SNS - Final Exam

Spring 2020

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Excellent work, well done

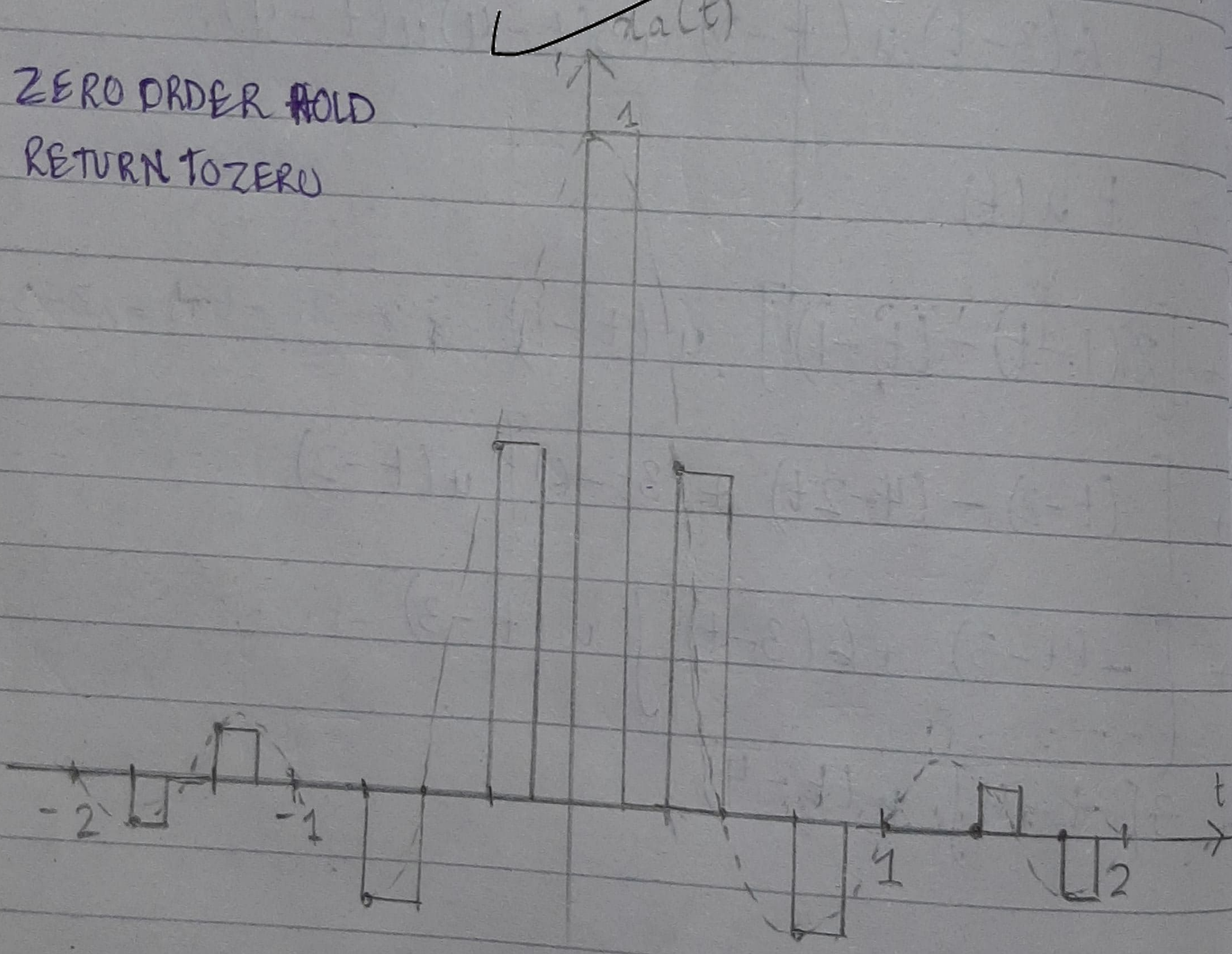
200/200

Question # 1

$x_a(t) = \text{sinc}(2t)$. $f_s = 1/T_s = 3\,Hz$

1.1 $T_s = 1/3 = 0.33\,s$

ZERO ORDER HOLD
RETURN TO ZERO

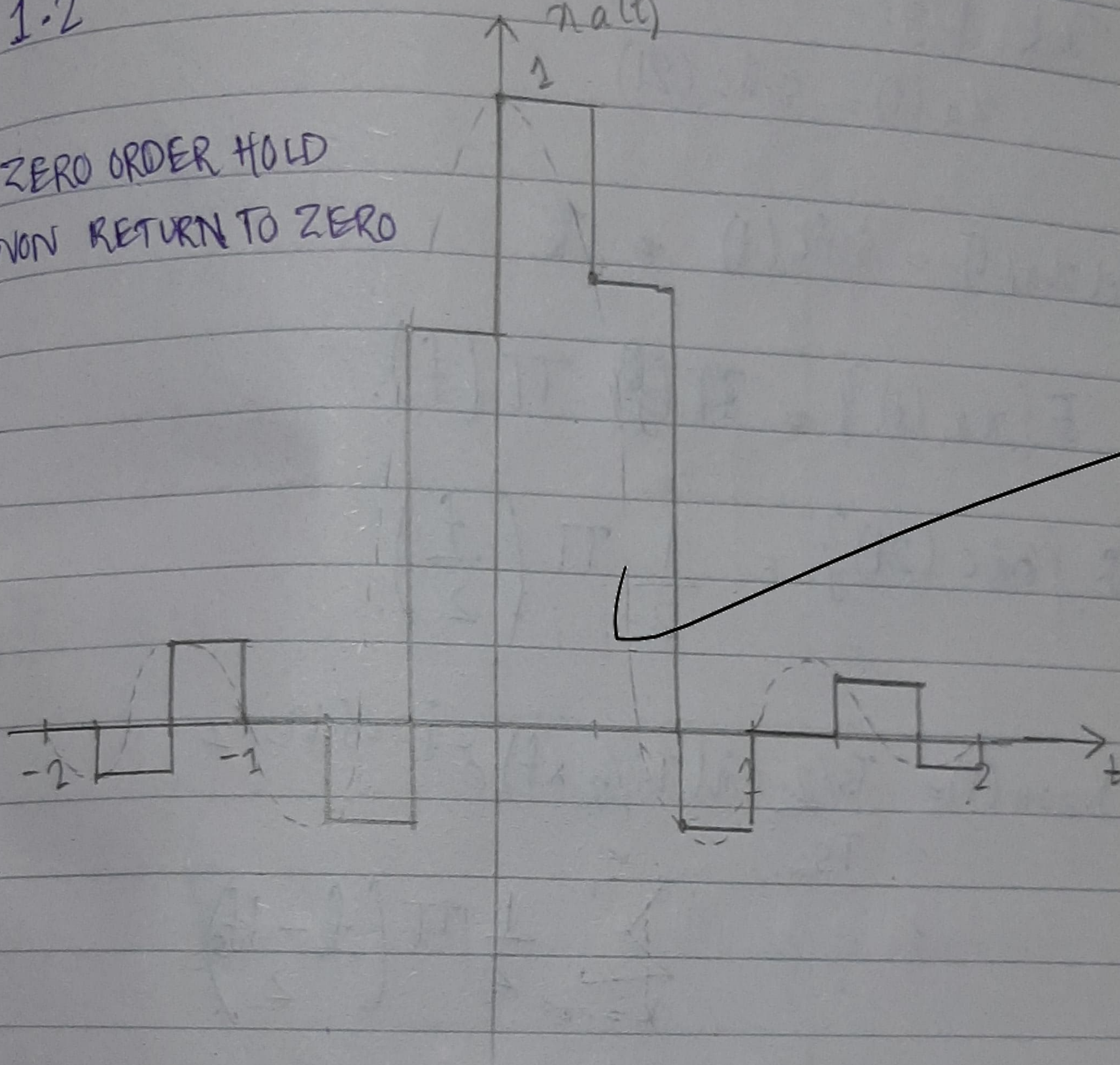


--- $x_a(t)$

—— $x_{ZOH}(t)$

1.2

ZERO ORDER HOLD
NON RETURN TO ZERO



--- $x_a(t)$

— $x_{ZOH}(t)$

Task 1.3.

$x_a(t) = \text{sinc}(2t)$.

let $x_{a_1}(t) = \text{sinc}(t)$

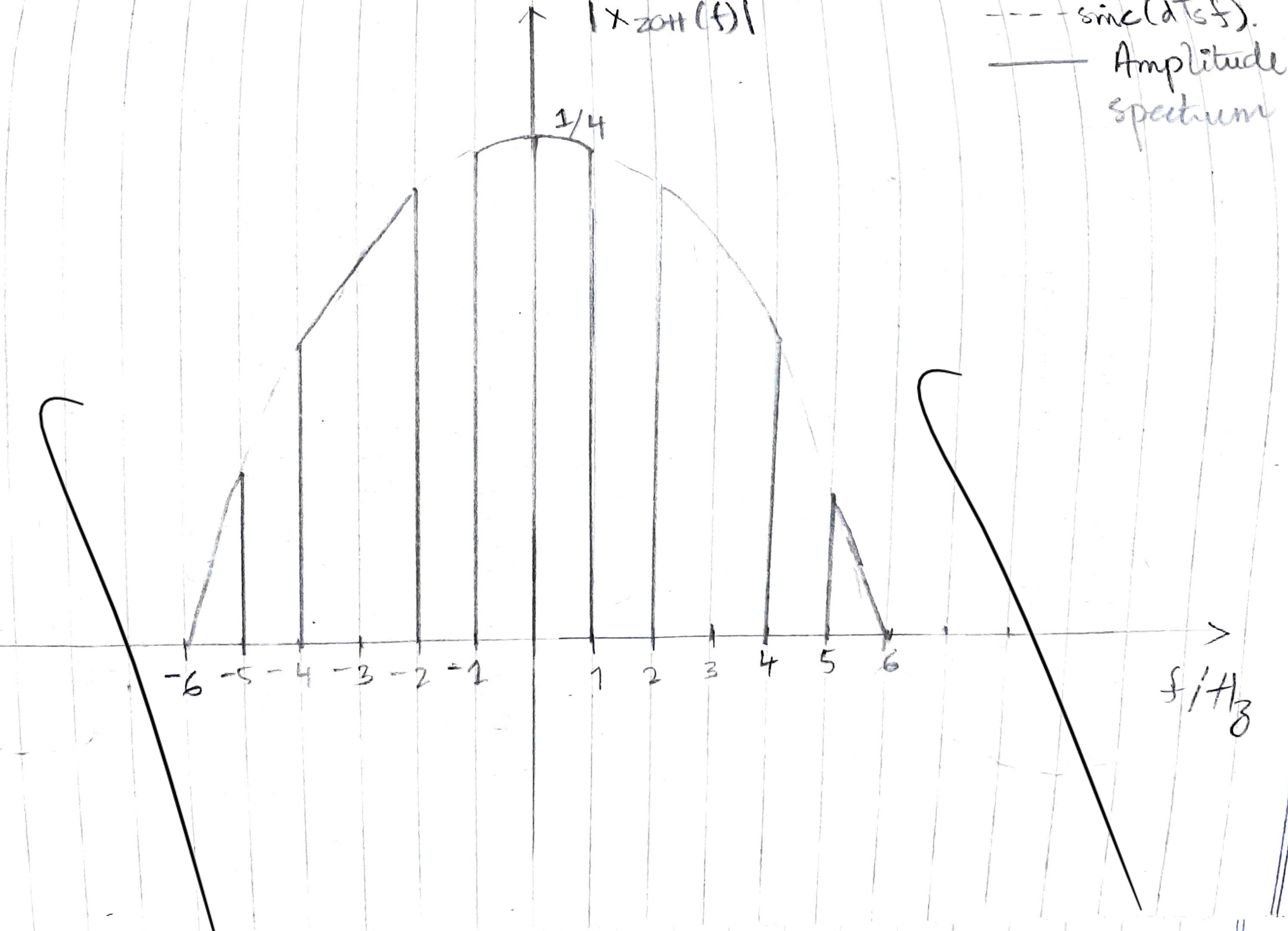
$\mathcal{F}\{x_{a_1}(t)\} = \Pi(f)$.

$\mathcal{F}\{\text{sinc}(2t)\} = \frac{1}{2}\Pi\left(\frac{f}{2}\right)$.

$X_{ZOH}(f) = \frac{T_s/2}{T_s}\,\text{sinc}(T_s/2 \times f)\,e^{-j2\pi f d T_s/2}$

$$\sum_{k=-\infty}^{\infty} \frac{1}{2}\Pi\left(\frac{f-kf_s}{2}\right).$$

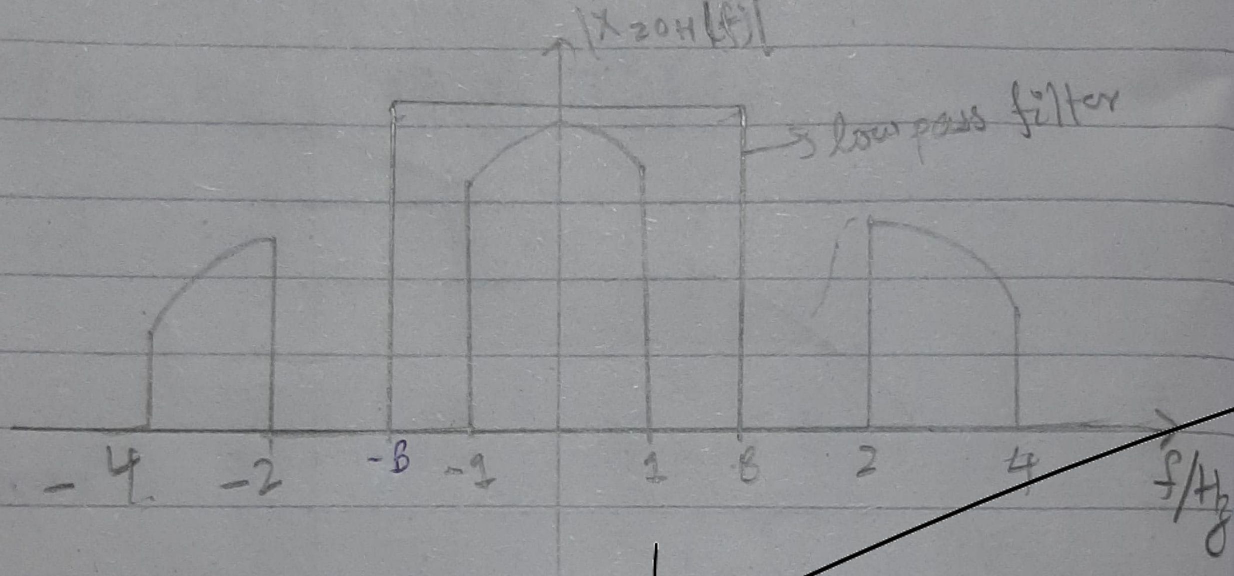
$$|X_{ZOH}(f)| = \frac{1}{4}\,\text{sinc}(1/6 \times f)\sum_{k=-\infty}^{\infty}\Pi\left(\frac{f-kf_s}{2}\right)$$



|x ZOH (f)|
sinc(αTsf).
Amplitude spectrum
1/4
-6 -5 -4 -3 -2 -1 1 2 3 4 5 6
f/Hz

Task 1-4.

Yes it is possible to reconstruct the original signal as the spectrums do not overlap.



Bandwidth of the filter.

B must be greater than 1 so that entire signal is covered.
B must be lesser than 2 so that only one signal is covered.

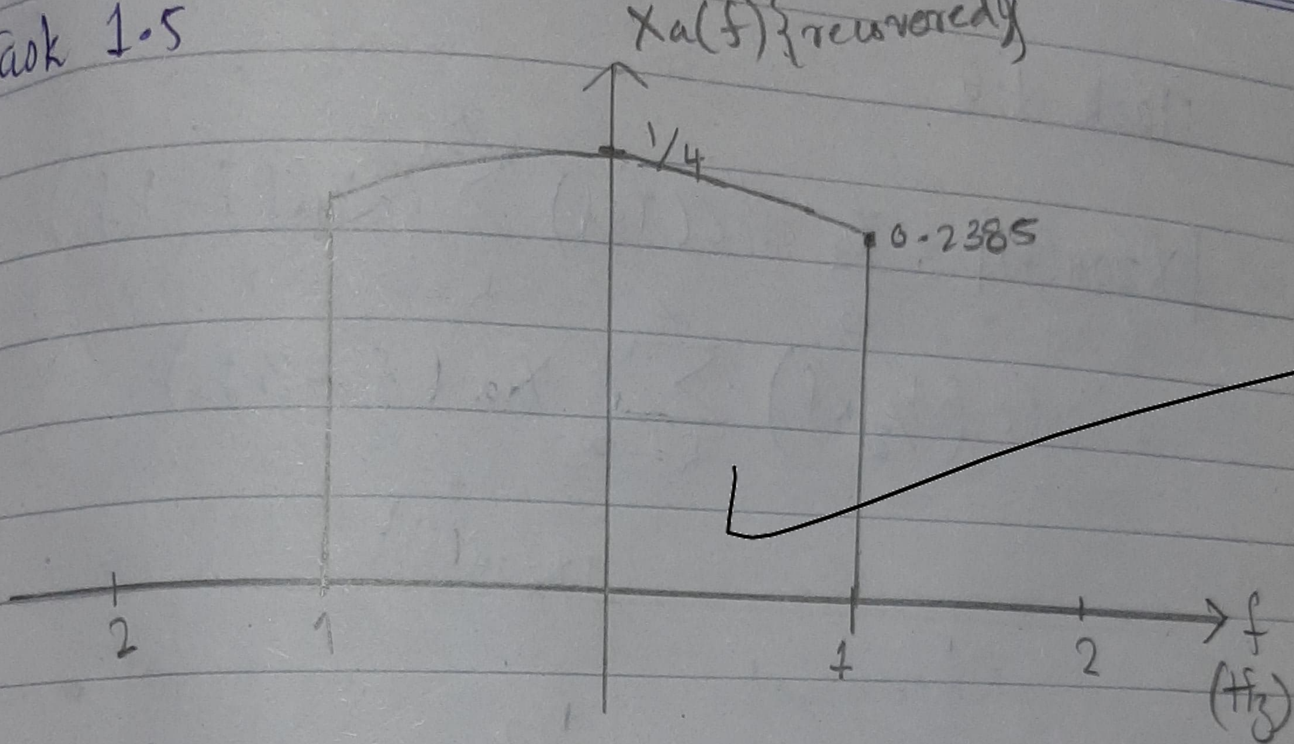
$1 \leq B \leq 2.$

$-2 \leq -B \leq -1$

Bandwidth $= 2B$

$2 \leq$ Bandwidth ≤ 4 Hz

Task 1.5



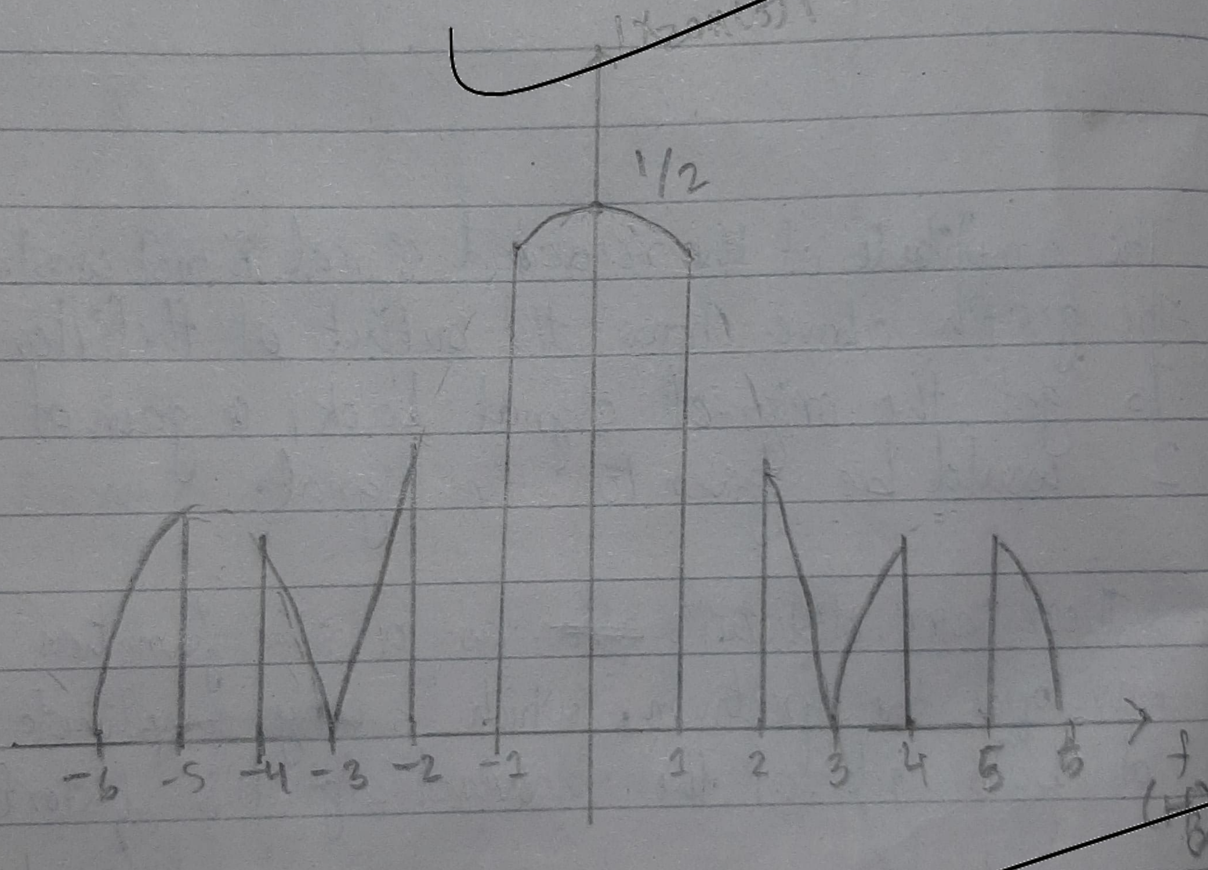
The amplitude of the recovered signal is not constant. The graph above shows the output of the filter. To get the original signal back, a gain of 2 would be given to the signal.

→ There are distortions ~~are~~ as a sinc function envelopes the spectrum, which is why amplitude of both ends of the recovered signal is distorted.

Task 1.6

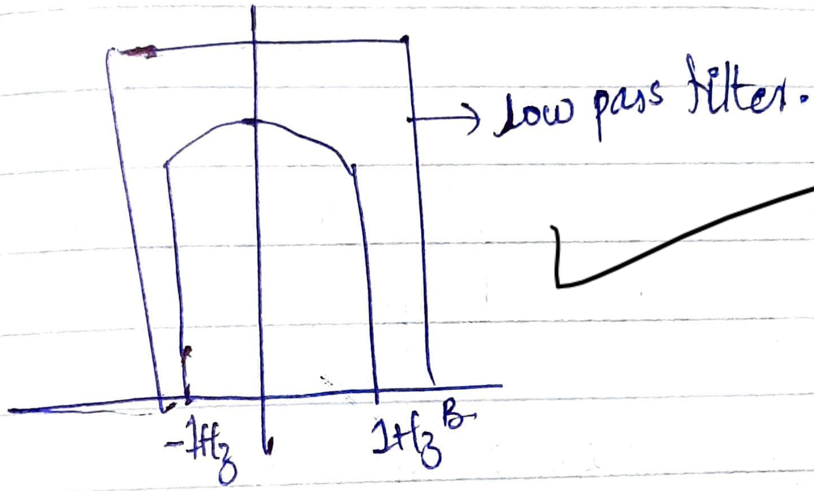
$$|X_{ZOH}(f)| = \frac{T_s}{T_s} \operatorname{sinc}(T_s f) \sum_{k=-\infty}^{\infty} X_a(f - k f_s)$$

$$= \operatorname{sinc}\left(\tfrac{1}{3} \times f\right) \sum_{k=-\infty}^{\infty} X_a(f - 3k)$$



Task 1.7

It is possible to reconstruct as the spectrums does not overlap each other. The band frequencies are the same as for ~~NRZ~~ return to zero scheme thus the bandwidth of the low pass filter must also be the same.



Bandwidth of low pass filter = 2B.

$B \geq 1Hz$.

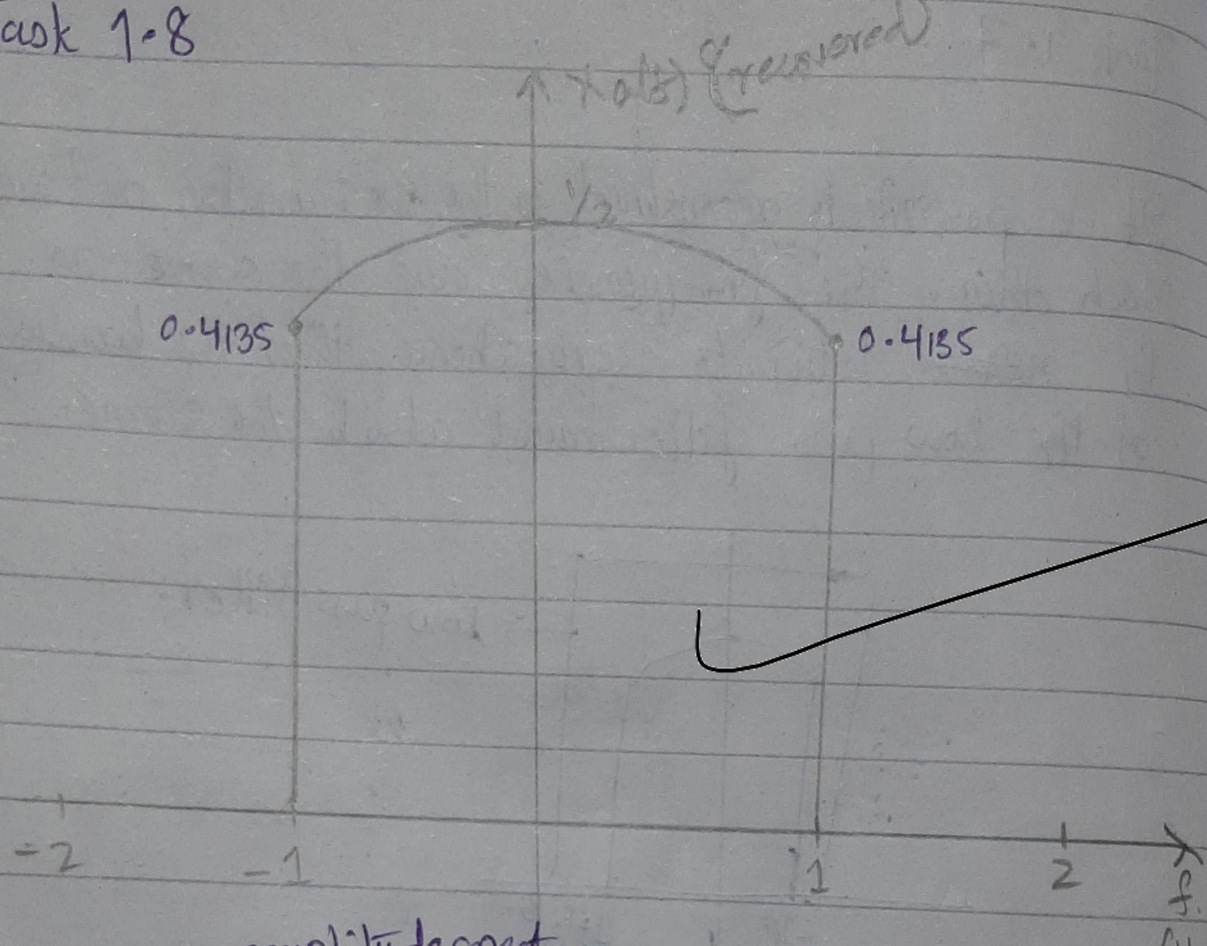
and $B \leq 2Hz$ (argument same as 1.4)

thus $1 \leq B \leq 2Hz$

$2 \leq$ Bandwidth ≤ 4

~~sinc function~~ impulse response of filter is sinc function

Task 1.8

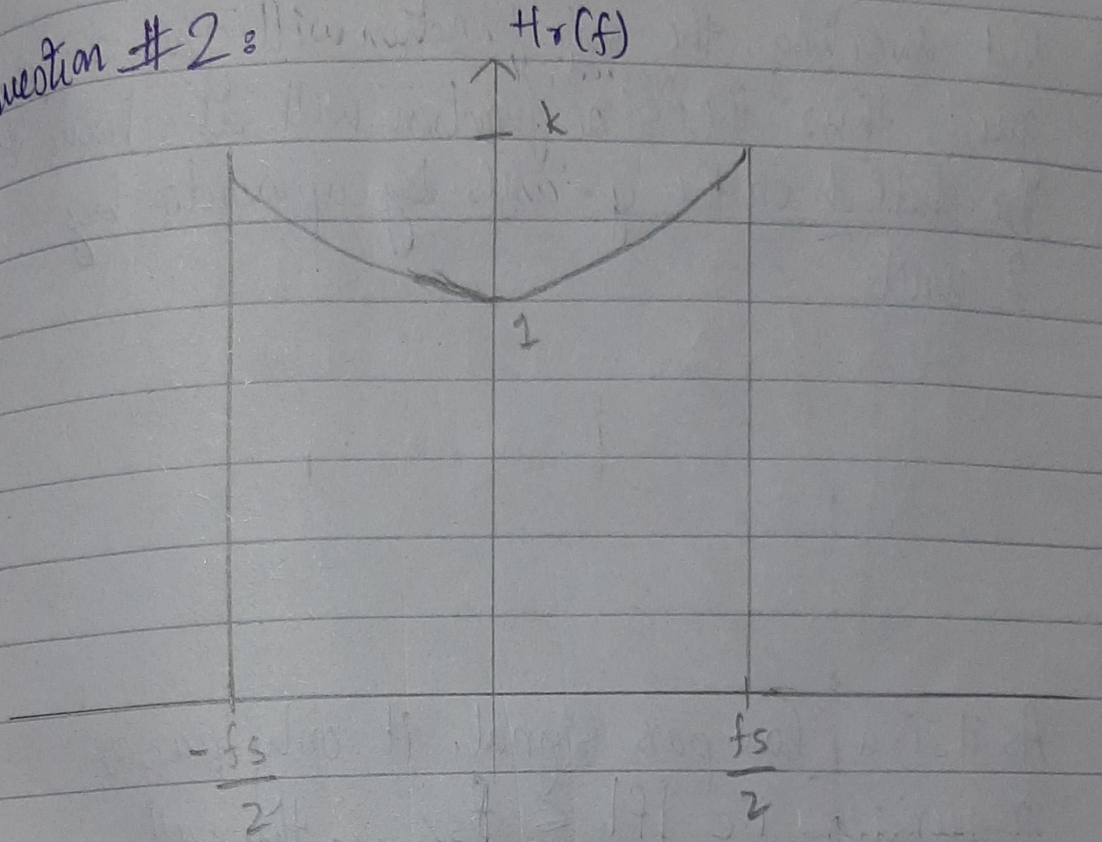


Xa(f) is the amplitude spectrum fourier transform of the recoved signal. It can be seen that the amplitude of the spectrum is distorted. The square pulse now looks eliptical from the top.

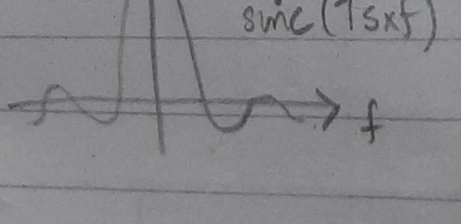
The distortion is greater that ~~Rt~~ Return to zero scheme. ~~Thes~~ As the sinc function (i.e the evelope) gets closer towards the y-axis.

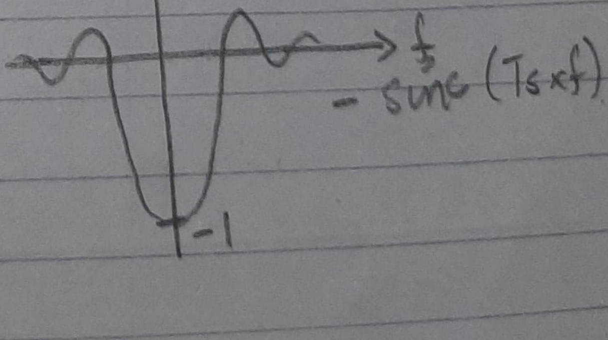
This also establishes that a lower duty cycle for the zero order hold sampling enables ~~a~~ the ~~reconstru~~ reconstructed signal to be of higher accuracy

Question #2:

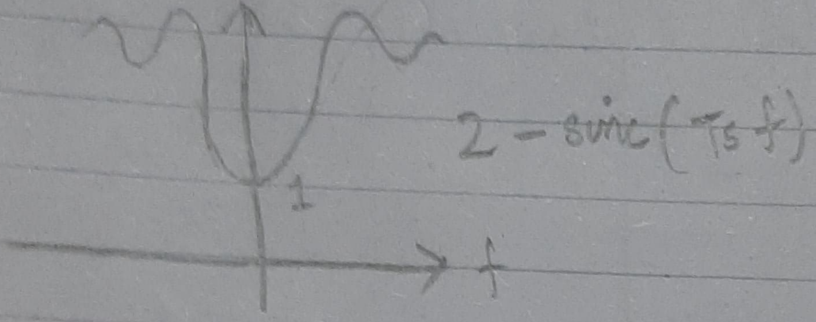


Since Hr(f) will reconstruct a signal distorted by the envelope of sinc($T_s \times f$), it's curve must be opposite to the curve of sinc($T_s \times f$) between $-\frac{f_s}{2}$ and $\frac{f_s}{2}$

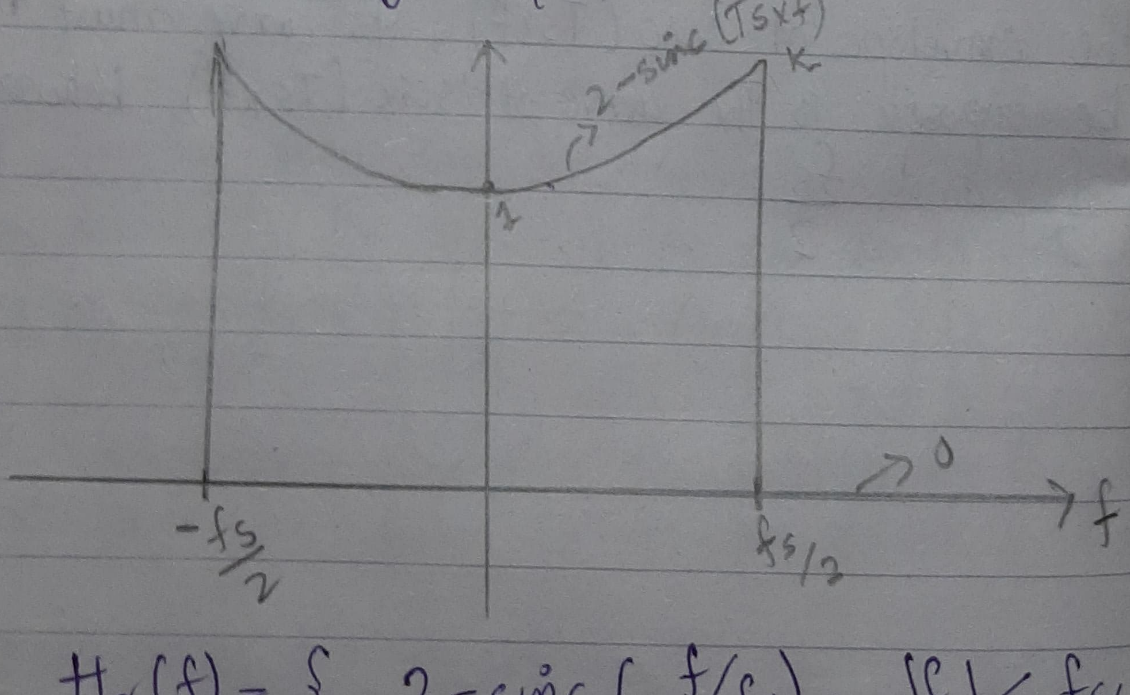




But inverting the sinc function will give a negative gain, thus the inverted sinc function will also have to be shifted on the y-axis upwards by 2 units.



As it is a low pass signal it only passes frequencies i.e $|f| \leq f_s/2$, thus all other frequency components are zero.



$$H_r(f) = \begin{cases} 2 - \text{sinc}(f/f_s), & |f| \leq f_s/2 \\ 0, & |f| > f_s/2 \end{cases}$$

Task 2.2

$$K = H_r(f_s/2) = 2 - \text{sinc}\left(\frac{f_s/2}{f_s}\right)$$

$$= 2 - \text{sinc}\left(\frac{1}{2}\right)$$

$$\cancel{= 2 - \frac{\sin(\pi/2)}{\pi/2}}$$

$$= 2 - \frac{\sin(\pi/2)}{\pi/2} = 2 - \frac{2}{\pi}$$

$$= 1.363$$

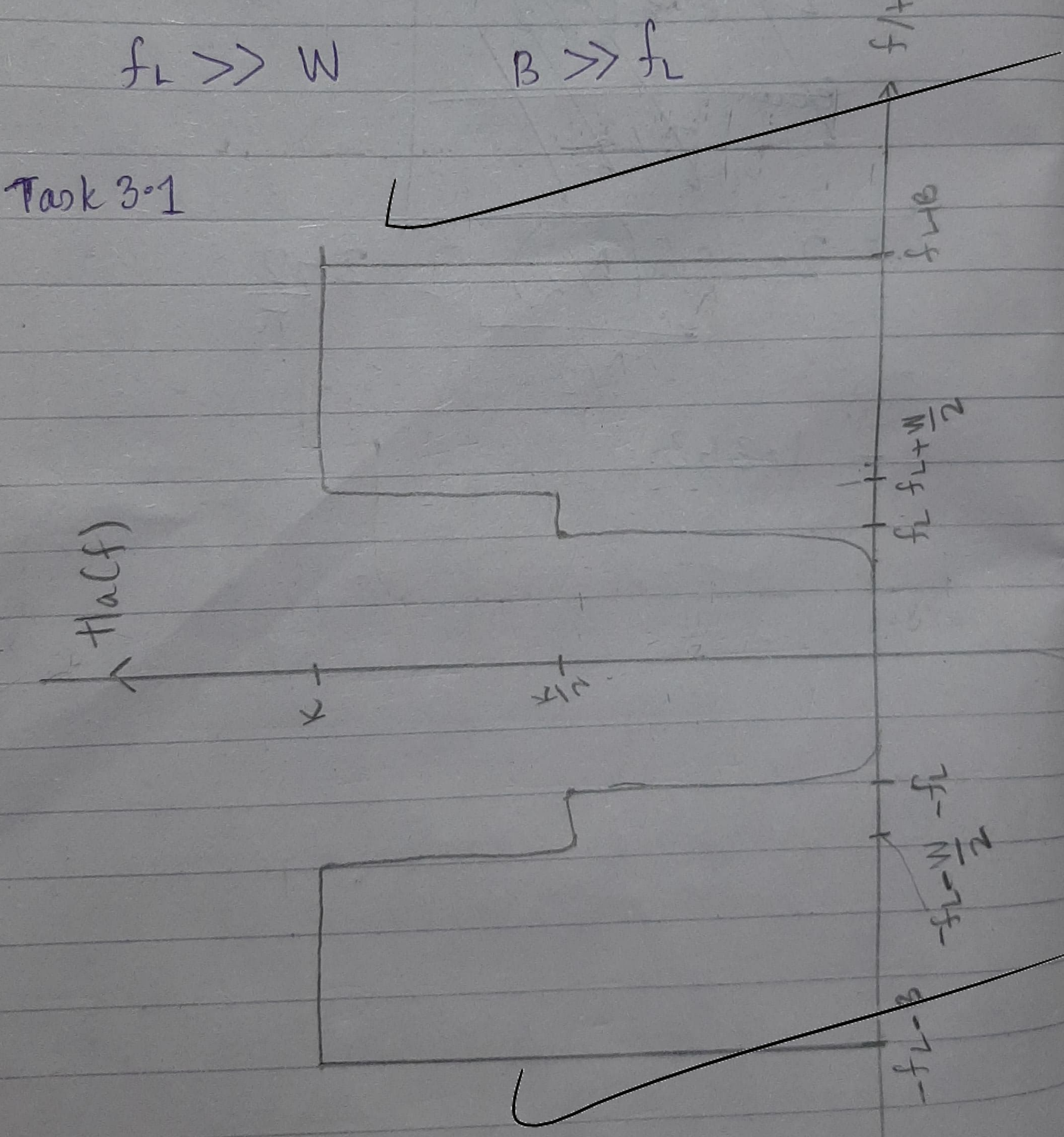
Good discussion; however a very simple solution for $H_r(f)$ is that it must be the reciprocal of $\text{sinc}(dT_sf)$ for $-f_s/2 \leq f \leq +f_s/2$

Question #3:

$$H_a(f) = \begin{cases} 0 & |f| < f_L - \epsilon \\ K/2 & f_L \le |f| \le f_L + W/2 \\ K & f_L + W/2 \le |f| \le f_L + B \\ 0 & |f| > f_L + B \end{cases}$$

$f_L >> W$ $\qquad$ $B >> f_L$

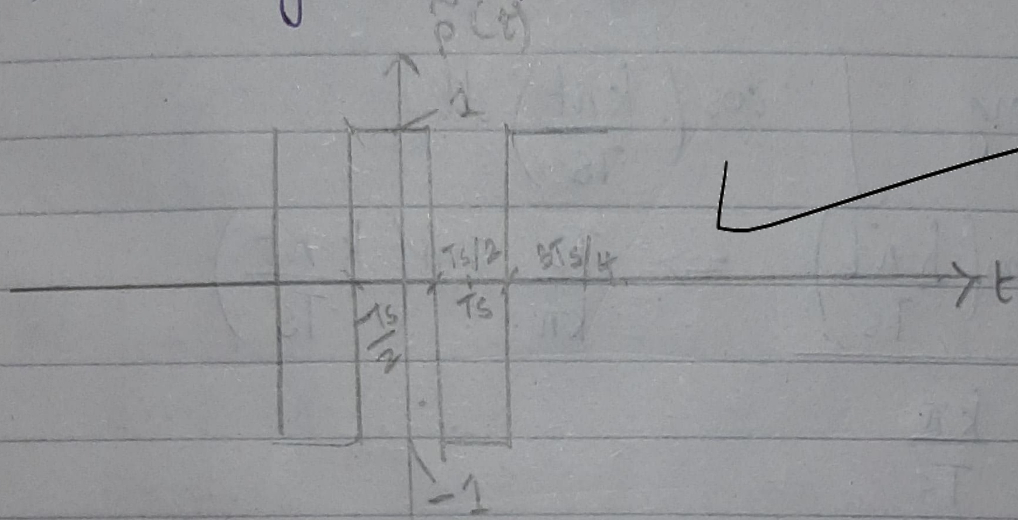
Task 3.1



Task 3.2

a) Time period of of $\tilde{p}(t) = 2T_s$

b) even symmetric $\tilde{p}(t)$



$$\tilde{p}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

avg value of signal $= 0$ i.e $a_0 = 0$

Signal is even symmetric $\Rightarrow b_k = 0$

$$a_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} \tilde{p}(t) \cos(k\omega_0 t)\, dt$$

$$a_k = \frac{2}{2T_s} \int_{t_0}^{t_0+2T_s} \tilde{p}(t) \cos\left(k \frac{\cancel{2}\pi}{\cancel{2}T_s} t\right) dt$$

$$a_k = \frac{1}{T_s}\left[\int_{-T_s/2}^{T_s/2} 1 \times \cos\left(\frac{k\pi t}{T_s}\right) dt + \int_{T_s/2}^{3T_s/2} -1 \times \cos\left(\frac{k\pi t}{T_s}\right) dt\right].$$

evaluating $\int \cos\left(\frac{k\pi t}{T_s}\right) dt$

$$\Rightarrow \frac{\sin\left(\frac{k\pi t}{T_s}\right)}{\frac{k\pi}{T_s}} \Rightarrow \frac{T_s}{k\pi} \sin\left(\frac{k\pi t}{T_s}\right).$$

$$a_k = \frac{1}{T_s}\left[\frac{T_s}{k\pi}\sin\left(\frac{k\pi t}{T_s}\right)\Big|_{-\frac{T_s}{2}}^{\frac{T_s}{2}} - \frac{T_s}{k\pi}\sin\left(\frac{k\pi t}{T_s}\right)\Big|_{\frac{T_s}{2}}^{\frac{3T_s}{2}}\right]$$

$$= \frac{\cancel{T_s}}{\cancel{T_s}\, k\pi}\left[\sin\left(\frac{k\pi \cancel{T_s}/2}{\cancel{T_s}}\right) + \sin\left(\frac{k\pi(+T_s/2)}{T_s}\right) - \sin\left(\frac{k\pi\, 3\cancel{T_s}/2}{\cancel{T_s}}\right) + \sin\left(\frac{k\pi\, T_s/2}{T_s}\right)\right].$$

$$= \frac{1}{k\pi}\left[\sin\left(\frac{k\pi}{2}\right) + \sin\left(\frac{k\pi}{2}\right) - \sin\left(\frac{k\,3\pi}{2}\right) + \sin\left(\frac{k\pi}{2}\right)\right]$$

$$= \frac{1}{k\pi}\left[3\sin\left(\frac{k\pi}{2}\right) - \sin\left(\frac{k3\pi}{2}\right)\right]$$

when $k = 0$.

$$\sin\left(\frac{k\pi}{2}\right) \ \& \ \sin\left(\frac{k3\pi}{2}\right) = 0$$

thus $a_k = 0$ for all even k.

for $k = 1, 5, 9 \ldots$ i.e

$$\frac{1}{k\pi}\left[3\sin\left(\frac{k\pi}{2}\right) - \sin\left(\frac{k3\pi}{2}\right)\right].$$

1st quadrant (+1) 3rd quadrant (−1)

$$= \frac{1}{k\pi}\left(3(1) - (-1)\right) = \frac{4}{k\pi}.$$

for $k = 3, 7, 11 \ldots$

$$\frac{1}{k\pi}\left[3\sin\left(\frac{k\pi}{2}\right) - \sin\left(\frac{k3\pi}{2}\right)\right)$$

3rd quadrant (−1) 1st quadrant (+1)

$$= \frac{1}{k\pi}\left(3(-1) - (1)\right] = -\frac{4}{k\pi}$$

Task 3.4

$$X_a(f) \longleftrightarrow x_a(t).$$

$$x_s(t) = \tilde{p}(t)\, x_a(t)$$

$$\tilde{p}(t) = \sum_{k=1,5,9,\ldots}^{\infty} \frac{4}{k\pi} \cos\left(\frac{\pi k t}{T_s}\right) + \sum_{k=3,7,11,\ldots}^{\infty} -\frac{4}{k\pi} \cos\left(\frac{\pi k t}{T_s}\right)$$

$$x_s(t) = \frac{4}{k\pi}\left[\sum_{k=1,5,9,\ldots}^{\infty} x_a(t) \cos\left(\frac{\pi k t}{T_s}\right) - \sum_{k=3,7,11,\ldots}^{\infty} x_a(t) \cos\left(\frac{\pi k t}{T_s}\right)\right]$$

$$\mathcal{F}\{x_s(t)\} = \mathcal{F}\left\{\frac{4}{k\pi}\left[\sum_{k=1,5,9}^{\infty} x_a(t) \cos\left(\frac{\pi k t}{T_s}\right) - \sum_{k=3,7,11,\ldots}^{\infty} x_a(t) \cos\left(\frac{\pi k t}{T_s}\right)\right]\right\}$$

$$2\pi f_0 \cancel{t} = \frac{\pi k \cancel{t}}{T_s} \Rightarrow f_0 = \frac{k}{2T_s}$$

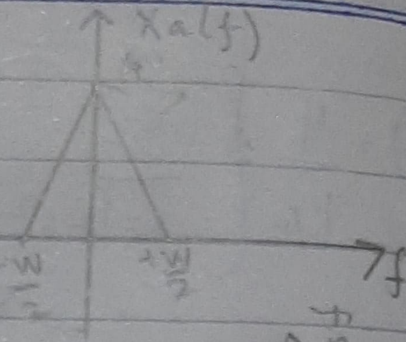
$$X_s(f) = \frac{4}{k\pi}\left[\sum_{k=1,5,9\ldots}^{\infty} \mathcal{F}\left\{x_a(t)\cos\left(\frac{\pi k t}{T_s}\right)\right\} - \sum_{k=3,7,11\ldots}^{\infty} \mathcal{F}\left\{x_a(t)\cos\left(\frac{\pi k t}{T_s}\right)\right\}\right]$$

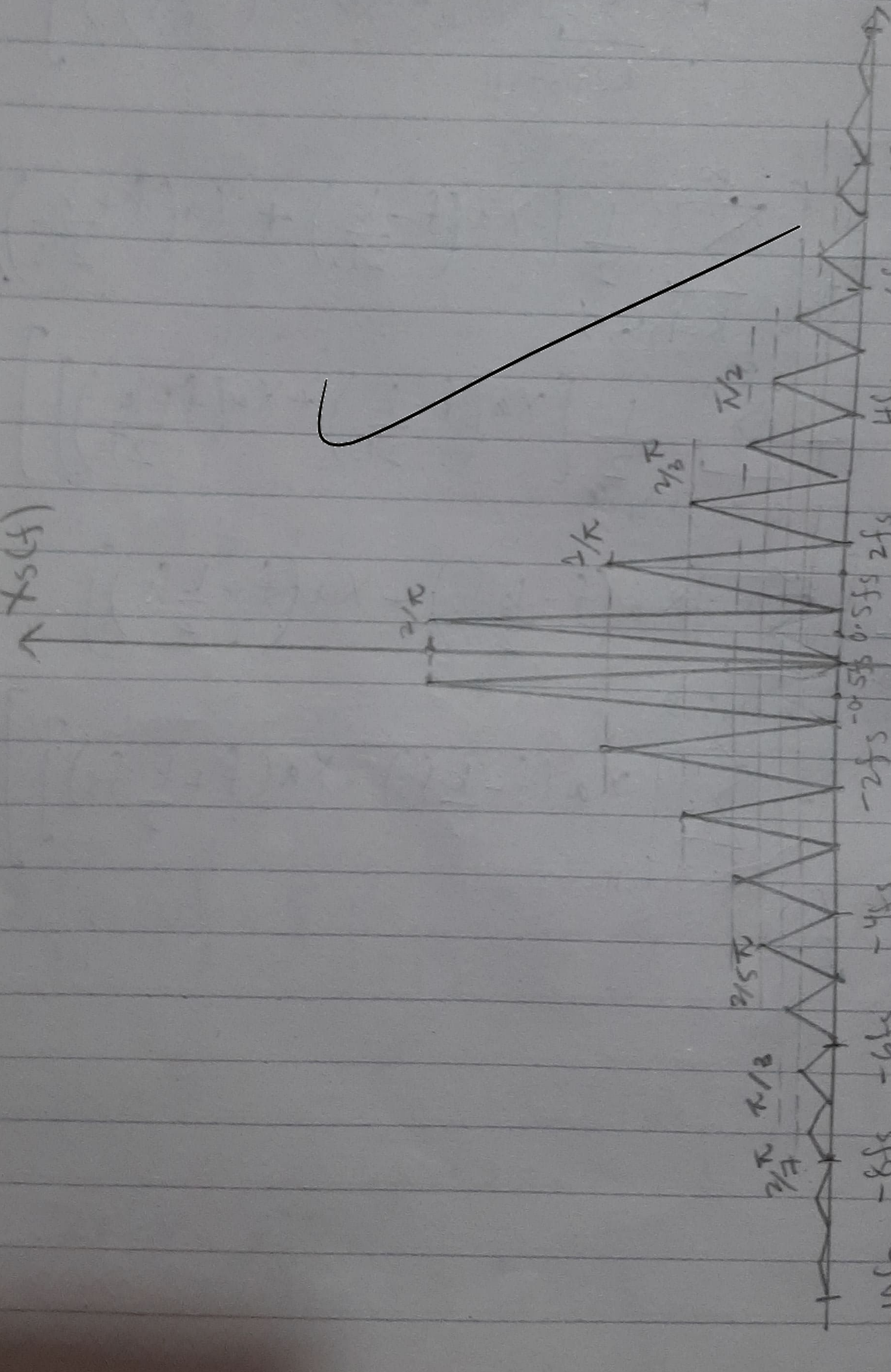
$$X_s(f) = \frac{4}{k\pi}\left[\sum_{k=1,5,9}^{\infty} \frac{1}{2}\left(X_a\left(f-\frac{k}{2T_s}\right) + X_a\left(f+\frac{k}{2T_s}\right)\right) - \sum_{k=3,7,11}^{\infty} \frac{1}{2}\left[X_a\left(f-\frac{k}{2T_s}\right) + X_a\left(f+\frac{k}{2T_s}\right)\right]\right]$$

$$= \frac{2}{k\pi}\left[\sum_{k=1,5,9\ldots}^{\infty}\left[X_a\left(f-\frac{k}{2}f_s\right) + X_a\left(f+\frac{k}{2}f_s\right)\right] - \sum_{k=3,7,11,\ldots}^{\infty}\left[X_a\left(f-\frac{k}{2}f_s\right) + X_a\left(f+\frac{k}{2}f_s\right)\right]\right]$$

Task 3.5

let $X_a(f)$ =





Task 3.6

$$0.5f_s - w \geq f_L + \frac{w}{2}$$

$$0.5 f_s \geq f_L + \frac{w}{2} + w$$

$$0.5 f_s \geq f_L + \frac{3w}{2}$$

$$f_s \geq \frac{f_L}{0.5} + \frac{3w}{0.5 \times 2}$$

$$f_s \geq 2f_L + 3w$$

Taking the least fs to save resources:

$$\Rightarrow f_s = 2f_L + 3w.$$

Question # 4:

$$x_s(t) = a_1\delta(t) + a_2\delta(t-T_s) + a_3\delta(t-2T_s).$$

selecting a_1, a_2 and a_3 randomly

$$x_s(t) = \delta(t) - \delta(t-T_s) + 3\delta(t-2T_s)$$

where $a_1 = 1$, $a_2 = -1$ and $a_3 = 3$

Task 4.1.

$h(t)$

$T_s = 1$.

$$h(t) = \begin{cases} t & 0 \leq t < 1 \\ 2-t & 1 \leq t < 2 \\ 0 & t < 0 \text{ and } t \geq 2 \end{cases}$$

$$\Rightarrow h(t) = t[u(t) - u(t-1)] + (2-t)[u(t-1) - u(t-2)]$$

$$= tu(t) - tu(t-1) + 2u(t-1) - 2u(t-2) - tu(t-1) + tu(t-2)$$

$$= t\,u(t) + (2 - 2t)\,u(t-1) + (t-2)\,u(t-2)$$

$$h(t) = t\,u(t) + 2(1-t)\,u(t-1) + (t-2)\,u(t-2)$$

Task 4.2

$x_R(t) = x_s(t) * h(t)$ as $T_s = 1$

$x_s(t) = \delta(t) - \delta(t-1) + 3\delta(t-2)$

$h(t) = t\,u(t) + 2(1-t)\,u(t-1) + (t-2)\,u(t-2)$

$x_R(t) = t\,u(t) + 2(1-t)\,u(t-1) + (t-2)\,u(t-2)$

$- \left[(t-1)\,u(t-1) + 2(1-t+1)\,u(t-1-1) + (t-1-2)\,u(t-1-2)\right]$

$+ 3\left[(t-2)\,u(t-2) + 2(1-t+2)\,u(t-2-1) + (t-2-2)\,u(t-2-2)\right]$

$\Rightarrow t\,u(t) + 2(1-t)\,u(t-1) + (t-2)\,u(t-2)$

$- \left[(t-1)\,u(t-1) + 2(2-t)\,u(t-2) + (t-3)\,u(t-3)\right]$

$+ 3\left[(t-2)\,u(t-2) + 2(3-t)\,u(t-3) + (t-4)\,u(t-4)\right]$

$$\Rightarrow t\,u(t) + 2(1-t)\,u(t-1) + (t-2)\,u(t-2)$$

$$-(t-1)\,u(t-1) - (4-2t)\,u(t-2)$$

$$-(t-3)\,u(t-3) + (3t-6)\,u(t-2)$$

$$+ 6(3-t)\,u(t-3) + 3(t-4)\,u(t-4).$$

$$\Rightarrow t\,u(t)$$

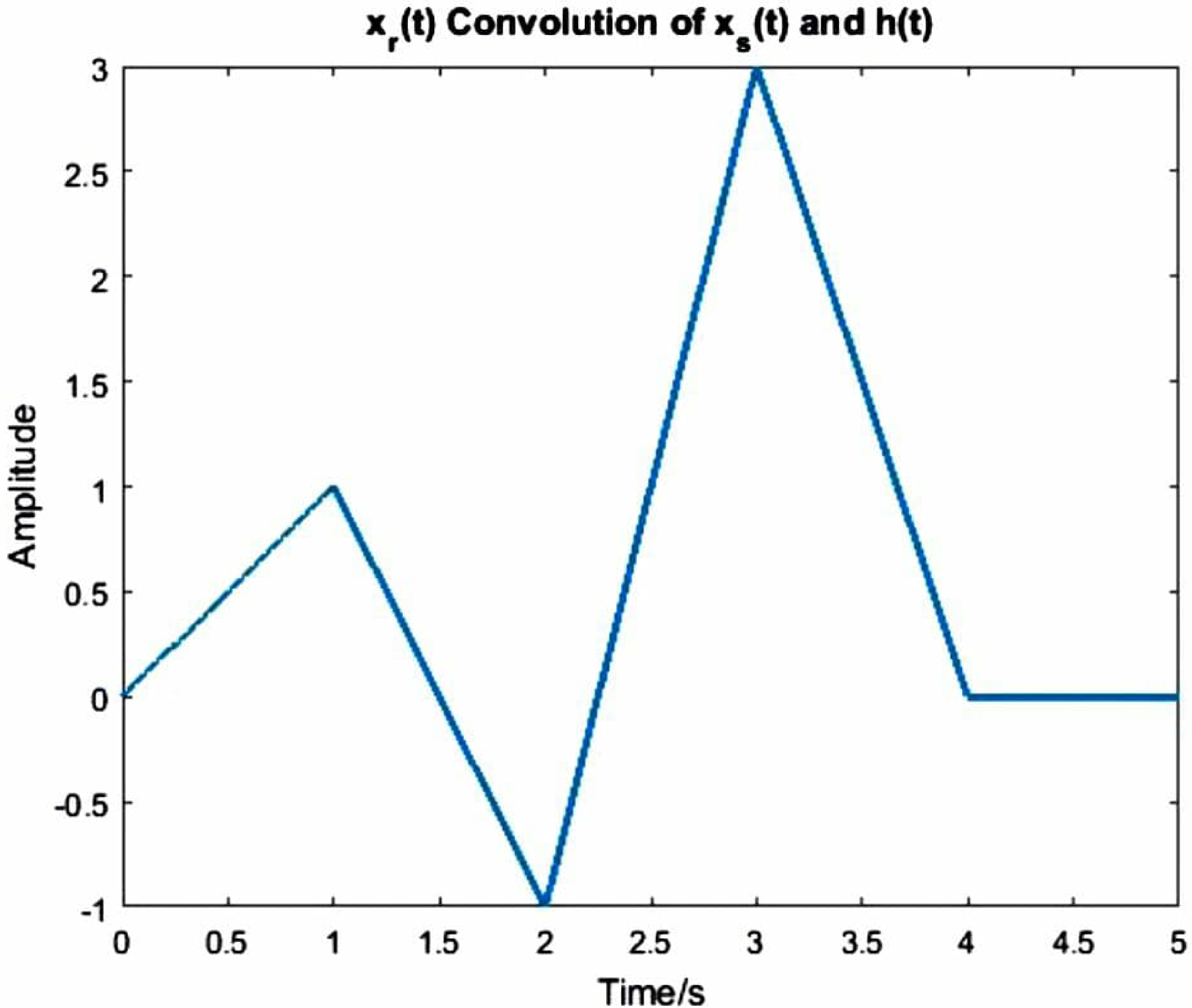
$$+ \left[2(1-t) - (t-1)\right] u(t-1) \Rightarrow 2-2t-t+1 \Rightarrow 3-3t$$

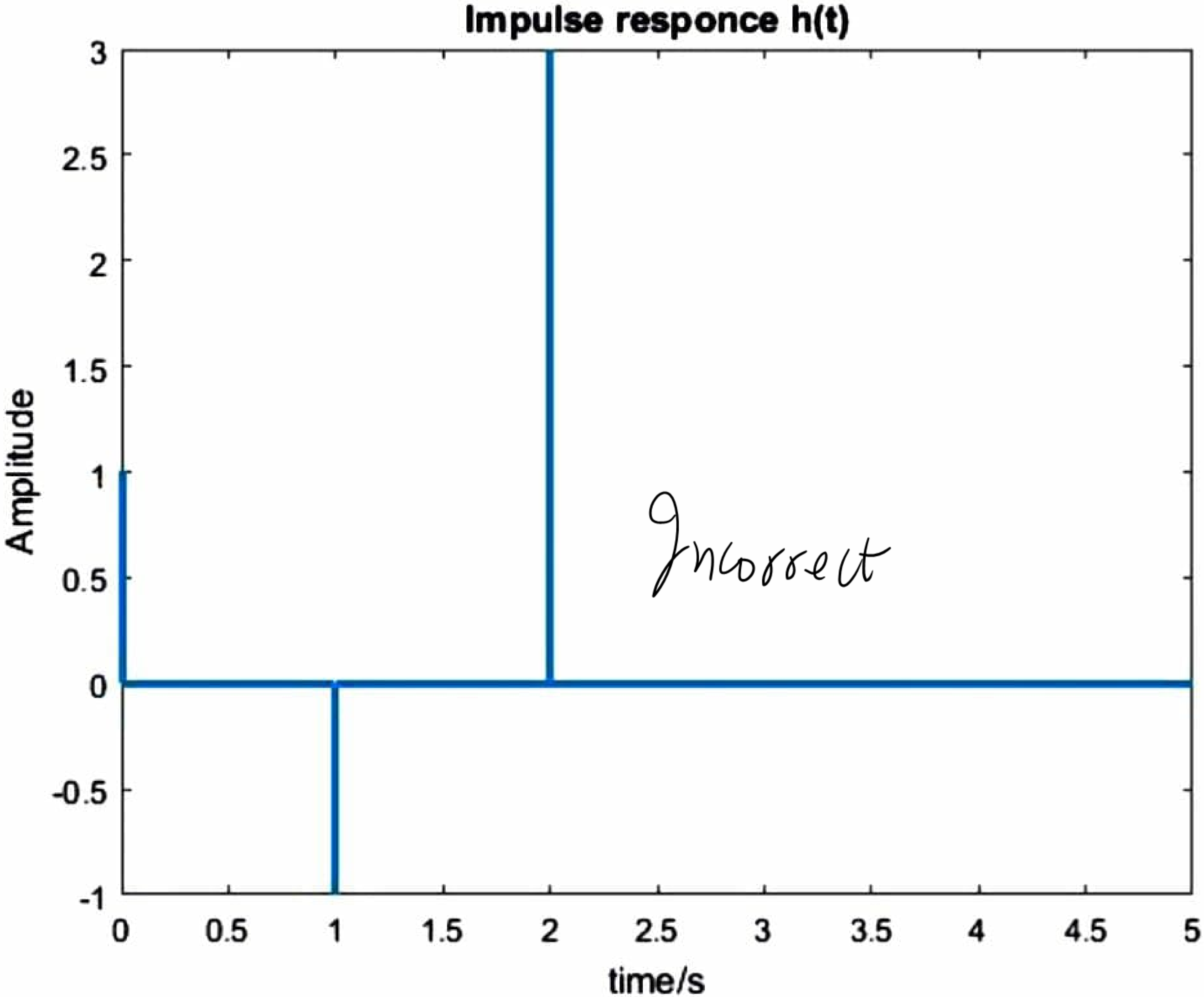
$$+ \left[(t-2) - (4-2t) + (3t-6)\right] u(t-2) \quad \begin{array}{l} t-2-4+2t \\ +3t-6 \\ 6t-12 \end{array}$$

$$+ \left[-(t-3) + 6(3-t)\right] u(t-3) \quad -t+3+18-6t = 21-7t$$

$$+ 3\left[t-4\right] u(t-4).$$

$$x_R(t) = t\,u(t) + (3-3t)\,u(t-1) + (6t-12)\,u(t-2) + (21-7t)\,u(t-3) + (3t-12)\,u(t-4)$$


$x_r(t)$ Convolution of $x_s(t)$ and h(t)
Amplitude
Time/s
3
2.5
2
1.5
1
0.5
0
-0.5
-1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


Impulse responce h(t)
Amplitude
time/s
3
2.5
2
1.5
1
0.5
0
-0.5
-1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Incorrect

Q # 5:

$$\tilde{x}[n] = [\cdots, 1, 2, 2, 1, \cdots]$$

$$\tilde{h}[n] = [\cdots, 0, 1, 1, 1, \cdots]$$

5.1:

$$\tilde{y}[n] = \tilde{x}[n] \circledast \tilde{h}[n]$$

$$\tilde{y}[n] = \sum_{k=-\infty}^{\infty} \tilde{x}[k]\tilde{h}[n-k]$$

$\tilde{y}[0] \Rightarrow$

$$\begin{aligned}
k=0 &\Rightarrow x[0]\,h[0] && \Rightarrow 1\times 0 \\
k=1 &\quad + x[1]\,h[3]. && + 2\times 1 \\
k=2 &\quad + x[2]\,h[2] && + 2\times 1 \\
k=3 &\quad + x[3]\,h[1] && + 1\times 1 \\
& && \overline{5}
\end{aligned}$$

$\tilde{y}[1] \Rightarrow$

$$\begin{aligned}
k=0 &\Rightarrow x[0]\,h[1] && \Rightarrow 1\times 1 \\
k=1 &\quad + x[1]\,h[0]. && + 2\times 0 \\
k=2 &\quad + x[2]\,h[3] && + 2\times 1 \\
k=3 &\quad + x[3]\,h[2] && + 1\times 1 \\
& && \overline{4}
\end{aligned}$$

$\tilde{y}[2] \Rightarrow$

$$
\begin{array}{llll}
k=0 & \Rightarrow x[0]\,h[2] & \Rightarrow & 1\times 1 \\
k=1 & +\, x[1]\,h[1] & & +\,2\times 1 \\
k=2 & +\, x[2]\,h[0] & & +\,2\times 0 \\
k=3 & +\, x[3]\,h[3] & & \underline{+\,1\times 1} \\
 & & & 4
\end{array}
$$

$\tilde{y}[3] \Rightarrow$

$$
\begin{array}{llll}
k=0 & \Rightarrow x[0]\,h[3] & \Rightarrow & 1\times 1 \\
k=1 & +\, x[1]\,h[2] & & +\,2\times 1 \\
k=2 & +\, x[2]\,h[1] & & +\,2\times 1 \\
k=3 & +\, x[3]\,h[0] & & \underline{+\,1\times 0} \\
 & & & 5
\end{array}
$$

$\tilde{y}[n] = [\ldots, 5, 4, 4, 5, \ldots]$

5.2

DTFS coefficients :

$$\cancel{c_k = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{x}[k] e^{j2\pi kn/N}}$$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi kn}{N}}$$

$\tilde{x}[n] = [1, 2, 2, 1]$

$$\Rightarrow c_k = \frac{1}{4}\left[1 \times e^{0} + 2 \times e^{-j\frac{2\pi k \times 1}{4}} + 2 \times e^{-j\frac{2\pi k \times 2}{4}} + 1 \times e^{-j\frac{2\pi k \times 3}{4}}\right]$$

$$\Rightarrow c_k = \frac{1}{4}\left[1 + 2e^{-jk\pi/2} + 2e^{-jk\pi} + e^{-jk\,3\pi/2}\right]$$

$$\Rightarrow c_k = \frac{1}{4}\left[1 + 2\left[\cos\left(-\frac{k\pi}{2}\right) + j\sin\left(-\frac{k\pi}{2}\right)\right] + 2\left[\cos(-k\pi) + j\sin(-k\pi)\right] + \cos\left(-k\frac{3\pi}{2}\right) + j\sin\left(-k\frac{3\pi}{2}\right)\right]$$

$$\Rightarrow c_k = \frac{1}{4}\left[1 + 2\cos\left(\frac{k\pi}{2}\right) + 2\cos(k\pi) + \cos\left(\frac{k3\pi}{2}\right)\right.$$
$$\left. - j\left[2\sin\left(\frac{k\pi}{2}\right) + 2\sin(k\pi) + \sin\left(\frac{k3\pi}{2}\right)\right]\right]$$

$$c_0 = \frac{1}{4}\left[1 + 2\cos 0 + 2\cos 0 + \cos 0 - j \times 0\right].$$

$$c_0 = \frac{1}{4}\left[1 + 2 + 2 + 1\right] = \frac{6}{4} = \frac{3}{2}$$

$$c_1 = \frac{1}{4}\left[1 + 2\cos\left(\frac{\pi}{2}\right) + 2\cos(\pi) + \cos\left(\frac{3\pi}{2}\right)\right.$$
$$\left. - j\left[2\sin\left(\frac{\pi}{2}\right) + 2\sin(\pi) + \sin\left(\frac{3\pi}{2}\right)\right]\right]$$

$$c_1 = \frac{1}{4}\left[1 + 0 - 2 + 0 - j(2 - 1)\right] = -\frac{1}{4} - j\frac{1}{4}$$

$$c_2 = \frac{1}{4}\left[1 + 2\cos(\pi) + 2\cos(2\pi) + \cos(3\pi) - j(0)\right]$$

$$= \frac{1}{4}\left[1 - 2 + 2 - 1\right] = 0$$

$$c_3 = \frac{1}{4}\left[1 + 2\cos\left(\frac{3\pi}{2}\right) + 2\cos(3\pi) + \cos\left(\frac{9\pi}{2}\right)\right.$$
$$\left. - j\left[2\sin\left(\frac{3\pi}{2}\right) + 2\sin(3\pi) + \sin\left(\frac{9\pi}{2}\right)\right]\right]$$

$$= \frac{1}{4}(-1 + j) = -\frac{1}{4} + j\frac{1}{4}$$

$h[n] = [0, 1, 1, 1]$

$$\Rightarrow d_k = \frac{1}{4}\left[0 \times e^0 + 1 \times e^{-j\frac{2\pi k}{4} \times 1} + 1 \times e^{-j\frac{2\pi k}{4} \times 2} + 1 \times e^{-j\frac{2\pi k}{4} \times 3} \right]$$

$$\Rightarrow d_k = \frac{1}{4}\left[\cos\left(\frac{k\pi}{2}\right) - j\sin\left(\frac{k\pi}{2}\right) + \cos(k\pi) - j\overset{\nearrow 0}{\sin(k\pi)} + \cos\left(\frac{k3\pi}{2}\right) - j\sin\left(\frac{k3\pi}{2}\right) \right]$$

$$d_0 = \frac{1}{4}\left[\cos 0 + \cos 0 + \cos 0 - j \sin \times 0 \right]$$

$$d_0 = \frac{3}{4}$$

$$d_1 = \frac{1}{4}\left[\cos\left(\frac{\pi}{2}\right) + \cos(\pi) + \cos\left(\frac{3\pi}{2}\right) - j\left\{ \sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{3\pi}{2}\right) \right\} \right]$$

$$d_1 = \frac{1}{4}\left[0 - 1 + 0 - j(1 - 1) \right] = -\frac{1}{4}$$

$$d_2 = \frac{1}{4}\Big[\cos(\pi) + \cos(2\pi) + \cos(3\pi)$$

$$-j\big(\sin(\pi) + \sin(3\pi)\big)\Big].$$

$$= \frac{1}{4}\left(-1+1-1-j\times 0\right) = -\frac{1}{4}$$

$$d_3 = \frac{1}{4}\Big[\cos\left(\frac{3\pi}{2}\right) + \cos(3\pi) + \cos\left(\frac{9\pi}{2}\right).$$

$$-j\left[\sin\left(\frac{3\pi}{2}\right) + \sin\left(\frac{9\pi}{2}\right)\right]\Big].$$

$$= \frac{1}{4}\left[0 - 1 + 0 - j(-1+1)\right] = -\frac{1}{4}$$

let DTFS coefficients of $y[n] = p_k$

by cyclic convolution property: $p_k = N c_k d_k$

k	c_k	d_k	p_k
0	$3/2$	$3/4$	$9/2$
1	$-1/4 - j1/4$	$-1/4$	$1/4 + j1/4$
2	0	$-1/4$	0
3	$-1/4 + j1/4$	$-1/4$	$1/4 - j1/4$

Finding DTFS coefficients for $y[n]$:

$y[n] = [5, 4, 4, 5]$

$$p_k = \frac{1}{4}\left[5 \times e^0 + 4 \times e^{-j\frac{2\pi k}{4} \times 1} + 4 \times e^{-j\frac{2\pi k}{4} \times 2} + 5 \times e^{-j\frac{2\pi k}{4} \times 3}\right]$$

$$\Rightarrow p_k = \frac{1}{4}\left[5 + 4\left(\cos\left(\frac{k\pi}{2}\right) - j\sin\left(\frac{k\pi}{2}\right)\right) + 4\left[\cos(k\pi) - j\sin(k\pi)\right] + 5\left[\cos\left(\frac{k3\pi}{2}\right) - j\sin\left(\frac{k3\pi}{2}\right)\right]\right]$$

$$\Rightarrow p_k = \frac{1}{4}\left[5 + 4\cos\left(\frac{k\pi}{2}\right) + 4\cos(k\pi) + 5\cos\left(\frac{k3\pi}{2}\right) - j\left[4\sin\left(\frac{k\pi}{2}\right) + 5\sin\left(\frac{k3\pi}{2}\right)\right]\right]$$

$$\Rightarrow p_0 = \frac{1}{4}\left[5 + 4 + 4 + 5 - j \times 0\right]$$

$$p_0 = \frac{18}{4} = \frac{9}{2}$$

$$\Rightarrow P_1 = \frac{1}{4}\left[5 + 4\cos\left(\frac{\pi}{2}\right) + 4\cos(\pi) + 5\cos\left(\frac{3\pi}{2}\right)\right.$$

$$\left. -j\left[4\sin\left(\frac{\pi}{2}\right) + 5\sin\left(\frac{3\pi}{2}\right)\right]\right].$$

$$= \frac{1}{4}\left(5 + 0 - 4 + 0 - j(4 - 5)\right].$$

$$P_1 = 1/4 + j\,1/4.$$

$$P_2 = \frac{1}{4}\left[5 + 4\cos(\pi) + 4\cos(2\pi) + 5\cos(3\pi)\right.$$

$$\left. -j\left(4\sin(\pi) + 5\sin(3\pi)\right)\right].$$

$$= \frac{1}{4}\left(5 - 4 + 4 - 5 - j \times 0\right] = 0$$

$$P_3 = \frac{1}{4}\left[5 + 4\cos\left(\frac{3\pi}{2}\right) + 4\cos(3\pi) + 5\cos\left(\frac{9\pi}{2}\right)\right.$$

$$\left. -j\left[4\sin\left(\frac{3\pi}{2}\right) + 5\sin\left(\frac{9\pi}{2}\right)\right]\right].$$

$$= \frac{1}{4}\left(5 + 0 - 4 + 0 - j(-4 + 5)\right)].$$

$$P_3 = 1/4 - j\,1/4$$

The calculated values match the one in the table, thus conv. property holds.

Question. #6

6·1 $x[n] = \frac{1}{2}\left(\frac{1}{2^n} + \frac{1}{4^n}\right) u[n]$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left(\frac{1}{2^n} + \frac{1}{4^n}\right) u[n] e^{-j\Omega n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{1}{4^n}\right) e^{-j\Omega n}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{1}{2^n} e^{-j\Omega n} + \sum_{n=0}^{\infty} \frac{1}{4^n} e^{-j\Omega n}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\Omega n} + \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\Omega n}\right]$$

by geometric series summation formulae:

$$\Rightarrow \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}e^{-j\Omega}} + \frac{1}{1-\frac{1}{4}e^{-j\Omega}}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{1-\frac{1}{4}e^{-j\Omega}+1-\frac{1}{2}e^{-j\Omega}}{\left(1-\frac{1}{2}e^{-j\Omega}\right)\left(1-\frac{1}{4}e^{-j\Omega}\right)}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{2-\frac{3}{4}e^{-j\Omega}}{1-\frac{1}{2}e^{-j\Omega}-\frac{1}{4}e^{-j\Omega}+\frac{1}{8}e^{-2j\Omega}}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{2-\frac{3}{4}e^{-j\Omega}}{1-\frac{3}{4}e^{-j\Omega}+\frac{1}{8}e^{-2j\Omega}}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{\frac{8-3e^{-j\Omega}}{\cancel{4}}}{\frac{8-6e^{-j\Omega}+e^{-2j\Omega}}{\cancel{8}\,2}}\right] = \frac{8-3e^{-j\Omega}}{8-6e^{-j\Omega}+e^{-2j\Omega}}$$

6.2

$$x[n] = (-0.8)^n \left(u[n-3] - u[n+5]\right)$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} (-0.8)^n \left(u[n-3] - u[n+5]\right) e^{-j\Omega n}$$

$$= \cancel{\sum_{n=-5}^{3} (-0.8)^n e^{-j\Omega n}}$$

$$= \sum_{n=-\infty}^{\infty} (-0.8)^n u[n-3] e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} (-0.8)^n u(n+5) e^{-j\Omega n}$$

$$= \sum_{n=3}^{\infty} (-0.8)^n e^{-j\Omega n} - \sum_{n=-5}^{\infty} (-0.8)^n e^{-j\Omega n}$$

$$\sum_{n=L_1}^{L_2} a^n = \frac{a^{L_1} - a^{L_2+1}}{1-a}$$

$$\sum_{n=3}^{\infty} \left[(-0.8) e^{-j\Omega}\right]^n = \frac{\left[(-0.8) e^{-j\Omega}\right]^3 - \left[(-0.8) e^{-j\Omega}\right]^{\infty+1}}{1 - (-0.8) e^{-j\Omega}}$$

$$= \frac{(-0.8)^3 e^{-j\Omega 3} - 0}{1 + 0.8 e^{-j\Omega}}$$

$$\sum_{n=-5}^{\infty} \left[(-0.8)e^{-j\Omega}\right]^n = \frac{\left[(-0.8)e^{-j\Omega}\right]^{-5} - 0}{1+0.8e^{-j\Omega}}$$

$$= \frac{(-0.8)^{-5} e^{j\Omega 5}}{1+0.8e^{-j\Omega}}$$

$$X(\Omega) = \frac{(-0.8)^3 e^{-j\Omega 3} - (-0.8)^{-5} e^{j\Omega 5}}{1+0.8e^{-j\Omega}}$$

Question #7

Task 7.1 $X(\Omega) = \begin{cases} e^{\Omega} & -\pi \le \Omega < 0 \\ e^{-\Omega} & 0 \le \Omega < \pi \end{cases}$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} e^{\Omega} \cdot e^{j\Omega n}\, d\Omega + \int_{0}^{\pi} e^{-\Omega} \cdot e^{j\Omega n}\, d\Omega \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} e^{\Omega(1+jn)}\, d\Omega + \int_{0}^{\pi} e^{\Omega(-1+jn)}\, d\Omega \right]$$

$$= \frac{1}{2\pi} \left[\frac{e^{\Omega(1+jn)}}{(1+jn)} \Bigg|_{-\pi}^{0} + \frac{e^{\Omega(-1+jn)}}{(-1+jn)} \Bigg|_{0}^{\pi} \right]$$

$$= \frac{1}{2\pi} \left[\frac{e^{0}}{(1+jn)} - \frac{e^{-\pi(1+jn)}}{(1+jn)} + \frac{e^{\pi(-1+jn)}}{(-1+jn)} - \frac{e^{0}}{(-1+jn)} \right]$$

$$= \frac{1}{2\pi}\left[\frac{1-e^{-\pi(1+jn)}}{(1+jn)} + \frac{e^{\pi(-1+jn)}-1}{(-1+jn)}\right].$$

$$= \frac{1}{2\pi}\left\{\frac{-1+j\pi + e^{-\pi(1+jn)} - jne^{-\pi(1+jn)} + e^{\pi(-1+jn)} + jne^{\pi(-1+jn)} - 1 - j\pi}{1+n^2}\right\}$$

$$= \frac{1}{2\pi(1+n^2)}\left[-2 + e^{-\pi(1+jn)}(1-jn) + e^{\pi(-1+jn)}(1+jn)\right]$$

Task 7.2

$$X(\Omega) = \frac{\alpha e^{-j\Omega}}{(1-\alpha e^{-j\Omega})^2}$$

using the known transform pair:

$$e^{-an} u[n] \overset{F}{\longleftrightarrow} \frac{1}{1-\alpha e^{-j\Omega}}$$

Differentiation property in DTFT

if $\quad x[n] \overset{F}{\longleftrightarrow} X(\Omega)$

then $\quad n x[n] \longleftrightarrow j\frac{d}{d\Omega}(X(\Omega))$

thus

$$j\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right) \longleftrightarrow n e^{-an} u[n]$$

calculating $j\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right)$

$$\Rightarrow j \times -1 \times (1 - ae^{-j\Omega})^{-2} \times (-j \times -ae^{-j\Omega})$$

$$\Rightarrow -j \times (1 - ae^{-j\Omega})^{-2} \times (jae^{-j\Omega})$$

$$\Rightarrow \frac{-j(jae^{-j\Omega})}{(1-ae^{-j\Omega})^2}$$

$$\Rightarrow \frac{-j^2 ae^{-j\Omega}}{(1-ae^{-j\Omega})^2}$$

$$\Rightarrow \frac{ae^{-j\Omega}}{(1-ae^{-j\Omega})^2}$$

thus $j\frac{d}{d\Omega}\left(\frac{1}{1-ae^{-j\Omega}}\right) = \frac{ae^{-j\Omega}}{(1-ae^{-j\Omega})^2} \overset{F}{\longleftrightarrow} ne^{-an}u[n]$

$$x[n] = ne^{-an}u[n]$$

Question #8.

$$x[n] = \{\underset{\uparrow}{1}, 2, 3, 0, 0\}$$

$X[k] = ?$ $\quad k = 0, \ldots, 4.$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

$$= \sum_{n=0}^{4} x[n]\, e^{-j\frac{2\pi k n}{5}}$$

$$X[k] = 1 \times e^{-0} + 2 \times e^{-j\frac{2\pi k}{5} \times 1} + 3 \times e^{-j\frac{2\pi k}{5} \times 2} + 0$$

$$= 1 + 2e^{-j\frac{2\pi k}{5}} + 3e^{-j\frac{4\pi k}{5}}$$

$$= 1 + 2\cos\left(\frac{2\pi k}{5}\right) - j2\sin\left(\frac{2\pi k}{5}\right)$$
$$+ 3\cos\left(\frac{4\pi k}{5}\right) - j3\sin\left(\frac{4\pi k}{5}\right)$$

$$= 1 + 2\cos\left(\frac{2\pi k}{5}\right) + 3\cos\left(\frac{4\pi k}{5}\right) - j\left[2\sin\left(\frac{2\pi k}{5}\right) + 3\sin\left(\frac{4\pi k}{5}\right)\right]$$

$X[0] = 1 + 2\cos 0 + 3\cos 0 - j \times 0 = 6.$

$$X[1] = 1 + 2\cos\left(\frac{2\pi}{5}\right) + 3\cos\left(\frac{4\pi}{5}\right) - j\left[2\sin\left(\frac{2\pi}{5}\right) + 3\sin\left(\frac{4\pi}{5}\right)\right]$$

$$= 1 + \overset{0.618}{\cancel{1.902}} - 2.427 - j(3.665).$$

$$= \cancel{0.4749} - j3.665$$

$$-0.808966$$

$X[1] \Rightarrow -0.809 - j3.665$

$$X[2] = 1 + 2\cos\left(\frac{4\pi}{5}\right) + 3\cos\left(\frac{8\pi}{5}\right) - j\left[2\sin\left(\frac{4\pi}{5}\right) + 3\sin\left(\frac{8\pi}{5}\right)\right]$$

$X[2] = 0.3090 + j\,1.678.$

Obtaining $X[3]$ and $X[4]$ from Matlab:

$X[3] = 0.3090 - j\,1.6776$

$X[4] = -0.8090 + j\,3.6655.$