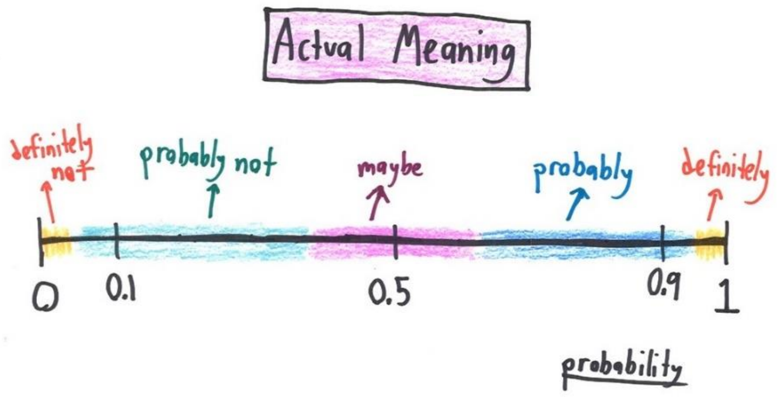
EE 354/CE 361/MATH 310 - Introduction to Probability and Statistics

EE 354/CE 361/MATH 310 – Introduction to Probability and Statistics

Fall 2023

Aamir Hasan

Lecture No 03 – 25th August 2023



Announcements!

- ➢ Lectures up to No 02 posted (PDF of text book posted)
- ➢ Attendance (up to lecture No 2) updated in PSCS
- ➢ Quiz No 01 – Wednesday August 30, 2023 – Good Luck

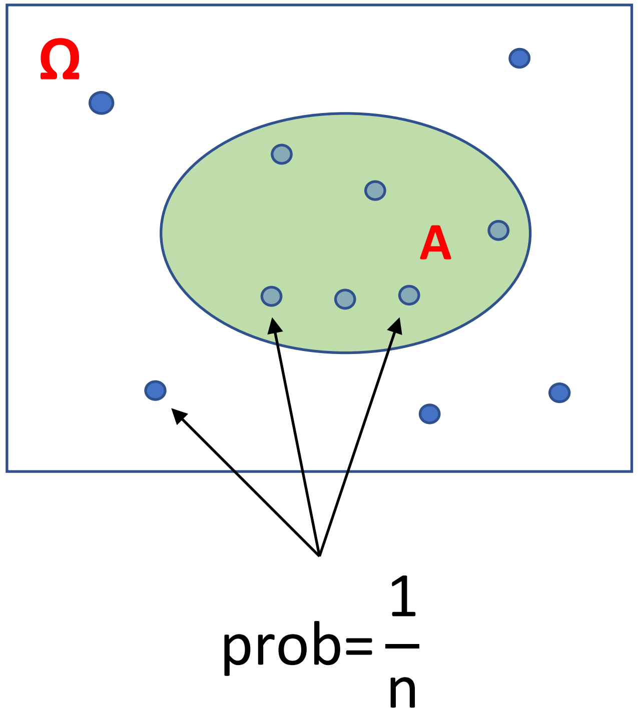
Agenda for today

- *Unit 1: Probability models and axioms Lecture outline (continue)*

Discrete uniform law

finite

- Assume Ω consists of n equally likely elements
- Assume A consists of k elements

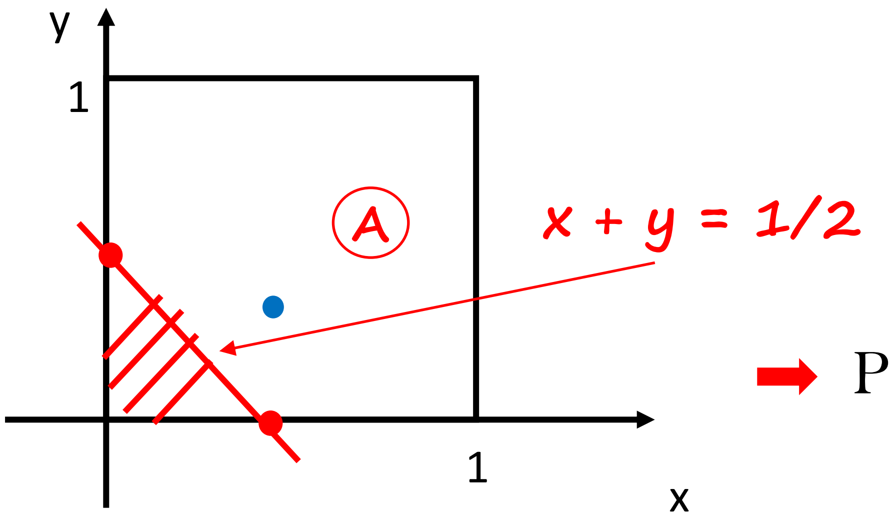


$$P(A) = \boldsymbol{k} * \frac{\mathbf{1}}{\boldsymbol{n}}$$

Probability calculation: continuous example

- (x, y) such that $0 \leq x , y \leq 1$
- **Uniform** probability law: Probability = Area

➡ $P(\{(x, y) \mid x + y \leq ½\}) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8}$



➡ $P(\{(0.5, 0.3)\}) = 0$

$$P(\{(x, y) \mid x + y = ½\}) = ?$$

Probability calculation steps (sequence of 4 steps)

1. Specify the sample space

2. Specify the probability law

3. Identify an event of interest

4. Calculate…

Probability calculation: discrete but infinite sample space

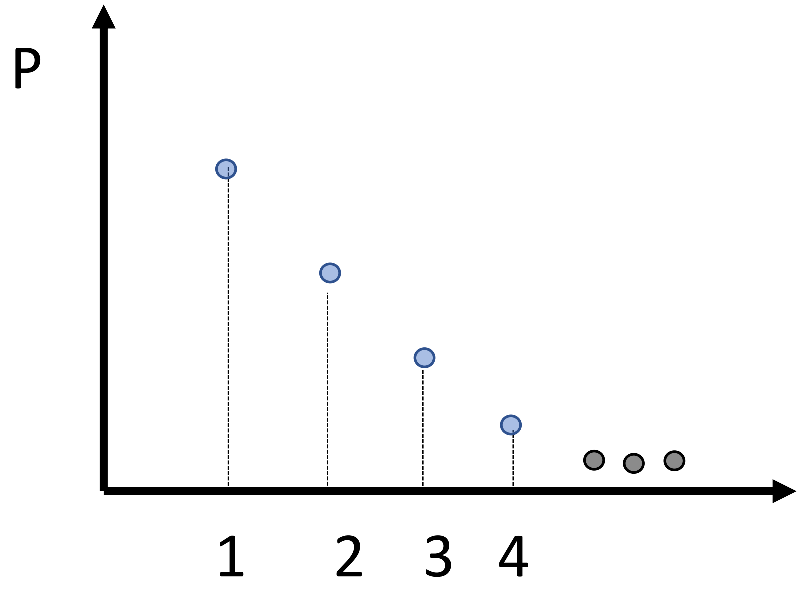
Specify the sample space

1. Sample space: {1, 2, . . . }

Specify the probability law

2. We are given $P(n) = \frac{1}{2^n}$, n=1, 2, . . .

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2} * \frac{1}{1-\frac{1}{2}} = 1$$



Identify an event of interest

3. P (outcome is even) = P ({2, 4, 6, . . . })

Calculate the probability

$$= P(\{2\} \cup \{4\} \cup \{6\} \cup \ldots) = P(2) + P(4) + P(6) + \ldots$$

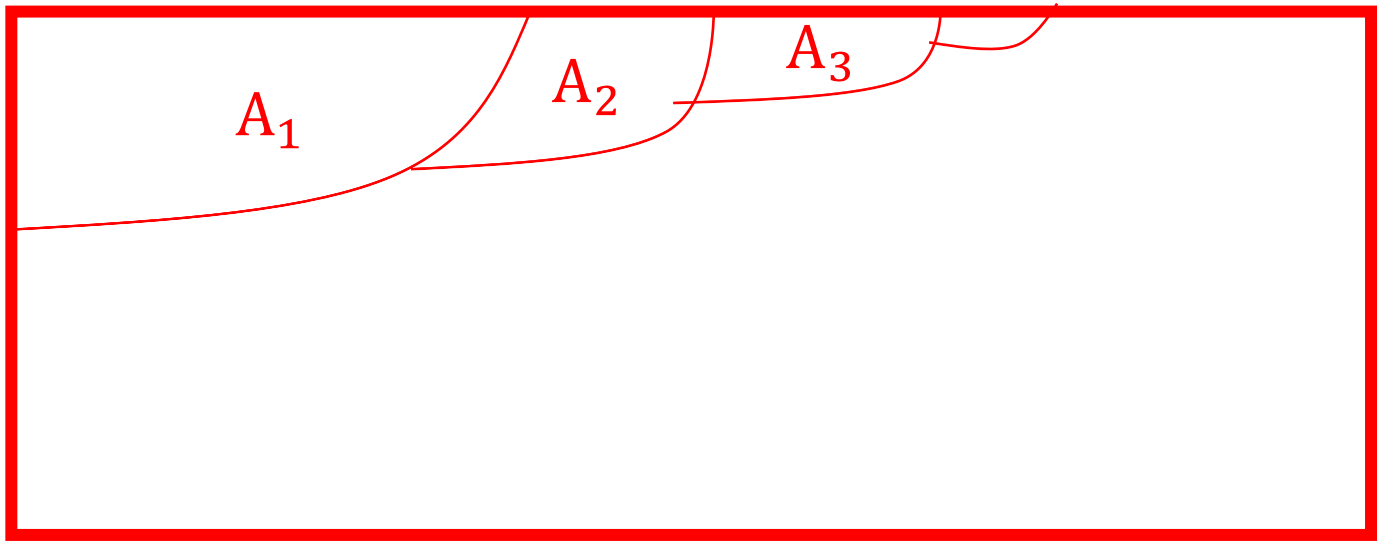
$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots = \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \ldots\right) = \frac{1}{4} * \frac{1}{1-\frac{1}{4}} = \frac{1}{3}$$

Countable additivity axiom

- Strengthens the finite additivity axiom

Countable Additivity Axiom:

If $A_1, A_2, A_3, \ldots$ is an infinite sequence of disjoint events,
then $P\,(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots.$



Countable additivity axiom

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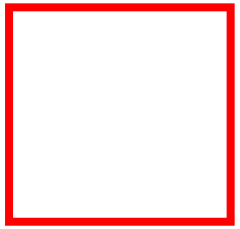
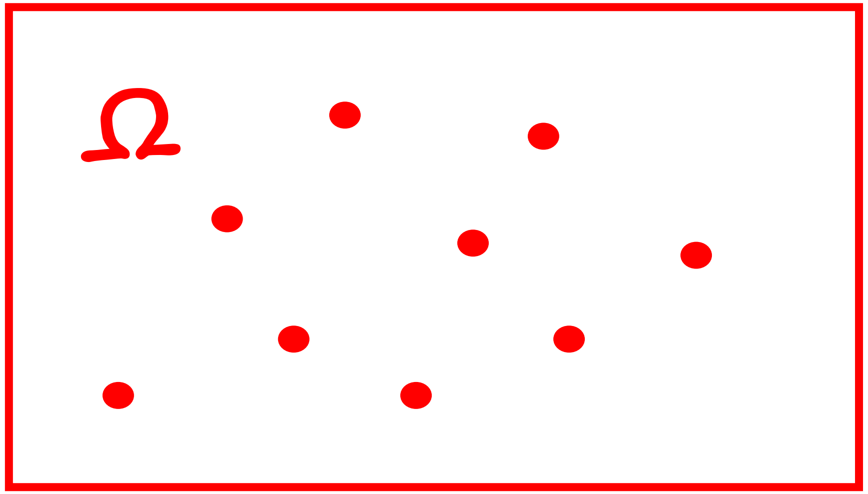


$$1 = P\,(\Omega) = P\,\left(\cup \{(x, y)\}\right) \overset{?}{=} \sum P\,(\{(x, y)\}) = \sum 0 = 0$$

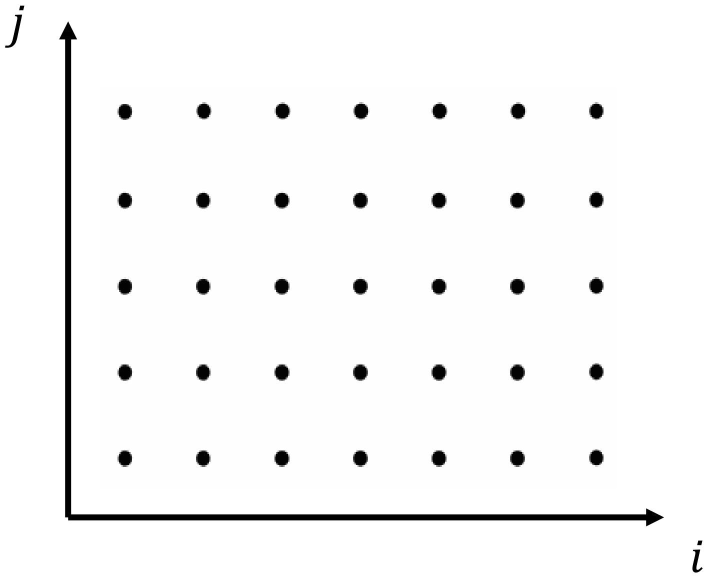
- Additivity holds only for **"countable"** sequences of events
- The unit square (similarly, the real line, etc.) is **not countable** (its elements cannot be arranged in a sequence)
- "Area" is a legitimate probability law on the unit square, as long as we do not try to assign probabilities/areas to "very strange" sets

Countable versus uncountable sets

- Countable – can be put in a 1-1 correspondence with positive integers



  - Positive integers?
  - All integers?
  - Pair of integers?



- Uncountable –
  - Unit interval [0, 1]

Unit 1: Mathematical Background

- Sets & De Morgan’s law
- Infinite Geometric series
- Sums with multiple indices
- Countable & uncountable sets
- Bonferroni inequality

Sets

- A collection of distinct elements

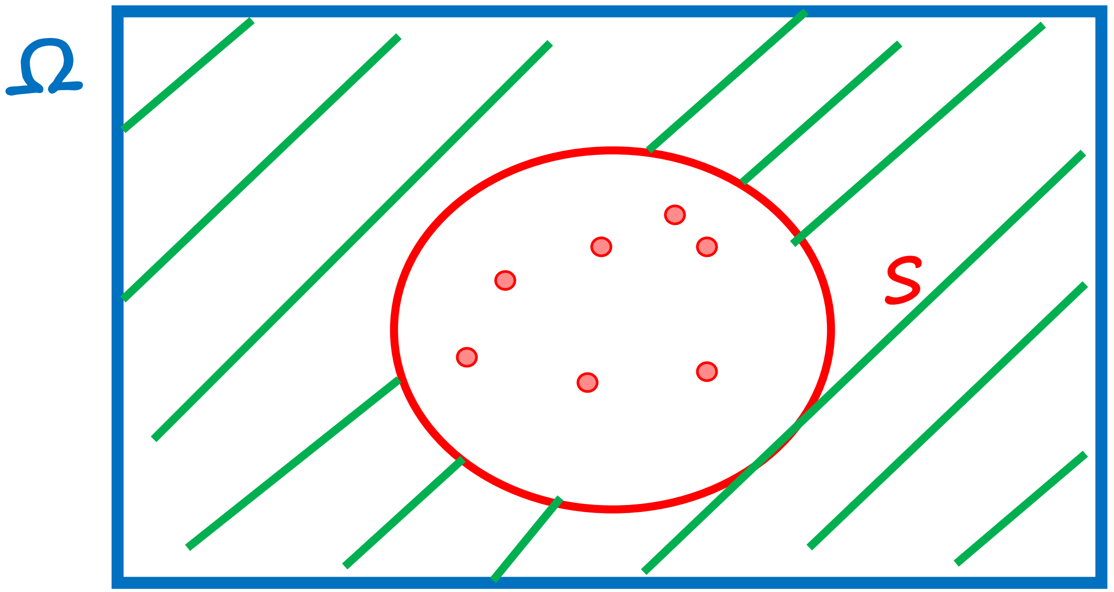
$\{a, b, c, d\}$ finite

$\mathbf{R}$: real numbers infinite

$\{ x \in \mathbf{R} : \cos(x) > 1/2 \}$

Ω = universal set

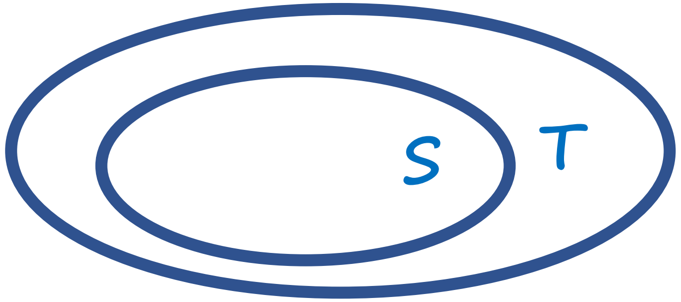
$\emptyset$ = empty set $\Omega^c = \emptyset$



$x \in S$ $x \notin S$

S^c $x \in S^c$ if $x \in \Omega$,
$x \notin S$

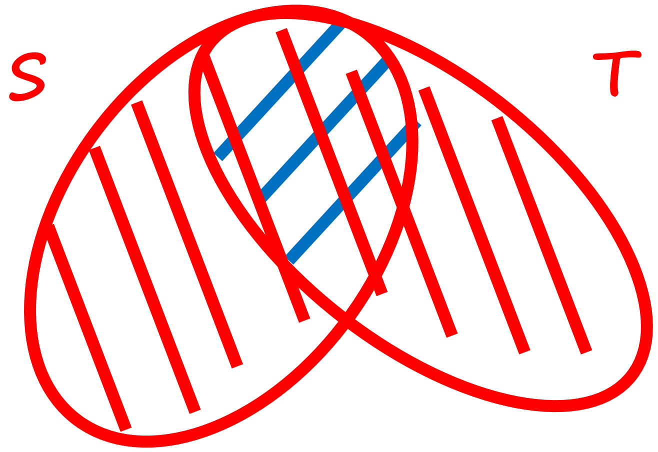
$(S^c)^c = S$



$S \subset T : x \in S \Rightarrow x \in T$

$\subseteq$

Unions and intersections

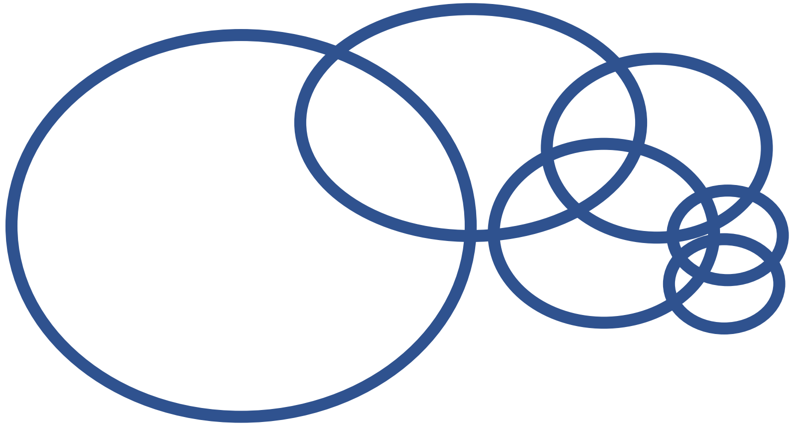


$S \cup T$

$S \cap T$

$x \in S \cup T \Leftrightarrow x \in S$ or $x \in T$

$x \in S \cap T \Leftrightarrow x \in S$ and $x \in T$



S_n $\quad$ $n=1,2,\ldots$

$x \in \bigcup_n S_n$ iff $x \in S_n$, for some n

$x \in \bigcup_n S_n$ iff $x \in S_n$, for all n

Set properties

$S \cup T \cup U$

$\shortparallel$

$S \cup T = T \cup S$

$S \cup (T \cup U) = (S \cup T) \cup U$

$S \cap (T \cup U) = (S \cap T) \cup (S \cap U)$

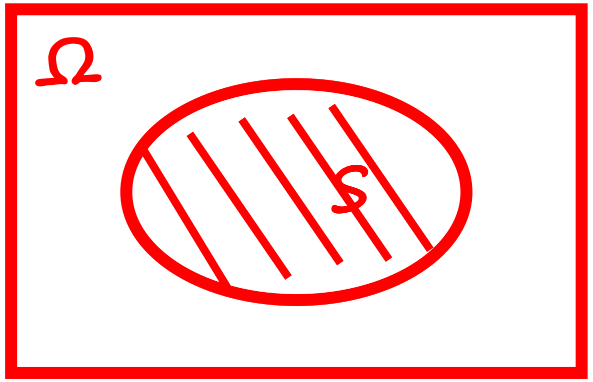
$S \cup (T \cap U) = (S \cup T) \cap (S \cup U)$

$(S^c)^c = S,$

$S \cap S^c = \emptyset,$

$S \cup \Omega = \Omega,$

$S \cap \Omega = S.$

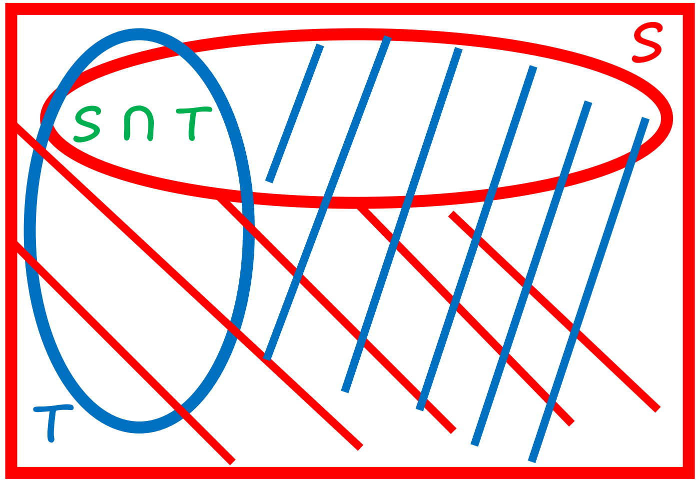


$S \cap (T \cap U) = (S \cap T) \cap U$

$$\left.\begin{array}{l} S \subset T \\ T \subset S \end{array}\right\} \Rightarrow S = T$$

De Morgan's laws

$$\boxed{(S \cap T)^c = S^c \cup T^c}$$



$$S \cap (T \cap U) = (S \cap T) \cap U$$

$S \to S^c \quad T \to T^c$

$S^c \to S \quad T^c \to T$

$$(S^c \cap T^c)^c = S \cup T$$

$$\boxed{S^c \cap T^c = (S \cup T)^c}$$

$$\left(\bigcap_n S_n\right)^c = \bigcup_n S_n{}^c$$

$$\left(\bigcup_n S_n\right)^c = \bigcap_n S_n{}^c$$

$$x \in (S \cap T)^c \Leftrightarrow x \notin S \cap T \Leftrightarrow \left\{\begin{matrix} x \notin S \\ \text{or} \\ x \notin T \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} x \in S^c \\ \text{or} \\ x \in T^c \end{matrix}\right\} \Leftrightarrow x \in S^c \cup T^c$$

Example - Geniuses + Chocolates

$P(G) = 0.6 \qquad P(C) = 0.7 \qquad P(G \cap C) = 0.4$

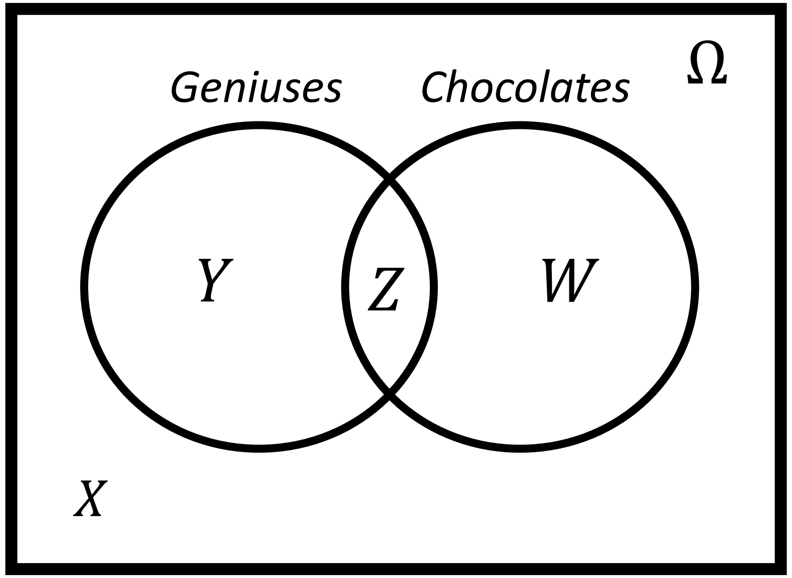
Probability that student picked up at random neither likes chocolate nor is a genius?

$$P(G) = P(Y \cup Z) = P(Y) + P(Z) = 0.6 => P(Y) = 0.2$$

$$P(C) = P(Z \cup W) = P(Z) + P(W) = 0.7 => P(W) = 0.3$$

$$P(Z) = 0.4$$

Class - EE 354



$$1 = P(\Omega) = P(X) + P(Y) + P(Z) + P(W)$$

$$= P(X) + 0.2 + 0.4 + 0.3$$

$$=> P(X) = 0.1$$

Mathematical background: Geometric Series

$S =$ $$\sum_{i=0}^{\infty} \propto^i = 1+\propto + \propto^2 + \ \dots \ = \frac{1}{1-\propto} \qquad |\propto| < 1$$

$$(1 - \propto)(1 + \propto + \dots + \propto^n) = 1 - \propto^{n+1}$$

$n \rightarrow \infty$

$$(1 - \propto) S = 1$$

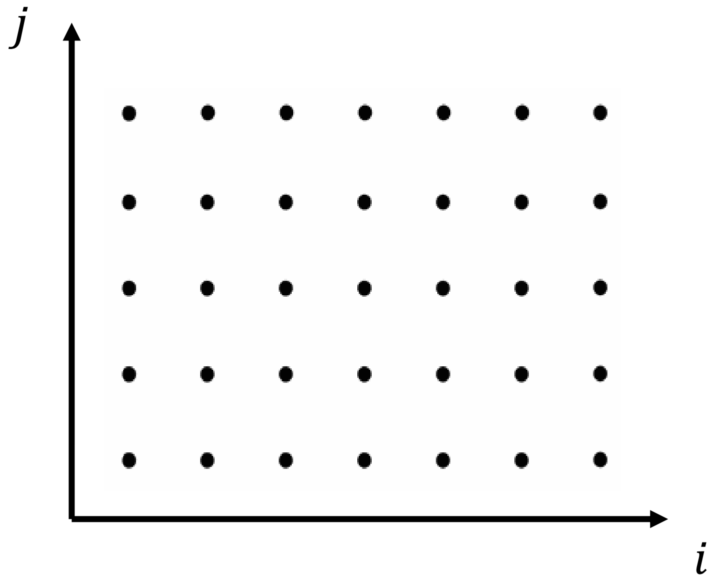
$$S = 1 + \sum_{i=1}^{\infty} \propto^i = 1 + \propto \sum_{i=0}^{\infty} \propto^i = 1 + \propto S \Rightarrow S(1 - \propto) = 1$$

$S < \infty$ taken for granted

About the order of summation in series with multiple indices

$$\sum_{i\geq 1, j\geq 1} a_{ij}$$

$n \rightarrow \infty$



(a) Fix i and let j => 1 - ∞

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right)$$

(b) Fix j and let i => 1 - ∞

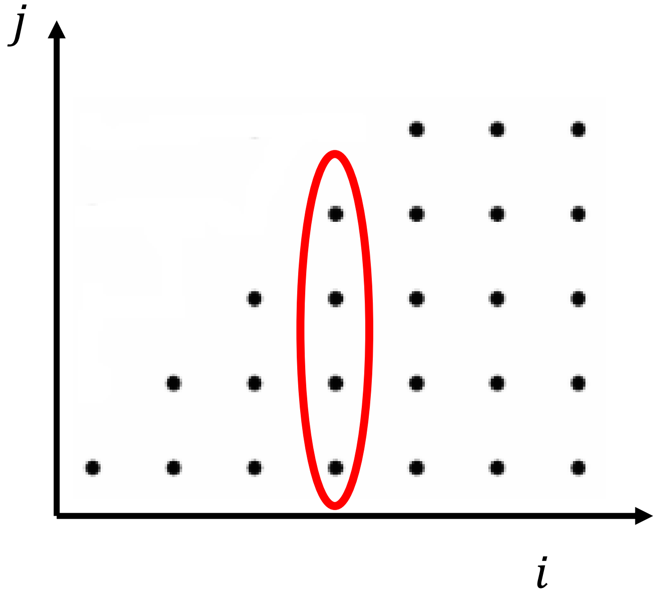
$$\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right)$$

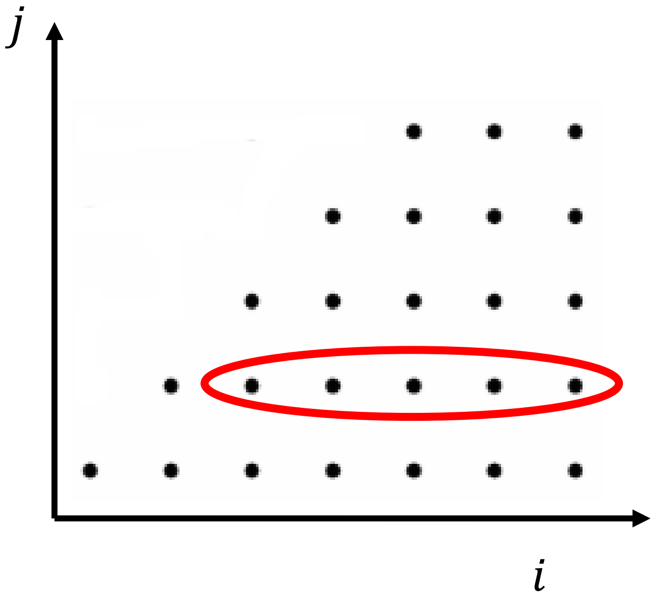
if $\sum |a_{ij}| < \infty$, then order does not matter

About the order of summation in series with multiple indices

if

$$\sum_{(i,j):j\le i} |a_{ij}| < \infty$$



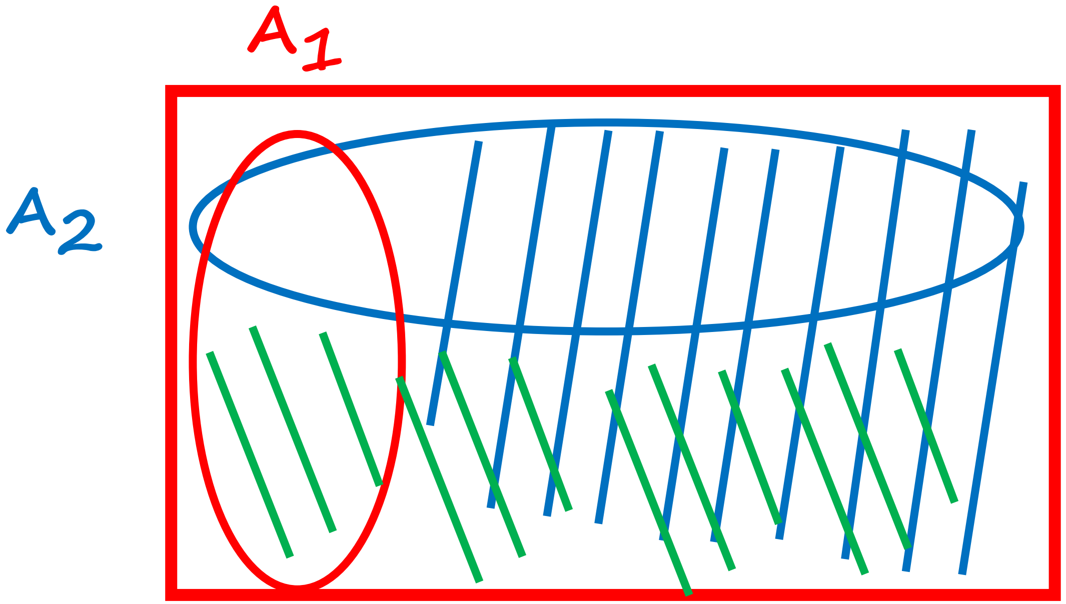


$$\sum_{i=1}^{\infty}\sum_{j=1}^{i} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=j}^{\infty} a_{ij}$$

Interpreting the union bound and the Bonferroni inequality

$$P\,(A_1 \cup A_2) \leq P(A_1) + P(A_2)$$

- Suppose that:
  - Very few students are smart A_1
  - Very few students are beautiful A_2
- Then: very few students are smart or beautiful



- Suppose that:
  - Most of the students are smart
  - Most students are beautiful
- Then: most students are smart and beautiful

$$P\,(A_1 \cap A_2) \geq \underline{P(A_1)} + \underline{P(A_2)} - 1$$