University of Oslo
Real Analysis - MAT2400
Spring 2015

Ivar Stangeby

March 25, 2015

Abstract

This document is going to be a way for me as a student of MAT2400 at the University of Oslo to gather my thoughts around the course Real Analysis. I've struggled with my intuition for this subject, and this is a last ditch effort to build it all up from scratch.
For this task, I've decided to use the text books written by Terence Tao, namely Analysis I and II. There is absolutely nothing wrong with the text book offered at my university, it is brilliantly written, but I have been exposed to the writings of Terence Tao before, and therefore I wish to give his books a try.
The first part of this document related to the book Analysis I, is going to be mostly involved in building a solid foundation for the concepts discussed in Analysis II. The material included from Analysis I is very similar to the curriculum of the subject MAT1140 at the University of Oslo.
The structure of this document is going to be me writing down the results encountered throughout the text books along the proofs I find extra intriguing. I'm going to attempt to prove the theorems myself, and if I find it reasonable I'm going to write down my own proof. Included will also be my attempted solutions to selected exercises.
This document is mainly for my own good and well being, but if anyone can find any use from them, then that is great.

Chapter 1

Introduction

Chapter 2

Starting at the beginning: the natural numbers.

In order for us to start exploring the various properties of the real numbers, which is what real analysis is concerned with, we are going to have to start from the very beginning. That is the natural numbers, denoted $\mathbb{N}$. From these natural numbers, we can construct the integers, $\mathbb{Z}$, the rationals $\mathbb{Q}$, the real numbers $\mathbb{R}$, and finally; the complex numbers $\mathbb{C}$. The latter being the main focus of the subject Complex Analysis.

2.1 The Peano axioms

One of the most standard ways of defining the natural numbers, is in terms of the *Peano axioms.* One can also define natural numbers through the notion of cardinality.

Definition 2.1.1 (Informal)**.** A *natural number* is any element of the set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\},$$

which is the set of all the numbers created by starting with 0 and then counting forward indefinitely. We call $\mathbb{N}$ the *set of all natural numbers.*

In order for us to rigorously define the set of natural numbers, we're going to use the two fundamental concepts of *the number 0* and the *increment operation.* These will be covered in the Peano Axioms. We will use $n++$ to denote the *successor* of n.

Starting with the first two:

Axiom 2.1. *0 is a natural number.*

Axiom 2.2. *If n is a natural number, then $n++$ is also a natural number.*

Now, in order to avoid having to deal with incredibly long strings of +'es. We're going to use an auxilliary definition.

Definition 2.1.2. We define 1 to be the number $0++$, 2 to be the number $(0++)++$, etc.

We can based off of this, propse the following:

Proposition 2.1.3. *3 is a natural number.*

Proof. By Axiom 1, 0 is a natural number. It then follows by Axiom 2 that both 1, 2, and 3 are natural numbers. □

In order for us to avoid the problem of having the successive numbers wrap around to previous numbers, we impose a new axiom, namely:

Axiom 2.3. *0 is not the successor of any natural number: i.e., we have $n++ \neq 0$ for every natural number n.*

We can, equipped with this new axiom, show for example the following:

Proposition 2.1.4. 4 *is not equal to* 0*.*

Proof. By definition, $4 = 3++$. By the first two axioms, 3 is a natural number. Thus, since 0 is not the successor of any natural number, $3++ \neq 0$, i.e., $4 \neq 0$. □

Assuming the following axiom allows us to rule out any behaviour where the successors wrap around, but not to 0, i.e., $5++ = 1$.

Axiom 2.4. *Different natural numbers must have different successors; i.e., if n, m are natural numbers and $n \neq m$, then $n{+}{+} \neq m{+}{+}$. Equivalently, if $n++ = m++$, then we must have $n = m$.*

We can now prove extensions of the previous proposition where we do not have zeroes on the right hand side of the equation.

Proposition 2.1.5. 6 *is not equal to* 2.

Proof. Assume for contradiction that $6 = 2$. By the previous axiom we must have $5++ = 1++$. Applying the same axiom again, we have $5 = 1$ so that $4++ = 0++$. But, this leads to a contradiction, because by the same axiom, $4 = 0$. This contradicts our previously proven proposition. □

Assume now that we are presented with a weird number system

$$\mathbb{N} = \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, \dots\}.$$

Even though this set contains real numbers, which we haven't defined or talked about yet, it satisfies all the previous axioms. But this is not the number system we're interested in. We want our set of natural numbers to only be containing all the numbers that can be directly derived from 0 just using the successor operation.

We want to introduce some axiom that does not allow other forms of successors to occur. Therefore we introduce the following:

Axiom 2.5 (Principle of mathematical induction)**.** *Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true for every natural number n.*

We're now equipped with the tools required to deal with propositions of the following form:

Proposition 2.1.6. *A certain property $P(n)$ is true for every natural number n.*

Proof. Using induction, we show the base case of $P(0)$. Assume, for the sake of induction, that $P(n)$ is true. We now want to show that it has to follow that $P(n{+}{+})$ also must be true. If this is the case, we have shown, using mathematical induction that $P(n)$ is true for every natural number n. □

The previous five axioms are known as the *Peano Axioms* for the natural numbers. We now want to more rigorously define the kind of number system we are to refer to as the *natural numbers*.

Assumption 2.1.7 (Informal)**.** *There exists a number system $\mathbb{N}$, whose elements we will call natural numbers, for which Axioms 1-5, are true.*

This number system is what we refer to as *the* natural number system. But one should not rule out the possibility that there are more than one natural number system. But as long as these are *isomorphic* one can consider them as equal.

With only this, rather simplistic definition of natural numbers, the five axioms, some axioms from set theory we can build all other number systems, create functions and do the algebra and calculus that we are used to.

An very common question now arises, and this is about the finiteness or infiniteness of the natural number system. How can something infinite come from something strictly finite? One can easily show that all the natural numbers are finite. It is clear that 0 is finite. If n is finite, then clearly $n++$ is finite. Therefore all natural numbers are finite. It then follows that infinity is not a natural numbers. There are other number systems that admit the infinite numbers.

It is an interesting fact that the definition of $\mathbb{N}$ is *axiomatic* rather than *constructive*. This means, that so far we're only conserned with what the natural numbers are, not what they do, what they measure or what they can be used for.

As long as a mathematical model obeys the previous axioms, it is of no concern whether which mathematical model is "true". It is this form of *abstractness* that makes mathematics so useful. One does not neccesarily need a concrete model, because the numbers can be understood abstractly through the use of axioms.

As a consequence of the axioms previously discussed, we can now define sequences *recursively*. That is, start with some base value and then building the next value in the sequence by means of a function. This leads to the following:

Proposition 2.1.8 (Recursive definitions)**.** *Suppose for each natural number n, we have some function $f_n : \mathbb{N} \to \mathbb{N}$ from the natural numbers to the natural numbers. Let c be a natural number.*

Then we can assign a unique natural number a_n to each natural number n, such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each natural number n.

Proof. Using induction, we verify the base case. We clearly see that this procedure gives a single value to a_0, namely c. (We know from axiom 3 that a_0 won't be redefined.) Suppose now inductively that the procedure gives a single value to a_n. Then it gives a single value to a_{n++}, namely $a_{n++} = f_n(a_n)$. (We know from axiom 4 that a_{n++} won't be redefined.) This completes the induction, since a_n is defined for every natural number n, with a single value assigned to each a_n. □

Equipped with the tools that are recursive definitions we can now define multiple operations on the set of natural numbers. Up until now, we've only dealt with one, being the increment operation.

2.2 Addition

Currently our number system does not support any more advanced operations than incrementing a number. We now turn our heads to addition. The operation is simple. To add 3 to 5, we simply increment 5 three times. This is one increment more than adding 2 to 5, which is one increment more than adding 1 to 5, which is one increment more than adding 0 to 5. We can therefore easily give a recursive definition of addition.

Definition 2.2.1 (Addition of natural numbers)**.** Let m be a natural number. To add zero to m, we define $0+m = m$. Now suppose inductively that we have defined how to add n to m. Then we can add $n++$ to m by defining $(n++)+m = (n+m)++$.

For example, $2+3 = (3++)++ = 4++ = 5$. By using the priciple of mathematical induction, we see that we now have defined $n+m$ for every natural number n. We are specializing the previous general discussion about recursive definitions to the setting where $a_n = n+m$, and $f_n(a_n) = a_n++$.

It's worth noting that this definition is actually *asymmetric*. That is, while yielding the same result, $3+5$ is incrementing 5 three times, where as $5+3$ is incrementing 3 five times. We shall soon see, that it is a general fact that $a+b = b+a$ for all natural numbers a, b.

One can easily prove, using the first two axioms and the principle of mathematical induction to show that the sum of two natural numbers us again a natural number.

At the present moment, we have only two facts about addition. However, this is perfectly sufficient to deduce everything else we know about addition. Starting with some basic lemmas.

Lemma 2.2.2. *For any natural number n, $n+0 = n$.*

This lemma is not obvious from our previous definition, since we still do not know that $a+b = b+a$.

Proof. Using induction. The base case $0+0 = 0$ follows from the definition of addition of natural numbers. $0+m = m$ for all natural numbers, and 0 is known to be a natural number. Suppose inductively that $n+0 = n$. We now wish to show that $(n++)+0 = n++$. By definition of addition yields that $(n++)+0$ is equal to $(n+0)++$, which is equal to $n++$ since $n+0 = n$. This closes the induction. □

Lemma 2.2.3. *For any natural numbers n and m, $n+(m++) = (n+m)++$.*

Again, this is not obvious.

Proof. We induct on n, keeping m fixed. Considering the base case $n = 0$. We therefore have to prove $0+(m++) = (0+m)++$. By definition of addition, we have $0+(m++) = m++$ and $0+m = m$. So, both sides are equal to $m++$ and are thus equal. Assuming now, that we have shown $n+(m++) = (n+m)++$, we want to show that $(n++)+(m++) = ((n++)+m)++$. Looking at the left hand side. By definition of addition it is equal to $(n+(m++))++$, which in turn, by the inductive hypothesis, is equal to $((n+m)++)++$. Now, examining the right hand side. By the definition of addition, it is equal to $((n+m)++)++$. The two sides are equal, and this closes the induction. □

We can now easily show the following result:

Corollary. *For all natural numbers n, $n++ = n+1$.*

Proof. This is a special case of the previous lemma, where $m = 0$. We have $n + (m{+}{+}) = (n + m){+}{+}$. Setting $m = 0$, we get $n + (0{+}{+}) = (n + 0){+}{+}$. Evaluating the left hand side we get $n+1$ and evaluating the right hand side, we get $(n){+}{+} = n{+}{+}$, which is what we wanted to show. □

Now for one of the first major results. We can now prove that $a + b = b + a$.

Proposition 2.2.4 (Addition is commutative)**.** *For any natural numbers n and m, $n+m = m+n$.*

Proof. Using induction on n keeping m fixed. First showing the base case, where $n = 0$. We want to show that $0 + m = m + 0$. By the definition of addition, the left hand side is equal to m. By lemma 2.2.2, the right hand side is equal to m. Therefore, the base case is true. Now, assuming that it is shown that $n+m = m+n$. We now want to show that $(n{+}{+}) + m = m + (n{+}{+})$. Looking at the left hand side, we see that it is equal to $(n+m){+}{+}$, by definition of addition. The right hand side, by lemma 2.2.3 must be equal to $(m+n){+}{+}$. Since we assumed $n+m = m+n$, the left and right hand side is equal and therefore the induction is closed. □

Proposition 2.2.5 (Addition is associative)**.** *For any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.*

Proof. See Exercise 2.2.1. □

Proposition 2.2.6 (Cancellation law)**.** *Let a, b, c be natural numbers such that $a + b = a + c$.*

Since we haven't explored the concept of subtraction or negative numbers yet we cannot use these properties to prove this law. This law is actually crucial in defining subtraction rigorously.

Proof. We prove this with induction on a, keeping b and c fixed. Showing the base case with $a = 0$. We have $0 + b = 0 + c$. Using the definition of addition, $b = c$. Now, for the inductive hypothesis. Assuming that it is shown that $a + b = a + c$, we want to show that $(a{+}{+}) + b = (a{+}{+}) + c$ implies $b = c$. Left hand side evaluates to $(a + b){+}{+}$ and the right hand side evaluates to $(a + c){+}{+}$ by the definition of addition. By Axiom 2.4 we see that $(a + b) = (a + c)$, and therefore by our assumption $b = c$. This closes the induction. □

We now want to look at how natural numbers interacts with positivity. First, a definition:

Definition 2.2.7 (Positive natural numbers)**.** A natural number n is said to be *positive* if and only if it is not equal to 0.

This leads to the following proposition.

Proposition 2.2.8. *If a is positive and b is a natural number, then $a+b$ is positive (and hence $b+a$ is also, by Proposition 2.2.4).*

Proof. Using induction on b. Showing the base case where $b = 0$. Assuming a a positive number and b a natural number. We then have $a + b = a + 0 = a$, and by assumption a is a positive number. Now for the inductive step. We assume shown that $a+b$ is a positive number. We want to show that $a + (b{+}{+})$ must also be a positive number. Using the commutativity of natural numbers and the definition of addition we can show that this must be equal to $(a + b){+}{+}$. Since we know that 0 is not the successor to any number, and that $(a + b)$ is positive, we must have $(a + b){+}{+} \neq 0$. This closes the induction. □

Corollary 2.2.9. *If a and b are natural numbers such that $a + b = 0$, then $a = 0$ and $b = 0$.*

Proof. Assume for contradiction that $a \neq 0$ and $b \neq 0$. Since $a \neq 0$ then it is positive by definition, and then it follows by 2.2.8 that $a + b$ is positive. The same argument for $b \neq 0$. Therefore, our assumptionds leads to contradictions. In other words, $a = 0, b = 0$. □

Lemma 2.2.10. *Let a be a positive number. Then there exists exactly one natural number b such that $b{+}{+} = a$.*

Proof. See Exercise 2.2.2. □

We can now, since we have a notion of addition, proceed with defining a notion of order on the natural numbers.

Definition 2.2.11 (Ordering of the natural numbers)**.** Let n and m be natural numbers. We say that n is *greater than or equal to* m, and write $n \geq m$ or $m \leq n$, if and only if we have $n = m + a$ for some natural number a. We say that n is *strictly greater than* m and write $n > m$ or $m < n$, if and only if $n \geq m$ or $m \leq n$ and $n \neq m$.

An example would be $8 > 5$ because $8 = 5 + 3$ and $8 \neq 5$. Another important thing to note is that $n{+}{+} > n$ for all natural numbers n. This means that there are no largest natural number n, because the next number $n{+}{+}$ is always larger.

Proposition 2.2.12 (Basic properties of order for natural numbers)**.** *Let a, b, c be natural numbers. Then*

(a) (Order is reflexive) $a \geq a$.

(b) (Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$.

(c) (Order is anti-symmetric) If $a \geq b$ and $b \geq a$, then $a = b$.

(d) (Addition preserves order) $a \geq b$ if and only if $a + c \geq b + c$.

(e) $a < b$ if and only if $a + + \leq b$.

(f) $a < b$ if and only if $b = a + d$ for some positive number d.

Proof. See Exercise 2.2.3. □

Proposition 2.2.13 (Trichotomy of order for natural numbers)**.** *Let a and b be natural numbers. Then exactly one of the following statements is true: $a < b, a = b$ or $a > b$.*

Proof. The gaps of this proof will be filled in Exercise 2.2.4.

The first step is going to be showing that no more than one of the statements can hold at any given time. That is, assuming $a < b$, then $a \neq b$ by definition. If $a > b$, then $a \neq b$ by definition. Assuming $a < b$ and $b > a$, we have $a = b + m$ and $b = a + n$. Substituting we get $a = a + n + m$, and using the cancellation law for addition we get $0 = n + m$. By 2.2.9 we get $n = 0$ and $m = 0$, thus $a = b$ which is a contradiction. Therefore only one of the statements may apply at any given time.

We must now show that at least one of the statemements must apply. Keeping b fixed we induct on a. Considering the base case where $a = 0$ yields $0 \leq b$ for all b. In other words, either $0 = b$ or $0 < b$. For the inductive hypothesis we assume shown that at least one of the three holds for two natural numbers a and b. We now want to show that it must neccesarily hold at least one for $a{+}{+}$ and b.

If $a < b$ then $a{+}{+} \leq b$, thus $a{+}{+} = b$ or $a{+}{+} < b$ by 2.2.12 If $a = b$ then $a{+}{+} = a{+}1 = b{+}1$ and thus by definition of the ordering of natural numbers, we have $a{+}{+} > b$. If $a > b$ then $a = b + n$ for some number n. It then follows that $a{+}{+} = (b + n){+}{+}$ which in turn is equal to $b + (n{+}{+}) = b + (n + 1)$. In other words, by definition $a{+}{+} > b$. This closes the induction. □

Armed with the previous proposition, we can now obtain a stronger version of the principle of induction.

Proposition 2.2.14 (Strong principle of induction)**.** *Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.*

Proof. See Exercise 2.2.5. □

Exercise 2.2.1. Prove 2.2.5. (Hint: fix two of the variables and induct on the third).

Solution: Fixing b and c we induct on a. Examining the base case where $a = 0$ yields $(0 + b) + c = 0 + (b + c)$. Evaluating both sides using the definition of addition of natural numbers yields $b + c$ on both sides. Therefore the base case holds.

For the inductive hypothesis we assume that we have shown that $(a + b) + c = a + (b + c)$ holds. We want to show that it also holds for $a{+}{+}$. We get $((a{+}{+}) + b) + c = (a{+}{+}) + (b + c)$. Evaluating the left hand side by applying the definition of addition twice, we get

$$\begin{aligned} ((a{+}{+}) + b) + c &= ((a + b){+}{+}) + c \\ &= ((a + b) + c){+}{+}. \end{aligned}$$

Similarily, evaluating the right hand side yields

$$(a{+}{+}) + (b + c) = (a + (b + c)){+}{+}.$$

By our assumption we know $a + (b + c) = (a + b) + c$, and therefore this closes the induction. □

Exercise 2.2.2. Prove 2.2.10. (Hint: use induction.)

SOLUTION: Let a be a positive number. We want to show that there exists exactly one natural number b such that $b{+}{+} = a$. Fixing b we induct on a. We first want to show the base case where $a = 1$. It then follows that if we set $b = 0$, then we have $b{+}{+} = 0{+}{+} = 1 = a$.

For the inductive hypothesis we assume that we have shown that there exists exactly one natural number b such that $b{+}{+} = a$. We want to show that it then follows that there exists exactly one natural number c such that $c{+}{+} = a{+}{+}$. Substituting this for a we get $c{+}{+} = (b{+}{+}){+}{+}$. From axiom 4 we know that $c = b{+}{+}$. This closes the induction.

□

Exercise 2.2.3. Prove 2.2.12. (Hint: You will need many of the preceding propositions, corollaries and lemmas.)

SOLUTION: Proving basic properties of order for natural numbers.

(a) $a \geq a$.

From the assumption, we have $a = a$ or $a > a$. For $a > a$ there must exist a positive number b such that $a = a + b$. Using the law of cancellation, we get $b = 0$, which contradicts the fact that b is positive. Therefore $a = a$, which always holds. In any case, we're done.

(b) If $a \geq b$ and $b \geq c$, then $a \geq c$.

From the assumption, there exists natural numbers m and n such that $a = b + m$ and $b = c + n$. Substituting we get $a = (c + n) + m$. Using the associativity of addition of we get $a = c + (n + m)$. Since $(n + m)$ is a natural number, by definition $a \geq c$.

(c) If $a \geq b$ and $b \geq a$ then $a = b$.

If $a \geq b$, then there exist a natural number c such that $a = b + c$. Also, if $b \geq a$ then there exists a natural number d such that $b = a + d$. Substituting we get $a = (a + d) + c$. Using the associativity of the natural numbers, we have $a = a + (d + c)$. The law of cancellation gives $0 = (d + c)$. From 2.2.9 we have $d = 0$, $c = 0$. Again, substituting yields $a = 0 + b = b$ and $b = 0 + a = a$.

(d) $a \geq b$ if and only if $a + c \geq b + c$.

We argue contrapositively. Assume that $a+c < b+c$. By definition, then there exists a natural number d such that $b + c = (a + c) + d$. Using associativity and the law of cancellation, this simplifies to $b = a + d$, which in turn means that $a < b$. We have now shown the contrapositive of the right implication. Thus the right implication holds.

Now to show the left implication. Assume that $a + c \geq b + c$. We see that there exist a natural number d such that $a + c = (b + c) + d$ (equality when $d = 0$). Using associativity and the law of cancellation, we get $a = b + d$. This is the definition of *greater than*, therefore we can conclude that $a \geq b$.

We have now shown both implications and the statement holds.

(e) $a < b$ if and only if $a{+}{+} \leq b$.

Assume $a < b$, then by definition there exists a positive number c such that $b = a + c$. Substituting this into the right hand side we achieve $a{+}{+} \leq a + c$. Rephrased, $a + 1 \leq a + c$. This clearly holds true for any positive number c. (Can be shown rigorously by mathematical induction).

Now, assume that $a{+}{+} \leq b$. In other words, there exists a natural number d such that $b = (a{+}{+}) + d$. (d is allowed to be zero). Using the law of associativity we can deduce that $b = a + (1 + d)$. In other words, the definition of $a < b$.

The two implications have been shown and the statement holds.

(f) $a < b$ if and only if $b = a + d$ for some positive number d.

Assume $a < b$. From the definition, we get $a \leq b$ and $a \neq b$. $a \leq b$ means that there must exist a natural number n such that $a + n = b$. In the case where $n = 0$, $a = b$ which is a contradiction of our assumption, therefore n must be positive by definition of positive since it is not equal to 0. Let $d = n$, and we have shown that $a < b$ if $b = a + d$ for some positive number d.

Now, assume that $b = a + d$ for some positive number d. Since d is not 0 by definition, we know that $b \neq a$. Therefore, we have both $a \leq b$ as well as $a \neq b$. This is the definition of $a < b$, and we are done.

$\square$