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Chapter summaries in MAT2400 - Real analysis

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Contents

Chapter 1

Preliminaries: Proofs, Sets, and Functions

1.1 Proofs

Implication

Many mathematical statements are on the form *If A, then B*. These are denoted $A \Rightarrow B$. If $A \Rightarrow B$ then it is not neccessarily so that $B \Rightarrow A$. That is, these two mean different things. Note that $A \Rightarrow B$ is logically equivalent to $\sim B \Rightarrow \sim A$.

Equivalence

If it is true that $A \Rightarrow B$ and $B \Rightarrow A$. Then we call these statements equivalent. This is denoted $A \Longleftrightarrow B$. When proving that two statements are equivalent, it is often easier to prove that they both imply eachother. That is, by proving $A \Rightarrow B$ and $B \Rightarrow A$, you have proven that $A \Longleftrightarrow B$.

Methods of proof

There are several ways of proving mathematical hypotheses.

1. Instead of proving $A \Rightarrow B$, prove $\sim B \Rightarrow \sim A$. This is called a *contrapositive proof.*
2. Another common method of proof is *proof by contradiction.* Assume the opposite of what you want to prove, and by showing that this leads to a contradiction, our assumption must be false, and hence the original hypothesis is true.
3. When dealing with natural numbers, the go-to method of proof is *proof by induction.* Show that a statement holds for a given number, and then show that if it holds for an arbitrary number n, it must also hold for the successor $n+1$.

This list is of course non-exhaustive, but these are some of the more common methods one should consider when faced with a problem to solve.

1.2 Sets and boolean operations

A set is a collection of mathematical objects. It may be finite, it may be infinite. A is a subset of B if all the elements in A are also in B. This is denoted $A \subseteq B$. Two sets are equal if they contain exactly the same elements. This is denoted $A = B$, where A and B are sets. If A is a subset of B and B is a subset of A, the sets are equal.
The set that contains no elements is called the empty set and is denoted $\emptyset$.

1.3 Families of sets

1.4 Functions

1.5 Relations and partitions

1.6 Countability

Chapter 2

Metric Spaces

2.1 Definitions and examples

Definition 1. A metric space (X,d) consists of a non-empty set X and a function $d : X \times X \to [0,\infty)$ such that:

1. (Positivity) $\forall x, y \in X, d(x, y) \geq 0$ with equality if and only if $x = y$
2. (Symmetry) $\forall x, y \in X, d(x, y) = d(y, x)$
3. (Triangle inequality) $\forall x, y, z \in X$

$$d(x, y) \leq d(x, z) + d(y, x)$$

A function d satisfying conditions 1-3, is called a metric on X.

Definition 2. Assume that (X, d_X) and (Y, d_Y) are metric spaces. An isometry from (X, d_X) to (Y, d_Y) is a bijection $i : X \to Y$ such that $d_X\left(x, y\right) = d_Y\left(i(x), i(y)\right)$ for all $x, y \in X$. We say that (X, d_X) and (Y, d_Y) are isometric if there exists an isometry from (X, d_X) to (Y, d_Y).

2.2 Convergence and Continuity

2.3 Open and closed sets

2.4 Complete spaces

2.5 Compact Sets

2.6 An alternative description of compactness

2.7 The completion of a metric space

Chapter 3

Space of continuous functions

3.1 Modes of continuity

3.2 Modes of convergence

3.3 The spaces $C(X, Y)$

3.4 Application to differential equations

3.5 Compact subsets of $C(X, \mathbb{R}^m)$

3.6 Differential equations revisited

3.7 Polynomials are dense in $C([a, b], \mathbb{R})$

3.8 Baire's Category Theorem

Chapter 4

Series of functions

4.1 $\limsup$ and $\liminf$

4.2 Integrating and differentiating sequences

4.3 Power series

4.4 Abel's Theorem

4.5 Normed spaces

4.6 Inner product spaces

4.7 Linear operators

Chapter 5

Measure and integration

5.1 Measure spaces

5.2 Complete measures

5.3 Measurable functions

5.4 Integration of simple functions

5.5 Integrals of nonnegative functions

5.6 Integrable functions

5.7 $L^1\left(X, A, \mu\right)$ and $L^2\left(X, A, \mu\right)$

Chapter 6

Constructing measures