PROBABILITY

TERMINOLOGY

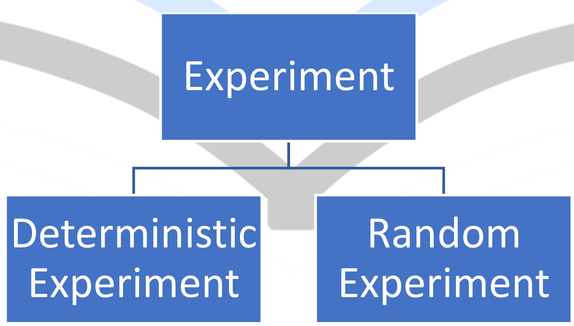
Experiment:

An experiment is a procedure which can repeated infinitely but it has a well-defined set of possible outcomes.

Tossing a coin is an example of experiment since it can be infinitely repeated and it has defined set of possible outcomes which are " heads" & " tails".

Rolling a dice is another example of experiment which can be repeated several times and a infinite set of possible outcomes, S = { 1, 2, 3, 4, 5, 6 }

Measuring the lifetime of an automobile is also an example of experiment and it's sample space consists of all nonnegative real numbers. That is,
S = [0, ∞)



- **Deterministic Experiment:**

The experiments which have only one possible result or outcome, that is, whose result is certain or unique are called deterministic or predictable experiments.

The results of these experiments are known with certainty and is known prior to its conduct.

An experiment conducted to verify the Newton's law of motion and an experiment conducted to verify the Economic Law of Demand are examples of deterministic or predictable experiment.

➢ Random Experiment:

A Random Experiment is an experiment, trial, or observation that can be repeated numerous times under the same conditions. The outcome of an individual random experiment must be independent and identically distributed. It must in no way be affected by any previous outcome and cannot be predicted with certainty.

Examples-

- Tossing a coin.
- Rolling a dice.
- The selection of a numbered ball (1-50) in an urn.
- The time difference between two messages arriving at a message center.
- The time difference between two different voice calls over a particular network.
- The number of calls to a communication system during a fixed length interval of time.

Sample Space:

The set of all possible outcomes of an experiment is known as the sample space of the experiment and it is denoted by S.

The sample space of rolling a dice experiment is, S = { 1, 2, 3, 4, 5, 6 }

The sample space of tossing a coin experiment is, S = { H, T }

Outcome:

An outcome is the result of an experiment. In other words, an outcome is a particular result of an experiment.

In a tossing coin experiment “Head” is an outcome of the experiment.

- A discrete sample space in in which there is a finite number of outcomes or a countably infinite number of outcomes.

- A continuous sample space has uncountable outcomes.

Event:

Any subset of the sample space S is defined as an event. An event is the set of outcomes of an experiment to which a probability is assigned.

In a rolling dice example, which has sample space, $S = \{ 1, 2, 3, 4, 5, 6 \}$; occurrence of even numbers, $E_1 = \{ 2, 4, 6 \}$ and occurrence of odd numbers $E_2 = \{ 1, 3, 5 \}$ can be two possible events.

Basic concepts of probability

Probability:

The probability of an event measures the likelihood of the occurrence of that event.

The classical probability approach is,

$$Probability\ of\ an\ event = \frac{Number\ of\ favorable\ outcomes}{Total\ number\ of\ possible\ outcomes}$$

- The probability of event A is denoted by P(A)
- Probability is quantified as a number between 0 and 1, where, loosely speaking, 0 indicates impossibility and 1 indicates certainty.
- The higher the probability of an event, the more likely it is that the event will occur.
- The sum of probabilities of all sample points in a sample space is equal to 1.
- The probability of event A is the sum of the probabilities of all the sample points in event A.

Example 1:

Suppose we draw a card from a deck of playing cards. What is the probability that we draw a spade?

Solution: The sample space of this experiment consists of 52 cards, and the probability of each sample point is 1/52. Since there are 13 spades in the deck, the probability of drawing a spade is,

$$P(\text{Spade}) = {}^{13}/_{52} = {}^{1}/_{4}$$

Example 2: Suppose a coin is flipped 3 times. What is the probability of getting two tails and one head?

Solution: For this experiment, the sample space consists of 8 sample points.

S = {TTT, TTH, THT, THH, HTT, HTH, HHT, HHH}

Each sample point is equally likely to occur, so the probability of getting any particular sample point is 1/8. The event "getting two tails and one head" consists of the following subset of the sample space.

A = {TTH, THT, HTT}

The probability of Event A is the sum of the probabilities of the sample points in A. Therefore,

P(A) = 1/8 + 1/8 + 1/8 = 3/8

Axioms of Probability:

Consider an experiment whose sample space is S. For each event E of the sample space, S, we assume that a number P(E) is defined and satisfies the following three conditions:

(i) (Axiom of positivizes): $0 \leq P(E) \leq 1$.

(ii) (Axiom of certainty): $P(S) = 1$.

(iii) (Axiom of additivity): For any sequence of events $E_1, E_2, \ldots$ that are mutually exclusive, that is, events for which $E_nE_m = \emptyset$ when $n \neq m$, then,

$$P\left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} P(E_n)$$

Types of events:

Events can be-

- Independent (independent event means each event is not affected by other events. Example: Tossing a coin, throwing a die etc.)

- Dependent (dependent events indicate that they can be influenced by the previous events. Example: After taking a card from a deck of card the probability changes since there are less cards available then.)

- Mutually Exclusive (mutually exclusive events mean both the events can not occur at the same time. For example, in a coin tossing experiment both head and tail can not occur at the same time so the occurrences of head or tail is are mutually exclusive events, Turning left or right at the same time are also mutually exclusive events.)

- Disjoint events (disjoint events do not have any common elements)

- Joint events (joint events have common elements. For example, hearts and kings are joint events.)

Probability Laws:

i. Addition laws of probability:

- **For disjoint events A and B**

The probability that, either event A or event B will occur is,

$$P(A \cup B) = P(A) + P(B)$$

- **For disjoint events A, B, C, …, Z**

The probability that, either event A or event B or event C or … or event Z will occur is,

$$P(A \cup B \cup C \cup \cdots \cup Z) = P(A) + P(B) + P(C) + \ldots + P(Z)$$

- **For joint events A and B**

The probability that, either event A or event B or both will occur is,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **For joint events A, B, and C**

The probability that, either event A or event B or event C or any two of them or all will occur is,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C)$$
$$+ P(A \cap B \cap C)$$

- **For joint events A and B**

The probability that, either event A or event B or both will occur is,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **For joint events A, B, and C**

  The probability that, either event A or event B or event C or any two of them or all will occur is,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

ii. Multiplication laws of probability:

- **For two independent events A and B**

  The probability that, both event A and event B will occur simultaneously is,

$$P(A \cap B) = P(A)\,P(B)$$

- **For two dependent events A and B**

  The probability that, both event A and event B will occur simultaneously is,

$$P(A \cap B) = P(A|B)\,P(B)$$

Here, occurrence of event A depends on occurrence of event B.

Independence:

Two events are known as independent events if the occurrence of one event does not affect the probability of occurring another event.

For two independent events A and B, the probability that A and B will both occur is found by multiplying the two probabilities.

$$P\ (A \text{ and } B) = P\ (A)\ P(\ B\ )$$

Similarly, for three independent events, A, B, and C, the special rule of multiplication used to determine the probability that all three events will occur is:

$$P\ (A \text{ and } B \text{ and } C) = P(A)\ P(B)\ P(C)$$

Conditional Probability:

If the probability of a particular event occurring, given that another event has occurred, then it is known as conditional probability. In other words **conditional probability** is the probability of one event occurring with some relationship to one or more other events.

The formula for conditional probability is:

$$P\ (B|A) = P\ (A \text{ and } B)\ /\ P(A)$$

which you can also rewrite as:

$$P\ (B|A) = P(A\cap B)\ /\ P(A)$$

Example 3:

Your neighbor has 2 children. You learn that he has a son, Joe. What is the probability that Joe's sibling is a brother?

Solution: Let us consider the experiment of selecting a random family having two children and recording whether they are boys or girls. Then, the sample space is S = {BB, BG, GB, GG}, where, e.g., outcome "BG" means that the first-born child is a boy and the second-born is a girl. Assuming boys and girls are equally likely to be born, the 4 elements of S are equally likely. The event, E, that the neighbor has a son is the set E = {BB, BG, GB}. The event, F, that the neighbor has two boys (i.e., Joe has a brother) is the set F = {BB}. We want to compute,

$$P(F|E) = \frac{P(F\cap E)}{P(E)}$$

$$= \frac{P(\{BB\})}{P(\{BB,BG,GB\})}$$

$$= \frac{1/4}{3/4}$$

$$= \frac{1}{3}$$

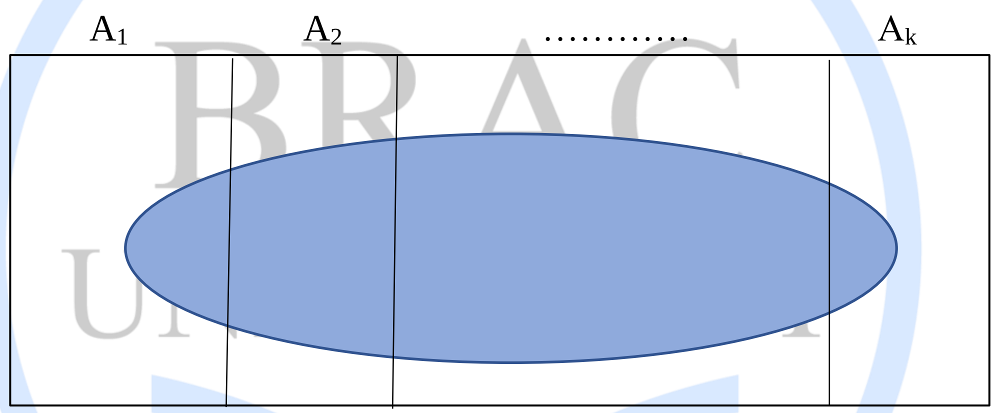
Total Probability Law:

Let, events A_1, A_2, …, A_k form partition of S. Let B be an event with P(B)>0. Then,

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots\ldots + P(A_k \cap B)$$

$$= P(A_1)\, P(B|A_1) + P(A_2)\, P(B|A_2) + \ldots.. + P(A_k)\, P(B|A_k)$$

$$= \sum_{i=1}^{k} P(A_i)\, P(B|A_i)$$



Example 4: A person has undertaken a mining job. The probabilities of completion of job on time with and without rain are 0.42 and 0.90 respectively. If the probability that it will rain is 0.45, then determine the probability that the mining job will be completed on time.

Solution: Let A be the event that the mining job will be completed on time and B be the event that it rains. We have,

$$P(B) = 0.45,$$

$$P\text{ (no rain)} = P(B') = 1 - P(B) = 1 - 0.45 = 0.55$$

By multiplication law of probability,

$$P(A|B) = 0.42$$

$$P(A|B') = 0.90$$

Since, events B and B′ form partitions of the sample space S, by total probability theorem, we have,

$$P(A) = P(B)\,P(A|B) + P(B')\,P(A|B')$$

$$=0.45 \times 0.42 + 0.55 \times 0.9$$

$$= 0.189 + 0.495 = 0.684$$

So, the probability that the job will be completed on time is 0.684.

Probability using contingency table:

Contingency table:

Contingency table is a power tool in data analysis for comparing categorical variables. Although it is designed for analyzing categorical variables, this approach can also be applied to other discrete variables and even continuous variables.

A general 2×2 contingency table will be like the follows:

X \ Y	Y_1	Y_2	Total
X_1	a	b	a + b
X_2	c	d	c + d
Total	a + c	b + d	a + b + c + d

Here the two variables are X and Y and each of them have two possible categories.

Example 5:

Suppose a study of speeding violations and drivers who use cell phones produced the following fictional data:

	Speeding violation in the last year	No speeding violation in the last year	Total
Cell phone user	25	280	305
Not a cell phone user	45	405	450
Total	70	685	755

a. Find *P* (Person is a car phone user).

Ans. P (person is a car phone user) = $\frac{\text{number of car phone users}}{\text{Total number of users in the study}} = \frac{305}{755}$

b. Find *P* (person had no violation in the last year)

Ans. P (person had no violation in the last year) = $\frac{\text{number of car phone users that had no violation}}{\text{Total number of users in the study}} = \frac{685}{755}$

c. Find *P* (Person is a car phone user | person had a violation in the last year)

Ans. *P* (Person is a car phone user | person had a violation in the last year)

$$= \frac{\text{number of car phone users that had violation in the last year}}{\text{Total number of users in the study that had violation in the last year}} = \frac{25}{70}$$