Midterm I | Signals and Systems | Spring 2022

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Question 1 [10]: Express the waveform of Figure 1 as a sum of step and ramp functions.

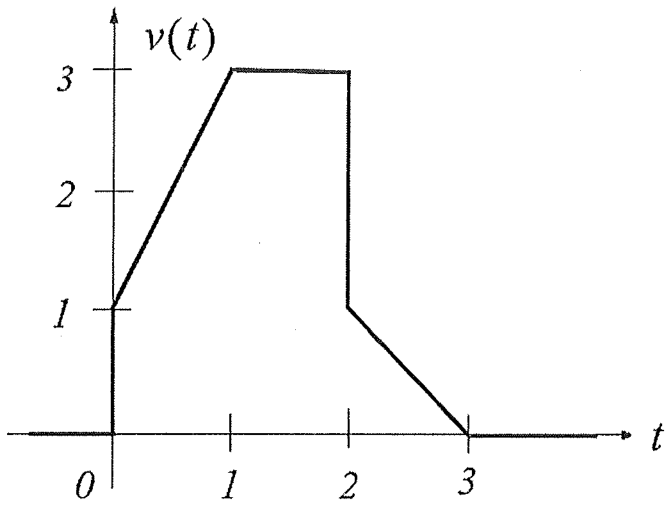


Fig. 1

Question 2 [20]: In the network of Fig. 2 $i_S(t)$ is a unit ramp current source and the switch is closed at time $t = 0$. Express the capacitor voltage, $v_C(t)$, as a function of the input signal. You are supposed to make a suitable assumption about the initial condition of $v_C(t)$.

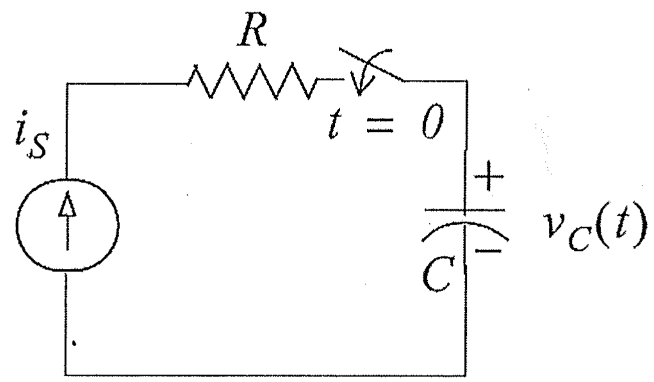


Fig. 2

Question 3 [50]: Consider an LTI system, whose impulse reesponse is h(t) as shown below in Fig. 3. Find the response of the system y(t), if x(t) is given as shown in Fig. 3. Show all steps

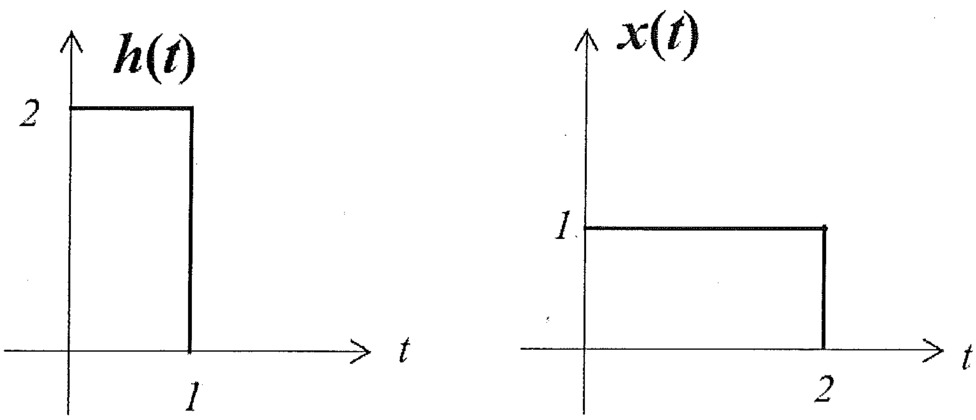


Fig. 3

Question 4 [50]: Find the Fourier series coefficients of the following periodic signal: Show all steps

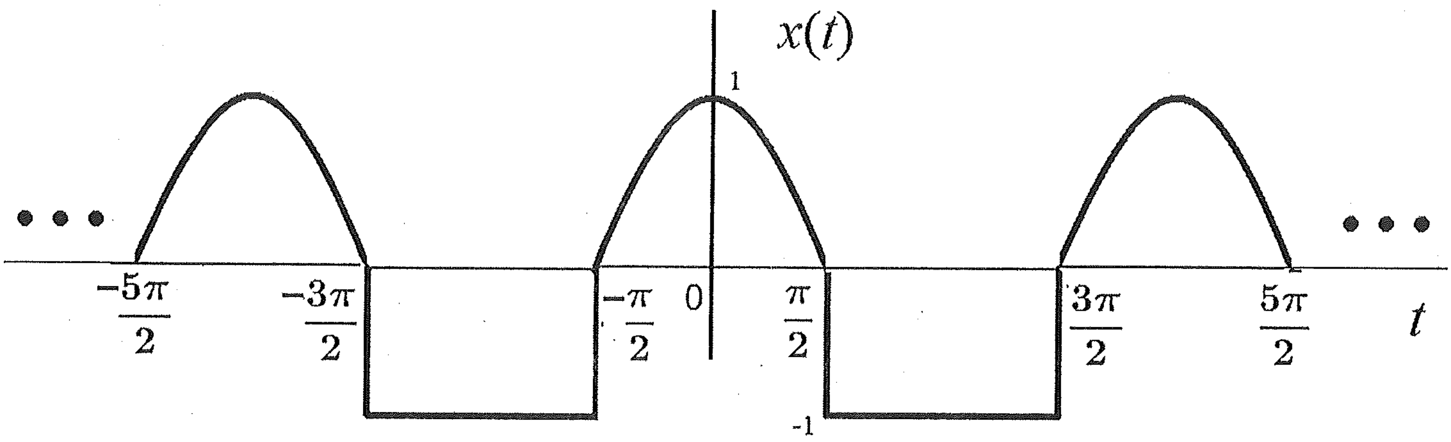


Fig. 4

Question 5 [20]: Find the energy and power of the following signals:

(a)

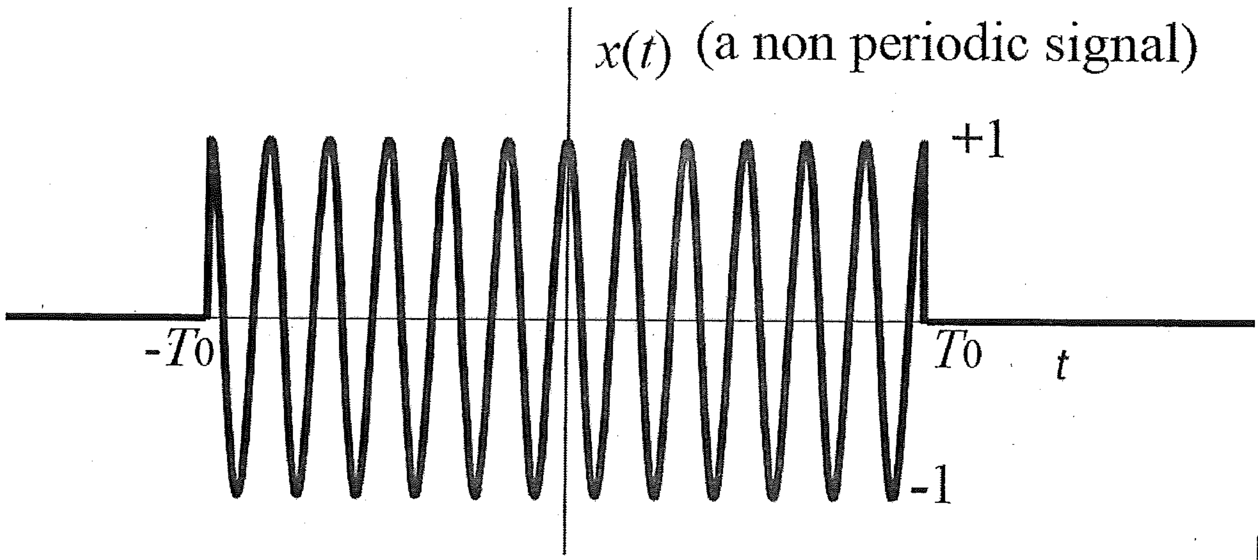


Fig. 5(a)

(b)

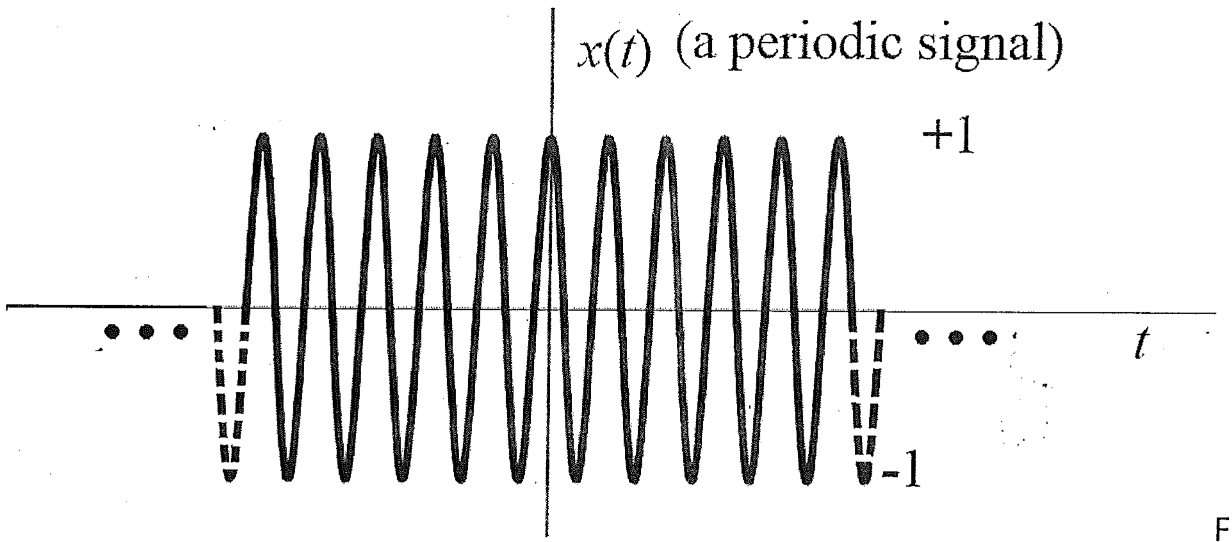
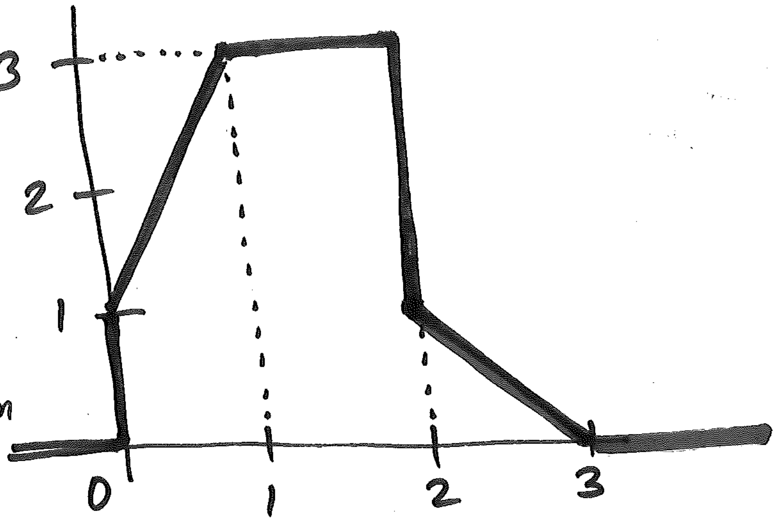


Fig. 5(b)

Midterm 1 Solution.
Dr. Shafayat Abrar.

Q1. Solution



@ $t=0$, two functions start, one is unit step function and the other is a ramp function with amplitude 2.0.

$x(t) = u(t) + 2\,r(t)$ for

$\boxed{x(t) = u(t) + 2t\,u(t)}$ $0 \leq t \leq 1.$

Check
$x(0) = 1 + 2*0 = 1$ ✓
$x(1) = 1 + 2*1 = 3$ ✓

@ $t=1$, the ramp function discontinues, it means we need to add another ramp function with amplitude "-2.0". Another step function appears with amplitude "2.0"

$x(t) = u(t) + 2\,r(t) - 2\,r(t)\,u(t-1) + 2u(t-1)$

$x(t) = u(t) + 2t\,u(t) - 2t\,u(t-1) + 2u(t-1)$ $1 \leq t \leq 2$

$\boxed{\text{or simply } x(t) = u(t) + 2u(t-1)}$

Check
$x(1) = 1 + 2*1*1 - 2*1*1 + 2*1$
$x(1) = 1 + \cancel{2} - \cancel{2} + 2 = 3$ ✓
$x(2) = 1 + 2*2*1 - 2*2*1 + 2*1$
$x(2) = 1 + 4 - 4 + 2 = 3$ ✓

at $t=2$ The step input with amplitude 2.0 discontinues, it means another step input with amplitude −2.0 starts.

Since, there is a decline it means a downward ramp starts at $t=2$.

$$x(t) = u(t) + \cancel{2t\,u(t)} - \cancel{2t\,u(t-1)} + \cancel{2u(t-1)}$$
$$\quad - \cancel{2u(t-2)} - (t-2)\,u(t-2)$$

$$\boxed{x(t) = u(t) - (t-2)\,u(t-2)} \qquad 2 \leq t \leq 3$$

Check.

$X(2) = 1 - (2-2) * 1 = 1$ ✓

$X(3) = 1 - (3-2) * 1 = 1 - 1 = 0$ ✓

at $t=3$ we need to eliminate every thing to ensure $x(t) = 0 \quad t \geq 3$.

$$x(t) = u(t) - (t-2)\,u(t-2)$$
$$\quad - u(t-3) + (t-2)\,u(t-3)$$

for $t \geq 3$

Check

$$x(3) = u(3) - u(3-2)(3-2)$$
$$\quad - u(3-3) + (3-2)\,u(3-3)$$
$$= 1 - 1*1 - 1 + 1*1 = 1-1+1-1 = 0.$$

Q2. Solution.

$$i_s(t) = i_c(t) = C\frac{dv_c(t)}{dt}$$

$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_s(\lambda)\,d\lambda$$

$$v_c(t) = \underbrace{\frac{1}{C}\int_{-\infty}^{0} i_s(\lambda)\,d\lambda}_{=\text{initial voltage across capacitor.}} + \frac{1}{C}\int_{0}^{t} i_s(\lambda)\,d\lambda$$

$$v_c(t) = v_c(0) + \frac{1}{C}\int_0^t i_s(\lambda)\,d\lambda$$

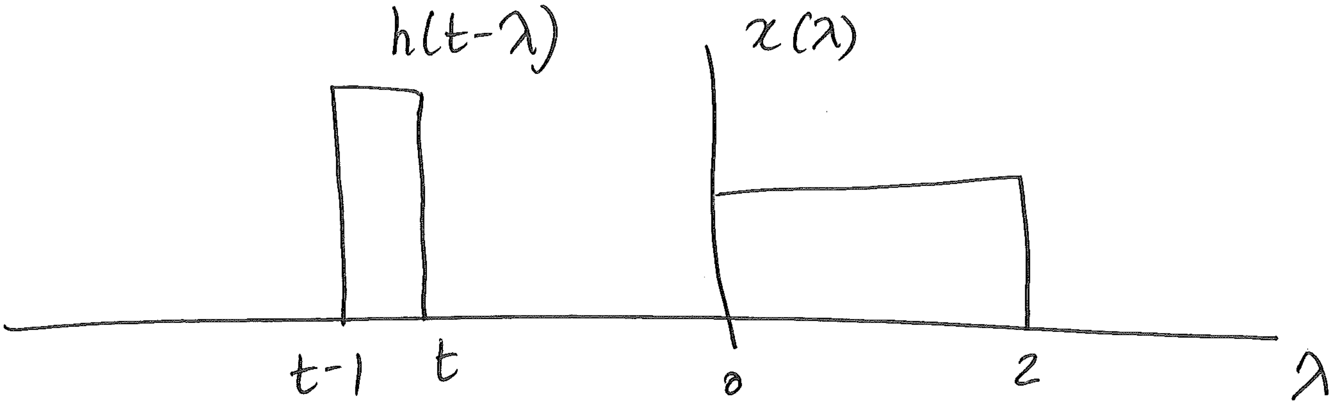
It is given that $i_s(t) = t\,u(t)$

$$v_c(t) = v_c(0) + \frac{1}{C}\int_0^t \lambda\,d\lambda =$$

$$= v_c(0) + \frac{1}{2C}\lambda^2\Big|_0^t$$

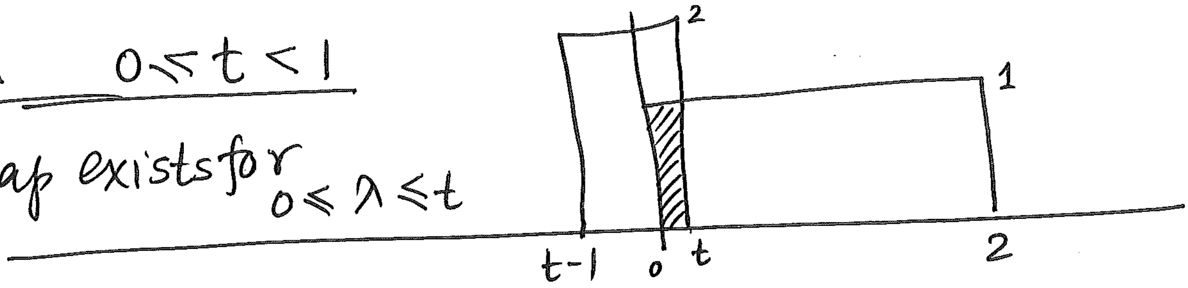
$$\boxed{v_c(t) = v_c(0) + \frac{t^2}{2C}}$$

Q3. solution



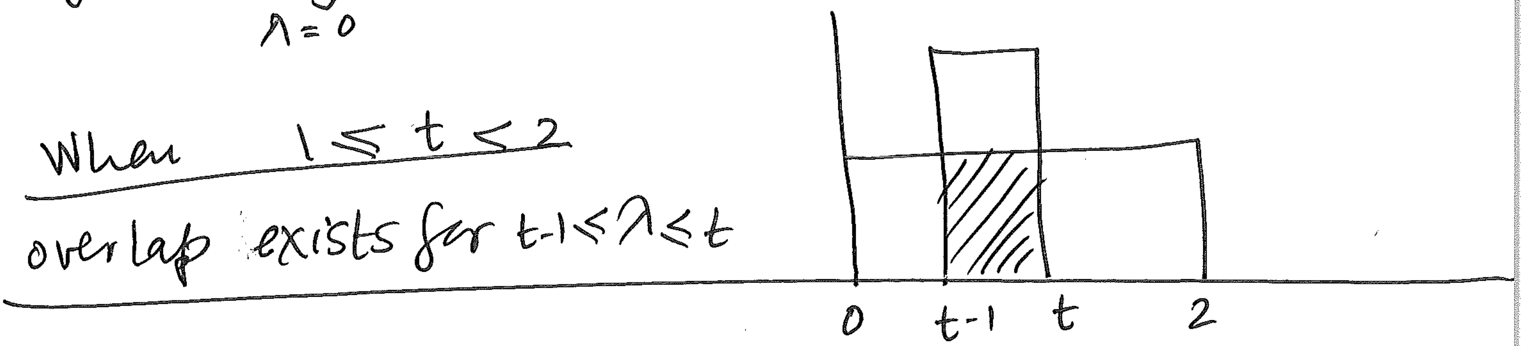
When $t \le 0$; we have no overlap, so $y(t) = 0$ for $t \le 0$.

When $0 \le t < 1$
overlap exists for $0 \le \lambda \le t$



$$y(t) = \int_{\lambda=0}^{t} (2)(1)\,d\lambda = 2\lambda\Big|_0^t = 2t. \quad 0 \le t \le 1$$

When $1 \le t < 2$
overlap exists for $t-1 \le \lambda \le t$



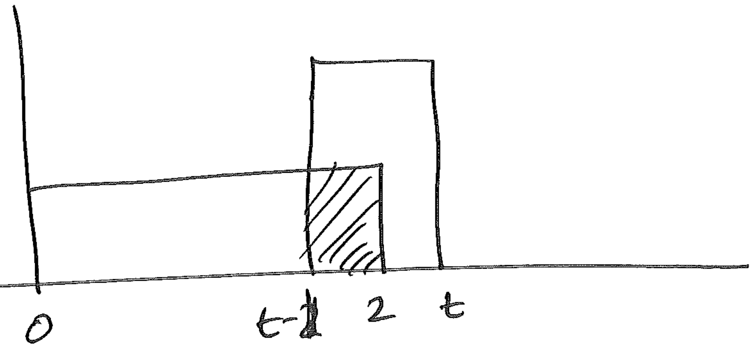
$$y(t) = \int_{t-1}^{t} (2)(1)\,d\lambda = 2\lambda\Big|_{t-1}^{t} = 2(\cancel{t} - \cancel{t} + 1)$$

$y(t) = 2 \quad$ for $1 \le t \le 2$

When $2 \leq t < 3$

Overlap exists for

$t-1 \leq \lambda < 2$



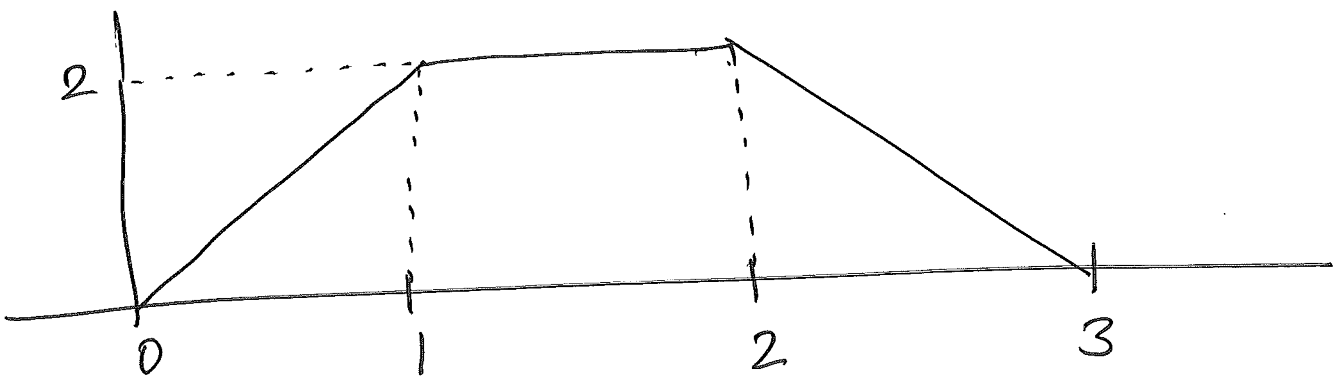
$$y(t) = 2\int_{t-1}^{2} d\lambda = 2\lambda\Big|_{t-1}^{2} = 2(2-t+1)$$

$$= 2(3-t)$$

When $t > 3$ No overlap exists.

$$y(t) = 0 \quad \forall t > 3.$$

Plot of $y(t)$



Q4. Solution:

$$\tilde{x}(t) = \begin{cases} \cos(t) & -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \\ -1 & \frac{\pi}{2} \leq t \leq \frac{3\pi}{2} \end{cases}$$

The time period is $\frac{3\pi}{2} - \left(\frac{-\pi}{2}\right) = \frac{3\pi}{2} + \frac{\pi}{2} = 2\pi.$

So, $T_0 = 2\pi.$

Since $\tilde{x}(t)$ exhibits even symmetry

$b_n = 0 \quad \forall n.$

$$a_0 = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)\,dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos(t)\,dt - \frac{1}{2\pi}\int_{+\pi/2}^{3\pi/2} dt$$

$$a_0 = \frac{1}{2\pi} \sin(t)\Big|_{-\pi/2}^{\pi/2} - \frac{1}{2\pi}\, t\,\Big|_{+\pi/2}^{3\pi/2}$$

$$= \frac{2}{2\pi} - \frac{(3\pi/2 - \pi/2)}{2\pi}$$

$$= \frac{1}{\pi} - \frac{\pi}{2\pi} = \frac{1}{\pi} - \frac{1}{2} < 0$$

$$= -0.1817$$

$$a_k = \frac{2}{T_0} \int_{T_0} \tilde{x}(t) \cos(k\omega_0 t)\, dt$$

$$= \frac{2}{2\pi} \int_{-\pi/2}^{\pi/2} \cos(t) \cos\left(\frac{k 2\pi t}{2\pi}\right) dt$$

$$+ \frac{2}{2\pi} \int_{\pi/2}^{3\pi/2} (-1) \cos\left(\frac{k 2\pi t}{2\pi}\right) dt$$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(t) \cos(kt)\, dt - \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} \cos(kt)\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \cos((k-1)t)\, dt + \frac{1}{2\pi} \int_{-\pi/2}^{+\pi/2} \cos((k+1)t)\, dt$$

$$- \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} \cos(kt)\, dt$$

$$= \left. \frac{\sin(k-1)t}{2\pi(k-1)} \right|_{-\pi/2}^{+\pi/2} + \left. \frac{\sin(k+1)t}{2\pi(k+1)} \right|_{-\pi/2}^{\pi/2}$$

$$- \frac{1}{\pi} \left. \frac{\sin(kt)}{k} \right|_{\pi/2}^{3\pi/2}$$

$$a_k = \frac{\sin(k-1)\frac{\pi}{2} + \sin(k-1)\frac{\pi}{2}}{2\pi(k-1)} + \frac{\sin(k+1)\pi/2 + \sin(k+1)\pi/2}{2\pi(k+1)}$$

$$- \frac{\sin\left(\frac{3\pi}{2}k\right) - \sin\left(\frac{\pi}{2}k\right)}{\pi k}$$

$$a_k = \frac{\sin[(k-1)\pi/2]}{\pi(k-1)} + \frac{\sin[(k+1)\pi/2]}{\pi(k+1)}$$

$$- \frac{\sin\left(\frac{\pi k}{2} + \pi k\right) - \sin\left(\frac{\pi}{2}k\right)}{\pi k}$$

Q5 (a)

Energy of one cycle $= \frac{T_0}{2}$.

Energy of 12 cycles $= 12 * \frac{T_0}{2} = 6T_0$.
$< \infty$

Since $x(t) = 0$, for $|t| > T_0$

Therefore $P = 0$.

Q5. (b)

Average Power: $P = \frac{1}{T_0} \int_{T_0} x^2(t)\, dt$

$$P = \frac{1}{T_0} \int_0^{T_0} \cos(\omega_0 t)^2\, dt$$

$$P = \frac{1}{T_0} \int_0^{T_0} \left[\frac{1}{2} + \frac{\cos(2\omega_0 t)}{2} \right] dt$$

$$P = \frac{1}{2} + 0 = \frac{1}{2} < \infty$$

Energy: of one cycle.

$$E_1 = \int_0^{T_0} x^2(t)\, dt = \frac{T_0}{2}$$

Total energy at infinite many cycles.

$$E = E_1 * \infty = \infty .$$