

Call a nonnegative integer *legal* if it is in [A377912](#). That is, a number is legal if it does not contain an even digit  $d$  immediately followed by a digit  $q \leq d$ . Let  $S$  denote the sequence [A342042](#), and  $a(n)$  the  $n$ -th term of this sequence. We know that every legal number appears in  $S$ .

**Theorem.** *Let  $d \in \{0, 1, \dots, 8\}$ . For every positive integer  $k$ , the  $k$ -digit number  $d99\dots 9$  appears in  $S$  before the  $k$ -digit number  $(d+1)99\dots 9$ .*

*Proof.* Suppose not, and that  $k$  and  $d$  are minimal such that this fails. Suppose that  $a(n_0) = (d+1)99\dots 9$  and  $a(n_1) = d99\dots 9$ , where  $n_1 > n_0$ .

**Case 1:** If  $d$  is odd, then  $d+1$  is even. Then,  $a(n_0 - 1)$  ends with a digit strictly less than  $d+1$ . However, this means that  $d99\dots 9$  would have been a possible candidate for  $a(n_0)$ . Since  $d99\dots 9 < (d+1)99\dots 9$ , we get a contradiction.

**Case 2:** When  $d$  is even.

**Claim:** Then,

$$\begin{aligned} & |\{n \leq n_0 : a(n) \text{ has } k \text{ digits and ends with } d\}| \\ & \geq |\{n \leq n_0 : a(n) \text{ has } k \text{ digits and starts with } d+1\}| \end{aligned} \tag{1}$$

Suppose not. Then some  $k$ -digit number  $(d+1)d_2\dots d_k$  appears directly after a  $k$ -digit number of the form  $d'_1\dots d'_{k-1}d'_k$ , where  $d'_k$  is either odd or less than  $d$ . In either case,  $d99\dots 9$  would have been possible in the place of  $(d+1)d_2\dots d_k$ , a contradiction.

Now, notice that

$$\begin{aligned} & |\{n \in \mathbb{N} : a(n) \text{ has } k \text{ digits and ends with } d\}| \\ & < |\{n \in \mathbb{N} : a(n) \text{ has } k \text{ digits and starts with } d+1\}| \end{aligned} \tag{2}$$

Since every  $k$ -digit legal number has appeared when we reach  $n_0$ , we get that (1) contradicts (2).  $\square$