Modal Logic as Algebra

Sebastian Karlsson

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1 Introduction

The goal of this essay is to give a glimpse of the relation between modal logic and algebra. The first part of the paper is devoted to giving an algebraic equivalence to normal modal logics. In the second part I discuss how this is related to frame semantics.

Let's start right away by defining boolean algebras. Boolean algebras are abstract models/constructions equivalent to ordinary propositional logic [1]. Consider an algebra $\mathfrak{A} = (A, +, -, 0)$ with one binary operation $+$, one unary operation $-$ and one nullary operation 0 on the set A (called the carrier set of $\mathfrak{A}$). Furthermore, define: $a \cdot b := -(-a + (-b))$ for $a, b \in A$ and $1 := -0$. $\mathfrak{A}$ is a boolean algebra if both $+$ and $\cdot$ are commutative, associative and distribute over each other and the following identities hold: $x + 0 = x$, $x \cdot 1 = x$, $x + (-x) = 1$ and $x \cdot (-x) = 0$. I will use the abbreviation $a - b := a + (-b)$.

In order to make a boolean algebra into a model for propositional modal logic (with one unary modality $\Diamond$ and its dual $\Box$), one extends it with a unary operation f, and requires that the operation is "normal" and "additive". f is additive if for $a, b \in A$, $f(a+b) = f(a) + f(b)$ and normal if $f(0) = 0$. f is often called an operator on a boolean algebra, and the algebra $\mathfrak{A} = (A, +, -, 0, f)$ is called a boolean algebra with an operator (BAO). The reason for these requirements will become evident later.

Let $ML(\Phi)$ be the language of modal logic built inductively using the propositional letters in Φ, disjunction, negation, diamond and box. Now, if $\varphi \in ML(\Phi)$, how can we interpret φ in an algebra $\mathfrak{A} = (A, +, -, 0, f)$? We start with an assignment $\theta : \Phi \to A$, a function which maps propositional letters to elements of our algebra. Then we define a so called meaning function $\tilde{\theta} : ML(\Phi) \to A$ inductively as follows (Let $\psi, \chi \in ML(\Phi)$

and $p \in \Phi$):

$$\begin{aligned}
\tilde{\theta}(p) &= \theta(p) \in A \\
\tilde{\theta}(\neg\psi) &= -\tilde{\theta}(\psi) \\
\tilde{\theta}(\psi \vee \chi) &= \tilde{\theta}(\psi) + \tilde{\theta}(\chi) \\
\tilde{\theta}(\Diamond\psi) &= f(\tilde{\theta}(\psi)) \\
\tilde{\theta}(\Box\psi) &= -f(-\tilde{\theta}(\psi))
\end{aligned}$$

It is worth noting the similarities between an assignment θ which allows modal formulas to be interpreted in an algebra, and a valuation V on a frame, making a frame into a model.

Furthermore, one says that an equation $\psi \approx \chi$ is true in $\mathfrak{A}$ if for all assignments θ, $\tilde{\theta}(\psi) = \tilde{\theta}(\chi)$. This is written as $\mathfrak{A} \models \psi \approx \chi$.

Example 1. If a is an element of a boolean algebra (with an operator), does $- - a = a$ hold? Let's see using the axioms: $a = a \cdot 1 = a \cdot (-a + (- - a)) = (a \cdot (-a)) + (a \cdot (- - a)) = 0 + (a \cdot (- - a)) = a \cdot (- - a)$.[1] Similarly, one gets that $- - a = - - a \cdot 1 = - - a \cdot (-a + a) = (- - a \cdot (-a)) + (- - a \cdot a) = 0 + (- - a \cdot a) = a \cdot (- - a)$. Hence, $a = - - a$. This is the algebraic equivalent of the tautology $p \leftrightarrow \neg\neg p$.

Another interesting identity in a boolean algebra is $a = a + a$ (compare it with the tautology $p \leftrightarrow (p \vee p)$). Here is a proof: $a = a + 0 = a + (a \cdot (-a)) = (a + a) \cdot (a - a) = (a + a) \cdot 1 = a + a$. One step might look weird: It was the distributivity of addition over multiplication that was used! From this identity it follows that $1 + a = 1$, since $1 + a = (-a + a) + a = -a + (a + a) = -a + a = 1$.

Example 2. Let's use the abbreviations $\psi \to \chi := \neg\psi \vee \chi$, $\bot := p \wedge \neg p$ and $\top := p \vee \neg p$. It is straightforward to check that $\tilde{\theta}(\psi \to \chi) = -\tilde{\theta}(\psi) + \tilde{\theta}(\chi)$, $\tilde{\theta}(\bot) = 0$ and $\tilde{\theta}(\top) = 1$. For example, $\tilde{\theta}(\top) = \tilde{\theta}(p \vee \neg p) = \tilde{\theta}(p) + \tilde{\theta}(\neg p) = \tilde{\theta}(p) - \tilde{\theta}(p) = 1$. An equation of the form $\varphi \approx \top$ is therefore true in a BAO if for all assignments θ, $\tilde{\theta}(\varphi) = 1$.

2 Algebraic Syntax for Normal Modal Logics

In this section I will show how one usually relates normal modal logics with classes of boolean algebras with operators. This can be done in such a way that a formula is derivable in a normal modal logic Λ if and only if it is true in every BAO in the corresponding class.

Let Σ be a set of formulas and V_Σ be the class of boolean algebras such that $\mathfrak{A} \in V_\Sigma$ if and only if $\mathfrak{A} \models \varphi \approx \top$ for every $\varphi \in \Sigma$. One writes $V_\Sigma \models \varphi \approx \top$, to say that for all

[1] In order to avoid superfluous parentheses, the "-" sign applies only to what is strictly in front of it, i.e. $-a + a$ means $(-a) + a$ and not $-(a + a)$.

$\mathfrak{A} \in V_\Sigma$, $\mathfrak{A} \models \varphi \approx \top$. Furthermore, let $K\Sigma$ be the normal modal logic axiomatized by Σ, i.e. the smallest normal modal logic containing K and the formulas Σ.

Theorem 1 (Algebraic Soundness). *If* $K\Sigma \vdash \varphi$ *then* $V_\Sigma \models \varphi \approx \top$.

Proof. The proof is by induction on the "proof length" of formulas in $K\Sigma$.

1. **Base case:** I will take it as given that any tautology is true in a boolean algebra, i.e. that if φ is a tautology, then $V_\Sigma \models \varphi \approx \top$. Also, it is assumed that if $\varphi \in \Sigma$, then $V_\Sigma \models \varphi \approx \top$. It is left to show that axiom K and the dual axiom $\Diamond p \leftrightarrow \neg\Box\neg p$ are true in any BAO. Apply the meaning function to axiom K for an arbitrary assignment θ:

$$\begin{aligned}
\tilde{\theta}(\text{Ax. K}) &= \tilde{\theta}(\Box(p \to q)) \to (\Box p \to \Box q)) \\
&= - - f[-(-\theta(p) + \theta(q))] - -f[-\theta(p)] - f[-\theta(q)] \\
&= f[\theta(p) \cdot (-\theta(q))] + f[-\theta(p)] - f[-\theta(q)] \\
&= f[\theta(p) \cdot (-\theta(q)) - \theta(p)] - f[-\theta(q)] \\
&= f[(\theta(p) - \theta(p)) \cdot (-\theta(q) - \theta(p))] - f[-\theta(q)] \\
&= f[1 \cdot (-\theta(q) - \theta(p))] - f[-\theta(q)] \\
&= f[-\theta(q)] + f[-\theta(p)] - f[-\theta(q)] \\
&= 1 + f[-\theta(p)] = 1
\end{aligned}$$

   Hence, $V_\Sigma \models$ Axiom K $\approx \top$. In the derivation, the axioms of a BAO along with the identities from example 1 were used freely. The additivity of f was also used! The dual axiom can be proved to hold in a similar way.

2. **Uniform Substitution:** Suppose that the theorem holds for φ, i.e. for any assignment θ, $\tilde{\theta}(\varphi) = 1$. Is the same true for $\varphi[\psi/p]$ for an arbitrary formula ψ? Yes; take any assignment θ, and define a new assignment θ_0 such that $\theta_0(p) = \tilde{\theta}(\psi)$. With induction one gets that $\tilde{\theta}_0(\varphi) = \tilde{\theta}(\varphi[\psi/p])$. So if $\tilde{\theta}_0(\varphi) = 1$ for any assignment θ_0, it follows that $\tilde{\theta}(\varphi[\psi/p]) = 1$ for any assignment θ. Hence, $V_\Sigma \models \varphi[\psi/p] \approx \top$.

3. **Modus Ponens:** Suppose that the theorem holds for ψ and $\psi \to \chi$. I.e. that $\tilde{\theta}(\psi) = 1$ and $\tilde{\theta}(\psi \to \chi) = 1$ for any θ. Is it then also true for χ, i.e. is $\tilde{\theta}(\chi) = 1$ for any θ? Yes: $1 = \tilde{\theta}(\psi \to \chi) = -\tilde{\theta}(\psi) + \tilde{\theta}(\chi) = -1 + \tilde{\theta}(\chi) = 0 + \tilde{\theta}(\chi) = \tilde{\theta}(\chi)$.

4. **Necessitation:** Suppose that the theorem holds for φ. One must show that it is also true for $\Box\varphi$. It is now that the normality of f is needed: $\tilde{\theta}(\Box\varphi) = -f(-\tilde{\theta}(\varphi)) = -f(-1) = -f(0) = -0 = 1$.

$\square$

One remark. For any normal modal logic Λ, the soundness result implies that we do not need to find a set of axioms Σ for Λ in order to talk about the corresponding class of BAO, V_Σ. Soundness implies that for any set Σ of axioms for Λ, $V_\Sigma = V_\Lambda$.

We are now in position to start working towards the completeness of normal modal logics with respect to their corresponding class of BAO. What we would like to achieve is that if $V_\Lambda \models \varphi \approx \top$, then $\vdash_\Lambda \varphi$. Equivalently, one can show that for every non-theorem φ of Λ, there is a $\mathfrak{A} \in V_\Lambda$ such that $\varphi \approx \top$ is not true in $\mathfrak{A}$. Our goal will be to construct such an algebra.

The idea will be to use sets of formulas that are provable equivalent in Λ as the elements of the algebra. Define an equivalence relation $\equiv_\Lambda$ on $ML(\Phi)$ by saying that $\psi \equiv_\Lambda \chi$ iff $\vdash_\Lambda \psi \leftrightarrow \chi$. It is not hard to see that it is an equivalence relation. E.g. it is symmetric since $(\psi \leftrightarrow \chi) \to (\chi \leftrightarrow \psi)$ is in Λ as it is a tautology, and since Λ is closed under MP, $\psi \equiv_\Lambda \chi$ implies that $\chi \equiv_\Lambda \psi$.

This relation is actually stronger than an equivalence relation, because it is closed under logical operations. More precisely, it can be shown that [2]:

$$\psi \equiv_\Lambda \chi \text{ and } \psi' \equiv_\Lambda \chi' \Rightarrow \psi \vee \psi' \equiv_\Lambda \chi \vee \chi'$$
$$\psi \equiv_\Lambda \chi \Rightarrow \neg\psi \equiv_\Lambda \neg\chi$$
$$\psi \equiv_\Lambda \chi \Rightarrow \Diamond\psi \equiv_\Lambda \Diamond\chi$$

We can now define $[\psi]$ to be the set of all formulas χ such that $\chi \equiv_\Lambda \psi$. The properties above assure that the following operations are well defined (i.e. independent of the representative of the equivalence class):

$$[\psi] + [\chi] := [\psi \vee \chi]$$
$$-[\psi] := [\neg\psi]$$
$$f([\psi]) := [\Diamond\psi]$$

Using these operations, one can define the *Lindenbaum-Tarski algebra* of Λ to be the algebra $\mathfrak{L}_\Lambda = (E, +, -, f)$, where $E = \{[\psi] \mid \psi \in ML(\Phi)\}$. This algebra has two very nice properties. First, it is a boolean algebra with an operator. Secondly, it turns out that a formula φ is provable in Λ if and only if $\varphi \approx \top$ is true in $\mathfrak{L}_\Lambda$. I will not go through the proof that it is a BAO (The proof is straightforward but quite long due to all the details that need to be verified.) but rather focus on the proof of the second statement. First, note that in $\mathfrak{L}_\Lambda$, $1 = [\top]$ and $0 = [\bot]$.

Theorem 2. $\vdash_\Lambda \varphi$ *if and only if* $\mathfrak{L}_\Lambda \models \varphi \approx \top$.

Proof. Consider first the direction from right to left. Suppose that $\nvdash_\Lambda \varphi$. Then, $\nvdash_\Lambda \varphi \leftrightarrow \top$ which gives that $[\varphi] \neq [\top]$. To prove that $\mathfrak{L}_\Lambda \models \varphi \not\approx \top$, one has to find an assignment θ such that $\tilde{\theta}(\varphi) \neq [\top]$. Let $\theta(p) := [p]$ for any $p \in \Phi$. By induction on the syntax, one has that $\tilde{\theta}(\varphi) = [\varphi]$. So there is an assignment θ such that $\tilde{\theta}(\varphi) \neq [\top] = 1$. Thus, $\mathfrak{L}_\Lambda \models \varphi \not\approx \top$.

For the direction from left to right, one has to show that if $\vdash_\Lambda \varphi$, then for every assignment θ, $\tilde{\theta}(\varphi) = [\top]$. Now, let θ be an arbitrary assignment. It maps propositional letters to elements of $\mathfrak{L}_\Lambda$, i.e. equivalence classes of formulas. Let $\alpha : \Phi \to ML(\Phi)$ be a function that gives representatives for these equivalence classes. That is, for each $p \in \Phi$, $\theta(p) = [\alpha(p)]$. For any formula φ, let $p_1, ..., p_n$ be the propositional letters occurring in φ. By an inductive argument on the syntax one can establish that $\tilde{\theta}(\varphi) = [\varphi[\alpha(p_1)/p_1, ..., \alpha(p_n)/p_n]]$.[2]

Suppose that φ is a theorem of Λ. Then $\vdash_\Lambda \varphi \leftrightarrow \top$ so $[\varphi] = [\top]$. Since Λ is closed under uniform substitution, one has that $[\varphi[\alpha(p_1)/p_1, ..., \alpha(p_n)/p_n]] = [\top]$ for any α. Hence, $\mathfrak{L}_\Lambda \models \varphi \approx \top$. □

Corollary 1 (Algebraic Completeness)**.** *If* $V_\Lambda \models \varphi \approx \top$ *then* $\vdash_\Lambda \varphi$.

Proof. By theorem 4, if $\nvdash_\Lambda \varphi$, then $\mathfrak{L}_\Lambda \not\models \varphi \approx \top$. Since $\mathfrak{L}_\Lambda$ is a BAO and since Λ is sound with respect to $\mathfrak{L}_\Lambda$, it follows that $\mathfrak{L}_\Lambda \in V_\Lambda$. Thus, $V_\Lambda \not\models \varphi \approx \top$. □

Example 3. Since any normal modal logic Λ has a Lindenbaum-Tarski algebra, V_Λ is always nonempty. This is also true for the inconsistent logic, i.e. the logic were $\bot \in \Lambda$. The inconsistent logic contains all formulas, so its Lindenbaum-Tarski algebra contains one single element $[\top] = [\varphi]$ for any formula φ. The class V_Λ contains all isomorphic copies of the trivial one element BAO.

For any normal modal logic Λ, if $\mathfrak{A} \in V_\Lambda$, then any subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is in V_Λ. This is because $\mathfrak{A} \models \varphi \approx \top$ means that $\tilde{\theta}(\varphi) = 1$ for any assignment θ on $\mathfrak{A}$, so if the range of θ is restricted to the carrier set of $\mathfrak{B}$, the same still holds.

Any BAO where f is the identity function is contained in the class defined by axiom T, $V_{\{\mathrm{T}\}} = \{\mathfrak{A} \mid \mathfrak{A} \models p \to \Diamond p \approx \top\}$. If f is the identity function, then for any assignment θ, $\tilde{\theta}(p \to \Diamond p) = -\tilde{\theta}(p) + f(\tilde{\theta}(p)) = -\tilde{\theta}(p) + \tilde{\theta}(p) = 1$.

3 Frames

We have now seen how each normal modal logic corresponds precisely to a class of boolean algebras with an operator. This raises a question: How is this related to frame semantics? In this section I will briefly explore this topic.

It turns out that one way is relatively easy; any frame can be turned into a boolean algebra. The main idea is this: If we have a model based on the frame, we can identify each formula by the set of worlds it is true in. For example, if we have two formulas and identify them with the set of worlds they are true in, then their disjunction corresponds to the union of those worlds. This leads us to the following definitions.

[2]For the inductive case of "$\vee$", one may also assume that the list $p_1, ..., p_n$ contains propositional letters not occurring in the formula. This is a technical detail and doesn't change the argument.

The full complex algebra $\mathfrak{F}^+$ of a frame $\mathfrak{F} = (W, R)$ is the BAO whose carrier set is the power set $\mathcal{P}(W)$ of the set of worlds, $+$ corresponds to union of sets $\cup$, $-$ corresponds to set complement and $0 = \emptyset$. The operator m_R is defined by:

$$m_R(X) := \{y \in W \mid \text{ there is } x \in X \text{ with } yRx\}$$

Intuitively, the operation m_R maps a set of worlds X to all those worlds that can "see" some world of X. It is immediate that $\mathfrak{F}^+$ is in fact a boolean algebra since all the axioms for a boolean algebra are elementary properties of sets. It is also immediate that m_R is an operator, i.e. that $m_R(\emptyset) = \emptyset$ and $m_R(X \cup Y) = m_R(X) \cup m_R(Y)$ for $X, Y \subseteq W$. A subalgebra of a full complex algebra is called a complex algebra.

Before going to the next theorem, it is first worth noting that a valuation on a frame $\mathfrak{F}$, making the frame into a model, is in fact also an assignment with respect to the full complex algebra $\mathfrak{F}^+$.

Theorem 3. *$(\mathfrak{F}, \theta), w \Vdash \varphi$ if and only if $w \in \tilde{\theta}(\varphi)$, where $\theta : ML(\Phi) \to \mathcal{P}(W)$ is an assignment with respect to the full complex algebra $\mathfrak{F}^+$.*

Proof. The proof is by induction on the syntax:

1. Base case, φ is a propositional letter p. Then, it is simply the definition that $(\mathfrak{F}, \theta), w \Vdash p$ **iff** $w \in \theta(p) = \tilde{\theta}(p)$.

2. If φ is of the form $\neg\psi$. Then $(\mathfrak{F}, \theta), w \Vdash \neg\psi$ **iff** $(\mathfrak{F}, \theta), w \nVdash \psi$ **iff** (I.A.) $w \notin \tilde{\theta}(\psi)$ **iff** $w \in -\tilde{\theta}(\psi)$ **iff** $w \in \tilde{\theta}(\neg\psi)$.

3. If φ is of the form $\psi \vee \chi$. Then $(\mathfrak{F}, \theta), w \Vdash \psi \vee \chi$ **iff** $(\mathfrak{F}, \theta), w \Vdash \psi$ or $(\mathfrak{F}, \theta), w \Vdash \chi$ **iff** (I.A.) $w \in \tilde{\theta}(\psi)$ or $w \in \tilde{\theta}(\chi)$ **iff** $w \in \tilde{\theta}(\psi) \cup \tilde{\theta}(\chi)$ **iff** $w \in \tilde{\theta}(\psi \vee \chi)$.

4. If φ is of the form $\Diamond\psi$. Then $(\mathfrak{F}, \theta), w \Vdash \Diamond\psi$ **iff** there is a $v \in W$ such that wRv and $(\mathfrak{F}, \theta), v \Vdash \psi$ **iff** (I.A.) there is a $v \in W$ such that wRv and $v \in \tilde{\theta}(\psi)$ **iff** $w \in m_R(\tilde{\theta}(\psi))$ **iff** $w \in \tilde{\theta}(\Diamond\psi)$.

5. If φ is of the form $\Box\psi$. Then $(\mathfrak{F}, \theta), w \Vdash \Box\psi$ **iff** for all $v \in W$ such that wRv, $(\mathfrak{F}, \theta), v \Vdash \psi$ holds **iff** (I.A.) for all $v \in W$ such that wRv, $v \in \tilde{\theta}(\psi)$ holds **iff** there is not a v such that wRv and $v \notin \tilde{\theta}(\psi)$ **iff** there is not a v such that wRv and $v \in -\tilde{\theta}(\psi)$ **iff** $w \notin m_R(-\tilde{\theta}(\psi))$ **iff** $w \in -m_R(-\tilde{\theta}(\psi))$ **iff** $w \in \tilde{\theta}(\Box\psi)$.

$\Box$

Corollary 2. *$\mathfrak{F} \models \varphi$ if and only if $\mathfrak{F}^+ \models \varphi \approx \top$.*

Proof. Immediate from Theorem 3, by noting that in $\mathfrak{F}^+$, $1 = W$. So $\varphi \approx \top$ is true iff for all $w \in W$, and all assignments θ, $w \in \tilde{\theta}(\varphi)$. $\Box$

Example 4. Consider the reflexive frame $\mathfrak{F} = (W, R)$ where $W = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 2), (1, 3), (3, 3)\}$. Its corresponding full complex algebra is $\mathfrak{F}^+ = (\mathcal{P}(W), \cup, -, m_R)$, where $\mathcal{P}(W) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ and m_R can be described pointwise as: $m_R(\emptyset) = \emptyset$, $m_R(\{1\}) = \{1\}$, $m_R(\{2\}) = \{1, 2\}$, $m_R(\{3\}) = \{1, 3\}$, $m_R(\{1, 2\}) = \{1, 2\}$, $m_R(\{1, 3\}) = \{1, 3\}$, $m_R(\{2, 3\}) = \{1, 2, 3\}$ and $m_R(\{1, 2, 3\}) = \{1, 2, 3\}$.

Since the frame is reflexive, it should validate axiom T: $p \to \Diamond p$. Does $\mathfrak{F}^+ \models p \to \Diamond p \approx \top$ hold? If one translates the axiom for an arbitrary assignment θ, one gets that $\tilde{\theta}(p \to \Diamond p) = -\theta(p) \cup m_R(\theta(p))$. From our above calculations we see that whatever $\theta(p)$ is, it is always the case that $\theta(p) \subseteq m_R(\theta(p))$. Hence, $-\theta(p) \cup m_R(\theta(p)) = W$ so axiom T is also valid in our full complex algebra $\mathfrak{F}^+$.

Let's now consider the harder direction. Can any BAO be converted into a frame such that precisely the same formulas are valid? The answer has to be no, because if Λ is a normal modal logic, then if any $\mathfrak{A} \in V_\Lambda$ could be converted into a Λ-frame that validates precisely the same formulas as $\mathfrak{A}$, algebraic completeness would imply that Λ is frame complete. This is because any formula valid on all Λ-frames would be valid on all Λ-frames constructed out of a BAO, and any such formula would by the algebraic completeness be a theorem of Λ. However, we know that there are frame incomplete normal modal logics [2].

There is still at least one very elegant way of turning a BAO into a frame. By viewing the elements of a BAO as propositions, one construct so called ultrafilters, which are a sort of algebraic equivalence to maximally consistent set of formulas. One then views these ultrafilters as worlds, and construct an accessibility relation quite similar to the relation on the canonical frame of a normal modal logic.

More precisely, a *filter* F of a boolean algebra (with an operator) $\mathfrak{A} = (A, +, -, 0, f)$, is a subset of A such that for $a, b \in A$:

1. $1 \in F$
2. If $a, b \in F$, then $a \cdot b \in F$
3. If $a \in F$ and $b = a + b$, then $b \in F$

A filter is *proper* if $0 \notin F$ and it is an *ultrafilter* if it is proper and for every $a \in A$, either a or $-a$ is in F. The third criterion might seem strange. Boolean algebras can in fact also be seen as certain types of partially ordered sets [3], in which case $b = a + b$ is equivalent to $a \leq b$. In such a context, the third criterion means that F must be *upwards closed*. The criteria ensures that when a formula φ is interpreted in the algebra, the interpretation of every formula that is a logical consequence of φ belongs to any filter containing φ. To see this, let u be an ultrafilter of any BAO and let ψ and χ be formulas and $\tilde{\theta}(\psi) \in u$. Then, if $\tilde{\theta}(\psi \to \chi) = 1$, i.e. $-\tilde{\theta}(\psi) + \tilde{\theta}(\chi) = 1$, it follows by lemma 1 that $\tilde{\theta}(\chi) = \tilde{\theta}(\psi) + \tilde{\theta}(\chi)$, so $\tilde{\theta}(\chi) \in u$.

Lemma 1. *For any elements a, b in a BAO, $b = a + b$ if and only if $-a + b = 1$.*

Proof. For the direction from right to left: $b = a+b \Rightarrow -a+b = -a+a+b \Rightarrow -a+b = 1$.

For the direction from left to right: $-a + b = 1 \Rightarrow (-a + b) \cdot a = 1 \cdot a \Rightarrow 0 + b \cdot a = a \Rightarrow (b \cdot a) + b = a + b \Rightarrow (a + 1) \cdot b = a + b \Rightarrow 1 \cdot b = a + b$. $\square$

Let $Uf\mathfrak{A}$ be the set of all ultrafilters of $\mathfrak{A}$. The *ultrafilter frame* of $\mathfrak{A}$ is the frame $\mathfrak{A}_+ := (Uf\mathfrak{A}, Q)$ such that for $u, v \in Uf\mathfrak{A}$,

$$uQv \text{ iff } \forall a \in v,\ f(a) \in u$$

The ultrafilter frame construction is very important, because it turns out that any abstract boolean algebra with an operator can be embedded into the full complex algebra of its ultrafilter frame. That is, there is a subalgebra $\mathfrak{B}$ of $(\mathfrak{A}_+)^+$ such that $\mathfrak{A} \cong \mathfrak{B}$. This is known as the Jonsson-Tarski theorem.

Theorem 4 (Jonsson-Tarski Theorem)**.** *Any BAO $\mathfrak{A}$ is isomorphic to a subalgebra of $(\mathfrak{A}_+)^+$.*

Proof. Unfortunately too long for this paper. See for example [2] or [4]. $\square$

It follows from the Jonsson-Tarski theorem that anything valid on the ultrafilter frame of a BAO $\mathfrak{A}$ is also true in $\mathfrak{A}$. If φ is valid on $\mathfrak{A}_+$, then by corollary 2 it is true in $(\mathfrak{A}_+)^+$. Hence, φ is true in every subalgebra of $(\mathfrak{A}_+)^+$, and in particular, true in the subalgebra isomorphic to $\mathfrak{A}$.

The Jonsson-Tarski theorem opens the door to study the relationship between normal modal logics and frames algebraically. An illustration of how this can be done is given in example 6. In example 5 I give a simple example of an ultrafilter frame.

Example 5. Let's take the complex algebra from example 4 and see what its ultrafilter frame looks like. Since the carrier set of $\mathfrak{F}$ is finite and very small, it is relatively easy to just extensively check what subsets of $\mathcal{P}(W)$ are ultrafilters and which are not. In general, it is possible to prove that for any nonzero element of a BAO, there is an ultrafilter containing it. Furthermore, every filter of a BAO can be extended into an ultrafilter [2].

Hence, there is an ultrafilter containing $\{1\}$. It does not contain $-\{1\} = \{2, 3\}$. Since $\{1, 2\} = \{1\} \cup \{1, 2\}$ and $\{1, 3\} = \{1\} \cup \{1, 3\}$, the ultrafilter must contain $\{1, 2\}$ and $\{1, 3\}$. It must also contain $\{1, 2, 3\}$ since it is the 1-element of $\mathfrak{F}$. It is now easy to see that no other elements are in it. Hence, the unique ultrafilter containing $\{1\}$ is $\{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. Similarly, it is possible to boil down what all the other ultrafilters are, and come to the conclusion that there are only three: One for every singleton set of a world of $\mathfrak{F}$.

Having constructed all the ultrafilters, it is straightforward to check what the accessibility relation must look like. The original frame and the ultrafilter frame of its full

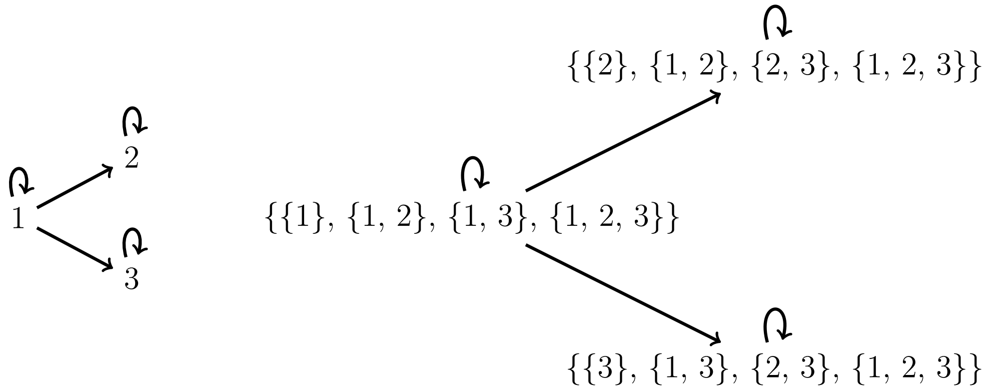

Figure 1: The frame in example 4 to the left and the ultrafilter frame of its full complex algebra to the right. They are clearly isomorphic.

complex algebra can be seen in figure 1. They are isomorphic! In general, whenever $\mathfrak{F}$ is finite, $\mathfrak{F}$ is isomorphic to $(\mathfrak{F}^+)_+$. This is not necessarily true when $\mathfrak{F}$ has infinitely many worlds [2].

Example 6. Suppose we wanted to give an algebraic proof that KB is frame complete with respect to all symmetric frames. I will take it as given that a frame validate axiom B iff it is symmetric. By corollary 1 we know that KB is complete with respect to $V_{\{B\}}$ and by corollary 2 we know that for any symmetric frame $\mathfrak{F}$, $\mathfrak{F}^+ \in V_{\{B\}}$. Therefore, if we can show that any BAO in $V_{\{B\}}$ can be embedded into the full complex algebra $\mathfrak{F}^+$ of a symmetric frame $\mathfrak{F}$, then anything valid on all symmetric frames is automatically true in $V_{\{B\}}$, and hence a theorem of KB.

By the Jonsson-Tarski theorem, if we can show that for any $\mathfrak{A} \in V_{\{B\}}$, $\mathfrak{A}_+$ is symmetric, then we know that $\mathfrak{A}$ can be embedded into the full complex algebra of a symmetric frame.

Let $\mathfrak{A} \in V_{\{B\}}$ and let u and v be ultrafilters of $\mathfrak{A}$ such that uQv. $\mathfrak{A}_+$ is symmetric if this implies that vQu. Now, $\mathfrak{A} \in V_{\{B\}}$ means that for any assignment θ on $\mathfrak{A}$, $\tilde{\theta}(p \to \Box\Diamond p) = 1$, i.e., $-\theta(p) - f(-f(\theta(p)) = 1$. This is true iff for all elements a of $\mathfrak{A}$, $-a - f(-f(a)) = 1$ iff (by lemma 1) for all elements a of $\mathfrak{A}$, $-f(-f(a)) = a - f(-f(a))$. (*)

uQv means that for all $a \in v$, $f(a) \in u$. We'd like to show that vQu, i.e., that for all $a \in u$, $f(a) \in v$. Now,

$$\begin{aligned} a \in u &\Rightarrow -f(-f(a)) \in u && \text{(By (*) and the 3rd ultrafilter criteria.)} \\ &\Rightarrow f(-f(a)) \notin u && \text{(Since } u \text{ is an ultrafilter.)} \\ &\Rightarrow -f(a) \notin v && \text{(Otherwise } f(-f(a)) \text{ would be in } u \text{ since } uQv.) \\ &\Rightarrow f(a) \in v && \text{(Since } v \text{ is an ultrafilter.)} \end{aligned}$$

Hence, if $a \in u$ then $f(a) \in v$, so vQu holds. Therefore, $\mathfrak{A}_+$ is a symmetric frame and KB is frame complete.

This proof was maybe a little long winded for the frame completeness of KB. However, I hope that it helps the understanding of all these concepts by showing them in action, and using them to prove something that usually is proved in another fashion.

4 Final Thoughts

In a way, section 1 and 2 introduced another language to discuss modal logic. The algebraic soundness and completeness justified this, by showing how each normal modal logic corresponds precisely to an equational class of algebras. Soundness and completeness with respect to frames is definitely philosophically more interesting, since frames give formulas an intuitive meaning whereas BAO:s don't. However, the importance of the results from section 2 are that they allow modal logic to be studied using techniques from universal algebra.

Classes of algebras defined by a set of equations is an important concept in universal algebra. Section 2 allows general theorems regarding equational classes to be applied to modal logic. For example, there's a theorem in universal algebra called Birkhoffs theorem [3], stating that a class of algebras closed under the taking of subalgebras, homomorphic images and products is also an equational class. When this is translated into modal logic, the Goldblatt-Thomason theorem is a relatively immediate consequence [2]. However, in order to do this translation, one would have to explore the relation between algebras and frames deeper than what was done in section 3.

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