Introduction to Probability and Statistics
(EE 354 / CE 361 / Math 310)

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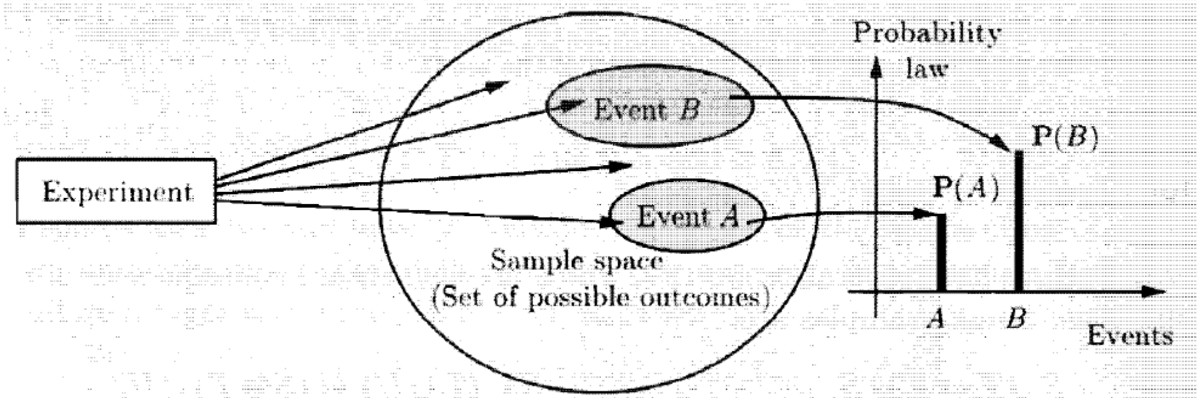
Outline: Unit 3

- Counting: Why?
- The Counting Principle
- Permutations
- Combinations
- Partitions
- Independent Bernoulli Trials

Probabilistic Model

Every probabilistic model involves an underlying process, called the ***Experiment,*** that will produce exactly one out of several possible outcomes.

- Sample Space (Ω)
  - The set of all possible outcomes of an experiment
- Probability Law
  - Assigns to a set A of possible outcomes a non-negative number P(A) that encodes our belief about the collective "likelihood" of the elements of A.
  - This probability law must satisfy certain properties, known as "Probability Axioms."



1. $P(A) \geq 0$
2. $P(A \cup B) = P(A) + P(B)$
3. $P(\Omega) = 1$

Counting: Why?

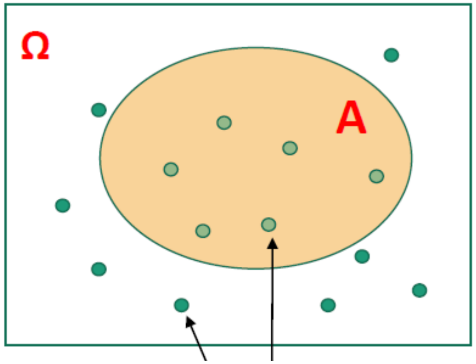
- The calculation of probabilities often involves counting the number of outcomes in various events

- For example:

When the sample space Ω has a finite number of equally likely outcomes, so that the discrete uniform probability law applies. Then, the probability of any event A is given by

$$\mathbf{P}(A) = \frac{\text{number of elements of } A}{\text{number of elements of } \Omega},$$

and involves counting the elements of A and of Ω.



The Counting Principle

The Counting Principle

Consider a process that consists of r stages. Suppose that:

(a) There are n_1 possible results at the first stage.

(b) For every possible result at the first stage, there are n_2 possible results at the second stage.

(c) More generally, for any sequence of possible results at the first $i-1$ stages, there are n_i possible results at the ith stage. Then, the total number of possible results of the r-stage process is

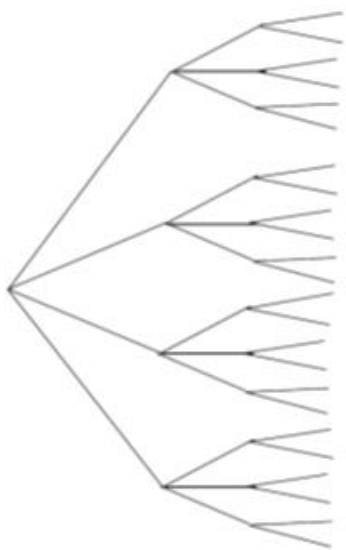
$$n_1 n_2 \cdots n_r.$$

Example:

- 4 shirts.
- 3 ties.
- 2 jackets.

Number of possible attires?

- r stages
- n_i choices at stage i



$24 = 4 \times 3 \times 2$

$r = 3$
$n_1 = 4$
$n_2 = 3$
$n_3 = 2$

The Counting Principle: Examples

- Number of license plates with 2 letters followed by 3 digits
  - What if repetition is prohibited?
- Number of ways of ordering n elements
- Number of distinct subsets of {1,2,3,…n} $= 2^n$

$\boxed{2 \mid 2 \mid 2 \mid 2 \mid \quad \mid \qquad} = 2^n$

The Counting Principle: Examples

- Number of license plates with 2 letters followed by 3 digits

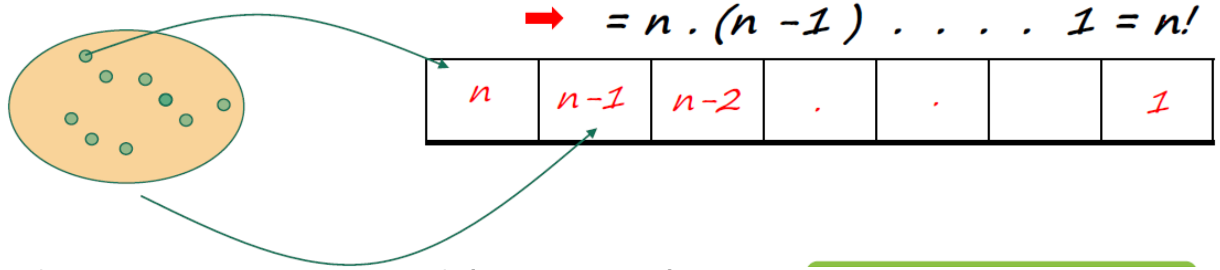
$\Rightarrow 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$

  - What if repetition is prohibited?

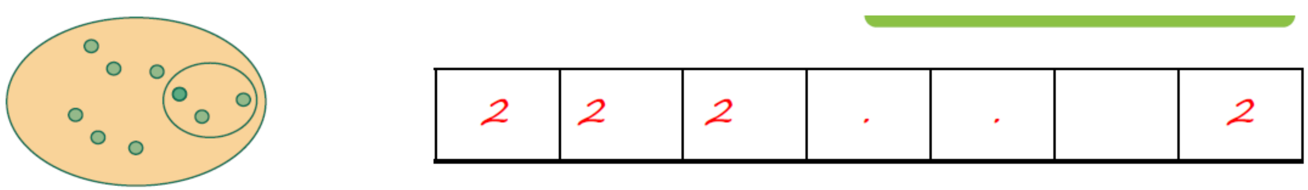
$26 \cdot 25 \cdot 10 \cdot 9 \cdot 8$

- Number of ways of ordering n elements

$\Rightarrow = n \cdot (n-1) \cdot \cdot \cdot \cdot 1 = n!$



- Number of distinct subsets of {1,2,3,…n}



The Counting Principle: Example

- Find the probability that six rolls of 6-sided die all give different numbers. Assume all outcomes are equally likely.

$$P(A) = \frac{\#\ in\ A}{\#\ possible\ outcomes}$$

$\Omega = \{(1,1,1,1,1,1), (1,1,1,1,1,2)$

The Counting Principle: Example

- Find the probability that "six rolls of 6-sided" die all give different numbers. Assume all outcomes are equally likely.

$$P(A) = \frac{\#\ in\ A}{\#\ possible\ outcomes} = \frac{6!}{6^6}$$

$=$

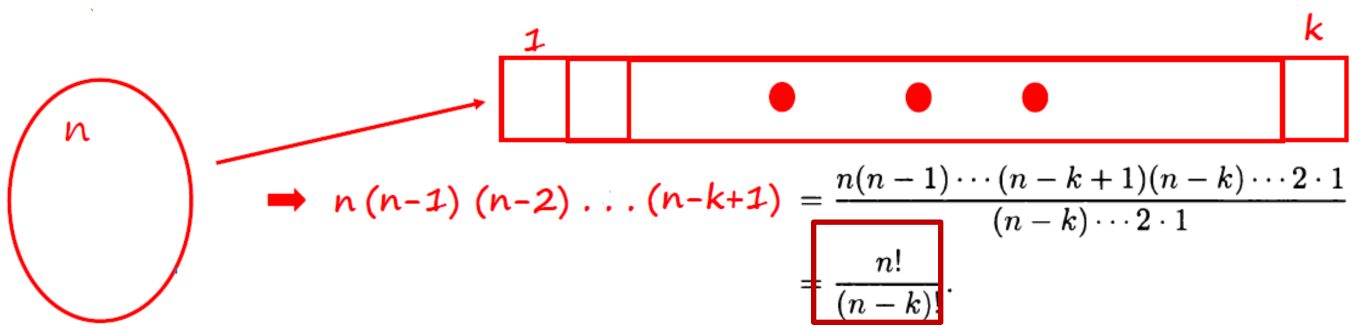
Selection of k Objects out of a Collection of n Objects

- Consider the selection of k objects out of a collection of n objects
  - If the order of the selection matters, the selection is called a **Permutation**.
  - If the order of the selection does not matter, the selection is called a **Combination**.

k-Permutations

We start with n distinct objects, and let k be some positive integer, with $k \leq n$. We wish to count the number of different ways that we can pick k out of these n objects and arrange them in a sequence, i.e., the number of distinct k-object sequences.

- Process
  - We can choose any of the n objects to be the first one
  - Having chosen the first one, there are only n-1 possible choices for the second
  - Having chosen the first two, there remain only n-2 available objects for the third stage.
  - When we are ready to select the last (kth object), we have already chosen k-1 objects, which leaves us with n-(k-1) choices for the last one. Consider the selection of k objects out of a collection of n objects



$$\Rightarrow n(n-1)(n-2)\ldots(n-k+1) = \frac{n(n-1)\cdots(n-k+1)(n-k)\cdots 2\cdot 1}{(n-k)\cdots 2\cdot 1} = \boxed{\frac{n!}{(n-k)!}}.$$

k-Permutations: Example

Example 1.28. Let us count the number of words that consist of four distinct letters.

k-Permutations: Example

Example 1.28. Let us count the number of words that consist of four distinct letters. This is the problem of counting the number of 4-permutations of the 26 letters in the alphabet. The desired number is

$$\frac{n!}{(n-k)!} = \frac{26!}{22!} = 26 \cdot 25 \cdot 24 \cdot 23 = 358,800.$$

Combinations

$\boxed{\text{counting the number of } k\text{-element subsets of a given } n\text{-element set.}}$

$\boxed{\binom{n}{k}}$ " n choose k "

- Example
  - There are n people and we are interested in forming a committee of k. How many different committees are possible?

Combinations

counting the number of k-element subsets of a given n-element set.

$$\binom{n}{k}$$

Notice that forming a combination is different than forming a k-permutation, because **in a combination there is no ordering of the selected elements**. For example, whereas the 2-permutations of the letters A, B, C, and D are

AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC,

the combinations of two out of these four letters are

AB, AC, AD, BC, BD, CD.

In the preceding example, the combinations are obtained from the permutations by grouping together "duplicates"; for example, AB and BA are not viewed as distinct, and are both associated with the combination AB. This reasoning can be generalized: each combination is associated with $k!$ "duplicate" k-permutations.

$$\binom{n}{k}k! = \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{k})!}$$

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

Combinations: Example

Example 1.30. The number of combinations of two out of the four letters A, B, C, and D

Combinations: Example

Example 1.30. The number of combinations of two out of the four letters A, B, C, and D is found by letting $n = 4$ and $k = 2$. It is

$$\binom{4}{2} = \frac{4!}{2!\,2!} = 6,$$

Combinations: Interesting Cases

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

$$\binom{n}{n} =$$

$$\binom{n}{0} =$$

$$\sum_{k=0}^{n} \binom{n}{k} =$$

Combinations: Interesting Cases

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

$\Rightarrow \binom{n}{n} = 1 \qquad \frac{n!}{n!\,0!} \qquad 0! = 1$ convention

$\Rightarrow \binom{n}{0} = \frac{n!}{0!\,n!} = 1 \qquad \varnothing$

$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \ldots \binom{n}{n} = \#\ all\ subsets = 2^n$

Partitions

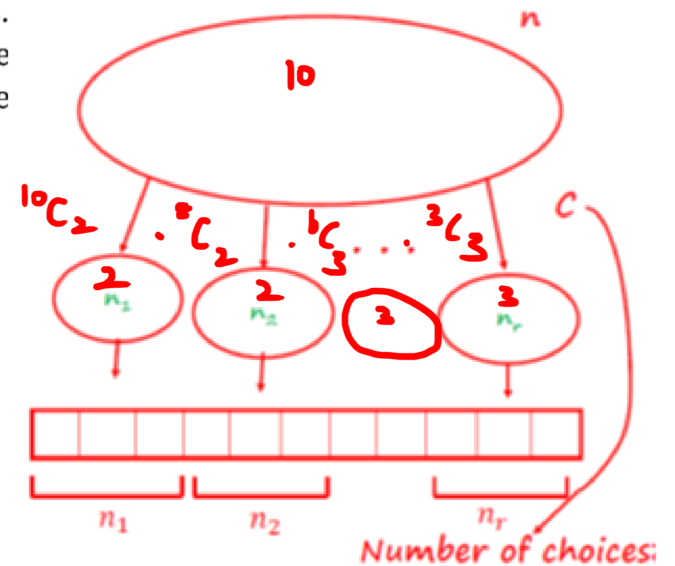
We are given an n-element set and nonnegative integers $n_1, n_2, \ldots, n_r$, whose sum is equal to n. We consider partitions of the set into r disjoint subsets, with the ith subset containing exactly n_i elements. Let us count in how many ways this can be done.

We form the subsets one at a time. We have $\binom{n}{n_1}$ ways of forming the first subset. Having formed the first subset, we are left with $n - n_1$ elements. We need to choose n_2 of them in order to form the second subset, and we have $\binom{n-n_1}{n_2}$ choices, etc. Using the Counting Principle for this r-stage process, the total number of choices is

$$\overset{(1)}{\underset{{}^nC_{n_1}}{\binom{n}{n_1}}}\overset{(2)}{\underset{{}^{n-1}C_{n_2}}{\binom{n-n_1}{n_2}}}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r},$$

which is equal to

$$\frac{n!}{n_1!\,\cancel{(n-n_1)!}}\cdot\frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}}\cdots\frac{\cancel{(n-n_1-\cdots-n_{r-1})!}}{(n-n_1-\cdots-n_{r-1}-n_r)!\,n_r!}.$$



We note that several terms cancel and we are left with

$$\boxed{\frac{n!}{n_1!\,n_2!\cdots n_r!}.}$$

$$\frac{10!}{2!\,2!\,3!\,3!}$$

Partitions

We are given an n-element set and nonnegative integers $n_1, n_2, \ldots, n_r$, whose sum is equal to n. We consider partitions of the set into r disjoint subsets, with the ith subset containing exactly n_i elements. Let us count in how many ways this can be done.

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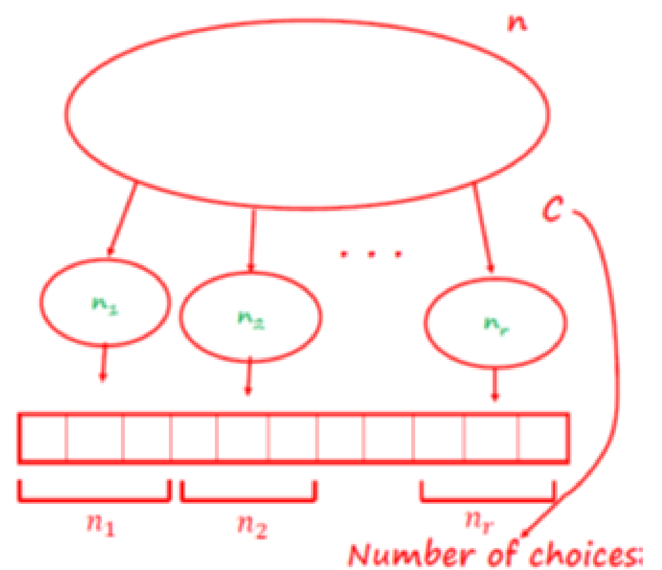
$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r},$$

which is equal to

$$\frac{n!}{n_1!\,(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!}\cdots\frac{(n-n_1-\cdots-n_{r-1})!}{(n-n_1-\cdots-n_{r-1}-n_r)!\,n_r!}.$$



We note that several terms cancel and we are left with

$$\frac{n!}{n_1!\,n_2!\cdots n_r!} = \textbf{multinomial coefficient} = \binom{n}{n_1, n_2, \ldots, n_r}$$

Partitions: Example

Example 1.33. A class consisting of 4 graduate and 12 undergraduate students is randomly divided into four groups of 4. What is the probability that each group includes a graduate student?

P(Each group includes a graduate student) $= \dfrac{\text{\# of elements in my event of interest}}{\text{\# of element in } \Omega}$

\# of elements in Ω = # of ways I can create 4 groups of 4 students $= \dfrac{16!}{4!\,4!\,4!\,4!} \rightarrow A$

\# of element event of interest = [# of ways to put 4 grad students in 4 different groups] [# of ways to put 12 undergrads in 4 groups of 3 each]

$= 4! \cdot \dfrac{12!}{3!\,3!\,3!\,3!} \rightarrow B$

P(Event of interest) $= B/A$ — Ans

Partitions: Example

Example 1.33. A class consisting of 4 graduate and 12 undergraduate students is randomly divided into four groups of 4. What is the probability that each group includes a graduate student?

According to our earlier discussion, there are

$$\binom{16}{4,4,4,4} = \frac{16!}{4!\,4!\,4!\,4!}$$

different partitions, and this is the size of the sample space.

Let us now focus on the event that each group contains a graduate student. Generating an outcome with this property can be accomplished in two stages:

(a) Take the four graduate students and distribute them to the four groups; there are four choices for the group of the first graduate student, three choices for the second, two for the third. Thus, there is a total of 4! choices for this stage.

(b) Take the remaining 12 undergraduate students and distribute them to the four groups (3 students in each). This can be done in

$$\binom{12}{3,3,3,3} = \frac{12!}{3!\,3!\,3!\,3!}$$

different ways.

By the Counting Principle, the event of interest can occur in

$$\frac{4!\,12!}{3!\,3!\,3!\,3!}$$

different ways. The probability of this event is

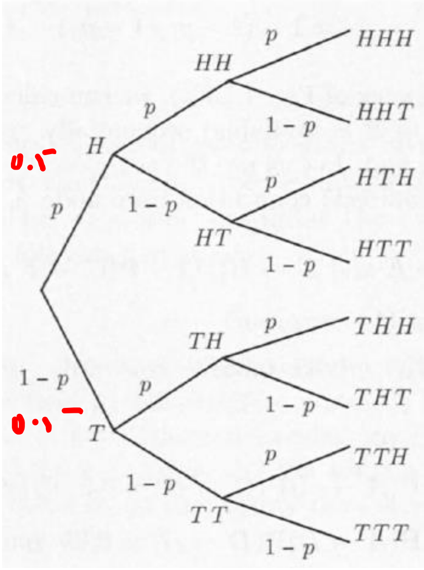
$$\frac{\dfrac{4!\,12!}{3!\,3!\,3!\,3!}}{\dfrac{16!}{4!\,4!\,4!\,4!}}.$$

Independent Bernoulli Trials

- Independent Trials
  - If an experiment involves a sequence of independent but identical stages, we say that we have a sequence of independent trials.

- Independent Bernoulli Trials
  - In the special case of independent trials where there are only two possible results at each stage, we say that we have a sequence of independent Bernoulli trials.

- Example
  - Consider the experiment that consists of 3 independent tosses of a coin



Head (p) Tail $= 1-p$

Toss 1, Toss 2, Toss 3, Toss 4, Toss 5, Toss 6

$P(A \cap B) = P(A) \cdot P(B)$

$P(A \cap B \cap C) = P(A) P(B) P(C)$

Independent Bernoulli Trials

Consider an experiment that consists of n independent tosses of a coin, in which the probability of heads is p

$n = 6$

$k = 2$

Let us now consider the probability

$$p(k) = \mathbf{P}(k \text{ heads come up in an } n\text{-toss sequence})$$

$p(2) = P(2\text{-heads come up in } 6\text{-toss sequence})$

$P(HHTTTT) = p \cdot p \cdot (1-p)(1-p)(1-p)(1-p) = p^2(1-p)^4$

$P(HTTTTH) = p(1-p) \cdot (1-p)(1-p)(1-p) \cdot p = p^2(1-p)^4$

$\vdots$

$P(2\text{-heads in } 6\text{-toss}) = {}^6C_2 \, p^2(1-p)^4$

$p(0)+p(1)+p(2)+p(3)+p(4)+p(5)+p(6) = 1$

Independent Bernoulli Trials

Consider an experiment that consists of n independent tosses of a coin, in which the probability of heads is p

$p(0)$

$p(1)$

$p(2)$

$p(3)$

$p(4)$

$p(5)$

$p(6)$

Let us now consider the probability

$$p(k) = \mathbf{P}(k \text{ heads come up in an } n\text{-toss sequence})$$

$n=6$

$\mathbf{P}$ (*H T T H H H*) = $P\,(1-p)\,(1-p)\,p\;p\;p = p^4\,(1-p)^2$

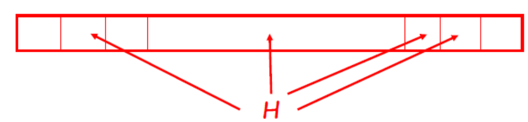
$\mathbf{P}$ (*particular sequence*) = $p^{\#heads}\,(1-p)^{\#tails}$

$\mathbf{P}$ (*particular k-head sequence*) = $p^k\,(1-p)^{n-k}$

$p(k) = \mathbf{P}(k \text{ heads come up in an } n\text{-toss sequence}),\; = $ (# of k-head sequence) * $p^k\,(1-p)^{n-k}$

6-toss

$$p(k) = \mathbf{P}(k \text{ heads come up in an } n\text{-toss sequence}), = \binom{n}{k} p^k (1-p)^{n-k}$$



The numbers $\binom{n}{k}$ (read as "n choose k") are known as the **binomial coefficients**, while the probabilities $p(k)$ are known as the **binomial probabilities**.

Independent Bernoulli Trials

Consider an experiment that consists of n independent tosses of a coin, in which the probability of heads is p

Let us now consider the probability

$$p(k) = \mathbf{P}(k \text{ heads come up in an } n\text{-toss sequence})$$

$p(k) = \mathbf{P}(k \text{ heads come up in an } n\text{-toss sequence}),$ $=$ (# of k-head sequence) * $p^k\,(1-p)^{n-k}$

$p(k) = \mathbf{P}(k \text{ heads come up in an } n\text{-toss sequence}),$ $= \binom{n}{k} p^k (1-p)^{n-k}$

The numbers $\binom{n}{k}$ (read as "n choose k") are known as the **binomial coefficients**, while the probabilities $p(k)$ are known as the **binomial probabilities**.

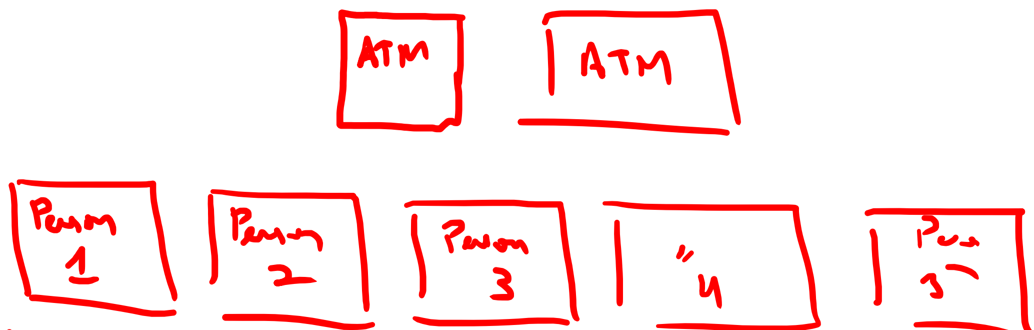
Note that the binomial probabilities $p(k)$ must add to 1, thus showing the **binomial formula**

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1.}$$

Binomial Probabilities: Example

Example 1.25. Grade of Service. An internet service provider has installed c modems to serve the needs of a population of n dialup customers. It is estimated that at a given time, each customer will need a connection with probability p, independent of the others. What is the probability that there are more customers needing a connection than there are modems?

P(You will have to wait outside the ATM) = ?



= + P(3 heads out of 5) + P(4 heads out of 5) + P(5 heads out of 5)

= 0.05

Binomial Probabilities: Example

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Binomial Probabilities: Example

Example 1.25. Grade of Service. An internet service provider has installed c modems to serve the needs of a population of n dialup customers. It is estimated that at a given time, each customer will need a connection with probability p, independent of the others. What is the probability that there are more customers needing a connection than there are modems?

Here we are interested in the probability that more than c customers simultaneously need a connection. It is equal to

$$\sum_{k=c+1}^{n} p(k),$$

where

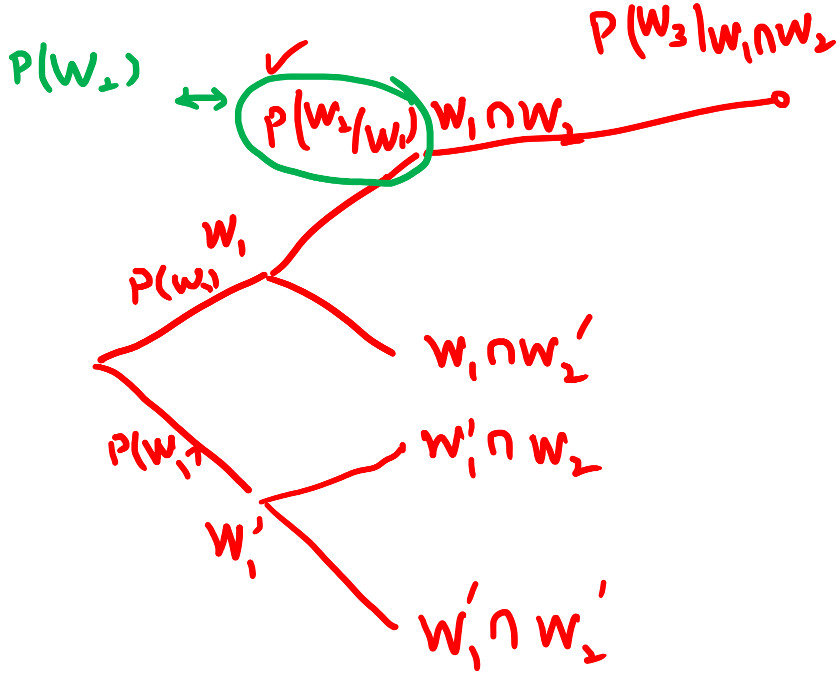
$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

are the binomial probabilities. For instance, if $n = 100$, $p = 0.1$, and $c = 15$, the probability of interest turns out to be 0.0399.

This example is typical of problems of sizing a facility to serve the needs of a homogeneous population, consisting of independently acting customers. The problem is to select the facility size to guarantee a certain probability (sometimes called **grade of service**) that no user is left unserved.

Mathematical Modeling: Using Probability

Imad Wasim : Number of wickets I.W. will take in an Over



" Independent Bernoulli Trials "

Prob. that he takes 2 wickets in an Over?

Assumption of Independence :

$$P(\text{2-heads in 6 coin tosses}) = {}^6C_2\, p^2 (1-p)^4$$