(k\frac{2\pi}{2}t\right)dt \;---$$
$$+ \int_0^1 t\cos\left(k\frac{2\pi}{2}t\right)dt.$$

$$a_k = -\int_{-1}^{0} t\cos(k\pi t)\,dt - \int_{-1}^{0}\cos(k\pi t)\,dt + \int_0^1 t\cos(k\pi t)\,dt$$

evaluating $-\int_{-1}^{0}\cos(k\pi t)\,dt = -\left[\frac{\sin(k\pi t)}{k\pi}\right]_{-1}^{0}$

$$= -\left[\frac{\sin(0)}{k\pi} - \frac{\sin(-k\pi)}{k\pi}\right] = 0.$$

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evaluating $\int t\cos(k\pi t)\,dt \Rightarrow u=t \quad u'=1 \quad v'=\cos(k\pi t)$

$$v = \frac{\sin(k\pi t)}{k\pi}$$

$$\Rightarrow \frac{t}{k\pi}\,\frac{\sin(k\pi t)}{\cancel{k\pi}} - \int \frac{\sin(k\pi t)}{k\pi}\,dt$$

$$\Rightarrow \frac{t}{k\pi}\sin(k\pi t) + \frac{\cos(k\pi t)}{k^2\pi^2}$$

$$\int_0^{-1} t\cos(k\pi t)\,dt \Rightarrow \cancel{\frac{-1}{k\pi}\cos(-k\pi)}$$

$$\Rightarrow \left[\cancelto{0}{\frac{-1}{k\pi}\sin(-k\pi)} + \frac{\cos(-k\pi)}{k^2\pi^2}\right] - \left[0 + \frac{\cos(0)}{k^2\pi^2}\right]$$

$$\Rightarrow \frac{\cos(k\pi)}{k^2\pi^2} - \frac{1}{k^2\pi^2}$$

$$\int_0^{1} t\cos(k\pi t)\,dt$$

$$\Rightarrow \left[\cancelto{0}{\frac{1}{k\pi}\sin(k\pi)} + \frac{\cos(k\pi)}{k^2\pi^2}\right] - \left[0 + \frac{\cos(0)}{k^2\pi^2}\right]$$

$$\Rightarrow \frac{\cos(k\pi)}{k^2\pi^2} - \cancel{\cos}\,\frac{1}{k^2\pi^2}$$

$$a_k = \frac{\cos(k\pi)}{k^2\pi^2} - \frac{1}{k^2\pi^2} + 0 + \frac{\cos(k\pi)}{k^2\pi^2} - \frac{1}{k^2\pi^2}$$

$$a_k = \frac{2}{k^2\pi^2}\left[\cos(k\pi) - 1\right]$$

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$$b_k = \frac{2}{2}\int_{T_0} x(t)\sin\left(k\frac{2\pi t}{2}\right)dt = \int_{T_0} x(t)\sin(k\pi t)\,dt$$

$$b_k = \int_{-1}^{0}(-t-1)\sin(k\pi t)\,dt + \int_0^1 t\sin(k\pi t)\,dt$$

$$b_k = \int_0^{-1} t\sin(k\pi t)\,dt + \int_0^1 t\sin(k\pi t)\,dt + \int_0^{-1}\sin(k\pi t)\,dt$$

evaluating $\int t\sin(k\pi t)\,dt \Rightarrow u = t \quad u' = 1 \quad v' = \sin(k\pi t) \quad v = -\frac{\cos(k\pi t)}{k\pi}$

$$-\frac{t}{k\pi}\cos(k\pi t) + \int \frac{\cos(k\pi t)}{k\pi} = -\frac{t}{k\pi}\cos(k\pi t) + \frac{\sin(k\pi t)}{k^2\pi^2}$$

$$\int_0^{-1} t\sin(k\pi t)\,dt = \left[\frac{-(-1)}{k\pi}\cos(-k\pi) + \cancel{\frac{\sin(-k\pi)}{k^2\pi^2}}^{\,0}\right] - 0 - 0$$

$$= \frac{1}{k\pi}\cos(k\pi) = \frac{\cos(k\pi)}{k\pi}$$

$$\int_0^1 t\sin(k\pi t)\,dt = \left[\frac{-1}{k\pi}\cos(k\pi) + \cancel{\frac{\sin(k\pi)}{k^2\pi^2}}^{\,0}\right] - 0 - 0$$

$$= \frac{-\cos(k\pi)}{k\pi}$$

evaluating $\int_0^{-1}\sin(k\pi t)\,dt \Rightarrow \left.\frac{-\cos(k\pi t)}{k\pi}\right|_0^{-1}$

$$\Rightarrow \frac{-\cos(-k\pi)}{k\pi} - \left(\frac{-\cos(0)}{k\pi}\right) \Rightarrow \frac{-\cos(k\pi)}{k\pi} + \frac{1}{k\pi}$$

$$\Rightarrow \frac{1-\cos(k\pi)}{k\pi}$$

$$b_k = \frac{\cos(k\pi)}{k\pi} - \frac{\cos(k\pi)}{k\pi} + \frac{1-\cos(k\pi)}{k\pi} = \frac{1-\cos(k\pi)}{k\pi}$$

$$d_k = \sqrt{a_k^2 + b_k^2}$$

$$a_k = \frac{2}{k^2\pi^2}\left[\cos(k\pi) - 1\right].$$ ~~for even~~

when k is even $a_k = \frac{2}{k^2\pi^2}(1-1) = 0$.

when k is odd. $a_k = \frac{2}{k^2\pi^2}(-1-1) = \frac{-4}{k^2\pi^2}$

$$b_k = \frac{1-\cos(k\pi)}{k\pi}$$

when k is even $\Rightarrow b_k = \frac{1-1}{k\pi} = 0$.

when k is odd $\Rightarrow b_k = \frac{1-(-1)}{k\pi} = \frac{2}{k\pi}$

when k is even $d_k = \sqrt{0^2+0^2} = 0$

when k is odd $d_k = \sqrt{\left(\frac{-4}{k^2\pi^2}\right)^2 + \left(\frac{2}{k\pi}\right)^2}$.

$$= \sqrt{\frac{16}{k^4\pi^4} + \frac{4}{k^2\pi^2}} = \sqrt{\frac{4}{k^2\pi^2}\left(\frac{4}{k^2\pi^2}+1\right)} = \frac{2}{k\pi}\sqrt{\frac{4}{k^2\pi^2}+1}$$

$$\phi_k = \tan^{-1}\left(\frac{b_k}{a_k}\right) =$$

for even k $\phi_k = \tan^{-1}\left(\frac{0}{0}\right) = 0$.

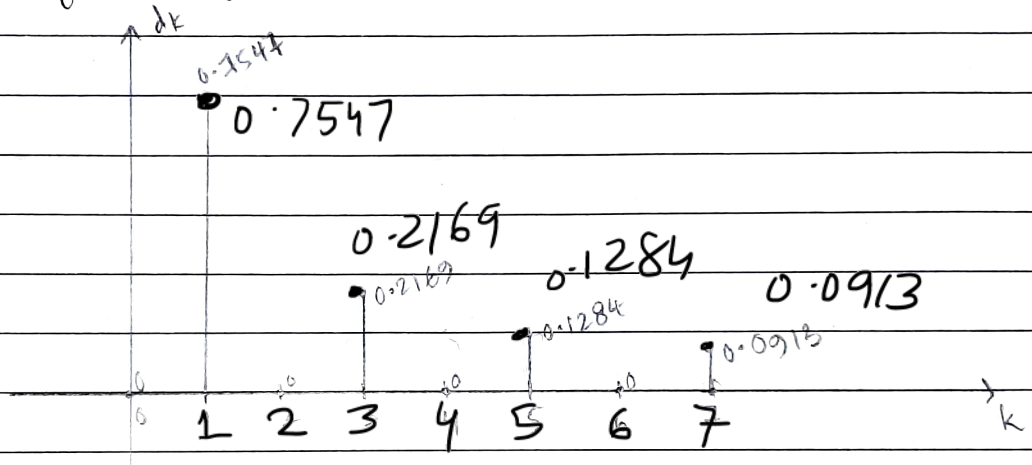
for odd k $\phi_k = \tan^{-1}\left(\frac{2/k\pi}{-4/k^2\pi^2}\right) = \tan^{-1}\left(\frac{-k\pi}{2}\right)$.

$$x(t) = d_0 + \sum_{k=1}^{\infty} d_k \cos(k\omega_0 t + \phi_k).$$

$$= 0 + \sum_{\frac{k+1}{2}=1}^{\infty} \frac{2}{k\pi}\sqrt{\frac{4}{k^2\pi^2}+1}\ \cos\left[k\pi t + \tan^{-1}\left(-\frac{k\pi}{2}\right)\right].$$

as k should be just odd

b). Amplitude spectrum



$k=1$

$$d_k = \frac{2}{\pi}\sqrt{\frac{4}{\pi^2}+1} = 0.7547$$

$$k=3 \Rightarrow d_k = \frac{2}{3\pi}\sqrt{\frac{4}{9\pi^2}+1} = 0.2169$$

$$k=5 \Rightarrow d_k = \frac{2}{5\pi}\sqrt{\frac{4}{25\pi^2}+1} = 0.1284$$

$$k=7 \Rightarrow d_k = \frac{2}{7\pi}\sqrt{\frac{4}{49\pi^2}+1} = 0.0913$$

ϕ_k @ $k=1$ $\tan^{-1}\left(\frac{-\pi}{2}\right) = -1.003.$

ϕ_k @ $k=3$ $\tan^{-1}\left(\frac{-3\pi}{2}\right) = -1.362$

ϕ_k @ $k=5$ $\tan^{-1}\left(\frac{-5\pi}{2}\right) = -1.444.$

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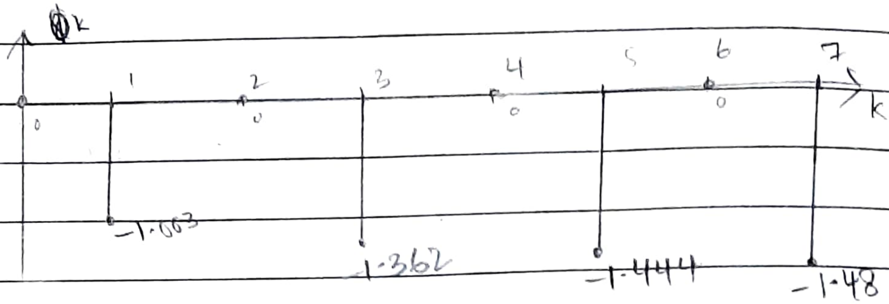
ϕ_k @ $k=7$ $\tan^{-1}\left(-\frac{7\pi}{2}\right) = -1.48$



c). Power of signal $x(t) = \frac{1}{T_0}\int_{T_0} x(t)^2\, dt$.

$$= \frac{1}{2}\left[\int_{-1}^{0} (-t-1)^2\, dt + \int_0^1 t^2\, dt\right].$$

$$= \frac{1}{2}\left[\int_{-1}^{0} \left(t^2+2t+1\right) dt + \frac{t^3}{3}\Big|_0^1\right].$$

$$= \frac{1}{2}\left[\frac{t^3}{3} + \frac{\cancel{2}t^2}{\cancel{2}} + t\Big|_{-1}^{0} + \frac{1}{3} - \cancel{\frac{0}{3}}\right].$$

$$= \frac{1}{2}\left[0+0+0-\left(\frac{-1}{3}\cancel{+1}\cancel{-1}\right)+\frac{1}{3}\right].$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}\right) = \frac{1}{3}$$

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$$\text{power of } k_{th} \text{ harmonic} = \frac{d_k^2}{2}$$

d) ~~x(t) = a_0 + a_1 cos(kt~~

power of 1st harmonic $\Rightarrow \frac{(0.7547)^2}{2} = 0.2848.$

power of second harmonic $\Rightarrow 0$

power of third harmonic $\Rightarrow \frac{(0.2169)^2}{2} = 0.0235$

power of fourth harmonic $\Rightarrow 0$

power of fifth harmonic $\Rightarrow \frac{(0.1284)^2}{2} = 0.0082$

total power of first five harmonics = 0.31654 W

e). $\frac{\text{Power of first five harmonics}}{\text{total power}} \times 100 =$

$\Rightarrow \frac{0.31654}{1/3} \times 100 = 94.963\%$

Q no #2 a) $x(t) = 1 + t\cos(t) + t^2\sin(t) + t^3\sin(t)\cos(t)$

$x(-t) = 1 - t\cos(-t) + (-t)^2\sin(-t) + (-t)^3\sin(-t)\cos(-t)$.

$x(-t) = 1 - t\cos(t) - t^2\sin(t) + t^3\sin(t)\cos(t)$

$$x_e(t) = \frac{x(t) + x(-t)}{2} = \frac{2 + 0 + 0 + 2t^3\sin(t)\cos(t)}{2}$$

$x_e(t) = 1 + t^3\sin(t)\cos(t)$

$$x_o(t) = \frac{x(t) - x(-t)}{2} = \frac{2t\cos(t) + 2t^2\sin(t)}{2}.$$

$x_o(t) = t\cos(t) + t^2\sin(t)$.

b). $x(t) = 1 + t + 4t^2$

$x(-t) = 1 - t + 4(-t)^2 = 1 - t + 4t^2$.

$$x_e(t) = \frac{x(t) + x(-t)}{2} = \frac{2 + 8t^2}{2} = 1 + 4t^2$$

$$x_o(t) = \frac{x(t) - x(-t)}{2} = \frac{2t}{2} = t.$$

$x(t) = x_e(t) + x_o(t) = 1 + 4t^2 + t = x(t)$.

Q3: $x(t) = u(t)$ step response

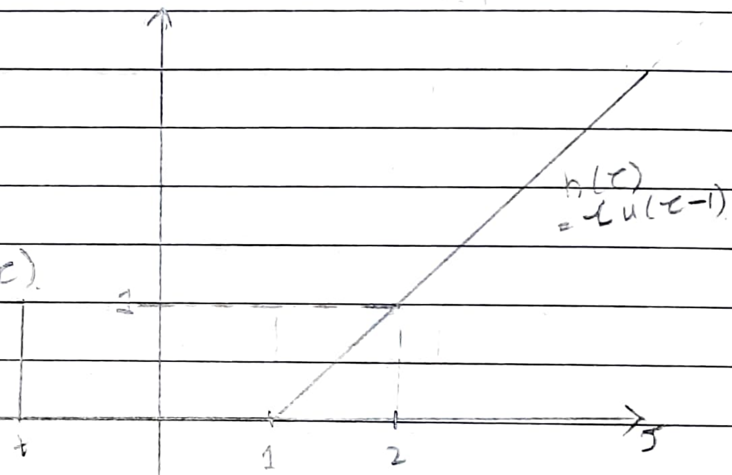
a) $h(t) = \delta(t) - \delta(t-3)$

$x(t) * h(t) = u(t) * (\delta(t) - \delta(t-3))$

$= u(t) * \delta(t) + u(t) * -\delta(t-3)$

$= u(t) - u(t-3)$

b) $h(t) = t\,u(t-1)$.



$$x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau$$

$t < 1$ no overlap.

$t > 1 \quad 1 < \tau < t$:

$$\int_{1}^{t} (\tau - 1)(1)\,d\tau = \frac{\tau^2}{2} - \tau \Big|_{1}^{t}$$

$$\frac{t^2}{2} - t - \left(\frac{1}{2} - 1\right) \Rightarrow \frac{t^2}{2} - t + \frac{1}{2}$$

$$u(t) * t\,u(t-1) = \begin{cases} 0 & t < 1 \\ \frac{t^2}{2} - t + \frac{1}{2} & t \geq 1 \end{cases}$$

Q4:

a) $y(t) = t^2 x(t)$

$y(t) = Sys\{x(t)\} = t^2 x(t) \rightarrow t^2$ multiplied with input.

$y'(t) = Sys\{x(t-\tau)\} = t^2 x(t-\tau)$

shifting time by τ in $y(t)$: $y(t) = t^2 x(t)$

$y(t-\tau) = (t-\tau)^2 x(t-\tau)$

$t^2 x(t-\tau) \neq (t-\tau)^2 x(t-\tau)$

thus system is time variant.

b) $y(t) = x(4-t)$.

$y'(t) = Sys\{x(t-\tau)\} = x(4-(t-\tau)) = x(4+\tau-t)$

Shifting time by τ in $y(t) \Rightarrow y(t) = x(4-t)$.

$y(t-\tau) = x(4-(t-\tau))$

$= x(4+\tau-t)$

$y'(t) = y(t-\tau)$

$x(4+\tau-t) = x(4+\tau-t)$

thus system is time invariant.

c) $y(t) = x(t)\sin(t)$ multiplying input by $\sin(t)$.

$y'(t) = Sys\{x(t-\tau)\} = x(t-\tau)\sin(t)$.

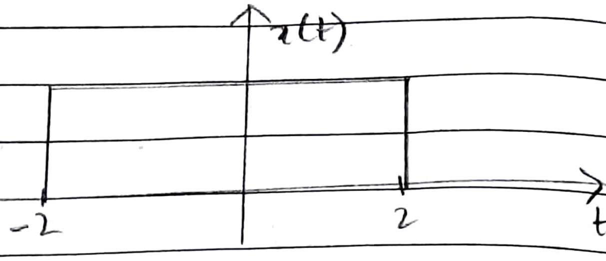
shifting time by τ in $y(t)$: $y(t) = x(t)\sin(t)$.

$y(t-\tau) = x(t-\tau)\sin(t-\tau)$

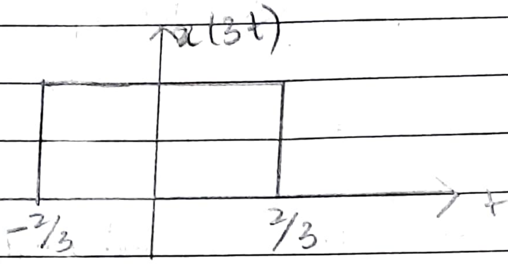
$x(t-\tau)\sin(t) \neq x(t-\tau)\sin(t-\tau)$

thus system is time variant.

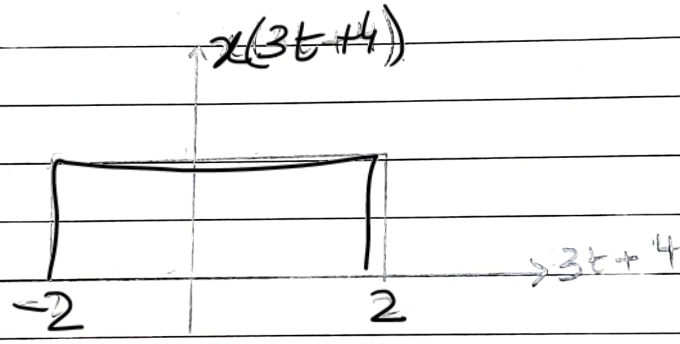
Q5: $x(t) = \begin{cases} 1 & |t| \le 2 \\ 0 & \text{otherwise} \end{cases}$

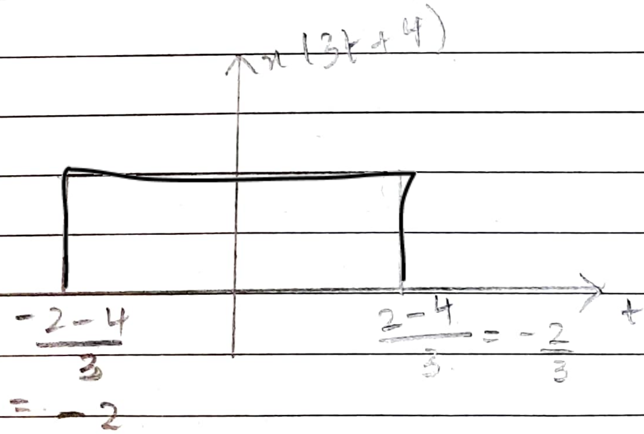


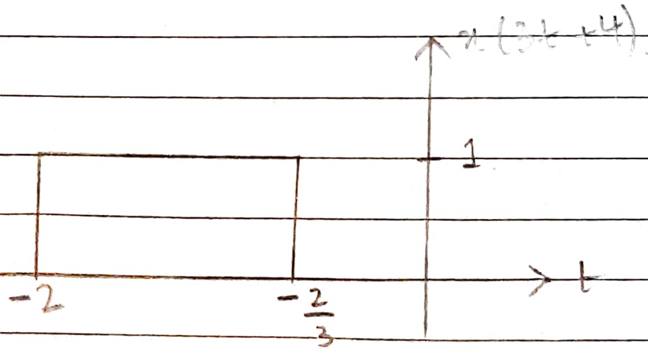
a) $x(3t)$



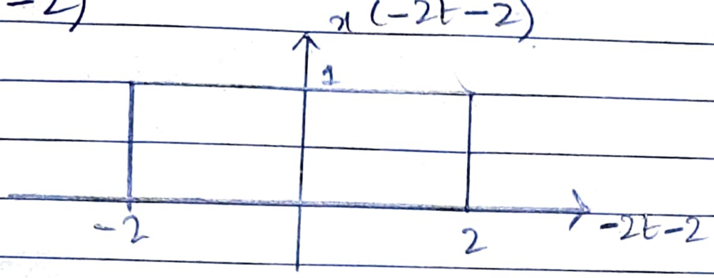
b) $x(3t+4)$

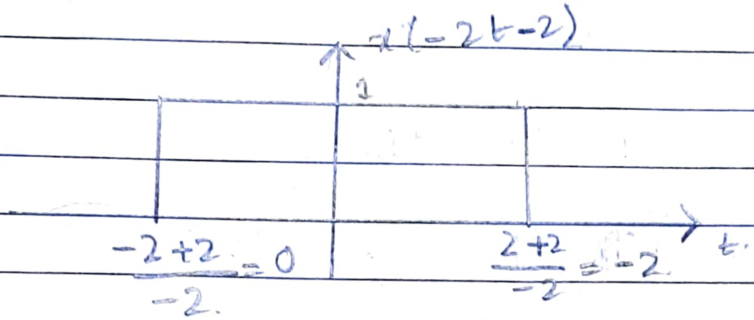


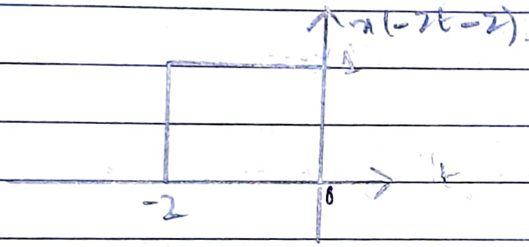




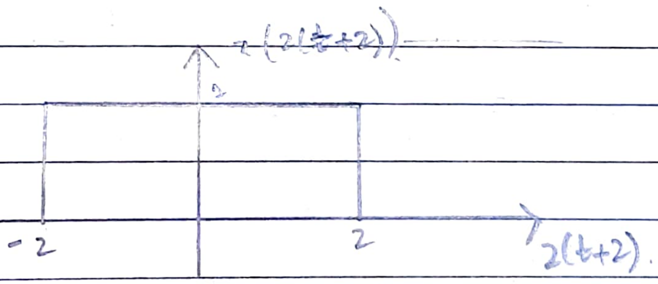
c). $x(-2t-2)$

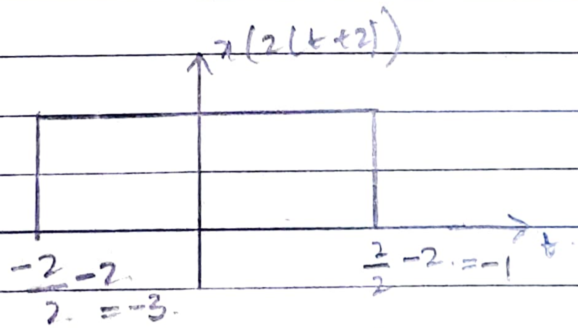

x(-2t-2)
1
-2
2
-2t-2

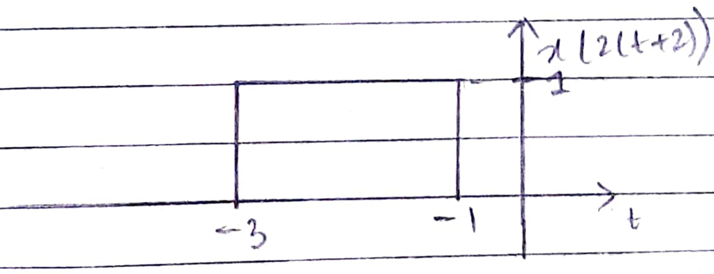

x(-2t-2)
1
(-2+2)/-2 = 0
(2+2)/-2 = -2
t


x(-2t-2)
1
-2
0
t

d) $x(2(t+2))$


x(2(t+2))
1
-2
2
2(t+2)


x(2(t+2))
-2/2 - 2 = -3
2/2 - 2 = -1
t


x(2(t+2))
1
-3
-1
t