MAT2400
Assignment 1

Ivar Stangeby

February 27, 2015

Problem 1. *Show that a strictly increasing function* $f : \mathbb{N} \to \mathbb{N}$ *must satisfy* $f(n) \geq n, \forall n \in \mathbb{N}$.

Solution: Assume that $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. By the definition of a strictly increasing function we know that

$$f(n+1) > f(n), \forall n \in \mathbb{N}.$$

We can now, since we are working with the natural numbers, easily show this inductively. Let us first show the base case.

$$f(1) \geq 1$$

This is intuitively true, because 1 is defined as the least element of the set of natural numbers. Assuming that we have verified this as true for all n up to and including some number k. We know want to show that it then follows that it must be true for $k+1$. By assumption:

$$f(k) \geq k$$

Using the standard metric in $\mathbb{R}$ we can see that for any two pairs of successive integer numbers,

$$\inf \{d(k, k+1) \mid k \in \mathbb{N}\} = 1$$

where,

$$d(x, y) = |x - y|$$

That is, the smallest distance possible with two different numbers is 1. It then follows that

$$\begin{aligned} f(k+1) > f(k) + 1 &\geq k + 1 \\ f(k+1) &\geq k + 1 \end{aligned}$$

as we wanted to show. Thus, by the induction principle, a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$, must neccessary satisfy $f(n) \geq n, \forall n \in \mathbb{N}$.

Problem 2. *Let* $(X, d))$ *be a complete metric space. Let* $B(x, r)$ *denote the open ball centered at* $x \in X$ *with radius* r*, i.e.,*

$$B(x, r) = \{y \in X \mid d(x, y) < r\},$$

and $\overline{B}(x, r)$ *the closed ball of radius* r*, i.e.,*

$$\overline{B}(x, r) = \{y \in X \mid d(x, y) \leq r\}.$$

For any set $C \in X$*, let* $\overline{C}$ *denote its closure. Is it true that for any complete metric space* X*,*

$$\overline{B(x, r)} = \overline{B}(x, r)? \tag{1}$$

Solution: Consider the discrete metric,

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

We can show that (1) does not neccessarily hold under the discrete metric. Lets assume we take the radius r to be 1. The open ball $B(x, r)$ is then any two points with less than a distance r between. Thus the open ball only contains the point x. In that case, taking the closure of this open ball changes nothing, and we're left with just the point x. However, the closed ball $\overline{B}(x, r)$ has to be the entirety of our space X, since the distance between two points are allowed to be 1. Thus, if we let our metric space be (X, d) with $X = \mathbb{R}$ and d the discrete metric (1) does not hold. We then have a complete metric space (R, d). We then have a complete metric space (R, d).

Problem 3. *Let* ℓ *be the set of sequences of real numbers where only a finite number of terms are different from zero,*

$$\ell = \{\{x_n\}_{n=1}^{\infty} \mid x_i = 0 \textit{ for all but a finite number of i's}\}.$$

For $x = \{x_n\}$ *and* $y = \{y_n\}$ *in* ℓ*, define*

$$d(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n|.$$

Solution:

a) To show that d is a metric on ℓ we must show the three properties of a metric function.

1. Positivity: Since the metric is defined as the biggest difference between to corresponding elemnts from $\{x_n\}$ and $\{y_n\}$, the metric must neccesarily satisfy the property of positivity since there does exists a finite number of non-zero elements in each sequence. Thus, $d(x, y) \geq 0$ with equality only if $x = y$.

2. Symmetry:

$$d(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n| = \sup_{n \in \mathbb{N}} |y_n - x_n| = d(y, x).$$

   Thus the metric is symmetric.

3. Triangle Inequality: Want to show that given three sequences x, y, z, the metric satisfies

$$d(x, z) \leq d(x, y) + d(y, z).$$

The trivial case, when $x = y = z$ is just that, trivial. Thus we assume that x, y and z are not equal.

b) Letting $u_k \in \ell$ be defined as

$$u_k = \left\{1, \frac{1}{2}, \ldots, \frac{1}{k}, 0, 0, 0, \ldots\right\} \tag{2}$$

we want to show that $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in (ℓ, d). We can do this using a traditional $\varepsilon - N$-proof. We want to show that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all integers $m, n > N$, $d(u_m, u_n) < \varepsilon$.

Assume that $m > n$. We're then going to have two sequences looking like this:

$$u_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, 0, \ldots\right\}$$
$$u_m = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots, \frac{1}{m-1}, \frac{1}{m}, 0, 0, 0, \ldots\right\}$$

By observation, we see that

$$\begin{aligned} d(u_m, u_n) &= \sup_{i \in \mathbb{N}} \{|u_{m_i} - u_{n_i}|\} \\ &= \left|u_{m_{n+1}} - u_{n_{n+1}}\right| \end{aligned}$$

Since given any $\varepsilon > 0$, we can always choose an $N \in \mathbb{N}$ such that we can get numbers on the form $\frac{1}{n}$ as close to 0 as we want. Certainly smaller than ε. Thus it follows that $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in (ℓ, d).

c) Let ℓ be a subset of a metric space (X, d). Let $x \in \ell$, and choose a subsequence x_s of x that only contains all the zero-elements of x. We can do this by shifting sufficiently far to the right in the sequence, so that all the non-zero elements are to the left. By observation we see that x_s must converge to the null sequence $x_0 \in \ell$. I've decided to interpret the condition that ℓ must contain a finite number of non-zero elements such that it can also contain zero non-zero elements.

Since we can chose such a subsequence for any sequence in ℓ, the metric space (ℓ, d) is compact. It then follows, that since every compact metric space is complete, and in a complete space all Cauchy sequences converge, thus $\{u_k\}$ must converge.

d) Let c_0 be defined as follows:

$$c_0 = \left\{\{x_n\}_{n=1}^{\infty} \mid \lim_{n\to\infty} x_n = 0\right\}$$

In order to show that c_0 is a compact space under the metric d we have to show that all Cauchy sequences in c_0 converge.

Assume then that $x \in c_0$ is Cauchy. This means that for all ε there exists an $N \in \mathbb{N}$ such that given $m, n > N, d(x_n, x_m) < \varepsilon$.

e) ℓ is dense in c_0 if and only if for each $x \in c_0$ there is a sequence $\{y_n\}$ from ℓ converging to x. Thus we want to show that given an x from c_0, we can always produce a sequence $\{y_n\}$ from ℓ converging to x. Observe that the sequence $\{0, 0, 0, \dots\}$ from ℓ satisfies this for all sequences $x \in c_0$. If $\{y_n\}$ are to converge to x means that

$$\lim_{n\to\infty} d(y_n, x_n) = 0$$

Since all sequences in c_0 converge to zero, the distance between elements in the sequence y_n and x must tend to zero. Thus, ℓ is dense in c_0.

Problem 4. *Let X denote the open interval $(0, \infty) \subset \mathbb{R}$. Let $d : X \times X \to \mathbb{R}$ be defined as*

$$d(x, y) = |\ln(x) - \ln(y)|\,.$$

Solution:

a) Again, to show that d is a metric we must show the three properties. The first two, positivity and symmetry are trivial. We need to show that the triangle inequality holds. That is,

$$\begin{aligned}
d(x, z) &= |\ln(x) - \ln(z)| \\
&= |\ln(x) - \ln(y) + \ln(y) - \ln(z)| \\
&= \left|\ln\left(\frac{x}{y}\right) + \ln\left(\frac{y}{z}\right)\right| \le \left|\ln\left(\frac{x}{y}\right)\right| + \left|\ln\left(\frac{y}{z}\right)\right| \\
&= |\ln(x) - \ln(y)| + |\ln(y) - \ln(z)| \\
&= d(x, y) + d(y, z)
\end{aligned}$$

Thus, d is a metric on X.