

Mandatory assignment 2

MAT2400

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Problem 1

a) We want to show that $h(A, B) = 0$ implies that $A = B$. We assume that A and B are two non-empty closed subsets of X . In other words, $A, B \in P(X)$. If we can show the contrapositive, then the original implication must hold. We therefore assume that $A \neq B$ and we want to show that then $h(A, B) \neq 0$.

Let $x \in X$ be such that $x \in A$ and $x \notin B$. By the definition of $\text{dist}(x, A)$ and $\text{dist}(x, B)$ we see that since $x \in A$, $\text{dist}(x, A) = 0$ and since x is not in B we have $\text{dist}(x, B) > 0$ by the definition of d . Since we take the supremum of all of these values we must have that

$$\sup_{x \in X} |\text{dist}(x, A) - \text{dist}(x, B)| \geq \text{dist}(x, B) > 0.$$

Hence, $h(A, B) = 0$ implies that $A = B$.

b) We now want to show that h is a metric on $P(X)$. Positivity and symmetry follows directly from the absolute values in the expression for $h(A, B)$. We therefore need to show that h obeys the triangle inequality.

$$\begin{aligned} h(A, B) &= \sup_{x \in X} |\text{dist}(x, A) - \text{dist}(x, B)| \\ &= \sup_{x \in X} |\text{dist}(x, A) - \text{dist}(x, C) + \text{dist}(x, C) - \text{dist}(x, B)| \\ &\leq \sup_{x \in X} (|\text{dist}(x, A) - \text{dist}(x, C)| + |\text{dist}(x, C) - \text{dist}(x, B)|) \\ &\leq \sup_{x \in X} |\text{dist}(x, A) - \text{dist}(x, C)| + \sup_{x \in X} |\text{dist}(x, C) - \text{dist}(x, B)| \\ &= h(A, C) + h(C, B) \end{aligned}$$

Hence h is a metric on $P(X)$.

c) We want to show the two inequalities

$$h(A, B) \geq \hat{h}(A, B), \tag{1}$$

$$\hat{h}(A, B) \geq h(A, B). \tag{2}$$

Problem 2

a)

Convergence

We want to show that the series

$$\Upsilon = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}$$

converges uniformly for all $x \in \mathbb{R}$ and that its sum equals $P_r(x)$. We observe that we can split Υ into three subseries:¹

$$\Upsilon = 1 + \sum_{n=1}^{\infty} r^n e^{inx} + \sum_{n=1}^{\infty} r^n e^{-inx}.$$

Applying Eulers formula, we can simplify this to the series²

$$\Upsilon = 1 + \sum_{n=1}^{\infty} r^n \cos(nx) + \sum_{n=1}^{\infty} r^n \cos(nx) = 1 + 2 \sum_{n=1}^{\infty} r^n \cos(nx).$$

We know that since we have $0 < r < 1$ and $\cos(nx) \leq 1$ that

$$\Upsilon \leq 1 + 2 \sum_{n=1}^{\infty} r^n,$$

but since this series converges, so must Υ .

Sum

We want to show that the sum of Υ is equal to

$$P_r(x) = \frac{1 - r^2}{1 - 2r \cos(x) + r^2}.$$

We rewrite Υ to the following form, in order for us to be able to apply the formula for finite geometric series:³

$$\Upsilon = \lim_{N \rightarrow \infty} \left(\sum_{n=0}^N (e^{ix} r)^n + \sum_{n=0}^N (e^{-ix} r)^n - 1 \right).$$

We then get that the sum of Υ is equal to

$$\Upsilon = \lim_{N \rightarrow \infty} \left(\frac{1 - (e^{ix} r)^{N+1}}{1 - e^{ix} r} + \frac{1 - (e^{-ix} r)^{N+1}}{1 - e^{-ix} r} - 1 \right).$$

Applying Eulers formula again which yields⁴

$$\Upsilon = \frac{1 - r^2}{1 - 2r \cos(x) + r^2} = P_r(x).$$

¹Note the change of summation index and the removal of absolute values

²The imaginary terms cancel out

³Notice again the change of summation index. That is where the -1 originates from.

⁴Some algebra required

b) We want to show that $P_r(x)$ is positive or zero for all x . We have

$$\begin{aligned} P_r(x) &= \frac{1-r^2}{1-r2\cos(x)+r^2} \\ &\geq \frac{1-r^2}{1-2r+r^2} \\ &= \frac{(1-r)(1+r)}{(1-r)^2} = \frac{1+r}{1-r} > 1. \end{aligned}$$

Therefore, we can conclude that no matter what x is, $P(x)$ is greater or equal to zero.

c) We want to show that $P_r(x) \rightarrow 0$ as $r \uparrow 1$ on the interval $X = [-\pi, -\delta] \cup [\delta, \pi]$. We first state the definition of uniform convergence.⁵

A sequence $\{f_n\}$ of functions converges uniformly to a function f if and only if for all $\varepsilon > 0$ there exists an $N > 0$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$ and for all $n \geq N$.

In our case, we need to rewrite $P_r(x)$ to a form which we can express in terms of a natural number n . We want to create the sequence $\{r_n\}$ of rational numbers defined as

$$r_n = \frac{n-1}{n}.$$

This series converges to 1 as $n \rightarrow \infty$. We now want to show that for all $\varepsilon > 0$ there exists an $N > 0$ such that what was stated above holds. That is

$$|P_{r_n}(x) - 0| < \varepsilon,$$

for all $x \in X$ and for all $n > N$. Writing it out and taking the limit we see that

$$\lim_{n \rightarrow \infty} \frac{1 - \left(\frac{n-1}{n}\right)^2}{1 - \left(\frac{n-1}{n}\right)\cos(x) + \left(\frac{n-1}{n}\right)^2} = \frac{0}{2 - 2\cos(x)} = 0 < \varepsilon.$$

We have $\cos(x) \neq 1$ for all $x \in X$ since $0 \notin X$, therefore our proof is done.

d) We want to show that

$$\int_{-\pi}^{\pi} P_r(x) dx = 2\pi.$$

Recall that

$$P_r(x) = 1 + 2 \sum_{n=1}^{\infty} r^n \cos(nx).$$

We can therefore write the integral as follows

$$\int_{-\pi}^{\pi} \left(1 + 2 \sum_{n=1}^{\infty} r^n \cos(nx) \right) dx = \int_{-\pi}^{\pi} 1 dx + 2 \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} (r^n \cos(nx)) \right) dx$$

⁵This is a bit informal, but used just as a reminder

We know that since $P_r(x)$ converges uniformly on all of \mathbb{R} the sum and integrals are interchangeable.⁶ Therefore we can rewrite this expression as

$$\int_{-\pi}^{\pi} 1 \, dx + 2 \sum_{n=1}^{\infty} \left(\int_{-\pi}^{\pi} r^n \cos(nx) \, dx \right).$$

The integral contained in the sum evaluates to zero, therefore we are left with

$$\int_{-\pi}^{\pi} P_r(x) \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi,$$

and we are done.

e) We want to show that

$$\Lambda = \lim_{r \uparrow 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) P_r(y) \, dy = f(x).$$

We observe that we are working with a *convolution*.⁷ Since the convolution is commutative (it does not matter which function we slide along the y axis) we can rewrite our

⁶Corollary 4.2.3

⁷<http://en.wikipedia.org/wiki/Convolution>