University of Oslo

Mandatory Assignment MAT2400

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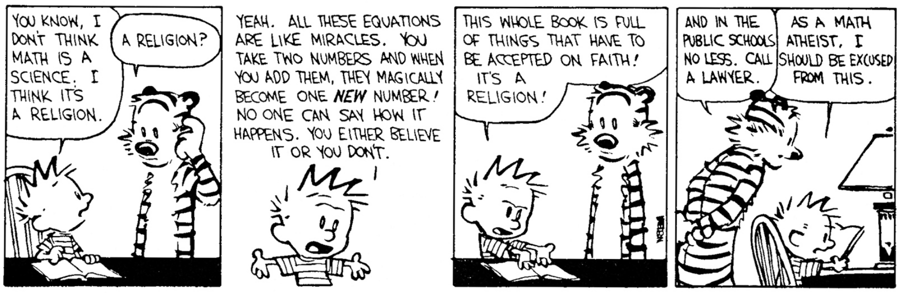

Problem 1. *Assume $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. Show that f satisfies $f(n) \geq n$ for all n.*

Solution 1. This obviously holds for $n = 1$, since 1 is the least element of $\mathbb{N}$. Assume now that we have shown $f(n) \geq n$ for all n up to some $n = k$. Since $f(k+1) > f(k)$, by the assumption that f is strictly increasing we can easily see that $f(k+1) - f(k) > 1$. This is because for two elements to be different in $\mathbb{N}$, they must differ by at least 1.

$$f(k+1) > f(k) \geq k \Longrightarrow f(k+1) \geq k+1.$$

Thus it must neccessarily hold for $n = k+1$ as well. By induction we have now shown that f satisfies $f(n+1) \geq n$ for all n.

Problem 2. *Assume (X, d) a complete metric space. Prove or disprove that the closure of an open ball $\overline{B(x, r)}$ is equal to the closed ball $\overline{B}(x, r)$ for all complete metric spaces.*

Solution 2. We can disprove the above by looking at a complete space under the discrete metric. Let us first introduce the metric $d : X \times X \to \{0, 1\}$, defined as

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

To construct a counter example, assume that we look at open at closed balls around a point x, with a radius $r = 1$. By observation we see that the closure of the open ball around x is just x itself,

$$\overline{B\left(x,r\right)} = x$$

where as the closed ball around x is the whole of X.

$$\overline{B}\left(x,r\right) = X.$$

We have now, using the discrete metric d and a radius $r = 1$ constructed a metric space where the statement initially given does not hold.

Problem 3. *Define ℓ to be the set of all sequences of real numbers where only a finite number of elements are non-zero. Furthermore, for $x = \{x_n\}$ and $y = \{y_n\}$ in ℓ, define*

$$d(x,y) = \sup_{n \in \mathbb{N}} \left| x_n - y_n \right|.$$

Solution 3.

a) To show that d defines a metric on ℓ we have to show the three properties a metric should satisfy. These are *positivity, symmetry* and *the triangle inequality.* The first two, positivity and symmetry follow directly from the definition of absolute value. The triangle inequality is a bit trickier to show. We want d to satisfy the following relation:

$$d(x,z) \leq d(x,y) + d(y,z).$$

Written out, this becomes:

$$\sup_{n \in \mathbb{N}} \left| x_n - z_n \right| \leq \sup_{n \in \mathbb{N}} \left| x_n - y_n \right| + \sup_{n \in \mathbb{N}} \left| y_n - z_n \right|.$$

We know that for some $i, j, k \in \mathbb{N}$:

$$\begin{aligned} d(x,z) &= \left| x_i - z_i \right| \\ d(x,y) &= \left| x_j - y_j \right| \\ d(y,z) &= \left| y_k - z_k \right|. \end{aligned}$$

Intuitively, these absolute values should obey the triangle inequality for real numbers, but I can't say I've managed to completely convince myself. I have therefore not been able to construct a full fledged proof for d being a metric on ℓ.

b) Let $u_k \in \ell$ be defined as

$$u_k = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{k}, 0, 0, 0, \cdots\right\}.$$

To show that $\{u_k\}_{k=1}^{\infty}$ is Cauchy we want to show that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m, n > N$, it is implied that $d(x_n, x_m) < \varepsilon$.

Given two sequences u_n and u_m, the distance between them is given as following (assuming $m > n$).

$$\begin{aligned} d(u_n, u_m) &= \sup_{i\in\mathbb{N}} |u_{n_i} - u_{m_i}| \\ &= |u_{n_{n+1}} - u_{m_{n+1}}| \\ &= \frac{1}{n+1}. \end{aligned}$$

Since a number on this form can me made as arbitrarily close to zero by chosing n arbitrarily large, we have shown that no matter what $\varepsilon > 0$ we are given, we can always find an N such that $m, n > N \implies d(u_m, u_n) < \varepsilon$. Because of this, $\{u_k\}_{n=1}^{\infty}$ is Cauchy.

c) For $\{u_k\}_{k=1}^{\infty}$ to converge we would have to find a sequence in ℓ it converges to. If $\{u_k\}_{k=1}^{\infty}$ were to converge it would have to be to the harmonic sequence, $\left\{\frac{1}{k}\right\}_{k=1}^{\infty}$ because it is the only sequence where elements are equal to the elements of the sequence $\{u_k\}_{k=1}^{\infty}$. Since the the sequence neccessarily has to converge to a single unique point we know that the harmonic sequence is the only such point. But since the harmonic sequence is not a member of ℓ it does not converge.

d) Define c_0 to be the space of sequences of real numbers whose limit are 0.

$$c_0 = \left\{\{x_n\}_{n=1}^{\infty} \mid \lim_{n\to\infty} x_n = 0\right\}.$$

To show that (c_0, d) is a complete metric space, we must show that all Cauchy sequences in (c_0, d) converge.

We know by definition of c_0 that all its elements are sequences of real numbers that converge to 0. If a sequence of real numbers converges to zero, it is also Cauchy. Therefore, it follows that c_0 is a complete metric space.[1]

e) By the definition of c_0 it is clear that $\ell \subset c_0$. ℓ is dense in c_0 if and only if for every $x \in c_0$ and for all $\varepsilon > 0$ there exists a $y \in \ell$ such that $d(x, y) < \varepsilon$. Intuitively this seems reasonable, since we know a sequence x converges to 0, and y contains an infinite number of 0's.

$$\begin{aligned} x &= \{x_1, x_2, x_3, \ldots\} \longrightarrow 0 \\ y &= \{y_1, y_2, y_3, \ldots, y_n, 0, 0, 0, \ldots\} \longrightarrow 0 \end{aligned}$$

[1] Captain Hindsight: What I've written here does not actually prove anything, but I haven't had time to correct it. I've looked at the convergence of the elements of the sequence, rather than the sequence itself.

Thus, we can always chose the sequences we need to satisfy the definition of *dense*. Therefore, ℓ is dense in c_0.[2]

The limit of the sequence u_k, containing an infinite amount of zeroes, is zero. That is $u_k \longrightarrow 0$. This means that $u_k \in c_0$.

[2]Captain Hindsight strikes again. This is also a bit vague, and could've used further elaboration.

Problem 4. *Let* $X = (0, \infty) \subset \mathbb{R}$ *and let* $d : X \times X \longrightarrow \mathbb{R}$ *be defined as*

$$d(x, y) = |\ln(x) - \ln(y)|.$$

Solution 4.

a) We want to show that d is a metric, and that (X, d) is complete. First, we need to show the three properties of a metric function. The first two, positivity and symmetry are trivial. For the triangle inequality, we observe the following:

$$\begin{aligned} d(x, z) &= |\ln(x) - \ln(z)| = |\ln(x) + \ln(y) - \ln(y) - \ln(z)| \\ &\leq |\ln x - \ln(y)| + |\ln(y) - \ln(z)| \\ &= d(x, y) + d(y, z). \end{aligned}$$

Thus d is a metric on X. To show that (X, d) is complete, we must show that all Cauchy sequences converge.

Assume that $x \in X$ is Cauchy. This means that given a $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $m, n > N \implies d(x_m, x_n) < \varepsilon$. In our metric d this means that the values of x_m, x_n become arbitrarily close to each other, which in turn means, that there exists a point a such that $d(x_m, a) < \varepsilon/2$ and $d(x_n, a) < \varepsilon/2$. Therefore x converges to a. Since x was any arbitrary Cauchy sequence in X this holds for all Cauchy sequences in X, and therefore X under the metric d is a complete space.

b) Given a differentiable function $f : X \longrightarrow X$ that satisfy the following:

$$x\,|f'(x)| \leq kf(x), \quad 0 < k < 1. \tag{1}$$

We want to show that f has a unique fixed point. If we can show f to be a contraction, it follows from the completeness of X that we can apply Banachs fix point theorem. For f to be a contraction, we want it to satisfy the following inequality,

$$d(f(x), f(y)) < kd(x, y), \quad 0 < k < 1$$

If we define an auxilliary function $h(t)$ as

$$h(t) = \frac{d}{dt} \ln(f(t)) = \frac{f'(x)}{f(x)},$$

we can use $h(t)$ to show the contraction property of f. Rearranging (1) we get

$$\frac{|f'(x)|}{f(x)} \leq \frac{k}{x}.$$

We then see that we can rephrase the contraction property of f in terms of $h(t)$.

$$\left| \int_x^y h(t)\,dt \right| \leq \left| \int_x^y \frac{k}{t}\,dt \right|.$$

Evaluating this, we get the inequality we wanted f to satisfy, namely

$$d(f(x), f(y)) < kd(x, y), \quad 0 < k < 1$$

Therefore, by Banach's fix point theorem, f has one unique fixed point.