Probability

Probability measures the likelihood of occurring an event.

Examples:

- Weather Forecasting
- Predicting batting average in cricket matches
- Politics
- Choosing appropriate insurance strategies
- Selecting sport strategies etc.

Terminologies:

Three keywords are used while studying probability:

- Experiment
- Outcome
- Event

Experiment:

Experiment is an act that can be repeated under given conditions.

Experiment may be Deterministic or predictable and Random or un predictable.

Deterministic or predictable experiment:

An experiment is called deterministic when the outcome or result is unique or certain. Everyone conducting the experiment will get the same result or outcome.

Examples:

- Predicting the amount of money in a bank account if you know the initial deposit and the interest rate.

- The relationship between a circumference and radius of a circle, or the area and radius of a circle.

Random or unpredictable experiment:

An experiment whose outcomes can not be predicted with certainty in advance is called random or unpredictable experiment.

Examples:

- An experiment of tossing a coin
- An experiment of rolling a die
- Number of defected items produced by a machine by an hour
- Drawing a card from a pack

Sample Space:

A set or collection of all possible outcomes of a random experiment is called sample space of that random experiment and it is denoted by S. Each outcome of an experiment is a sample point or element in the sample space.

For example,

- Tossing a coin: $S = \{ H, T \}$
- Throwing a dice: $S = \{1, 2, 3, 4, 5, 6 \}$
- Lifetime of a lightbulb: $S= \{x|0 \leq x < \infty\} = [0,\infty)$

- Consider an experiment that consists of rolling two balanced dice, one white and one red are thrown and number of dots on their upper faces are noted, also if b be the outcomes of the white die and r be the outcomes of the red die. If we let denote the outcome in which white dice has value w and red dice has value r, then the sample space of this experiment is:

		White Die					
		1	2	3	4	5	6
Red Die	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

<u>Event:</u>

An event, E can be defined as a set of outcomes of an experiment or a subset of a sample space, S.

For example, in throwing a die experiment, the sample space is S={1, 2, 3, 4, 5, 6}.

E={2, 4, 6} is an event, which can be described in words as ”the number is even”.

Mutually exclusive events:

Two events are called mutually exclusive if the occurrence of one event means that none of the other events can occur simultaneously in a single trial. In other words, if one of those events occur, then the other evets will not occur.

For example,

In tossing a coin experiment, event E_1 = { Head } and event

E_2 = { Tail } are mutually exclusive events as both of the events E_1 & E_2 can not occur at the same time.

On a day, Event E_1= {Rain} & event E_2 = {Sunny} may occur simultaneously. These are not mutually exclusive events.

Collectively Exhaustive events:

Events of an experiment are said to be collectively exhaustive events if they include all possible outcomes.

For example; In a coin tossing experiment events,

E_1 = {Head} and event E_2 ={Tail} are collectively exhaustive, because together they comprise the all the outcomes that are possible in a coin tossing experiment. There are no other possible outcomes of this experiment than these two.

Equally likely events:

Equally likely events are events that have the same probability or likelihood of occurring.

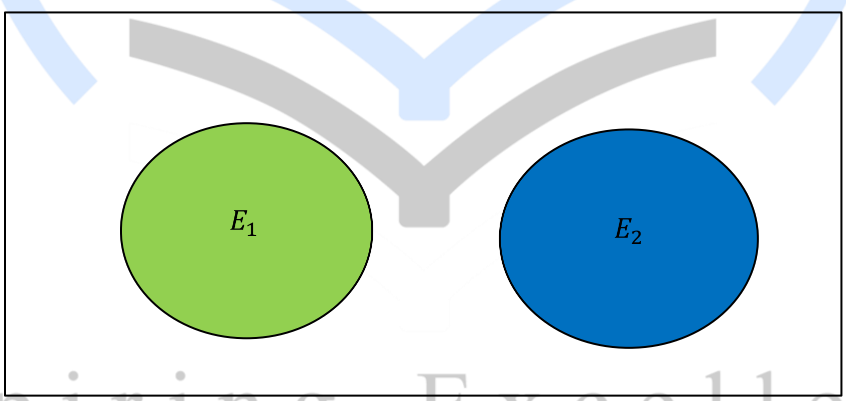
Each numeral on a die is equally likely to occur when the die is tossed.

The sample space of throwing a die is, S = { 1, 2, 3, 4, 5, 6 } and the probability of getting a chosen numeral = $\frac{1}{6}$. Here the chance of occurring each numeral is the same and so they are equally likely events.

Disjoint events:

Two events are called disjoint, if they have no common elements between them.

Mutually exclusive events are disjoint events.

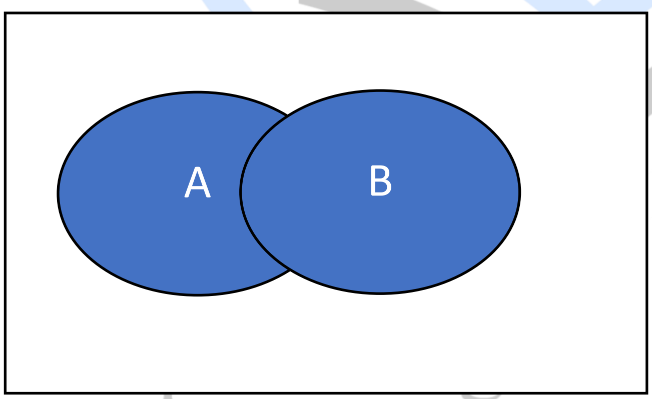


Dependent Events:

If two events have some common elements then both of these events are referred as dependent events.

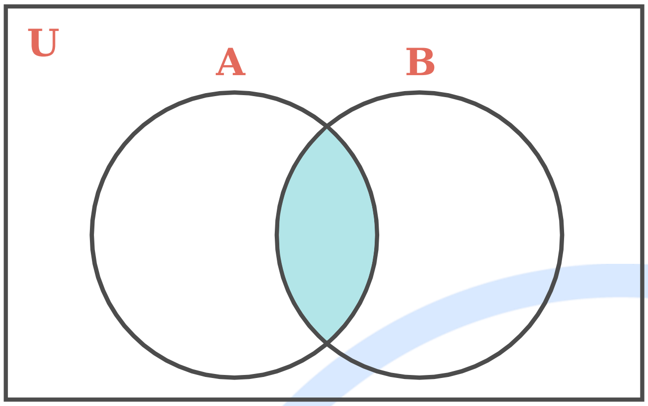
For example, in a deck of 52 cards, if E_1 be an event of selecting 'Red' card and E_2 be another event of selecting 'Queen' card then these events are joint events since there are two 'Queen' cards in a set of 'Red' cards.

- **Union**: The union of two sets contains all the elements contained in either set (or both sets). The union is notated $A \cup B$, *where A and B are two sets.*
- **Intersection**: The intersection of two sets contains only the elements that are in both sets. The intersection is notated $A \cap B$.
- **Compliment**: The complement of a set *A* contains everything that is *not* in the set *A*. The complement is notated *A'*, or *Ac*, or sometimes *~A*.



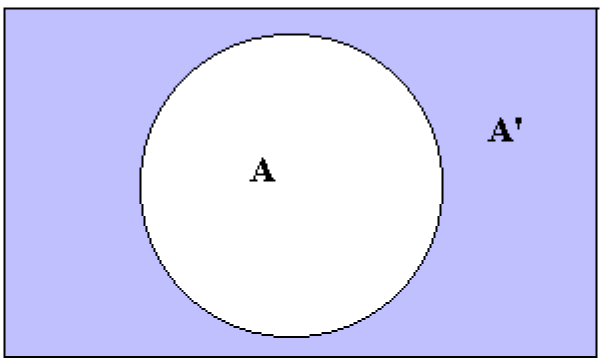
A union B

Elements that belong to either A, B or both



A intersect B

Elements that belong to both A & B



A compliment

Elements that don't belong to A

Independent Events:

Two events are known as independent events if the occurrence of one event does not affect the probability of occurring another event.

For two independent events A and B, the probability that A and B will both occur is found by multiplying the two probabilities.

$$P(A \text{ and } B) = P(A)\,P(B)$$

For example; In a coin tossing experiment events,

E_1 = {Head} and event E_2 ={Tail} are independent events because in this case occurrence of event, E_1 does not affect event, E_2.

Approaches of Assigning Probability

- At first we identify the sample space S of the random experiment.
- We then define our favorable event and assign probability to the event using one of the following 3 basic approaches:

- Classical approach
- Frequency approach
- Subjective approach

Classical approach:

If a random experiment has a total of n(S) possible outcomes, all of which are mutually exclusive, equally likely and collectively exhaustive, such that n(A) of the outcomes are favorable to an event A, then the probability of the event A is defined by,

$$P(A) = \frac{n(A)}{n(S)}$$

Frequency approach:

If an experiment is repeated n times under the same conditions and event E occurs f times out of n times, then

$$P(E) = \lim_{n\to\infty} \frac{f}{n}$$

That is, when n is very large, P(E) is very close to the relative frequency of event E.

For example;

In a dice throwing experiment- S= {1, 2, 3, 4, 5, 6}

And our favorable event is E= {2}

Let, 2 occurred a total of 998 times out of total 6000 trials. Therefore $P(E) = \lim_{n\to\propto} \frac{998}{6000} \approx \frac{1}{6}$

Subjective approach:

Subjective probability is the probability that an individual assigns to an event, E of a random experiment on the basis of his/her experience, judgement, concept, intuition, information and beliefs.

For example; on a day of summer someone made a statement on probability that rain will occur on that day is 0.70, based on his previous experience.

Axioms of Probability

Valid probabilities will follow 3 axioms-

Axiom 1: (Axiom of positivizes): $0 \leq P(E) \leq 1$

Axiom 2: (Axiom of certainty): P(S) = 1

Axiom 3: (Axiom of additivity): For a sequence of disjoint events $E_1, E_2, \ldots, E_n$ -

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

Example 1:

A person holds ticket in a lottery that offers 10 prizes and sells 120 tickets. What is the probability that the person will not win a prize?

Solution:

Let A be the event of winning a prize.

$$\text{Here, } P(A) = \frac{n(A)}{n(S)} = \frac{10}{120} = \frac{1}{12}$$

$$\text{Thus, } P(A^c) = 1 - \frac{1}{12} = \frac{11}{12}$$

So, the probability that the person will not win a prize is $\frac{11}{12}$.

Example 2:

A dice is thrown in an experiment. What is the probability that an even no will occur?

Solution:

The sample space for the experiment is, S = { 1, 2, 3, 4, 5, 6 }

Let the event is E = {2, 4, 6 }

Here, n(E) = 3 and n(S) = 6

Therefore, the probability of occurring the event, E

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2} = 0.510$$

Addition Laws

- **For disjoint events A and B-**

The probability that, either event A or event B will occur is,

$$P(A \cup B) = P(A) + P\ (B\ )$$

- **For disjoint events A, B, C, … , and Z**

The probability that, either event A or event B or event C or … or event Z will occur is,

$$P(A \cup B \cup C \cup \cdots \cup Z) = P(A) + P(B) + P(C) + \ldots + P(Z)$$

- **For joint events A and B-**

The probability that, either event A or event B or both will occur is,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **For joint events A, B, and C**

The probability that, either event A or event B or event C or any two of them or all will occur is,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C)$$

$$+ P(A \cap B \cap C)$$

Example 3:

In a company, 60% of the employees have motorcycle, 40% has private car and 20% has both.

If an employee is selected randomly from that company, then

a) What is the probability that the employee has either motorcycle or private car?

b) What is the probability that the employee has neither motorcycle nor private car?

Solution:

Let, M= the randomly selected employ has motorcycle

C= the randomly selected employee has car

$$\text{Here, } P\ (M) = \frac{60}{100} = 0.6,$$

$$P(C) = \frac{40}{100} = 0.4$$

$$P(M \cap C) = \frac{20}{100} = 0.2$$

a. Probability that the person has either motorcycle or private car is,

$$P(M \cup C) = P(M) + P(C) - P(M \cap C)$$

$$= 0.6 + 0.4 - 0.2$$

$$= 0.8$$

b. Probability that the person has neither motorcycle nor private car is

$$P(M \cup C)^c = 1 - P(M \cup C)$$

$$= 1 - 0.8$$

$$= 0.2$$

Conditional probability:

Conditional probability is defined as the likelihood of an event or outcome occurring, based on the occurrence of a previous event or outcome. The probability that event A occurs, given that event B has occurred, is called a conditional probability.

The conditional probability of A, given B, is denoted by the symbol P(A|B).

$$P(\mathrm{A}|\mathrm{B}) = \frac{P(\mathrm{A} \cap \mathrm{B})}{P(\mathrm{B})}; \quad for\ P(\mathrm{B}) > 0$$

So, we can write, $P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{A}|\mathrm{B})\, P(\mathrm{B})$ (Product Rule)

Or, $P(\mathrm{A} \cap \mathrm{B}) = \mathrm{P(B|A)\ P(A)}$ ((Product Rule)

Example 4:

In a class of 120 students, 60 are studying English, 50 are studying French and 20 are studying both English and French. If a student is selected at random from this class, what is the probability that he or she is studying English given that he is studying French.

Solution:

$$\text{Here, } P(E) = \frac{60}{120} = 0.5$$

$$P(F) = \frac{50}{120} = 0.42$$

$$P(E \cap F) = \frac{20}{120} = 0.17$$

The probability that s/he is studying English given that s/he is studying French is,

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{20}{50} = 0.4$$

Conditional Probability: Chain Rule

- The chain rule permits the calculation of any member of the joint distribution of a set of random variables using only conditional probabilities.
- The chain rule is useful in the study of Bayesian networks which describe a probability distribution in terms of conditional probabilities.

The conditional probability of A, given B is,

$$P(\mathrm{A}|\mathrm{B}) = \frac{P(\mathrm{A} \cap \mathrm{B})}{P(\mathrm{B})}; \quad for\ P(\mathrm{B}) > 0$$

So we can write,

$$P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{A}|\mathrm{B})\ P(\mathrm{B})$$

Similarly, for events A, B & C,

$$\mathrm{P}(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C}) = P(\mathrm{A})\ P(\mathrm{B}|\mathrm{A})\ \mathrm{P}(\mathrm{C}|\mathrm{A} \cap \mathrm{B})$$

The general form of the chain rule for $E_1,\ E_2,\ E_3, \ldots.. E_{n-1}, E_n$ events is,

$$\mathrm{P}\,(E_1 \cap E_2 \cap \ldots \ldots \cap E_n) = \mathrm{P}(E_1)\ \mathrm{P}(E_2|\ E_1) \ldots..\ \mathrm{P}(E_n|\ E_1\ E_2 \ldots..\ E_{n-1})$$

Multiplication Laws

- **For two dependent events A and B**

The probability that, both event A and event B will occur simultaneously is,

$$P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{A}|\mathrm{B})\ P(\mathrm{B})$$

Here, occurrence of event A depends on occurrence of event B.

- **For two independent events A and B**

The probability that, both event A and event B will occur simultaneously is,

$$P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{A})\ P(\mathrm{B})$$

Example 5:

In rainy season, it rains 70% of the days in Bangladesh. When it rains, 80% times it makes thunderstorms. What is the probability that, in a particular day of rainy season, it will rain and it will thunderstorm?

Solution:

Let, R= it will rain on that particular day

T= it will thunderstorm on that particular day

Here, given that, $P(R) = \frac{70}{100} = 0.7$ and

$$P(T|R) = \frac{80}{100} = 0.8$$

Therefore, the probability that, on that particular day of rainy season, it will rain and it will thunderstorm is-

$$P(R \cap T) = P(T|R)\, P(R)$$

$$= 0.8 * 0.7$$

$$= 0.56$$

Example 6:

A jar contains 3 red, 5 green, 2 blue and 6 yellow marbles. A marble is chosen at random from the jar. After replacing it, a second marble is chosen. What is the probability of choosing a green and then a yellow marble?

Solution:

Let,

G = Green marble will be chosen

Y = Yellow marble will be chosen

Here,

$P(G) = \frac{5}{16}$

$P(Y) = \frac{6}{16}$

Then, the probability of choosing a green and then a yellow marble is,

$$P(G \cap Y) = P(G) * P(Y)$$

$$= \frac{5}{16} * \frac{6}{16}$$

$$= \frac{30}{256}$$

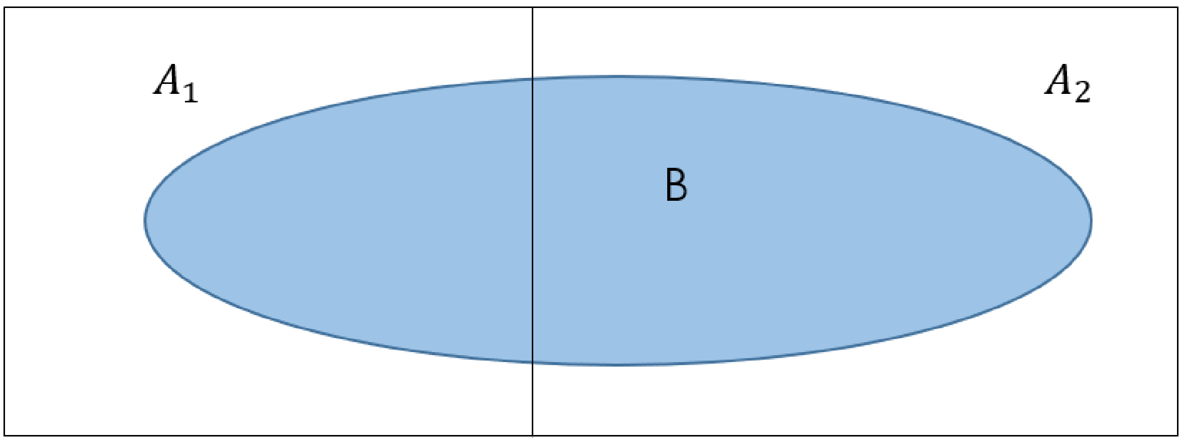
$$= \frac{15}{128}$$

Law of Total Probability

Let, events A_1 and A_2 form partition of S. Let B be an event with P(B)>0. Then,

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B)$$

$$= P(A_1)\, P(B|A_1) + P(A_2)\, P(B|A_2)$$

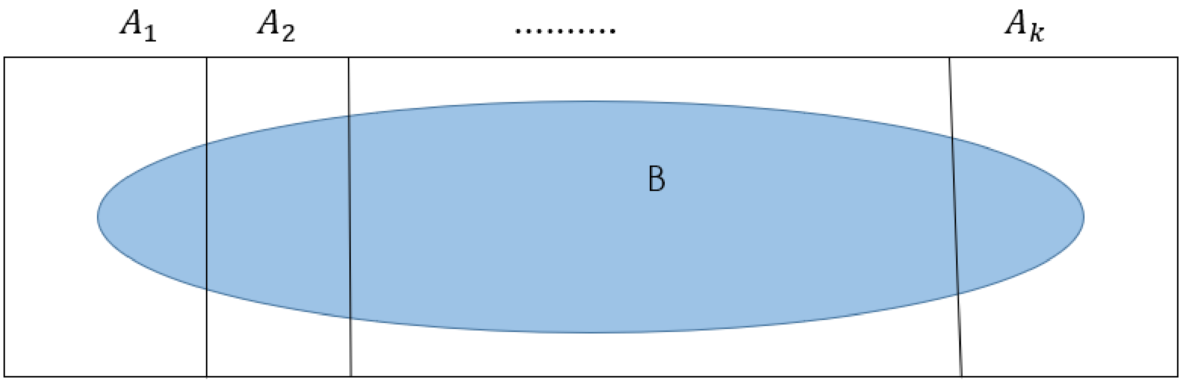


Let, events $A_1, A_2, \ldots, A_k$ form partition of S. Let B be an event with P(B)>0. Then,

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots\ldots + P(A_k \cap B)$$

$$= P(A_1)\, P(B|A_1) + P(A_2)\, P(B|A_2) + \ldots\ldots + P(A_k)\, P(B|A_k)$$

$$= \sum_{i=1}^{k} P(A_i)\, P(B|A_i)$$


A_1
A_2
....
A_k
B