CSE330 Assignment 02 Solution

1. Let $f(x) = \tan(x)$. In the following we would like to calculate the truncation errors.
   (a) (3 marks) First write down the approximate polynomial, $p_3(x)$, for the function $f(x)$ and identify the Taylor coefficients, $a_0, \cdots, a_3$.
   (b) (2 marks) Compute the percentage relative error at $x = \pi/4$ if $f(x)$ is approximated by $p_3(x)$ polynomial.
   (c) (5 marks) Use the Lagrange reminder form to evaluate the upper bound of truncation error at $x = \pi/4$ for some $\xi \in [0, \pi/4]$.

$f(x) = \tan x$

(a)

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots$$

$$f'(x) = \sec^2 x$$

$$f''(x) = 2 \sec x \frac{d}{dx} \sec x$$

$$= 2 \sec x (\sec x \tan x)$$

$$= 2 \sec^2 x \tan x$$

$$f'''(x) = 2 \left[\frac{d}{dx}(\sec^2 x) \tan x + \frac{d}{dx}(\tan x) \sec^2 x\right]$$

$$= 2 (2 \sec^2 x \tan x \tan x + \sec^2 x \sec^2 x)$$

$$= 2 (2 \sec^2 x \tan^2 x + \sec^4 x)$$

$$= 4 \sec^2 x \tan^2 x + 2 \sec^4 x$$

$$\therefore f(x) = 0 + 1 x + \frac{0}{2!} x^2 + \frac{2}{3!} x^3 + \cdots$$

$$= x + \frac{1}{3} x^3 + \cdots \quad [\text{Taylor expansion of } \tan x]$$

$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Comparing,

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = \frac{1}{3}$$

(b)

$$\text{Relative error} = \frac{|f(x) - P_3(x)|}{f(x)}$$

For, $x = \pi/4$,

$$f(x) = \tan x$$

$$\Rightarrow f(\pi/4) = \tan \pi/4 = 1$$

$$P_3(x) = x + \frac{1}{3} x^3$$

$\Rightarrow f(\pi/4) = \tan \pi/4 = 1$

$P_3(x) = x + \frac{1}{3}x^3$

$\Rightarrow P_3(\pi/4) = \pi/4 + \frac{1}{3}(\pi/4)^3 = 0.9469$

$\therefore$ relative error $= \frac{|1 - 0.9469|}{1} = 0.0531$

(c) $f(x) = P_3(x) + \frac{f^4 \xi}{4!}(x - x_0)^4$

$f''''(x) = -8\sec^2 x \tan x + 24 \sec^4 x \tan x$

Lagrange form of reminder $= \frac{f^4 \xi}{4!}(\pi/4 - 0)^4$

$= \frac{80}{4!}(\pi/4)^4$

$= 1.268$

$\xi \in [0, \pi/4]$

$\Rightarrow f^4(\xi = 0) = 0$

$\Rightarrow f^4(\xi = \pi/4) = 80$

2. Consider the function $f(x) = e^{x} - e^{-x}$ and the nodes are at -1, 0, and 1. Now answer the following questions using 3 significant figures:

(a) (1 mark) Write down the matrices b and V used in Vandermonde method.

(b) (2 marks) Compute the determinant of the Vandermonde matrix V .

(c) (3 marks) Using The results of the previous two parts, calculate the Taylor coefficients a_0, a_1 and a_2; and finally find the interpolating polynomial.

(d) (4 marks) Evaluate the upper bound of interpolation error for the given function for the interval $\xi \in [-2.1, 2.1]$.

a: (a)

$$V = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \begin{aligned} x_0 &= -1 \\ x &= 0 \\ x &= 1 \end{aligned}$$

$$b = \begin{bmatrix} -2.35 \\ 0 \\ 2.35 \end{bmatrix}$$

(b)

$$\det(V) = 1\times 0 - (-1)\times 1 + 1 \times 1$$
$$= 2$$

(c)

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2.35 \\ 0 \\ 2.35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} -2.35 \\ 0 \\ 2.35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2.35 \\ 0 \end{bmatrix}$$

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$
$$= 0 + 2.35x + 0$$
$$= 2.35x$$

(d)

$$|f(x) - P_2(x)| = \frac{f^3(\xi)}{3!}(x+1)\,x\,(x-1)$$

$$\omega(x) = (x+1)x(x-1)$$
$$= x^3 - x$$
$$\Rightarrow \omega'(x) = 3x^2 - 1 = 0$$
$$\Rightarrow x = \pm 0.577$$

x	$\omega(x)$
0.577	-0.385
-0.577	0.385
2.1	7.16 ✓
-2.1	-7.16

$$f(x) = e^{x} - e^{-x}$$
$$\Rightarrow f'(x) = e^{x} + e^{-x}$$
$$\Rightarrow f^2(x) = e^{x} - e^{-x}$$
$$\Rightarrow f^3(x) = e^{x} + e^{-x}$$
$$\Rightarrow f^3(-2.1) = 8.29$$
$$\Rightarrow f^3(2.1) = 8.29$$

$$\therefore |f(x) - P_2(x)| = \frac{8.29}{3!} \times 7.16$$
$$= 9.89$$

3. Consider the function $f(x) = e^x + e^{-x}$ and the nodes are at -1, 0, and 1. Now answer the following questions using 3 significant figures:

(a) (4 marks) Evaluate the Lagrange bases for the given function and nodes.

(b) (3 marks) Compute the Lagrange interpolation polynomial for the given function, and express the result in the natural basis. Also use this polynomial to find an approximate value of $f(6)$.

(c) (3 marks) Evaluate the relative error in percentage form at $x = 1.5$.

3. (a)

$$l_0(x) = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x(x-1)}{2}$$

$$l_1(x) = \frac{(x+1)(x-1)}{(0+1)(0-1)} = -x^2+1$$

$$l_2(x) = \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{x(x+1)}{2}$$

(b)

$$P_2(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + f(x_2)\, l_2(x)$$

$$= 3.09\ \frac{x(x-1)}{2} + 2\,(-x^2+1) + 3.09\ \frac{x(x+1)}{2}$$

$$= 1.545\, x(x-1) + 2(-x^2+1) + 1.545\, x(x+1)$$

$$= 1.545x^2 - 1.545x - 2x^2 + 2 + 1.545x^2 + 1.545x$$

$$= 1.09x^2 + 2$$

$$= 2 + 1.09x^2$$

(c) Relative error = $\left|\frac{f(x) - P_2(x)}{f(x)}\right| \times 100\%$

$f(x) = e^x + e^{-x}$

$\Rightarrow f(1.5) = 4.70$

$P_2(x) = 2 + 1.09x^2$

$\Rightarrow P_2(1.5) = 4.45$

$= \left|\frac{4.70 - 4.45}{4.70}\right| \times 100\%$

$= 5.32\%$

4. Consider the function $f(x) = e^x - e^{-x}$ and the nodes are at -2, 0, and 2. Now answer the following questions using 3 significant figures:

   (a) (4 marks) Evaluate the Newton coefficients $a_k = f[x_0, \cdots, x_k]$ using Newton's divided-difference method for the given function and nodes.

   (b) (3 marks) Compute the Newton interpolation polynomial for the given function, and express the result in the natural basis. Also use this polynomial to find an approximate value of $f(6)$.

   (c) (3 marks) Evaluate the relative error in percentage form at $x = 1.5$.

3. (a)

$x_0 = -2 \quad f[x_0] = -7.25$

$f[x_0, x_1] = 3.63$

$x_1 = 0 \quad f[x_1] = 0$

$f[x_0, x_1, x_2] = 0$

$f[x_1, x_2] = 3.63$

$x_2 = 2 \quad f[x_2] = 7.25$

$a_0 = -7.25, \quad a_1 = 3.63, \quad a_2 = 0$

(b) $P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$

$= -7.25 + 3.63(x+2) + 0$

$= -7.25 + 3.63x + 7.26$

$= 3.63x + 0.01$

(c) $f(x) = e^x - e^{-x}$

$\Rightarrow f(1.5) = 4.26$

$P_2(x) = 3.63x + 0.01$

$\Rightarrow P_2(1.5) = 5.46$

$\therefore$ Relative error = $\left|\frac{f(x) - P_2(x)}{f(x)}\right| \times 100\%$

$= \left|\frac{4.26 - 5.46}{4.26}\right| \times 100\%$

$= 28.2\%$