On the use of Trinomial Trees and forward shooting grids to price popular options

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Contents

1 European Option

1.1 The algorithm to price a European Option using a trinomial tree

For a trinomial tree, the formula for the stock price is given by:

$$S_k^n = S_0 u^{N_u(k)} d^{N_d(k)} m^{N_m(k)},$$
$$n = 0,1,...,N \quad k = 0,1,...,n, \quad N_u + N_d + Nm = n$$

Here, $m = 1$, thus, the formula for the stock price is given by

$$S_k^n = S_0 u^{N_u(k)} d^{N_d(k)}, n = 0,1,...,N \quad k = 0,1,...,n, \quad N_u + N_d + Nm = n$$

This is the k^{th} possible stock price level at time n, and V_k^n is the time-n option value when the stock price is S_k^n.
Here, we have set $d = 1/u$, $m = 1$ and the final states are, beginning by the top:

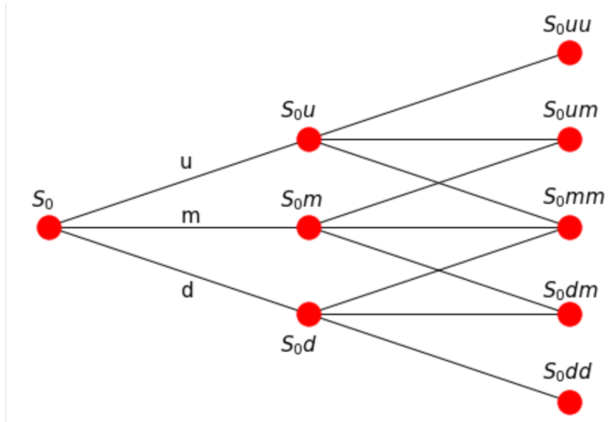

Figure 1: Trinomial tree

$$u^2 d^0, u^2 d, u^2 d^2, ..., u^2 d^4$$

and the formula becomes equivalent to writing

$$S_{n,k} = S_0 u^n d^k$$

with k varying from 0 to 2n[1].

The Risk Neutral Pricing formula to a simple European Option with payoff $g(S_T)$ at time t=0 is given by : $E^Q(e^{-r\triangle t} g(S_N))$. We may not be able to compute this explicitly for some complicated payoff structures which is why the lattice (tree) method allows to approximate the continuous price process S by a simple discrete process to facilitate the expectation computation.

$$Euro.option : V_k^n = e^{-r\triangle t}[quV_k^{n+1} + qmV_{k+1}^{n+1} + qdV_{k+2}^{n+1}] \quad \text{for each } k = 0,1,...,n$$

$$Amer.option : V_k^n = max(g(S_k^n), e^{-r\triangle t}[quV_k^{n+1} + qmV_{k+1}^{n+1} + qdV_{k+2}^{n+1}]) \quad \text{for each } k = 0,1,...,n$$

The algorithm works as follows:

[1] (At time n, the tree has 2n+1 values, as showed in the graph where n=2 and we have 5 final values)

(i) Define $s_k^n = S_0 u^{n-k}$ for n=0,1,...,N, k=0,1,...,2N.

(ii) $Let V_k^N$ be the n-option value when the stock price is s_k^n. The option price at terminal time n = N is given by the payoff function i.e $V^N = g(S_N)$ and in particular:

$$V_k^N = g(S_0 u^{n-k} d^k)$$

for each k=0,1,...,2N.

(iii) Loop backward[2]in time for each n=N-1,N-2,...,0 and compute:

$$V_k^N = e^{-r\Delta t}(q_u V_k^{n+1} + q_m V_{k+1}^{n+1} + q_d V_{k+2}^{n+1})$$

(iv) The required time-zero option value is $V_{0,0}^0$

Note: If the option is american, then (iii) is slightly modified (as we will see later when pricing american options) and the the recursive equation will become:

$$V_k^N = max\big(g(s_k^n), e^{-r\Delta t}(q_u V_{k)}^{n+1} + q_m V_{k+1}^{n+1})$$

1.2 Trinomial Model and Black Scholes: do they converge with N?

Let us recall the Black-Scholes formula for a European call:

$$C(S_0, T, K, r, \sigma) = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

where N is the standard normal cdf.

As N $\to \infty$, the random walk on the trinomial tree converges to a geometric Brownian motion, a similar correspondence was established in the notes between the Black-Scholes and Cox-Ross-Rubinstein theory. Therefore, we expect our trinomial option prices to converge to the corresponding prices given by Black-Scholes pricing. The aim of this section is to investigate this phenomenon.
We have plotted below the price of a Call Option with S0, r, T, sigma, K = 100, 0.01, 1, 0.2, 100 for varying N and $\lambda \in \{1, 1.25, 1.5, 1.75\}$
As we can see on the four graphs below for each lambda parameter, the price obtained through the trinomial seem to converge for a sufficient number of steps, to the exact price obtained by the Black-Scholes formula for a European Call.
In Figure 7, we have plotted the option price for a fixed N=500 and $\lambda = 1$ to study the results when S_0 increases/decreases. As expected, the Call price increases with the spot price. This is usually described by the positive delta on a Call Option (bounded between 0 and 1).

[2]The backward induction algorithm can be derived from the risk-neutral principle (martingal property of the discounted price under the risk neutral probability), and holds for options whose payoffs are in the form given in the question which is a function of the stock. When applied in the context of a trinomial tree (using the exact same methodology as the binomial tree), we can calculate the option value at interior nodes of the tree by considering it as a weighting of the option value at the future nodes, discounted by one time step. Thus we can calculate the option price at time n V_n, as the option price of an up move $q_u V_{n+1}$ plus the option price of the middle move by $q_m V_{n+1}$ plus the option price of a down move by $q_d V_{n+1}$, discounted by one time step, $e^{-r\Delta t}$

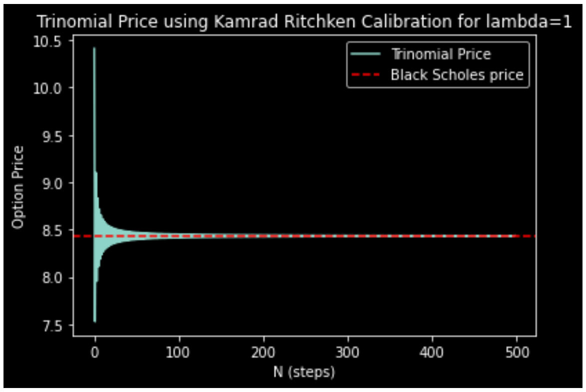

Figure 2: Trinomial price with λ =1

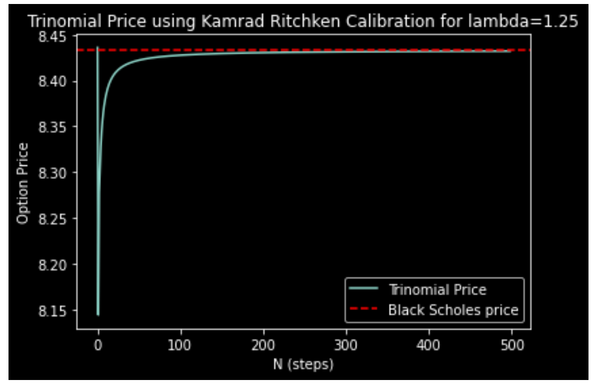

Figure 3: Trinomial price using λ = 1.25

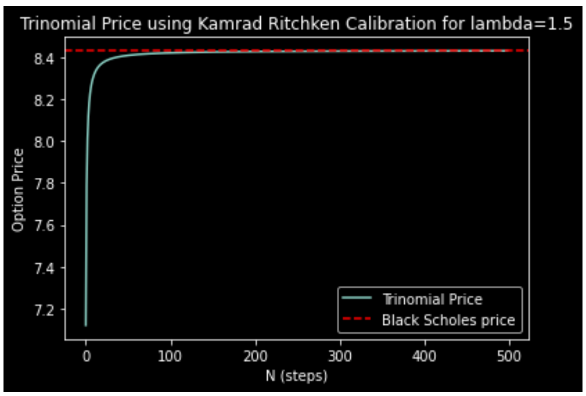

Figure 4: Trinomial price using λ = 1.5

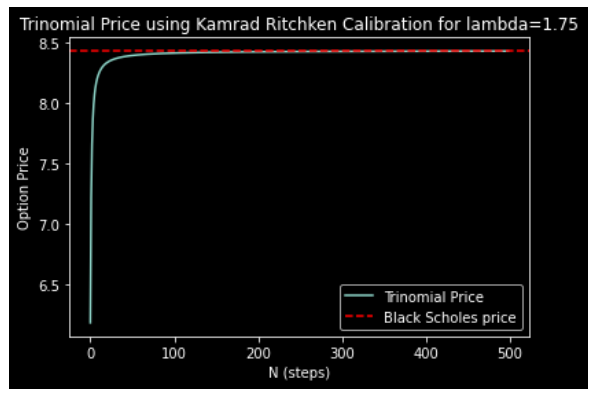

Figure 5: Trinomial price using λ = 1.75

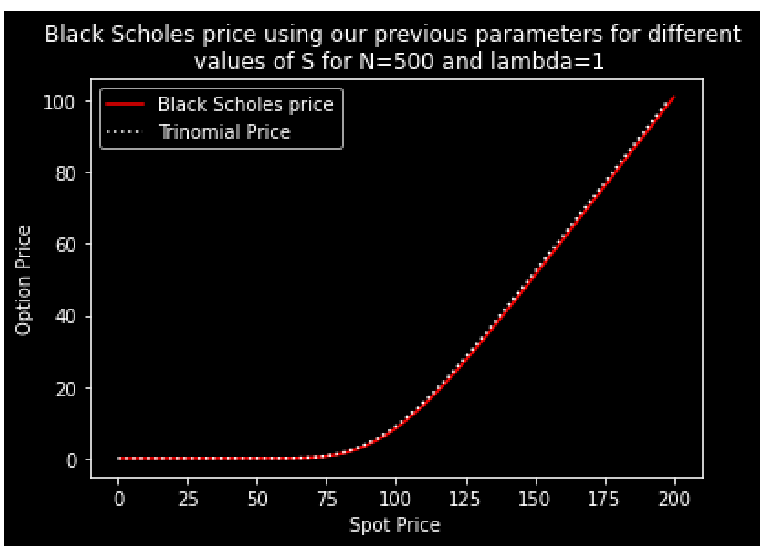

Figure 6: Trinomial tree price final value for different values of S0, and λ = 1

1.3 The differences between the Trinomial Tree and the BS Approach

We test for different values of S(0), we have to choose the λ such that we always have a good estimate of the Black Scholes price no matter what the initial price is. To evaluate the best λ, we average the rate of error made compared to the Black-Scholes price, for each initial value chosen for the underlying. It appears that the best λ given by this method is $\lambda = 1.75$ that yields the lowest difference in % (after taking the mean).

Stock Prices	Price Trinomial	Price Black Scholes	Difference in %
70	0.278311	0.279521	-0.433121
80	1.303312	1.302245	0.081942
90	3.860321	3.861404	-0.028057
100	8.429346	8.433319	-0.047104
110	14.947178	14.944704	0.016557
120	22.941777	22.941877	-0.000434
130	31.896781	31.896319	0.001449

Stock Prices	Price Trinomial	Price Black Scholes	Difference in %
70	0.278926	0.279521	-0.212943
80	1.301272	1.302245	-0.074759
90	3.862726	3.861404	0.034231
100	8.432182	8.433319	-0.013475
110	14.946319	14.944704	0.010804
120	22.942402	22.941877	0.002291
130	31.896500	31.896319	0.000569

Stock Prices	Price Trinomial	Price Black Scholes	Difference in %
70	0.279585	0.279521	0.022770
80	1.302914	1.302245	0.051379
90	3.876481	3.861404	0.390462
100	8.408380	8.433319	-0.295712
110	14.950654	14.944704	0.039814
120	22.947005	22.941877	0.022356
130	31.892741	31.896319	-0.011218

Stock Prices	Price Trinomial	Price Black Scholes	Difference in %
70	0.279559	0.279521	0.013621
80	1.302503	1.302245	0.019767
90	3.861996	3.861404	0.015321
100	8.429223	8.433319	-0.048563
110	14.942738	14.944704	-0.013156
120	22.942899	22.941877	0.004455
130	31.896507	31.896319	0.000591

Figure 7: Tables of the prices found with the trinomial tree and the Black Scholes Formula, for different initial values of S and for $\lambda = 1, 1.25, 1.5, 1.75$

2 American options

2.1 The algorithm to price an American Option using a trinomial tree

Recall the algorithm when we studied the European option. For the American option, the algorithm stays intact with the exception of point iii) which is altered as follows:

$$V_k^N = max\big(g(s_k^n), e^{-r\Delta t}(q_u V_k^{n+1} + q_m V_{k+1}^{n+1})$$

The intuition behind this change is that at each time point n, there are two possibilities:

- If it is optimal to exercise the option now, the option holder immediately receives the payoff $g(S_n)$.
- If it is not optimal to exercise the option now, then the option continues to exist and its value in the next period will be V^{n+1} (which is random from perspective of time n).

2.2 Early exercise for an American Call Option

We have to find the optimal early exercise strategy at time zero. We define the early exercise to be optimal at node (k,n) if the intrinsic value is greater than the continuation value:

$$V_{k,cont}^n = e^{-r\Delta t}(q_u V_k^{n+1} + q_m V_{k+1}^{n+1} + q_d V_{k+2}^{n+1})$$

These quantities can be computed at time 0. Thus, at time zero, we have for each node several possible continuation values (happening with the three probabilities of the model) and the possible intrinsic values, all depending on s_k^n. The algorithm in simple terms works as follows:

- Exercise if intrinsic value is higher than the continuation value.
- Else: don't exercise.

We have plotted the value of the American Call and the intrinsic value of the Call, both at t=0, for several values of S_0 in the x axis, fixing K=100, N=500 to price the American call at time zero. We can tell from this graph two things in particular:

- The price of the option (American or European) is above the intrinsic value.
- The American and European price seem very close if not equal.

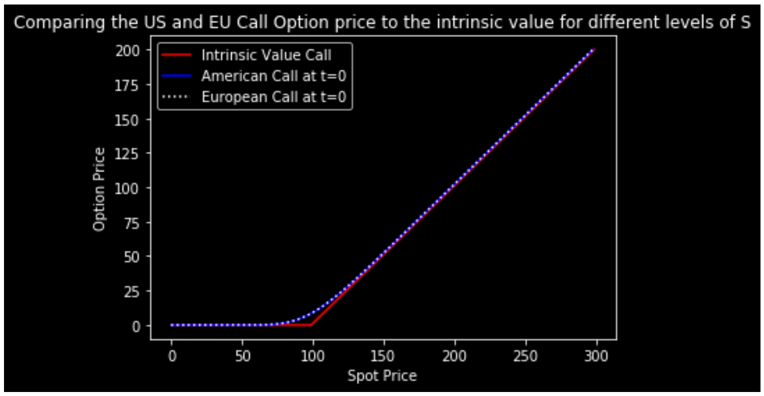

Figure 8: Time-zero value of an American call option and the intrinsic value of the option against initial stock price

Call American=Call European?

On the graph below, we have plotted: Call Option Price - Intrinsic Value for both American and European options. What we see is that the call price is the same for both European and American types which suggests that it is not optimal to exercise the call option at t=0. This is not surprising and is explained by Hull in his book *Options, Futures and Other Derivatives*. The rationale being that an american call that pays no dividend (as per our assumption) has the same value as an european call option because it is never optimal to exercise an american option before maturity. There are different approaches to explain this result. We will here briefly prove that the American Call Option cannot be superior to the European one. Suppose, the American call is ITM before maturity (i.e S_T>K). If you exercise the option, you will receive S_T - K. However, if you sell the call, you will receive the value of the call.

The Put Call Parity tells us that C>S_T-K and hence an arbitrage opportunity would occur (long the call, short the stock and invest the proceeds).

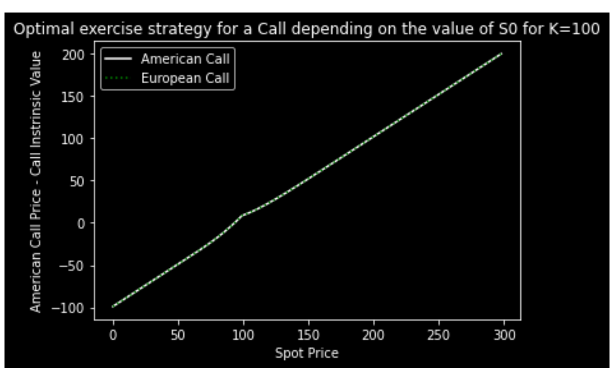

Figure 9: Optimal early exercise at time zero for American Call

2.3 Early exercise for an American Put Option

The Put option however tells a different story. This time, for a non-dividend stock, the optimal early exercise is non-trivial. Hull suggests that American puts should be equal or more valuable than European ones. One of the reason is the time-value of the put option.
Indeed, let us recall that the definition of a put is a contract giving the option buyer the right, but not the obligation, to sell an asset, at a specified price, by a specified date to the writer of the put. If you exercise this option at t=0, you receive K, the strike price and can invest it a the risk-free rate and receive at maturity of the option Ke^{rT} and hence you would be better off.

At time zero, we compute the values $V_{n,k}$ using the American option algorithm we have previously defined where the payoff is given by: $(K - S_T)^+$

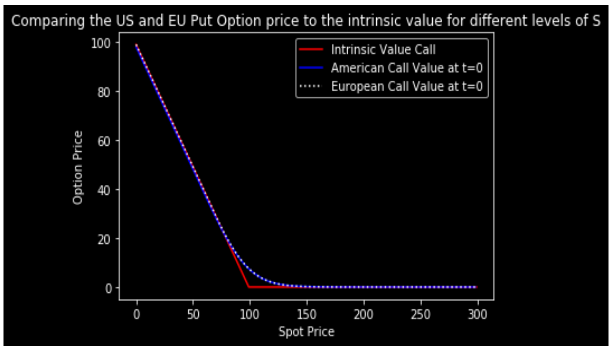

Figure 10: Computational time for European look-back floating strike options

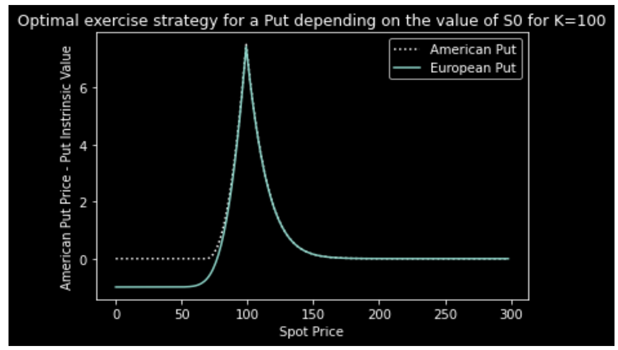

Figure 11: Computational time for American look-back floating strike options

We've plotted in 13 the difference between the American Put price and the Put intrinsic value. We see that for both European and American Option Price the biggest difference between Intrinsic and Option price is reached around the money. This is something we expect and could understand using different approaches (e.g Gamma of the option at its peak ATM/Payoff function jumping around the money).

What we see as well in 13 is that the American Put diverges from the European Option Price for low Strikes and converges to the intrinsic value (see 13 for low K, the y axis tends to 0). Indeed, if we remember what we said earlier, if we exercise early, we receive the Strike that we can invest at the risk-free rate i.e the deeper the ITM option is, the higher the early exercise risk premium as exhibited by our graph.

Now let's look at how the early exercice strategy changes with rates and volatility.
Coval and Shumway (2001) have shown that the less volatile the asset is, the higher the early exercise risk premium. Similarly, the shorter the time-to-maturity, the higher the early exercise risk premium. In this section, we will investigate whether this holds true from a mathematical perspective using the continuation value we have derived previously.

The higher r, the more likely that an optimal strategy will happen in a short time. That is because the exponential makes the continuation value a decreasing function of the discount factor r.
On the other hand, the conclusion is the opposite when r goes to zero.

Now let's explore what a change of volatility implies: when σ goes to infinity, u goes to infinity and d goes to zero. Taking the limit of infinity is a bit wrong, the limit for σ is the value such that $q_u = 0$. Solving this equation leads to a polynomial, with non-zero solution $\sigma = 1 + \frac{1}{\lambda\sqrt{\Delta t}}$ and we can assume Δt close to zero, which gives very high bound for the volatility.

The difference between q_u and q_d is $O(\sigma\sqrt{\Delta t})$ (simple calculation, the r/σ doesn't count for big values) thus, in an expectation, $q_u * u$ will count for less, because u is an exponential of $\sigma\sqrt{\Delta t}$ and by a Taylor expansion, if we write $q_u = A - B\sigma\sqrt{\Delta t}$, $q_d = A + B\sigma\sqrt{\Delta t}$, and $exp(\sigma\sqrt{\Delta t}) = 1 + \sigma\sqrt{\Delta t} + \sigma^2\sqrt{\Delta t}^2 + ...$ we see this fact, B being approximately equal to A with $\lambda = 1.75$.

Our belief is thus that the higher σ is, the higher the continuation values will be (let's recall that q_m doesn't depend on the volatility) because the $V_{n,k}$ is a decreasing function of $S_{n,k}$, so the later the exercise will be: the intrinsic value is lower than the continuation value for a longer time.

Figure 13 shows that the lower r is, the higher the price of the American Put (ceteris paribus) relative to the European one, this is motivated by the fact that the put is long rho. What else, the American put price is almost equal to the European one except for low values of r where the American Put Option diverges and become more expensive.

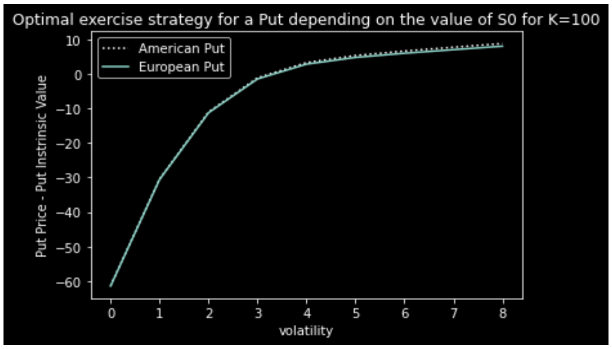

Figure 12: Optimal early exercise at time zero when S0 = K =100 for American Put, with volatility in x axis

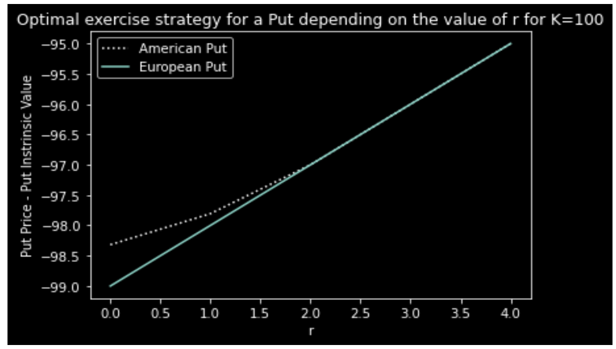

Figure 13: Difference between Put price and intrinsic value with r in x axis, K=100=S_0

We see that the difference between the put price and the intrinsic value increases with respect to the volatility, this is consistent with our remark for the early exercise strategy, and we've seen that u increases exponentially with σ while d doesn't really vary (but close to zero) and the price of the American put will indeed increase with volatilty.

3 Lookback option

3.1 Pricing the floating strike lookback put option using the forward shooting grid method

At time n, the possible values of Sn are $\{S_0u^n, S_0u^{n-1}, ..., S_0, ..., S_0u^{-n}\}$. Hence we set:

$$s_k^n = S_0u^{n-k}$$

where n=0,1,...,N and k=0,1,...,2n.

Since the option involves the maximum value of the stock price path, it is natural to choose as the auxiliary variable:

$$M_n := \max_{i=0,1,...,n} S_i$$

which represents the running maximum of the stock price level from time zero up to time n. At time n, the best possible historical maximal value of the stock price is S_0u^n (which occurs if the stock price goes up in every period) and the worst historical maximum value is S_0 (which occurs if the stock price never goes above its initial value S0).
In general, the possible values of M_n for some n are: $\{S_0, S_0u, S_0u^n\}$. Hence we define a grid for the process M_n as:

$$m_j^n = S_0u^{n-j}$$

where n=0,1,...,N and j=0,1,...,n.

As M_n+1 is the running maximum of the stock price from time zero up to time n+1, we expect:

$$M_{n+1} = max(M_n, S_{n+1})$$

Suppose at time n the stock price process moves from state s_k^n to $s_{k_{new}}^{n+1}$ for some new state k_{new}.

We need to identify the corresponding transition associated with the auxiliary variable. Starting with:

$$\begin{aligned}
M_{n+1} = max(M_n, S_{n+1}) &\iff m_j^{n+1}new = max(m_j^n, s_{k_{new}}^{n+1}) \\
&\iff S_0u^{n+1-j_{new}} = max(S_0u^{n-j}, S_0u^{n+1-k_{new}}) \\
&\iff n+1-j_{new} = max(n-j, n+1-k_{new}) \\
&\iff j_{new} = n+1-max(n-j, n+1-k_{new}) \\
&\iff j_{new} = min(j+1, k_{new}) \\
&\iff j_{new} := \phi(k_{new}, j)
\end{aligned}$$

Hence we can define the shooting function as:

$$\phi(n, k_{new}, j) = min(j+1, k_{new})$$

which describes the new state of the auxiliary variable M when stock price moves to a k_{new} state knew and that the current state of M is j.

3.2 The algorithm

As we have now defined the shooting function associated to our auxiliary variable, we can now write the algorithm to price the floating strike lookback put as follow:

(i) Define $s_k^n = S_0u^{n-k}$ and $m_j^n = S_0u^{n-j}$ for n=0,1,...,N, k=0,1,...,2N and j =0,1,...,N.

(ii) $Let V_{k,j}^N$ be the n-option value when the stock price is s_k^n and the running maximum of the stock price is m_j^n. The option price at terminal time n = N is given by the payoff function:

$$g(s_k^n, m_j^n) = m_j^n - s_k^n$$

for all k = 0,1, ...,2N and j =0,1, ...,N.

(iii) Loop backward in time that for each n=N-1,N-2,...,0 and compute:

$$V_{k,j}^{N} = e^{-r\Delta t}(q_u V_{k,\phi(n,k,j)}^{n+1} + q_m V_{k+1,\phi(n,k+1,j)}^{n+1} + q_d V_{k+2,\phi(n,k+2,j)}^{n+1})$$

(iv) The required time-zero option value is $V_{0,0}^{0}$

Note: If the option is american, then (iii) is slightly modified and the the recursive equation will become:

$$V_{k,j}^{N} = max\{g(s_k^n, m_j^n), e^{-r\Delta t}(q_u V_{k,\phi(n,k,j)}^{n+1} + q_m V_{k+1,\phi(n,k+1,j)}^{n+1} + q_d V_{k+2,\phi(n,k+2,j)}^{n+1})\}$$

3.3 The Analysis

The code complexity of our algorithm is $\mathcal{O}(n^3)$[3]which makes the algorithm very time-consuming for big simulations because of the emphasized convexity of n^3 (as we will show below for N>100). Hence it is necessary that we find a reasonable compromise between N and computational time which leads us to study the convergence of the price for different Ns and report their computational time. In this sentence we

The two tables below exhibit this for both European and American lookback floating strike options:

N	EU Prices	Computational time (seconds)
10	13.204753	0.003953
20	14.861644	0.022972
50	14.861644	0.317141
100	15.294589	2.549420
200	15.610695	21.463724
500	15.898234	338.356436

Figure 14: Computational time for European lookback floating strike options

N	AM Prices	Computational time (seconds)
10	13.393334	0.004087
20	15.008448	0.025280
50	15.008448	0.368047
100	15.434824	3.146343
200	15.747517	24.155889
500	16.032977	402.013110

Figure 15: Computational time for American lookback floating strike options

As we can see from these tables, any N>200 leads to prices taking over 20seconds which in the case of a lookback is very time-consuming and seems unreasonable. On the other hand, as shown in the figure below, the price seems to converge for N>100 and 2 seconds to compute an option that is not vanilla seems acceptable as long as the purpose is not to build a systematic strategy which needs to be very time-efficient.

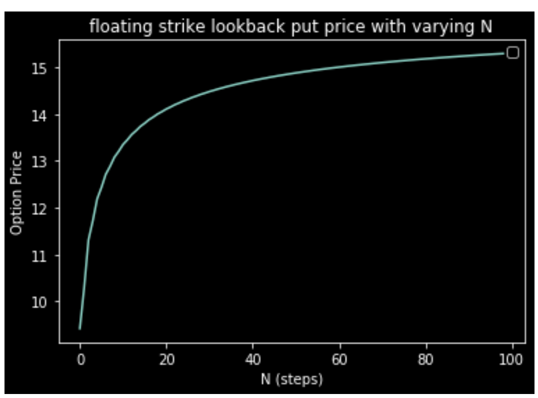

Figure 16: Floating strike lookback put price with varying N

[3]One could greatly reduce the code complexity by introducing a vectorized solution to handle the backward recursion equation

4 Code Link

https://github.com/BassamSINAN/Lattice-Methods-for-Option-Pricing

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