University of Oslo
Real Analysis - MAT2400
Assignment 2

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Problem 1. Let (X, d) be a bounded metric space, and let $P(X)$ denote the collection of non-empty closed subsets of X. For A and B in $P(X)$, let

$$h(A, B) = \sup_{x \in X} |\operatorname{dist}(x, A) - \operatorname{dist}(x, B)|,$$

where $\operatorname{dist}(x, C)$ is given by

$$\operatorname{dist}(x, C) = \inf_{c \in C} d(x, c).$$

The function h is called the *Hausdorff metric.*

a) Show that if $h(A, B) = 0$ then $A = B$. Here A and B are two non-empty closed subsets of X.

b) Show that h is a metric on $P(X)$.

c) For A and B in $P(X)$, let $\hat{h}$ be defined as

$$\hat{h}(A, B) = \max\left\{\sup_{a \in A} \operatorname{dist}(a, B), \sup_{b \in B} \operatorname{dist}(b, A)\right\}.$$

Show that

$$\hat{h}(A, B) = h(A, B) \text{ for all } A, B \text{ in } P(X).$$

(**Hint:** Show the two inequalities $h(A, B) \geq \hat{h}(A, B)$ and $\hat{h}(A, B) = h(A, B)$.)

Solution 1. Before I start, I want to jot down the properties of the various mathematical objects presented. We are given a metric space (X, d) and it is said to be bounded. What this means, is that there exists some number r such that $d(x, y) \leq r$ for all $x, y \in X$. $P(X)$ is a collection of non-empty closed subsets of X. In other words, the elements of $P(X)$ are sets that contain their own boundary.

We want to show that $h(A, B) = 0 \implies A = B$. Let us therefore assume that $h(A, B) = 0$. This gives us the following equation:

$$\sup_{x \in X} |\operatorname{dist}(x, A) - \operatorname{dist}(x, B)| = 0.$$

This means that the largest difference we can have between $\operatorname{dist}(x, A)$ and $\operatorname{dist}(x, B)$ is 0, however, since we are working with absolute values this actually means that the $\operatorname{dist}(x, A) - \operatorname{dist}(x, B) = 0$ for all $x \in X$. We can therefore meaningfully examine the equation,

$$\operatorname{dist}(x, A) = \operatorname{dist}(x, B).$$

This equation being true means that if you pick the element in A that is the smallest distance away from x then this distance is exactly equal to the distance between the element in B that is the smallest distance away from x. Or, strictly speaking

$$\inf_{a \in A} d(x, a) = \inf_{b \in B} d(x, b).$$

Problem 2. Let $0 < r < 1$ and consider the series

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}.$$

Show that the series converges uniformly for all $x \in \mathbb{R}$, and that its sum equals

$$P_r(x) = \frac{1 - r^2}{1 - 2r\cos(x) + r^2}.$$