

Reconstruction of k Impulse Functions from k Measurements of a Linear Dynamical System

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I. MATHEMATICAL FRAMEWORK

A. Expected Value of Stochastic Discrete Maps

Consider a network represented by directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = 1, \dots, n$, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the sets of network vertices and directed edges. Let $A = [a_{ij}]$ be the weighted and directed adjacency matrix of \mathcal{G} . At each time point $t \in \mathbb{Z}_{\geq 0}$, we associate each node i with a discrete non-negative random variable X_i^t .

The evolution of the dynamics in our system follow a hierarchical stochastic process. Starting at some non-random initial state \mathbf{x}^0 , we define discrete non-negative random variable X_j^t that represents the number of successful transmissions received by node j at time t . Initially, we define

$$X_j^{t+1} \sim \sum_{i=1}^n B(X_i^t, a_{ji}),$$

which is the sum of binomial distributions conditioned on random variables that are the states at time t . The expectation of this conditional distribution is given by

$$\begin{aligned} \mathbb{E}[X_j^{t+1}] &= \mathbb{E}[\mathbb{E}[X_j^{t+1} | \mathbf{X}^t]] \\ &= \mathbb{E}[\mathbb{E}[\sum_{i=1}^n B(X_i^t, a_{ji})]] \\ &= \mathbb{E}[\sum_{i=1}^n \mathbb{E}[B(X_i^t, a_{ji})]] \\ &= \mathbb{E}[\sum_{i=1}^n a_{ji} X_i^t] \\ &= \sum_{i=1}^n a_{ji} \mathbb{E}[X_i^t] \\ &= \mathbf{a}_j \mathbf{X}^t. \end{aligned}$$

In vector form, this expectation becomes

$$\mathbb{E}[\mathbf{X}^{t+1}] = A\mathbb{E}[\mathbf{X}^t],$$

which we see is a simple discrete linear map.

B. Upper Bound on Variance

As exciting as linearity in these stochastic linear maps are, the distributional variances provide more trouble. First, we establish a useful inequality for our purposes. If $\alpha_1, \dots, \alpha_n$ are real numbers such that $\alpha_i \geq 0$ where $\sum_{i=1}^n \alpha_i = c$, and X_1, \dots, X_n are random variables. Then

$$\begin{aligned} Var\left(\sum_{i=1}^n \alpha_i X_i\right) &\leq \left(\sum_{i=1}^n \alpha_i\right) \sum_{i=1}^n \alpha_i Var(X_i) \\ &\leq c \sum_{i=1}^n \alpha_i Var(X_i). \end{aligned}$$

For hierarchical mixture models, we have the following formula

$$\begin{aligned}
Var[X_j^{t+1}] &= \mathbb{E} [Var [X_j^{t+1} | \mathbf{X}^t]] + Var [\mathbb{E} [X_j^{t+1} | \mathbf{X}^t]] \\
&= \mathbb{E} \left[\sum_{i=1}^n a_{ji}(1 - a_{ji})X_i^t \right] + Var \left[\sum_{i=1}^n a_{ji}X_i^t \right] \\
&= \sum_{i=1}^n a_{ji}(1 - a_{ji})\mathbb{E} [X_i^t] + Var \left[\sum_{i=1}^n a_{ji}X_i^t \right] \\
&\leq \sum_{i=1}^n a_{ji}(1 - a_{ji})\mathbb{E} [X_i^t] + c \sum_{i=1}^n a_{ji}Var [X_i^t],
\end{aligned}$$

which, we see is linear in the expected value and variance of X_i^t , and therefore can write

$$Var [\mathbf{X}^{t+1}] \leq (A - A_2)\mathbb{E} [\mathbf{X}^t] + cAVar [\mathbf{X}^t],$$

where A_2 is a matrix with entries $[a_2]_{ij} = [a]_{ij}^2$. We notice this relationship is recurrent, and can propagate all the way back to some initial condition

$$Var [\mathbf{X}^{t+1}] \leq \left(\sum_{i=0}^k c^i [A^{k+1} - A^i A_2 A^{k-i}] \right) \mathbb{E} [\mathbf{X}^0] + c^{k+1} A^k A_2 Var [\mathbf{X}^0],$$

and when the branching parameter $c = 1$:

$$Var [\mathbf{X}^{t+1}] \leq \left((k+1)A^{k+1} - \sum_{i=0}^k A^i A_2 A^{k-i} \right) \mathbb{E} [\mathbf{X}^0] + A^k A_2 Var [\mathbf{X}^0].$$

Another way to represent this inequality is

$$Var[\mathbf{X}^{t+1}] \leq \left(\sum_{i=0}^k c^i A^i A^* A^{k-i} \right) \mathbb{E}[\mathbf{X}^0] + c^{k+1} A^k A_2 Var[\mathbf{X}^0].$$

Here we see that $A^* = A - A_2$ acts as a sort of dispersion matrix, where the more A^* projects the eigenvector with largest eigenvalue of A away from itself, the lower the variance will be. One trivial way to decrease the variance is to make the entries $[a]_{ij}$ close to 0 or 1 such that the entries of $[a(1-a)]_{ij}$ become very small. We also see that the variance of the original distribution propagates linearly with the time step, implying that for critical systems with branching parameter = 1, the variance of the original distribution does not factor into the growth of the state variance beyond a constant value.

C. Intuition of Variance Bound and Slope From 1 Eigenvector Case

Suppose our matrix A contained only a single non-zero eigenvalue $\lambda = 1$ composed of some eigenvector \mathbf{v} such that $A = \mathbf{v}\mathbf{v}^T$. Then for \mathbf{v}_2 containing squared elements of \mathbf{v} , we have

$$\begin{aligned}
Var[\mathbf{X}^{t+1}] &\leq \left((k+1)A^{k+1} - \sum_{i=0}^k A^i A_2 A^{k-i} \right) \mathbb{E}[\mathbf{X}^0] + A^k A_2 Var[\mathbf{X}^0] \\
&\leq \left((k+1)(\mathbf{v}\mathbf{v}^T)^{k+1} - \sum_{i=0}^k (\mathbf{v}\mathbf{v}^T)^i (\mathbf{v}_2 \mathbf{v}_2^T) (\mathbf{v}\mathbf{v}^T)^{k-i} \right) \mathbb{E}[\mathbf{X}^0] + (\mathbf{v}\mathbf{v}^T)^k \mathbf{v}_2 \mathbf{v}_2^T Var[\mathbf{X}^0] \\
&\leq ((k+1)(\mathbf{v}\mathbf{v}^T) - (k-1)(\mathbf{v}\mathbf{v}^T)(\mathbf{v}_2 \mathbf{v}_2^T)(\mathbf{v}\mathbf{v}^T) - (\mathbf{v}_2 \mathbf{v}_2^T)(\mathbf{v}\mathbf{v}^T) - (\mathbf{v}\mathbf{v}^T)(\mathbf{v}_2 \mathbf{v}_2^T)) \mathbb{E}[\mathbf{X}^0] + (\mathbf{v}\mathbf{v}^T) \mathbf{v}_2 \mathbf{v}_2^T Var[\mathbf{X}^0] \\
&\leq ((k+1)(\mathbf{v}\mathbf{v}^T) - (k-1)(\mathbf{v}^T \mathbf{v}_2)^2 (\mathbf{v}\mathbf{v}^T) - (\mathbf{v}_2 \mathbf{v}_2^T)(\mathbf{v}\mathbf{v}^T) - (\mathbf{v}\mathbf{v}^T)(\mathbf{v}_2 \mathbf{v}_2^T)) \mathbb{E}[\mathbf{X}^0] + (\mathbf{v}\mathbf{v}^T) \mathbf{v}_2 \mathbf{v}_2^T Var[\mathbf{X}^0] \\
&\leq (k(1 - (\mathbf{v}^T \mathbf{v}_2)^2) + 1 + (\mathbf{v}^T \mathbf{v}_2)^2) (\mathbf{v}\mathbf{v}^T) \mathbb{E}[\mathbf{X}^0] - (\mathbf{v}^T \mathbf{v}_2)(\mathbf{v}_2 \mathbf{v}^T + \mathbf{v}\mathbf{v}_2^T) \mathbb{E}[\mathbf{X}^0] + (\mathbf{v}\mathbf{v}^T) \mathbf{v}_2 \mathbf{v}_2^T Var[\mathbf{X}^0].
\end{aligned}$$

While this expression may look very confusing, the key aspect to notice is that the only term that depends on the time step k is

$$1 - (\mathbf{v}^T \mathbf{v}_2)^2,$$

where $\mathbf{v}^T \mathbf{v}_2$ is simply a sum of each element of \mathbf{v} raised to the third power. Hence, the closer the entries of \mathbf{v} are to 1, the smaller this slope will be. To maximize this slope, we just require that every element of \mathbf{v} is equal to $\frac{1}{n}$. Hence, for the extreme case where our matrix has rank 1 with one non-zero eigenvalue and corresponding eigenvector, we see that the increase in variance over time is given by a very simple expression.

D. Exact Variance for First Step

Suppose our initial inputs are drawn from independent distributions with mean $\mathbb{E}[\mathbf{X}^0]$ and variance $\text{Var}[\mathbf{X}^0]$. Then the firing rates immediately after have exact variance

$$\begin{aligned} \text{Var}[\mathbf{X}^1] &= \mathbb{E}[\text{Var}[\mathbf{X}^1 | \mathbf{X}^0]] + \text{Var}[\mathbb{E}[\mathbf{X}^1 | \mathbf{X}^0]] \\ &= \mathbb{E}[A^* \mathbf{X}^0] + \text{Var}[A \mathbf{X}^0] \\ &= A^* \mathbb{E}[\mathbf{X}^0] + A_2 \text{Var}[\mathbf{X}^0] \\ &= A \mathbb{E}[\mathbf{X}^0] + A_2 (\text{Var}[\mathbf{X}^0] - \mathbb{E}[\mathbf{X}^0]) \\ &= \mathbb{E}[\mathbf{X}^1] + A_2 (\text{Var}[\mathbf{X}^0] - \mathbb{E}[\mathbf{X}^0]). \end{aligned}$$

We see that there is a linear aspect in the growth of the variance, where one component of $\text{Var}[\mathbf{X}^1]$ is $\mathbb{E}[\mathbf{X}^1]$. This term increases the variance as quickly as the mean, which is bad. Alternatively, the second term can reduce the variance of \mathbf{X}^1 if the variance of the initial distribution is lower than the expected value. Suppose very simply that there is no variance in the initial distribution. Then $\text{Var}[\mathbf{X}^1] = \mathbb{E}[\mathbf{X}^1] - A_2 \mathbb{E}[\mathbf{X}^0]$. From this relation alone, we see that to minimize some measure of variance to expected value, we want our initial condition to excite the largest eigenmode of A_2 , not necessarily A .

E. Bayes Theorem and Posteriors

In the quest to understand the question of input reconstruction, suppose we have some initial state \mathbf{X}^0 drawn from some distribution. The question we are interested in is how to reconstruct this initial state given a measurement later on. Suppose this later measurement occurs at time $T = 1$. Then $X_j^1 \sim \sum_{i=1}^n B(X_i^0, a_{ji})$. If all inputs to the j th neuron are identical, then $X_j^1 \sim \sum_{i=1}^n B(X_i^0, a_j) = B(\mathbf{o}_j^T \mathbf{X}^0, a_j)$. Hence,

$$P(X_j^1 = x_j^1 | \mathbf{X}^0 = \mathbf{x}_0) = \binom{\mathbf{o}_j^T \mathbf{X}^0}{x_j^1} a_j^{x_j^1} (1 - a_j)^{\mathbf{o}_j^T \mathbf{X}^0 - x_j^1}.$$

Via Bayes Theorem, we can invert this conditional probability

$$P(\mathbf{X}^0 = \mathbf{x}^0 | X_j^1 = x_j^1) = \frac{P(X_j^1 = x_j^1 | \mathbf{X}^0 = \mathbf{x}_0) P(\mathbf{X}^0 = \mathbf{x}^0)}{P(X_j^1 = x_j^1)}.$$

We note that because what we measure is X_j^1 , across all guesses of initial state, $P(X_j^1 = x_j^1)$ remains constant. We also note that if the initial states are drawn independently of each other, $P(\mathbf{X}^0 = \mathbf{x}^0) = \prod_{i=1}^n P(X_i^0 = x_i^0)$. Hence, we can write

$$P(\mathbf{X}^0 = \mathbf{x}^0 | X_j^1 = x_j^1) \propto \binom{\mathbf{o}_j^T \mathbf{X}^0}{x_j^1} a_j^{x_j^1} (1 - a_j)^{\mathbf{o}_j^T \mathbf{X}^0 - x_j^1} \prod_{i=1}^n P(X_i^0 = x_i^0),$$

which we can write as the log-likelihood

$$\begin{aligned} \log(P(\mathbf{X}^0 = \mathbf{x}^0 | X_j^1 = x_j^1)) &\propto \log\left(\binom{\mathbf{o}_j^T \mathbf{x}^0}{x_j^1}\right) + \log\left(a_j^{x_j^1} (1 - a_j)^{\mathbf{o}_j^T \mathbf{x}^0 - x_j^1}\right) + \log\left(\prod_{i=1}^n P(X_i^0 = x_i^0)\right) \\ &\propto \log\left(\binom{\mathbf{o}_j^T \mathbf{x}^0}{x_j^1}\right) + x_j^1 \log(a_j) + (\mathbf{o}_j^T \mathbf{x}^0 - x_j^1) \log(1 - a_j) + \sum_{i=1}^n \log(P(X_i^0 = x_i^0)) \\ &\propto \log\left(\binom{\mathbf{o}_j^T \mathbf{x}^0}{x_j^1}\right) + x_j^1 \log\left(\frac{a_j}{1 - a_j}\right) + \mathbf{o}_j^T \mathbf{x}^0 \log(1 - a_j) + \sum_{i=1}^n \log(P(X_i^0 = x_i^0)), \end{aligned}$$

which in theory we can use to find the most likely initial condition given what we know about the stimulus distribution and measured data by setting $\nabla_{\mathbf{x}^0} \log(P(\mathbf{X}^0 = \mathbf{x}^0 | X_j^1 = x_j^1)) = \mathbf{0}$.

F. State-Space and Statistical Mechanics

The next approach is to use statistical mechanics to enumerate the tree of probabilities for transitioning from one state to another. Consider a vector \mathbf{X}^t of n random variables at time t , drawn from $X_j^{k+1} \sim \sum_{i=1}^n B(X_i^k, a_{ji})$. Because the state transition process does not depend on system memory, the probability of transitioning from any state \mathbf{x}_m to any other state \mathbf{x}_p is completely governed by a non-hierarchical sum of binomial distributions. Hence, we can enumerate all possible discrete states \mathbf{x}_i , as

$$\mathbf{x}_i \in \left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \dots, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \dots, \begin{bmatrix} d \\ d \\ \vdots \\ d \end{bmatrix} \right),$$

and the probability of transitioning from any state \mathbf{x}_m and \mathbf{x}_p is given simply as the product of sums of binomial distributions.