Introduction to Probability and Statistics
(EE 354 / CE 361 / Math 310)

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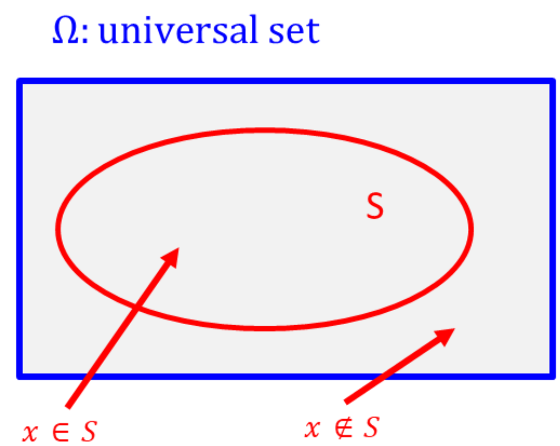
Outline: Unit 1

- Sets: A Brief Review
- Deeper Dive into Components of a Probabilistic Model
  - Experiment
  - Sample Space
  - Probability Law
  - Probability Axioms
- Consequences Of Probability Axioms
  - Discrete Probability Law
  - Discrete Uniform Probability Law
- Two Stages of Probabilistic Modeling
- Probabilistic Model Examples: Discrete and Continuous

Sets: A Brief Review



Sets

- A Collection of distinct elements

$\{a, b, c, d\}$ Finite

$R: real\ numbers$ Infinite

$\{\, x \in R: \cos(x) > \frac{1}{2} \}$

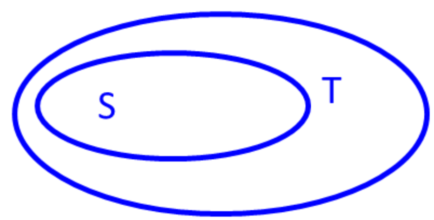
$S^c: x \in S^c \ if \ x \in \Omega, x \notin S$

$(S^c)^c = S$

$\Omega^c = \phi$ $\qquad \phi: empty\ set$

$S \subset T: x \in S \rightarrow x \in T$

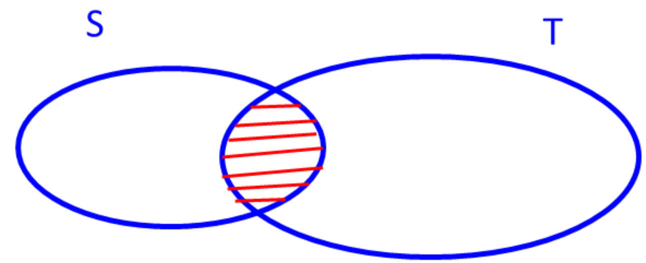
implies



Sets: A Brief Review

Unions and Intersections

$S_1 \cup S_2 \cup S_3 \cup S_4 \qquad \bigcup_{n=1}^{4} S_n$



S_n n=1,2,...

$x \in \bigcup_n S_n \;\; iff \; x \in S_n, for \; some \; n$

$x \in \bigcap_n S_n \;\; iff \; x \in S_n, for \; all \; n$

$S \cup T : x \in S \cup T \leftrightarrow x \in S \; or \; S \in T$

$S \cap T : x \in S \cap T \leftrightarrow x \in S \; and \; x \in T$

- Two sets are said to be **disjoint** if their intersection is empty
- A collection of sets is said to be **disjoint** if no two of them have a common element
- A collection of sets is said to be a **partition** of a set S is if the sets in the collection are disjoint and their union is S.

Sets: A Brief Review

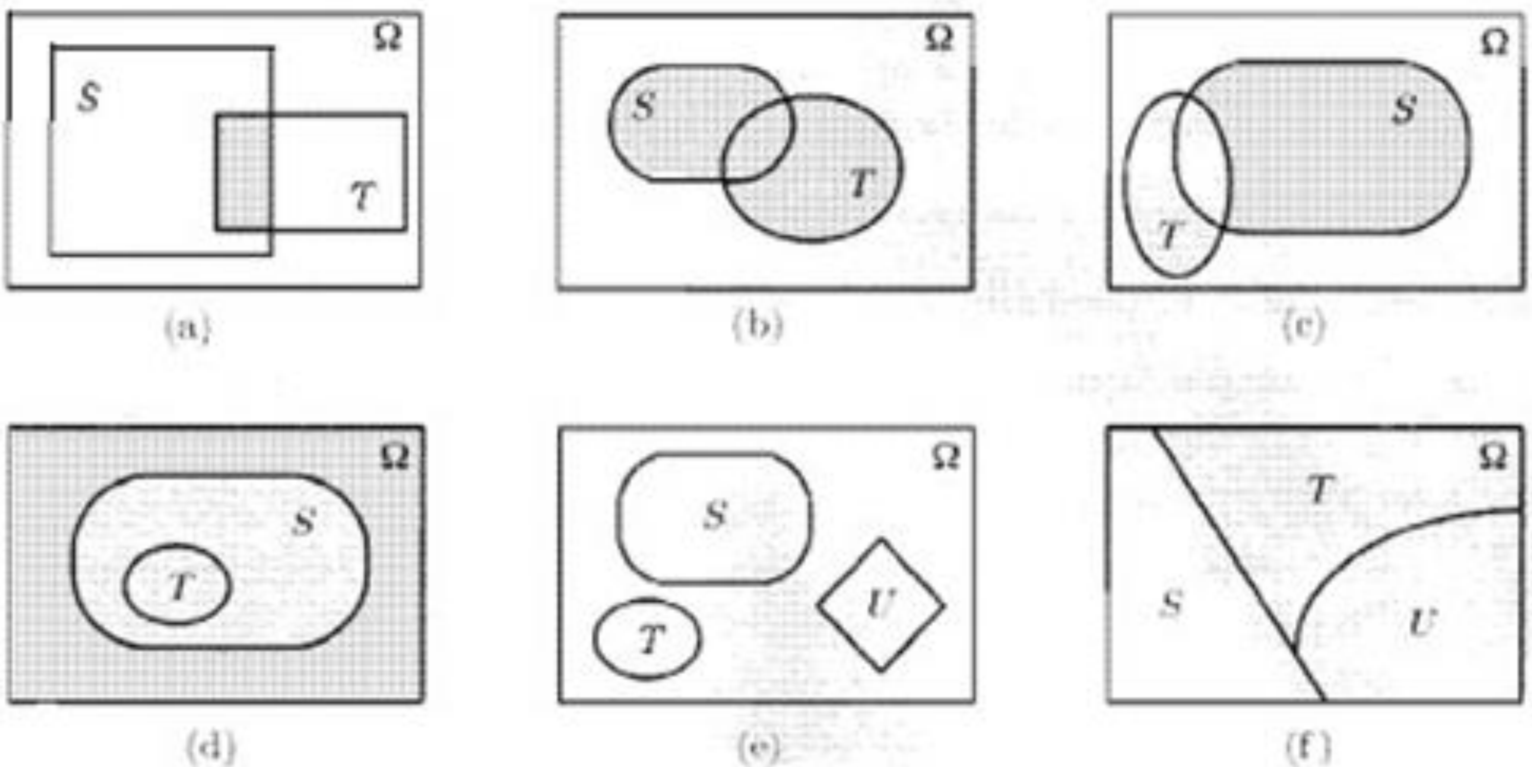


Figure 1.1: Examples of Venn diagrams (a) The shaded region is $S \cap T$. (b) The shaded region is $S \cup T$. (c) The shaded region is $S \cap T^c$. (d) Here, $T \subset S$. The shaded region is the complement of S. (e) The sets S, T, and U are disjoint. (f) The sets S, T, and U form a partition of the set Ω.

Sets: A Brief Review

Examples: Algebra of Sets

$S \cup T = T \cup S$

$S \cup (T \cup U) = (S \cup T) \cup U$

Distributive

$S \cap (T \cup U) = (S \cap T) \cup (S \cap U)$

$S \cup (T \cap U) = (S \cup T) \cap (S \cup U)$

$(S^c)^c = S$

$S \cap S^c = \phi$

$S \cup \Omega = \Omega$

$S \cap \Omega = S$

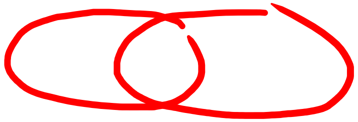
$A = B$

1) $A \subset B$ | 2) $B \subset A$

$x \in A$

$\vdots$

$x \in B$



Sets: A Brief Review

De Morgans's laws

$$\left(\bigcap_n S_n\right)^c = \bigcup_n S_n^c$$

$$\left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c$$

$(S_1 \cap S_2)^c = S_1^c \cup S_2^c$

$(S_1 \cup S_2)^c = S_1^c \cap S_2^c$

PROOF:-

A) $(S_1 \cap S_2)^c \subset S_1^c \cup S_2^c$

$x \in (S_1 \cap S_2)^c$

$x \notin (S_1 \cap S_2)$

$x \notin S_1$ OR $x \notin S_2$

$x \in S_1^c$ OR $x \in S_2^c$

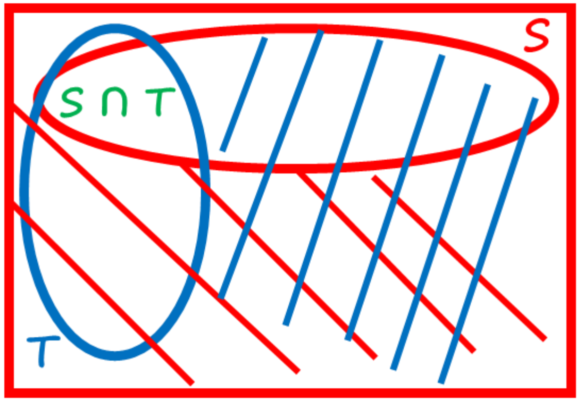
$x \in S_1^c \cup S_2^c$

B) $S_1^c \cup S_2^c \subset (S_1 \cap S_2)^c$

Sets: A Brief Review

De Morgan's laws

$$\boxed{(S \cap T)^c = S^c \cup T^c}$$



$$x \in (S \cap T)^c \Leftrightarrow x \notin S \cap T \Leftrightarrow \left\{\begin{matrix} x \notin S \\ \text{or} \\ x \notin T \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} x \in S^c \\ \text{or} \\ x \in T^c \end{matrix}\right\} \Leftrightarrow x \in S^c \cup T^c$$

EXPERIMENT: "Flip a coin"

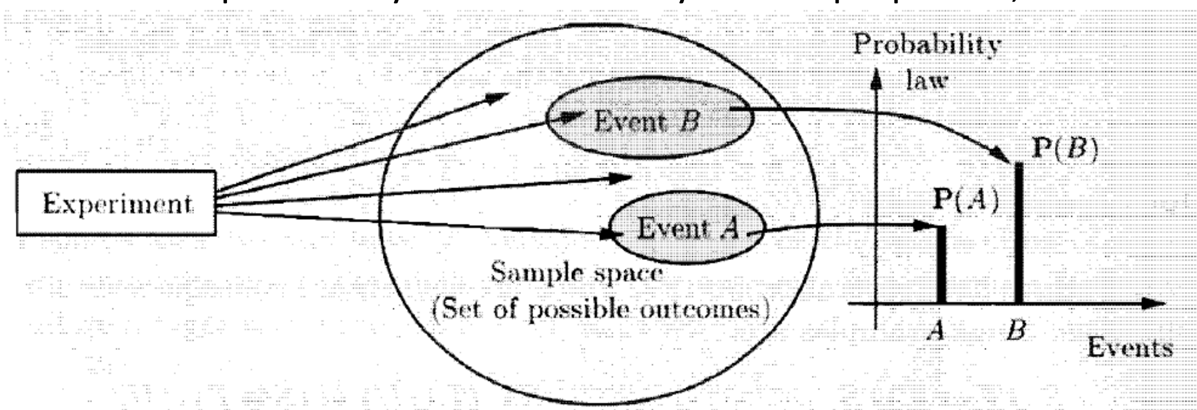
Probabilistic Model

Sample Space: Ω = { H,T }

Every probabilistic model involves an underlying process, called the "***Experiment,***" that will produce exactly one out of several possible outcomes.

∅	A {H}	B {T}	AUB {H,T}	
0	0.5	0.5	1	
0	0.2	0.3	0.5	X
0	0.3	0.7	1.0	
0	0	1	1	

- "Sample Space" (Ω)
  - The set of all possible outcomes of an experiment
- Probability Law subset of Ω
  - Assigns to a "set A of possible outcomes" a non-negative number P(A) that encodes our belief about the collective "likelihood" of the elements of A.
  - This probability law must satisfy certain properties, known as "Probability Axioms."



1. $P(A) \geq 0$
2. $P(AUB) = P(A) + P(B)$
3. $P(\Omega) = 1$

4. $P(A) \leq 1$
5. $P(\emptyset) = 0$
6. $P(AUBUC) = P(A) + P(B) + P(C)$

Probabilistic Model

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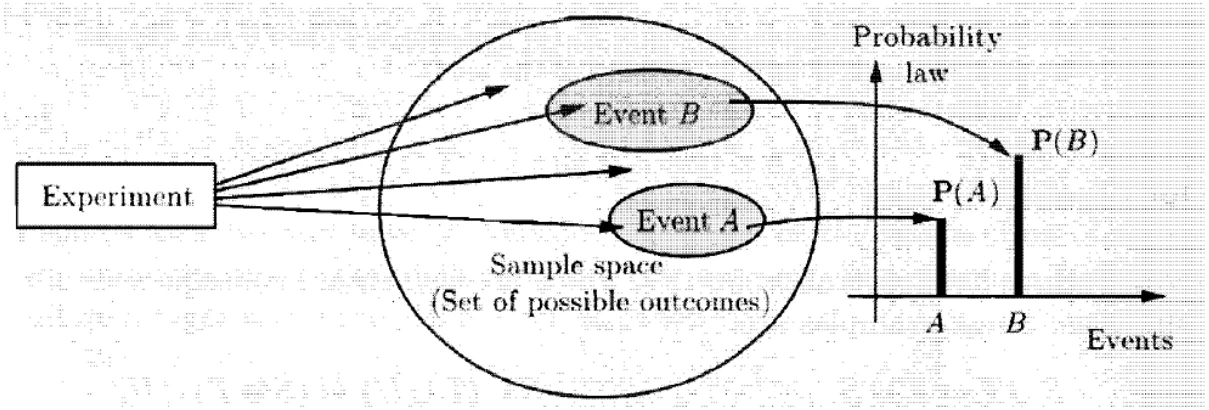

1. $P(A) \geq 0$
2. $P(A \cup B) = P(A) + P(B)$
3. $P(\Omega) = 1$

Experiment

- In our formulation of a probabilistic model, there is ONLY ONE Experiment.

- Examples:
  - Analyzing one toss of a coin $\Omega = \{H, T\}$
  - Analyzing two tosses of a coin $\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$

One Probabilistic Model ⟷ One Experiment

Probabilistic Model

Every probabilistic model involves an underlying process, called the ***Experiment,*** that will produce exactly one out of several possible outcomes.

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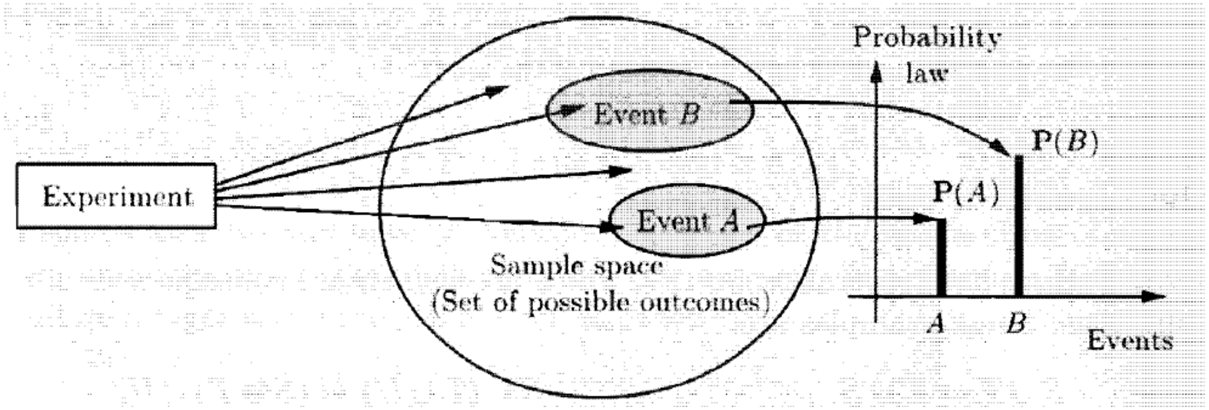

1. $P(A) \geq 0$
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Sample Space

- Elements of a sample space must be “MUTUALLY EXCLUSIVE”
  - When the experiment is carried out, there is a unique outcome.
  - Roll of a die: outcome cannot be “1 or 3”.

- Elements of a sample space must be “COLLECTIVELY EXHAUSTIVE”
  - No matter what happens in the experiment, we always obtain an outcome that has been included in the sample space.

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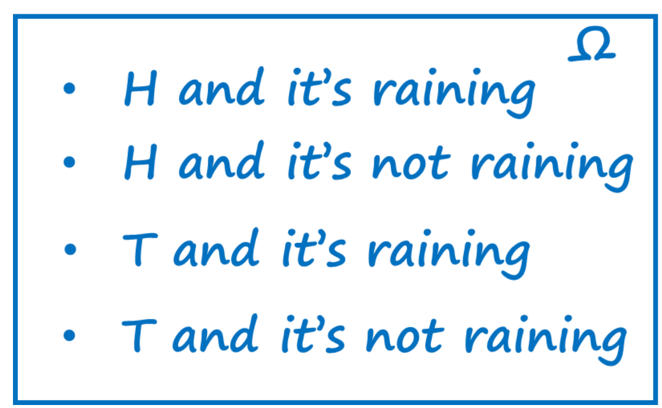
- <u>**Sample space should have enough details to include all outcomes of interest to the modeler but avoid irrelevant details.**</u>



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- Elements of a sample space must be "COLLECTIVELY EXHAUSTIVE"
  - No matter what happens in the experiment, we always obtain an outcome that has been included in the sample space.

- **<u>May consist of finite or infinite number of possible outcomes</u>**

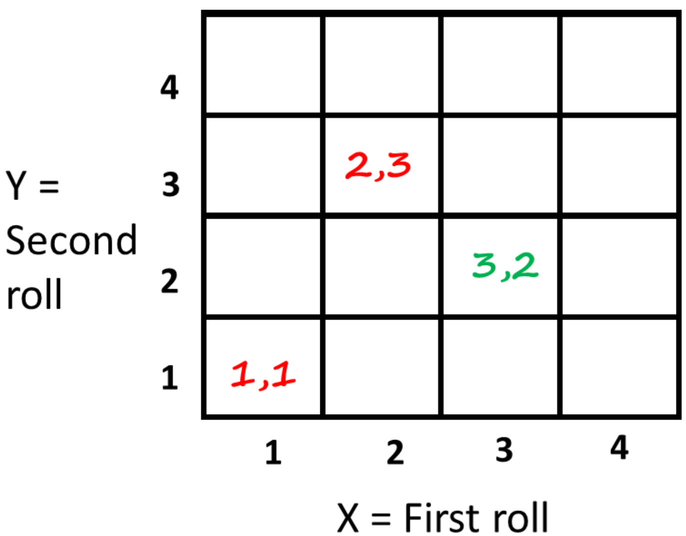
Sample Space: Finite/Discrete Example

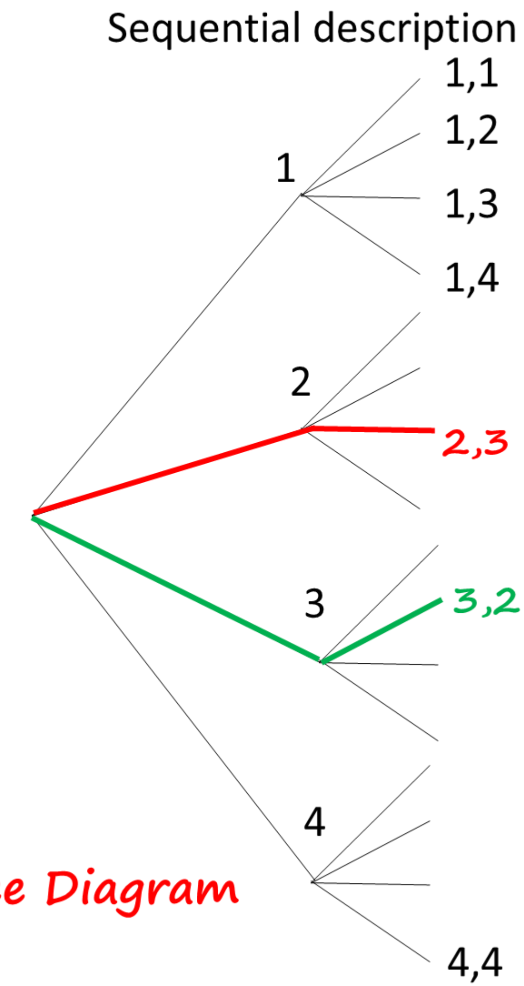
3) Tree Diagram

4-sided

- Two rolls of a tetrahedral die

2)





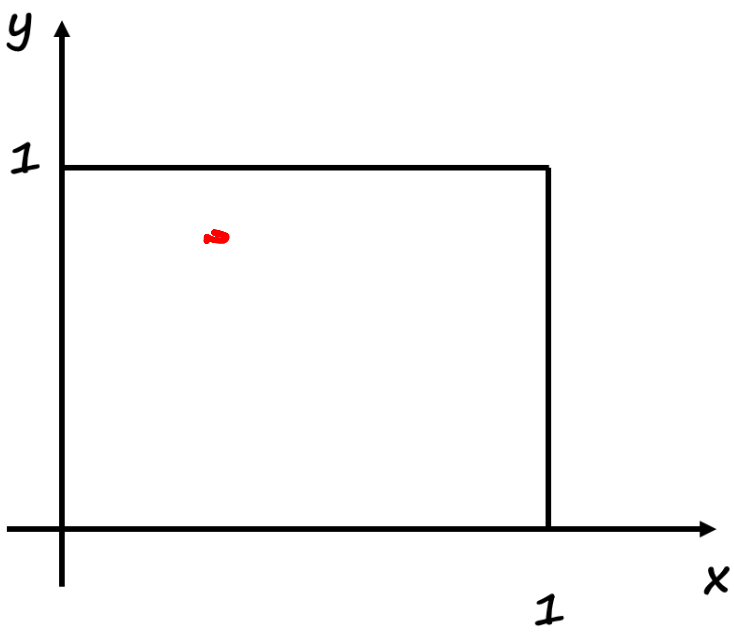
Tree Diagram

1)

S.Space = { (1,1), (1,2), (1,3), (1,4)
(2,1) (2,4)
(3,1) (3,4)
(4,1) (4,4) }

Sample Space: Infinite/Continuous Example

- Ω : $\{$(x,y) such that 0 ≤ x,y ≤ 1$\}$



Probabilistic Model

Every probabilistic model involves an underlying process, called the ***Experiment,*** that will produce exactly one out of several possible outcomes.

- Sample Space (Ω)
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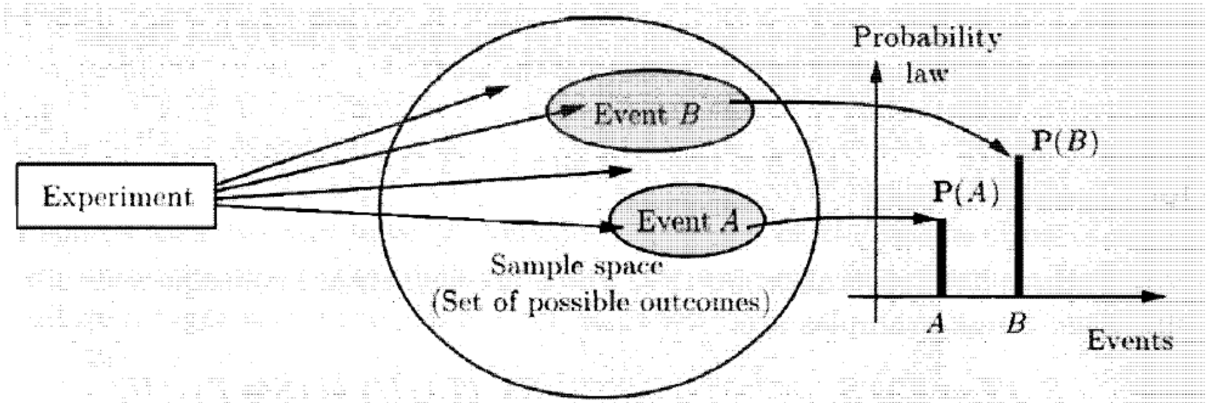


1. $P(A) \geq 0$
2. $P(A \cup B) = P(A) + P(B)$
3. $P(\Omega) = 1$

$A = \{H\}$ 0.5 0.2

$B = \{T\}$ 0.5 0.8

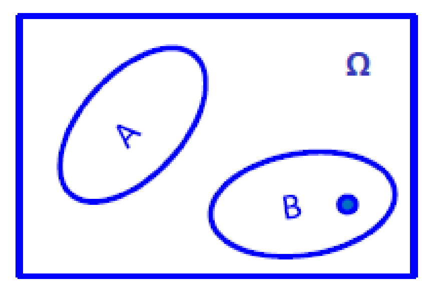
$A \cup B = \{H, T\}$ 1 1

Probability Law

- "Event"
  - A subset of the sample space
  - A probability law assigns a probability to events

- Probability "Axioms"
  - While assigning probabilities to events, a probability law must follow certain rules known as "Probability Axioms".
  - Why? Because if these probability axioms are followed, the resulting probabilistic model can be used for certain interesting calculations (using the Probability theory tools that have been built on the basis of these axioms.)

- Nonnegativity: $\mathbf{P}(A) \geq 0$
- Normalization: $\mathbf{P}(\Omega) = 1$
- (Finite) additivity: (to be strengthened later)
  If $A \cap B = \emptyset$, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$



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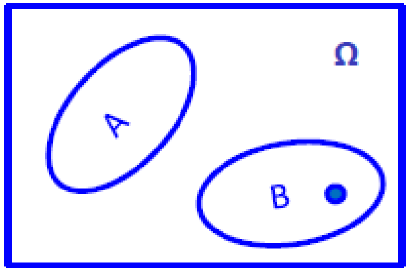


Probability Axioms: Some Simple Consequences

Axioms

a) $P(A) \geq 0$

b) $P(\Omega) = 1$

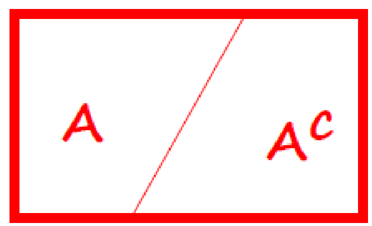
For disjoint events:

c) $P(A \cup B) = P(A) + P(B)$

<u>Consequence</u>

$$\boxed{P(A) \leq 1}$$

<u>Proof:</u>



$A \cup A^c = \Omega$

$A \cap A^c = \emptyset$

$$\Rightarrow 1 \overset{(b)}{=} P(\Omega) = P(A \cup A^c)$$

$$\Rightarrow \overset{(c)}{=} P(A) + P(A^c)$$

$$\Rightarrow P(A) = 1 - P(A^c) \overset{(a)}{\leq} 1$$

Probability Axioms: Some Simple Consequences

Axioms

a) $P\ (A) \geq 0$

b) $P\ (\Omega) = 1$

For disjoint events:

c) $P\ (A \cup B) = P\ (A) + P\ (B)$

<u>Consequence</u>

$$\boxed{P\ (\emptyset) = 0}$$

<u>Proof:</u>

$\phi = \Omega^c$

$\Omega = \Omega \cup \phi = \Omega \cup \Omega^c$

$1 = P\ (\Omega) + P\ (\Omega^c)$

$1 = 1 + P\ (\emptyset) \Rightarrow P\ (\emptyset) = 0$

Probability Axioms: Some Simple Consequences

Axioms

a) $P(A) \geq 0$

b) $P(\Omega) = 1$

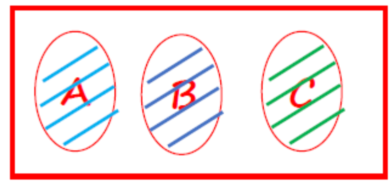
For disjoint events:

c) $P(A \cup B) = P(A) + P(B)$

<u>Consequence</u>

$\boxed{\begin{array}{l} \text{A, B, C disjoint:} \\ \qquad P(A \cup B \cup C) = P(A) + P(B) + P(C) \end{array}}$

<u>Proof:</u>



$$P(A \cup B \cup C) = P(\underbrace{(A \cup B)}_{D} \cup C) = P(A \cup B) + P(C) \quad = P(D) + P(C)$$

$$\Rightarrow \quad = P(A) + P(B) + P(C)$$

<u>Corollary:</u>

$\boxed{\text{If } A_1, \ldots A_k \text{ disjoint} \Rightarrow P(A_1 \cup \ldots \cup A_k) = \sum_{i=1}^{k} P(A_i)}$

Outcome A Single element event $\{A\}$

Discrete Probability Law

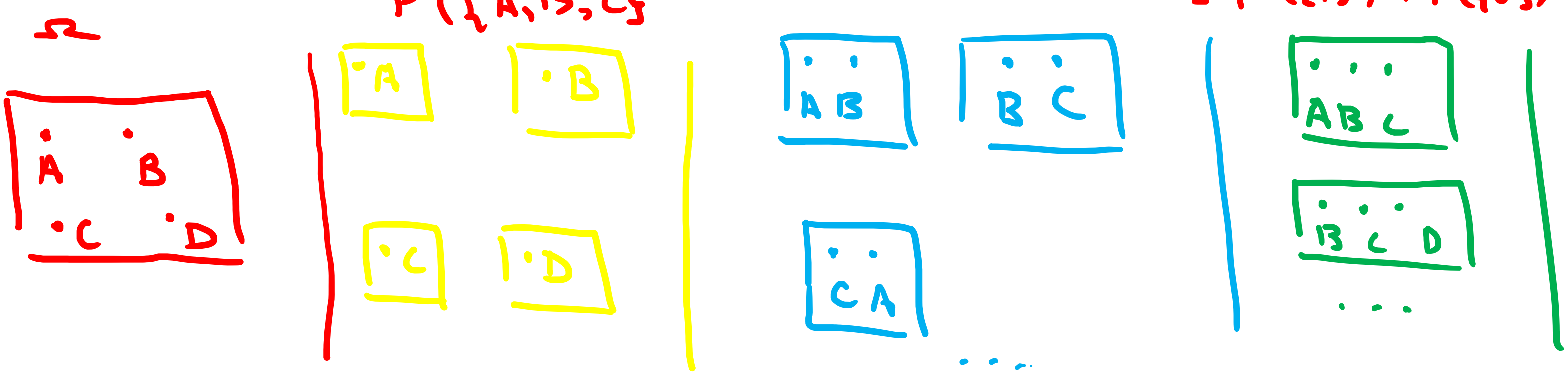
Discrete Probability Law

If the sample space consists of a finite number of possible outcomes, then the probability law is specified by the probabilities of the events that consist of a single element. In particular, the probability of any event $\{s_1, s_2, \ldots, s_n\}$ is the sum of the probabilities of its elements:

$$\mathbf{P}(\{s_1, s_2, \ldots, s_n\}) = \mathbf{P}(s_1) + \mathbf{P}(s_2) + \cdots + \mathbf{P}(s_n).$$

$P(\{A, B\}) = P(\{A\} \cup \{B\}) = P(\{A\}) + P(\{B\})$

$P(\{A, B, C\}) = P(\{A\}) + P(\{B\}) + P(\{C\})$



Discrete Probability Law: Proof

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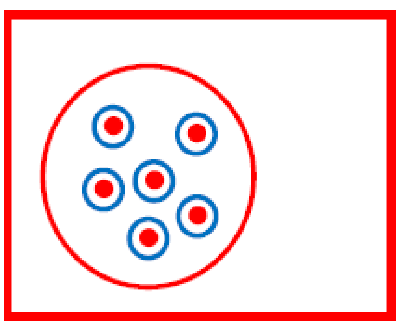
Previously-discussed consequence of Probability Axioms:

$$\boxed{\text{If } A_1, \ldots A_k \text{ disjoint} \Rightarrow P(A_1 \cup \ldots \cup A_k) = \textstyle\sum_{i=1}^{k} P(A_i)}$$

- $P(\{s_1, s_2, \ldots, s_k\}) = P(\{s_1\} \cup \{s_2\} \cup \ldots \cup \{s_k\})$



$= P(\{s_1\} + \{s_2\} + \ldots + \{s_k\})$

$= P(s_1) + P(s_2) + \ldots + P(s_k)$

Discrete Probability Law

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$$\mathbf{P}\big(\{s_1, s_2, \ldots, s_n\}\big) = \mathbf{P}(s_1) + \mathbf{P}(s_2) + \cdots + \mathbf{P}(s_n).$$

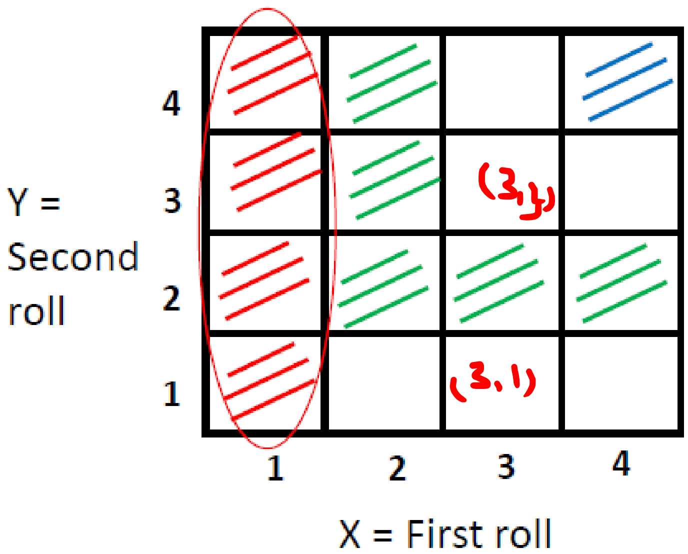
Discrete Probability Law: Example

$P(\{(1,1)\}) = \frac{1}{16}$ $\quad P(\{(1,2)\}) = \frac{1}{16}$ $\quad \ldots \quad P\{(4,4)\} = \frac{1}{16}$

Experiment:-

(4-sided)

- Two rolls of a tetrahedral die



$\Omega = \{(1,1), (1,2), (1,3), (1,4)$

$\ldots$

$(4,4)\}$

- Let every possible outcome have probability 1/16

$P(\text{"First roll is 1"})$

- P (X = 1) = 1) IDENTIFY EVENT OF INTEREST

$E = \{(1,1), (1,2), (1,3), (1,4)\}$

$P(E) = P\{(1,1)\} + P(\{(1,2)\})$

$+ P\{(1,3)\} + P(\{(1,4)\})$

$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{4}{16}$

Let $Z = \min(X, Y)$

$X=2, Y=3, Z=2$

- P (Z = 4) =

- P (Z = 2) = $P(\text{'Min of Two Rolls is 2'})$

$E = \{(3,2), (4,2), (2,2), (2,3), (2,4)\}$

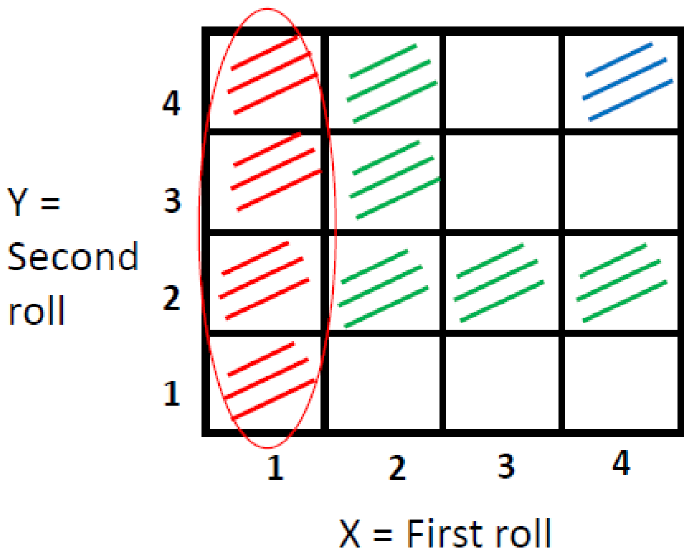
$P(E) = P\{(3,2)\} + P\{(4,2)\} + \ldots$

$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$

$= 5/16$

Discrete Probability Law: Example

- Two rolls of a tetrahedral die



- Let every possible outcome have probability 1/16

- P (X = 1) =

Let $Z = \min(X, Y)$

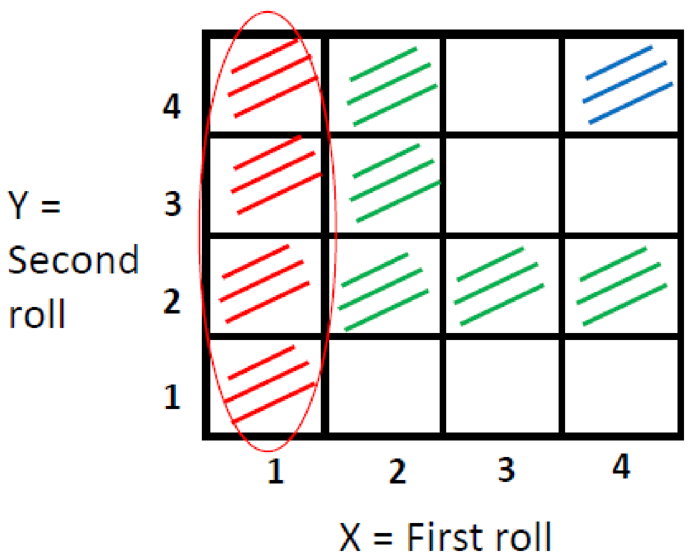
X=2, Y=3, Z=2

- P (Z = 4) =

- P (Z = 2) =

Discrete Probability Law: Example

- Two rolls of a tetrahedral die



- Let every possible outcome have probability 1/16

- P (X = 1) = $4 * \frac{1}{16} = \frac{1}{4}$

Let $Z = \min(X, Y)$

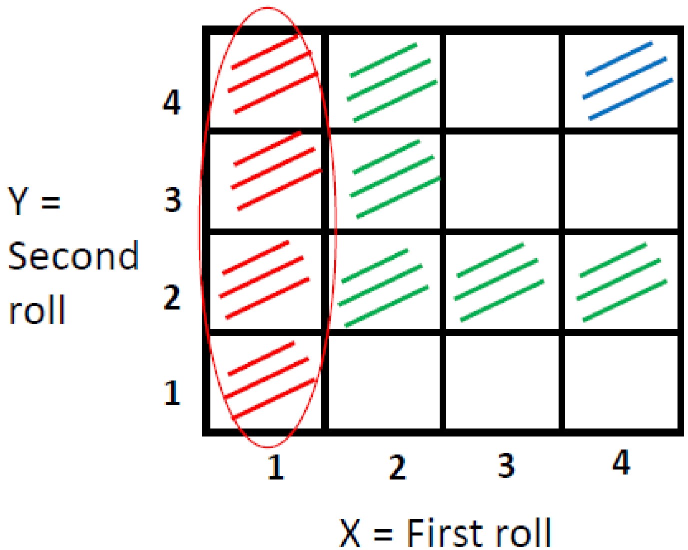
X=2, Y=3, Z=2

- P (Z = 4) = $\frac{1}{16}$

- P (Z = 2) = $5 * \frac{1}{16}$

Discrete Probability Law: Example

- Two rolls of a tetrahedral die



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Let $Z = \min(X, Y)$

X=2, Y=3, Z=2

- P (Z = 4) = $\frac{1}{16}$

- P (Z = 2) = $5 * \frac{1}{16}$

Discrete Probability Law → Discrete Uniform Probability Law

- For the special scenario when all the elements of a finite sample space are equally likely, we can further say the following:

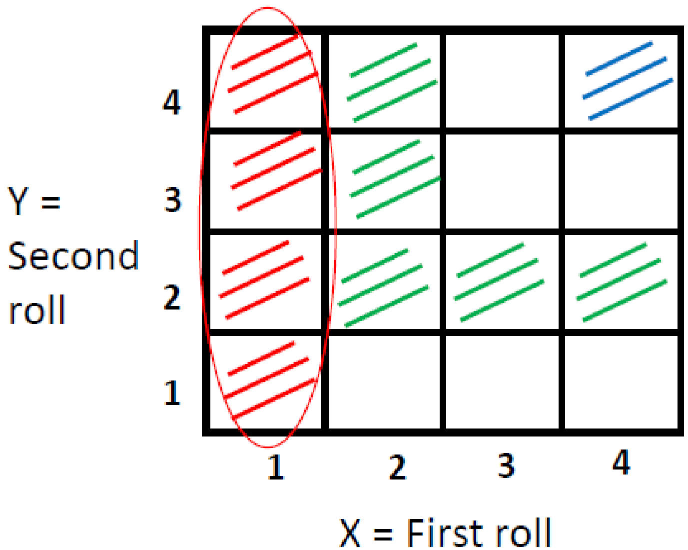
Discrete Uniform Probability Law

If the sample space consists of n possible outcomes which are equally likely (i.e., all single-element events have the same probability), then the probability of any event A is given by

$$\mathbf{P}(A) = \frac{\text{number of elements of } A}{n}.$$

Discrete Uniform Probability Law: Example

- Two rolls of a tetrahedral die



- Let every possible outcome have probability 1/16
- $P(X = 1) = 4 * \frac{1}{16} = \frac{1}{4}$

Let $Z = \min(X, Y)$

$X=2, Y=3, Z=2$

- $P(Z = 4) = \frac{1}{16}$
- $P(Z = 2) = 5 * \frac{1}{16}$

Probability Law: Explicit vs Implicit/Indirect Specification

- Explicit Specification $\Omega = \{H, T\}$ $\Omega = \{1, 2, 3, 4\}$
  - List out the assigned probability of each subset of sample space

ϕ	$\{H\}$	$\{T\}$	$\{H, T\}$
0.	0.5	0.5	1.0

$\{1, 2\}$ $\{1, 3\}$

- Implicit Specification ✓
  - For the experiments with finite sample spaces, Discrete Probability Law shows that in probabilities of single-element events are sufficient to characterize (or specify) the probability law.
  - We can use the probabilities of single-element events and Discrete Probability Law to figure out the probability of any other event (i.e the events with more than one element).

$\{H\}$	$\{T\}$
0.2	0.8

$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
0.25	0.25	0.25	0.25

Utilizing Probability Theory for Dealing with Uncertainty: Two Stages

1. Construct a probabilistic model (PROBABILISTIC MODELING)
   - Specify the sample space
   - Specify the probability law implicitly/indirectly (Refer to the last slide)

2. Utilize the probabilistic model to derive the probabilities of certain events of interest (PROBABILISTIC REASONING)
   - Identify an event of interest
   - Calculate its probability (using various probability theory tools/laws)

Example A: Two Stages of Utilizing Probability Theory

Tim has a four-sided dice. When he rolls the dice, all the possible outcomes (1, 2, 3,4) are equally likely.
What is the probability of outcome being even. ?

1. Probabilistic Model

   1- S.Space = $\{1, 2, 3, 4\}$

   2) Implicit Specification: $P(\{1\}) = \frac{1}{4}$ $P(\{2\}) = \frac{1}{4}$ $P(\{3\}) = \frac{1}{4}$ $P(\{4\}) = \frac{1}{4}$

2. Calculate the Probability of Event of Interest

   2) a) Identify E of Interest

   $E = \{2, 4\}$

   b) Discrete Uniform Prob. Law = $\frac{2}{4}$ =

1 2 3 4

$x + 2x + 3x + 4x = 1 \Rightarrow x = 1$

Example B: Two Stages of Utilizing Probability Theory

Tim has a peculiar four-sided dice. When he rolls the dice, the probability of getting any particular outcome is proportional to result of the dice.

(a) What is the probability of the number being even?

1. Probabilistic Model

1) $\Omega = \{1, 2, 3, 4\}$

2) = Implicit Specification of Prob. Law: $P(\{1\}) = 0.1$ $P(\{2\}) = 0.2$ $P(\{3\}) = 0.3$ $P(\{4\}) = 0.4$

2. Calculate the Probability of Event of Interest ("PROBABILISTIC REASONING")

3) Identify Event of Interest $E = \{2, 4\}$

4) ~~Dis~~ Tool/Law?

Discrete Prob. Law:

$P(\{2,4\}) = P(\{2\}) + P(\{4\}) = 0.2 + 0.4 = 0.6$

Example B: Two Stages of Utilizing Probability Theory

Tim has a peculiar four-sided dice. When he rolls the dice, the probability of getting any particular outcome is proportional to result of the dice.

(a) What is the probability of the number being even?

1. <u>Probabilistic Model</u>

Sample Space	Probability Assignment
No	Probability
1	*p*
2	*2p*
3	*3p*
4	*4p*

2. <u>Calculate the Probability of Event of Interest</u>

Probability Axioms: Some More Consequences

Axioms

a) $P(A) \geq 0$

b) $P(\Omega) = 1$

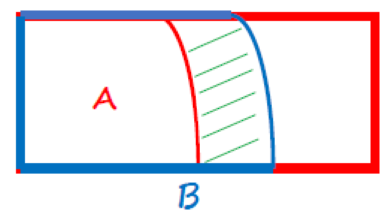
For disjoint events:

c) $P(A \cup B) = P(A) + P(B)$

Consequence

$\boxed{\text{If } A \subset B, \text{ then } P(A) \leq P(B)}$

Proof:



$B = A \cup (B \cap A^c)$

$P(B) = P(A) + \underline{P(B \cap A^c)} \geq P(A)$

Probability Axioms: Some More Consequences

Axioms

a) $P(A) \geq 0$

b) $P(\Omega) = 1$

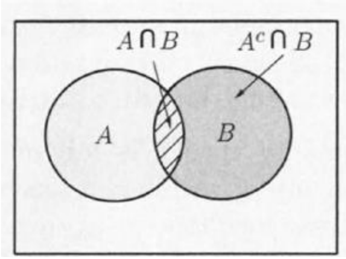
For disjoint events:

c) $P(A \cup B) = P(A) + P(B)$

Consequence

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Proof:



From diagram , we can express the events $A \cup B$ and B as unions of disjoint events:

$$A \cup B = A \cup (A^c \cap B), \qquad B = (A \cap B) \cup (A^c \cap B).$$

Using the additivity axiom, we have

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(A^c \cap B), \qquad \mathbf{P}(B) = \mathbf{P}(A \cap B) + \mathbf{P}(A^c \cap B).$$

Subtracting the second equality from the first and rearranging terms, we obtain $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$

Corollary:

$$\boxed{P(A \cup B) \leq P(A) + P(B)}$$

EXPERIMENT: ASK A RANDOMLY SELECTED STUDENT: Do you like cricket? Do you like soccer?

Example A: Two Stages of Utilizing Probability Theory

Out of the students in a class, 60% love Cricket, 70% love Soccer, and 40% love both Cricket and Soccer. What is the probability that a randomly selected student loves neither Cricket nor Soccer.

1. Probabilistic Model

1) Sample Space: $\{CS, CS', C'S, C'S'\}$

2) Implicit spec. of Prob. Law:

$P(\text{'Loves cricket'}) = P(\underbrace{\{CS, CS'\}}_{A}) = 0.6$

$P(\text{'Loves Soccer'}) = P(\underbrace{\{CS, C'S\}}_{B}) = 0.7$

$P(\text{'Loves Both'}) = P(\{CS\}) = 0.4$

2. Calculate the Probability of Event of Interest

a. Identify Event of Interest: $E = \{C'S'\}$ $\quad P(E) = ?$

b) TOOLS / LAWS?

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

$$P(\{CS, CS', C'S\}) = P(\{CS, CS'\}) + P(\{CS, C'S\}) - P(\{CS\})$$

$$= 0.6 + 0.7 - 0.4 = 0.9$$

$$P(\{C'S'\}) = 1 - P(\{CS, CS', C'S\}) = 1 - 0.9 = 0.1$$

$P(\Omega) = 1, \quad \Omega = A \cup A^c \Rightarrow P(A \cup A^c) = 1$

$P(A) + P(A^c) = 1 \Rightarrow \boxed{P(A^c) = 1 - P(A)}$

$S, S_{even} = \{2, 4, 6, 8, \ldots\ldots\}$

1 2 3 4

Possibilities for Sample Spaces

Finite Infinite

1. Discrete and Finite

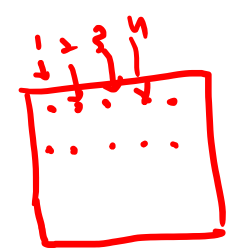
$\Omega = \{H, T\}$

$\Omega = \{1, 2, 3, 4\}$

2. Discrete and Infinite (Countably Infinite)
   - Elements can be arranged in a sequence (i.e. there can be 1-to-1 mapping between elements and integers)
   - Example: Consider repeated coin tosses. After how many coin tosses will you get the first tail.

     Sample Space = {1, 2, 3, 4, ...}

3. Continuous (Uncountably Infinite)
   - Elements cannot be arranged in a sequence
   - Example: Throwing a dart on a unit square.



$\Omega : [0, 1]$

$P(A_1 \cup A_2 \cup A_3 \cup A_4) = P(A_1)+P(A_2)+P(A_3)+P(A_4)$

$P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup \ldots\ldots) = P(A_1)+P(A_2)+P(A_3)+\ldots\ldots$

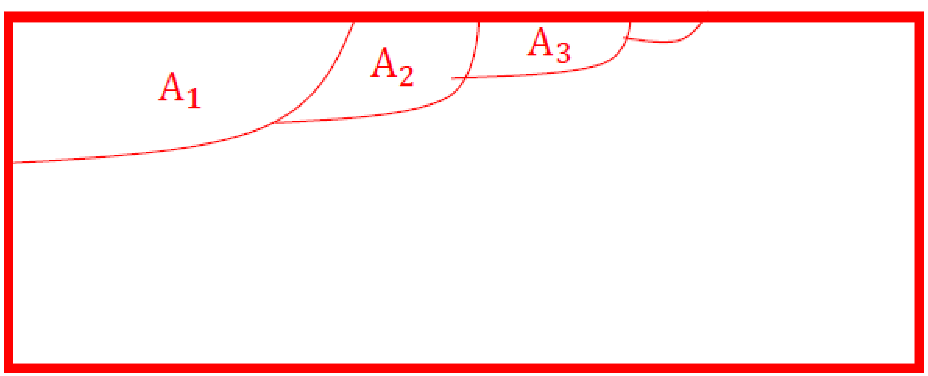
Dealing with Countably Infinite Sample Spaces

- Countable Additivity Axiom
  - Strengthens Axiom 3 (Finite Additivity Axiom)

> – (Finite) additivity:
> If $A \cap B = \emptyset$, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$

Countable Additivity Axiom:

If $A_1, A_2, A_3, \ldots$ is an infinite sequence of disjoint events,
then $P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$.

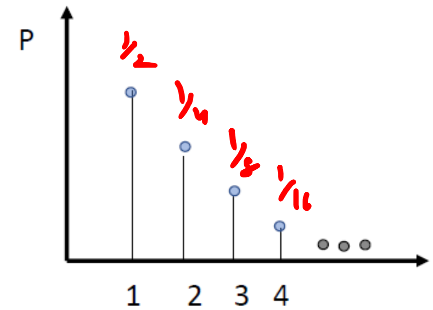


$P(\{1,2\}) = P\{1\} + P\{2\} =$ $P\{1,2,3\} = P\{1\} + P\{2\} + P\{3\}$

$P(A \cup B) = P(A) + P(B)$ $P\{A_1 \cup A_2 \cup A_3\} = P(A_1) + P(A_2) + P(A_3)$

Example: Countably Infinite Sample Spaces

$P\{$



- Sample space: $\{1, 2, \ldots\}$
  - We are given $\mathrm{P}(n) = 2^{-n}$, $n = 1, 2, \ldots$
  - Find P(outcome is even)

$P(\{2, 4, 6, 8, \ldots\})$.

Through Countable Additivity Axiom

- P (outcome is even) = $P(\{2, 4, 6, \ldots\})$

$= P(\{2\} \cup \{4\} \cup \{6\} \cup \ldots) = P(2) + P(4) + P(6) + \ldots$

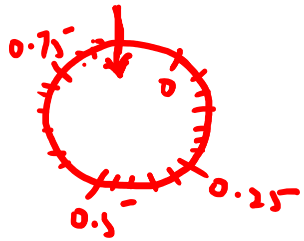
$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots = \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \ldots\right) = \frac{1}{4} * \frac{1}{1 - \frac{1}{4}} = \frac{1}{3}$

* Sum of a Geometric Series

$$\sum_{i=0}^{\infty} \alpha^i = 1 + \alpha + \alpha^2 + \ldots = \frac{1}{1-\alpha} \qquad |\alpha| < 1$$

$P(0.4) = 1/10$ $P(0.4) = 0$

Dealing with Continuous Sample Spaces



S. Space = $\Omega = \{ \quad \}$

$\Omega = [0,1]$

All outcomes are equally likely.

$P(\{0.4\}) = ?$

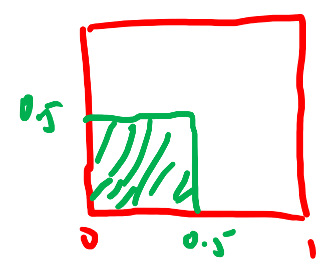
$P(0.4) = 0$ $P(0.5) = 0$

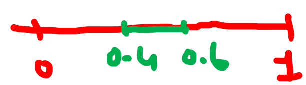
$P([0.4, 0.6]) =$

$P([0.39995, 0.40005])$

$P(A) = \dfrac{\text{"Length" of } A}{\text{"Length" of } \Omega}$

$P(A) = \dfrac{\text{"Area" of Event } A}{\text{"Area" of } \Omega}$





Dealing with Continuous Sample Spaces

- Major difference from discrete sample spaces
  - Probabilities of single-element events are not sufficient to characterize the probability law
  - Probabilities of single element events must be zero

Example 1.4. A wheel of fortune is continuously calibrated from 0 to 1, so the possible outcomes of an experiment consisting of a single spin are the numbers in the interval $\Omega = [0, 1]$. Assuming a fair wheel, it is appropriate to consider all outcomes equally likely, but what is the probability of the event consisting of a single element? It cannot be positive, because then, using the additivity axiom, it would follow that events with a sufficiently large number of elements would have probability larger than 1. Therefore, the probability of any event that consists of a single element must be 0.

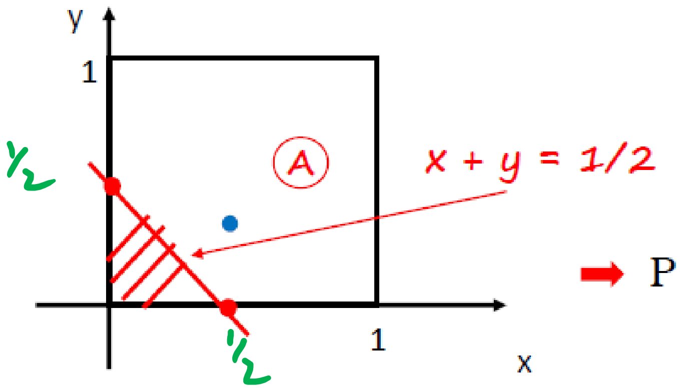
In this example, it makes sense to assign probability $b - a$ to any subinterval $[a, b]$ of $[0, 1]$, and to calculate the probability of a more complicated set by evaluating its "length."† This assignment satisfies the three probability axioms and qualifies as a legitimate probability law.

Example: Continuous Sample Spaces

(x, y) such that $0 \le x , y \le 1$

- ~~Uniform~~ probability law: Probability = Area

➡ $P(\{(x, y) \mid x + y \le ½\}) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8}$



➡ $P(\{(0.5, 0.3)\}) = 0$