Mandatory assignment 2
MAT2400

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Problem 1

a) We want to show that $h(A,B) = 0$ implies that $A = B$. We assume that A and B are two non-empty closed subsets of X. In other words, $A, B \in P(X)$. If we can show the contrapositive, then the original implication must hold. We therefore assume that $A \neq B$ and we want to show that then $h(A,B) \neq 0$.

Let $x \in X$ be such that $x \in A$ and $x \neq B$. By the definition of $\text{dist}(x,A)$ and $\text{dist}(x,B)$ we see that since $x \in A$, $\text{dist}(x,A) = 0$ and since x is not in B we have $\text{dist}(x,B) > 0$ by the definition of d. Since we take the supremum of all of these values we must have that

$$\sup_{x\in X} |\text{dist}(x,A) - \text{dist}(x,B)| \geq \text{dist}(x,B) > 0.$$

Hence, $h(A,B) = 0$ implies that $A = B$.

b) We now want to show that h is a metric on $P(X)$. Positivity and symmetry follows directly from the absolute values in the expression for $h(A,B)$. We therefore need to show that h obeys the triangle inequality.

$$\begin{aligned}
h(A,B) &= \sup_{x\in X} |\text{dist}(x,A) - \text{dist}(x,B)| \\
&= \sup_{x\in X} |\text{dist}(x,A) - \text{dist}(x,C) + \text{dist}(x,C) - \text{dist}(x,B)| \\
&\leq \sup_{x\in X} \left(|\text{dist}(x,A) - \text{dist}(x,C)| + |\text{dist}(x,C) - \text{dist}(x,B)|\right) \\
&\leq \sup_{x\in X} |\text{dist}(x,A) - \text{dist}(x,C)| + \sup_{x\in X} |\text{dist}(x,C) - \text{dist}(x,B)| \\
&= h(A,C) + h(C,B)
\end{aligned}$$

Hence h is a metric on $P(X)$.

c) We want to show the two inequalities

$$h(A,B) \geq \hat{h}(A,B), \tag{1}$$

$$\hat{h}(A,B) \geq h(A,B). \tag{2}$$

Problem 2

a)

Convergence

We want to show that the series

$$\Upsilon = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}$$

converges uniformly for all $x \in \mathbb{R}$ and that its sum equals $P_r(x)$. We observe that we can split Υ into three subseries:[1]

$$\Upsilon = 1 + \sum_{n=1}^{\infty} r^n e^{inx} + \sum_{n=1}^{\infty} r^n e^{-inx}.$$

Applying Eulers formula, we can simplify this to the series[2]

$$\Upsilon = 1 + \sum_{n=1}^{\infty} r^n \cos(nx) + \sum_{n=1}^{\infty} r^n \cos(nx) = 1 + 2\sum_{n=1}^{\infty} r^n \cos(nx).$$

We know that since we have $0 < r < 1$ and $\cos(nx) \leq 1$ that

$$\Upsilon \leq 1 + 2\sum_{n=1}^{\infty} r^n,$$

but since this series converges, so must Υ.

Sum

We want to show that the sum of Υ is equal to

$$P_r(x) = \frac{1 - r^2}{1 - 2r\cos(x) + r^2}.$$

We rewrite Υ to the following form, in order for us to be able to apply the formula for finite geometric series:[3]

$$\Upsilon = \lim_{N\to\infty} \left(\sum_{n=0}^{N} \left(e^{ix} r\right)^n + \sum_{n=0}^{N} \left(e^{-ix} r\right)^n - 1 \right).$$

We then get that the sum of Υ is equal to

$$\Upsilon = \lim_{N\to\infty} \left(\frac{1 - (e^{ix} r)^{N+1}}{1 - e^{ix} r} + \frac{1 - \left(e^{-ix} r\right)^{N+1}}{1 - e^{-ix} r} - 1 \right).$$

Applying Eulers formula again which yields[4]

$$\Upsilon = \frac{1 - r^2}{1 - 2r\cos(x) + r^2} = P_r(x).$$

[1] Note the change of summation index and the removal of absolute values

[2] The imaginary terms cancel out

[3] Notice again the change of summation index. That is where the -1 originates from.

[4] Some algebra required

b) We want to show that $P_r(x)$ is positive or zero for all x. We have

$$\begin{aligned}
P_r(x) &= \frac{1-r^2}{1-r2\cos(x)+r^2} \\
&\geq \frac{1-r^2}{1-2r+r^2} \\
&= \frac{(1-r)(1+r)}{(1-r)^2} = \frac{1+r}{1-r} > 1.
\end{aligned}$$

Therefore, we can conclude that no matter what x is, $P(x)$ is greater or equal to zero.

c) We want to show that $P_r(x) \to 0$ as $r \uparrow 1$ on the interval $X = [-\pi, -\delta] \cup [\delta, \pi]$. We first state the definition of uniform convergence.[5]

> A sequence $\{f_n\}$ of functions converges uniformly to a function f if and only if for all $\varepsilon > 0$ there exists an $N > 0$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$ and for all $n \geq N$.

In our case, we need to rewrite $P_r(x)$ to a form which we can express in terms of a natural number n. We want to create the sequence $\{r_n\}$ of rational numbers defined as

$$r_n = \frac{n-1}{n}.$$

This series converges to 1 as $n \to \infty$. We now want to show that for all $\varepsilon > 0$ there exists an $N > 0$ such that what was stated above holds. That is

$$|P_{r_n}(x) - 0| < \varepsilon,$$

for all $x \in X$ and for all $n > N$. Writing it out and taking the limit we see that

$$\lim_{n\to\infty} \frac{1-\left(\frac{n-1}{n}\right)^2}{1-\left(\frac{n-1}{n}\right)\cos(x)+\left(\frac{n-1}{n}\right)^2} = \frac{0}{2-2\cos(x)} = 0 < \varepsilon.$$

We have $\cos(x) \neq 1$ for all $x \in X$ since $0 \notin X$, therefore our proof is done.

d) We want to show that

$$\int_{-\pi}^{\pi} P_r(x)\,dx = 2\pi.$$

Recall that

$$P_r(x) = 1 + 2\sum_{n=1}^{\infty} r^n \cos(nx).$$

We can therefore write the integral as follows

$$\int_{-\pi}^{\pi} \left(1 + 2\sum_{n=1}^{\infty} r^n \cos(nx)\right) dx = \int_{-\pi}^{\pi} 1\,dx + 2\int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} \left(r^n \cos(nx)\right)\right) dx$$

[5] This is a bit informal, but used just as a reminder

We know that since $P_r(x)$ converges uniformly on all of $\mathbb{R}$ the sum and integrals are interchangeable.[6] Therefore we can rewrite this expression as

$$\int_{-\pi}^{\pi} 1\,dx + 2\sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} r^n \cos(nx)\,dx\right).$$

The integral contained in the sum evaluates to zero, therefore we are left with

$$\int_{-\pi}^{\pi} P_r(x)\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi,$$

and we are done.

e) We want to show that

$$\Omega = \lim_{r\uparrow 1}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)P_r(y)\,dy = f(x).$$

Seeing that this expression looks a lot like a convolution we're going to look at the properties of the Dirichlet kernel in order to show that $\Omega = f(x)$. The Dirichlet kernel is defined as follows:

$$D_N(y) = \sum_{n=-N}^{N} e^{iny}.$$

We observe that

$$\lim_{r\uparrow 1} P_r(y) = \sum_{n=-\infty}^{\infty} e^{inx} = \lim_{N\to\infty}\sum_{n=-N}^{N} e^{inx} = \lim_{N\to\infty} D_N(y).$$

It can be shown that

$$\lim_{N\to\infty} D_N(y) = \delta(y),$$

where δ is the Dirac Delta Function, which is what we call a *unit impulse*. That is, $\delta(x) = 0$ over the entire real line, except for $x = 0$, however it has the curious property that its integral over the entire real line is equal to one. Multiplying δ by 2π and examining the fourier series we get

$$2\pi\delta = \sum_{n=-\infty}^{\infty} e^{inx} = \lim_{r\uparrow 1} P_r(x)$$

If we look at the *convolution* between f and $2\pi\delta$ over the interval $[-\pi,\pi]$ it can be written as

$$f * 2\pi\delta = \int_{-\pi}^{\pi} f(x-y)2\pi\delta(y)\,dy.$$

however since δ is a unit impulse, the convolution $f * 2\pi\delta$ is what we call a unit impulse response. This means that

$$f * (2\pi\delta) = \int_{-\pi}^{\pi} f(x-y)2\pi\delta(y)\,dy = 2\pi f(x).$$

This becomes evident if we first look at the case where $x = 0$. Since the integral of $\delta(y) = 0$ everywhere except at $y = 0$ where the integral is one, the only contribution to the whole integral is going to be at $y = 0$, at that point we get the term

$$f(0)\delta(0) = f(0)\cdot 1 = f(0).$$

[6] Corollary 4.2.3

If we instead let x be any arbitrary number, we see that at the spike we get

$$f(x-0)\delta(0) = f(x)\cdot 1 = f(x).$$

We can therefore look back at our original expression, we know that the integrand is equal to zero at every point except at the point $y=0$ where the integrand is equal to $2\pi f(x)$.

$$\frac{1}{2\pi}(f*(2\pi\delta) = \frac{2\pi f(x)}{2\pi} = f(x)$$

g) We want to show that

$$\lim_{r\uparrow 1}\sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{inx} = f(x).$$

The big question here is under what conditions we are allowed to interchange limits and sums. We know from (f) that the series converges uniformly and absolutely. We are therefore allowed to interchange the limit such that we get

$$\sum_{n=-\infty}^{\infty}\lim_{r\uparrow 1} c_n r^{|n|} e^{inx} = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

which we recognise as the Fourier series for $f(x)$.