Intro to Probability and Stats Date: 22-Nov

Assignment 3

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Problem 1:- let $X \sim \text{Poisson}(\lambda)$. find $E[X]$ and $\text{var}(X)$

Ans:- Pmf of Poisson: $P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}$

$$E[X] = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!}$$

let $\rightarrow z = k-1$

$$\sum_{k=1}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \lambda \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!} = \lambda \sum_{q=0}^{\infty} e^{-\lambda} \frac{\lambda^z}{z!}$$

$$\Downarrow$$

$$E[X] = \lambda$$

Ans//

Now for $\text{Var}[X]$:

$$E[x(x-1)] = \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= e^{-x} \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x}{x!}$$

$$x! = x(x-1)(x-2)!$$

taking limit $x=2$

$$= e^{-x} \sum_{x=2}^{\infty} \frac{\cancel{x(x-1)}\lambda^x}{\cancel{x(x-1)}(x-2)!} = e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!}$$

let $y = x-2$

$\hookrightarrow$ $y=0$ when x will be 2

$$= \lambda^2 e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$$

$$= \lambda^2 . e^{-\lambda} . e^{\lambda} = \lambda^2$$

$$E[x(x-1)] = \lambda^2$$

$$E[x^2 - x] = \lambda$$

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$E[x^2] - E[x] = \lambda^2$

$E[x^2] - \lambda = \lambda^2$

$E[x^2] = \lambda^2 + \lambda$

$\because var(x) = E[x^2] - (E[x])^2 = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \lambda$

$var(x) = \lambda$ Ans//

Problem 2) :-

Sol: given:

$var(2x-y) = 6$ and $var(x+2y) = 9$

Applying properties of variance;

$var(2x-y) = 6$

$2^2 \, var(x) + -1^2 . \, var(y) = 6$

$4 var(x) + var(y) = 6 \Rightarrow (1)$

$var(x+2y) = 9$

$var(x) + 2^2 \, var(y) = 9$

$var(x) + 4 \, var(y) = 9 \Rightarrow (2)$

$4 var(x) + var(y) = 6 \longrightarrow (1)$

$var(x) + 4 var(y) = 9 \longrightarrow (2)$

Solving directly through calculator

$var(x) = 1$ and $var(y) = 2$

Ans//

Problem 3)

Part i) find P.

$$P_x(k) = \frac{P}{k+1}, \quad k \in S_x$$

given: $S_x = \{0, 1, 2, 3\}$

for $k = 0 \rightarrow P_x(0) = \frac{P}{1} = P$

for $k = 1 \rightarrow P_x(1) = \frac{P}{1+1} = \frac{P}{2}$

for $k = 2 \rightarrow P_x(2) = \frac{P}{2+1} = \frac{P}{3}$

for $k = 3 \rightarrow P_x(3) = \frac{P}{3+1} = \frac{P}{4}$

$\because P_x(0) + P_x(1) + P_x(2) + P_x(3) = 1$

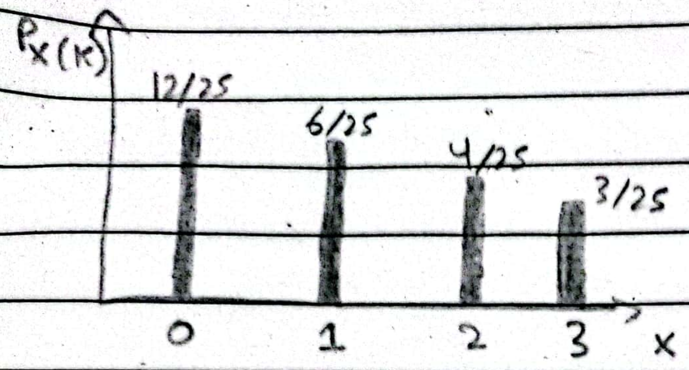
$$P + \frac{P}{2} + \frac{P}{3} + \frac{P}{4} = 1$$

$$\frac{12P + 6P + 4P + 3P}{12} = 1$$

$$25P = 12$$

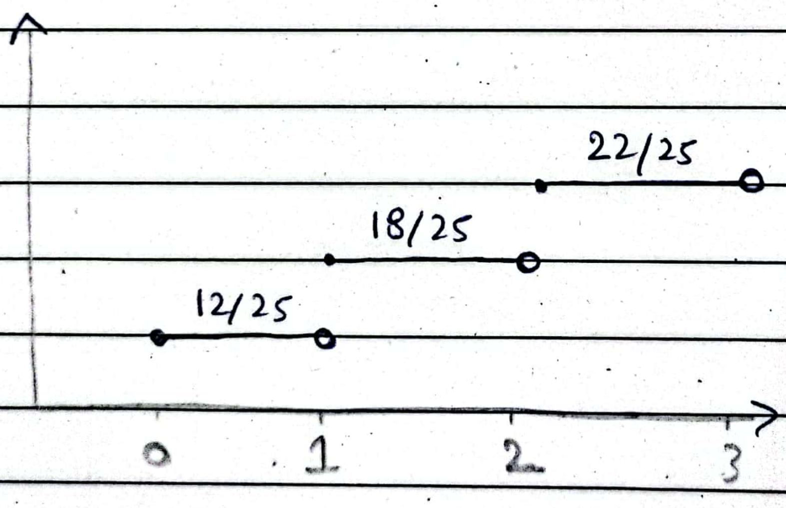
$P = \frac{12}{25}$ Ans//

ii) find and plot the CDF of x.



$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 12/25 & \text{if } 0 \le x < 1 \\ 18/25 & \text{if } 1 \le x < 2 \qquad \because 12/25 + 6/25 = 18/25 \\ 22/25 & \text{if } 2 \le x < 3 \qquad \because 18/25 + 4/25 = 22/25 \end{cases}$$

CDF :



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iii)

$P\{x \leq 0.5\} = 12/25$ $\rightarrow$ $\because$ it lies in the Second entry of CDF

$P\{x < 1.5\} = 18/25$ $\rightarrow$ Since it lies in the 3^{rd} entry of CDF

$P\{1 < x \leq 2\} = P\{2 \leq x < 3\} - P\{1 \leq x < 2\}$

$P\{1 < x \leq 2\} = \frac{22}{25} - \frac{18}{25} = \frac{4}{25}$ Ans//

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Problem 4:

Supposing z as a discrete random variable, because it either takes one of these three $0, -1, 1$.

Now assuming a clock of radius 1.

Area of circle $= \pi r^2$

z will be 1 when in xy plane both x and y are $+ve$ otherwise z will be -1 if xy are both $-ve$.

Since, xy are both $+ve$ only in 1^{st} and 3^{rd} quadrant:

total area of $1+3 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{2\pi}{4}$

$= \frac{\pi}{2}$

total circumference $= \pi$ of a circle

$P(z=1) = \frac{\pi}{2} \div \pi = \frac{\cancel{\pi}}{2} \times \frac{1}{\cancel{\pi}}$

$= \frac{1}{2}$

PMF of z:

$$= \begin{cases} \frac{1}{2} & \text{if } z = 1, -1 \\ 0 & \text{otherwise} \end{cases}$$

Now for finding the CDF, using the Summation method,

$$F_x(z) = \begin{cases} 0 & z < -1 \\ 1/2 & \text{if } z < 1 \text{ and } z \geq -1 \\ 1 & \text{if } z \geq 1 \end{cases}$$

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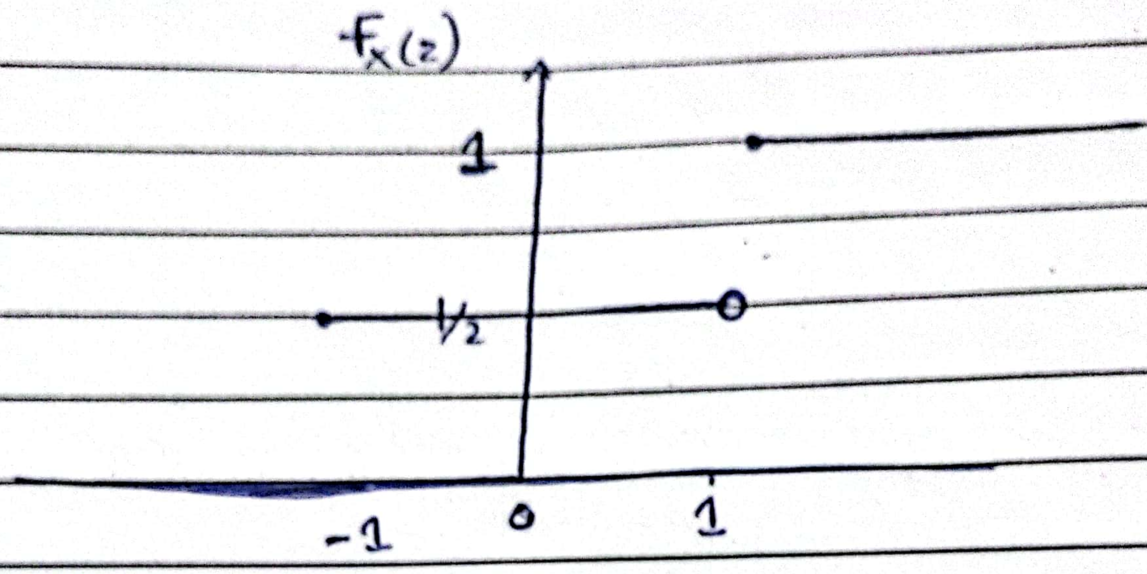
CDF Plot:



Problem 5)

$$f_X(x) = \begin{cases} ce^{-4x} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

i) find c

$$= \int_0^\infty ce^{-4x} dx = c \int_0^\infty e^{-4x} dx = c \int_0^\infty -\frac{1}{4} \cdot e^{-4x}$$

$$= -\frac{c}{4} \left[e^{-4x} \right]_0^\infty = e^{-4\infty} - e^{-4(0)}$$

$$= -\frac{c}{4}(0-1) = \frac{c}{4}$$

$\because$ integral in this case will always be equal to 1, hence

$$\frac{c}{4} = 1$$

$c = 4$ Ans//

ii) find and Plot the CDF of x.

$$F_x = \int_0^x 4e^{-4x}\,dx \qquad \because C=4$$

$$\cancel{= \int_0^x 4 \cdot \frac{1}{4} e^{-4x}\,dx = \left[2e^{-4x}\right]_0^x}$$

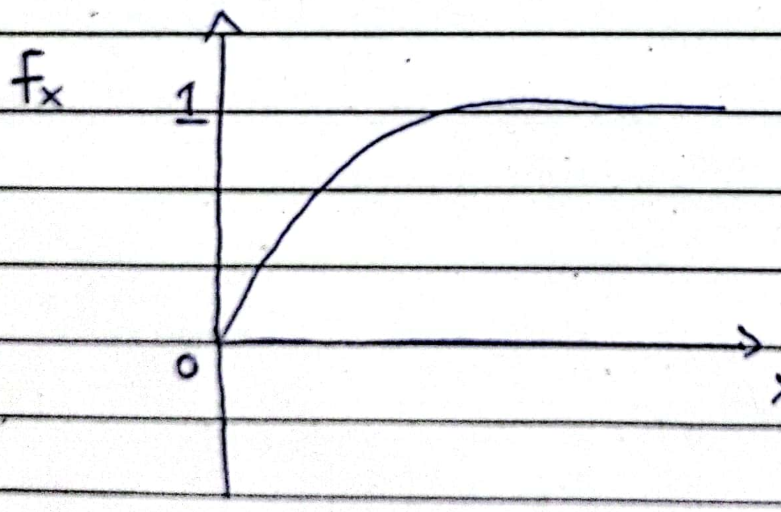
$$4\int_0^x e^{-4x} = 4\left[-\frac{1}{4}e^{-4x}\right]_0^x$$

$$= 4\left[\frac{-e^{-4x}}{4} - \left(\frac{-1}{4}\right)\right]$$

$$= 4\left[\frac{-e^{-4x}}{4} + \frac{1}{4}\right] = \cancel{4}\left[\frac{-e^{-4x}+1}{\cancel{4}}\right]$$

$$F_x = 1 - e^{-4x} \quad \text{Ans}$$

CDF Plot: Made from online software (graph).



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iii) find the Probability $P[2<x<5]$

$$P[2<x<5] = \int_2^5 4e^{-4x}\,dx \qquad \because c=4$$

$$= 4\int_2^5 e^{-4x} = 4x \frac{-1}{4}\left[e^{-4x}\right]_2^5$$

$$P[2<x<5] = -e^{-4(5)} - \left[-e^{-4\times 2}\right]$$

$$P[2<x<5] = -e^{-20} + e^{-8}$$

$$P[2<x<5] = 0.000335 \quad \text{Ans//}$$

iv) find $E[x]$

$$E[x] = \int_0^\infty x\, 4e^{-4x}\,dx$$

Since this is a improper integral

$$\lim_{a\to\infty} \int_0^a x\,4e^{-x}\,dx = \lim_{a\to\infty}\left(-4ae^{-a} - 4e^{-a} + 4\right)$$

after evaluating the limit:-

$$\lim_{a\to\infty} -4ae^{-a} = -4\times \lim_{a\to\infty}(ae^{-a})$$

$$= -4\times 0 = 0$$

$$\lim_{a\to\infty} -4e^{-a} = -4\times 0$$

$$\lim_{a\to\infty} 4 = 4$$

hence

$$E[x] = 0+0+4 = 4 \quad \text{Ans//}$$

Q6)

Given :-

Mean of Random variable = 5

variance of Random variable = 16

$X \sim N(5, 16)$

$\sigma = \sqrt{16} = 4$

i)

$P[X > 4] = 1 - P(X \le 4)$

$= 1 - P\left(\frac{X-\mu}{\sigma} \le \frac{4-\mu}{\sigma}\right)$

$\Rightarrow \frac{X-5}{4} \le \frac{4-5}{4}$

$\frac{X-5}{4} \le -\frac{1}{4}$

let $\frac{X-5}{4} = y$

$1 - P\left(y \le -\frac{1}{4}\right) = 1 - (1 - P(y \le 1/4))$

$= 1 - (1 - 0.5987)$

$= 0.5987$ Ans//

ii) $P[X \ge 7] = 1 - P[X < 7]$

$= 1 - P\left[\frac{X-4}{6} \le \frac{7-5}{4}\right]$

$= 1 - P[Y \le 1/2]$

$= 1 - 0.69$

$= 0.31$ Ans

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$$P[2<x<7] = P\left[\frac{2-\mu}{6} < \frac{x-\mu}{6} < \frac{7-\mu}{6}\right]$$

$$P[2<x<7] = P\left[\frac{2-5}{4} < \frac{x-5}{4} < \frac{7-5}{4}\right]$$

$$= P\left[-\frac{3}{4} < \frac{x-5}{4} < \frac{2}{4}\right]$$

$$= P\left[\frac{-3}{4} < \frac{x-5}{4} < \frac{1}{2}\right]$$

$$= P(y<1/2) - P(y<-3/4)$$

$$= 0.6915 - (1 - P(y<3/4))$$

$$= 0.4649 \text{ Ans}$$

$$P[6 \leq x \leq 9] = P\left[\frac{6-5}{4} < \frac{x-5}{4} < \frac{9-5}{4}\right]$$

$$P[6 \leq x \leq 9] = P\left[\frac{1}{4} < \frac{x-5}{4} < 1\right]$$

$$= P\left(y < \frac{1}{4}\right) - P(y < 0.25)$$

$$= 0.841 - 0.5987 = 0.2423$$ Ans//

$P(0 \le x \le 9) =$

ii) find a if $P[x < a] = 0.887$ $\hookrightarrow =$

$$P\left[y < \frac{a-5}{4}\right] = 0.887$$

$$\frac{a-5}{4} = 1.21$$

$a - 5 = 4.84 \Rightarrow a = 9.84$ Ans//

iii) find b if $P[x > b] = 0.1113$

$$1 - P\left(\frac{b-\mu}{\sigma}\right) = 0.1113$$

$$\cancel{1 - 0.1113 = \frac{b-5}{4}}$$

$$\cancel{0.8887 = \frac{b-5}{4}}$$

$$\cancel{3.5548 = b-5}$$

$$\cancel{b = 3.5548 + 5 = 8.5548} \text{ ~~Ans//~~}$$

$$1 - 0.1113 = P\left(y < \frac{b-5}{4}\right)$$

$$0.8887 = P\left(y < \frac{b-5}{4}\right)$$

$$1.22 = \frac{b-5}{4}$$

$$4.88 = b-5$$

$$b = 4.88 + 5 = 9.88 \quad \text{Ans//}$$

iv) find c if $P[4 < x \leq c] = 0.123$

$$\Rightarrow P[x \leq c] - P[x \leq 4] = 0.123$$

$$P\left[Y \leq \frac{c-5}{4}\right] - P\left[Y < -\frac{1}{4}\right] = 0.123$$

$$P[Y \leq (c-5)/4] - [1 - P(y < -1/4)] = 0.123$$

$$P\left[Y \leq \frac{c-5}{4}\right] - [1 - 0.5987] = 0.123$$

$$P\left[Y \leq \frac{c-5}{4}\right] - 0.4013 = 0.123$$

$$\frac{C-5}{4} = 0.06$$

$$C-5 = 0.24$$

$$C = 0.24 + 5 = 5.24 \text{ Ans}$$