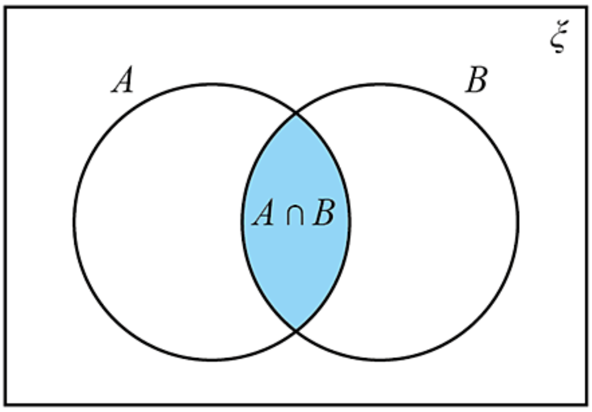
# Bayes' Theorem Lecture Notes

## Conditional Probability (Review):

Conditional probability is the probability of one event occurring based on the occurrence of a previous event.




From the diagram above we get two equations:

- $P(A|B) = \frac{P(A \cap B)}{P(B)}$ (1.1)

- $P(B|A) = \frac{P(B \cap A)}{P(A)}$ (1.2)

Rearranging equation **1.1** we get:

$$P(A \cap B) = P(A|B) * P(B) \quad (1.3)$$

Rearranging equation **1.2** we get:

$$P(B \cap A) = P(B|A) * P(A) \quad (1.4)$$

The LHS of both equations 1.3 and 1.4 are equal. Equating them gives:

$$P(A|B) * P(B) = P(B|A) * P(A) \quad (1.5)$$

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B)} \quad (1.6)$$

Equation 1.6 represents the equation of Bayes' Theorem.

- Bayes' Theorem provides a mathematical rule for revising an estimate or forecast in light of experiences and observations.
- Suppose we have estimated prior probabilities for events, and then we obtain new information/evidence. Bayes' Theorem gives us a way to determine the updated or posterior probability based on the new evidence.

Bayes' Theorem is a method of revising a probability given that additional information is obtained. For two events A and B, the Bayes' Theorem is as follows:

likelihood term

posterior probability

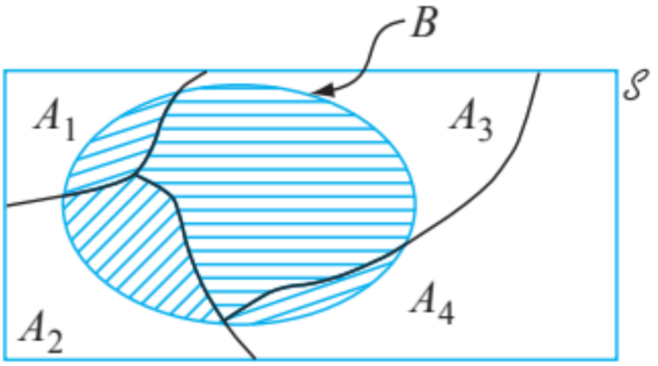
prior probability

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

evidence

- **Posterior Probability:** A revised probability based on additional information
- **Prior Probability:** Initial probability based on the present level of information
- **Likelihood Term**
- **Evidence**

**Total Probability (Review):**




- Let the rectangle be a sample space
- $A_1$, $A_2$, $A_3$ and $A_4$ are mutually exclusive and collectively exhaustive events
- Let B be another event in the sample space

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) + P(B \cap A_4)$$

$$= P(B|A_1) * P(A_1) + P(B|A_2) * P(A_2) + P(B|A_3) * P(A_3) + P(B|A_4) * P(A_4)$$

$$= \sum_{i=1}^{4} P(B|A_i) * P(A_i)$$

In general: $\boldsymbol{P(B) = \sum_i^k P(B|A_i) * P(A_i)}$

If there are three events A1, A2 & B where A1 & A2 are mutually exclusive and collectively exhaustive events then the workable form of Bayes' theorem is,

$$\boldsymbol{P(A_1|B) = \frac{P(B|A_1) * P(A_1)}{P(B)} = \frac{P(B|A_1) * P(A_1)}{P(B|A_1) * P(A_1) + P(B|A_2) * P(A_2)}}$$

Thus, for n events A1, A2, ...., Ai, ..., An, the form of Bayes' theorem is,

$$\boldsymbol{P(A_i|B) = \frac{P(B|A_i) * P(A_i)}{P(B)} = \frac{P(B|A_i) * P(A_i)}{P(B|A_1) * P(A_1) + P(B|A_2) * P(A_2) + \cdots P(B|A_n)P(A_n)}}$$

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# Examples

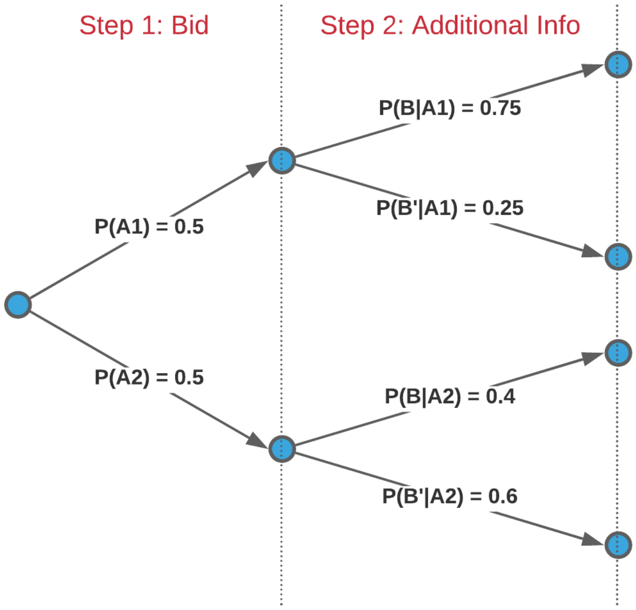
1. A consulting firm submitted a bid for a large consulting contract. The firm's management felt it had a 50-50 chance of landing the project. However, the agency to which the bid was submitted subsequently asked for additional information. Past experiences indicate that for 75% of successful bids and 40% of unsuccessful bids the agency asked for additional information.
a) What is the prior probability of the bid being successful (that is, prior to the request for additional information)?
b) What is the conditional probability of a request for additional information given that the bid will be ultimately successful?
c) Compute the posterior probability that the bid will be successful given a request for additional information.

## Answer

Let $A_1$ = the bid is successful
Let $A_2$ = the bid is unsuccessful
Let B = additional information requested

$P(A_1) = 0.5$
$P(A_2) = 0.5$
$P(B|A_1) = 0.75$
$P(B|A_2) = 0.4$

We can represent this information using a tree diagram:




$P(B \cap A_1)=P(B|A_1)*P(A_1)=0.5*0.75=0.375$

$P(B' \cap A_1)=P(B'|A_1)*P(A_1)=0.5*0.25=0.125$

$P(B \cap A_2)=P(B|A_2)*P(A_2)=0.5*0.4=0.2$

$P(B' \cap A_2)=P(B'|A_2)*P(A_2)=0.5*0.6=0.3$

a) $P(A_1) = 0.5$
b) $P(B|A_1) = 0.75$
c) $P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B)} = \frac{P(B|A_1)P(A_1)}{\sum P(B|Ai)*P(Ai)} = \frac{P(B|A_1)P(A_1)}{P(B|A_1)*P(A_1)+P(B|A_2)*P(A_2)}$
$= \frac{0.75*0.5}{(0.75*0.5)+(0.4*0.5)}$
$= 0.652$

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2. A particular study showed that 12% of men will likely develop prostate cancer at some point in their lives. A man with prostate cancer has 95% chance of a positive test result from a medical screening exam. A man without prostate cancer has a 6% chance of getting a false positive test result. What is the probability that a man has cancer given he has a positive test result?

**Answer**

Let $A_1$ = has prostate cancer
Let $A_2$ = does not have prostate cancer
Let B = positive test result

$P(A_1) = 0.12$
$P(A_2) = 0.88$
$P(B|A_1) = 0.95$
$P(B|A_2) = 0.06$

$P(A_1|B) =?$

$$P(A_1|B) = \frac{P(B|A_1) * P(A_1)}{P(B)} = \frac{P(B|A_1) * P(A_1)}{\sum P(B|Ai) * P(Ai)} = \frac{0.95 * 0.12}{(0.95 * 0.12) + (0.06 * 0.88)} = 0.683$$

The probability that a man has cancer given he has a positive test result is 68.3%.

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3. Suppose that an insurance company classifies people into one of the three classes – good risks, average risks and bad risks. Their records indicate that the probabilities that good, average, and bad risk persons will be involved in an accident over a 1-year span are 0.05, 0.15 and 0.3 respectively. 20% of the population are "good risks", 50% are "average risks", and 30% are "bad risks”.
a) What proportion of people have accidents in a fixed year?
b) If policy holder A had no accidents in 1987, what is the probability that he or she is a good risk person?

**Answer**

Let $A_1$ = good risk
Let $A_2$ = average risk
Let $A_3$ = bad risk
Let B = accident occurs

$P(A_1) = 0.2$
$P(A_2) = 0.5$
$P(A_3) = 0.3$
$P(B|A_1) = 0.05$
$P(B|A_2) = 0.15$
$P(B|A_3) = 0.3$

a) $$P(B) = \sum P(B|Ai)P(Ai) = P(B|A_1) * P(A_1) + P(B|A_2) * P(A_2) + P(B|A_3) * P(A_3)$$
$$= (0.05 * 0.2) + (0.15 * 0.5) + (0.3 * 0.3) = 0.175$$

b) Let B' = accident does not occur

$P(B') = 1 - P(B)$

$\therefore P(B') = 1 - 0.175 = 0.825$

and $P(B'|A_1) = 1 - P(B|A_1)$

$= 1 - 0.05 = 0.95$

$$P(A_1|B') = \frac{P(B'|A_1) * P(A_1)}{P(B')} = \frac{0.95 * 0.20}{0.825} = 0.23$$

If policy holder A had no accidents in 1987, the probability that he or she is a good risk is 23%.

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**<u>Practice Problems</u>**

Textbook: Probability & Statistics for Engineering and the Sciences (Devore)

*Conditional Probability and Bayes' Theorem*

*Page 80-83*: 45, 49, 51, 53, 55, 59, 61, 65, 67