APPLIQUÉES
ROUEN

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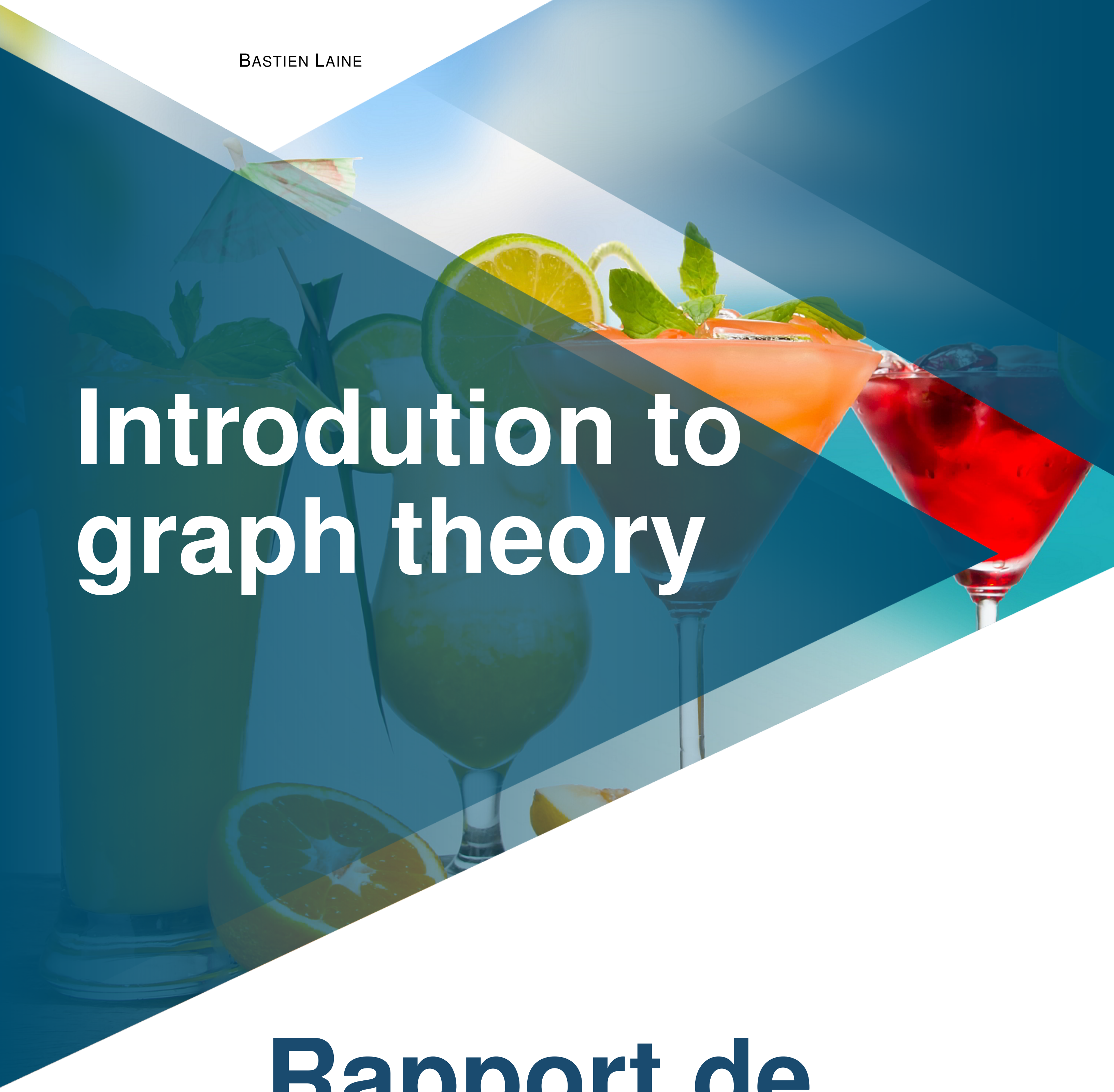

Introdution to graph theory

Rapport de BDD

Table des matières

Chapitre 1
Definitions

1.1 General definitions

1.1.1 Graph

A graph is a pair (V, E) with a incidence relation $\sim$ such that :

1. V, E are finite sets
2. $\forall e \in E, \exists!$ 1 or 2 element(s) $v \in V | e \sim v$

1.1.2 Vertex

A vertex of G is an element of V=V(G)

1.1.3 Edge

A edge of G is an element of E=E(G)

1.1.4 End of an edge

$$\begin{aligned} v \sim e \quad &\iff \quad v \text{ is incident with } e \\ &\iff \quad v \text{ is an end of } e \end{aligned}$$

1.1.5 Parallelism

If 2 edges e and f have the same set of ends, e and f are parallels

1.1.6 Simple graph

A simple graph is a graph without any parallels edges or loop

1.1.7 Adjacent

Two vertex v and w are adjacent if $\exists$ one edge whose set of its ends is equal to v, w

1.2 Type of graph

1.2.1 Isomorphism

2 graphs G and H are isomorphic is $\exists \phi : V(G) \cup E(G) \to V(H) \cup E(H)$ bijective, such that :

1. $\phi(v) \in V(H)$ with $v \in V(G)$
2. $\phi(e) \in E(H)$ with $e \in E(G)$
3. $e \underset{G}{\sim} v \iff \phi(e) \underset{H}{\sim} \phi(v)$

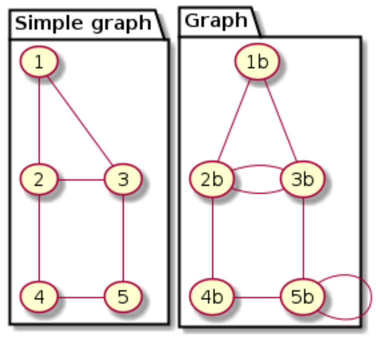

FIGURE 1.1 – Simple graph ⇔ Graph

1.2.2 Complete graph (K_n)

A complete graph is a simple graph with n vertex (for K_n) suc that every pair of distinct vertex are adjacent

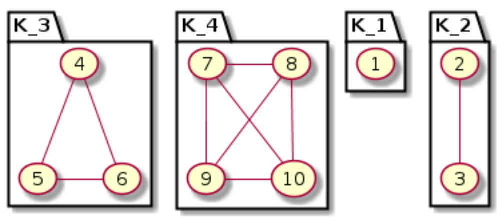

FIGURE 1.2 – Every complete graph from K_0 to K_4

1.2.3 Subgraph

A graph H is a subgraph of G if :

1. $V(H) \subseteq V(G)$
2. $E(H) \subseteq E(G)$
3. $\forall e \in E(H)$, the set of ends of e in H is equals to the set of ends in G

1.2.4 Spanning subgraph

A subgraph H of G is spanning if $V(H) = V(G)$

1.2.5 subgraph

A subgraph H of G is spanning if $V(H) = V(G)$

1.3 Operations

1.3.1 Deletetion

1.3.2 Contraction

G/e is the subgraph obtained from G bydeleting e and identitfying the 2 ends of e
$|E(G)| = |E(G/e)| - 1$
$|V(G)| = |V(G/e)| - 1$ (if e is not a loop)

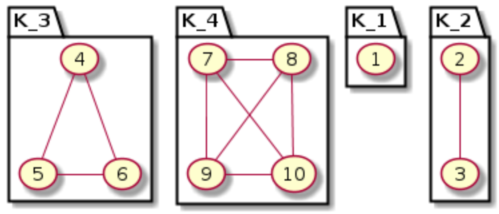

FIGURE 1.3 – Every complete graph from K_0 to K_4

1.3.3 Minor

H is a **minor** of G if H can be obtained from G by deleting edges/vertices and contracting edges

1.3.4 Subdivision

H is a **subdivision** of G if H can be obtain by subdivising somme edges of G
H is a **subdivision** of G if H can be obtained from G by replacing each edge with a path of length ≥ 1

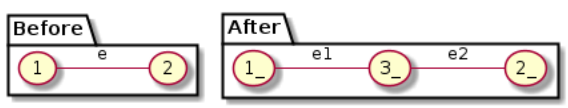

FIGURE 1.4 – Every complete graph from K_0 to K_4

1.3.5 Topological minor

H is a **topological minor** of G if a subdivision of H is a subgraph of G

$$\begin{array}{ccccc} \text{Subgraph} & \Rightarrow & \text{topological minor} & \Rightarrow & \text{minor} \\ & \not\Leftarrow & & \not\Leftarrow & \end{array}$$

Chapitre 2
Connectedness

2.1 Definitions

2.1.1 Walk

For 2 vertices v and w, a **walk** from v to w is an alternating sequence $v_0e_1v_ae_1 \cdots e_nv_n$ of vertices $v_0, \cdots, v_n$ and edges $e_1, \cdots, e_n$. such that the sets of ends of e_i is v_{i-1}, v_i and $v_0 = v, v_n = w, n \geq 0$

2.1.2 Trail

A **trail** is a walk without using the same edge twice

2.1.3 Path

A **path** is a walk not using any vertex twice

2.1.4 Closed trail

A walk from v to w is **closed** if $v = w$ A **circuit** is also a closed trail

2.1.5 Cycle

A **cycle** is a circuit $v_0e_1v_ae_1 \cdots e_nv_n$ such that $v_0, v_1, \cdots, v_{n-1}$ are distinct
$Circuit \Rightarrow cycle$
$Circuit \not\Leftarrow cycle$

2.2 Theorems

2.2.1 Lemma

Definition

If G has a walk from x to y, then G has a path from x to y

Proof

Induction on n
Let's call n the length of the walk W from x to y

1. If $n = 0$: trival
2. If $n \neq 0$: We may assume W is not a path (Else, trivial)
   W has a vertex z that is visited more that once.
   $x = v_0e_1v_1e_2 \cdots z \cdots z \cdots e_ny$
   $\Downarrow$
   $W' = v_0e_1v_1e_2 \cdots z \cdots e_ny$
   W' is a walk from x to y whose length is $< n$

By the induction hypothesis, G has a path from x to y
G is connected $\Leftrightarrow \forall x, y \in V(G), G$ has a path from x to y

2.2.2 $\sim$ relation

Definition

$\forall x, y \in V(G), x \sim y \Leftrightarrow G$ has a path from x to y

Equivalent ?

$$\begin{cases} \textbf{symmetric} : x \sim y \Rightarrow y \sim x \\ \textbf{reflexive} : x \sim x \\ \textbf{transitive} : x \sim y, y \sim z \Rightarrow x \sim z \end{cases}$$

2.2.3 Connected component graph

A **connected component** of a graph G is a subgraph induced on an equivalence class of $(V(G), \sim)$
i.e. : a component is a maximal connected subgraph

- If C, D are component, then $C = D$ or $V(C) \cap V(D) = \emptyset$
- G is disconnected $\Leftrightarrow$ $V(G)$ can be partitionned into A and B (i.e. $A \cup B = V(G), A \cap B = \emptyset$) such that $A, B \neq \emptyset$ such that G has no edge having one end in A and another on B

2.2.4 Forest and tree

"Minimally connected" graph $\sim$ tree

Tree

A **tree** is a connected graph with no cycle

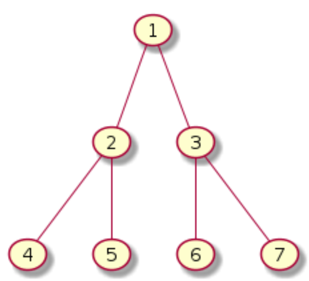

FIGURE 2.1 – A tree

Forest

A **forest** is a graph with no cycle

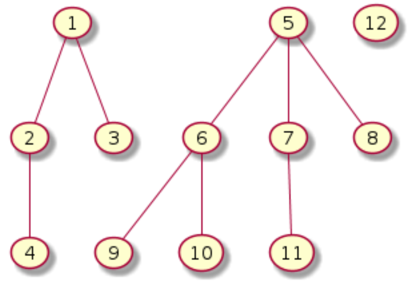

FIGURE 2.2 – A forest

Equivalence on trees

The following are equivalent :

1. T is a tree
2. T is loopless and for $v, w \in V(T)$, T has a UNIQUE path from v to w
3. T is connected and T
   e is disconnected from all $e \in E(T)$
4. T has no cycle and $T + xy$ (adding a new edge xy to T) has a cycle for any $x, y \in V(G)$

Lemma

If T is a tree with at least 1 vertex, then $|E(T)| = |V(T)| - 1$

Proof

Induction on $|E(T)|$
If $E(T) = \emptyset, \Rightarrow |V(T)| = 1 \Rightarrow 1 - 1 = 0$
Now let $e \in E(T)$
$T \backslash e$ has exactly 2 components T_1, T_2
$|E(T_1)| = |V(T_1)| - 1$
$|E(T_2)| = |V(T_2)| - 1$

FIGURE 2.3 – A bipartite graph (With its 2 parts)

Corrolary

If T is a tree with at least 2 vertices, then T has a leaf

2.2.5 Bipartite graphs

G is a **bipartite** graph if G has a **bipartition** (A, B) such that $A \cup B = V(G), A \cap B = \emptyset$and evey edge as 1 end on A, and another end on B

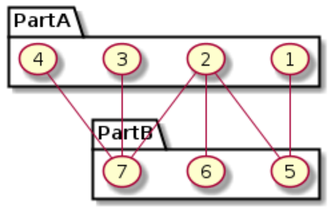

FIGURE 2.4 – A bipartite graph (With its 2 parts)

2.2.6 Bipartite ⇔ no odd cycle

Definition

G is bipartite ⇔ G has no odd cycle

Proof

- ⇒ : trivial
- ⇐ : Induction on $|E(G)|$
  If $|E(G)| = 0$, trivial
  Let $e \in E(G)$
  $G\backslash e$ has no odd cycle ⇒ $G\ e$ is bipartite
  $G\backslash e$ has a bipartition (A, B)
  We may assume both ends are in A
  If$G\backslash e$ has a path P from x to y, length of P is even.
  $P + e :=$ odd cycle

Chapitre 3
Bipartite graph

3.1 Complete bipartite graph

Let's call $K_{m,n}$a complete bipartite graph

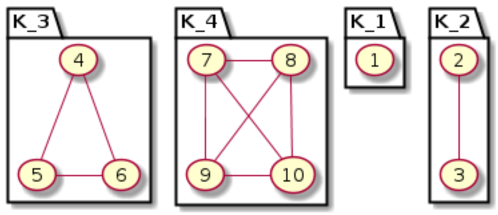

FIGURE 3.1 – Every complete graph from K_0 to K_4

3.2 Euler tours

3.2.1 Definition

A circuit is **Eulerian** if it contains every edge

3.2.2 Theorem

G has a Eulerian circuit $\Leftrightarrow$ $G\backslash$ (isolated vertices) is coonnected and every vertex has even degree

3.2.3 Proof

— $\Rightarrow$: We may assume G has no isolated vertices $\Rightarrow$ C contains a walk from v to w
— $\Leftarrow$: Induction on $|V(G)| + |E(G)|$
We may assume G is connected (by removing isolated vertices)
We may assume $|V(G)| > 1$
G contains a cycle C (Else, it would be a tree)
By the induction hyp, each non trivial component of H has an Eulerian circuit

Chapitre 4
Matchings

A **matching** of a graph G is a set of non-loop edges such that no 2 edges share an end.
$\emptyset$ is a matching

4.1 Matching in a bipartite graph

$\nu(G)$: max size (number of edge) of a matching of G

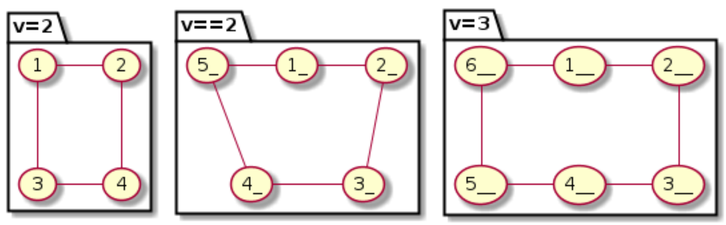

FIGURE 4.1 – Examples of $\nu(G)$

$\tau(G)$: min number vertices that meet every edge
($min|x| : x \subseteq V(G)$ every edge has at least 1 end in X)

M**-alternating path** is a path $P = v_0e_1v_1e_2v_2e_3v_3e_4\cdots e_nv_n$ such that $e_2, e_3, \cdots \in M$

M**-augmenting path** is a path $P = v_0e_1v_1e_2v_2e_3v_3e_4\cdots e_nv_n$ such that P is M-alternating and V_0V_n are not incident with any edge in M

4.1.1 König's theorem

G bipartite $\Rightarrow \nu(G) = \tau(G)$

4.1.2 Proof

4.1.3 Lemma

Let M be a matching of a graph G
M is a maximum matching $\Leftrightarrow G$ has no M-augmenting path

4.1.4 Proof

$\Rightarrow$ If P is an M-augmenting path
Let $M' = M\Delta E(P)$ $(X\Delta Y = (X - Y) \cup (Y - X))$

$\Leftarrow$: $\tau(G) \geq \nu(G)$: Trivial (Even works for all graphs)
$\tau(G) \leq \nu(G)$: Let $k = \nu(G)$
Let M be a matching of size k.

We want to find a set of k vertices meeting every edges Let (A, B) be a bipartition of H
G has no M-augmenting path
Let $A' = A - V(M)$
Let B_0 be the subset of B that can be reached from A' by an M-alternating path.
Let A_0 =subset of A incident with an edge in M
Let $A_1 = A - A_0 - A'$
CLAIM : Every edge is incident with a vertex in $A_1 \cap B_0 \Rightarrow$ **TRUE**

4.1.5 Hall's theorem

Let G be a bipartite grpah with a bipartition (A, B)
G has a matching M covering $A \Leftrightarrow \forall S \subseteq A, |N_G(S)| \geq |S|$ (With $N_G(S)$=set of all neighbours of G vertices in S)

4.1.6 Proof

$\Leftarrow$: König's theorem show that $\nu(G) = |A| \Rightarrow \tau(G) = |A|$

$\Rightarrow$:

4.1.7 Other proof

Induction on $|A|$

Case 1 : $|N_G(S)| > |S|$
$\forall \phi \in S \subsetneq A$, let's choose $x \in A, y \in B$ such that x and y are adjacents
Let $H = G \backslash x \backslash y$
For $\phi \neq S \subseteq A - \{x\}$
$$\begin{aligned} |N_H(S)| &\geq |N_G(S)| - 1 \\ &\geq (|S| + 1) - 1 = |S| \end{aligned}$$ By induction, H has a matching M covering $A - \{x\}$
$M \cup \{xy\}$ is a matching of G covering A

Case 2 : There is a set $A_0 \subsetneq A$ such that $|N_G(A_0)| = |A_0|$
By induction, there is a matching M_1 covering A_0
Let $S \subseteq A - A_0$
Let $H = G \backslash A_0 \backslash N_G(A_0)$
Goal : $|NH(S)| \geq |S|$?
$$\begin{aligned} |N_G(A_0 \cup S)| &= |N - G(A_0)| \\ &= |A_0| + |N_H(S)| \\ &\geq |A_0| + |S| \end{aligned}$$

4.1.8 Def : k-regular

G is **k-regular** if every vertex has degree k

4.1.9 Def : Perfect matching

A matching is **perfect** if it covers every vertex

4.1.10 Corrolary

Every k-regular bipartite graph has a parfect matching

4.1.11 Proof

4.1.12 Petersen's corrolary

k : even, ≥ 2
Every k-regular graph has a spanning 2-regular subgraph.

4.1.13 Proof

G has an Eulerian circuit C.
We orient edges according to C.
Construct a bipartite graph on $V \cup V'$ (Where V' is a copy of V)

4.2 Matching in a general graph

When do we have a perfect matching ?

— $|V(G)|$ is even
— No isolated vertices
— No odd components

4.2.1 Tutte theorem

G has a perfect matching $\Leftrightarrow odd(G\backslash S) \leq |S| \forall S \subseteq V(G)$
(Where $odd(G)$ is the number of odd component of G)

4.2.2 Proof

$\Rightarrow$: trivial

$\Leftarrow$: Induction on $|V(G)|$

1. $|V(G)|$is even
   Proof : $odd(G) \leq 0$
   Let's call X critical if $odd(G\backslash X) = |X|$
   G has a critical set ($\emptyset$)
2. If $odd(G\backslash X) \geq |X| - 1$, then X is critical
   Proof : $0 \cong |V(G| \cong odd(G\backslash X) + |X|$ (mod 2)
   Choose a maximal ciritcal set X
   $C_1 \cdots C_K$: odd components of $G\backslash X$
   $D_1 \cdots D_K$: even components of $G\backslash X$
   $k = |X|$
3. $G\backslash X$ has no even component
   Proof : Otherwise, let $v \in V($even componenent $D)$
   $odd(G\backslash(X \cup v)) \geq |X|$
4. $\forall i \in 1, \cdots, k$, each $v \in V(C_i)$, $C_i\backslash v$ has a perfect matching.
   Proof : If not, then there is a set $Y \subseteq V(e_i) - \{v\}$
   $odd(V_i - v - Y) > |Y|$
   Let $X' = X \cup Y \cup v$
   $odd(G\backslash X') \geq (|X| - 1) + (|Y|_1)$

4.2.3 Cut edge

An edge e is a **cut-edge** if $G\backslash e$ has more components than G

4.2.4 Petersen's Corrolary (1891)

Q : Does every 3-regular graph have a perfect matching ?
A : False, we can found some counter-example
But... If G is a **simple** 3-regular graph, with at most 2 cut-edges, then G has a perfect matching

4.2.5 Proof

Suppose $X \subseteq V(G)$.
$odd(G\backslash X) > |X|$

- Each component of G has even number of vertices.
- Let's take a look a one particular componenet.
  If C is an odd component of $G\backslash X$, then the number of edges from X to C is odd
  $3|V(C)| = 2|E(C)| +$(number of edges from X to C)
  Let's go back to every component.
  $3(K-2)+2 \leq$ (Number of edges from X to odd components) $\leq 3|X|$
  $\Leftrightarrow |X| \leq k \leq |X+1| \Leftrightarrow k = |X+1|$
  But, we also have :
  $odd(G\backslash X) + |X| \cong |V(G)|$ (mod 2)
  $\Leftrightarrow |V(G)| \cong 0$ (mod 2)
  $\Leftrightarrow |V(G)| \cong |X|$ (mod 2)
  Contradiction

4.2.6 Tutte-Berge formula

$|V(G)| - 2\nu(G) = \max_{X \in V(G)}(odd(G\backslash X) - |X|)$

4.2.7 Proof

- $\geq$: trivial
- $\leq$: Let $k = \max(odd(G\backslash X) - |X|)$
  Goal : find a matching of size $\frac{|V(G)|-k}{2}$
  G' is obtained from G by adding a complete graph on k vertices and making new vertices adjacent to all new vertices.
  Enough to show that G' has a perfect matching.
  $odd(G'\backslash X) \leq 1$
  If X does not contain all new vertices, then $odd(G') = 0$
  If X contains all new vertices,

$$\begin{aligned} odd(G'\backslash X) &= odd(G\backslash(X \cap V(G))) \\ &\leq |X \cap V(G)| + k \\ &\leq |X| \end{aligned}$$

4.2.8 Other proof

Let $f_G(U)$:=number of vertices in odd components of $G\backslash U$
Choose U such that :

— $|V(G)| + 2\nu(G) = odd(G\backslash U) - |U|$

— Minimizing $f_G(U)$

4.2.9 Properties

— All even componenets of $G\backslash U$ have perfect matchings.

— **CLAIM** : If C is a component of $G\backslash U$, and $v \in V(C)$, then$C\backslash v$ has a perfect matching.
If not, then there is $X \subseteq V(C) - \{v\}$ such that $odd(C\backslash X\backslash V) > |X|$

— By the parity, $odd(C\backslash X\backslash V) \geq |X| + 2$.

4.2.10 Claim

Fr every non empty subset W of U, number of odd componenet of $G\backslash U$ having neighbours in > $\geq |W| + 1$

4.2.11 Gallait-Edmond Sctructure Theorem

Define D, A, C as :

— D(G) = Vertices in odd component of $G\backslash U$
— A(G) = U
— C(G) = vertices in even components of $G\backslash U$

Then :

1. For each component C of $G[D(G)]$ and each vertex v of C, $C\backslash v$ has a perfect matching.

2. $G[C(G)]$ has a perfect matching

3. For all $\phi \neq S \leq A(G)$, S has $\geq |S| + 1$ components of $G[D(G)]$ having neighbours of S.
4. $|V(G)| - 2\nu(G) = odd(G\backslash A(G)) - |A(G)|$

4.2.12 Lemma

Let P be a path from x to y in G, where y is adjacent to a vertex z not covered by M
Then one of the followings holds :

1. G has a M-augmenting path from x to z

2. G has a M-flower from x

A M-flower is an M-alternating walk $P = v_0v_1 \cdots v_n$ such that $v_0, \cdots, v_{n-i}$ are distinct, and $v_n = v_i$ for some $i < n$

4.2.13 Cycle shrinking Lemma

Let G be a simple graph.
M be a matching.
C be a cycle with $2k + 1$ edges such that exactly k edges of C are in M
And 1 vertex is not covered by M
Then :

$$M \text{ is a maximum matching of } G \Leftrightarrow M - E(C) \text{ is a maximum matching of } G\backslash E(C)$$

4.2.14 Stable matching

For each vertex x of G, let us give a linear ordering $\leq_v$ of the edges incident with v (i.e. : $e \leq_v f \Rightarrow v$ prefers f to e)
M is **stable** if there is no edge $e \in E(G) \backslash M$ such that for each end v, there is $f \in M$ incident with v and $f \leq_v e$

4.2.15 Gale-Shapley theorem

Every simple bipartite graph has a stable matching

4.2.16 $\Rightarrow$ Algorithm for Gale-Shapley theorem

Step 2i Every single boy proposes to the best girl tha he never proposed to

Step 2i+1 Each girl chooses the best boy among all boys who proposed to her at state $2i$, and the current partener

And we loop to step $2i + 2$

Chapitre 5

Connectivity

5.1 Definition

5.1.1 k-connected

A graph is k-**connected** if :

1. $|V(G)| > k$
2. $G\backslash X$ is connected for all $X \subseteq V(G)$ with $|X| < k$

5.1.2 Cut edge

If $G\backslash v$ has more components than G, then v is a **cut-vertex** if G

5.1.3 k-edge-connected

$\delta_G(X)$ = set of all edges having one end in X and another end in $V(G)\backslash X$ If G is a k-**edge-connected** graph, then $|\delta_G(X)| \geq k \forall \emptyset \neq X \subsetneq V(G)$

5.1.4 2-edge-connected

Lemma

Let $x, y \in E(G)$
If G has a cycle D containing y and z, then G has a cycle containing x and z

Theorem

Let G be a connected graph with >2 vertices
The following are equivalent :

1. G is 2-connected
2. If e, f are non-loop edges, then G has a cycle containing e ad f
3. If v, w are vertices of G, then G has a cycle containing v and w

Property

Let e be an edge of a 2-connected graph G
Then, either $G\backslash e$pr G/e is 2-connected, unless $|V(G)| = 3$

Theorem

Menger's theorem

G gas at least k disjoint paths from S to T
$\Leftrightarrow$
G has no $x \subseteq V(G)$ such that $|x| < k$ and $G\backslash x$ has no path from $s - x$ to $t - x$

Menger's theorem corrolary

Let a, b be non-adjacent pair of dinstinct vertices of G.
G has k iternally distinct path from a to b
$\Leftrightarrow$
G has no set S, $|S| < k, S \subseteq V(G) - \{a, b\}$, such that $G \backslash S$ has no path from a to b

Corrolary

Let G be a graph with >k vertices :
G is k-connected
$\Leftrightarrow$
$\forall$ pairs a, b of distinct vertices, G has k internally paths from a to b

5.1.5 3-connected

Theorem

If G is 3-connected and $|V(G)| > 4$, then G has an edge e such that G/e is 3-connected

Tutte's chain theorem

Let G be a simple 3-connected graph.
Then, there exists an edge e such that either $G \backslash e$ or G/e is simple 3-connected (unless G is the "Wheel")
(The "Wheel" is a cycle + a vertex adjacent to every other vertex)

Seymour's splitter theorem

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