Chapter 1

Metric Spaces

1.1 Definitions and examples

Proposition 1 (Inverse Triangle Inequality). *For all elements x, y, z in a metric space (X, d), we have:*

$$|d(x, y) - d(x, z)| \leq |d(y, z)|$$

Proof. Let (X, d) be a metric space, and let x, y, z be three aribtrary elements in X. From the triangle inequality, that we know d satisfy from the definition of a metric space, we have:

$$d(x, y) \leq d(x, z) + d(z, y).$$

Now, in order to show that the inverse triangle inequality holds, note that the absolute value involved is going to be the largest of the two numbers $d(x, y) - d(x, z)$ and $d(x, z) - d(x, y)$. It therefore suffices to show that both of these must be smaller than $d(y, z)$.

Simply rearranging the triangle inequality shows us that the first inequality holds.

$$d(x, y) - d(x, z) \leq d(y, z)$$

Applying the triangle inequality to the points x, z with y as an intermediate point gives us

$$d(x, z) \leq d(x, y) + d(y, z),$$

and this can be rearranged to give the second inequality. $\square$

1.2 Convergence and continuity

Definition 1. Let (X, d) be a metric space. A sequence $\{x_n\}$ in X converges to a point $a \in X$ if there for every $\varepsilon > 0$ exists an $N \in \mathbb{N}$ such that $d(x_n, a) < \varepsilon$ for all $n \geq N$. We write $\lim_{n\to\infty} x_n = a$ or $x_n \to a$.

Lemma 1. *A sequence $\{x_n\}$ in a metric space (X, d) converges to a if and only if $\lim_{n\to\infty} d(x_n, a) = 0$.*

Proof. This is simply a reformulation of the previous definition, but we prove it rigorously by showing that these statements are equivalent. Let (X, d) be a metric space, and let $\{x_n\}$ be a sequence in this metric space.

Let $a \in X$ and assume that $\{x_n\}$ converges to a. By definition of convergence, for every $\varepsilon > 0$ we can chose an $N \in \mathbb{N}$ such that $d(x_n, a) < \varepsilon$ for all $n \geq N$. This simply means that we can force the distance between the elements in $\{x_n\}$ and a to be arbitrarily close to zero by picking elements far out in the sequence.

We can generate a new sequence of the distances between the elements of $\{x_n\}$ and a, namely the sequence $\{d(x_n, a)\}$. Since we have $\lim_{n\to\infty} x_n = a$ we know that the following limit must equate to zero:

$$\lim_{n\to\infty} d(x_n, a) = d(a, a) = 0.$$

Now, assume that $\lim_{n\to\infty} d(x_n, a) = 0$. For this equation to be true, we must have $\lim_{n\to\infty} x_n = a$, because we have equality only when $x_n = a$, by the definition of a metric. But then, by definition, the sequence $\{x_n\}$ converges to a. □

Proposition 2. *A sequence in a metric space can not converge to more than one point.*

Proof. Let $\{x_n\}$ be a sequence in a metric space (X, d). Assume for contradiction that $\{x_n\}$ converges to both the point a and the point a'. By definition of convergence, we have $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n = a'$. Using the triangle inequality we have:

$$d(a, a') \leq d(a, x_n) + d(x_n, a').$$

Taking the limits we get the following inequality:

$$d(a, a') \leq \lim_{n\to\infty} d(a, x_n) + \lim_{n\to\infty} d(x_n, a') = 0 + 0 = 0,$$

but this is only possible if $a = a'$. (The limits equal zero, due to the previous lemma.) □

Definition 2. Assume that (X, d_X) and (Y, d_Y) are two metric spaces. A function $f : X \to Y$ is *continuous* at a point $a \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $d_Y(f(x), f(a)) < \varepsilon$ whenever $d_X(x, a) < \delta$.

Proposition 3. *The following are equivalent for a function $f : X \to Y$ between metric spaces:*

(i) f is continuous at a point $a \in X$.

(ii) For all sequences $\{x_n\}$ converging to a, the sequence $\{f(x_n)\}$ converges to $f(a)$.

Proof. We show that this is true by showing both the left and right implication. Let us first assume that f is continuous at a point $a \in X$. By definition of continuity we have that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $d(f(x), f(a)) < \varepsilon$ whenever $d(x, a) < \delta$.

Assume now that an arbitrary sequence $\{x_n\}$ converges to a. This means that $\lim_{n\to\infty} x_n = a$. Therefore, we must have $\lim_{n\to\infty} f(x_n) = f(a)$ since functions obey the axiom of substitiom from zet theory. Thus the right implication is shown. We now need to show the left implication.

Assume that for all sequences $\{x_n\}$ converging to a, the sequence $\{f(x_n)\}$ converges to $f(a)$. We must now show that this implies that f is continuous at a. Since we have $f(x_n) \to f(a)$ there exists, for all $\varepsilon > 0$ an $N \in \mathbb{N}$ such that picking elements farther than N into the sequence, we can get the distance between these elements and $f(a)$ arbitrarily small. But this is the definition of f being continuous at the point a. $\square$

Chapter 2

Preliminaries

Chapter 3

Spaces of continuous functions

3.1 The spaces $C(X, Y)$

Lemma 2. *Let (X, d_X) and (Y, d_Y) be metric spaces, and assume that X is compact. If $f, g : X \to Y$ are continuous functions, then*

$$\rho(f, g) = \sup \{d_Y (f(x), g(x)) \mid x \in X\}^{1}.$$

is finite, and there is a point $x \in X$ such that $d_Y (f(x), g(x)) = \rho(f, g)$.

Proof. The condition that X is compact is crucial in this situation, because it allows us to make use of the extreme value theorem. This theorem states that if X is compact, and if the function with X as its domain is continuous, then said function has a maximum and minimum point in X.

We introduce the function

$$h(x) = d_Y(f(x), g(x)).$$

We need to show that this function is continuous. We do this by showing that $|h(x) - h(y)|$ can be forced to be less than ε. Using the triangle inequality and the inverse triangle inequality we get

$$\begin{aligned} |h(x) - h(y)| &= |d_Y (f(x), g(x)) - d_Y (f(y), g(y))| \\ &= |d_Y (f(x), g(x)) - d_Y (f(x), g(y)) + d_Y (f(x), g(y)) - d_Y (f(y), g(y))| \\ &\leq |d_Y (f(x), g(x)) - d_Y (f(x), g(y))| + |d_Y (f(x), g(y)) - d_Y (f(y), g(y))| \\ &\leq d_Y (g(x), g(y)) + d_Y (f(x), f(y)) \end{aligned}$$

Since both f and g are continuous functions, we can always chose a $\delta > 0$ such that $\leq d_Y (g(x), g(y)) < \varepsilon/2$ and $d_Y (f(x), f(y)) < \varepsilon/2$.

We therefore have

$$|h(x) - h(y)| \leq d_Y (g(x), g(y)) + d_Y (f(x), f(y)) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

[1] The basic idea is to measure the distance between two functions by looking at the point they are the furthest apart

and by the extreme value theorem $p(f, g)$ is finite and there is a point $x \in X$ such that $d_Y\left(f(x), g(x)\right) = \rho(f, g)$.

$\square$

We can now show that ρ is a metric on $C(X, Y)$.

Proposition 4. *Let (X, d_X) and (Y, d_Y) be metric spaces, and assume that X is compact. Then*

$$\rho(f, g) = \sup\left\{d_Y\left(f(x), g(x)\right) \mid x \in X\right\}$$

defines a metric on $C\left(X, Y\right)$.

Proof. We need to show symmetry, positivity and the triangle inequality.

1. Symmetry

   We need to show that $\rho(f, g) = \rho(g, f)$. Since d_Y is a metric, we have $d_Y(f(x), g(x)) = d_Y(g(x), f(x))$ therefore this property follows from the d_Y metric.

2. Positivity

   Again, this follows from the metric d_Y.

3. Triangle inequality

   Need to show that $\rho(f, g) \leq \rho(f, h) + \rho(h, g)$. By Lemma 2, there is an $x \in X$ such that $\rho(f, g) = d_Y(f(x), g(x))$. We therefore have

$$\begin{aligned}
\rho(f, g) &= d_Y(f(x), g(x)) \\
&\leq d_Y(f(x), h(x)) + d_Y(h(x), g(x)) \\
&= \rho(f, h) + \rho(h, g).
\end{aligned}$$

   as we wanted to show.

$\square$

Proposition 5. *A sequence $\{f_n\}$ converges to f in $(C(X, Y), \rho)$ if and only if it converges uniformly to f.*

Proof. By Proposition 3.2.3, $\{f_n\}$ converges uniformly to f if and only if

$$\sup\left\{d_Y(f_n(x), f(x))\right\} \to 0,$$

as $n \to \infty$. But this is just saying that $\rho(f_n, f) \to 0$ as $n \to \infty$. $\square$

Theorem 1. *Assume that (X, d_X) is a compact and (Y, d_Y) a complete metric space. Then $(C(X, Y), \rho)$ is complete.*

Proof. Assume that $\{f_n\}$ is a Cauchy sequence in $C(X, Y)$. We must prove that it converges, by the definition of a complete space, to a function $f \in C(X, Y)$. $\square$

3.2 Applications to differential equations

3.3 Compact subsets of $C(X, \mathbb{R}^m)$

Definition 3. Let (X, d) be a metric space and assume that A is a subset of X. We say that A is *dense* in X if for each $x \in X$ there is a sequence from A converging to x.

Definition 4. A metric set (X, d) is called *separable* if it has a countable, dense subset A.

Proposition 6. *All compact metric spaces (X, d) are separable. We can choose the countable dense set A in such a way that for any $\delta > 0$, there is a finite subset A_δ of A such that all elements of X are within distance less than δ of A_δ, i.e., for all $x \in X$ there is an $a \in A_\delta$ such that $d(x, a) < \delta$.*

Proof. We need to show that (X, d) has a countable dense subset A. We know that by definition, a compact set X is totally bounded. This means that there exists a finite number of subsets of X whose union contains X. More formally, for all $n \in \mathbb{N}$ there is a finite number of balls of radius $\frac{1}{n}$ that cover X. The centers of these balls constitute a countable subset of X, lets call it A.

All we need to do now, is show that this countable subset A of X is also dense. For A to be dense, we need that for each $x \in X$ there is a sequence from A converging to x. Let x be an element of X. We need to find a sequence $\{a_n\}$ that converges to x in A.

That is $d(\{a_n\}, x) \to 0$ as $n \to \infty$. First we pick the center a_1 of one of the balls of radius 1 that x belongs to. Then we pick the center a_2 of one of the balls of radius $\frac{1}{2}$ that x belongs to, and so on and so forth. To find the set A_δ just chose $m \in \mathbb{N}$ so big that $\frac{1}{m} < \delta$ and let A_δ consist of the centers of the balls of radius $\frac{1}{m}$. $\square$