Mandatory Assignment
MAT2400
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Problem 1. *Assume* $f : \mathbb{N} \to \mathbb{N}$ *is a strictly increasing function. Show that* f *satisfies* $f(n) \geq n$ *for all* n.

Solution 1. This obviously holds for $n = 1$, since 1 is the least element of $\mathbb{N}$. Assume now that we have shown $f(n) \geq n$ for all n up to some $n = k$. Since $f(k+1) > f(k)$, by the assumption that f is strictly increasing we can easily see that $f(k+1) - f(k) > 1$. This is because for two elements to be different in $\mathbb{N}$, they must differ by at least 1.

$$f(k+1) > f(k) \geq k \Longrightarrow f(k+1) \geq k+1.$$

Thus it must neccessarily hold for $n = k+1$ as well. By induction we have now shown that f satisfies $f(n+1) \geq n$ for all n.

Problem 2. *Assume* (X, d) *a complete metric space. Prove or disprove that the closure of an open ball* $\overline{B(x, r)}$ *is equal to the closed ball* $\overline{B}(x, r)$ *for all complete metric spaces.*

Solution 2. We can disprove the above by looking at a complete space under the discrete metric. Let us first introduce the metric $d : X \times X \to \{0, 1\}$, defined as

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

To construct a counter example, assume that we look at open at closed balls around a point x, with a radius $r = 1$. By observation we see that the closure of the open ball around x is just x itself,

$$\overline{B(x, r)} = x$$

where as the closed ball around x is the whole of X.

$$\overline{B}(x, r) = X.$$

We have now, using the discrete metric d and a radius $r = 1$ constructed a metric space where the statement initially given does not hold.

Problem 3. *Define ℓ to be the set of all sequences of real numbers where only a finite number of elements are non-zero. Furthermore, for $x = \{x_n\}$ and $y = \{y_n\}$ in ℓ, define*

$$d(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n|.$$

Solution 3.

a) To show that d defines a metric on ℓ we have to show the three properties a metric should satisfy. These are *positivity, symmetry* and *the triangle inequality.* The first two, positivity and symmetry follow directly from the definition of absolute value. The triangle inequality is a bit trickier to show. We want d to satisfy the following relation:

$$d(x, z) \leq d(x, y) + d(y, z).$$

Written out, this becomes:

$$\sup_{n \in \mathbb{N}} |x_n - z_n| \leq \sup_{n \in \mathbb{N}} |x_n - y_n| + \sup_{n \in \mathbb{N}} |y_n - z_n|.$$

We know that for some $i, j, k \in \mathbb{N}$:

$$\begin{aligned} d(x, z) &= |x_i - z_i| \\ d(x, y) &= |x_j - y_j| \\ d(y, z) &= |y_k - z_k|. \end{aligned}$$

Intuitively, these absolute values should obey the triangle inequality for real numbers, but I can't say I've managed to completely convince myself. I have therefore not been able to construct a full fledged proof for d being a metric on ℓ.

b) Let $u_k \in \ell$ be defined as

$$u_k = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{k}, 0, 0, 0, \cdots\right\}.$$

To show that $\{u_k\}_{k=1}^{\infty}$ is Cauchy we want to show that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m, n > N$, it is implied that $d(x_n, x_m) < \varepsilon$.

Given two sequences u_n and u_m, the distance between them is given as following (assuming $m > n$).

$$
\begin{aligned}
d(u_n, u_m) &= \sup_{i\in\mathbb{N}} |u_{n_i} - u_{m_i}| \\
&= |u_{n_{n+1}} - u_{m_{n+1}}| \\
&= \frac{1}{n+1}.
\end{aligned}
$$

Since a number on this form can me made as arbitrarily close to zero by chosing n arbitrarily large, we have shown that no matter what $\varepsilon > 0$ we are given, we can always find an N such that $m, n > N \implies d(u_m, u_n) < \varepsilon$. Because of this, $\{u_k\}_{n=1}^{\infty}$ is Cauchy.

c) For $\{u_k\}_{k=1}^{\infty}$ to converge we would have to find a sequence in ℓ it converges to. If $\{u_k\}_{k=1}^{\infty}$ were to converge it would have to be to the harmonic sequence, $\left\{\frac{1}{k}\right\}_{k=1}^{\infty}$ because it is the only sequence where elements are equal to the elements of the sequence $\{u_k\}_{k=1}^{\infty}$. Since the the sequence neccessarily has to converge to a single unique point we know that the harmonic sequence is the only such point. But since the harmonic sequence is not a member of ℓ it does not converge.

d) Define c_0 to be the space of sequences of real numbers whose limit are 0.

$$
c_0 = \left\{ \{x_n\}_{n=1}^{\infty} \mid \lim_{n\to\infty} x_n = 0 \right\}.
$$

To show that (c_0, d) is a complete metric space, we must show that all Cauchy sequences in (c_0, d) converge.

We know by definition of c_0 that all its elements are sequences of real numbers that converge to 0. If a sequence of real numbers converges to zero, it is also Cauchy. Therefore, it follows that c_0 is a complete metric space.[1]

e) By the definition of c_0 it is clear that $\ell \subset c_0$. ℓ is dense in c_0 if and only if for every $x \in c_0$ and for all $\varepsilon > 0$ there exists a $y \in \ell$ such that $d(x, y) < \varepsilon$. Intuitively this seems reasonable, since we know a sequence x converges to 0, and y contains an infinite number of 0's.

$$
\begin{aligned}
x &= \{x_1, x_2, x_3, \ldots\} \longrightarrow 0 \\
y &= \{y_1, y_2, y_3, \ldots, y_n, 0, 0, 0, \ldots\} \longrightarrow 0
\end{aligned}
$$

Thus, we can always chose the sequences we need to satisfy the definition of *dense*. Therefore, ℓ is dense in c_0.[2]

The limit of the sequence u_k, containing an infinite amount of zeroes, is zero. That is $u_k \longrightarrow 0$. This means that $u_k \in c_0$.

[1] Captain Hindsight: What I've written here does not actually prove anything, but I haven't had time to correct it. I've looked at the convergence of the elements of the sequence, rather than the sequence itself.

[2] Captain Hindsight strikes again. This is also a bit vague, and could've used further elaboration.

Problem 4. *Let $X = (0, \infty) \subset \mathbb{R}$ and let $d : X \times X \longrightarrow \mathbb{R}$ be defined as*

$$d(x, y) = |\ln(x) - \ln(y)|.$$

Solution 4.

a) We want to show that d is a metric, and that (X, d) is complete. First, we need to show the three properties of a metric function. The first two, positivity and symmetry are trivial. For the triangle inequality, we observe the following:

$$\begin{aligned} d(x, z) &= |\ln(x) - \ln(z)| = |\ln(x) + \ln(y) - \ln(y) - \ln(z)| \\ &\leq |\ln x - \ln(y)| + |\ln(y) - \ln(z)| \\ &= d(x, y) + d(y, z). \end{aligned}$$

Thus d is a metric on X. To show that (X, d) is complete, we must show that all Cauchy sequences converge.

Assume that $x \in X$ is Cauchy. This means that given a $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $m, n > N \implies d(x_m, x_n) < \varepsilon$. In our metric d this means that the values of x_m, x_n become arbitrarily close to each other, which in turn means, that there exists a point a such that $d(x_m, a) < \varepsilon/2$ and $d(x_n, a) < \varepsilon/2$. Therefore x converges to a. Since x was any arbitrary Cauchy sequence in X this holds for all Cauchy sequences in X, and therefore X under the metric d is a complete space.

b) Given a differentiable function $f : X \longrightarrow X$ that satisfy the following:

$$x\,|f'(x)| \leq kf(x), \quad 0 < k < 1. \tag{1}$$

We want to show that f has a unique fixed point. If we can show f to be a contraction, it follows from the completeness of X that we can apply Banachs fix point theorem. For f to be a contraction, we want it to satisfy the following inequality,

$$d(f(x), f(y)) < kd(x, y), \quad 0 < k < 1$$

If we define an auxilliary function $h(t)$ as

$$h(t) = \frac{d}{dt} \ln(f(t)) = \frac{f'(x)}{f(x)},$$

we can use $h(t)$ to show the contraction property of f. Rearranging (1) we get

$$\frac{|f'(x)|}{f(x)} \leq \frac{k}{x}.$$

We then see that we can rephrase the contraction property of f in terms of $h(t)$.

$$\left| \int_x^y h(t)\,dt \right| \leq \left| \int_x^y \frac{k}{t}\,dt \right|.$$

Evaluating this, we get the inequality we wanted f to satisfy, namely

$$d(f(x), f(y)) < kd(x, y), \quad 0 < k < 1$$

Therefore, by Banach's fix point theorem, f has one unique fixed point.