

UNIVERSITY OF OSLO

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# Chapter summaries in MAT2400 - Real analysis

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# Chapter 1

## Preliminaries: Proofs, Sets, and Functions

### 1.1 Proofs

#### Implication

Many mathematical statements are on the form *If A, then B*. These are denoted  $A \Rightarrow B$ . If  $A \Rightarrow B$  then it is not necessarily so that  $B \Rightarrow A$ . That is, these two mean different things. Note that  $A \Rightarrow B$  is logically equivalent to  $\sim B \Rightarrow \sim A$ .

#### Equivalence

If it is true that  $A \Rightarrow B$  and  $B \Rightarrow A$ . Then we call these statements equivalent. This is denoted  $A \Longleftrightarrow B$ . When proving that two statements are equivalent, it is often easier to prove that they both imply each other. That is, by proving  $A \Rightarrow B$  and  $B \Rightarrow A$ , you have proven that  $A \Longleftrightarrow B$ .

#### Methods of proof

There are several ways of proving mathematical hypotheses.

1. Instead of proving  $A \Rightarrow B$ , prove  $\sim B \Rightarrow \sim A$ . This is called a *contrapositive proof*.
2. Another common method of proof is *proof by contradiction*. Assume the opposite of what you want to prove, and by showing that this leads to a contradiction, our assumption must be false, and hence the original hypothesis is true.
3. When dealing with natural numbers, the go-to method of proof is *proof by induction*. Show that a statement holds for a given number, and then show that if it holds for an arbitrary number  $n$ , it must also hold for the successor  $n + 1$ .

This list is of course non-exhaustive, but these are some of the more common methods one should consider when faced with a problem to solve.

## **1.2 Sets and boolean operations**

A set is a collection of mathematical objects. It may be finite, it may be infinite.  $A$  is a subset of  $B$  if all the elements in  $A$  are also in  $B$ . This is denoted  $A \subseteq B$ . Two sets are equal if they contain exactly the same elements. This is denoted  $A = B$ , where  $A$  and  $B$  are sets. If  $A$  is a subset of  $B$  and  $B$  is a subset of  $A$ , the sets are equal. The set that contains no elements is called the empty set and is denoted  $\emptyset$ .

## **1.3 Families of sets**

## **1.4 Functions**

## **1.5 Relations and partitions**

## **1.6 Countability**

## Chapter 2

# Metric Spaces

### 2.1 Definitions and examples

**Definition 1.** A metric space  $(X, d)$  consists of a non-empty set  $X$  and a function  $d : X \times X \rightarrow [0, \infty)$  such that:

1. (Positivity)  $\forall x, y \in X, d(x, y) \geq 0$  with equality if and only if  $x = y$
2. (Symmetry)  $\forall x, y \in X, d(x, y) = d(y, x)$
3. (Triangle inequality)  $\forall x, y, z \in X$

$$d(x, y) \leq d(x, z) + d(y, x)$$

A function  $d$  satisfying conditions 1-3, is called a metric on  $X$ .

**Definition 2.** Assume that  $(X, d_X)$  and  $(Y, d_Y)$  are metric spaces. An isometry from  $(X, d_X)$  to  $(Y, d_Y)$  is a bijection  $i : X \rightarrow Y$  such that  $d_X(x, y) = d_Y(i(x), i(y))$  for all  $x, y \in X$ . We say that  $(X, d_X)$  and  $(Y, d_Y)$  are isometric if there exists an isometry from  $(X, d_X)$  to  $(Y, d_Y)$ .

### 2.2 Convergence and Continuity

### 2.3 Open and closed sets

### 2.4 Complete spaces

### 2.5 Compact Sets

### 2.6 An alternative description of compactness

### 2.7 The completion of a metric space

## **Chapter 3**

# **Space of continuous functions**

**3.1 Modes of continuity**

**3.2 Modes of convergence**

**3.3 The spaces  $C(X, Y)$**

**3.4 Application to differential equations**

**3.5 Compact subsets of  $C(X, \mathbb{R}^m)$**

**3.6 Differential equations revisited**

**3.7 Polynomials are dense in  $C([a, b], \mathbb{R})$**

**3.8 Baire's Category Theorem**

## **Chapter 4**

# **Series of functions**

**4.1**  $\limsup$  and  $\liminf$

**4.2** Integrating and differentiating sequences

**4.3** Power series

**4.4** Abel's Theorem

**4.5** Normed spaces

**4.6** Inner product spaces

**4.7** Linear operators



## **Chapter 5**

# **Measure and integration**

**5.1 Measure spaces**

**5.2 Complete measures**

**5.3 Measurable functions**

**5.4 Integration of simple functions**

**5.5 Integrals of nonnegative functions**

**5.6 Integrable functions**

**5.7  $L^1(X, A, \mu)$  and  $L^2(X, A, \mu)$**

## **Chapter 6**

# **Constructing measures**

**6.1 Outer measure**

**6.2 Measurable sets**

**6.3 Carathéodory's Theorem**

**6.4 Lebesgue measure on  $\mathbb{R}$**

**6.5 Approximation results**

**6.6 The coin tossing measure**

**6.7 Product measures**

**6.8 Fubini's Theorem**