Tuning Avalanche Exponent with Spontaneous Activity

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I. MATHEMATICAL FRAMEWORK

Consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = 1, \cdots, n$, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the sets of network vertices and directed edges. Let $A = [a_{ij}]$ be the weighted and directed adjacency matrix of $\mathcal{G}$. At each time point $t \in \mathbb{Z}_{\geq 0}$, we associate each node i with a discrete non-negative random variable X_i^t.

The evolution of the dynamics in our system follow a hierarchical stochastic process. Starting at some non-random initial state $\boldsymbol{x}^0$, we define discrete non-negative random variable X_j^t that represents the number of successful transmissions received by node j at time t. Initially, we define

$$X_j^{t+1} \sim \sum_{i=1}^{n} B(X_i^t, a_{ji}),$$

which is the sum of binomial distributions conditioned on random variables that are the states at time t.

II. JUDICIOUS SELECTION OF SPONTANEOUS ACTIVITY TO GENERATE AVALANCHES

We can enumerate all possible discrete states $\boldsymbol{x}_i$, as

$$\boldsymbol{x}_i \in \left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \cdots, \begin{bmatrix} d \\ d \\ \vdots \\ d \end{bmatrix} \right),$$

and the probability of transitioning from any state $\boldsymbol{x}_m$ and $\boldsymbol{x}_p$ is given simply as the product of sums of binomial distributions. From this system, we can construct infinite-dimensional Markov system

$$\boldsymbol{p}(k) = \mathbb{T}\boldsymbol{p}(k-1),$$

where $A \to \mathbb{T}$ is a map determined by the binomial evolution, and $\mathbb{T} = PDP^{-1}$. From $\boldsymbol{p}(0)$, we can decompose

$$\boldsymbol{p}(0) = \boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3 + \cdots = P\boldsymbol{c},$$

to yield

$$\boldsymbol{p}(t) = \boldsymbol{e}_1 + c_2\lambda_2^t\boldsymbol{e}_2 + c_3\lambda_3^t\boldsymbol{e}_3 + \cdots = PD^tP^{-1}\boldsymbol{p}(0),$$

where avalanche durations are given simply by the first entry of $\boldsymbol{p}(t)$. We find coefficients

$$\boldsymbol{c} = P^{-1}\boldsymbol{p}(0).$$

A. Given

By estimating $\mathbb{T}$, we obtain

$$\lambda(\mathbb{T}) \qquad\qquad P = [\boldsymbol{e}_1, \cdots],$$

from which we know that $P(\boldsymbol{X}^t = \boldsymbol{0}) = 1 - c_2\lambda_2^t e_{21} - c_3\lambda_3^t e_{31} + \cdots$. Of course, we can't pick any $\boldsymbol{c}$. We require

$$\boldsymbol{1}^T P\boldsymbol{c} = 1 \qquad\qquad P\boldsymbol{c} \geq \boldsymbol{0}$$

B. Statement 1

Hence, if we desire some $\boldsymbol{c}^*$, we seek

$$\arg\min_{\boldsymbol{c}} (\boldsymbol{c} - \boldsymbol{c}^*)^T(\boldsymbol{c} - \boldsymbol{c}^*),$$

such that

$$\boldsymbol{1}^T P\boldsymbol{c} = 1 \qquad\qquad P\boldsymbol{c} \geq \boldsymbol{0}.$$

C. Statement 2

A more direct statement is given $P(\boldsymbol{X}^t = \boldsymbol{0}) = 1 - |c_2||e_{21}||\lambda_2|^t - |c_3||e_{31}||\lambda_3|^t + \cdots$, find

$$\arg\min_{\boldsymbol{p}(0)} \||P^{-1}\boldsymbol{p}(0)| - |\boldsymbol{c}^*|\|_2^2,$$

such that

$$\sum \boldsymbol{p}(0) = 1 \qquad\qquad \boldsymbol{p}(0) \geq 0.$$