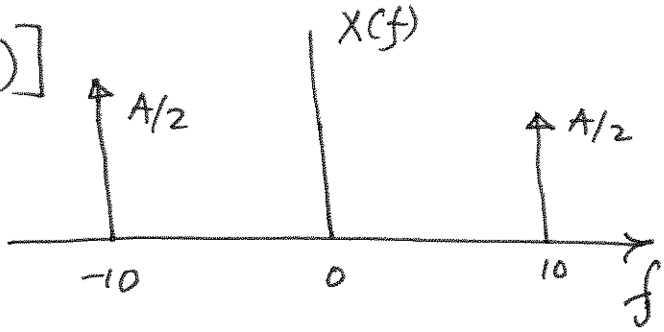
Final Exam Solution:

$$x(t) = A\cos(20\pi t) = A\cos(2\cdot 10\cdot \pi t)$$

$\Rightarrow f_m = 10\,\text{Hz}$. (message signal frequency).

(a) $X(f) = \frac{A}{2}\left[\delta(f-10) + \delta(f+10)\right]$

$$\boxed{x(t) \longleftrightarrow X(f)}$$

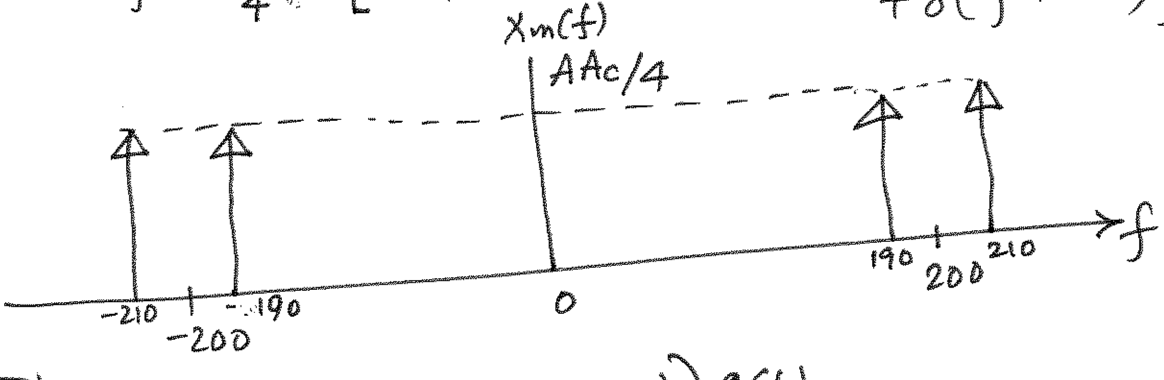


(b) $\boxed{\text{Method } 1}$

$$\begin{aligned} x_m(t) &= x(t)\,c(t) \\ &= AA_c\cos(2\cdot 10\cdot \pi t)\cos(2\pi\cdot 200t) \\ &= \frac{AA_c}{2}\left[\cos(2\pi\cdot 210t) + \cos(2\pi\cdot 190t)\right] \end{aligned}$$

$$\boxed{x_m(t) \longleftrightarrow X_m(f)}$$

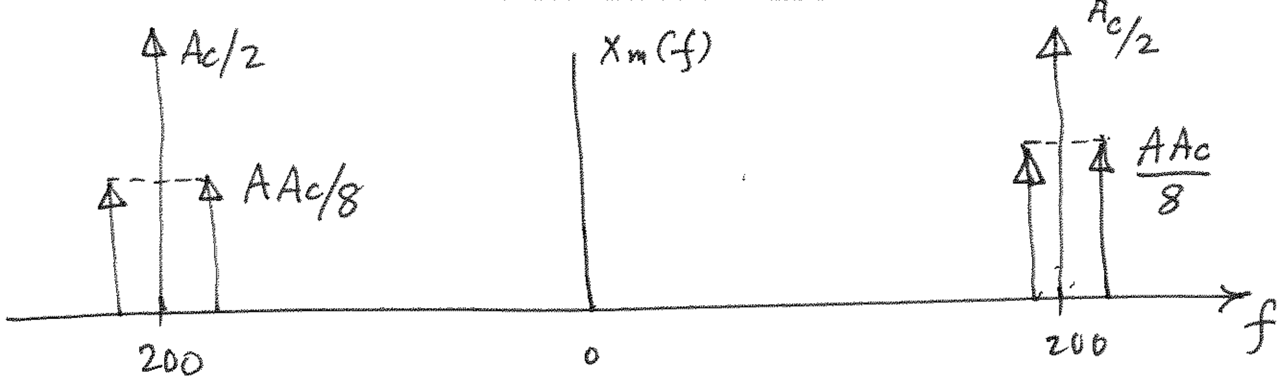
$$X_m(f) = \frac{AA_c}{4}\left[\delta(f-210) + \delta(f+210) + \delta(f-190) + \delta(f+190)\right]$$



(c) $\boxed{\text{Method 2}}$

$$\begin{aligned} x_m(t) &= (1 + g\,x(t))\,c(t) \\ &= g\frac{AA_c}{2}\left(\cos(2\pi\cdot 210t) + \cos(2\pi\cdot 190t)\right) \\ &\quad + A_c\cos(2\pi\cdot 200t) \end{aligned}$$

$$X_m(f) = g\frac{AA_c}{2*2}\left[\delta(f-210) + \delta(f+210) + \delta(f-190) + \delta(f+190)\right] + \frac{A_c}{2}\left[\delta(f-200) + \delta(f+200)\right], \text{ where } g = \tfrac{1}{2}.$$



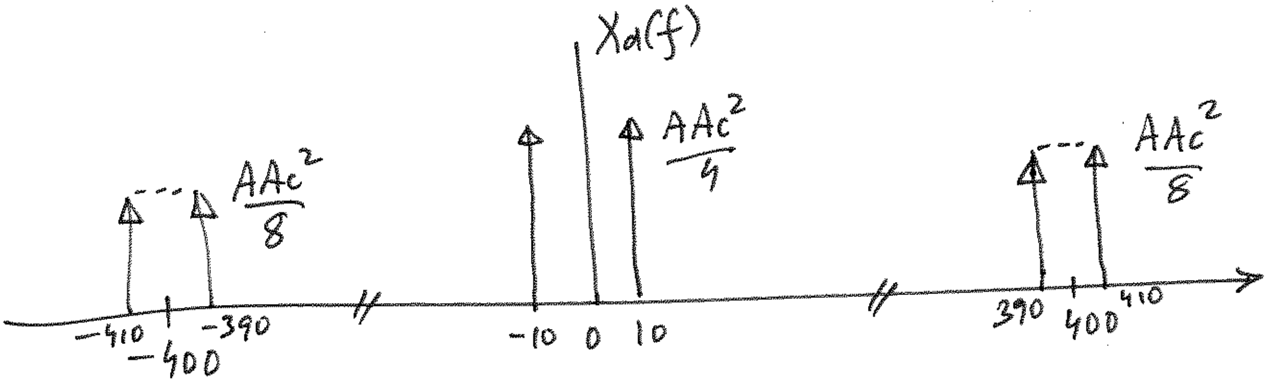
(d) $x_d(t) = x_m(t)\, c(t)$ $\boxed{\text{Method 1}}$

$$= x(t)\, c^2(t)$$

$$= A\cos(2\pi\cdot 10t)\Big(1+\cos(2\pi\cdot 400t)\Big)\frac{A_c^2}{2}$$

$$= \frac{AA_c^2}{2}\Big[\cos(2\pi 10t) + \frac{1}{2}\cos(2\pi\cdot 410t) + \frac{1}{2}\cos(2\pi\cdot 390t)\Big]$$

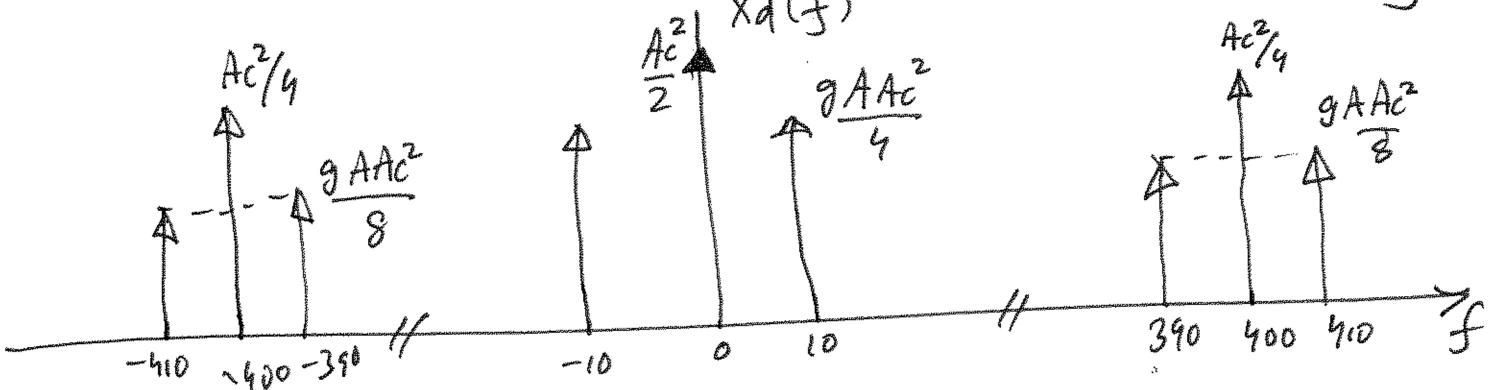
$$X_d(f) = \frac{AA_c^2}{4}\Big[\delta(f+10) + \delta(f-10)\Big] + \frac{AA_c^2}{8}\Big[\delta(f+410) + \delta(f-410) + \delta(f+390) + \delta(f-390)\Big]$$



(e) Method 2

$$x_d(t) = x_m(t)\, c(t)$$
$$= \left(1 + g\,x(t)\right) c^2(t)$$
$$= \left(1 + gA\cos(2\pi\cdot 10t)\right) \frac{A_c^2}{2}\left(1 + \cos(2\pi\cdot 400t)\right)$$
$$= \frac{A_c^2}{2} + \frac{gAA_c^2}{2}\cos(2\pi\cdot 10t) + \frac{gAA_c^2}{2*2}\left[\cos(2\pi\cdot 390t) + \cos(2\pi\, 410t)\right] + \frac{A_c^2}{2}\cos(2\pi\, 400t)$$

$$X_d(f) = \frac{A_c^2}{2}\delta(f) + \frac{gAA_c^2}{4}\left[\delta(f-10) + \delta(f+10)\right] + \frac{A_c^2}{4}\left[\delta(f-400) + \delta(f+400)\right]$$
$$+ \frac{gAA_c^2}{8}\left[\delta(f-390) + \delta(f+390) + \delta(f-410) + \delta(f+410)\right]$$



(f) For both methods, we first need a lowpass filter with bandwidth slightly greater than 10 Hz. One may choose BW = 20 Hz so that the required signal ($\neq$ 10 Hz) is extracted comfortably. The gain should be $\frac{4}{gA_c^2}$ so that the extracted signal has amplitude equal to A.
For method 2, however, we need to remove dc component. This may be attained by using a coupling capacitor.
The T.F. of L.P.F. is $\frac{4}{gA_c^2}\Pi\left(\frac{f}{40}\right)$.

CODE TO FIND DTFS of DISCRETE-TIME SIGNALS

```
function c = DTFS(x,idx)
% idx denotes the time index of signal x
  c = zeros(size(idx)); % Create all-zero vector.
  N = length(x);        % Period of the signal.
  for kk = 1:length(idx),
    k = idx(kk);
    tmp = 0;
    for nn = 1:length(x),
      n = nn-1;         % MATLAB indices start with 1.
      tmp = tmp+x(nn)*exp(-j*2*pi/N*k*n);
    end;
    c(kk) = tmp/N;
  end;
end
```

CODE TO FIND INVERSE DTFS of DISCRETE-TIME SIGNALS

```
function x = invDTFS(c,idx)
  x = zeros(size(idx)); % Create all-zero vector.
  N = length(c);        % Period of the coefficient set.
  for nn = 1:length(idx),
    n = idx(nn);
    tmp = 0;
    for kk = 1:length(c),
      k = kk-1;         % MATLAB indices start with 1.
      tmp = tmp+c(kk)*exp(j*2*pi/N*k*n);
    end;
    x(nn) = tmp;
  end;
end
```

CODE TO FIND PSUEDO-CONVOLUTION of DISCRETE-TIME SIGNALS

```
function y = pconv(x,h)
  N = length(x);        % Period for all three signals.
  y = zeros(size(x));   % Create all-zero vector.
  for n = 0:N-1,
    tmp = 0;
    for k = 0:N-1,
      tmp = tmp + per(x,k) * per(h,n-k);
    end;
    nn = n+1;
    y(nn) = tmp;
  end;
end

function xtilde = per(x,idx)
  N = length(x);   % Period of the signal.
  n = mod(idx,N);  % Modulo indexing.
  nn = n+1;        % MATLAB indices start with 1.
  xtilde = x(nn);
end
```

$$y_n = \text{inverse\_DTFS}\ (\mathrm{Y_k})$$
$$\text{where } \mathrm{Y_k} = N \mathrm{X}_k \mathrm{H}_k$$
$$\mathrm{H_k} = \mathrm{DTFS}\ (h_\mathrm{n}) \text{ and}$$
$$\mathrm{X_k} = \mathrm{DTFS}\ (x_\mathrm{n})$$

```
xn = [1  1  1  1  1  0  0  0];
hn = [1  1  1  0  0  0  0  0];
N = 8;
idx = 0 : N - 1;
Xk = DTFS(xn, idx);
Hk = DTFS(hn, idx);
Yk = N * Xk .* Hk;
yn = real(invDTFS(Yk, idx)) '
```

(real is used to avoid zero-valued imaginary parts from appearing in the display)

```
yn =
        1.00
        2.00
        3.00
        3.00
        3.00
        2.00
        1.00
        0.00
```

This must be equal to pseudo convolution of x_n and h_n, as shown below:

```
yn = pconv(xn, hn).'

yn =
        1.00
        2.00
        3.00
        3.00
        3.00
        2.00
        1.00
        0.00
```

Q3. (a) $x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$

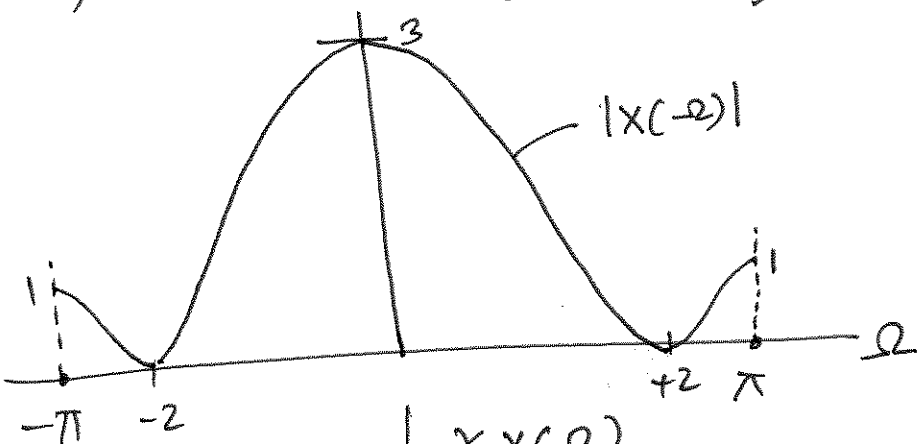
$$X(\Omega) = e^{j\Omega} + 1 + e^{-j\Omega}$$

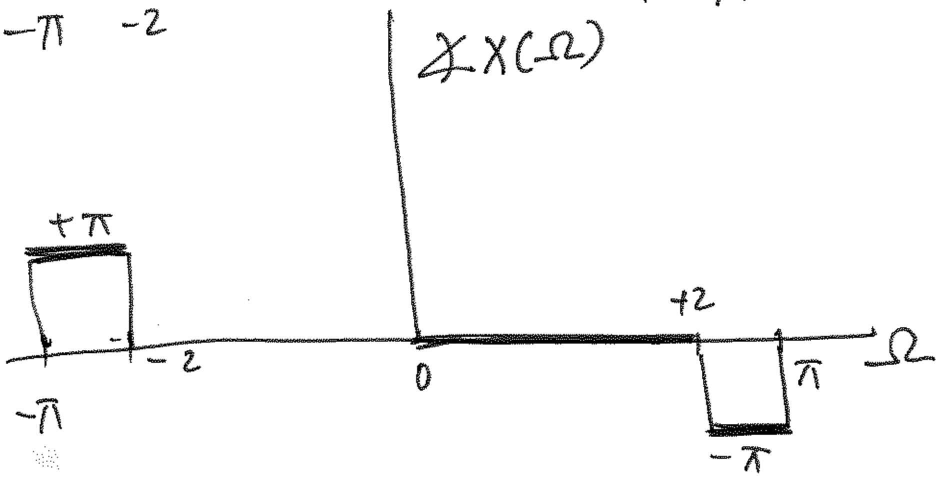
$$= \cos(-\Omega) + j\cancel{\sin(-\Omega)} + 1 + \cos(-\Omega) - j\cancel{\sin(-\Omega)}$$

$$= 2\cos(\Omega) + 1$$

$$|X(\Omega)| = |1 + 2\cos(\Omega)|$$

$$\measuredangle X(\Omega) = \tan^{-1}\left(\frac{0}{1 + 2\cos\Omega}\right) = \begin{cases} 0 & \text{if } 1 + 2\cos\Omega > 0 \\ \pm\pi & \text{if } 1 + 2\cos\Omega < 0 \end{cases}$$





Q3 (b)

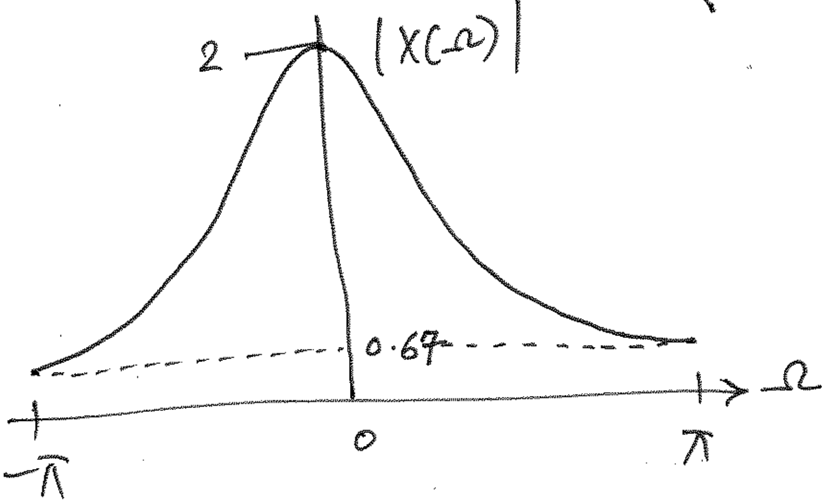
$$x[n] = (0.5)^n u[n]$$

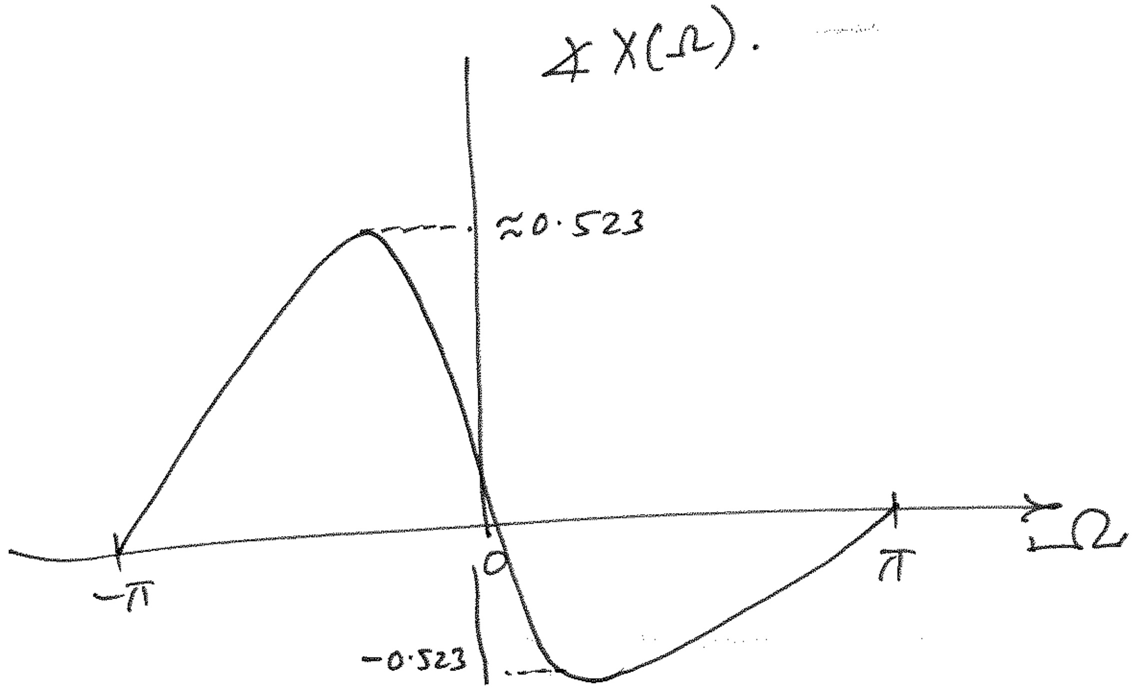
$$X(\Omega) = \sum_{n=0}^{\infty} (0.5)^n e^{-j\Omega n} = \frac{1}{1 - 0.5e^{-j\Omega}}$$

$$X(\Omega) = \frac{1}{1 - 0.5\cos(\Omega) + 0.5j\sin(\Omega)}$$

$$|X(\Omega)| = \frac{1}{\sqrt{(1 - 0.5\cos(\Omega))^2 + (0.5\sin(\Omega))^2}}$$

$$\measuredangle X(\Omega) = -\tan^{-1}\left(\frac{0.5\sin(\Omega)}{1 - 0.5\cos(\Omega)}\right)$$



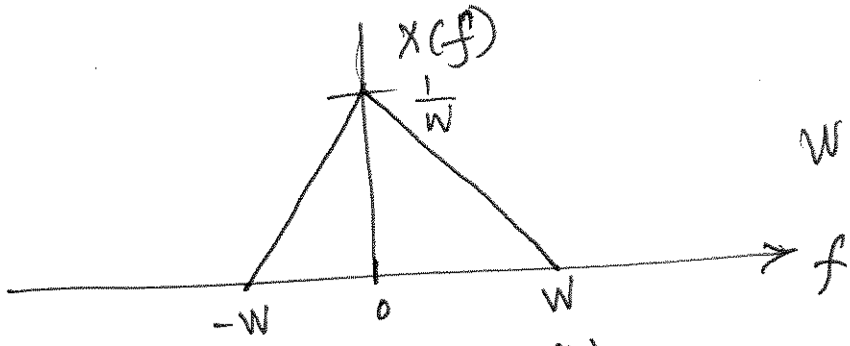


Q4. $x(t) = \text{sinc}^2\left(\frac{2}{3}t\right)$

$$\Rightarrow X(f) = \frac{3}{2}\Lambda\left(\frac{f}{2/3}\right)$$

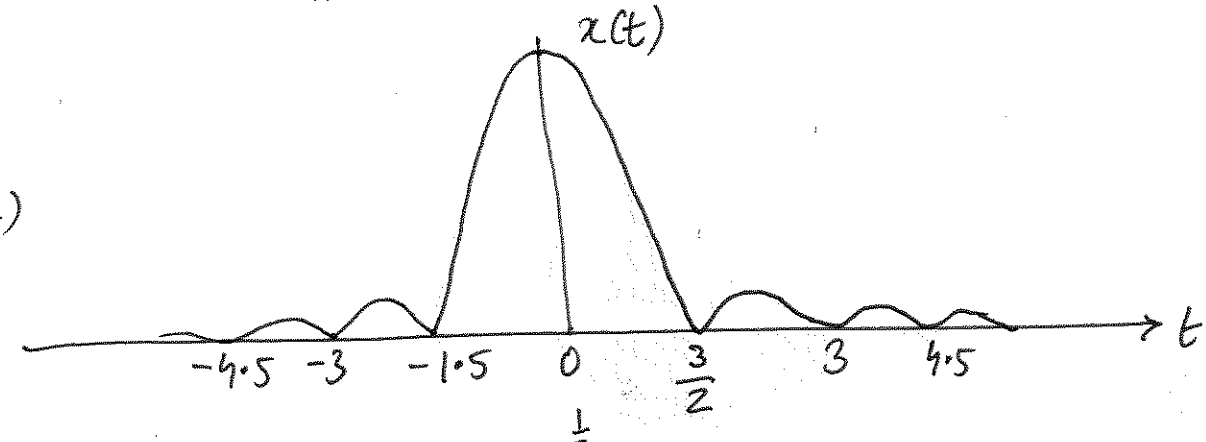
$$\text{sinc}^2(Wt) \longleftrightarrow \frac{1}{W}\Lambda\left(f/W\right)$$

(b)

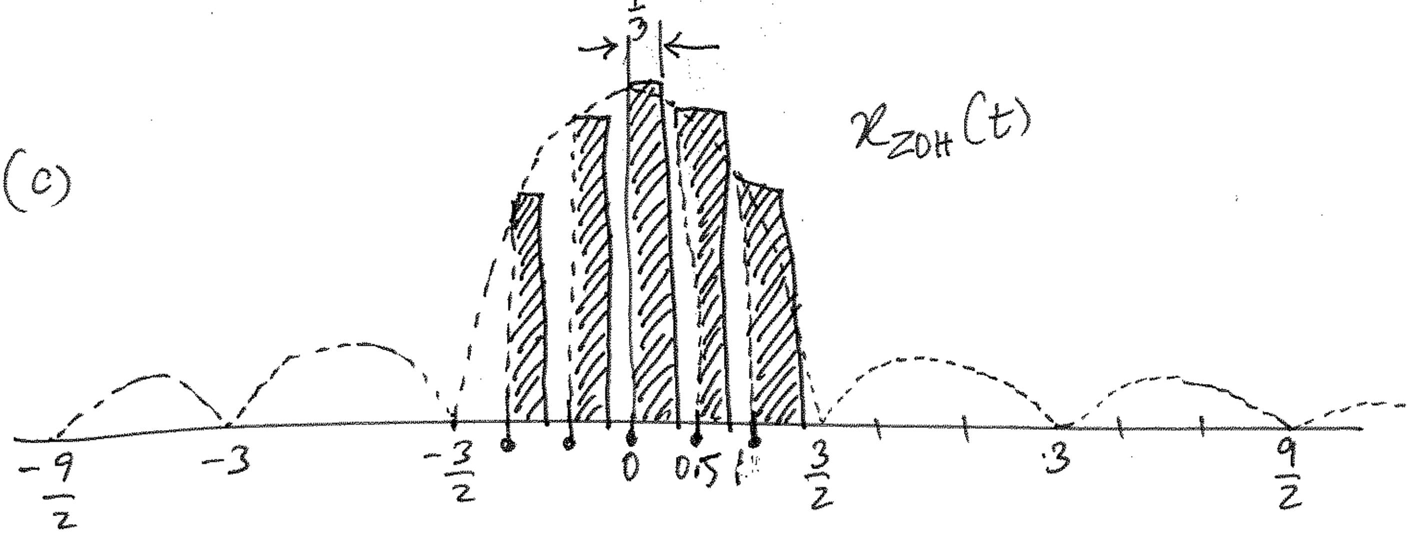


$W = 2/3$ Hz

(a)



(c)



Sampling Frequency is required to be $2W = \frac{4}{3}$ Hz (at least) $= 1.33$ Hz

Let us take $f_s = 2\text{ Hz} \gg 1.3\text{ Hz}$.

$T_s = \frac{1}{2} = 0.5$ sec

$$d\,\mathrm{sinc}\left(\frac{\omega d T_s}{2\pi}\right) = \frac{1}{3}\,\mathrm{sinc}\left(\frac{\cancel{2\pi} f\cdot\frac{1}{3}\cdot\frac{1}{2}}{\cancel{2\pi}}\right) = \frac{1}{3}\,\mathrm{sinc}(f/6)$$

(d)