MAT2400
Assignment 1

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Problem 1. *Show that a strictly increasing function* $f : \mathbb{N} \to \mathbb{N}$ *must satisfy* $f(n) \geq n, \forall n \in \mathbb{N}$.

Solution: Assume that $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. By the definition of a strictly increasing function we know that

$$f(n+1) > f(n), \forall n \in \mathbb{N}.$$

We can now, since we are working with the natural numbers, easily show this inductively. Let us first show the base case.

$$f(1) \geq 1$$

This is intuitively true, because 1 is defined as the least element of the set of natural numbers. Assuming that we have verified this as true for all n up to and including some number k. We know want to show that it then follows that it must be true for $k+1$. By assumption:

$$f(k) \geq k$$

Using the standard metric in $\mathbb{R}$ we can see that for any two pairs of successive integer numbers,

$$\inf \{d(k, k+1) \mid k \in \mathbb{N}\} = 1$$

where,

$$d(x, y) = |x - y|$$

That is, the smallest distance possible with two different numbers is 1. It then follows that

$$\begin{aligned} f(k+1) > f(k) + 1 &\geq k + 1 \\ f(k+1) &\geq k + 1 \end{aligned}$$

as we wanted to show. Thus, by the induction principle, a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$, must neccessary satisfy $f(n) \geq n, \forall n \in \mathbb{N}$.

Problem 2. *Let $(X, d))$ be a complete metric space. Let $B(x, r)$ denote the open ball centered at $x \in X$ with radius r, i.e.,*

$$B(x, r) = \{y \in X \mid d(x, y) < r\},$$

and $\overline{B}(x, r)$ the closed ball of radius r, i.e.,

$$\overline{B}(x, r) = \{y \in X \mid d(x, y) \leq r\}.$$

For any set $C \in X$, let $\overline{C}$ denote its closure. Is it true that for any complete metric space X,

$$\overline{B(x, r)} = \overline{B}(x, r)? \tag{1}$$

Solution: Consider the discrete metric,

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

We can show that (1) does not neccessarily hold under the discrete metric. Lets assume we take the radius r to be 1. The open ball $B(x, r)$ is then any two points with less than a distance r between. Thus the open ball only contains the point x. In that case, taking the closure of this open ball changes nothing, and we're left with just the point x. However, the closed ball $\overline{B}(x, r)$ has to be the entirety of our space X, since the distance between two points are allowed to be 1. Thus, if we let our metric space be (X, d) with $X = \mathbb{R}$ and d the discrete metric (1) does not hold. We then have a complete metric space (R, d). We then have a complete metric space (R, d).

Problem 3. *Let ℓ be the set of sequences of real numbers where only a finite number of terms are different from zero,*

$$\ell = \{\{x_n\}_{n=1}^{\infty} \mid x_i = 0 \textit{ for all but a finite number of } i\textit{'s}\}.$$

For $x = \{x_n\}$ and $y = \{y_n\}$ in ℓ, define

$$d(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n|.$$

a) To show that d is a metric on ℓ we must show the three properties of a metric function.

1. Positivity: Since the metric is defined as the biggest difference between to corresponding elemnts from $\{x_n\}$ and $\{y_n\}$, the metric must neccesarily satisfy the property of positivity since there does exists a finite number of non-zero elements in each sequence. Thus, $d(x, y) \geq 0$ with equality only if $x = y$.

2. Symmetry:

$$d(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n| = \sup_{n \in \mathbb{N}} |y_n - x_n| = d(y, x).$$

   Thus the metric is symmetric.

3. Triangle Inequality: Want to show that given three sequences x, y, z, the metric satisfies

$$d(x, z) \leq d(x, y) + d(y, z).$$

The trivial case, when $x = y = z$ is just that, trivial. Thus we assume that x, y and z are not equal.

b) Letting $u_k \in \ell$ be defined as

$$u_k = \left\{1, \frac{1}{2}, \dots, \frac{1}{k}, 0, 0, 0, \dots\right\} \tag{2}$$

we want to show that $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in (ℓ, d). We can do this using a traditional $\varepsilon - N$-proof. We want to show that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all integers $m, n > N, d(u_m, u_n) \leq \varepsilon$.