Mandatory Assignment — MAT2400

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Problem 1

In this problem we consider the series given by

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2x}. \tag{1}$$

First and foremost, we are interested in under what values of x this series converge. We see that by setting $x = 0$, then the series read $1 + 1 + 1 + \dots$ which clearly sums to infinity. So we can conclude that for $x = 0$ the series diverges. For $x > 0$ we now have a contribution from the n's again. We know that $1/(1+n^2x)$ is certainly smaller than $1/(n^2x)$ for all $x > 0$. Hence we have the following inequality:

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2x} < \sum_{n=1}^{\infty} \frac{1}{n^2x} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{M}{x}.$$

Hence no matter what x we chose, the limit is always smaller than M/x.[1] Hence the series converges for any $x > 0$.

We are now interested in whether the convergence is uniform or not on the interval $[a, \infty)$. Weierstrass' M-test immediately tells us that the convergence is uniform for the interval under the condition that $a \geq 1$. The problem lies in the area between zero and one.

[1] M was shown by Euler to be equal to $\pi^2/6$.