Chapter 1

Metric Spaces

1.1 Definitions and examples

Proposition 1 (Inverse Triangle Inequality)**.** *For all elements x, y, z in a metric space (X, d), we have:*

$$|d(x, y) - d(x, z)| \leq |d(y, z)|$$

Proof. Let (X, d) be a metric space, and let x, y, z be three aribtrary elements in X. From the triangle inequality, that we know d satisfy from the definition of a metric space, we have:

$$d(x, y) \leq d(x, z) + d(z, y).$$

Now, in order to show that the inverse triangle inequality holds, note that the absolute value involved is going to be the largest of the two numbers $d(x, y) - d(x, z)$ and $d(x, z) - d(x, y)$. It therefore suffices to show that both of these must be smaller than $d(y, z)$.

Simply rearranging the triangle inequality shows us that the first inequality holds.

$$d(x, y) - d(x, z) \leq d(y, z)$$

Applying the triangle inequality to the points x, z with y as an intermediate point gives us

$$d(x, z) \leq d(x, y) + d(y, z),$$

and this can be rearranged to give the second inequality. $\square$

1.2 Convergence and continuity

Definition 1. Let (X, d) be a metric space. A sequence $\{x_n\}$ in X converges to a point $a \in X$ if there for every $\varepsilon > 0$ exists an $N \in \mathbb{N}$ such that $d(x_n, a) < \varepsilon$ for all $n \geq N$. We write $\lim\limits_{n \to \infty} x_n = a$ or $x_n \to a$.

Lemma 1. *A sequence $\{x_n\}$ in a metric space (X, d) converges to a if and only if $\lim\limits_{n \to \infty} d(x_n, a) = 0$.*

Proof. This is simply a reformulation of the previous definition, but we prove it rigorously by showing that these statements are equivalent. Let (X, d) be a metric space, and let $\{x_n\}$ be a sequence in this metric space.

Let $a \in X$ and assume that $\{x_n\}$ converges to a. By definition of convergence, for every $\varepsilon > 0$ we can chose an $N \in \mathbb{N}$ such that $d(x_n, a) < \varepsilon$ for all $n \geq N$. This simply means that we can force the distance between the elements in $\{x_n\}$ and a to be arbitrarily close to zero by picking elements far out in the sequence.

We can generate a new sequence of the distances between the elements of $\{x_n\}$ and a, namely the sequence $\{d(x_n, a)\}$. Since we have $\lim_{n\to\infty} x_n = a$ we know that the following limit must equate to zero:

$$\lim_{n\to\infty} d(x_n, a) = d(a, a) = 0.$$

Now, assume that $\lim_{n\to\infty} d(x_n, a) = 0$. For this equation to be true, we must have $\lim_{n\to\infty} x_n = a$, because we have equality only when $x_n = a$, by the definition of a metric. But then, by definition, the sequence $\{x_n\}$ converges to a. $\square$

Proposition 2. *A sequence in a metric space can not converge to more than one point.*

Proof. Let $\{x_n\}$ be a sequence in a metric space (X, d). Assume for contradiction that $\{x_n\}$ converges to both the point a and the point a'. By definition of convergence, we have $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n = a'$. Using the triangle inequality we have:

$$d(a, a') \leq d(a, x_n) + d(x_n, a').$$

Taking the limits we get the following inequality:

$$d(a, a') \leq \lim_{n\to\infty} d(a, x_n) + \lim_{n\to\infty} d(x_n, a') = 0 + 0 = 0,$$

but this is only possible if $a = a'$. (The limits equal zero, due to the previous lemma.) $\square$

Definition 2. Assume that (X, d_X) and (Y, d_Y) are two metric spaces. A function $f : X \to Y$ is *continuous* at a point $a \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $d_Y(f(x), f(a)) < \varepsilon$ whenever $d_X(x, a) < \delta$.

Proposition 3. *The following are equivalent for a function $f : X \to Y$ between metric spaces:*

(i) f is continuous at a point $a \in X$.

(ii) For all sequences $\{x_n\}$ converging to a, the sequence $\{f(x_n)\}$ converges to $f(a)$.

Proof. We show that this is true by showing both the left and right implication. Let us first assume that f is continuous at a point $a \in X$. By definition of continuity we have that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $d(f(x), f(a)) < \varepsilon$ whenever $d(x, a) < \delta$.

Assume now that an arbitrary sequence $\{x_n\}$ converges to a. This means that $\lim_{n\to\infty} x_n = a$. Therefore, we must have $\lim_{n\to\infty} f(x_n) = f(a)$ since functions obey the axiom of substitiom from zet theory. Thus the right implication is shown. We now need to show the left implication.

Assume that for all sequences $\{x_n\}$ converging to a, the sequence $\{f(x_n)\}$ converges to $f(a)$. We must now show that this implies that f is continuous at a. $\square$