Introduction to Probability and Statistics
(EE 354 / CE 361 / Math 310)

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Outline: Unit 2

- Conditional Probability
  - Intuition
  - Definition
  - Examples

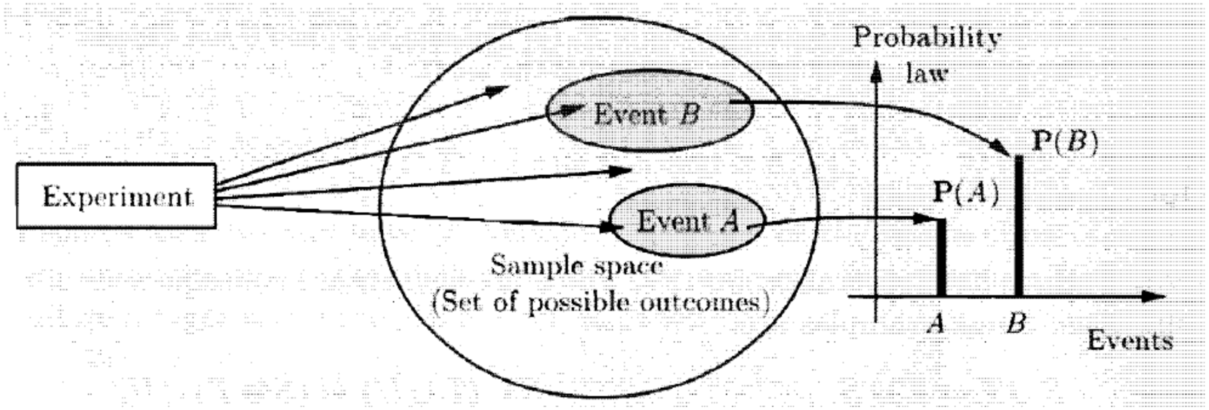
- Utilizing Conditional Probability for Modeling
  - Multiplication Rule
  - Total Probability Theorem
  - Bayes’ Rule

- Independence

Probabilistic Model

Every probabilistic model involves an underlying process, called the ***Experiment,*** that will produce exactly one out of several possible outcomes.

- Sample Space (Ω)
  - The set of all possible outcomes of an experiment
- Probability Law
  - Assigns to a set A of possible outcomes a non-negative number P(A) that encodes our belief about the collective "likelihood" of the elements of A.
  - This probability law must satisfy certain properties, known as "Probability Axioms."



1. $P(A) \geq 0$
2. $P(A \cup B) = P(A) + P(B)$
3. $P(\Omega) = 1$

Conditional Probability: Intuition

- Probability Law:
  - Encodes our "belief" about the likelihood of an event in a random experiment.
- Conditional Probability Law:
  - Encodes our "updated belief" about the likelihood of an event **IF** we have "additional partial information" about the results of the random experiment.

FAIR 4-SIDED DIE:

$P\{1\} = 0.25$ $P\{2\} = 0.25$ $P\{3\} = 0.25$ $P\{4\} = 0.25$

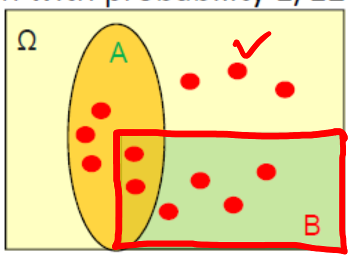
Additional Partial Info: Outcome is Even : Outcome belongs to event $B = \{2,4\}$

$P\{1|\text{Even}\} = 0$ $P\{2|\text{Even}\} = 0.5$ $P\{3|\text{Even}\} = 0$ $P\{4|\text{Even}\} = 0.5$

$P\{1|B\} = 0$ $P\{2|B\} = 0.5$ $P\{3|B\} = 0$ $P\{4|B\} = 0.5$

Conditional Probability: Moving towards Definition

Assume 12 equally likely outcomes, each with probability 1/12



$\mathbf{P}(A) = \frac{5}{12}$ $\mathbf{P}(B) = \frac{6}{12}$

$$P(A) = \frac{\#\text{ of elements in } A}{\#\text{ of elements in } \Omega} = 5/12$$

$$P(B) = \frac{6}{12} \qquad P(A \cap B) = 2/12$$

$P(A|B) = ?$

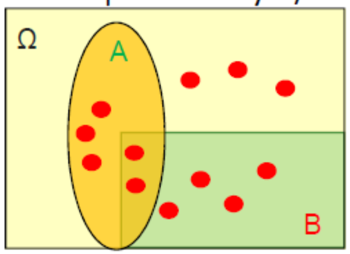
*"B is our new "UNIVERSE."

$P(A|B) = 2/6$

"Prob of Event A given the additional, partial info that outcome will belong to Event B."

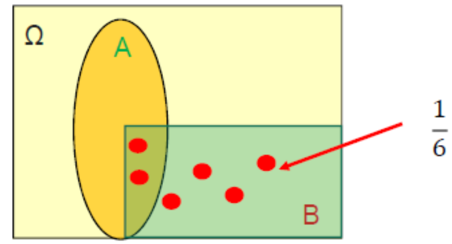
Conditional Probability: Moving towards Definition

Assume 12 equally likely outcomes,
each with probability 1/12



$\mathbf{P}(A) = \frac{5}{12}$ $\mathbf{P}(B) = \frac{6}{12}$

If we are told that B occurred

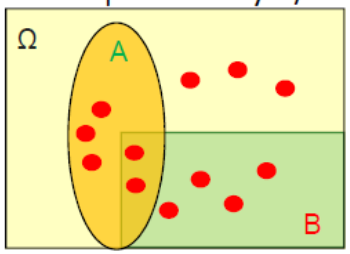


$\mathbf{P}(A|B) = \frac{2}{6} = \frac{1}{3}$

- $\mathbf{P}(A \mid B) =$ probability of A,
  given that B ~~occurred~~ will occur
  - B is our new universe

Conditional Probability: Moving towards Definition

Assume 12 equally likely outcomes,
each with probability 1/12



$\mathbf{P}(A) = \frac{5}{12}$ $\mathbf{P}(B) = \frac{6}{12}$

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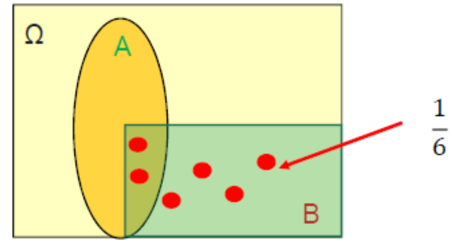


$\mathbf{P}(A|B) = \frac{2}{6} = \frac{1}{3}$

$P(A \cap B) = 2/12$

$P(B) = 6/12$

- $\mathrm{P}(A \mid B) =$ probability of A, given that B occurred
  - B is our new universe

- **Definition:** Assuming $\mathrm{P}(B) \neq 0$,

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

$\mathrm{P}(A \mid B)$ undefined if $\mathrm{P}(B) = 0$

$$= \frac{2/12}{6/12} = \frac{2}{6} = 1/3$$

Conditional Probability: Example

Example 1.6. We toss a fair coin three successive times. We wish to find the conditional probability $\mathbf{P}(A \mid B)$ when A and B are the events

$$A = \{\text{more heads than tails come up}\}, \qquad B = \{\text{1st toss is a head}\}.$$

Conditional Probability: Example

Example 1.6. We toss a fair coin three successive times. We wish to find the conditional probability $\mathbf{P}(A \mid B)$ when A and B are the events

$$A = \{\text{more heads than tails come up}\}, \qquad B = \{\text{1st toss is a head}\}.$$

$= \{HHH, HHT, HTH, THH\}$ $\qquad$ $B = \{HHH, HHT, HTH, HTT\}$

The sample space consists of eight sequences,

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

$P(A) = \frac{4}{8} = \frac{1}{2}$ $\qquad$ $P(B) = \frac{4}{8} = \frac{1}{2}$ $\qquad$ $P(A \cap B) = 3/8$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/8}{4/8} = 3/4$$

Conditional Probability: Example

Example 1.6. We toss a fair coin three successive times. We wish to find the conditional probability $\mathbf{P}(A \mid B)$ when A and B are the events

$$A = \{\text{more heads than tails come up}\}, \qquad B = \{\text{1st toss is a head}\}.$$

The sample space consists of eight sequences,

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

which we assume to be equally likely. The event B consists of the four elements HHH, HHT, HTH, HTT, so its probability is

$$\mathbf{P}(B) = \frac{4}{8}.$$

The event $A \cap B$ consists of the three elements HHH, HHT, HTH, so its probability is

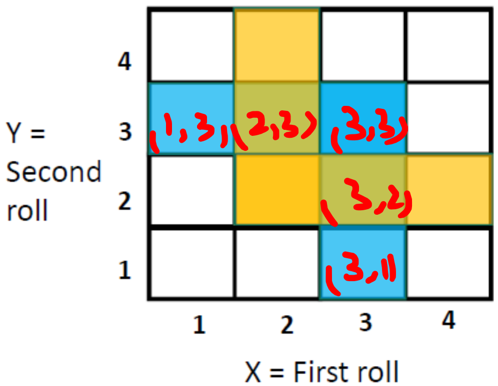
$$\mathbf{P}(A \cap B) = \frac{3}{8}.$$

Thus, the conditional probability $\mathbf{P}(A \mid B)$ is

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{3/8}{4/8} = \frac{3}{4}.$$

Conditional Probability: Example

- Two rolls of a 4-sided die



- Let every possible outcome have probability 1/16
- Let B be the event: $\min(X, Y) = 2$

  Let $M = \max(X, Y)$

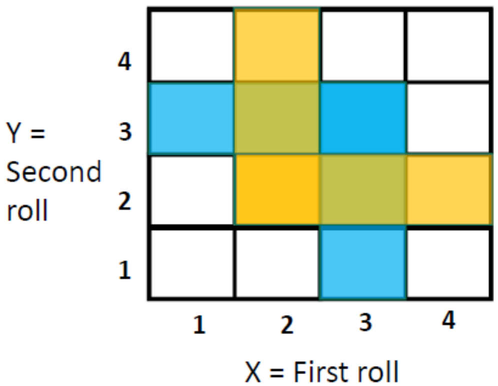
A = M=3 = "Max of two rolls is 3"

$P(A \cap B) = \frac{2}{16}$

$$P(\overbrace{M = 3}^{A} \mid B) = \quad ?$$

Conditional Probability: Example

- Two rolls of a 4-sided die



- Let every possible outcome have probability 1/16
- Let B be the event: $\min(\mathrm{X}, \mathrm{Y}) = 2$

  Let $\mathrm{M} = \max(\mathrm{X}, \mathrm{Y})$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\mathbf{P}\,(M = 3 \mid B) = \frac{P(\,M = 3 \text{ and } B\,)}{P(\,B\,)}$$

$$= \frac{2/16}{5/16} = \frac{2}{5}$$

Conditional Probabilities Satisfy Probability Axioms

- **Definition:** Assuming $\mathbf{P}(B) \neq 0$,

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}$$

$\mathbf{P}(A \mid B)$ undefined if $\mathbf{P}(B) = 0$

1. Non-negativity Axiom $P(A) \geq 0$
   - From definition of conditional probability

$$\boxed{\mathbf{P}(A \mid B) \geq 0}$$

2. Normalization Axiom $P(\Omega) = 1$

$$\mathbf{P}(\Omega \mid B) = \frac{\mathbf{P}(\Omega \cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(B)}{\mathbf{P}(B)} = 1,$$

3. Additivity Axiom $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

for any two disjoint events A_1 and A_2,

$$\begin{aligned}
\mathbf{P}(A_1 \cup A_2 \mid B) &= \frac{\mathbf{P}((A_1 \cup A_2) \cap B)}{\mathbf{P}(B)} \\
&= \frac{\mathbf{P}((A_1 \cap B) \cup (A_2 \cap B))}{\mathbf{P}(B)} \\
&= \frac{\mathbf{P}(A_1 \cap B) + \mathbf{P}(A_2 \cap B)}{\mathbf{P}(B)} \\
&= \frac{\mathbf{P}(A_1 \cap B)}{\mathbf{P}(B)} + \frac{\mathbf{P}(A_2 \cap B)}{\mathbf{P}(B)}
\end{aligned}$$

$$\boxed{P(A_1 \cup A_2 \mid B) = \mathbf{P}(A_1 \mid B) + \mathbf{P}(A_2 \mid B),}$$

Conditional Probability: Two Scenarios

1. Unconditional Probability → Conditional Probability
   - All the examples seen in previous slides fit this scenario

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

2. Conditional Probability → Unconditional Probability
   - While developing a probabilistic model for uncertain scenarios with sequential character, it is often natural and convenient to first specify conditional probabilities and then use them to determine unconditional probabilities.
   - This scenario can also be thought of as “Using conditional probability of modeling”

$$P(A \cap B) = P(A|B)\,P(B)$$

Utilizing Probability Theory for Modeling Uncertainty: Two Stages (Reference to Unit 1)

1. Construct a probabilistic model
   - Specify the sample space
   - Specify the probability law implicitly/indirectly (Refer to the last slide)

Conditional Probability

2. Utilize the probabilistic model to derive the probabilities of certain events of interest
   - Identify an event of interest
   - Calculate its probability (using various probability theory tools/laws)

$\Omega = \{PD, PD', P'D, P'D'\}$ "SEQUENTIAL EXPERIMENT"

Example: Using Conditional Probability for Modeling

Example 1.9. Radar Detection. If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present. the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05.

- **Q1**: *What is the probability that aircraft is present and is detected by radar.*
- Let's first think about a tree-diagram based description of the Sample Space

$A = \{\text{Aircraft is Present}\} = \{PD, PD'\}$

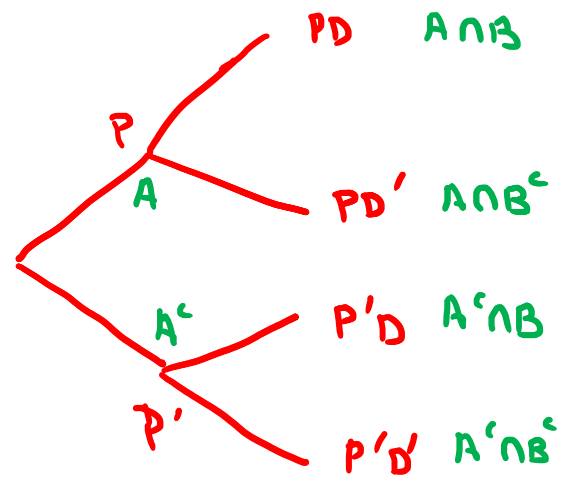
$B = \{\text{Radar generates an alarm}\} = \{PD, P'D\}$

$A \cap B = \{PD\}$

$A \cap B^c = \{PD'\}$

$A^c \cap B = \{P'D\}$

$A^c \cap B^c = \{P'D'\}$



~~$P(A \cap B) = 0.99$~~

$P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow P(B|A) = \frac{P(A \cap B)}{P(A)}$

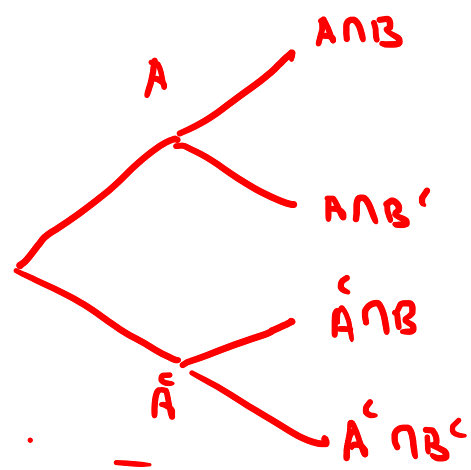
Example: Using Conditional Probability for Modeling

Example 1.9. Radar Detection. If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present. the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05.

- **Q1**: *What is the probability that aircraft is present and is detected by radar.*

$A = \{\text{Aircraft is Present}\}$

$B = \{\text{Radar generates an alarm}\} = \{\text{Aircraft is Detected}\}$



$P(B/A) = 0.99$

$P(B|A^c) = 0.10$ $\quad P(A \cap B) = ?$

$P(A) = 0.05$

$P(B|A) = \frac{P(A \cap B)}{P(A)} \rightarrow P(A \cap B) = P(A).P(B|A)$

$= 0.05 \times 0.99$

Example: Using Conditional Probability for Modeling

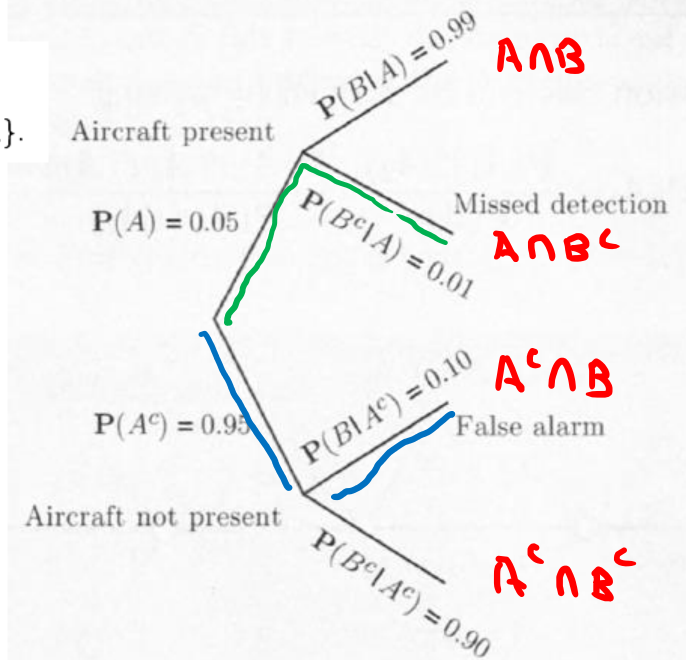
Example 1.9. Radar Detection. If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present. the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05. 1) What is the probability of no aircraft presence and a false alarm? 2) What is the probability of aircraft presence and no detection?

Q:-
1) $P(A^c \cap B)$
2) $P(A \cap B^c)$

Let A and B be the events

$A = \{\text{an aircraft is present}\}$, $A^c = \{\text{an aircraft is not present}\}$.

$B = \{\text{the radar generates an alarm}\}$, $B^c = \{\text{the radar does not generate an alarm}\}$.



$P(B|A) = 0.99 \Rightarrow P(B^c|A) = 0.01$

$P(B|A^c) = 0.10$

$P(A) = 0.05$

$P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(B|A)\,P(A)$

$P(B^c|A) = \frac{P(A \cap B^c)}{P(A)} \Rightarrow P(A \cap B^c) = P(B^c/A)\,P(A)$

$P(B|A^c) = P(A^c \cap B) / P(A^c) \Rightarrow P(A^c \cap B) = P(B|A^c)\,P(A^c)$

$P(B^c/A^c) = P(A^c \cap B^c) / P(A^c) \Rightarrow P(A^c \cap B^c) = P(B^c/A^c)\,P(A^c)$

Example: Using Conditional Probability for Modeling

Example 1.9. Radar Detection. If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present. the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of no aircraft presence and a false alarm? What is the probability of aircraft presence and no detection?

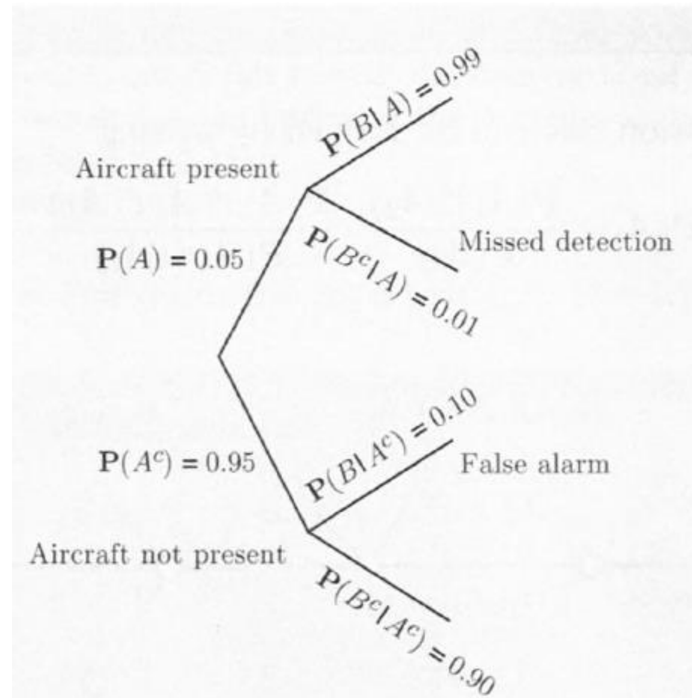
Let A and B be the events

$$A = \{\text{an aircraft is present}\},$$
$$B = \{\text{the radar generates an alarm}\},$$

and consider also their complements

$$A^c = \{\text{an aircraft is not present}\}.$$
$$B^c = \{\text{the radar does not generate an alarm}\}.$$

- In this scenario, it is more natural to provide conditional probabilities that are then recorded along the corresponding branches of the tree diagram. The desired (ordinary/unconditional) probabilities can then be computed:



$$\mathbf{P}(\text{not present, false alarm}) = \mathbf{P}(A^c \cap B) = \mathbf{P}(A^c)\mathbf{P}(B \mid A^c) = 0.95 \cdot 0.10 = 0.095,$$
$$\mathbf{P}(\text{present, no detection}) = \mathbf{P}(A \cap B^c) = \mathbf{P}(A)\mathbf{P}(B^c \mid A) = 0.05 \cdot 0.01 = 0.0005.$$

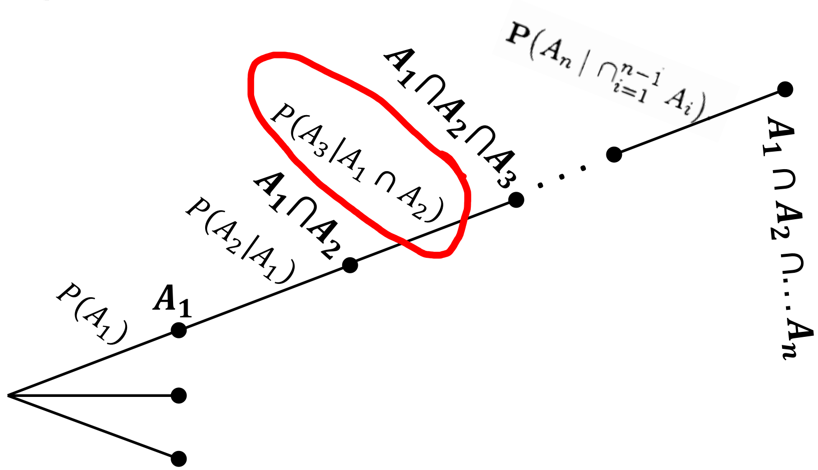
Sequential Experiments: General Structure of Moving from Conditional Probability to Unconditional Probability

(a) We set up the tree so that an event of interest is associated with a leaf. We view the occurrence of the event as a sequence of steps, namely, the traversals of the branches along the path from the root to the leaf.

(b) We record the conditional probabilities associated with the branches of the tree.

(c) We obtain the probability of a leaf by multiplying the probabilities recorded along the corresponding path of the tree.

In mathematical terms, we are dealing with an event A which occurs if and only if each one of several events $A_1, \ldots, A_n$ has occurred, i.e., $A = A_1 \cap A_2 \cap \cdots \cap A_n$.



Multiplication Rule

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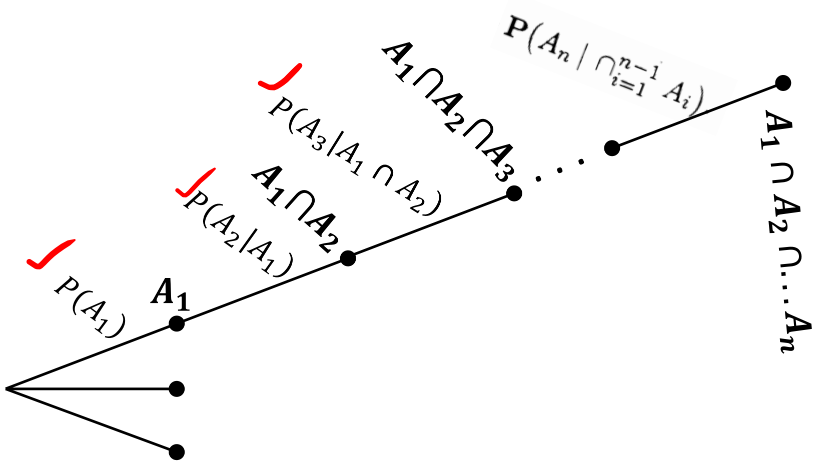
The occurrence of A is viewed as an occurrence of A_1, followed by the occurrence of A_2, then of A_3, etc., and it is visualized as a path with n branches, corresponding to the events $A_1, \ldots, A_n$. The probability of A is given by the following rule

Multiplication Rule

Assuming that all of the conditioning events have positive probability, we have

$$\mathbf{P}\big(\cap_{i=1}^{n} A_i\big) = \mathbf{P}(A_1)\mathbf{P}(A_2 \mid A_1)\mathbf{P}(A_3 \mid A_1 \cap A_2) \cdots \mathbf{P}\big(A_n \mid \cap_{i=1}^{n-1} A_i\big).$$



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Multiplication Rule

Assuming that all of the conditioning events have positive probability, we have

$$\mathbf{P}\left(\cap_{i=1}^{n} A_i\right) = \mathbf{P}(A_1)\mathbf{P}(A_2 \mid A_1)\mathbf{P}(A_3 \mid A_1 \cap A_2) \cdots \mathbf{P}\left(A_n \mid \cap_{i=1}^{n-1} A_i\right).$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid A_1 \cap A_2)$$

$$= \cancel{P(A_1)} \cdot \frac{\cancel{P(A_1 \cap A_2)}}{\cancel{P(A_1)}} \cdot \frac{P(A_1 \cap A_2 \cap A_3)}{P(\cancel{A_1 \cap A_2})}$$

52 card → "13" are "hearts"

Multiplication Rule: Example

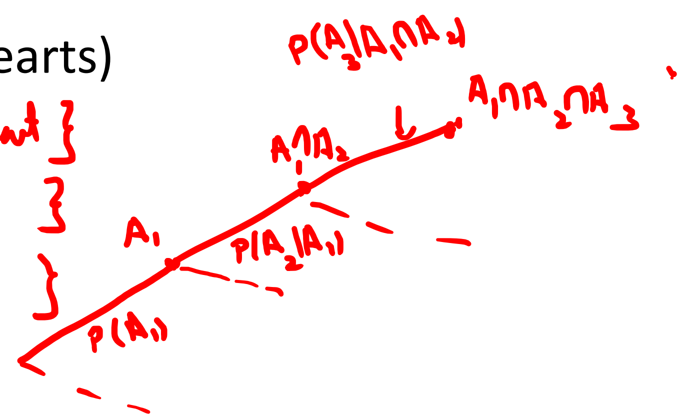
Example 1.10. Three cards are drawn from an ordinary 52-card deck without replacement (drawn cards are not placed back in the deck). We wish to find the probability that "none of the three cards is a heart." We assume that at each step, each one of the remaining cards is equally likely to be picked.

(Note: Of the 52 cards, 39 are not hearts)

$A_1 = \{$ 1st drawn card is NOT a heart $\}$

$A_2 = \{$ 2nd " " NOT " " $\}$

$A_3 = \{$ 3RD " " NOT " " $\}$



MULTIPLICATION RULE: –

$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2|A_1)\, P(A_3|A_1 \cap A_2)$

=

$P(A_1) = \frac{39}{52}$ $P(A_2|A_1) = \frac{38}{51}$ $P(A_3|A_1 \cap A_2) = 37/50$

Multiplication Rule: Example

Example 1.10. Three cards are drawn from an ordinary 52-card deck without replacement (drawn cards are not placed back in the deck). We wish to find the probability that none of the three cards is a heart. We assume that at each step, each one of the remaining cards is equally likely to be picked.

(Note: Of the 52 cards, 39 are not hearts)

Define the events

$$A_i = \{\text{the } i\text{th card is not a heart}\}, \qquad i = 1, 2, 3.$$

We will calculate $\mathbf{P}(A_1 \cap A_2 \cap A_3)$, the probability that none of the three cards is a heart, using the multiplication rule

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1)\mathbf{P}(A_2 \mid A_1)\mathbf{P}(A_3 \mid A_1 \cap A_2).$$

We have

$$\mathbf{P}(A_1) = \frac{39}{52},$$

since there are 39 cards that are not hearts in the 52-card deck. Given that the first card is not a heart, we are left with 51 cards, 38 of which are not hearts, and

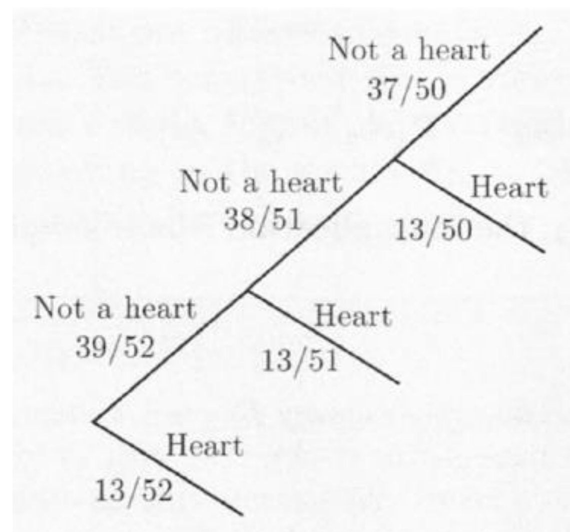
$$\mathbf{P}(A_2 \mid A_1) = \frac{38}{51}.$$

Finally, given that the first two cards drawn are not hearts, there are 37 cards which are not hearts in the remaining 50-card deck, and

$$\mathbf{P}(A_3 \mid A_1 \cap A_2) = \frac{37}{50}.$$

These probabilities are recorded along the corresponding branches of the tree describing the sample space, as shown in Fig. 1.11. The desired probability is now obtained by multiplying the probabilities recorded along the corresponding path of the tree:

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50}.$$

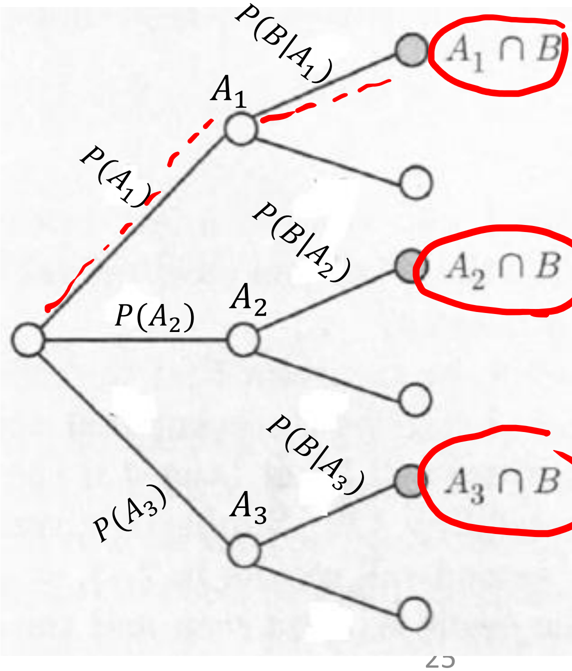


Total Probability Theorem

$P(B) = ?$

$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$

$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$



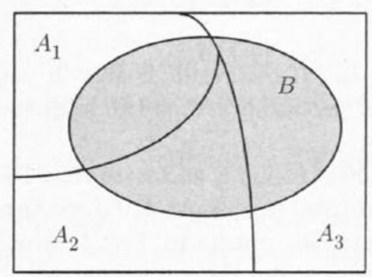
Total Probability Theorem

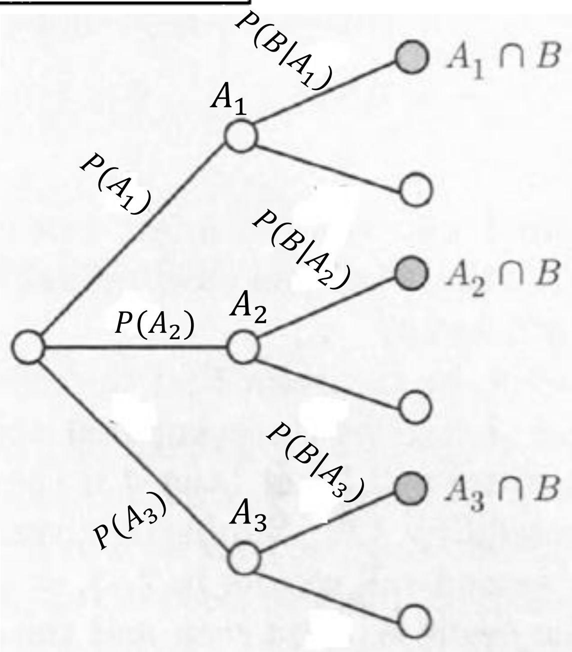
Total Probability Theorem

Let $A_1, \ldots, A_n$ be disjoint events that form a partition of the sample space (each possible outcome is included in exactly one of the events $A_1, \ldots, A_n$) and assume that $\mathbf{P}(A_i) > 0$, for all i. Then, for any event B, we have

$$\boxed{\mathbf{P}(B) = \mathbf{P}(A_1 \cap B) + \cdots + \mathbf{P}(A_n \cap B)}$$
$$\boxed{P(B) = \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n).}$$

- Intuition
  - We are partitioning the sample space into a number of scenarios (events) A_i
  - The probability that B occurs is a weighted average of its conditional probability under each scenario, where each scenario is weighted according to its (unconditional probability).
  - The key is to choose appropriately the partition/scenarios, suggested by the problem structure.





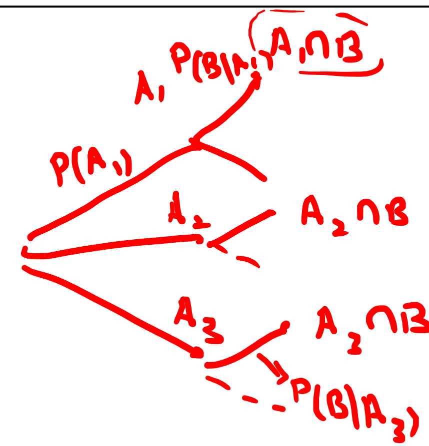
Total Probability Theorem: Example

Example 1.13. You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against "a randomly chosen opponent." What is the probability of winning?

$A_1 = \{$You play a player of Type 1$\}$

$A_2 = \{$ " " " Type 2 $\}$

$A_3 = \{$ " " " Type 3 $\}$



$B = \{$You win$\}$ $\quad P(B) = ?$

By Total Prob Theorem,

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$

$$= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$$

$P(A_1) = 0.5 \quad P(A_2) = 0.25 \quad P(A_3) = 0.25$

$P(B|A_1) = 0.3 \quad P(B|A_2) = 0.4 \quad P(B|A_3) = 0.5$

Total Probability Theorem: Example

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Let A_i be the event of playing with an opponent of type i. We have

$$\mathbf{P}(A_1) = 0.5, \qquad \mathbf{P}(A_2) = 0.25, \qquad \mathbf{P}(A_3) = 0.25.$$

Also, let B be the event of winning. We have

$$\mathbf{P}(B \mid A_1) = 0.3, \qquad \mathbf{P}(B \mid A_2) = 0.4, \qquad \mathbf{P}(B \mid A_3) = 0.5.$$

Thus, by the total probability theorem, the probability of winning is

$$\begin{aligned}\mathbf{P}(B) &= \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \mathbf{P}(A_2)\mathbf{P}(B \mid A_2) + \mathbf{P}(A_3)\mathbf{P}(B \mid A_3) \\ &= 0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5 \\ &= 0.375.\end{aligned}$$

$P(A_1|B) = \frac{P(A_1)\,P(B|A_1)}{P(B)}$

Bayes' Rule

Bayes' Rule

Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space, and assume that $\mathbf{P}(A_i) > 0$, for all i. Then, for any event B such that $\mathbf{P}(B) > 0$, we have

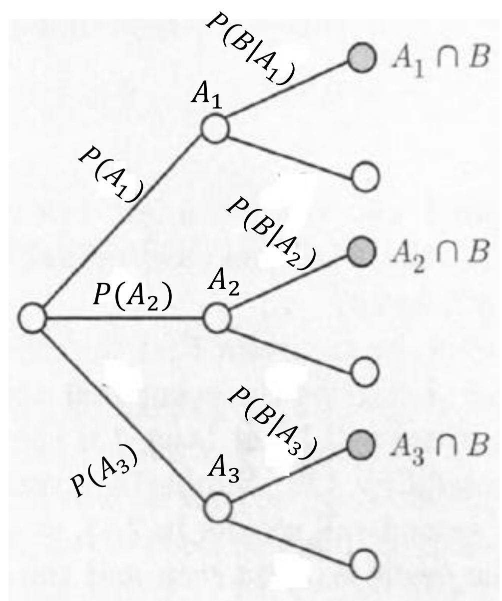
$$\boxed{\mathbf{P}(A_i \,|\, B) = \frac{\mathbf{P}(A_i)\mathbf{P}(B \,|\, A_i)}{\mathbf{P}(B)}}$$
$$= \frac{\mathbf{P}(A_i)\mathbf{P}(B \,|\, A_i)}{\mathbf{P}(A_1)\mathbf{P}(B \,|\, A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \,|\, A_n)}.$$

- Verification

$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)}$

$P(B|A_1) = \frac{P(A_1 \cap B)}{P(A_1)} \Rightarrow P(A_1 \cap B) = P(A_1)P(B|A_1)$

$$\boxed{P(A_1|B) = \frac{P(A_1)\,P(B|A_1)}{P(B)}}$$



Bayes' Rule

A_1, A_2, A_3 (CAUSES)

B (EFFECT)

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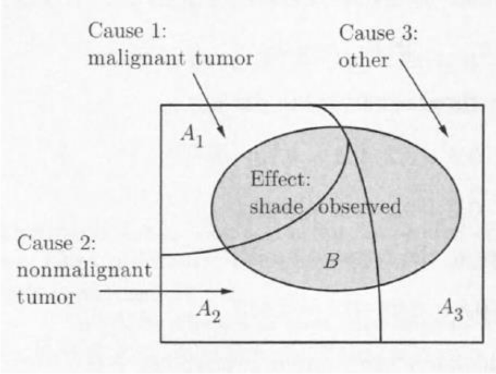
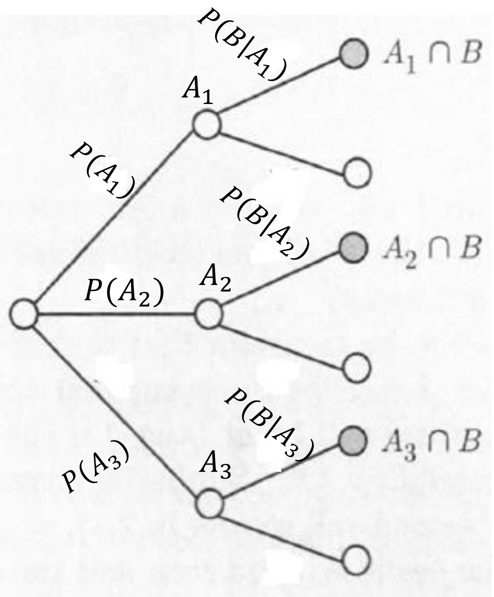
- "Inference" Context
  - Consider a set of mutually-exclusive and collectively-exhaustive potential "causes"{A_1, A_2, A_3} that result in a certain "effect" B.
  - We observe the effect B and wish to infer the cause (out of A_1, A_2, A_3) that made B happen.
  - This is possible by evaluating following probabilities: $\mathbf{P}(A_i \mid B)$

    i.e conditional probabilities of A_1, A_2, A_3, given that B has occurred.





Bayes' Rule

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↓

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Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space, and assume that $\mathbf{P}(A_i) > 0$, for all i. Then, for any event B such that $\mathbf{P}(B) > 0$, we have

$$\begin{aligned}\mathbf{P}(A_i \mid B) &= \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\mathbf{P}(B)} \\ &= \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n)}.\end{aligned}$$

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  - Consider a set of mutually-exclusive and collectively-exhaustive potential "causes"$\{A_1, A_2, A_3\}$ that result in a certain "effect" B.
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  - This is possible by evaluating following probabilities: $\mathbf{P}(A_i \mid B)$

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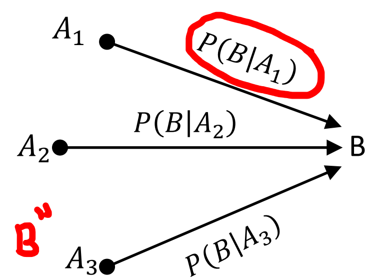
"If the cause A_1 is present,

"Cause-Effect Modeling"

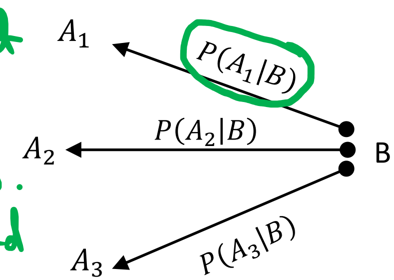
what is the prob. that we will see effect B"



If we see the effect B,

"Inference"

what is the prob. that it was caused by A_1"



Bayes' Rule: Example

Example 1.13. You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1). 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3).

Let A_i be the event of playing with an opponent of type i. We have

$$\mathbf{P}(A_1) = 0.5, \qquad \mathbf{P}(A_2) = 0.25, \qquad \mathbf{P}(A_3) = 0.25.$$

Also, let B be the event of winning. We have

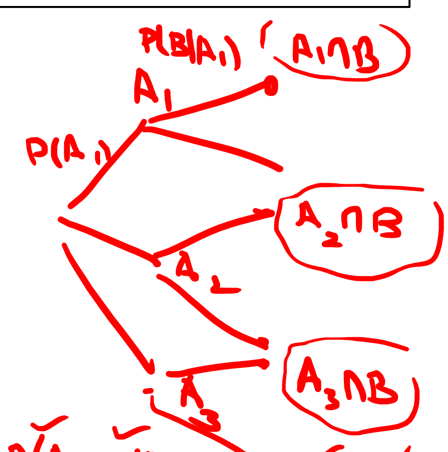
$$\mathbf{P}(B \mid A_1) = 0.3, \qquad \mathbf{P}(B \mid A_2) = 0.4, \qquad \mathbf{P}(B \mid A_3) = 0.5.$$

Suppose that you win. What is the probability $\boxed{P(A_1|B)}$ that you had an opponent of type 1?

BAYES' RULE:

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(B)}$$

$=$



$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$$

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Bayes' Rule: Example

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Also, let B be the event of winning. We have

$$\mathbf{P}(B \mid A_1) = 0.3, \qquad \mathbf{P}(B \mid A_2) = 0.4, \qquad \mathbf{P}(B \mid A_3) = 0.5.$$

Suppose that you win. What is the probability $\mathbf{P}(A_1 \mid B)$ that you had an opponent of type 1?

Using Bayes' rule, we have

$$\begin{aligned}\mathbf{P}(A_1 \mid B) &= \frac{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1)}{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \mathbf{P}(A_2)\mathbf{P}(B \mid A_2) + \mathbf{P}(A_3)\mathbf{P}(B \mid A_3)} \\ &= \frac{0.5 \cdot 0.3}{0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5} \\ &= 0.4.\end{aligned}$$

Bayes' Rule: Example

The incidence rate of a certain disease is 15/100000. There is a test for the disease which is 95% accurate(i.e. If a person has the disease, the test comes back positive with probability 0.95. If a person does not have the disease, it comes back negative with probability 0.95). Given that a person tested positive, what is the probability they have the disease.

$A_1 = \{\text{Person has the disease}\}$

$B = \{\text{Test is Positive}\}$

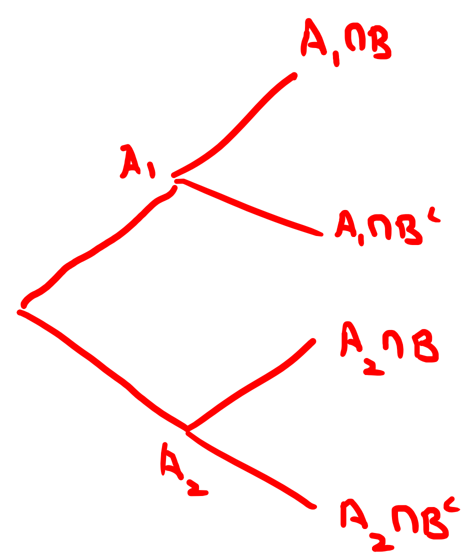
$A_2 = \{\text{Person DOES NOT have the disease}\}$

$P(A_1|B) = ?$

$P(A_1) = \frac{15}{100000} = 0.00015 \Rightarrow P(A_2) = 0.99985$

$P(B|A_1) = 0.95$

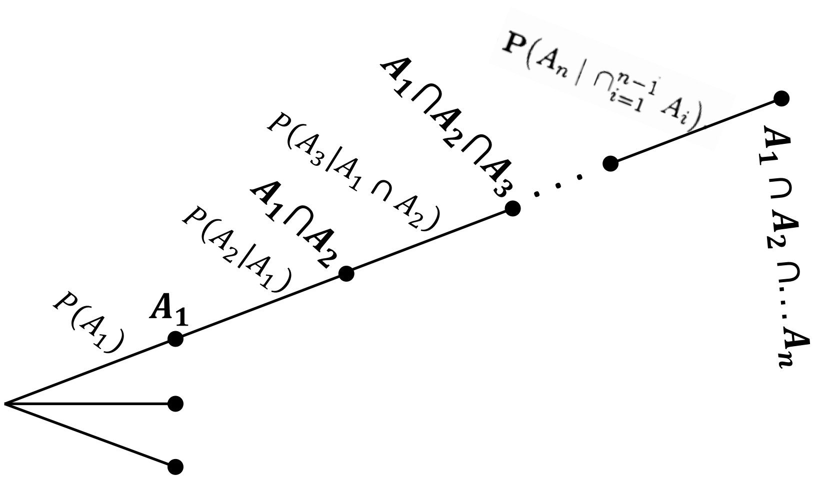
$P(B^c|A_2) = 0.95 \Rightarrow P(B|A_2) = 0.05$

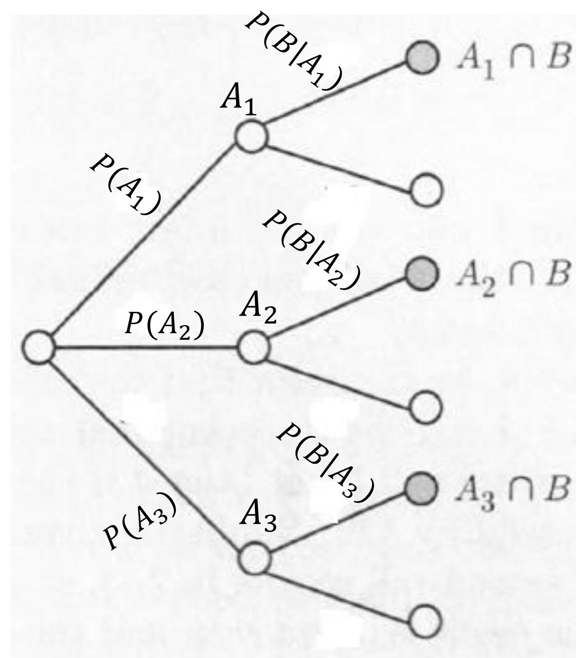


$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(B)}$$

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2)$$

Integrated View: Multiplication Rule, Total Probability Theorem, and Bayes' Rule





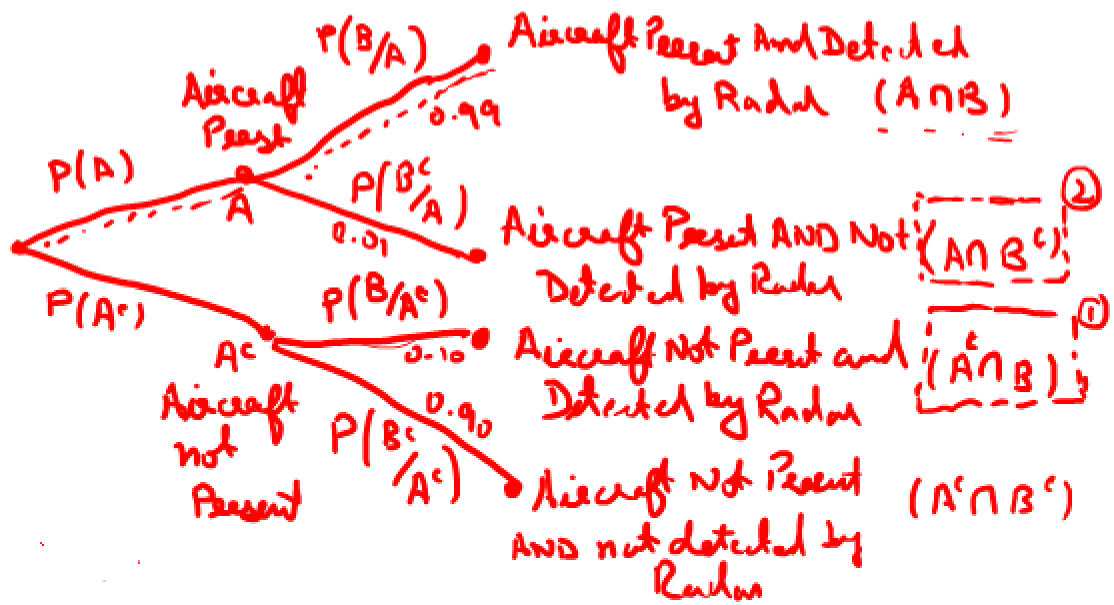
- Multiplication Rule : $P(A_1 \cap B)$
- Total Probability Theorem: P(B)
- Bayes' Rule: $P(A_1 \backslash B)$

Think about applying all three rules to this tree diagram (RHS)

Example of Integrated View: Multiplication Rule, Total Probability Theorem, and Bayes' Rule

Example 1.9. Radar Detection. If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present. the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05. Q ______ ?

Multiplication Rule:-



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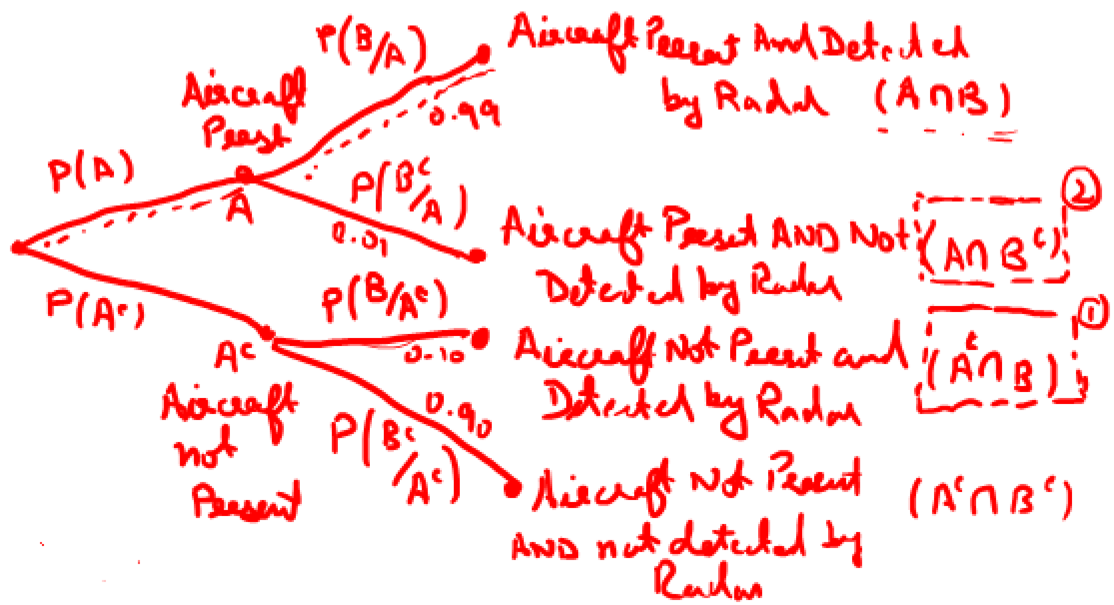
Total Prob Theorem:-



Example of Integrated View: Multiplication Rule, Total Probability Theorem, and Bayes' Rule

Example 1.9. Radar Detection. If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present. the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05. 1) Given a radar detection/alarm, what is the probability that an aircraft is present?

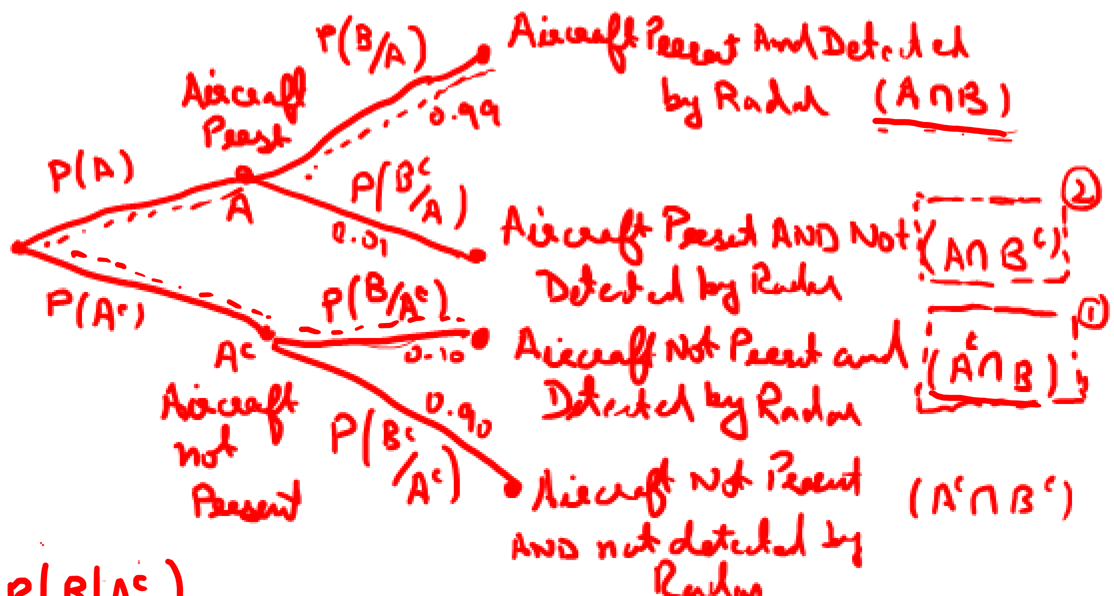
Bayes', ~~Problem~~ Rule 2) Given no radar detection, what is the prob that an aircraft is present

1) $P(A|B) = \frac{P(A)P(B|A)}{P(B)}$

2) $P(A|B^c) = \frac{P(A)\,P(B^c|A)}{P(B^c)}$

$$P(B) = P(A\cap B) + P(A^c \cap B)$$

$$= P(A)\,P(B|A) + P(A^c)\,P(B|A^c)$$



Independence

- Definition:

> If
> $$\mathbf{P}(A \mid B) = \mathbf{P}(A)$$
> We say that A is independent of B

- Intuition
  - Remember that conditional probability encodes our “updated belief” about the likelihood of an event **IF** we have “additional partial information” about the results of the random experiment.
  - If A is independent of B, occurrence of B provides no new information about occurrence of A.
  - If A is independent of B, our beliefs are not “updated” about the likelihood of A if we have the additional information that B will ~~has~~ occurr~~ed~~.

$P(A|B) = P(A)$

Independence: Example

Example 1.19. Consider an experiment involving two successive rolls of a 4-sided die in which all 16 possible outcomes are equally likely and have probability 1/16.

Are the following events independent?

A = {1st roll results in 1} B = {2nd roll results in 3}

$\Omega = \{(1,1), (1,2), (1,3), (1,4)$
$(2,1), (2,2), (2,3), (2,4)$
$(3,1), (3,2), (3,3), (3,4)$
$(4,1), (4,2), (4,3), (4,4)\}$

$P(A) = 4/16 = 1/4$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/16}{4/16} = 1/4$

$P(B) = 4/16$

$P(A \cap B) = 1/16$

Independence

- An Equivalent Definition

If
$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$$
We say that A is independent of B

- Verification

$$\boxed{P(A|B) = P(A)}$$

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

$$\boxed{P(A \cap B) = P(A)P(B)}$$

$P(A|B) = P(A)$ $P(B|A) = P(B)$

Independence

- An Equivalent Definition

> If
> $$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$$
> We say that A is independent of B

- We adopt this as the more widely used definition of independence, because it can be used even when P(B) = 0 in which case P(A|B) is undefined.

- The symmetry of this definition also implies that independence is a symmetric property
  - If A is independent of B, then B is independent of A
  - We say that A and B are independent events.
  - Also implies that P(B|A) = P(B)

Independence: Example

Example 1.19. Consider an experiment involving two successive rolls of a 4-sided die in which all 16 possible outcomes are equally likely and have probability 1/16.

Are the following events independent?

A = {1st roll results in 1}   B = {2nd roll results in 3}

$P(A \cap B) \stackrel{?}{=} P(A)P(B)$

{

$P(A) = 4/16 = 1/4$ $P(B) = 4/16 = 1/4$

$P(A \cap B) = 1/16$

$P(A|B) = P(A)$

$P(A \cap B) = P(A) . P(B)$

Independence: Example

Example 1.19. Consider an experiment involving two successive rolls of a 4-sided die in which all 16 possible outcomes are equally likely and have probability 1/16.

Are the following events independent?

A = {Maximum of two rolls is 2}   B = {Minimum of two rolls is 2}

$P(A \cap B) \overset{?}{=} P(A)P(B)$

$P(A) = 3/16$

$P(B) = 5/16$

$P(A \cap B) = 1/16$

{(1,1), (1,2), (1,3), (1,4)
(2,1), (2,2), (2,3), (2,4)
(3,1), (3,2), (3,3), (3,4)
(4,1), (4,2), (4,3), (4,4)}

$P(A \cap B) \neq P(A)P(B)$

$\frac{1}{16} \neq (3/16)(5/16)$

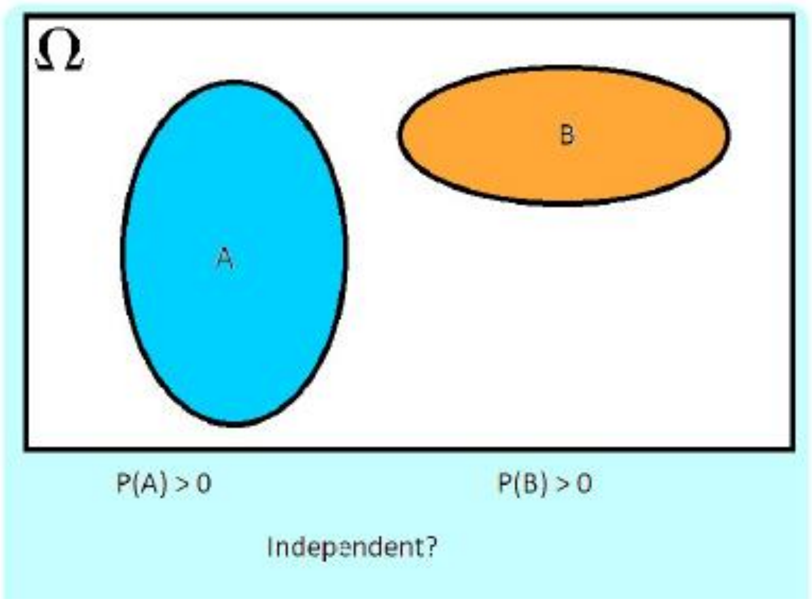
Are Two Disjoint Events Independent?

No

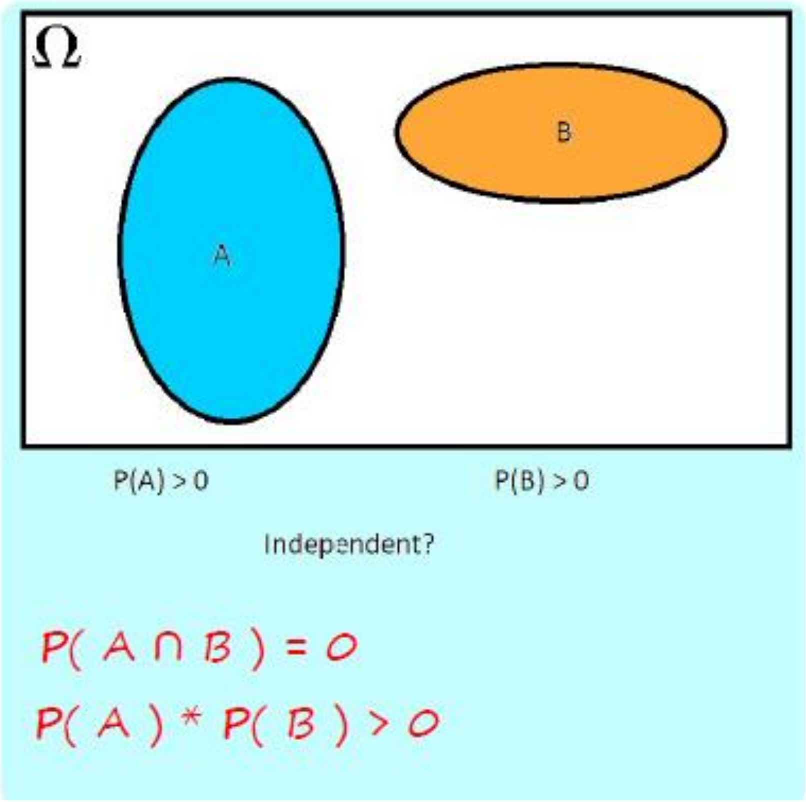
$P(A|B) = P(A)$

$P(A \cap B) = P(A)P(B)$

$P(A \cap B) = P(\phi) = 0$



Are Two Disjoint Events Independent?



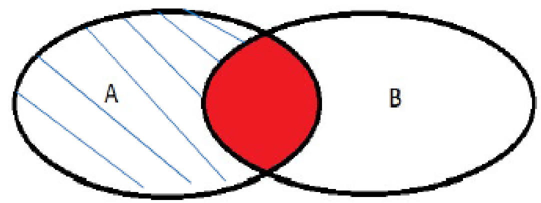
A is independent $\iff$ B is independent of A $\qquad P(A \cap B) = P(A).P(B)$

Independence of Event Complements

If A and B are independent then A and B^C are also independent.

A^c and B " " "
A^c and B^c " " "

- Intuitive Argument
  - If A and B are independent, the occurrence of B does not provide any new information on the probability of A occurring.
  - It is then intuitive that non-occurrence of B (i.e. B^C) should also provide no information on the probability of A. occurred.



➢ **Formal proof**

$$A = (A \cap B) \cup (A \cap B^c)$$
$$P(A) = P(A \cap B) + P(A \cap B^c)$$
$$= P(A)P(B) + P(A \cap Bc)$$

$$P(A \cap B^c) = P(A) - P(A)P(B)$$
$$= P(A)(1 - P(B))$$
$$= P(A)P(B^c)$$

Conditional Independence

- Conditional independence, given an event C, is defined as independence under the probability law $\mathbf{P}(\cdot \mid C)$

Given an event C, the events A and B are called <u>conditionally independent</u> if

$$\mathbf{P}(A \cap B \mid C) = \mathbf{P}(A \mid C)\mathbf{P}(B \mid C)$$

$P(A \cap B) = P(A).P(B)$

$P(A|B) = P(A)$

Given an event C, the events A and B are called <u>conditionally independent</u> if

$$\mathbf{P}(A \mid B \cap C) = \mathbf{P}(A \mid C)$$

Conditional Independence

- Conditional independence, given an event C, is defined as independence under the probability law $\mathbf{P}(\cdot \mid C)$

Given an event C, the events A and B are called <u>conditionally independent</u> if

$$\mathbf{P}(A \cap B \mid C) = \mathbf{P}(A \mid C)\mathbf{P}(B \mid C)$$

To derive an alternative characterization of conditional independence, we use the definition of the conditional probability and the multiplication rule, to write

$$\begin{aligned}
\mathbf{P}(A \cap B \mid C) &= \frac{\mathbf{P}(A \cap B \cap C)}{\mathbf{P}(C)} \\
&= \frac{\mathbf{P}(C)\mathbf{P}(B \mid C)\mathbf{P}(A \mid B \cap C)}{\mathbf{P}(C)} \\
&= \mathbf{P}(B \mid C)\mathbf{P}(A \mid B \cap C).
\end{aligned}$$

Given an event C, the events A and B are called <u>conditionally independent</u> if

$$\mathbf{P}(A \mid B \cap C) = \mathbf{P}(A \mid C)$$

Independence vs Conditional Independence: Intuition

If

$$\mathbf{P}(A \mid B) = \mathbf{P}(A)$$

We say that A is independent of B

- Intuition
  - If A is independent of B, knowledge about the occurrence of B provides no new information about occurrence of A.

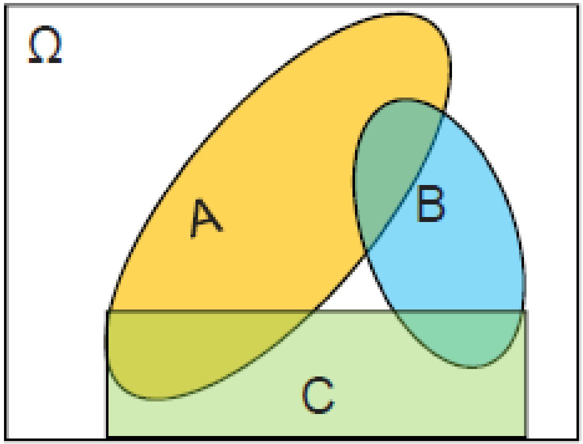
Given an event C, the events A and B are called <u>conditionally independent</u> if

$$\mathbf{P}(A \mid B \cap C) = \mathbf{P}(A \mid C)$$

- Intuition
  - If we have information about the occurrence of C, the additional knowledge that B will also occur does not change the probability of A.

Does Independence Imply Conditional Independence?

Assume A and B are independent

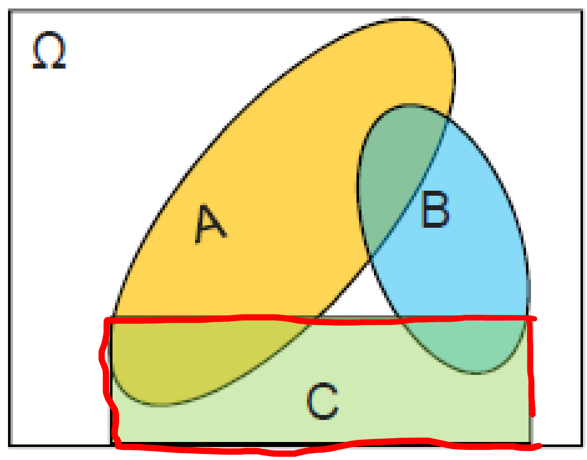


- If we are told that C occurred, are A and B independent?

Does Independence Imply Conditional Independence?

- No

Assume A and B are independent



- If we are told that C occurred, are A and B independent? *No*

Does <u>Independence</u> Imply <u>Conditional Independence</u>?

- No

Example 1.20. Consider two independent fair coin tosses, in which all four possible outcomes are equally likely. Let

$$H_1 = \{\text{1st toss is a head}\},$$

$$H_2 = \{\text{2nd toss is a head}\},$$

$$D = \{\text{the two tosses have different results}\}.$$

The events H_1 and H_2 are (unconditionally) independent. But

Are H_1 and H_2 conditionally independent with respect to event D.

Does Independence Imply Conditional Independence?

- No

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$$D = \{\text{the two tosses have different results}\}.$$

The events H_1 and H_2 are (unconditionally) independent. But

Are H_1 and H_2 conditionally independent with respect to event D.

$$\mathbf{P}(H_1 \mid D) = \frac{1}{2}, \qquad \mathbf{P}(H_2 \mid D) = \frac{1}{2}, \qquad \mathbf{P}(H_1 \cap H_2 \mid D) = 0,$$

so that $\mathbf{P}(H_1 \cap H_2 \mid D) \neq \mathbf{P}(H_1 \mid D)\mathbf{P}(H_2 \mid D)$, and H_1, H_2 are not conditionally independent.

Independence of a Collection of Events

We say that the events $A_1, A_2, \ldots, A_n$ are **independent** if

$$\mathbf{P}\left(\bigcap_{i\in S} A_i\right) = \prod_{i\in S} \mathbf{P}(A_i), \qquad \text{for every subset } S \text{ of } \{1, 2, \ldots, n\}.$$

Or

Events $A_1, A_2, \ldots, A_n$
are called **independent** if:

$$\mathrm{P}(A_i \cap A_j \cap \cdots \cap A_q) = \mathrm{P}(A_i)\mathrm{P}(A_j)\cdots\mathrm{P}(A_q)$$

for any distinct indices $i, j, \ldots, q$,
(chosen from $\{1, \ldots, n\}$)

- Intuition

  Information on some of the events tells us nothing about probabilities related to the remaining events

$P(A \cap B) = P(A) \cdot P(B)$

Independence of a Collection of Events

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for any distinct indices $i, j, \ldots, q$,
(chosen from $\{1, \ldots, n\}$)

For the case of three events, A_1, A_2, and A_3, independence amounts to satisfying the four conditions

$$\left.\begin{aligned}
\mathbf{P}(A_1 \cap A_2) &= \mathbf{P}(A_1)\,\mathbf{P}(A_2), \\
\mathbf{P}(A_1 \cap A_3) &= \mathbf{P}(A_1)\,\mathbf{P}(A_3), \\
\mathbf{P}(A_2 \cap A_3) &= \mathbf{P}(A_2)\,\mathbf{P}(A_3),
\end{aligned}\right\} \text{"Pairwise Independent"}$$

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1)\,\mathbf{P}(A_2)\,\mathbf{P}(A_3).$$

Independence of a Collection of Events

We say that the events $A_1, A_2, \ldots, A_n$ are **independent** if

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for any distinct indices $i, j, \ldots, q$,
(chosen from $\{1, \ldots, n\}$)

For the case of three events, A_1, A_2, and A_3, independence amounts to satisfying the four conditions

$$\left.\begin{aligned}\mathbf{P}(A_1 \cap A_2) &= \mathbf{P}(A_1)\,\mathbf{P}(A_2),\\ \mathbf{P}(A_1 \cap A_3) &= \mathbf{P}(A_1)\,\mathbf{P}(A_3),\\ \mathbf{P}(A_2 \cap A_3) &= \mathbf{P}(A_2)\,\mathbf{P}(A_3),\end{aligned}\right\} \text{Pairwise independence}$$

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1)\,\mathbf{P}(A_2)\,\mathbf{P}(A_3).$$

Does Pairwise Independence Imply Independence?

- No

Consider two independent fair coin tosses, and the following events:

$$H_1 = \{\text{1st toss is a head}\},$$
$$H_2 = \{\text{2nd toss is a head}\},$$
$$D = \{\text{the two tosses have different results}\}.$$

HH	HT
TH	TT

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$$H_1 = \{\text{1st toss is a head}\},$$
$$H_2 = \{\text{2nd toss is a head}\},$$
$$D = \{\text{the two tosses have different results}\}.$$

HH	HT
TH	TT

The events H_1 and H_2 are independent, by definition. To see that H_1 and D are independent, we note that

$$\mathbf{P}(D \mid H_1) = \frac{\mathbf{P}(H_1 \cap D)}{\mathbf{P}(H_1)} = \frac{1/4}{1/2} = \frac{1}{2} = \mathbf{P}(D).$$

Similarly, H_2 and D are independent. On the other hand, we have

$$\mathbf{P}(H_1 \cap H_2 \cap D) = 0 \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \mathbf{P}(H_1)\mathbf{P}(H_2)\mathbf{P}(D),$$

and these three events are not independent.

Is $P(A \cap B \cap C) = P(A)P(B)\ P(C)$ Enough for Independence of a Collection of 3 Events A, B, C?

- No

Consider two independent rolls of a fair six-sided die, and the following events:

$$A = \{\text{1st roll is 1, 2, or 3}\},$$
$$B = \{\text{1st roll is 3, 4, or 5}\},$$
$$C = \{\text{the sum of the two rolls is 9}\}.$$

Is $P(A \cap B \cap C) = P(A)P(B)\,P(C)$ Enough for Independence of a Collection of 3 Events A, B, C?

- No

> Consider two independent rolls of a fair six-sided die, and the following events:
>
> $$A = \{\text{1st roll is 1, 2, or 3}\},$$
> $$B = \{\text{1st roll is 3, 4, or 5}\},$$
> $$C = \{\text{the sum of the two rolls is 9}\}.$$

$$\mathbf{P}(A \cap B \cap C) = \frac{1}{36} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{4}{36} = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C).$$

- But

$$\mathbf{P}(A \cap B) = \frac{1}{6} \neq \frac{1}{2} \cdot \frac{1}{2} = \mathbf{P}(A)\mathbf{P}(B),$$

$$\mathbf{P}(A \cap C) = \frac{1}{36} \neq \frac{1}{2} \cdot \frac{4}{36} = \mathbf{P}(A)\mathbf{P}(C),$$

$$\mathbf{P}(B \cap C) = \frac{1}{12} \neq \frac{1}{2} \cdot \frac{4}{36} = \mathbf{P}(B)\mathbf{P}(C).$$

$P(\text{Link } A \to B \text{ is Up}) =$ (x) ? ~~Parallel: $p_1 + p_2 + p_3$~~ ✓

Series: $p_1 p_2 p_3$

$\boxed{P(A \cap B) = P(A).P(B)}$

Application of Independence: Reliability

- In probabilistic models of complex systems involving several components, it is often convenient to assume that behaviors of the components are uncoupled/independent. This typically simplifies the calculations and the analysis.

- Systems in Series

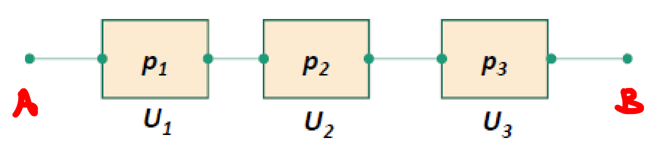
UP

Unit i is serviceable w.p p_i & denoted by U_i

Unit i is unserviceable w.p $(1 - p_i)$ & denoted by F_i

DOWN

➢ *$U_{i's}$ are independent, & therefore $F_{i's}$ are also independent*



System is up if U_1, U_2, U_3 happens

➡ $P(\textit{System is up}) = P(U_1 \text{ is Up AND } U_2 \text{ is Up AND } U_3 \text{ is Up}) = P(U_1 \cap U_2 \cap U_3) = P(U_1).P(U_2).P(U_3)$
$= p_1 p_2 p_3$

Application of Independence: Reliability

- In probabilistic models of complex systems involving several components, it is often convenient to assume that behaviors of the components are uncoupled/independent. This typically simplifies the calculations and the analysis.

- <u>Systems in Series</u>

Unit i is serviceable w.p p_i & denoted by U_i

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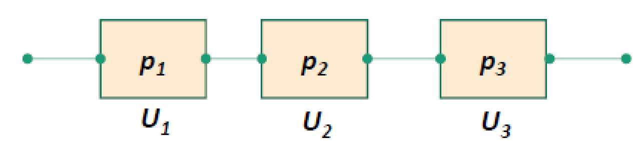
➢ *$U_{i's}$ are independent, & therefore $F_{i's}$ are also independent*



System is up if U_1, U_2, U_3 happens

➡ $P(\textit{System is up}) = P(U_1 \cap U_2 \cap U_3) = P(U_1) \cdot P(U_2) \cdot P(U_3)$

$= p_1 \cdot p_2 \cdot p_3$

P(System is Up)

$P(\text{System is Up}) = P(\text{L1 is Up OR L2 is Up OR L3 is Up}) = P(U_1 \cup U_2 \cup U_3) \overset{?}{=}$?

Application of Independence: Reliability

- In probabilistic models of complex systems involving several components, it is often convenient to assume that behaviors of the components are uncoupled/independent. This typically simplifies the calculations and the analysis. $P(\text{System is Up}) = 1 - P(\text{System is Down}) = 1 - P(\text{L1 is Down AND L2 is Down AND L3 is Down})$

$= 1 - P(F_1 \cap F_2 \cap F_3)$

$= 1 - P(F_1) \cdot P(F_2) \cdot P(F_3)$

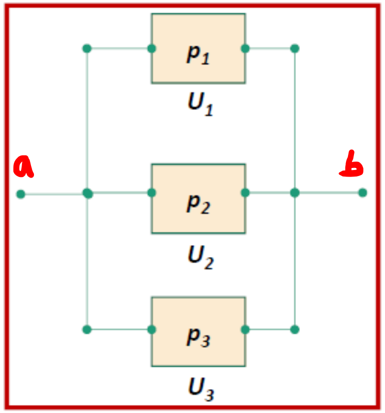
$= 1 - (1-p_1)(1-p_2)(1-p_3)$

- <u>Systems in Parallel</u>

Unit i is serviceable w.p p_i & denoted by U_i

Unit i is unserviceable w.p $(1 - p_i)$ & denoted by F_i

➢ *$U_{i's}$ are independent, & therefore $F_{i's}$ are also independent*



System is up if Path exists between A & B

$$P(\text{System is up}) = P(U_1 \cup U_2 \cup U_3)$$

$$= 1 - P(U_1 \cup U_2 \cup U_3)^c$$

$$= 1 - P(U_1^c \cap U_2^c \cap U_3^c)$$

$$= 1 - P(F_1 \cap F_2 \cap F_3)$$

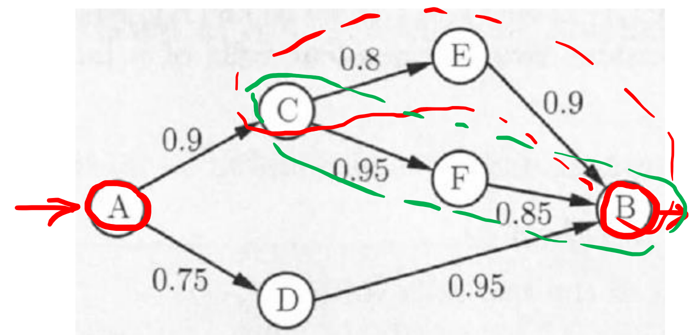
$$= 1 - P(F_1) \cdot P(F_2) \cdot P(F_3)$$

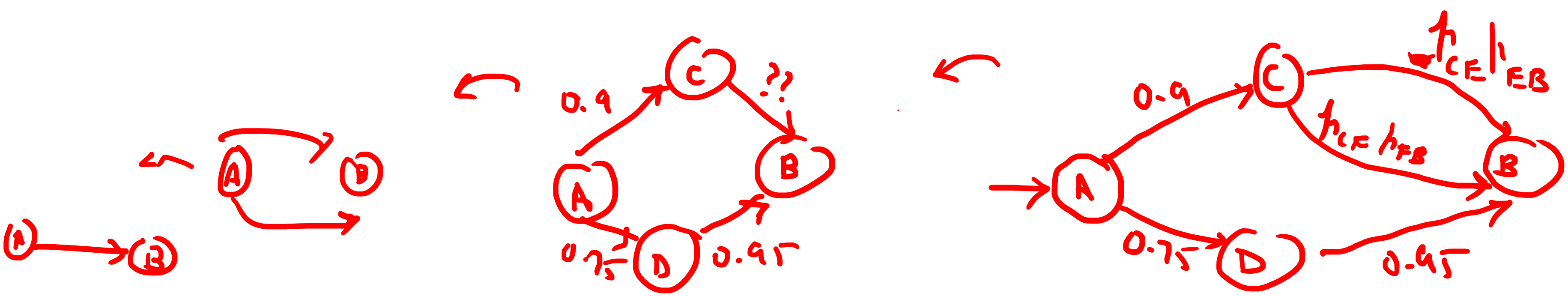
$$\boxed{P(\text{System is Up}) = 1 - (1 - p_1) \cdot (1 - p_2) \cdot (1 - p_3)}$$

Application of Independence: Reliability

Example 1.24. Network Connectivity. A computer network connects two nodes A and B through intermediate nodes C, D, E, F, as shown in Fig. 1.15(a). For every pair of directly connected nodes, say i and j, there is a given probability p_{ij} that the link from i to j is up. We assume that link failures are independent of each other. What is the probability that there is a path connecting A and B in which all links are up?

$\mathbf{P}(C \rightarrow B)$





Application of Independence: Reliability

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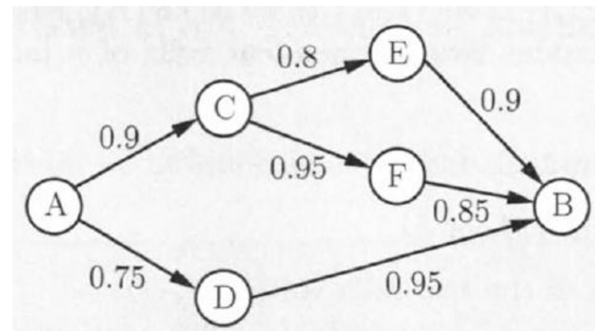
$$\begin{aligned}
\mathbf{P}(C \to B) &= 1 - \big(1 - \mathbf{P}(C \to E \text{ and } E \to B)\big)\big(1 - \mathbf{P}(C \to F \text{ and } F \to B)\big) \\
&= 1 - (1 - p_{CE}p_{EB})(1 - p_{CF}p_{FB}) \\
&= 1 - (1 - 0.8 \cdot 0.9)(1 - 0.95 \cdot 0.85) \\
&= 0.946,
\end{aligned}$$



Application of Independence: Reliability

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$$\begin{aligned}
\mathbf{P}(C \to B) &= 1 - \big(1 - \mathbf{P}(C \to E \text{ and } E \to B)\big)\big(1 - \mathbf{P}(C \to F \text{ and } F \to B)\big) \\
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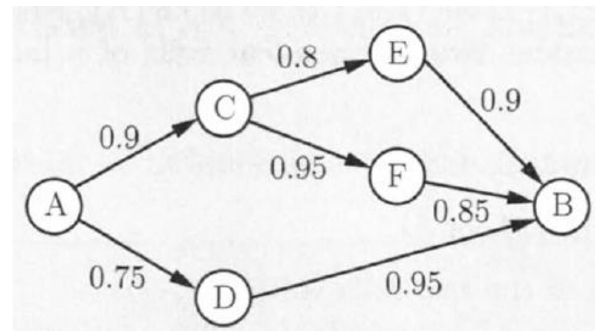
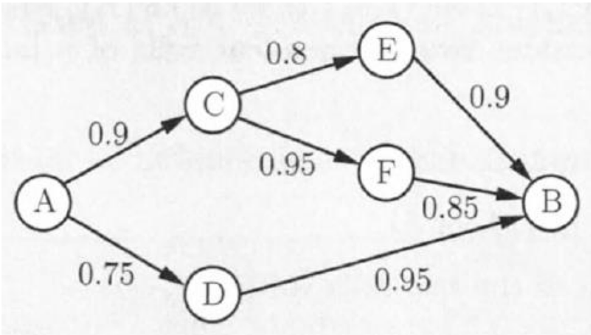


$\mathbf{P}(A \to B)$

Application of Independence: Reliability

Example 1.24. Network Connectivity. A computer network connects two nodes A and B through intermediate nodes C, D, E, F, as shown in Fig. 1.15(a). For every pair of directly connected nodes, say i and j, there is a given probability p_{ij} that the link from i to j is up. We assume that link failures are independent of each other. What is the probability that there is a path connecting A and B in which all links are up?



$$\begin{aligned}
\mathbf{P}(C \to B) &= 1 - \big(1 - \mathbf{P}(C \to E \text{ and } E \to B)\big)\big(1 - \mathbf{P}(C \to F \text{ and } F \to B)\big) \\
&= 1 - (1 - p_{CE}p_{EB})(1 - p_{CF}p_{FB}) \\
&= 1 - (1 - 0.8 \cdot 0.9)(1 - 0.95 \cdot 0.85) \\
&= 0.946,
\end{aligned}$$

$$\mathbf{P}(A \to C \text{ and } C \to B) = \mathbf{P}(A \to C)\mathbf{P}(C \to B) = 0.9 \cdot 0.946 = 0.851.$$

$$\mathbf{P}(A \to D \text{ and } D \to B) = \mathbf{P}(A \to D)\mathbf{P}(D \to B) = 0.75 \cdot 0.95 = 0.712,$$

and finally we obtain the desired probability

$$\begin{aligned}
\mathbf{P}(A \to B) &= 1 - \big(1 - \mathbf{P}(A \to C \text{ and } C \to B)\big)\big(1 - \mathbf{P}(A \to D \text{ and } D \to B)\big) \\
&= 1 - (1 - 0.851)(1 - 0.712) \\
&= 0.957.
\end{aligned}$$