EE 354/CE 361/MATH 310 - Introduction to Probability and Statistics

I have never seen a greener tree than You.
I have never seen a brighter moon than You.
I have never seen the dawn rise from
the night, or sweetness filled with
such delight as You!

~Rumi~

EE 354/CE 361/MATH 310 –
Introduction to Probability and Statistics

Fall 2023

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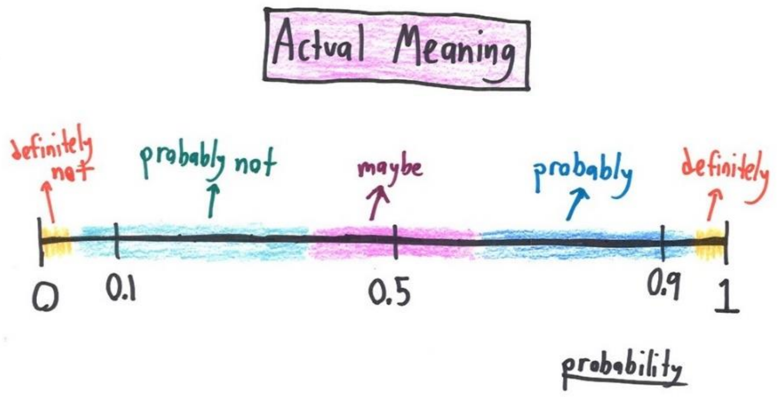
Lecture No 02 – 23rd August 2023



Announcements!

- ➢ Lecture 01 slides posted
- ➢ PDF of text book posted
- ➢ Attendance (lecture No 1) in PSCS updated
- ➢ Quiz No 01 – Wednesday August 30, 2023

Agenda for today

- *Unit 1: Probability models and axioms Lecture outline*

Unit 1: Probability models and axioms Lecture outline

- Sample space
- Probability laws
  - Axioms
  - Some properties
- Examples
  - Discrete
  - Continuous
- Interpretations of probabilities

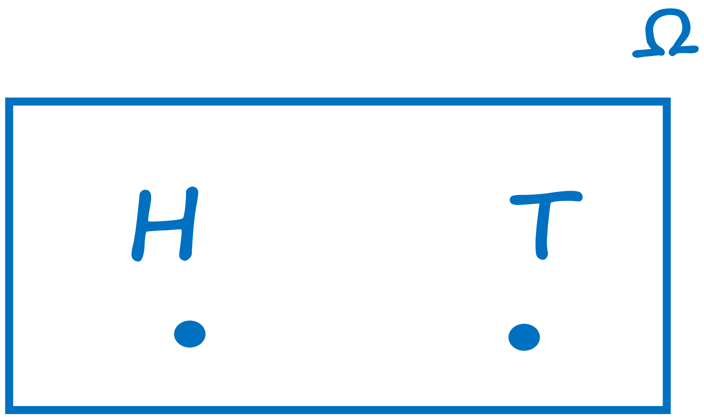
Sample space

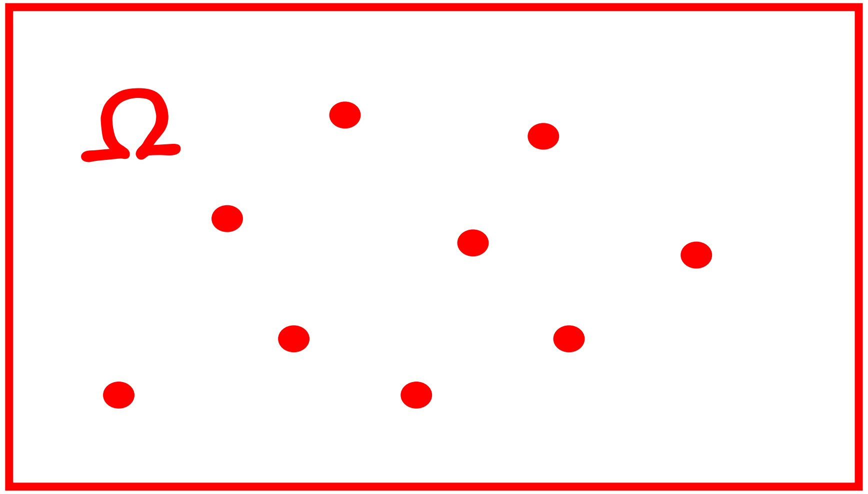
- Two steps
  - Describe possible outcomes
  - Describe beliefs about likelihood of outcomes

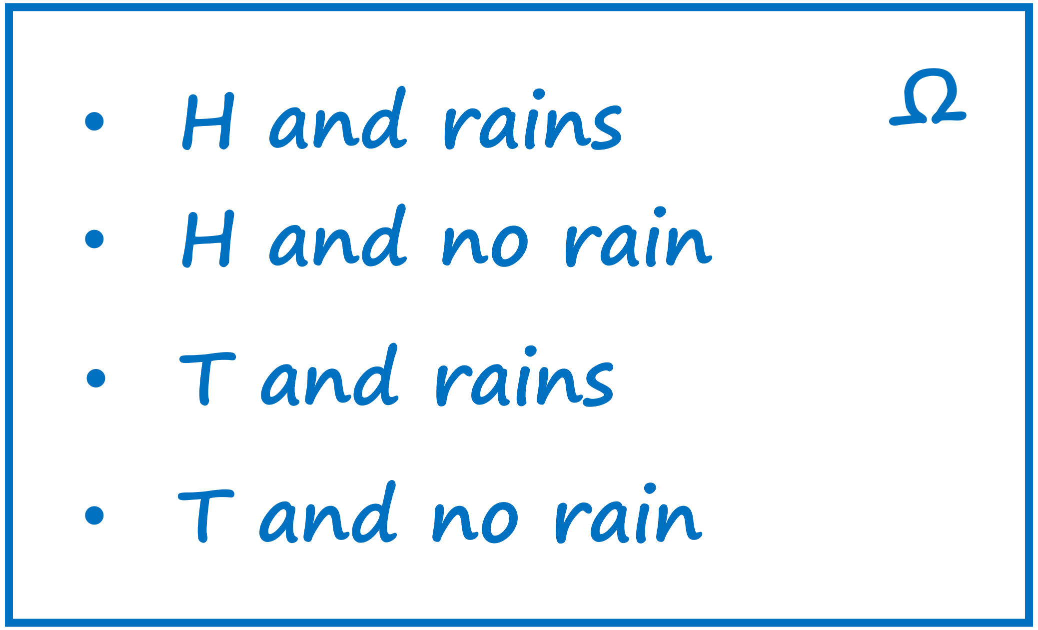
Sample space

- List (set) of possible outcomes, Ω

- List must be:
  - Mutually exclusive
  - Collectively exhaustive
  - At the “right granularity”

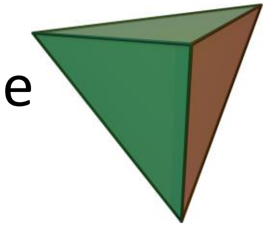


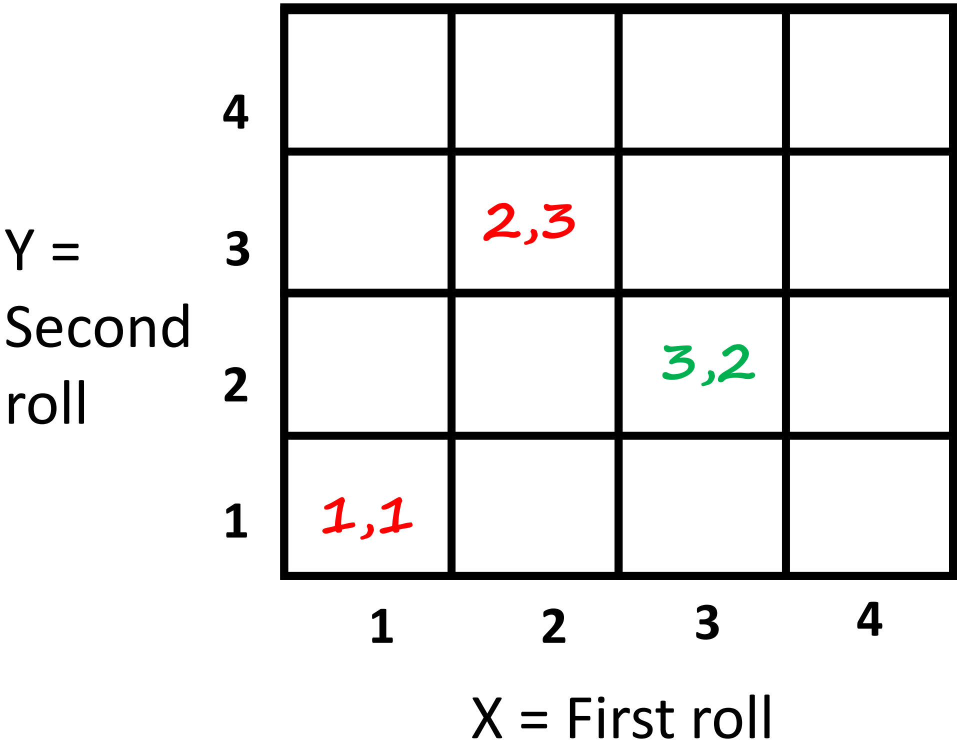


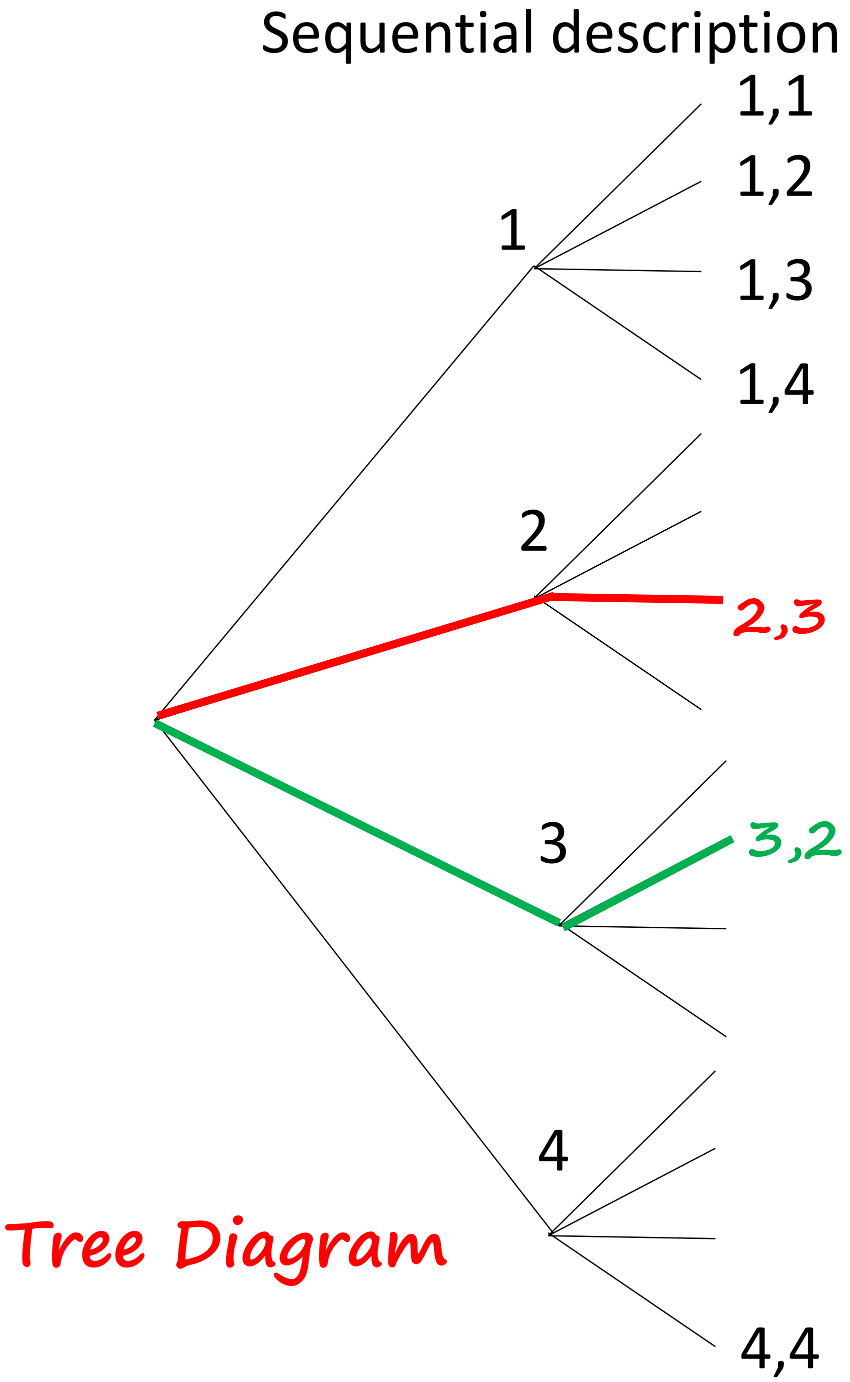


Sample space: discrete/finite example

- Two rolls of a tetrahedral die

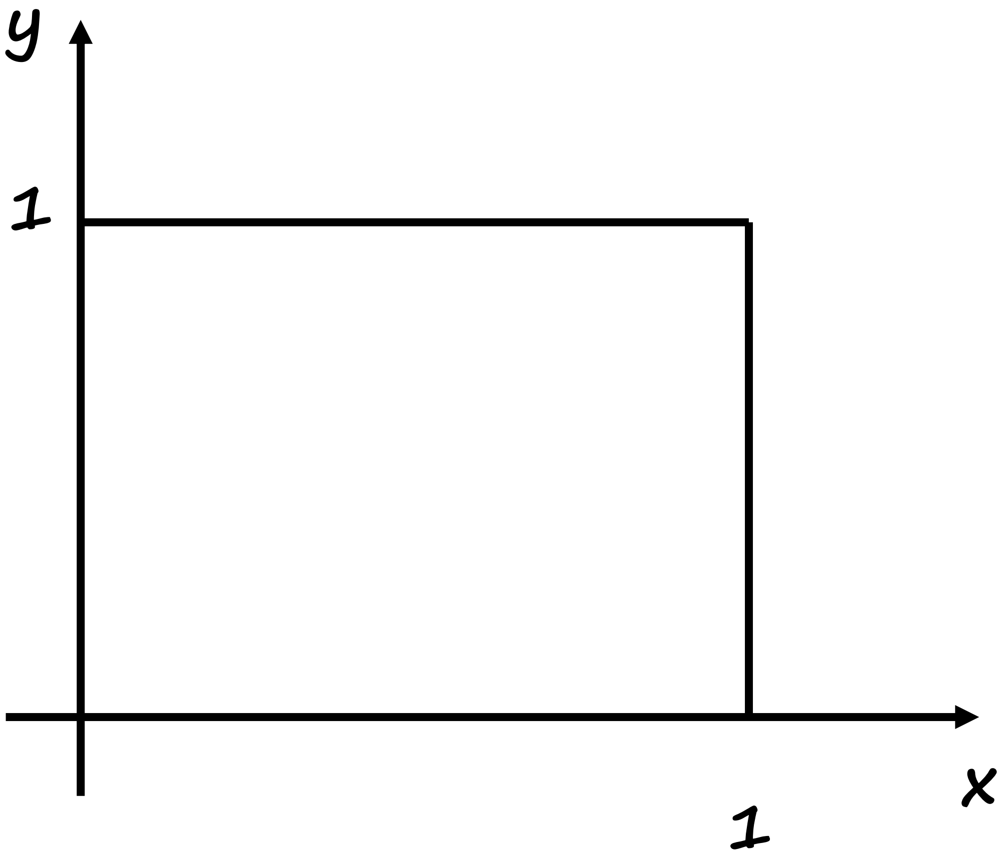






Sample space: continuous example

- (x,y) such that $0 \le x,y \le 1$



Probabilistic Model

- Two steps

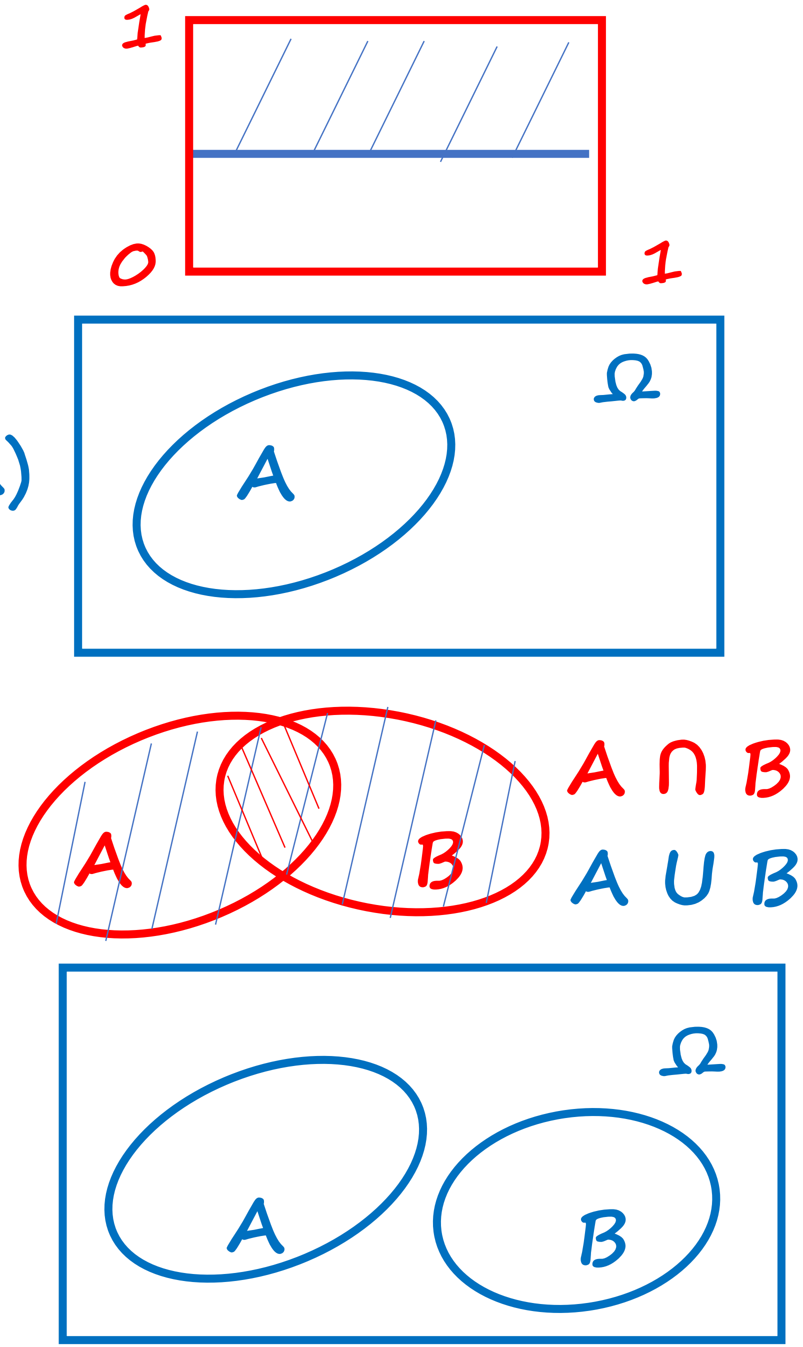
  - Describe possible outcomes – Sample Space

  - **Describe beliefs about likelihood of outcomes -**

Probability axioms

- Event: a subset of the sample space
  - Probability is assigned to events

P (A)



- Axioms:
  - Nonnegativity: $\mathbf{P}(A) \geq 0$
  - Normalization: $\mathbf{P}(\Omega) = 1$
  - (Finite) additivity: (to be strengthened later)

    If $A \cap B = \emptyset$, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$

empty set

Some simple consequences of the axioms

Axioms

$P(A) \geq 0$

$P(\Omega) = 1$

For disjoint events:

$P(A \cup B) = P(A) + P(B)$

Consequences

$P(A) \leq 1$

$P(\emptyset) = 0$

$P(A) + P(A^c) = 1$

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$

and similarly for *k* disjoint events

$$P(\{s_1, s_2, \ldots, s_k\}) = P(\{s_1\}) + \ldots + P(\{s_k\})$$

$$= P(s_1) + \ldots + P(s_k)$$

Some simple consequences of the axioms

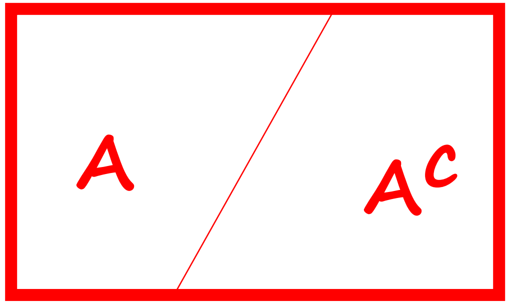
Axioms

a) $P(A) \geq 0$

b) $P(\Omega) = 1$

For disjoint events:

c) $P(A \cup B) = P(A) + P(B)$



$A \cup A^c = \Omega$

$A \cap A^c = \emptyset$

$\Rightarrow 1 \overset{(b)}{=} P(\Omega) = P(A \cup A^c)$

$\Rightarrow \overset{(c)}{=} P(A) + P(A^c)$

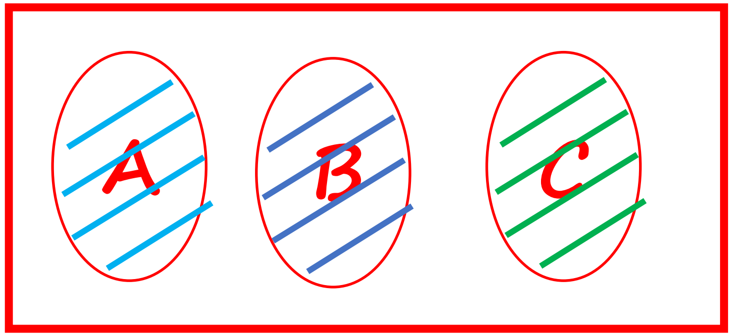
$\Rightarrow P(A) = 1 - P(A^c) \overset{(a)}{\leq} 1$

$\Rightarrow 1 = P(\Omega) + P(\Omega^c)$

$\Rightarrow 1 = 1 + P(\emptyset) \Rightarrow P(\emptyset) = 0$

Some simple consequences of the axioms

- A, B, C disjoint: $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

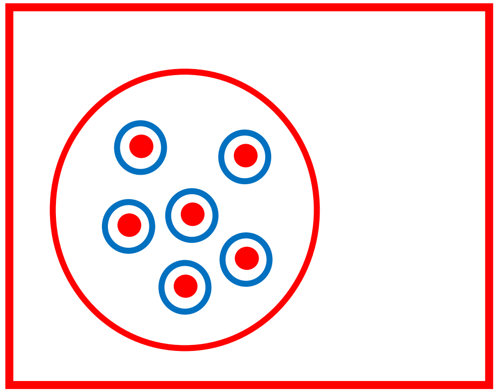


$$P(A \cup B \cup C) = P((A \cup B) \cup C) = P(A \cup B) + P(C)$$

$$\Rightarrow \quad = P(A) + P(B) + P(C)$$

$\Rightarrow$ If $A_1, \ldots A_k$ disjoint $\Rightarrow P\left(A_1 \cup \ldots \cup A_k\right) = \sum_{i=1}^{k} P(A_i)$

- $P(\{s_1, s_2, \ldots, s_k\}) = P(\{s_1\} \cup \{s_2\} \cup \ldots \cup \{s_k\})$

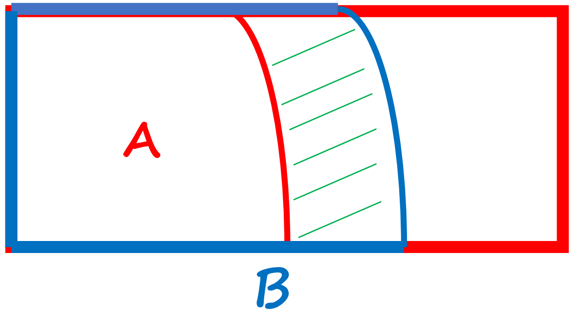


$$= P(\{s_1\} + \{s_2\} + \ldots + \{s_k\})$$

$$= P(s_1) + P(s_2) + \ldots + P(s_k)$$

More consequences of the axioms

- If $A \subset B$, then $P(A) \leq P(B)$

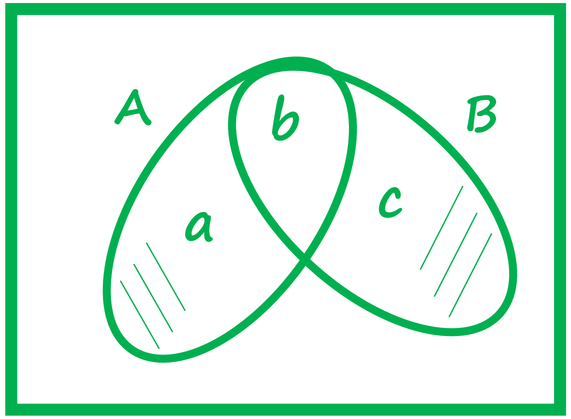


$\Rightarrow B = A \cup (B \cap A^c)$

$P(B) = P(A) + \underline{P(B \cap A^c)} \geq P(A)$

≥ 0

- $P(A \cup B) = P(A) + P(B) - \overline{P(A \cap B)}$



$a = P(A \cap B^c) \qquad b = P(A \cap B) \qquad c = P(B \cap A^c)$

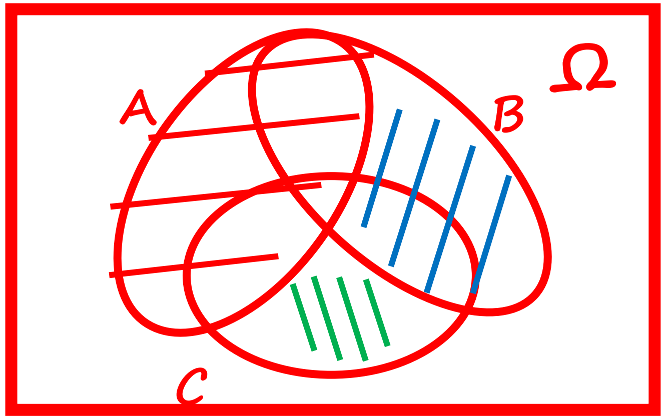
$P(A \cup B) = a + b + c$

$$P(A) + P(B) - P(A \cap B) = (a + b) + (b + c) - b$$
$$= a + b + c$$

- $P(A \cup B) \leq P(A) + P(B)$ *union bound*

More consequences of the axioms

- $P(A \cup B \cup C) = P(A) + P(A^C \cap B) + P(A^C \cap B^C \cap C)$



$\Rightarrow P(A \cup B \cup C) =$

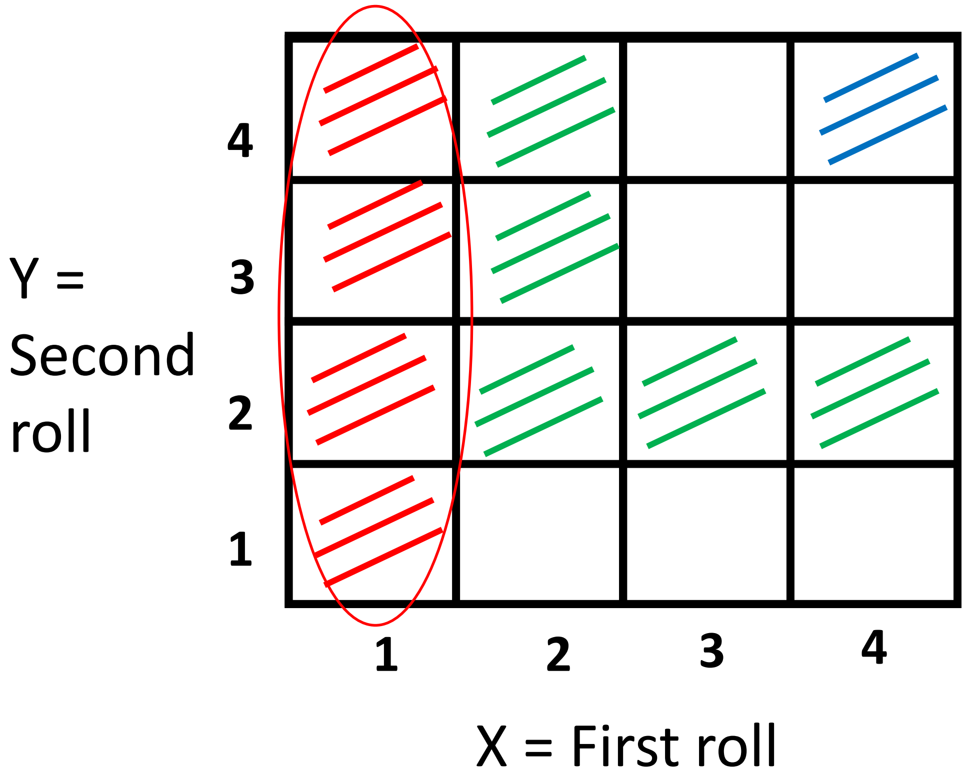
$\Rightarrow \quad = P(A \cup (B \cap A^C) \cup (C \cap A^C \cap B^C))$

Probability calculation steps (sequence of 4 steps)

1. Specify the sample space
2. Specify the probability law
3. Identify an event of interest
4. Calculate…

Probability calculation: discrete/finite example

- Two rolls of a tetrahedral die



- Let every possible outcome have probability 1/16

- $\mathrm{P}\,(X = 1) = 4 * \frac{1}{16} = \frac{1}{4}$

Let $Z = \min\,(X, Y)$

X=2, Y=3, Z=2

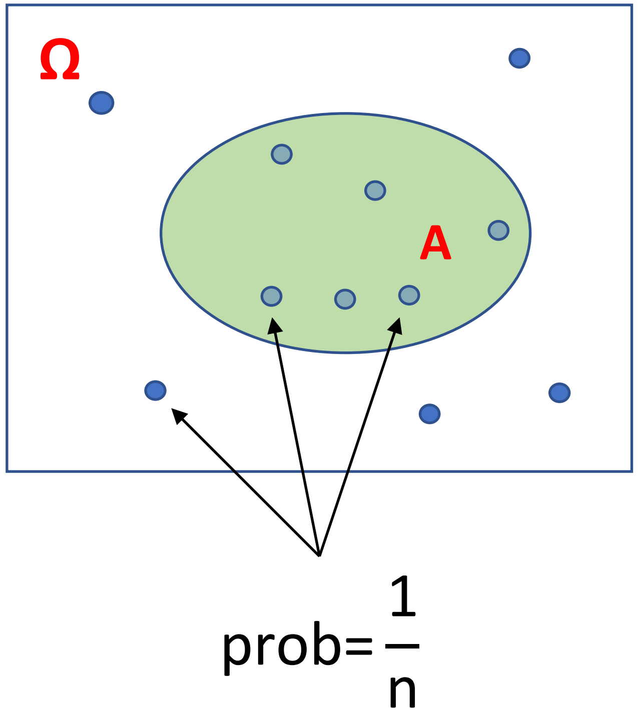
- $\mathrm{P}\,(Z = 4) = \frac{1}{16}$

- $\mathrm{P}\,(Z = 2) = 5 * \frac{1}{16}$

Discrete uniform law

finite

- Assume Ω consists of n equally likely elements
- Assume A consists of k elements



$$P(A) = \boldsymbol{k} * \frac{\mathbf{1}}{\boldsymbol{n}}$$