Mandatory assignment 2
MAT2400

Ivar Stangeby

April 28, 2015

Problem 1

a) We want to show that $h(A,B) = 0$ implies that $A = B$. We assume that A and B are two non-empty closed subsets of X. In other words, $A, B \in P(X)$. If we can show the contrapositive, then the original implication must hold. We therefore assume that $A \neq B$ and we want to show that then $h(A,B) \neq 0$.

Let $x \in X$ be such that $x \in A$ and $x \neq B$. By the definition of $\text{dist}(x, A)$ and $\text{dist}(x, B)$ we see that since $x \in A$, $\text{dist}(x, A) = 0$ and since x is not in B we have $\text{dist}(x, B) > 0$ by the definition of d. Since we take the supremum of all of these values we must have that

$$\sup_{x \in X} |\text{dist}(x, A) - \text{dist}(x, B)| \geq \text{dist}(x, B) > 0.$$

Hence, $h(A,B) = 0$ implies that $A = B$.

b) We now want to show that h is a metric on $P(X)$. Positivity and symmetry follows directly from the absolute values in the expression for $h(A,B)$. We therefore need to show that h obeys the triangle inequality.

$$\begin{aligned}
h(A,B) &= \sup_{x \in X} |\text{dist}(x, A) - \text{dist}(x, B)| \\
&= \sup_{x \in X} |\text{dist}(x, A) - \text{dist}(x, C) + \text{dist}(x, C) - \text{dist}(x, B)| \\
&\leq \sup_{x \in X} (|\text{dist}(x, A) - \text{dist}(x, C)| + |\text{dist}(x, C) - \text{dist}(x, B)|) \\
&\leq \sup_{x \in X} |\text{dist}(x, A) - \text{dist}(x, C)| + \sup_{x \in X} |\text{dist}(x, C) - \text{dist}(x, B)| \\
&= h(A,C) + h(C,B)
\end{aligned}$$

Hence h is a metric on $P(X)$.

c) We want to show the two inequalities

$$h(A,B) \geq \hat{h}(A,B), \tag{1}$$

$$\hat{h}(A,B) \geq h(A,B). \tag{2}$$

Problem 2

a)

Convergence

We want to show that the series

$$\Upsilon = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}$$

converges uniformly for all $x \in \mathbb{R}$ and that its sum equals $P_r(x)$. We observe that we can split Υ into three subseries:[1]

$$\Upsilon = 1 + \sum_{n=1}^{\infty} r^n e^{inx} + \sum_{n=1}^{\infty} r^n e^{-inx}.$$

Applying Eulers formula, we can simplify this to the series[2]

$$\Upsilon = 1 + \sum_{n=1}^{\infty} r^n \cos(nx) + \sum_{n=1}^{\infty} r^n \cos(nx) = 1 + 2\sum_{n=1}^{\infty} r^n \cos(nx).$$

We know that since we have $0 < r < 1$ and $\cos(nx) \leq 1$ that

$$\Upsilon \leq 1 + 2\sum_{n=1}^{\infty} r^n,$$

but since this series converges, so must Υ.

Sum

We want to show that the sum of Υ is equal to

$$P_r(x) = \frac{1-r^2}{1-2r\cos(x)+r^2}.$$

We rewrite Υ to the following form, in order for us to be able to apply the formula for finite geometric series:[3]

$$\Upsilon = \lim_{N\to\infty}\left(\sum_{n=0}^{N}\left(e^{ix}r\right)^n + \sum_{n=0}^{N}\left(e^{-ix}r\right)^n - 1\right).$$

We then get that the sum of Υ is equal to

$$\Upsilon = \lim_{N\to\infty}\left(\frac{1-(e^{ix}r)^{N+1}}{1-e^{ix}r} + \frac{1-\left(e^{-ix}r\right)^{N+1}}{1-e^{-ix}r} - 1\right).$$

Applying Eulers formula again which yields[4]

$$\Upsilon = \frac{1-r^2}{1-2r\cos(x)+r^2} = P_r(x).$$

[1] Note the change of summation index and the removal of absolute values
[2] The imaginary terms cancel out
[3] Notice again the change of summation index. That is where the -1 originates from.
[4] Some algebra required

b) We want to show that $P_r(x)$ is positive or zero for all x. We have

$$\begin{aligned} P_r(x) &= \frac{1-r^2}{1-r2\cos(x)+r^2} \\ &\geq \frac{1-r^2}{1-2r+r^2} \\ &= \frac{(1-r)(1+r)}{(1-r)^2} = \frac{1+r}{1-r} > 1. \end{aligned}$$

Therefore, we can conclude that no matter what x is, $P(x)$ is greater or equal to zero.

c) We want to show that $P_r(x) \to 0$ as $r \uparrow 1$ on the interval $X = [-\pi, -\delta] \cup [\delta, \pi]$. We first state the definition of uniform convergence.[5]

> A sequence $\{f_n\}$ of functions converges uniformly to a function f if and only if for all $\varepsilon > 0$ there exists an $N > 0$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$ and for all $n \geq N$.

In our case, we need to rewrite $P_r(x)$ to a form which we can express in terms of a natural number n. We want to create the sequence $\{r_n\}$ of rational numbers defined as

$$r_n = \frac{n-1}{n}.$$

This series converges to 1 as $n \to \infty$. We now want to show that for all $\varepsilon > 0$ there exists an $N > 0$ such that what was stated above holds. That is

$$|P_{r_n}(x) - 0| < \varepsilon,$$

for all $x \in X$ and for all $n > N$. Writing it out and taking the limit we see that

$$\lim_{n\to\infty} \frac{1-\left(\frac{n-1}{n}\right)^2}{1-\left(\frac{n-1}{n}\right)\cos(x)+\left(\frac{n-1}{n}\right)^2} = \frac{0}{2-2\cos(x)} = 0 < \varepsilon.$$

We have $\cos(x) \neq 1$ for all $x \in X$ since $0 \notin X$, therefore our proof is done.

d) We want to show that

$$\int_{-\pi}^{\pi} P_r(x)\,dx = 2\pi.$$

Recall that

$$P_r(x) = 1 + 2\sum_{n=1}^{\infty} r^n \cos(nx).$$

We can therefore write the integral as follows

$$\int_{-\pi}^{\pi} \left(1 + 2\sum_{n=1}^{\infty} r^n \cos(nx)\right) dx = \int_{-\pi}^{\pi} 1\,dx + 2\int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} \left(r^n \cos(nx)\right)\right) dx$$

[5] This is a bit informal, but used just as a reminder

We know that since $P_r(x)$ converges uniformly on all of $\mathbb{R}$ the sum and integrals are interchangeable.[6] Therefore we can rewrite this expression as

$$\int_{-\pi}^{\pi} 1\,dx + 2\sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} r^n \cos(nx)\,dx\right).$$

The integral contained in the sum evaluates to zero, therefore we are left with

$$\int_{-\pi}^{\pi} P_r(x)\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi,$$

and we are done.

e) We want to show that

$$\Omega = \lim_{r\uparrow 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)P_r(y)\,dy = f(x).$$

Taking the limit as $r \uparrow 1$, our expression for $P_r(y)$ turns into something we recognise as the Dirichlet kernel, $D_N(y) = \sum_{n=-N}^{N} e^{iny}$.[7]

$$\lim_{r\uparrow 1} P_r(y) = \lim_{N\to\infty} \sum_{n=-N}^{N} e^{iny} = \lim_{N\to\infty} D_N(y).$$

We can now look at the partial sums of Ω:

$$s_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)D_N(y)\,dy.$$

If we can show that these partial sums converge to $f(x)$ we're done. Using Dirichlet's Theorem we know that if f has a finite number of minima and maxima, then the Fourier series of f converges pointwise to f. Since f is 2π periodic, it has a finite number of min and max. Therefore, we have

$$\Omega = \lim_{N\to\infty} s_N(x) = \lim_{r\uparrow 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)P_r(y)\,dy = f(x).$$

This holds because we actually looking at partial sums s_N of the Fourier series for f,

$$s_N(x) = \sum_{n=-N}^{N} \langle f, e_n\rangle e_n(x).$$

[6] Corollary 4.2.3

[7] I am here following the derivation at page 138 and 139 in the compendium