Proof of formula for A339384(p^n)

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Peter Bala suggested:

$$a(p^{2n+2}) = A339384(p^{2n+2}) = 1 + \frac{1}{2}p^2(p-1)\left(\frac{p^{3n+3}-1}{p^3-1} + p^{3n+1}\frac{p^n-1}{p-1}\right) \tag{1}$$

Here is an attempt of proving this formula and the general formula for A339384 for prime powers:

Proposition 1. *For $n \in \mathbb{N}$ and p a prime number:*

$$a(p^{2n}) = 1 + \frac{1}{2}p^2(p-1)\left(\frac{p^{3n}-1}{p^3-1} + p^{3n-2}\frac{p^{n-1}-1}{p-1}\right) \tag{2}$$

$$a(p^{2n+1}) = 1 + \frac{1}{2}p^2(p-1)\left(\frac{p^{3n}-1}{p^3-1} + p^{3n-1}\frac{p^n-1}{p-1}\right) \tag{3}$$

Proof. We have:

$$a(p^n) = \sum_{k=1}^{n} \frac{\mathrm{lcm}(p^n,k)}{\gcd(p^n,k)} \mod p^n \tag{4}$$

Clearly, an arbitrary term corresponding to k is 0 if p doesn't divide it, hence we can write:

$$a(p^n) = 1 + \sum_{1\le r<p^{n-1},\gcd(p,r)=1} \frac{\mathrm{lcm}(p^n,pr)}{\gcd(p^n,pr)} \mod p^n + ... \tag{5}$$

$$+ \sum_{1\le r<p,\gcd(p,r)=1} \frac{\mathrm{lcm}(p^n,p^{n-1}r)}{\gcd(p^n,p^{n-1}r)} \mod p^n \tag{6}$$

$$= 1 + \sum_{1\le r<p^{n-1},\gcd(p,r)=1} p^{n-1}r \mod p^n + ... \tag{7}$$

$$+ \sum_{1\le r<p,\gcd(p,r)=1} pr \mod p^n \tag{8}$$

$$= 1 + \sum_{1\le r<p^{n-1},\gcd(p,r)=1} \left(p^{n-1}r - p^n\left\lfloor\frac{r}{p}\right\rfloor\right) + ... \tag{9}$$

$$+ \sum_{1\le r<p,\gcd(p,r)=1} \left(pr - p^n\left\lfloor\frac{r}{p^{n-1}}\right\rfloor\right) \tag{10}$$

This sum consists of "smaller sums" of the form

$$\sum_{1\le r<p^{n-m},\gcd(p,r)=1} \left(p^{n-m}r - p^n\left\lfloor\frac{r}{p^m}\right\rfloor\right) \tag{11}$$

In order to evaluate this expression, we first evaluate the part containing the floor-function:

$$\begin{aligned}
\sum_{1\le r<p^{n-m},\gcd(p,r)=1} \left\lfloor \frac{r}{p^m} \right\rfloor &= \sum_{p^m\le r<p^{n-m}} \left\lfloor \frac{r}{p^m} \right\rfloor - \sum_{p^{m-1}\le r<p^{n-m-1}} \left\lfloor \frac{pr}{p^m} \right\rfloor \\
&= p^m \sum_{l=1}^{p^{n-2m}-1} l - p^m \sum_{l=1}^{p^{n-2m-1}-1} l \\
&= \frac{1}{2}p^{n-m}(p^{n-2m}-1) - \frac{1}{2}p^{n-m-1}(p^{n-2m}-1) \\
&= \frac{1}{2}p^{n-m-1}(p^{n-2m}-1)(p-1)
\end{aligned}$$

Using this, we can rewrite (11):

$$\begin{aligned}
&p^{n-m}\left(\sum_{1\le r<p^{n-m},\gcd(p,r)=1} r\right) - p^n\left(\sum_{1\le r<p^{n-m},\gcd(p,r)=1} \left\lfloor \frac{r}{p^m} \right\rfloor\right) \\
&= \frac{1}{2}p^{2n-2m}\phi(p^{n-m}) - p^n\frac{1}{2}p^{n-m-1}(p^{n-2m}-1)(p-1) \\
&= \frac{1}{2}(p-1)p^{n-m-1}(p^{2n-2m} - p^n(p^{n-2m}-1)) \\
&= \frac{1}{2}(p-1)p^{2n-m-1}
\end{aligned}$$

Where ϕ is Euler totient function. Now, note that in the sums of (9) and (10) the floor-function (or mod) only affects the "smaller sums" where $2m < n$ and where m is as in (11). Thus, for even numbers we can split (9) and (10) like this:

$$\begin{aligned}
a(p^{2n}) &= 1 + \sum_{i=1}^{n} \sum_{1\le r<p^i,\gcd(p,r)=1} p^i r + \sum_{m=1}^{n-1} \frac{1}{2}(p-1)p^{4n-m-1} \\
&= 1 + \sum_{i=1}^{n} \frac{1}{2}p^{2i}\phi(p^i) + \frac{1}{2}(p-1)p^2p^{3n-2}\frac{p^{n-1}-1}{p-1} \\
&= 1 + \sum_{i=1}^{n} \frac{1}{2}p^{3i-1}(p-1) + \frac{1}{2}(p-1)p^2p^{3n-2}\frac{p^{n-1}-1}{p-1} \\
&= 1 + \frac{1}{2}p^2(p-1)\frac{p^{3n}-1}{p^3-1} + \frac{1}{2}(p-1)p^2p^{3n-2}\frac{p^{n-1}-1}{p-1} \\
&= 1 + \frac{1}{2}p^2(p-1)\left(\frac{p^{3n}-1}{p^3-1} + p^{3n-2}\frac{p^{n-1}-1}{p-1}\right)
\end{aligned}$$

If we for n substitute $n+1$, we get Peter Bala's formula. For odd numbers we have:

$$\begin{aligned}
a(p^{2n+1}) &= 1 + \sum_{i=1}^{n} \sum_{1 \le r < p^i, \gcd(p,r)=1} p^i r + \sum_{m=1}^{n} \frac{1}{2}(p-1)p^{2(2n+1)-m-1} \\
&= 1 + \sum_{i=1}^{n} \frac{1}{2} p^{2i} \phi(p^i) + \frac{1}{2}(p-1)p^2 p^{3n-1} \frac{p^n - 1}{p-1} \\
&= 1 + \sum_{i=1}^{n} \frac{1}{2} p^{3i-1}(p-1) + \frac{1}{2}(p-1)p^2 p^{3n-1} \frac{p^n - 1}{p-1} \\
&= 1 + \frac{1}{2} p^2 (p-1) \frac{p^{3n} - 1}{p^3 - 1} + \frac{1}{2}(p-1)p^2 p^{3n-1} \frac{p^n - 1}{p-1} \\
&= 1 + \frac{1}{2} p^2 (p-1) \left(\frac{p^{3n} - 1}{p^3 - 1} + p^{3n-1} \frac{p^n - 1}{p-1} \right)
\end{aligned}$$

This completes the proof. $\square$