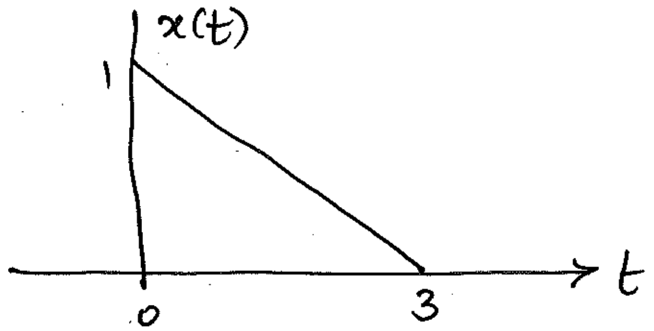
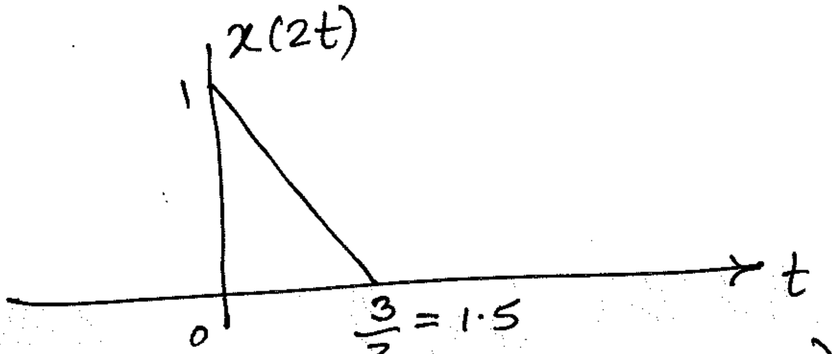
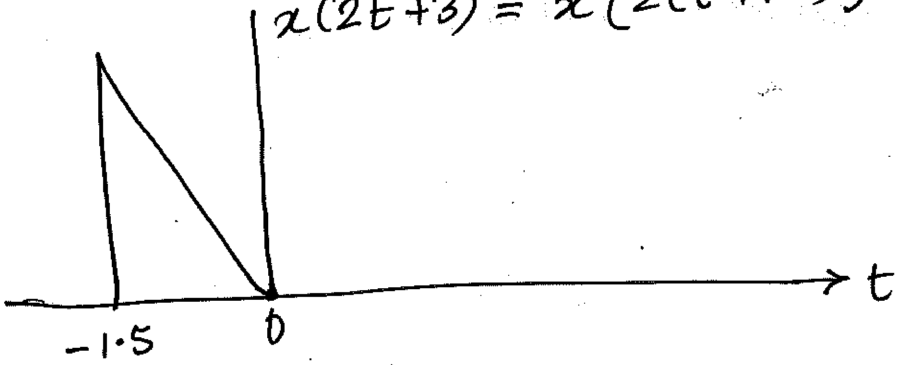
Q1 (a) $x(2t+3)$



<u>Method 1</u>

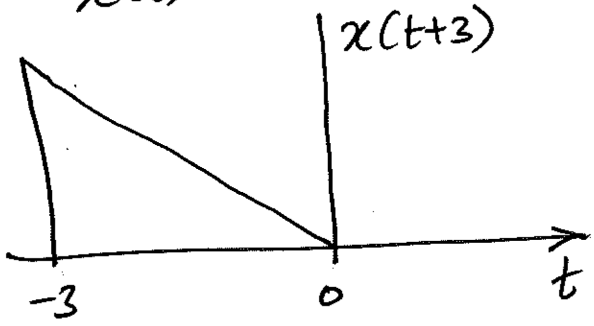
$$x(t) \xrightarrow{\text{scaling}} x(2t) \xrightarrow{\text{shifting}} x(2t+3) = x\left(2\left(t+3/2\right)\right)$$
$$= x\left(2(t+1.5)\right)$$

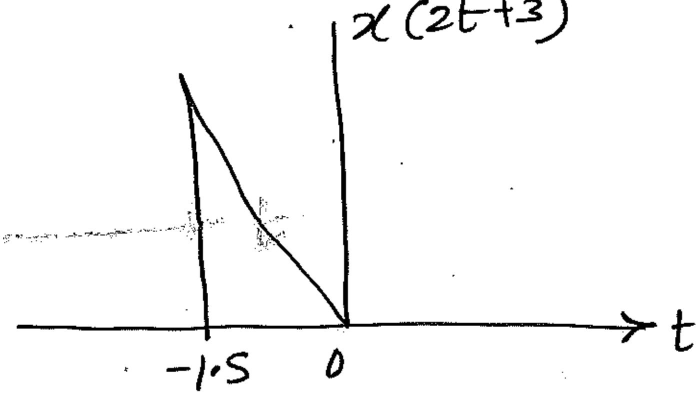




<u>Method 2</u>

$$x(t) \xrightarrow{\text{shifting}} x(t+3) \xrightarrow{\text{scaling}} x(2t+3)$$

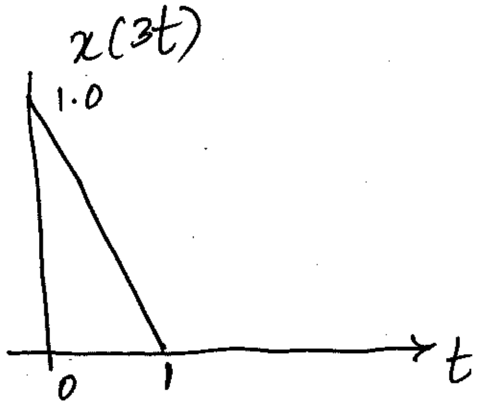


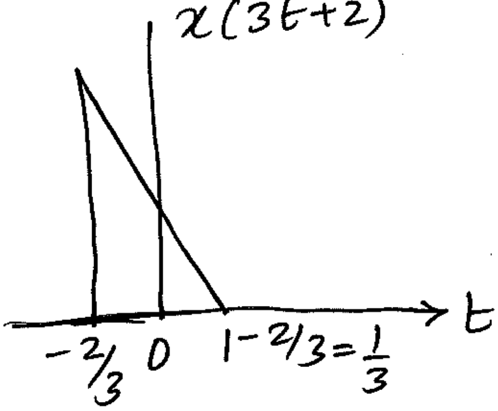


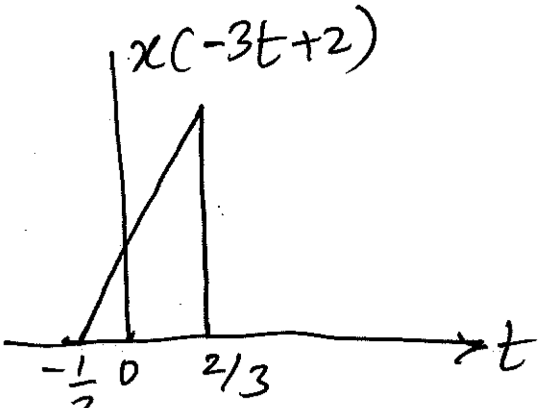
Q1(b) $x(2-3t)$

<u>Method 1</u> $x(t) \xrightarrow{\text{scaling}} x(3t) \xrightarrow{\text{shifting}} x(3t+2)$
$= x(3(t+2/3))$

$\xrightarrow{\text{time reversal}} x(-3t+2)$

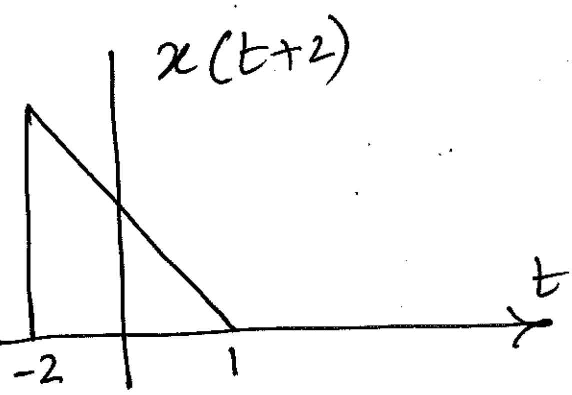


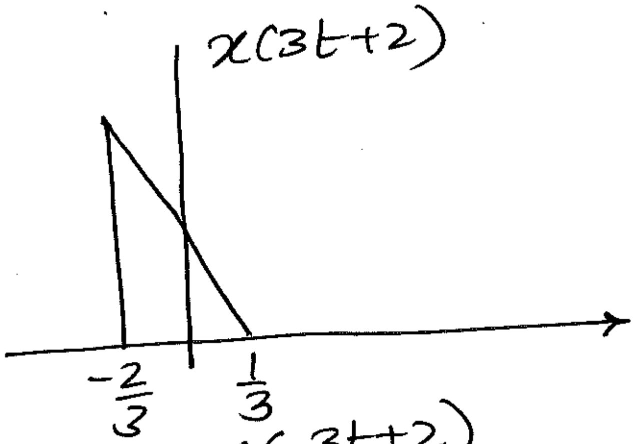


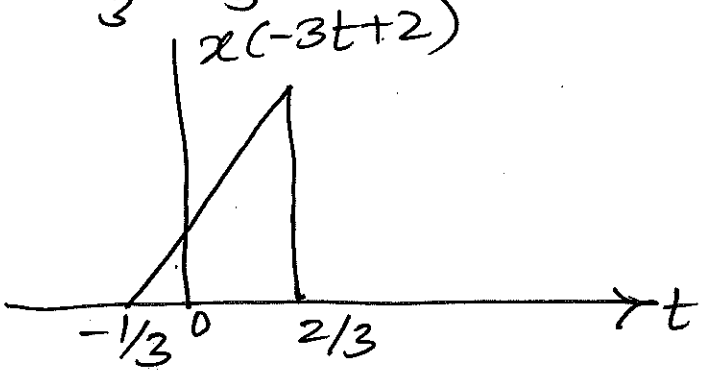


<u>Method 2</u> $x(t) \xrightarrow{\text{shifting}} x(t+2) \xrightarrow{\text{scaling}} x(3t+2)$

$\xrightarrow{\text{reversal}} x(-3t+2)$







Q2.

$$1 - e^{j\alpha} = e^{j\alpha/2} e^{-j\alpha/2} - e^{j\alpha/2} e^{j\alpha/2}$$

$$= e^{j\alpha/2}\left(e^{-j\alpha/2} - e^{+j\alpha/2}\right)$$

$$= e^{j\alpha/2}\left(\cancel{\cos\frac{\alpha}{2}} - j\sin\frac{\alpha}{2} - \cancel{\cos\frac{\alpha}{2}} - j\sin\frac{\alpha}{2}\right)$$

$$= -2j\sin\frac{\alpha}{2}\, e^{j\alpha/2}$$

Where we have exploited the facts that

(1) $1 = e^{j\theta} e^{-j\theta}$

(2) $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$.

Next not that

$$e^{-j\pi/2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = 0 - j(1) = -j$$

There fore, we obtain

$$1 - e^{j\alpha} = 2\sin\frac{\alpha}{2}\, e^{j\frac{\alpha}{2}} e^{-j\pi/2} = 2\sin\frac{\alpha}{2}\, e^{j(\alpha-\pi)/2}$$

Hence proved.

Q3.

(i) $x(t) = \cos\left(\frac{\pi}{3}(t+0) + 2\pi\right) = \cos\left(\frac{\pi t}{3}\right)$

$y(t) = \sin\left(\frac{\pi}{3}(t+1) + \frac{-\pi}{3}\right) = \sin\left(\frac{\pi t}{3}\right)$

$$x(t) \neq y(t)$$

$x(t)$ and $y(t)$ differ by a phase of $\pi/2$ radians.

(ii) $x(t) = \cos\left(\frac{3\pi}{4}\left(t+\frac{1}{2}\right) + \frac{\pi}{4}\right) = \cos\left(\frac{3\pi t}{4} + \frac{3\pi}{8} + \frac{\pi}{4}\right)$

$= \cos\left(\frac{3\pi t}{4} + \frac{5\pi}{8}\right)$

$y(t) = \sin\left(\frac{11\pi}{4}(t+1) + \frac{3\pi}{8}\right) = \sin\left(\frac{11\pi}{4}t + \frac{25\pi}{8}\right)$

$x(t) \neq y(t)$ $\quad \frac{11\pi}{4} \gg \frac{3\pi}{4}$ (frequencies are not same)

(iii) $x(t) = \cos\left(\frac{3}{4}\left(t+\frac{1}{2}\right) + \frac{1}{4}\right) = \cos\left(\frac{3t}{4} + \frac{5}{8}\right)$

$y(t) = \sin\left(\frac{3}{4}(t+1) + \frac{3}{8}\right) = \sin\left(\frac{3t}{4} + \frac{9}{8}\right)$

$$x(t) \neq y(t)$$

Q4

$x[n] = \cos\left(\Omega_x (n + P_x) + \theta_x\right)$

$y[n] = \cos\left(\Omega_y (n + P_y) + \theta_y\right)$

(i) $x[n] = \cos\left(\frac{\pi}{3}(n+0) + 2\pi\right) = \cos\left(\frac{\pi n}{3} + 2\pi\right)$

$y[n] = \cos\left(\frac{8\pi}{3}(n+0) + 0\right) = \cos\left(\frac{8\pi n}{3}\right)$

Frequencies are different $x[n] \neq y[n]$.

(ii) $x[n] = \cos\left(\frac{3\pi}{4}(n+2) + \frac{\pi}{4}\right) = \cos\left(\frac{3\pi n}{4} + \frac{7\pi}{4}\right)$

$y[n] = \cos\left(\frac{3\pi}{4}(n+1) - \pi\right) = \cos\left(\frac{3\pi n}{4} - \frac{\pi}{4}\right)$

Note that $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$

So $x[n] = y[n]$.

(iii) $x[n] = \cos\left(\frac{3}{4}(n+1) + \frac{1}{4}\right) = \cos\left(\frac{3n}{4} + 1\right)$

$y[n] = \cos\left(\frac{3}{4}(n+0) + 1\right) = \cos\left(\frac{3n}{4} + 1\right)$

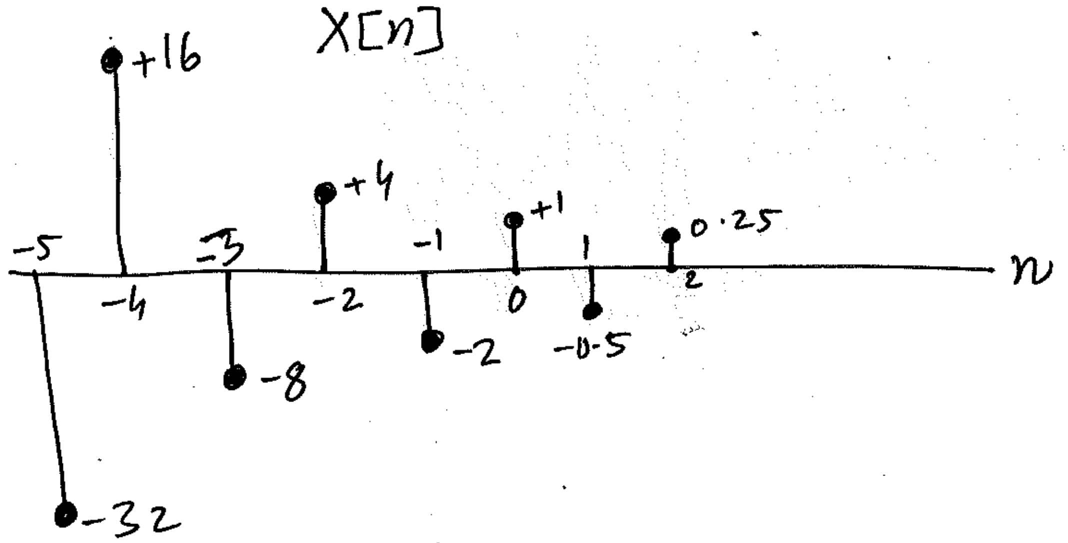
So $x[n] = y[n]$.

Q5

(a) $x[n] = \alpha^n \qquad -1 < \alpha < 0$

Consider $\alpha = -\frac{1}{2}$

n	$x[n]$	n	$x[n]$
-5	-32	0	1
-4	+16	1	-0.5
-3	-8	2	+0.25
-2	+4	3	-0.125
-1	-2	4	+0.0625



(b) $y(t) = e^{\beta t}$

we have to find β such that

$$y(n) = e^{\beta n} = (-e^{-1})^n = \left(\frac{-1}{e}\right)^n$$

There is a hint given in This question that

"β is Complex"

Let us assume $\quad \beta = a + jb.$

$$e^{\beta n} = e^{(a+jb)n} = \left(-\frac{1}{e}\right)^n = (-1)^n e^{-n}$$

Comparing both sides, we obtain

$$e^{a} = e^{-n} \quad \text{——— (1)}$$

and

$$e^{jbn} = (-1)^n \quad \text{——— (2)}$$

From (1), we obtain $\boxed{a=-1}$

Next, we have to find "b" such that eq (2) is satisfied.

note that $e^{jbn} = \cos(bn) + j\sin(bn)$

Now, we have to find "b" such that

$$\cos(bn) = (-1)^n \quad \forall n$$

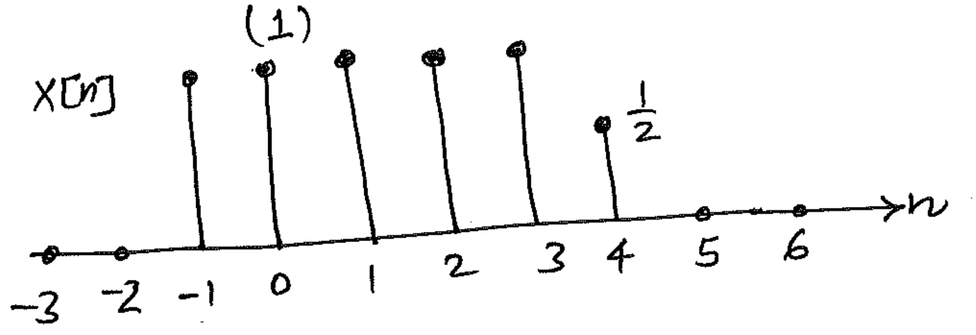
$$\sin(bn) = 0 \quad \forall n.$$

Also note that $(-1)^n = \begin{cases} +1 & \text{for even } n \\ -1 & \text{for odd value of } n \end{cases}$

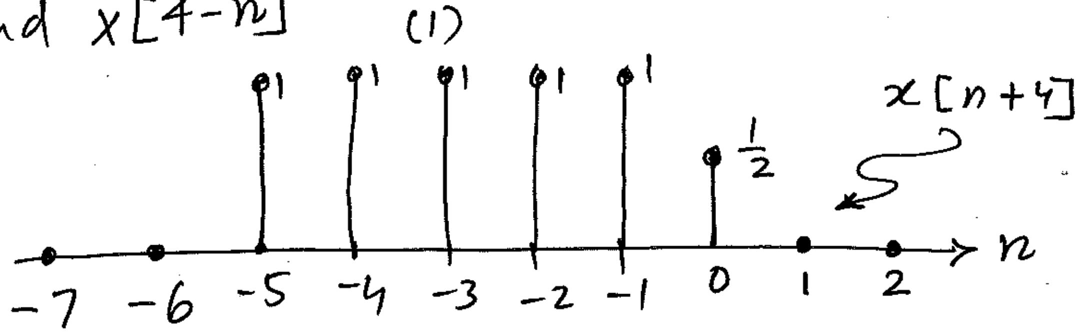
The value of b is immediately comes out to be $b = \pi$.

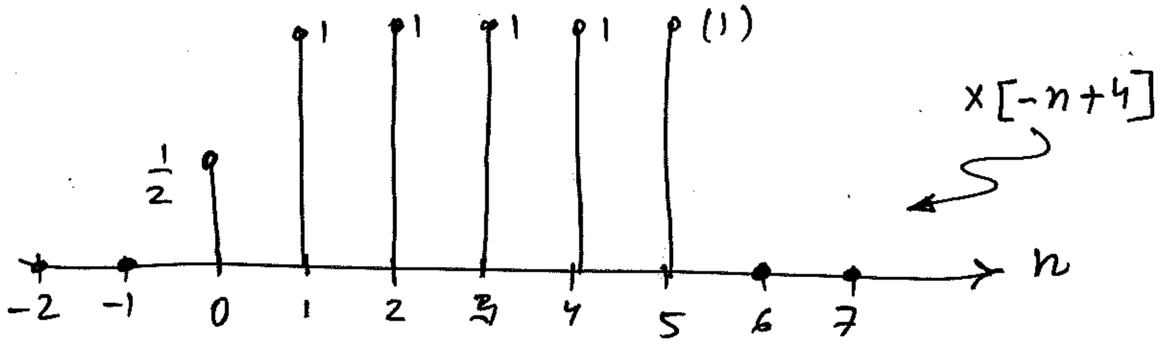
So $\boxed{\beta = -1 + j\pi}$

Q6.

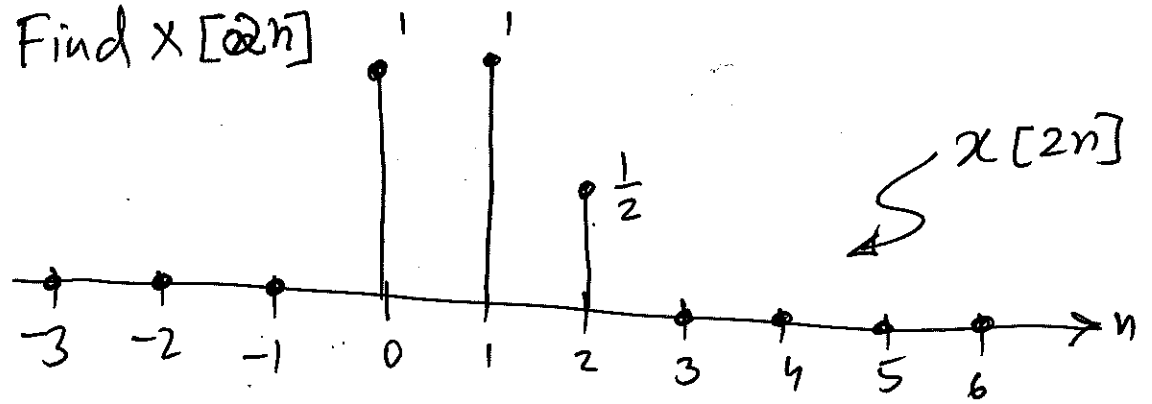


(i) Find $x[4-n]$

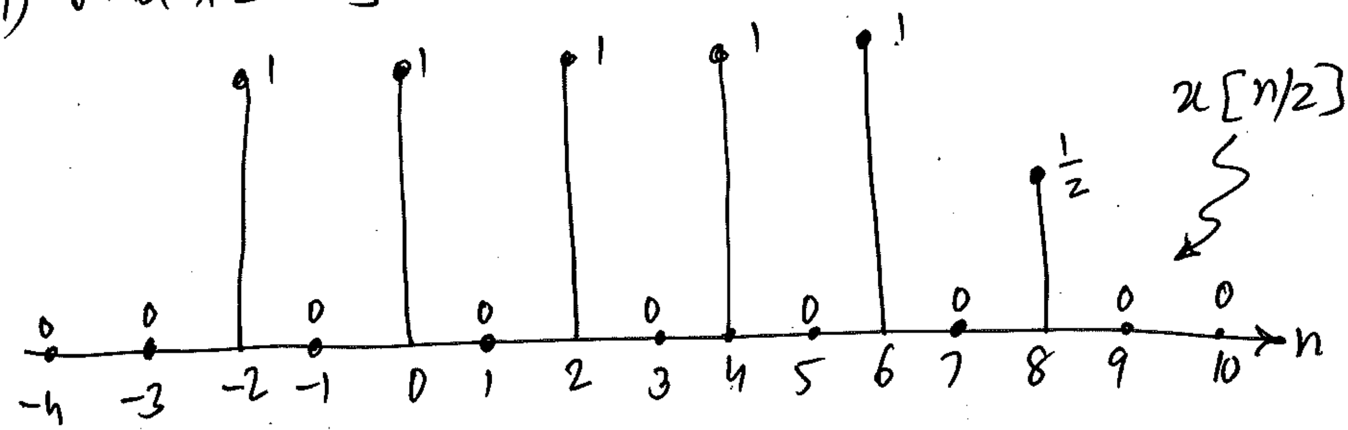




(ii) Find $x[2n]$



(iii) Find $x[n/2]$



Q7. $x(t) = \cos\left(\frac{2\pi t}{3}\right) + 2\sin\left(\frac{16\pi t}{3}\right)$

$y(t) = \sin(\pi t)$

using Euler notations.

$$x(t) = \frac{1}{2}e^{j2\pi t/3} + \frac{1}{2}e^{-j2\pi t/3} + \frac{1}{j}e^{j16\pi t/3} - \frac{1}{j}e^{-j16\pi t/3}.$$

$$y(t) = \frac{1}{2j}e^{j\pi t} - \frac{1}{2j}e^{-j\pi t}$$

$$z(t) = x(t)\,y(t) = \frac{1}{4j}e^{j5\pi t/3} + \frac{1}{4j}e^{j\pi t/3} - \frac{1}{2}e^{j19\pi t/3} + \frac{1}{2}e^{-j13\pi t/3} - \frac{1}{4j}e^{-j\pi t/3} - \frac{1}{4j}e^{-j5\pi t/3} + \frac{1}{2}e^{+j13\pi t/3} - \frac{1}{2}e^{-j19\pi t/3}$$

Let $2\pi f_0 = \frac{\pi}{3} \Rightarrow f_0 = \frac{1}{6}$ Hz.

$$z(t) = \frac{1}{2}\left(\frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}\right) + \frac{1}{2}\left(\frac{e^{j10\pi f_0 t} - e^{-j10\pi f_0 t}}{2j}\right) - \left(\frac{e^{j38\pi f_0 t} + e^{-j38\pi f_0 t}}{2}\right) + \frac{1}{2}\left(e^{j26\pi f_0 t} + e^{-j26\pi f_0 t}\right)$$

$$z(t) = \frac{1}{2}\sin(2\pi f_0 t) + \frac{1}{2}\cos(5\times 2\pi f_0 t) - \cos(19\times 2\pi f_0 t) + \cos(13\times 2\pi f_0 t)$$

where $f_0 = \frac{1}{6}$ Hz.

So, the time period is $T_0 = \frac{1}{f_0} = 6$ sec.

Q8.

Let $I = \int_{-\infty}^{\infty} x_e(t)\, x_o(t)\, dt$

$$= \underbrace{\int_{-\infty}^{0} x_e(t)\, x_o(t)\, dt}_{=:\, I_1} + \underbrace{\int_{0}^{+\infty} x_e(t)\, x_o(t)\, dt}_{=:\, I_2}$$

$I_1 := \int_{-\infty}^{0} x_e(t)\, x_o(t)\, dt$

Let $u = -t \Rightarrow du = -dt$

$$I_1 = \int_{\infty}^{0} x_e(-u)\, x_o(-u)\, (-du) = \int_{0}^{+\infty} x_e(-u)\, x_o(-u)\, du$$

$$\begin{cases} x_e(-u) = x_e(+u) \\ x_o(-u) = -x_o(+u) \end{cases}$$

$I_1 = -\int_0^{+\infty} x_e(u)\, x_o(u)\, du$

Let $u = t$

$\Rightarrow I_1 = -\int_0^{+\infty} x_e(t)\, x_o(t)\, dt = -I_2$

Therefore $I = I_1 + I_2 = -I_2 + I_2 = 0$.

Hence proved.

Q8 (b)

$$\int_{-\infty}^{+\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} (x_e(t) + x_o(t))^2\,dt$$

$$= \int_{-\infty}^{\infty} \left(x_e^2(t) + x_o^2(t) + 2x_e(t)x_o(t) \right) dt$$

$$= \int_{-\infty}^{\infty} x_e^2(t)\,dt + \int_{-\infty}^{\infty} x_o^2(t)\,dt + 2\underbrace{\int_{-\infty}^{\infty} x_e(t)x_o(t)\,dt}_{\substack{=0 \\ \text{as shown in} \\ \text{part (a)}}}$$

$$= \int_{-\infty}^{\infty} x_e^2(t)\,dt + \int_{-\infty}^{2} x_o^2(t)\,dt.$$

Hence proved.

Q9.

* $y(t) = \big(2 + \sin(t)\big)\, x(t)$

Let $t = t_0$

$y(t_0) = \big(2 + \sin(t_0)\big)\, x(t_0)$

The response $y(t_0)$ only depends on quantities determined at $t = t_0$.

The system is memoryless.

* Let $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$

$y(t) = Sys\{x(t)\} = \big(2 + \sin(t)\big)\, x(t)$

$y(t) = Sys\{\alpha_1 x_1 + \alpha_2 x_2\} = (2 + \sin t)(\alpha_1 x_1 + \alpha_2 x_2)$

$y_1(t) = Sys\{x_1(t)\} = \big(2 + \sin(t)\big)\, x_1(t)$

$y_2(t) = Sys\{x_2(t)\} = \big(2 + \sin(t)\big)\, x_2(t)$

$\tilde{y}(t) = \alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 (2 + \sin t) x_1 + \alpha_2 (2 + \sin t) x_2$

$= (2 + \sin t)(\alpha_1 x_1 + \alpha_2 x_2)$

$= y(t)$

The system is Linear.

* system is stable. See below:

Let $|x(t)| \leq B_x < \infty$

$|y(t)| = |(2 + \sin t)\, x(t)| \leq |2 + \sin t|\, B_x \leq 2 B_x < \infty$

Q9. For an invariant system we need.

$$y(t-t_0) = \big(2 + \sin(t-t_0)\big)\, x(t-t_0)$$

System after delay gives.

$$y_1(t) = (2+\sin(t))\, x(t-t_0)$$

Delay after system gives

$$y_2(t) = \big(2 + \sin(t-t_0)\big)\, x(t-t_0)$$

since $y_1(t) \neq y_2(t)$

The system is NOT time invariant.

(b) $x(2t)$

* This system is <u>linear</u>, proof is trivial and is left for you to establish, or you may refer to lecture notes.

* $x(t) \xrightarrow{} \boxed{S} \xrightarrow{x(bt)} \boxed{D} \xrightarrow[\text{(Replace } t \text{ by } t\pm m)]{x(b(t\pm m)) = x(bt \pm bm)} = y_1(t)$

$x(t) \xrightarrow{} \boxed{D} \xrightarrow{x(t\pm m)} \boxed{S} \xrightarrow[\text{(Replace } t \text{ by } bt)]{x(bt\pm m) = x(bt \pm m)} = y_2(t)$

Since $y_1(t) \neq y_2(t)$, System is NOT time-invariant.

* Causality

$$y(t) = x(2t)$$

@ $t = 1$.

$$y(t=1) = y(1) = x(2)$$

The response at $t=1$ depends on the future value of excitation -

NOT Causal.

* Invertible.

$$x(t) \longrightarrow \boxed{\text{Sys-1}} \xrightarrow{y_1(t) = x(2t)}$$

Consider another system.

$$x(t) \longrightarrow \boxed{\text{Sys-2.}} \xrightarrow{y_2(t) = x(t/2)}$$

$$x(t) \longrightarrow \boxed{\text{Sys-1}} \xrightarrow{x(2t)} \boxed{\text{Sys-2}} \xrightarrow{x(t)}$$

Yes this system is invertible.

* Stability

If $|x(t)| \leq B_x < \infty \quad \forall t$

then $|y(t)| = |x(2t)| \leq B_X < \infty \ \forall t$

System is STABLE.

(c) $\sum_{k=-\infty}^{\infty} x[k]$

* System is linear.

$$y[k] = \sum_{k=-\infty}^{\infty} (\alpha_1 x_1[k] + \alpha_2 x_2[k])$$

$$y_1[k] = \sum_{k=-\infty}^{\infty} x_1[k]$$

$$y_2[k] = \sum_{k=-\infty}^{\infty} x_2[k]$$

it is easy to show that $y[k] = \alpha_1 y_1[k] + \alpha_2 y_2[k]$

* it is easy to show

$$\sum_{k=-\infty}^{\infty} x[k-k_0] = \left. \sum_{k=-\infty}^{\infty} x[k] \right|_{k=k-k_0}$$

Sys is time invariant.

* Sys is not invertible
* Sys is not stable.

Even if $x[k]$ is bounded. say $x[k] = u[k]$

$$y[k] = \sum_{k=-\infty}^{\infty} x[k] \longrightarrow \infty$$

$y[k]$ is unbounded.

(d) $\sum_{k=-\infty}^{n} x[k] = y[n]$.

* System is Linear.
* System has memory.
* System is time-invariant.
* System is causal. (only depends on present and past values of $x[k]$).
* System is unbounded and unstable.

Let $x[n] = u[n]$

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=0}^{n} x[k] = \sum_{k=0}^{n} u[n]$$

$$y[n] = (n+1)$$

as $\lim_{n \to \infty} y[n] \to \infty$.

(e) $y(t) = \frac{d}{dt} x(t)$

This system is linear, time invariant, as well as invertible within an (unknown) constant.

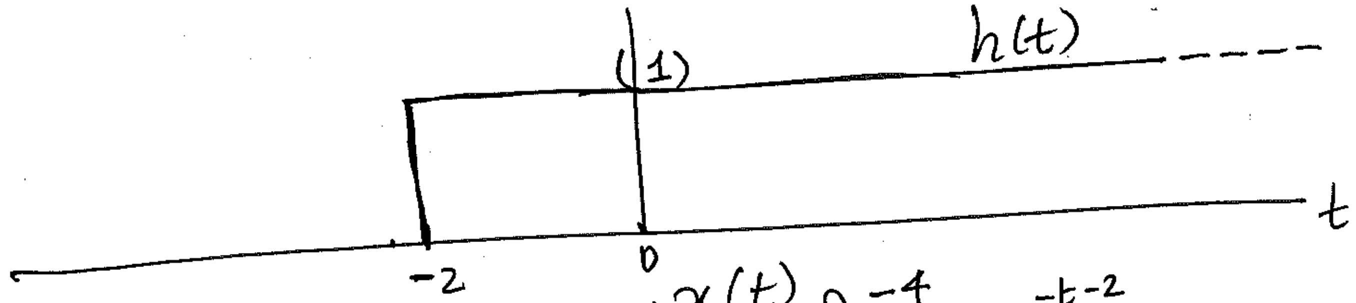
$$g(t) = \int_{-\infty}^{t} y(\tilde{t})\,d\tilde{t} = x(t) + c$$

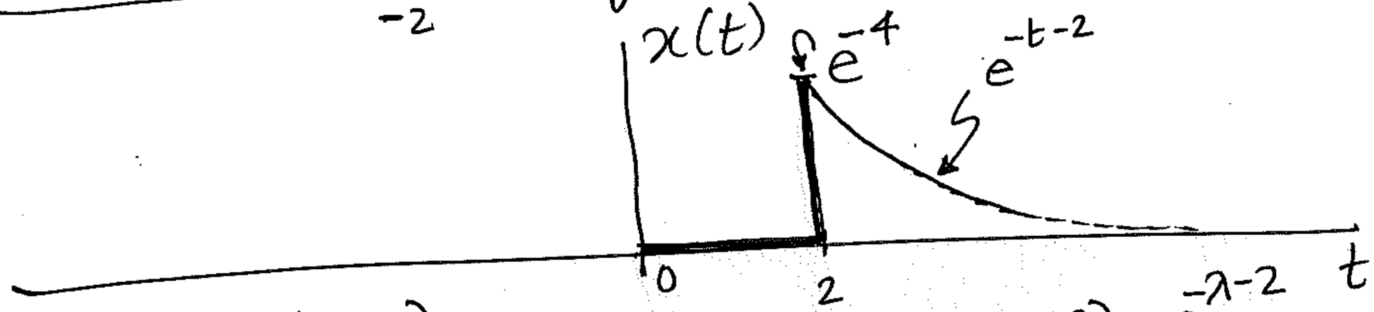
Q10

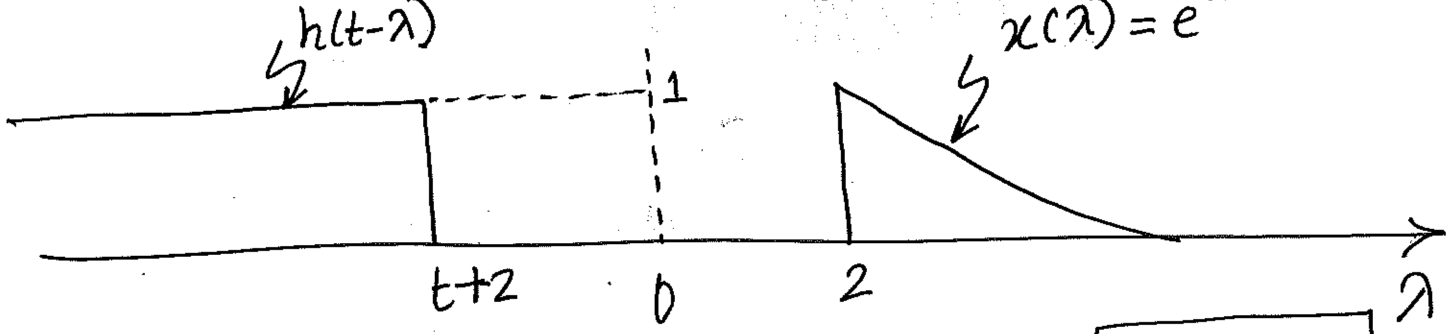
$$h(t) = u(t+2) = \begin{cases} 1 & t \geq -2 \\ 0 & t < -2 \end{cases}$$

$$x(t) = e^{-t-2}\, u(t-2)$$

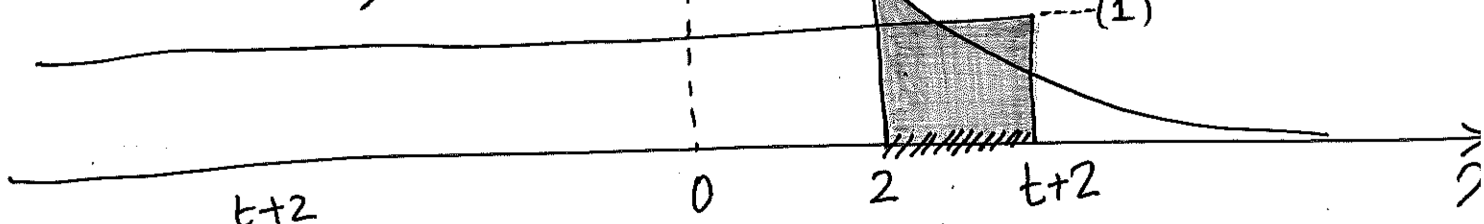
$$x(t) = \begin{cases} e^{-t-2} & t \geq 2 \\ 0 & t < 2 \end{cases}$$







When $t+2 \geq 2$, overlap occurs for $\boxed{2 < \lambda < t+2}$
or $t \geq 0$



$$y(t) = \int_2^{t+2} e^{-\lambda-2}\, d\lambda = e^{-2} \left. \frac{e^{-\lambda}}{-1} \right|_2^{t+2}$$

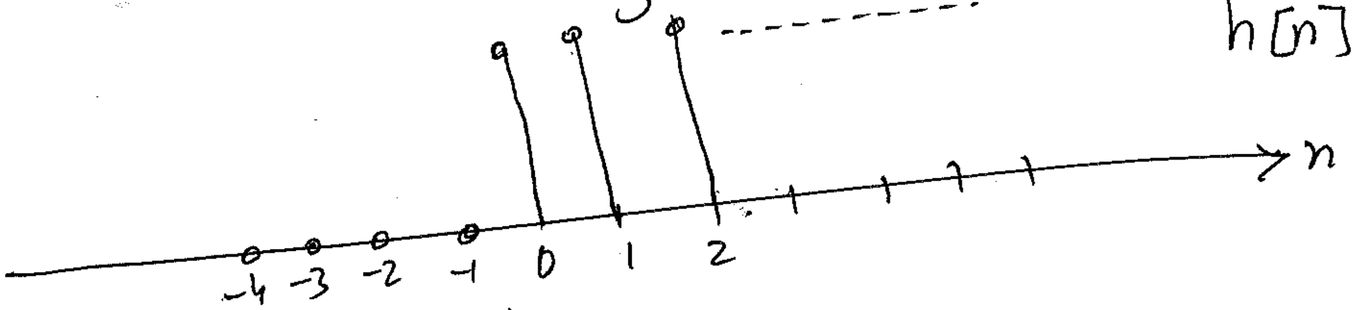
$$= e^{-2}\left(e^{-2} - e^{-(t+2)}\right)$$

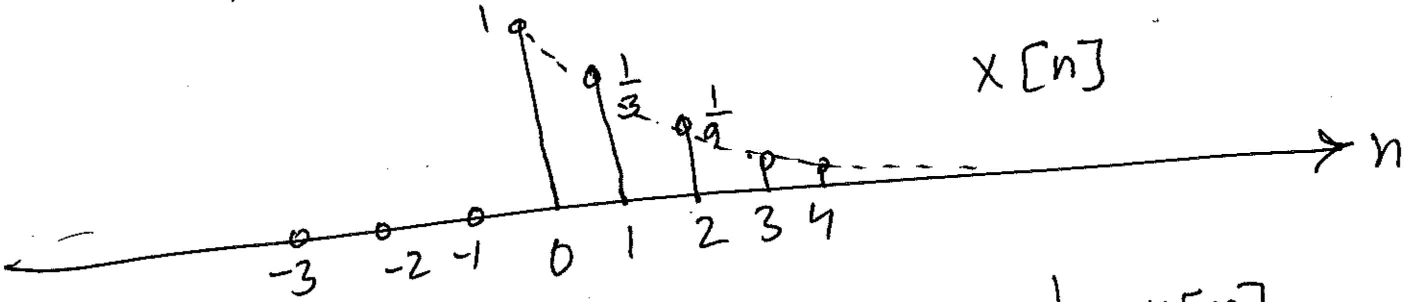
$$= e^{-4}\left(1 - e^{-t}\right) \qquad t \geq 0$$

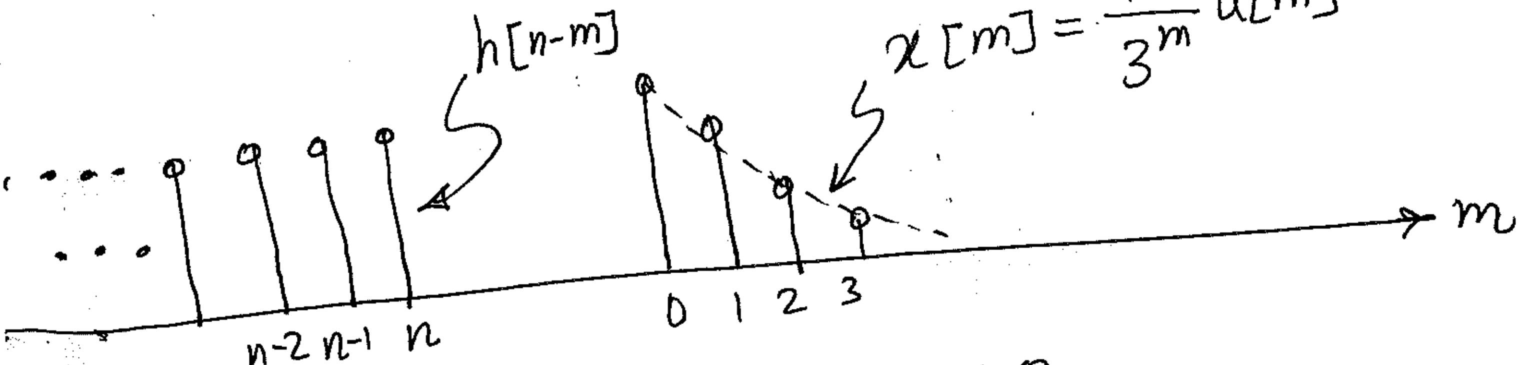
Q11

$$h[n] = u[n]$$

$$x[n] = \frac{1}{3^n} u[n]$$







Over laps occurs when $n \geq 0$
or in terms of "m" $0 \leq m \leq n$

for $n < 0$, $y[n] = x[n] * h[n] = 0$

for $n \geq 0$ $\quad y[n] = \sum_{m=0}^{n} \frac{1}{3^m} = \frac{1}{3^0} + \frac{1}{3^1} + \dots + \frac{1}{3^n}$

$y[\infty] = \frac{3}{2} = 1.5.$

$\Rightarrow y[0] = 1$

$y[1] = 1 + 1/3$

$y[2] = 1 + 1/3 + 1/3^2$

$y[3] = 1 + 1/3 + 1/3^2 + 1/3^3$ and so on