DEI
DIPARTIMENTO DI
INGEGNERIA DELL'INFORMAZIONE

UNIVERSITÀ
DEGLI STUDI
DI PADOVA

Digital Systems

Exercises on Boolean Algebra and Karnaugh Maps

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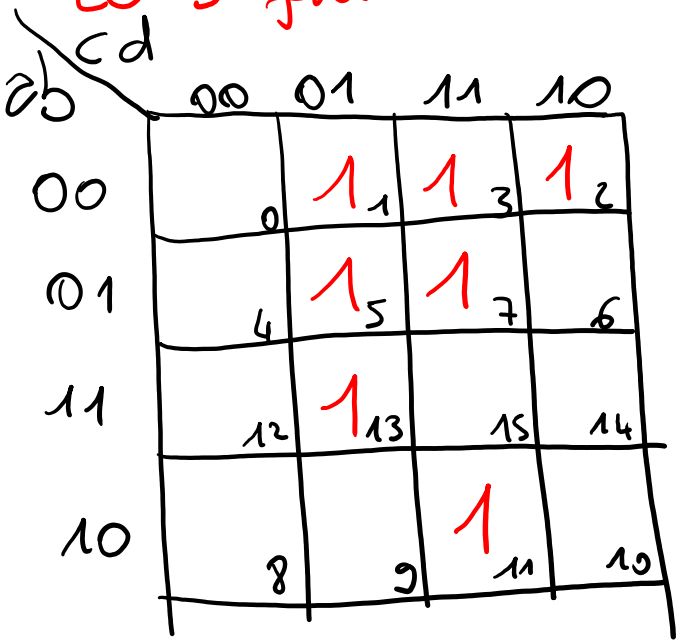
Exercise 1: Express the following function F in optimized SOP form

$$F(a,b,c,d) = \begin{cases} 1 \text{ if } abcd \text{ is a prime number (or if } abcd \text{ is equal to 1)} \\ 0 \text{ otherwise} \end{cases}$$

The function can be represented as a sum of minterms (standard SOP form), including the minterms with prime subscripts:

$$F(a,b,c,d) = \sum m(1,2,3,5,7,11,13)$$

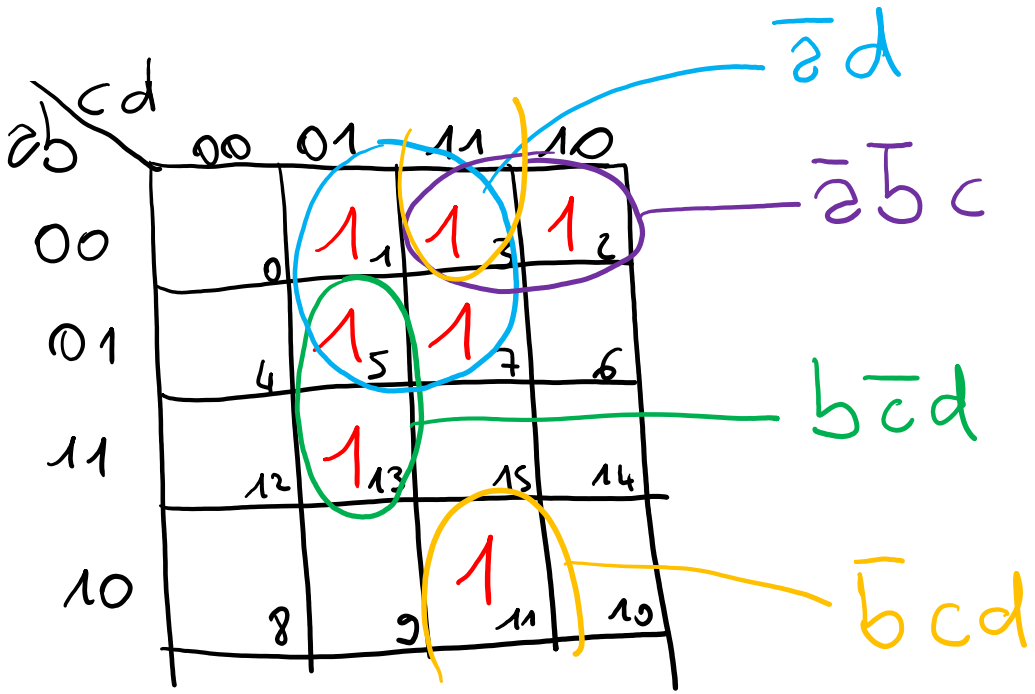
Let's fill the Karnaugh map, with '1' placed in the minterms of F:



Exercise 1: Express the following function F in optimized SOP form

$$F(a,b,c,d) = \begin{cases} 1 \text{ if } abcd \text{ is a prime number} \\ 0 \text{ otherwise} \end{cases}$$

Let's find the largest implicants (rectangles of 1-2-4-8-16 adjacent "1"s)

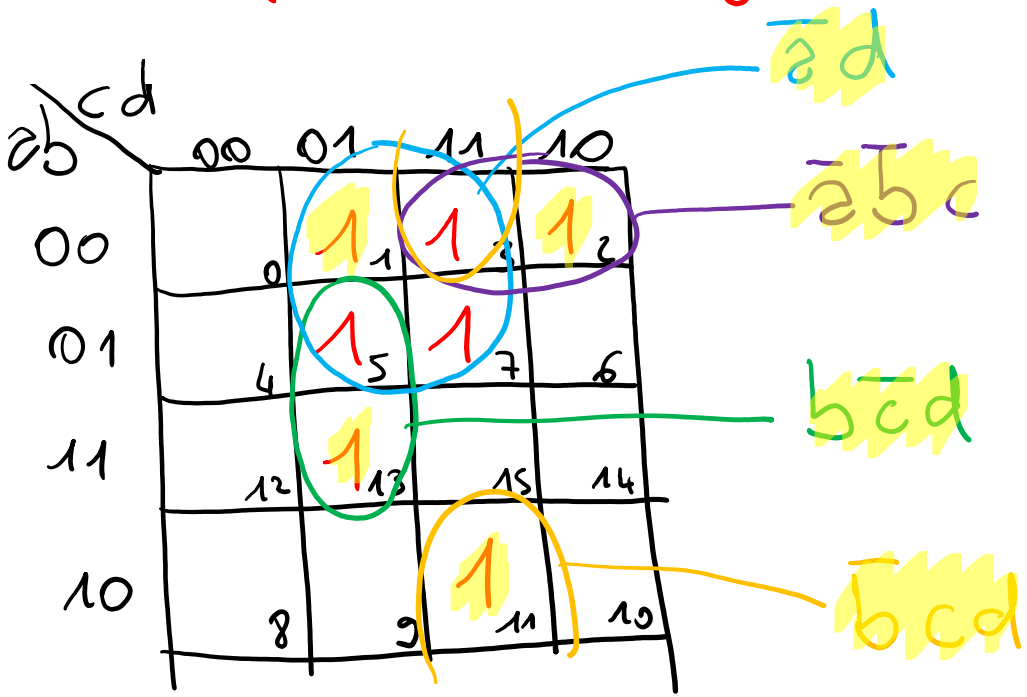


⟹ There are 4 PRIME IMPLICANTS (P.I): implicants that cannot be made larger

Exercise 1: Express the following function F in optimized SOP form

$$F(a,b,c,d) = \begin{cases} 1 \text{ if } abcd \text{ is a prime number} \\ 0 \text{ otherwise} \end{cases}$$

Of these prime implicants, let's find the ESSENTIAL PRIME IMPLICANTS, i.e. implicants including at least a "1" that is not covered by any other implicant



All the prime implicants (P.I.) are also essential prime implicants (E.P.I.). We need all E.P.I. to cover the function and in this case they are enough to include all "1"s in the map. The optimized SOP is:

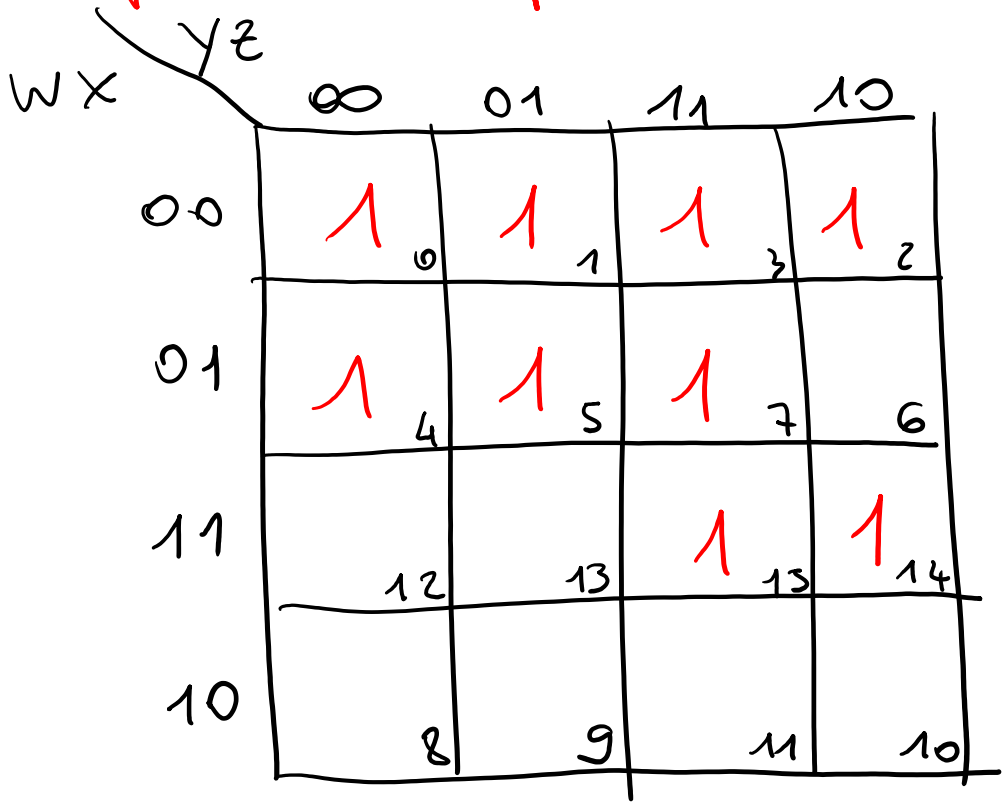
$$F = \bar{a}\bar{d} + \bar{a}\bar{b}c + b\bar{c}\bar{d} + \bar{b}cd$$

#

Exercise 2: Find the 2-levels optimized SOP form for the following function G

$$G(w,x,y,z) = \sum m(0,1,2,3,4,5,7,14,15)$$

Let's fill K-map with the minterms of function G :



Exercise 2: Find the 2-levels optimized SOP form for the following function G

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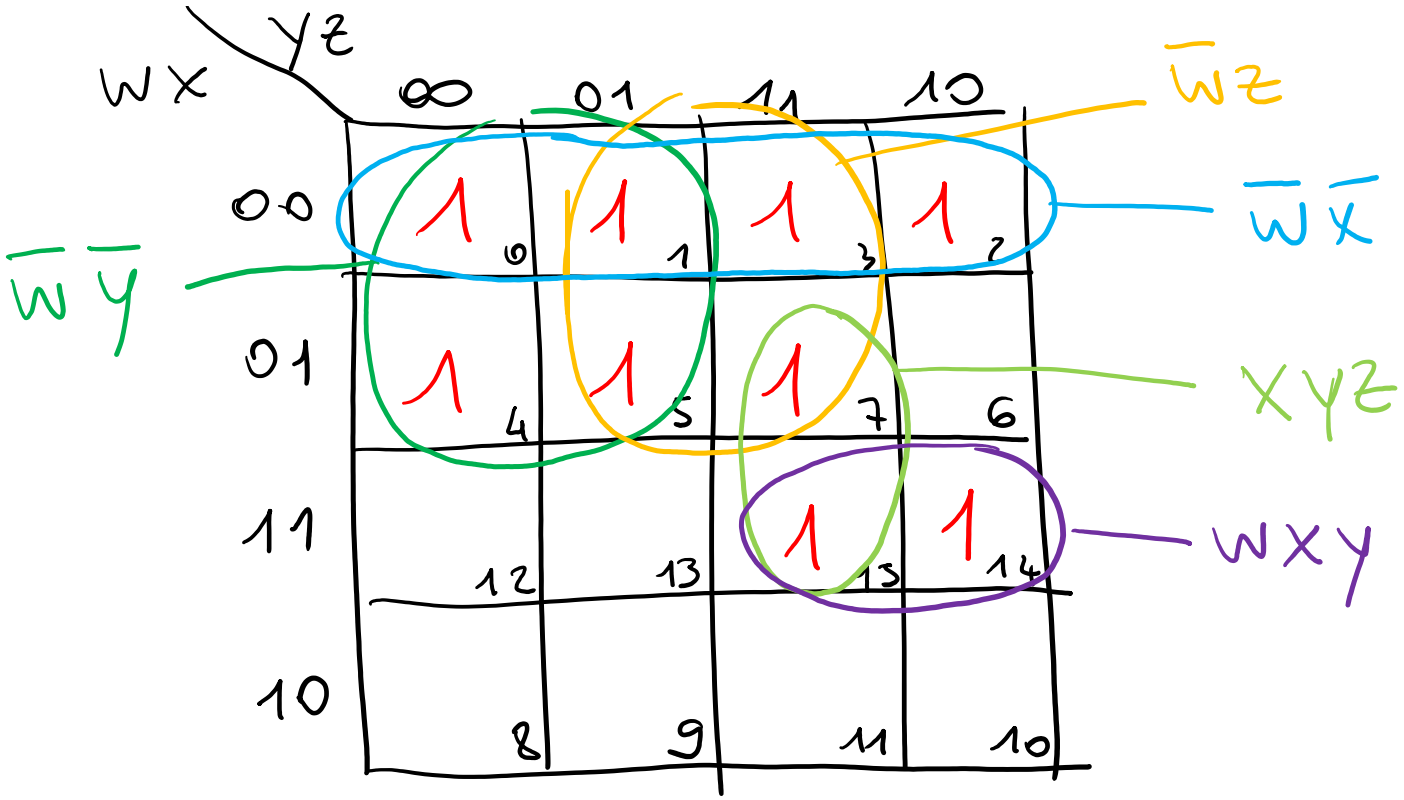
Let's find the prime implicants:

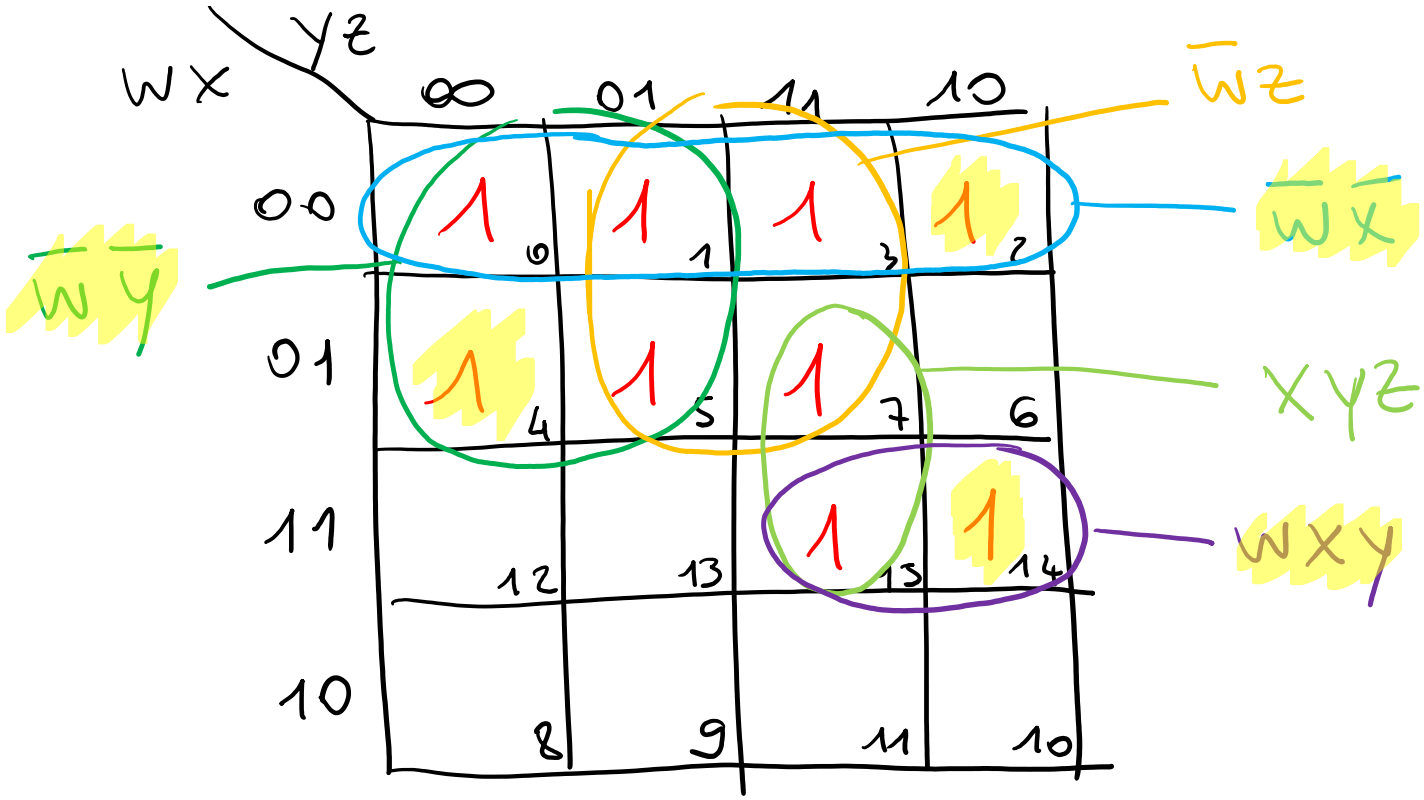


⇒ There are 5 prime implicants

Exercise 2: Find the 2-levels optimized SOP form for the following function G

$$G(w,x,y,z) = \sum m(0,1,2,3,4,5,7,14,15)$$

Let's see which of the P.I. are E.P.I :



⇒ Out of the 5 P.I.s, 3 of them are essential

Exercise 2: Find the 2-levels optimized SOP form for the following function G

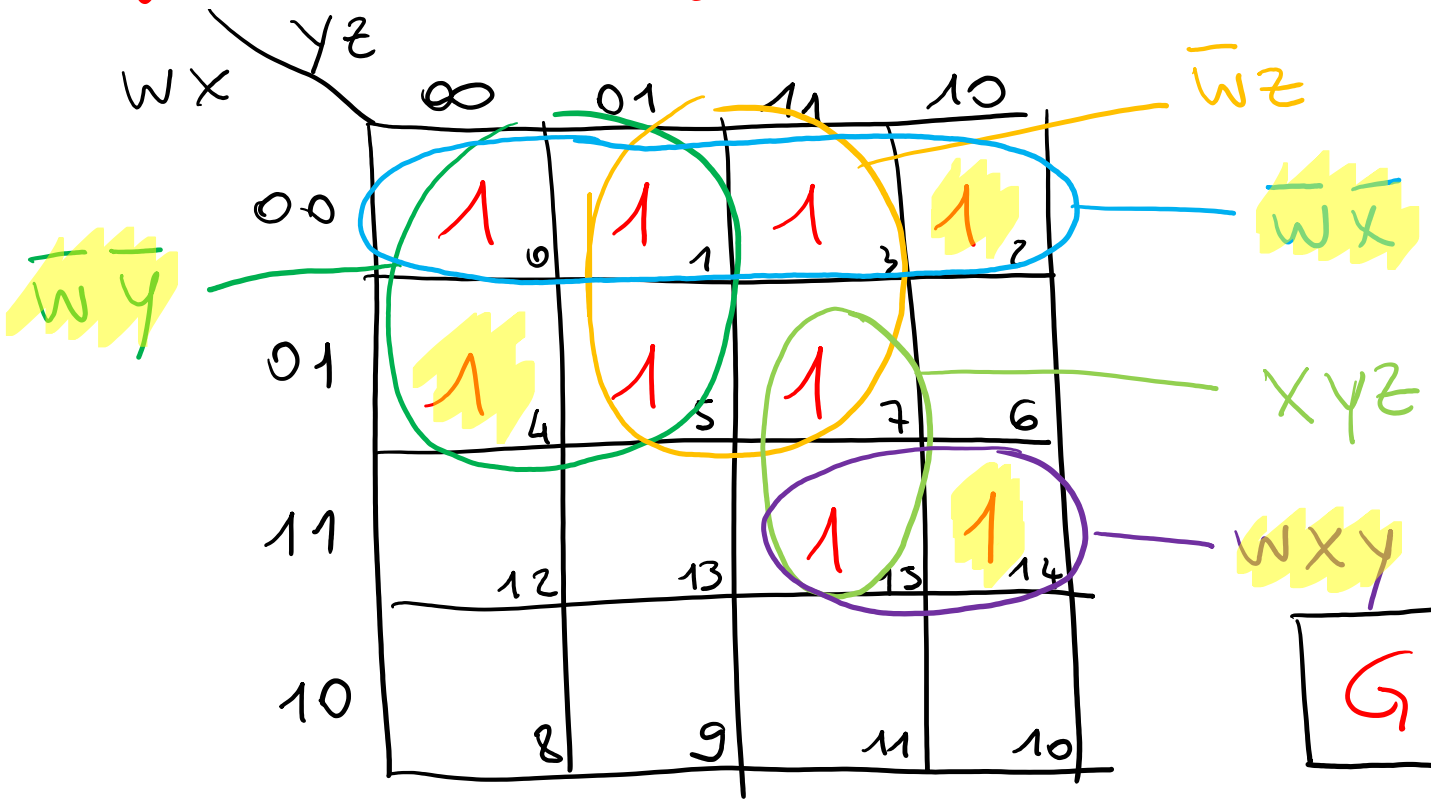
$$G(w,x,y,z) = \sum m(0,1,2,3,4,5,7,14,15)$$

The 3 E.P.I.s that we found are needed to cover the function, but they are not enough! In fact, m_7 is not covered by any of the EPIs.

⇓

Then we choose, in addition to the 3 E.P.I.s, the largest implicant covering m_7, i.e. the orange one.

⇒ The optimized SOP is:



$$G = \bar{w}\bar{y} + \bar{w}\bar{x} + wxy + \bar{w}z$$

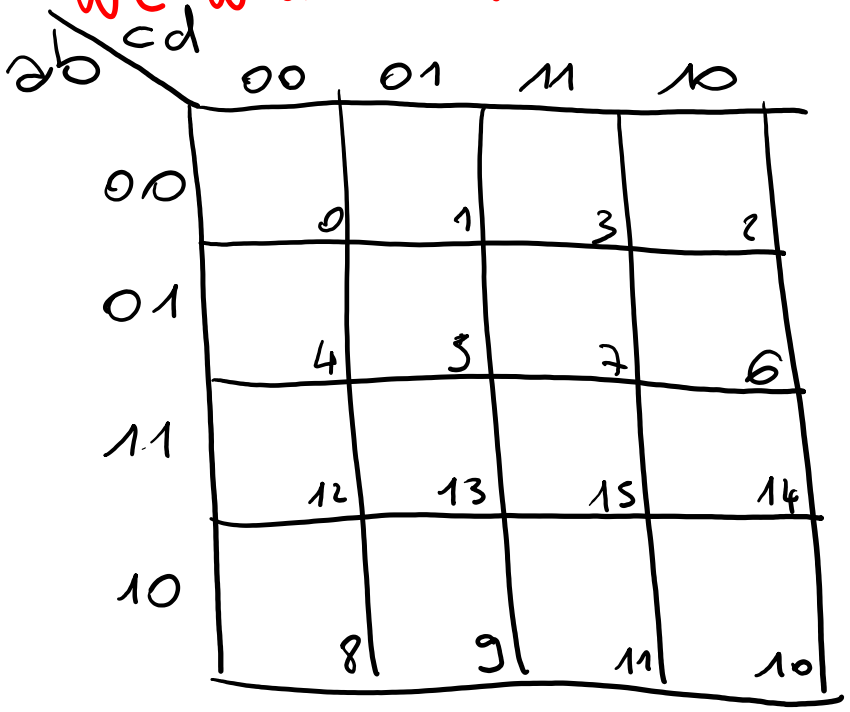
#

Exercise 3: 1) Is the following function expressed in 2-level optimized SOP form?

$$F(a,b,c,d) = \bar{a}c + \bar{a}b + bd + cd$$

2) Find the 2-level optimized POS form of the function F

The first step consists in determining the minterms of the function (i.e, the combinations of inputs for which F is equal to '1').
A first way to do this is to derive the truth table; a second way is to locate the implicants of the given expression directly on the K-map.
We will use the second way:



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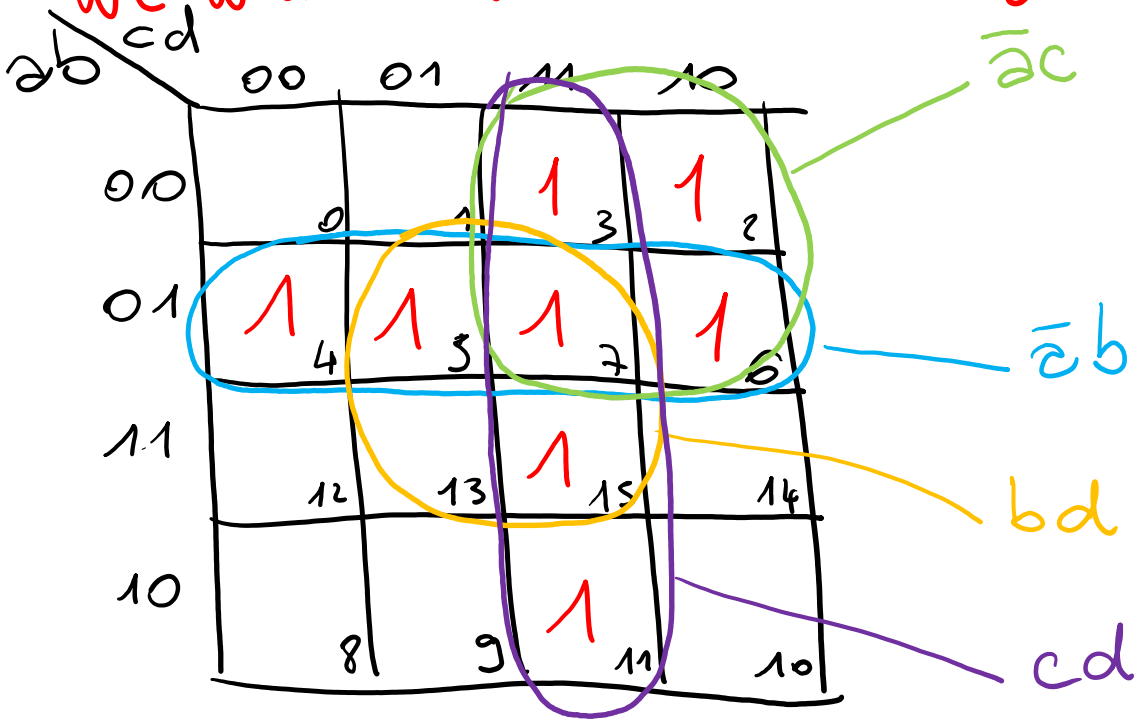


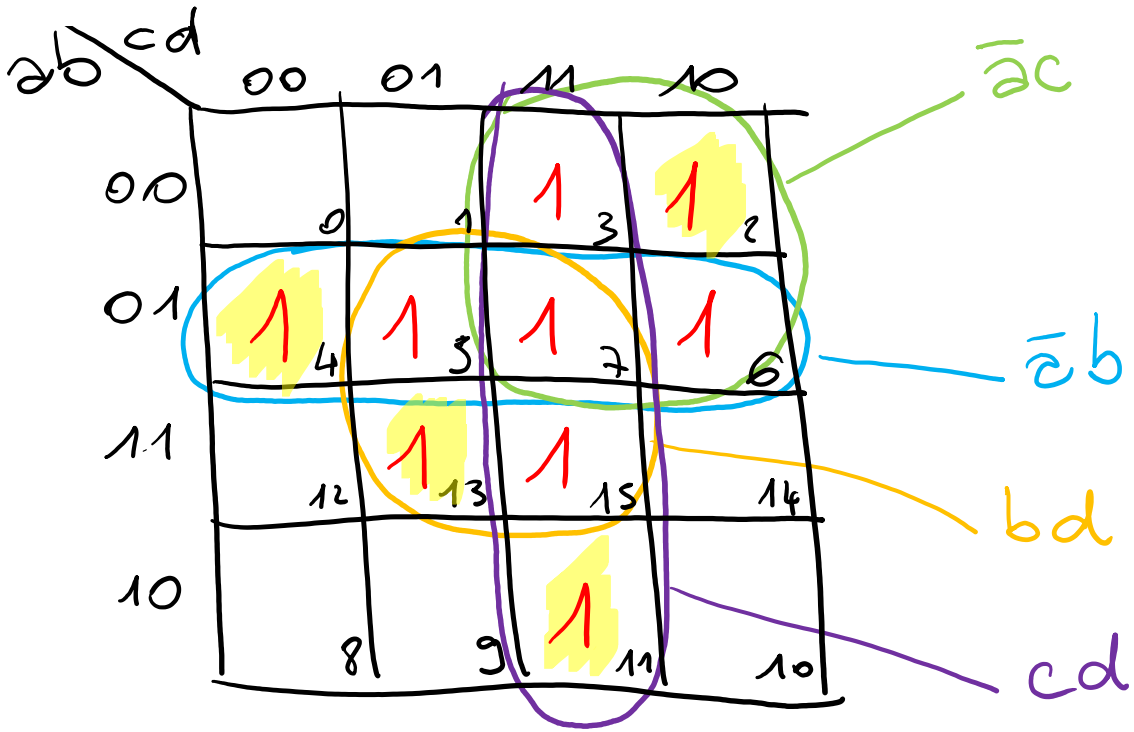
⇒ We found the rectangles on the map and we filled them with 1's as they are implicants of F

Exercise 3: 1) Is the following function expressed in 2-level optimized SOP form?

$$F(a,b,c,d) = \bar{a}c + \bar{a}b + bd + cd$$

2) Find the 2-level optimized POS form of the function F

Of the implicants that we found on the map, all of them are essential. In fact, each of them contains at least one square that is not covered by any other implicant.

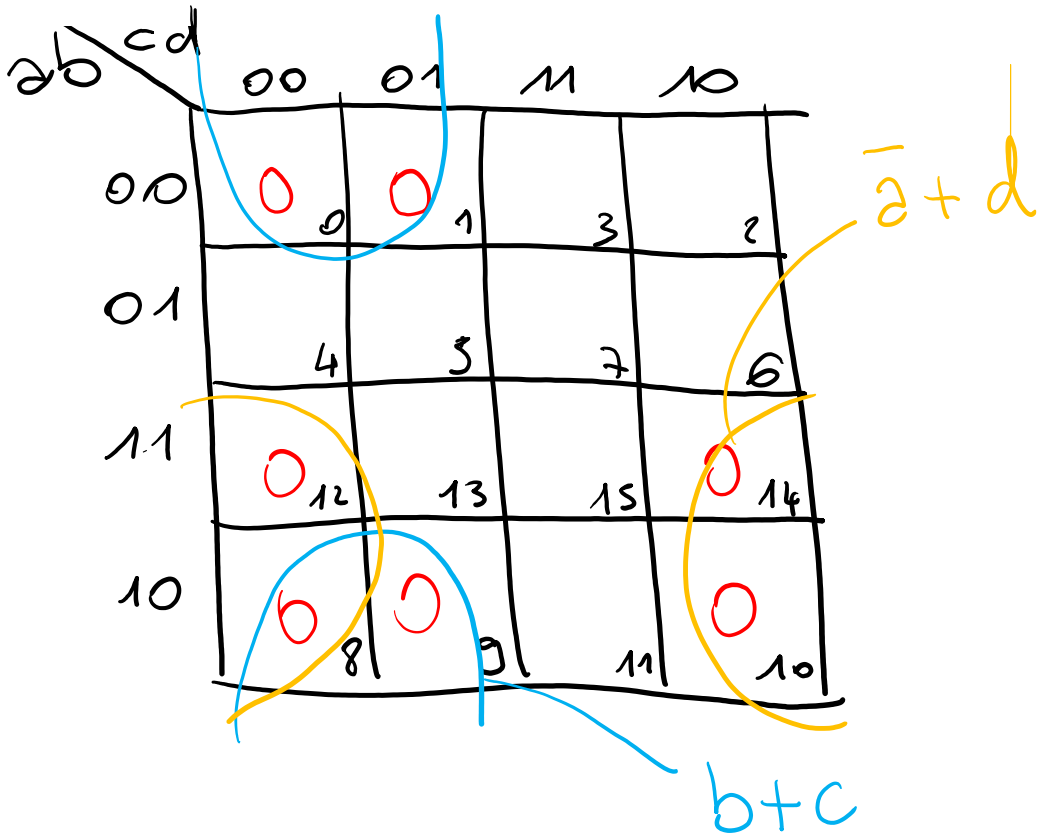


All the implicants in the given expression are PRIME and ESSENTIAL, so necessarily the given expression is an optimized SOP form!

Exercise 3: 1) Is the following function expressed in 2-level optimized SOP form?

$$F(a,b,c,d) = \bar{a}c + \bar{a}b + bd + cd$$

2) Find the 2-level optimized POS form of the function F



The minimized POS form is

$$F = (\bar{a} + d).(b + c)$$

#

Exercise 4: Use 4 binary digits (*dcba*) to represent decimal numbers from 0 to 9 (BCD). Implement the function:

$$F(d,c,b,a) = \begin{cases} 1 \text{ if } dcba \text{ is not zero and multiple of 3} \\ 0 \text{ otherwise} \end{cases}$$

Let's find the truth table of function F:

[note the order of inputs: 'd' is the MSB, 'a' is the LSB]

Decimal	d	c	b	a	F
0	0	0	0	0	0
1	0	0	0	1	0
2	0	0	1	0	0
3	0	0	1	1	1
4	0	1	0	0	0
5	0	1	0	1	0
6	0	1	1	0	1
7	0	1	1	1	0
8	1	0	0	0	0
9	1	0	0	1	1
	1	0	1	0	X
	1	0	1	1	X
	1	1	0	0	X
	1	1	0	1	X
	1	1	1	0	X
	1	1	1	1	X

6 don't care conditions!

Exercise 4: Use 4 binary digits (*dcba*) to represent decimal numbers from 0 to 9 (BCD). Implement the function:

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Decimal	d	c	b	a	F
0	0	0	0	0	0
1	0	0	0	1	0
2	0	0	1	0	0
3	0	0	1	1	1
4	0	1	0	0	0
5	0	1	0	1	0
6	0	1	1	0	1
7	0	1	1	1	0
8	1	0	0	0	0
9	1	0	0	1	1
	1	0	1	0	X
	1	0	1	1	X
	1	1	0	0	X
	1	1	0	1	X
	1	1	1	0	X
	1	1	1	1	X

→ From the truth table let's make the K-map: (attention to inputs order!)

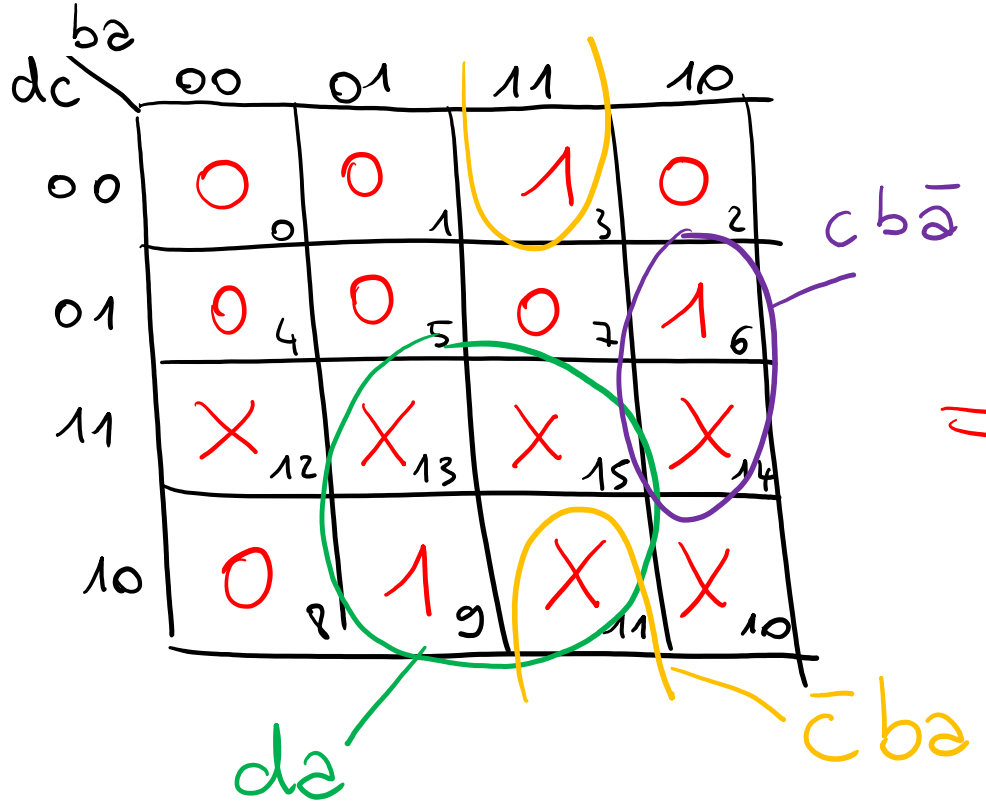
dc \ ba	00	01	11	10
00	0 (0)	0 (1)	1 (3)	0 (2)
01	0 (4)	0 (5)	0 (7)	1 (6)
11	X (12)	X (13)	X (15)	X (14)
10	0 (8)	1 (9)	X (11)	X (10)

Exercise 4: Use 4 binary digits (*dcba*) to represent decimal numbers from 0 to 9 (BCD). Implement the function:

$$F(d,c,b,a) = \begin{cases} 1 \text{ if } dcba \text{ is not zero and multiple of 3} \\ 0 \text{ otherwise} \end{cases}$$

Decimal	d	c	b	a	F
0	0	0	0	0	0
1	0	0	0	1	0
2	0	0	1	0	0
3	0	0	1	1	1
4	0	1	0	0	0
5	0	1	0	1	0
6	0	1	1	0	1
7	0	1	1	1	0
8	1	0	0	0	0
9	1	0	0	1	1
	1	0	1	0	X
	1	0	1	1	X
	1	1	0	0	X
	1	1	0	1	X
	1	1	1	0	X
	1	1	1	1	X

Let's find the P.I., covering also the squares with X, if they are useful to obtain larger implicants:



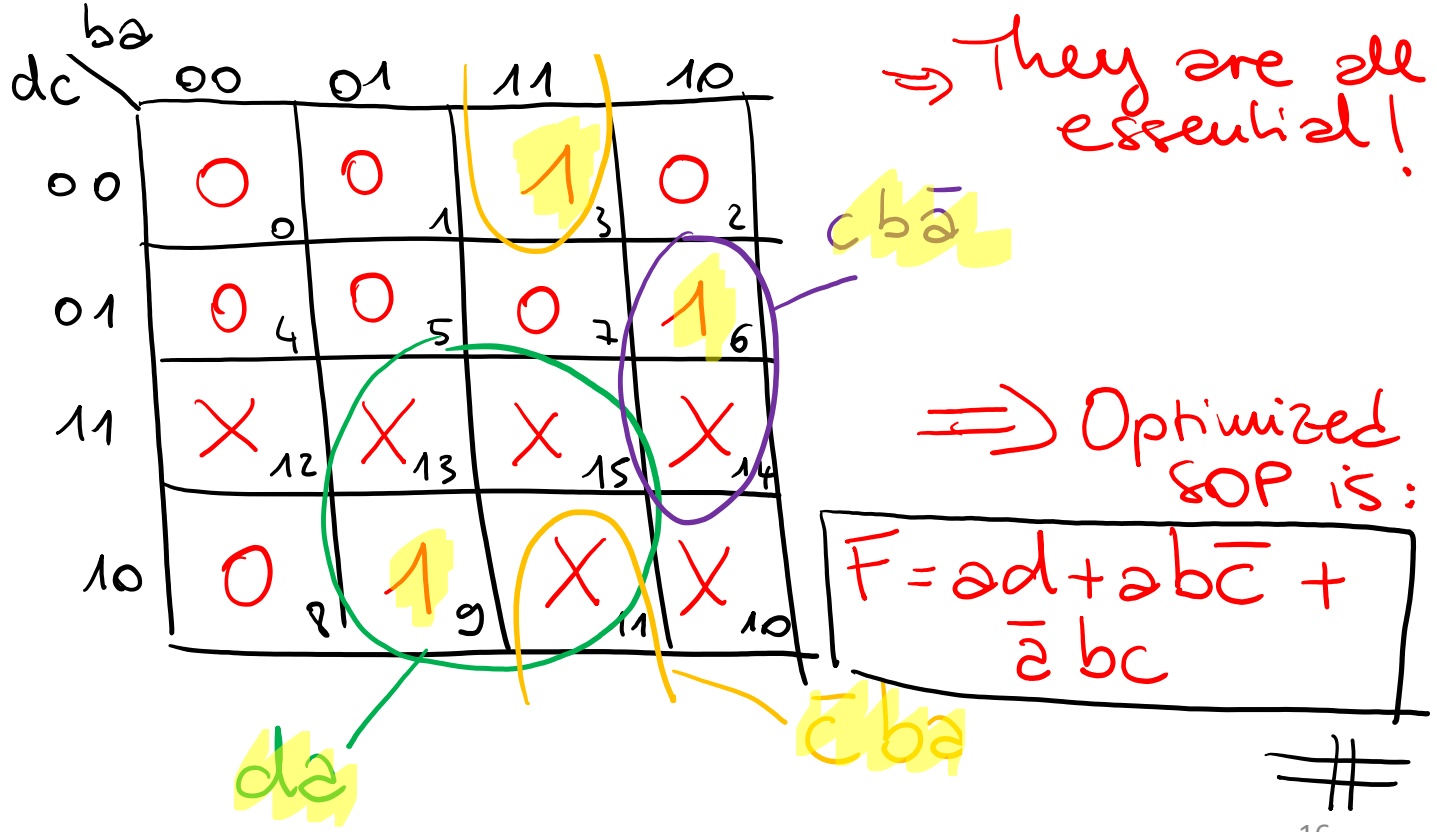
⇒ We have 3 I.P.

Exercise 4: Use 4 binary digits (*dcba*) to represent decimal numbers from 0 to 9 (BCD). Implement the function:

$$F(d,c,b,a) = \begin{cases} 1 \text{ if } dcba \text{ is not zero and multiple of 3} \\ 0 \text{ otherwise} \end{cases}$$

Decimal	d	c	b	a	F
0	0	0	0	0	0
1	0	0	0	1	0
2	0	0	1	0	0
3	0	0	1	1	1
4	0	1	0	0	0
5	0	1	0	1	0
6	0	1	1	0	1
7	0	1	1	1	0
8	1	0	0	0	0
9	1	0	0	1	1
	1	0	1	0	X
	1	0	1	1	X
	1	1	0	0	X
	1	1	0	1	X
	1	1	1	0	X
	1	1	1	1	X

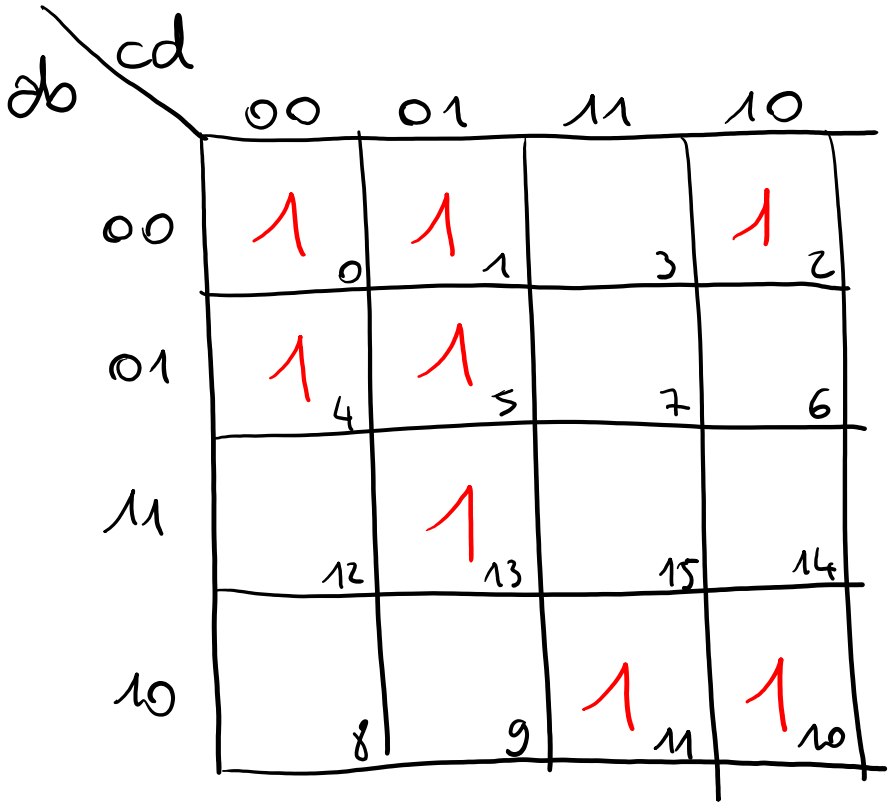
Let's see which of these P.I. are E.P.I.:



Esercizio 5: Find the optimized SOP form for the 4-input function with the following minterms:

$m_0, m_1, m_2, m_4, m_5, m_{10}, m_{11}, m_{13}, m_{15}$

Let's fill the K-map with the minterms of the function:



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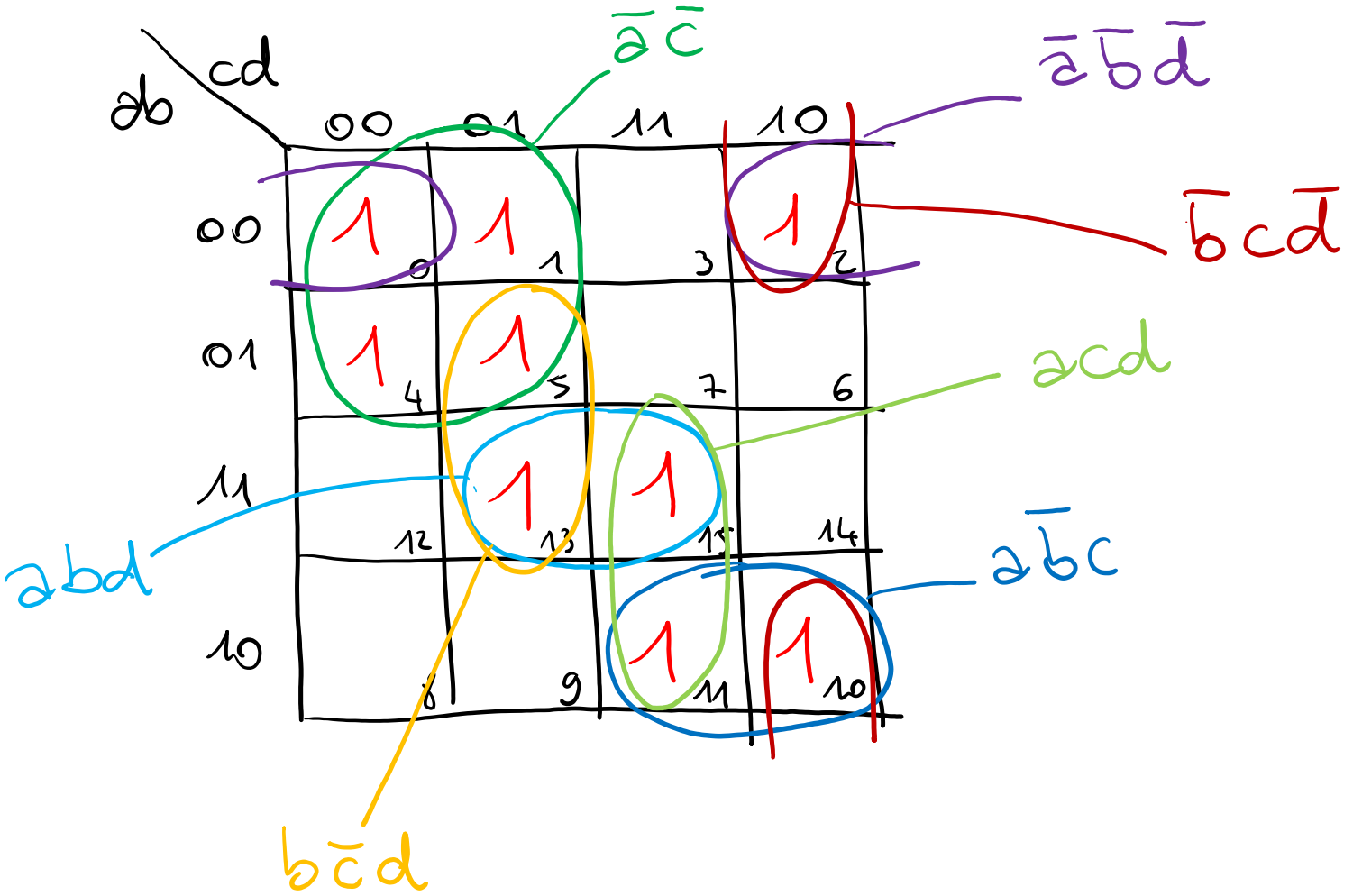
$m_0, m_1, m_2, m_4, m_5, m_{10}, m_{11}, m_{13}, m_{15}$

Let's find the prime implicants:

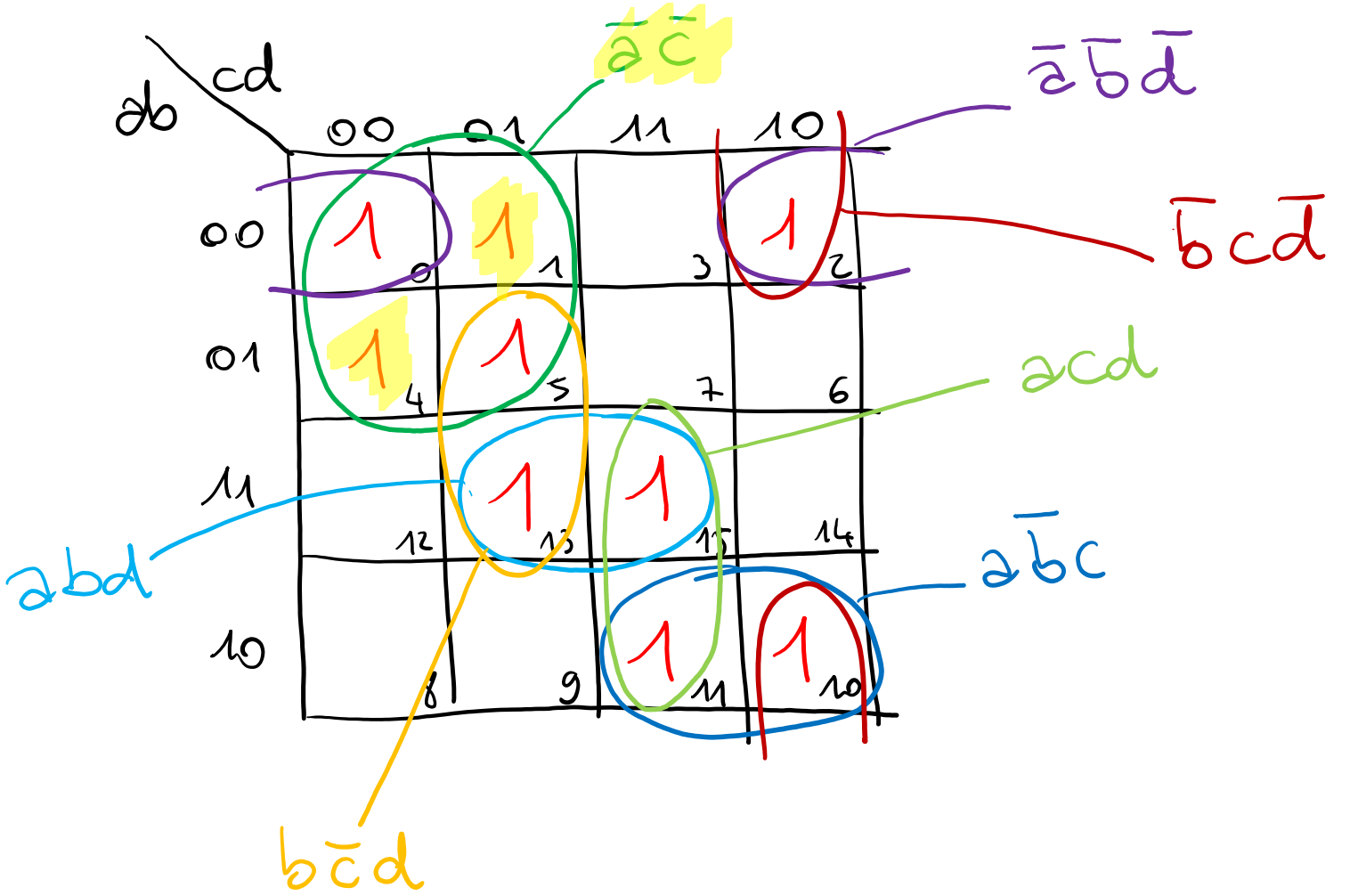


⇒ There are 7 IPs!

Esercizio 5: Find the optimized SOP form for the 4-input function with the following minterms:

$m_0, m_1, m_2, m_4, m_5, m_{10}, m_{11}, m_{13}, m_{15}$

Let's see which P.I.s are also E.P.Is:

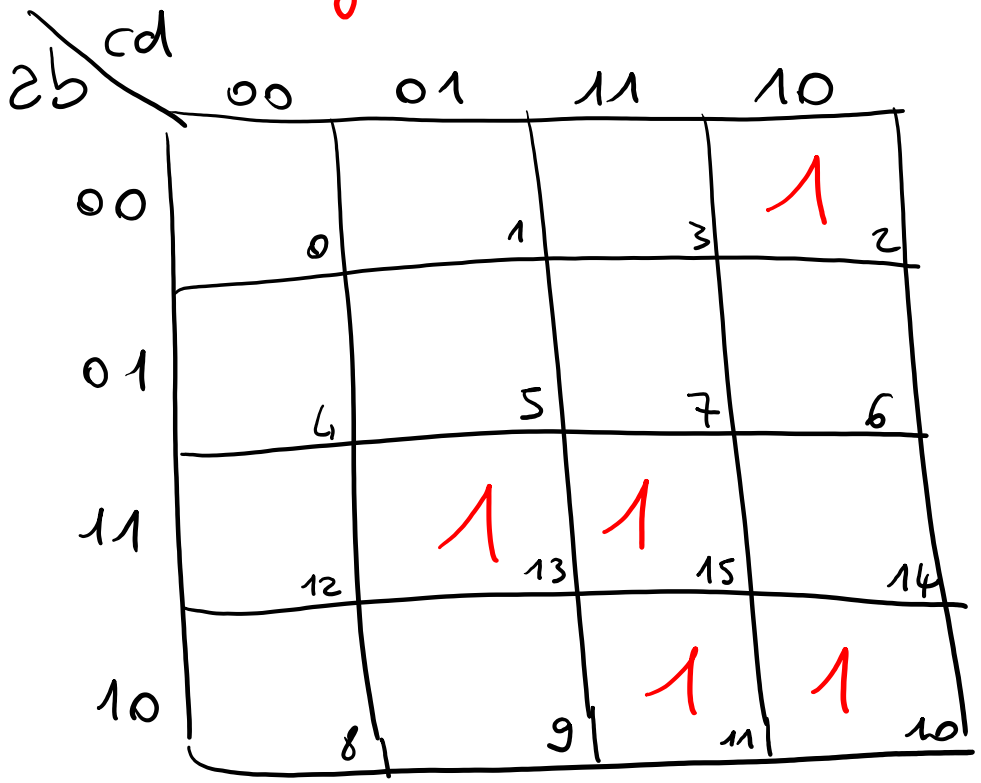


⇒ Only 1 IP is EPI (for sure we will need it to cover the function, but this is not enough)

Esercizio 5: Find the optimized SOP form for the 4-input function with the following minterms:

$m_0, m_1, m_2, m_4, m_5, m_{10}, m_{11}, m_{13}, m_{15}$

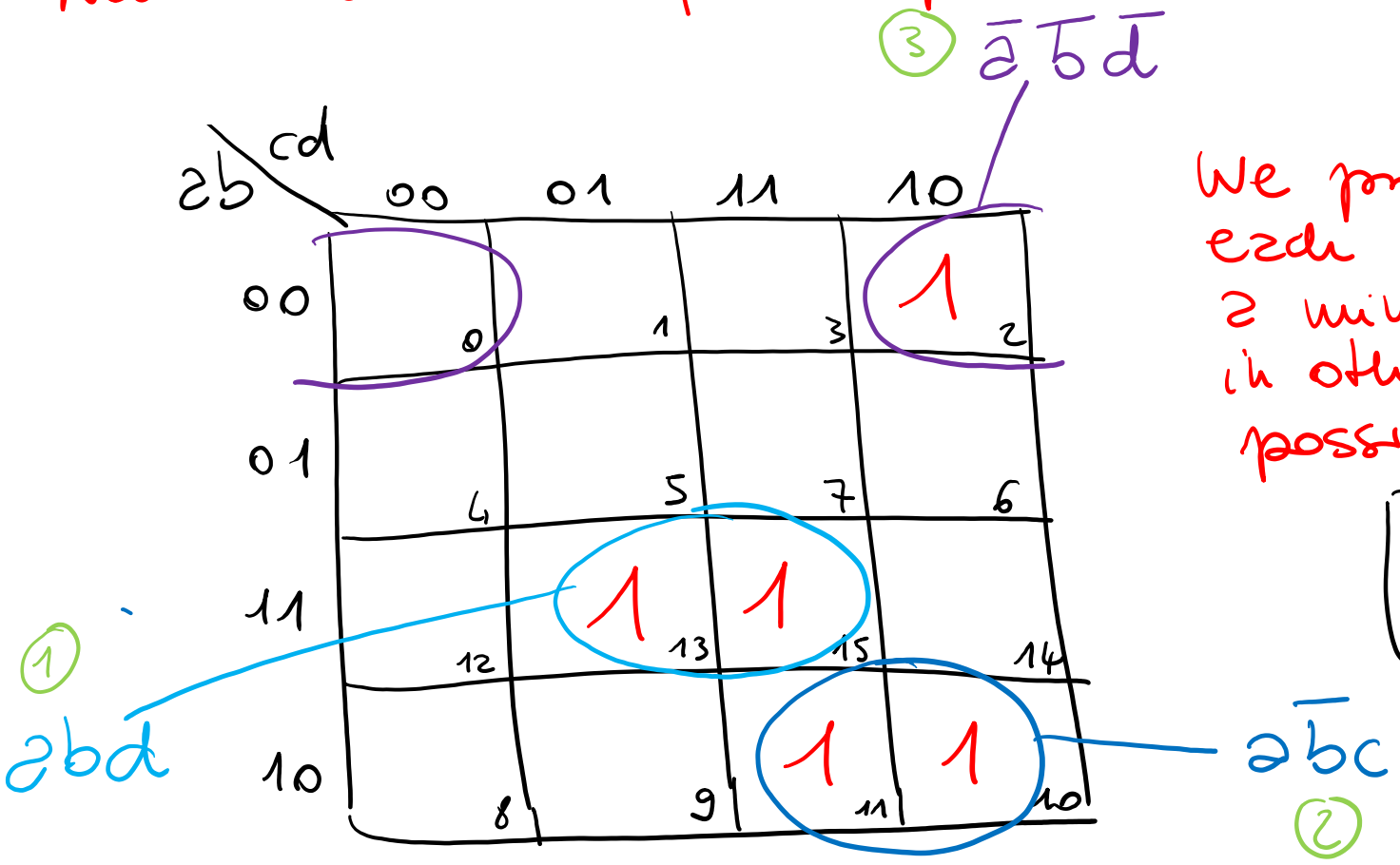
Let's draw another K-map, containing the '1's that are not covered by the E.P.I. $\bar{a}\bar{c}$:



Esercizio 5: Find the optimized SOP form for the 4-input function with the following minterms:

$m_0, m_1, m_2, m_4, m_5, m_{10}, m_{11}, m_{13}, m_{15}$

Now we choose as few as possible IPs to cover the remaining "1"s:



We proceed in order, selecting each IP containing at least 2 minterm that is not contained in other selected IPs, so one possible optimized SOP is:

$$\bar{a}\bar{c} + ab\bar{d} + a\bar{b}c + \bar{a}\bar{b}\bar{d}$$

#

i) ii)

Exercise 6: Optimize the following function in 2-level SOP and POS forms

$$F(a,b,c,d) = \prod M(2,5,6,7,8,9,10,11,14)$$

i) To find the optimized SOP, let's express the function as a sum of minterms:

$$F(a,b,c,d) = \sum m(0,1,3,4,12,13,15)$$

Let's fill the K-map with '1':

$\Rightarrow$ 6 I.P.s, 2 I.P.E.s

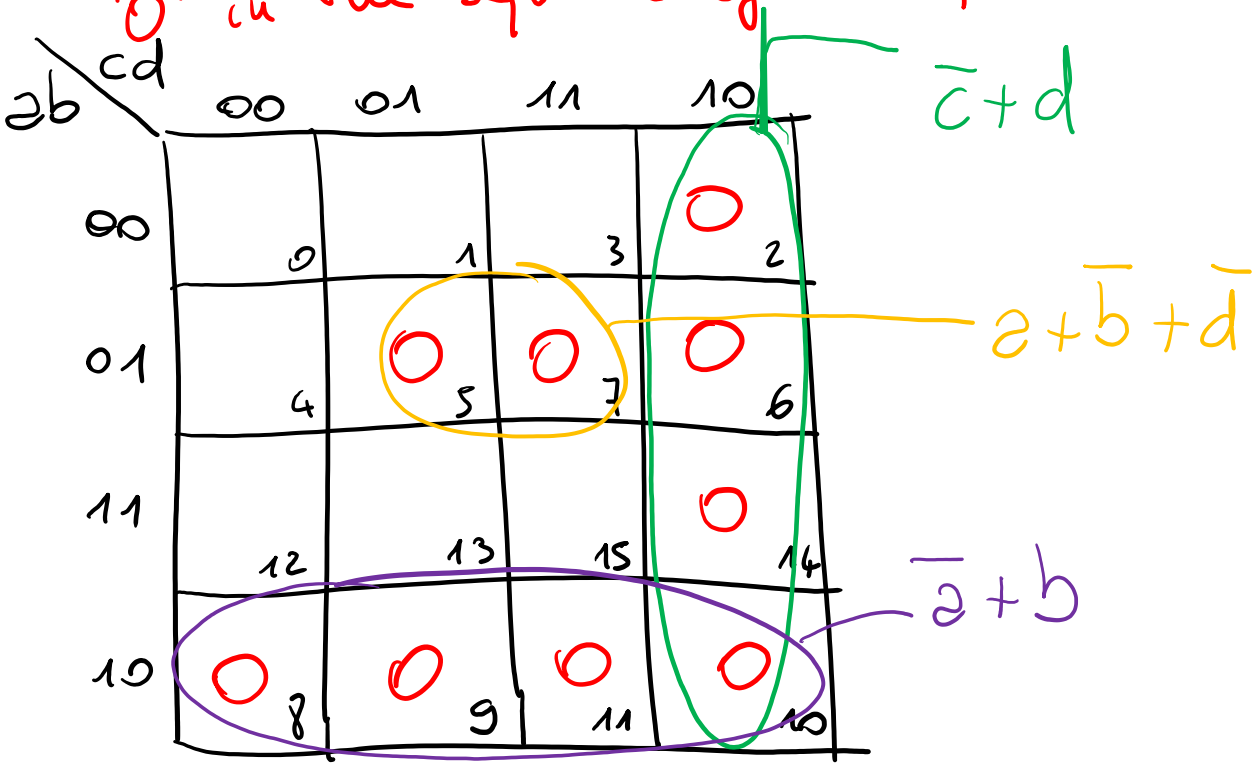
A possible minimized SOP is:

$$F = abd + \bar{a}\bar{b}d + \bar{a}\bar{c}\bar{d} + ab\bar{c}$$

Exercise 6: Optimize the following function in 2-level SOP and POS forms

$$F(a,b,c,d) = \prod M(2,5,6,7,8,9,10,11,14)$$

ii) To find the optimized POS, it is convenient to start from the standard POS form, given by the exercize. We fill the K-map with '0' in the squares of the function maxterms:



We found the greatest groupings of '0' and we can then write the optimized POS form:

$$F = (\bar{a}+b)\cdot(\bar{c}+d)\cdot(a+\bar{b}+\bar{d})$$

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