University of Oslo
Real Analysis - MAT2400
Assignment 2

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Problem 1. Let (X, d) be a bounded metric space, and let $P(X)$ denote the collection of non-empty closed subsets of X. For A and B in $P(X)$, let

$$h(A, B) = \sup_{x \in X} |\text{dist}(x, A) - \text{dist}(x, B)|,$$

where $\text{dist}(x, C)$ is given by

$$\text{dist}(x, C) = \inf_{c \in C} d(x, c).$$

The function h is called the *Hausdorff metric.*

a) Show that if $h(A, B) = 0$ then $A = B$. Here A and B are two non-empty closed subsets of X.

b) Show that h is a metric on $P(X)$.

c) For A and B in $P(X)$, let $\hat{h}$ be defined as

$$\hat{h}(A, B) = \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\}.$$

Show that

$$\hat{h}(A, B) = h(A, B) \text{ for all } A, B \text{ in } P(X).$$

(**Hint:** Show the two inequalities $h(A, B) \geq \hat{h}(A, B)$ and $\hat{h}(A, B) = h(A, B)$.)

Solution 1. Before I start, I want to jot down the properties of the various mathematical objects presented. We are given a metric space (X, d) and it is said to be bounded. What this means, is that there exists some number r such that $d(x, y) \leq r$ for all $x, y \in X$. $P(X)$ is a collection of non-empty closed subsets of X. In other words, the elements of $P(X)$ are sets that contain their own boundary.

We want to show $h(A, B) = 0 \implies A = B$. We want to show $A \neq B \implies h(A, B) \neq 0$. From the left side of our implication we have

$$\exists y \in X : y \notin A \lor y \notin B$$

Let us now look at $|\text{dist}(y, A) - \text{dist}(y, B)|$.

$$|\text{dist}(y, A) - \text{dist}(y, B)| = |\inf_{a \in A} d(y, a) - \inf_{b \in B} d(y, b)| = |d(y, a') - d(y, b')|$$

, for some $a' \in A$ and some $b' \in B$.

Since $y \notin A$ and A is closed we can construct a ball $B(y, \varepsilon_2)$ such that $B(y, \varepsilon) \cap A = \emptyset$. Therefore, $d(y, a) > \varepsilon_1$. Similarly we can construct a ball around y whose intersection with B is empty, and therefore $\text{dist}(y, A) > 0$ and $\text{dist}(y, B) > 0$.

Problem 2. Let $0 < r < 1$ and consider the series

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}.$$

Show that the series converges uniformly for all $x \in \mathbb{R}$, and that its sum equals

$$P_r(x) = \frac{1 - r^2}{1 - 2r\cos(x) + r^2}.$$