

UNIVERSITY OF OSLO  
REAL ANALYSIS - MAT2400  
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**Abstract**

This document is going to be a way for me as a student of MAT2400 at the University of Oslo to gather my thoughts around the course Real Analysis. I've struggled with my intuition for this subject, and this is a last ditch effort to build it all up from scratch.

For this task, I've decided to use the text books written by Terence Tao, namely Analysis I and II. There is absolutely nothing wrong with the text book offered at my university, it is brilliantly written, but I have been exposed to the writings of Terence Tao before, and therefore I wish to give his books a try.

The structure of this document is going to be me writing down the results encountered throughout the text books along the proofs I find extra intriguing. I'm going to attempt to prove the theorems myself, and if I find it reasonable I'm going to write down my own proof. Included will also be my attempted solutions to selected exercises.

This document is mainly for my own good and well being, but if anyone can find any use from them, then that is great.

## Chapter 1

# Introduction

## Chapter 2

# Starting at the beginning: the natural numbers.

In order for us to start exploring the various properties of the real numbers, which is what real analysis is concerned with, we are going to have to start from the very beginning. That is the natural numbers, denoted  $\mathbb{N}$ . From these natural numbers, we can construct the integers,  $\mathbb{Z}$ , the rationals  $\mathbb{Q}$ , the real numbers  $\mathbb{R}$ , and finally; the complex numbers  $\mathbb{C}$ . The latter being the main focus of the subject Complex Analysis.

### 2.1 The Peano axioms

One of the most standard ways of defining the natural numbers, is in terms of the *Peano axioms*. One can also define natural numbers through the notion of cardinality.

**Definition 2.1.1** (Informal). *A natural number is any element of the set*

$$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\},$$

*which is the set of all the numbers created by starting with 0 and then counting forward indefinitely. We call  $\mathbb{N}$  the set of all natural numbers.*

In order for us to rigorously define the set of natural numbers, we're going to use the two fundamental concepts of *the number 0* and the *increment operation*. These will be covered in the Peano Axioms. We will use  $n++$  to denote the *successor* of  $n$ .

Starting with the first two:

**Axiom 2.1.** *0 is a natural number.*

**Axiom 2.2.** *If  $n$  is a natural number, then  $n++$  is also a natural number.*

Now, in order to avoid having to deal with incredibly long strings of  $++$ 's. We're going to use an auxiliary definition.

**Definition 2.1.2.** *We define 1 to be the number  $0++$ , 2 to be the number  $(0++)++$ , etc.*

We can based off of this, propose the following:

**Proposition 2.1.3.** *3 is a natural number.*

*Proof.* By Axiom 1, 0 is a natural number. It then follows by Axiom 2 that both 1, 2, and 3 are natural numbers.  $\square$

In order for us to avoid the problem of having the successive numbers wrap around to previous numbers, we impose a new axiom, namely:

**Axiom 2.3.** *0 is not the successor of any natural number: i.e., we have  $n++ \neq 0$  for every natural number  $n$ .*

We can, equipped with this new axiom, show for example the following:

**Proposition 2.1.4.** *4 is not equal to 0.*

*Proof.* By definition,  $4 = 3++$ . By the first two axioms, 3 is a natural number. Thus, since 0 is not the successor of any natural number,  $3++ \neq 0$ , i.e.,  $4 \neq 0$ .  $\square$

Assuming the following axiom allows us to rule out any behaviour where the successors wrap around, but not to 0, i.e.,  $5++ = 1$ .

Replace all occurrences of  $++$  with a proper unary plus sign. To remove wonky spacing.

**Axiom 2.4.** *Different natural numbers must have different successors; i.e., if  $n, m$  are natural numbers and  $n \neq m$ , then  $n++ \neq m++$ . Equivalently, if  $n++ = m++$ , then we must have  $n = m$ .*

We can now prove extensions of the previous proposition where we do not have zeroes on the right hand side of the equation.

**Proposition 2.1.5.** *6 is not equal to 2.*

*Proof.* Assume for contradiction that  $6 = 2$ . By the previous axiom we must have  $5++ = 1++$ . Applying the same axiom again, we have  $5 = 1$  so that  $4++ = 0++$ . But, this leads to a contradiction, because by the same axiom,  $4 = 0$ . This contradicts our previously proven proposition.  $\square$

Assume now that we are presented with a weird number system

$$\mathbb{N} = \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, \dots\}.$$

Even though this set contains real numbers, which we haven't defined or talked about yet, it satisfies all the previous axioms. But this is not the number system we're interested in. We want our set of natural numbers to only be containing all the numbers that can be directly derived from 0 just using the successor operation.

We want to introduce some axiom that does not allow other forms of successors to occur. Therefore we introduce the following:

**Axiom 2.5** (Principle of mathematical induction). *Let  $P(n)$  be any property pertaining to a natural number  $n$ . Suppose that  $P(0)$  is true, and suppose that whenever  $P(n)$  is true,  $P(n++)$  is also true. Then  $P(n)$  is true for every natural number  $n$ .*

We're now equipped with the tools required to deal with propositions of the following form:

**Proposition 2.1.6.** *A certain property  $P(n)$  is true for every natural number  $n$ .*

*Proof.* Using induction, we show the base case of  $P(0)$ . Assume, for the sake of induction, that  $P(n)$  is true. We now want to show that it has to follow that  $P(n++)$  also must be true. If this is the case, we have shown, using mathematical induction that  $P(n)$  is true for every natural number  $n$ .  $\square$

The previous five axioms are known as the *Peano Axioms* for the natural numbers. We now want to more rigorously define the kind of number system we are to refer to as the *natural numbers*.

**Assumption 2.1.7** (Informal). *There exists a number system  $\mathbb{N}$ , whose elements we will call natural numbers, for which Axioms 1-5, are true.*

This number system is what we refer to as *the* natural number system. But one should not rule out the possibility that there are more than one natural number system. But as long as these are *isomorphic* one can consider them as equal.

With only this, rather simplistic definition of natural numbers, the five axioms, some axioms from set theory we can build all other number systems, create functions and do the algebra and calculus that we are used to.

An very common question now arises, and this is about the finiteness or infiniteness of the natural number system. How can something infinite come from something strictly finite? One can easily show that all the natural numbers are finite. It is clear that 0 is finite. If  $n$  is finite, then clearly  $n++$  is finite. Therefore all natural numbers are finite. It then follows that infinity is not a natural numbers. There are other number systems that admit the infinite numbers.

It is an interesting fact that the definition of  $\mathbb{N}$  is *axiomatic* rather than *constructive*. This means, that so far we're only concerned with what the natural numbers are, not what they do, what they measure or what they can be used for.

As long as a mathematical model obeys the previous axioms, it is of no concern whether which mathematical model is "true". It is this form of *abstractness* that makes mathematics so useful. One does not necessarily need a concrete model, because the numbers can be understood abstractly through the use of axioms.

As a consequence of the axioms previously discussed, we can now define sequences *recursively*. That is, start with some base value and then building the next value in the sequence by means of a function. This leads to the following:

**Proposition 2.1.8** (Recursive definitions). *Suppose for each natural number  $n$ , we have some function  $f_n : \mathbb{N} \rightarrow \mathbb{N}$  from the natural numbers to the natural numbers. Let  $c$  be a natural number.*

Strictly speaking, this axiom is not an axiom but an axiom schema.

Then we can assign a unique natural number  $a_n$  to each natural number  $n$ , such that  $a_0 = c$  and  $a_{n++} = f_n(a_n)$  for each natural number  $n$ .

*Proof.* Using induction, we verify the base case. We clearly see that this procedure gives a single value to  $a_0$ , namely  $c$ . (We know from axiom 3 that  $a_0$  won't be redefined.) Suppose now inductively that the procedure gives a single value to  $a_n$ . Then it gives a single value to  $a_{n++}$ , namely  $a_{n++} = f_n(a_n)$ . (We know from axiom 4 that  $a_{n++}$  won't be redefined.) This completes the induction, since  $a_n$  is defined for every natural number  $n$ , with a single value assigned to each  $a_n$ .  $\square$

Equipped with the tools that are recursive definitions we can now define multiple operations on the set of natural numbers. Up until now, we've only dealt with one, being the increment operation.

## 2.2 Addition

Currently our number system does not support any more advanced operations than incrementing a number. We now turn our heads to addition. The operation is simple. To add 3 to 5, we simply increment 5 three times. This is one increment more than adding 2 to 5, which is one increment more than adding 1 to 5, which is one increment more than adding 0 to 5. We can therefore easily give a recursive definition of addition.

**Definition 2.2.1** (Addition of natural numbers). *Let  $m$  be a natural number. To add zero to  $m$ , we define  $0 + m = m$ . Now suppose inductively that we have defined how to add  $n$  to  $m$ . Then we can add  $n++$  to  $m$  by defining  $(n++) + m = (n + m)++$ .*

For example,  $2 + 3 = (3++)++ = 4++ = 5$ . By using the principle of mathematical induction, we see that we now have defined  $n + m$  for every natural number  $n$ . We are specializing the previous general discussion about recursive definitions to the setting where  $a_n = n + m$ , and  $f_n(a_n) = a_n++$ .

It's worth noting that this definition is actually *asymmetric*. That is, while yielding the same result,  $3 + 5$  is incrementing 5 three times, whereas  $5 + 3$  is incrementing 3 five times. We shall soon see, that it is a general fact that  $a + b = b + a$  for all natural numbers  $a, b$ .

One can easily prove, using the first two axioms and the principle of mathematical induction to show that the sum of two natural numbers is again a natural number.

At the present moment, we have only two facts about addition. However, this is perfectly sufficient to deduce everything else we know about addition. Starting with some basic lemmas.

**Lemma 2.2.2.** *For any natural number  $n$ ,  $n + 0 = n$ .*

This lemma is not obvious from our previous definition, since we still do not know that  $a + b = b + a$ .

*Proof.* Using induction. The base case  $0 + 0 = 0$  follows from the definition of addition of natural numbers.  $0 + m = m$  for all natural numbers, and 0 is known to be a natural number. Suppose inductively that  $n + 0 = n$ . We now wish to show that  $(n++) + 0 = n++$ . By definition of addition yields that  $(n++) + 0$  is equal to  $(n + 0)++$ , which is equal to  $n++$  since  $n + 0 = n$ . This closes the induction.  $\square$