# Chapter 1

# Probability

## 1.1 Sample Spaces and Events

**The Sample Space of an Experiment:** The **sample space** of an experiment, denoted by $S$, is the set of all possible outcomes of that experiment.

**Events:** An **event** is any collection (subset) of outcomes contained in the sample space $S$. An event is said to be **simple** if it consists of exactly one outcome and **compound** if it consists of more than one outcome.

**Some Relations from Set Theory:**

1. The **union** of two events $A$ and $B$, denoted by $A \cup B$ and read "$A$ or $B$" is the event consisting of all outcomes that are *either in* $A$ *or in* $B$ *or in both events* (so that the union includes outcomes for which both $A$ and $B$ occur as well as the outcomes for which exactly one occurs) - that is, all outcomes in at least one of the events.

2. The **intersection** of two events $A$ and $B$, denoted by $A \cap B$ and read " $A$ *and* $B$ " is the event consisting of all outcomes that are in *both* $A$ *and* $B$.

3. The **complement** of an event $A$, denoted by $A'$, is the set of all outcomes in $S$ that are not contained in $A$.

**Mutually exlusive events:** When $A$ and $B$ have no outcomes in common, they are said to be **disjoint** or **mutually exclusive** events. Mathematicians write this compactly as $A \cup B = \emptyset$ where $\emptyset$ denotes the event consisting of no outcomes whatsoever (the "null" or "empty" event).

## 1.2 Axioms, Interpretations, and Properties of Probability

**Basic properties of probability:**

1. For any event $A, P(A) \geq 0$.

2. $P(S) = 1$.

3. If $A_1, A_2, A_3, \cdots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup A_2 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i).$$

### 1.2.1 Proposition

$P(\emptyset) = 0$ where $\emptyset$ is the null event. This in turn implies that the property contained in Axiom 3 is valid for a *finite* collection of events.[1]

### 1.2.2 Proposition

For any event $A, P(A) = 1 - P(A')$.[2]

### 1.2.3 Proposition

For any event $A, P(A) \leq 1$.

### 1.2.4 Proposition

For any events $A$ and $B$,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B).^3 \\ P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) \\ &\quad - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$

## 1.3 Counting Techniques

**The Product Rule for Ordered Pairs**

### 1.3.1 Proposition

If the first element or object of an ordered pair can be selected in $n_1$ ways, and for each of these $n_1$ ways the second element of the pair can be selected in $n_2$ ways, then the number of pairs is $n_1 n_2$.

**Product Rule for $K$-Tuples:**

Suppose a set consists of ordered collections of $k$ elements ($k$-tuples) and that there are $n_1$ possible choices for the first element; for each choice of the first element, there are $n_2$ possible choices of the second element;...; for each possible of the first $k-1$ elements, there are $n_k$ choices for $k$th element. Then there are $n_1 n_2 \cdot \cdots \cdot n_k$ possible $k$-tuples.

[1]Proof page 57.
[2]Proof page 60.

**Permutations:**

Any ordered sequence $k$ objects taken from a set of $n$ distinct objects is called a **permutation** of size $k$ of the objects. The numbers of permutations of size $k$ that can be constructed from the $n$ objects is denoted $P_{k,n}$.