

# MANDATORY ASSIGNMENT

## MAT2400

### UNIVERSITY OF OSLO

Ivar Haugaløkken Stangeby

March 5, 2015

**Problem 1.** Assume  $f : \mathbb{N} \rightarrow \mathbb{N}$  is a strictly increasing function. Show that  $f$  satisfies  $f(n) \geq n$  for all  $n$ .

**Solution 1.** This obviously holds for  $n = 1$ , since 1 is the least element of  $\mathbb{N}$ . Assume now that we have shown  $f(n) \geq n$  for all  $n$  up to some  $n = k$ . Since  $f(k+1) > f(k)$ , by the assumption that  $f$  is strictly increasing we can easily see that  $f(k+1) - f(k) > 1$ . This is because for two elements to be unequal in  $\mathbb{N}$ , they must differ by at least 1.

$$f(k+1) > f(k) \geq k \implies f(k+1) \geq k+1.$$

Thus it must necessarily hold for  $n = k+1$  as well. By induction we have now shown that  $f$  satisfies  $f(n) \geq n$  for all  $n$ .

**Problem 2.** Assume  $(X, d)$  a complete metric space. Prove or disprove that the closure of an open ball  $B(x, r)$  is equal to the closed ball  $\overline{B}(x, r)$  for all complete metric spaces.

**Solution 2.** We can disprove the above by looking at a complete space under the discrete metric. Let us first introduce the metric  $d : X \times X \rightarrow \{0, 1\}$ , defined as

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

To construct a counter example, assume that we look at open and closed balls around a point  $x$ , with a radius  $r = 1$ . By observation we see that the closure of the open ball around  $x$  is just  $x$  itself,

$$\overline{B(x, r)} = x$$

where as the closed ball around  $x$  is the whole of  $X$ .

$$\overline{B}(x, r) = X.$$

We have now, using the discrete metric  $d$  and a radius  $r = 1$  constructed a metric space where the statement initially given does not hold.

**Problem 3.** Define  $\ell$  to be the set of all sequences of real numbers where only a finite number of elements are non-zero. Furthermore, for  $x = \{x_n\}$  and  $y = \{y_n\}$  in  $\ell$ , define

$$d(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n|.$$

**Solution 3.**

a) To show that  $d$  defines a metric on  $\ell$  we have to show the three properties a metric should satisfy. These are *positivity*, *symmetry* and *the triangle inequality*. The first two, positivity and symmetry follow directly from the definition of absolute value. The triangle inequality is a bit trickier to show. We want  $d$  to satisfy the following relation:

$$d(x, z) \leq d(x, y) + d(y, z).$$

Written out, this becomes:

$$\sup_{n \in \mathbb{N}} |x_n - z_n| \leq \sup_{n \in \mathbb{N}} |x_n - y_n| + \sup_{n \in \mathbb{N}} |y_n - z_n|.$$

We know that for some  $i, j, k \in \mathbb{N}$ :

$$\begin{aligned} d(x, z) &= |x_i - z_i| \\ d(x, y) &= |x_j - y_j| \\ d(y, z) &= |y_k - z_k| \end{aligned}$$

Intuitively, these absolute values should obey the triangle inequality for real numbers, but I can't say I've managed to completely convince myself. I have therefore not been able to construct a full fledged proof for  $d$  being a metric on  $\ell$ .

b) Let  $u_k \in \ell$  be defined as

$$u_k = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, 0, 0, 0, \dots \right\}.$$

To show that  $\{u_k\}_{k=1}^{\infty}$  is Cauchy we want to show that for all  $\varepsilon > 0$  there exists an  $N \in \mathbb{N}$  such that for all  $m, n > N$ , it is implied that  $d(x_n, x_m) < \varepsilon$ .

Given two sequences  $u_n$  and  $u_m$ , the distance between them is given as following (assuming  $m > n$ ).

$$\begin{aligned} d(u_n, u_m) &= \sup_{i \in \mathbb{N}} |u_{n_i} - u_{m_i}| \\ &= |u_{n_{n+1}} - u_{m_{n+1}}| \\ &= \left| 0 - \frac{1}{n+1} \right| \\ &= \frac{1}{n+1} \end{aligned}$$

Since a number on this form can be made as arbitrarily close to zero by choosing  $n$  arbitrarily large, we have shown that no matter what  $\varepsilon > 0$  we are given, we can always find an  $N$  such that  $m, n > N \implies d(u_m, u_n) < \varepsilon$ . Because of this,  $\{u_k\}_{k=1}^\infty$  is Cauchy.

c) For  $\{u_k\}_{k=1}^\infty$  to converge we would have to find a sequence in  $\ell$  it converges to. If  $\{u_k\}_{k=1}^\infty$  were to converge it would have to be to the harmonic sequence,  $\{\frac{1}{k}\}_{k=1}^\infty$  because it is the only sequence where elements are equal to the elements of the sequence  $\{u_k\}_{k=1}^\infty$ . Since the sequence necessarily has to converge to a single unique point we know that the harmonic sequence is the only such point. But since the harmonic sequence is not a member of  $\ell$  it does not converge.

d) Define  $c_0$  to be the space of sequences of real numbers whose limit are 0.

$$c_0 = \left\{ \{x_n\}_{n=1}^\infty \mid \lim_{n \rightarrow \infty} x_n = 0 \right\}.$$

To show that  $(c_0, d)$  is a complete metric space, we must show that all Cauchy sequences in  $(c_0, d)$  converge.

We know by definition of  $c_0$  that all its elements are sequences of real numbers that converge to 0. If a sequence of real numbers converges to zero, it is also Cauchy. Therefore, it follows that  $c_0$  is a complete metric space.

e) By the definition of  $c_0$  it is clear that  $\ell \subset c_0$ .  $\ell$  is dense in  $c_0$  if and only if for every  $x \in c_0$  and for all  $\varepsilon > 0$  there exists a  $y \in \ell$  such that  $d(x, y) < \varepsilon$ . Intuitively this seems reasonable, since we know a sequence  $x$  converges to 0, and  $y$  contains an infinite number of 0's.

$$\begin{aligned} x &= \{x_1, x_2, x_3, \dots\} \longrightarrow 0 \\ y &= \{y_1, y_2, y_3, \dots, y_n, 0, 0, 0, \dots\} \longrightarrow 0 \end{aligned}$$

Thus, we can always choose the sequences we need to satisfy the definition of *dense*. Therefore,  $\ell$  is dense in  $c_0$ .

The limit of the sequence  $u_k$ , containing an infinite amount of zeroes, is zero. That is  $u_k \longrightarrow 0$ . This means that  $u_k \in c_0$ .

**Problem 4.** Let  $X = (0, \infty) \subset \mathbb{R}$  and let  $d : X \times X \longrightarrow \mathbb{R}$  be defined as

$$d(x, y) = |\ln(x) - \ln(y)|.$$

**Solution 4.**

a) We want to show that  $d$  is a metric, and that  $(X, d)$  is complete. First, we need to show the three properties of a metric function. The first two, positivity and symmetry are trivial. For the triangle inequality, we observe the following:

$$\begin{aligned} d(x, z) &= |\ln(x) - \ln(z)| = |\ln(x) + \ln(y) - \ln(y) - \ln(z)| \\ &\leq |\ln(x) - \ln(y)| + |\ln(y) - \ln(z)| \\ &= d(x, y) + d(y, z). \end{aligned}$$

Thus  $d$  is a metric on  $X$ . To show that  $(X, d)$  is complete, we must show that all Cauchy sequences converge.

**b)** Given a differentiable function  $f : X \rightarrow X$  that satisfy the following:

$$x |f'(x)| \leq kf(x), \quad 0 < k < 1. \quad (1)$$

We want to show that  $f$  has a unique fixed point. If we can show  $f$  to be a contraction, it follows from the completeness of  $X$  that we can apply Banach's fix point theorem. For  $f$  to be a contraction, we want it to satisfy the following inequality,

$$d(f(x), f(y)) < kd(x, y), \quad 0 < k < 1$$

If we define an auxilliary function  $h(t)$  as

$$h(t) = \frac{d}{dt} \ln(f(t)) = \frac{f'(t)}{f(t)}$$

we can use this to show the contraction property. Rearranging (1) we get

$$\frac{|f'(x)|}{f(x)} \leq \frac{k}{x}.$$

We then see that we can rephrase the contraction property of  $f$  in terms of  $h(t)$ .

$$\left| \int_x^y h(t) dt \right| \leq \left| \int_x^y \frac{k}{t} dt \right|.$$

Evaluating this, we get the inequality we wanted  $f$  to satisfy, namely

$$d(f(x), f(y)) < kd(x, y), \quad 0 < k < 1$$

Therefore, by Banach's fix point theorem,  $f$  has one unique fixed point.