

# Rigid Motions of $K_5$ Obtained from $D_6$ Action on Planar Rooted Trees

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## **Abstract**

This study investigates associahedra - polytopes which are interesting in various mathematical disciplines such as topology and combinatorics. We prove that the  $D_6$  action on the planar rooted trees correspond to rigid motions of a certain realization of  $K_5$ . We also offer an explicit description of which element in  $D_6$  corresponds to what type of rigid motion of  $K_5$ . In further research this could be generalized to  $K_n$ .

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# 1 Introduction

The mathematical structures called associahedra of dimension  $n$ , denoted  $K_{n+2}$ , is a certain kind of polytope in  $n$ -dimensional space. It was first described in 1951 by the mathematician Dov Tamari in his doctoral thesis [1]. The associahedron has been used to investigate the property of associativity in topology by among others, J. Stasheff [2]. The structure is interesting in many areas of mathematics, e.g. in the theory of operads and homotopy theory.

The associahedron is closely related to combinatorics since its vertices are in one-to-one correspondence with a type of graph called planar rooted binary trees. These graphs have symmetry and this symmetry can be described with a mathematical structure called a group. The aim of this article is to describe how these symmetries can be seen on the associahedron.

The symmetries of the associahedra has been studied on an abstract level by Carl W. Lee [3]. We offer a more hands on and explicit presentation.

## 2 Basic Definitions and Examples

### 2.1 Basic Concepts in Group Theory

We begin by giving definitions of some basic concepts necessary for understanding the investigation of the symmetric properties of the associahedra. Symmetry is described by a mathematical concept called a *group*.

**Definition 1** (Group). A group  $G$  is a set together with an operation  $\odot$  which takes any two elements of  $G$  and produces a third. The operation should be associative, i.e. for all  $a, b, c \in G$ ,  $a \odot (b \odot c) = (a \odot b) \odot c$ . There should exist an element  $e$  such that  $e \odot a = a \odot e = a$ . Lastly, every element in  $G$  should have an inverse element, that is, for all  $a \in G$  there exist  $\acute{a} \in G$  such that  $a \odot \acute{a} = \acute{a} \odot a = e$ .

The element  $e$  is called the identity element. Note that it is not necessary that  $a \odot b =$

$b \odot a$  for all  $a, b \in G$ . If that is the case, then the group is an *abelian* group. Often,  $a \odot b$  is just written as  $ab$  and  $\underbrace{a \odot a \odot \dots \odot a}_{n \text{ times}}$  as  $a^n$ . It is easy to show the uniqueness of the identity element and the inverse element of an element  $a$ . We give a basic example in order to make the concept of a group more easy to understand.

**Example 1.** The dihedral group  $D_n$  is the group consisting of all rotations and reflections of a regular polygon of  $n$  sides.  $D_3$  is the rotations and reflections of a triangle and it consists of 6 elements. The identity element leaves the triangle unchanged<sup>1</sup>. If one labels

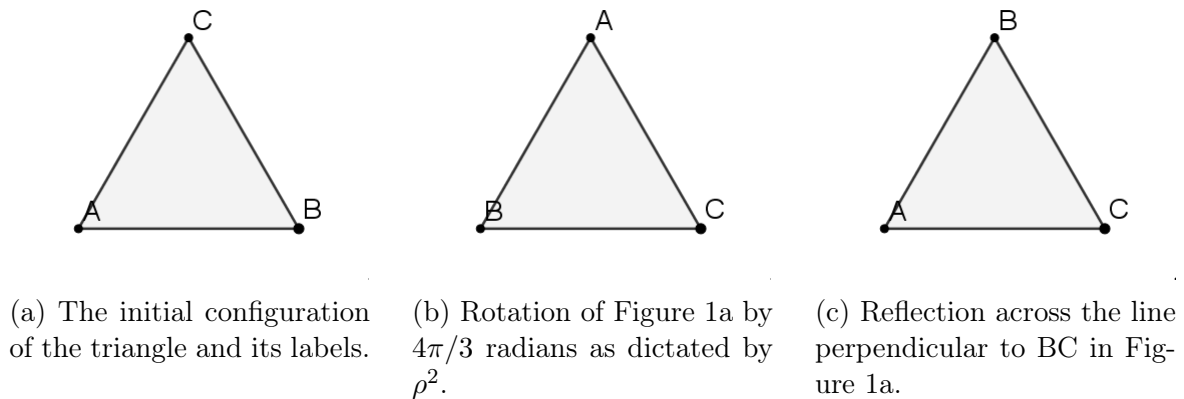


Figure 1: Illustration of how the elements of  $D_3$  affects a triangle with labeled vertices.

the vertices of the triangle as in Figure 1a, then a rotation of  $4\pi/3$  radians produces the triangle in Figure 1b. That rotation is denoted  $\rho^2$  since it can be seen as the initial rotation  $\rho$  of  $2\pi/3$  radians applied twice. Furthermore, a reflection along a line perpendicular to BC in Figure 1a, produces the triangle in Figure 1c. The group operation  $a \odot b$  could be seen as first rotating or reflecting the triangle in the way  $b$  describes, then rotating or reflecting the triangle in the way  $a$  describes.

A *subgroup* is a subset of the elements of a group such that the subset itself forms a group under the same operation. The *order* of a group is the number of elements a group contains. The order of a subgroup must always divide the order of the group. This is a central result in elementary group theory known as Lagrange's theorem.

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<sup>1</sup>Can be seen as rotating the triangle three times.

**Definition 2** (Group action). Consider a non-empty set  $X$  and a group  $G$ , a *group action* on the set  $X$  is a function  $\theta : G \times X \rightarrow X$ . Let  $e$  be the identity element of  $G$  and  $g, h \in G$ . Let  $x \in X$ . The group action should satisfy the following:  $\theta(e, x) = x$  and  $\theta(g, \theta(h, x)) = \theta(gh, x)$ .

The triangles in Figure 1 can be viewed as elements in a set which  $D_3$  acts upon.

The *generators* of a group  $G$  is any set of elements of  $G$  such that all other elements in  $G$  can be viewed as the generators composed with each other in some way. Additionally, among the generators there should not exist some element which can be viewed as the other generators composed with each other.

**Definition 3** (Orbit). The orbit of an element  $x$  in a set  $X$  under the action of a group  $G$  is the set  $\{\theta(g, x) : g \in G\}$ .

In this paper we will consider action on sets by the dihedral group. When talking about orbits, we will mostly mean the set  $\{\theta(g, x) : g \text{ is a generator of } G\}$ . For the one interested in groups or would like to know more, we recommend the reader to pick up any book on abstract algebra such as *Contemporary Abstract Algebra* by J. A. Gallian.

## 2.2 Basic Concepts in Graph Theory

A *multiset* is a set-like structure in which duplicates of elements are allowed but no order is defined between the elements. A multiset can be denoted by square-brackets. We now define *graphs* - a central concept in combinatorics.

**Definition 4** (Graph). A graph,  $H$ , is an ordered pair  $H = (V, E)$  of a set  $V$ , called the *vertices* and a multiset of pairs of vertices,  $E$ , called the *edges*. If the pairs of the vertices are ordered pairs, then the graph is an *ordered* graph. If the pairs of the vertices are unordered, then the graph is an *unordered* graph.

In this article a graph - if not explicitly pointed out - is an unordered graph. Note that in combinatorics, a graph is not a way of representing functions, but a way of describing

a relation between things. The "things" are visualized by dots, called nodes or vertices, and the relations are visualized by connecting two nodes with an edge.

**Example 2.** Let  $H$  be a graph,  $H = (V, E)$ , where  $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$  and  $E = [\{v_1, v_2\}, \{v_1, v_2\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_3, v_3\}]$ . Then the graph is represented by:

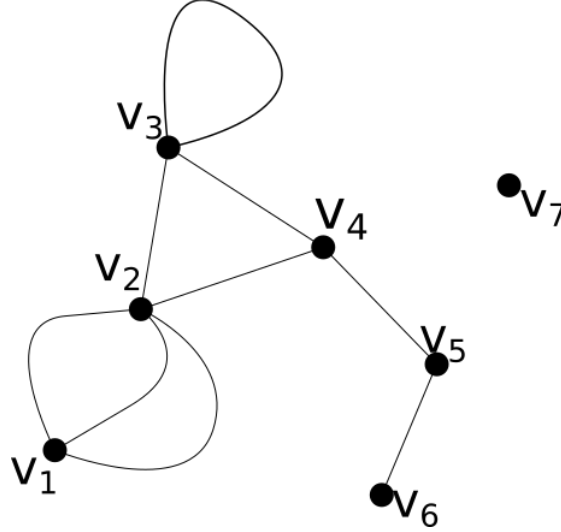


Figure 2: An example of a graph.

We now move on to define the type of graphs we will work with in this paper.

**Definition 5** (Tree). A *tree* is a graph in which any two vertices are connected by a unique path.

The graph in Figure 2 is not a tree since not every vertex is connected to another one by a unique path.

**Definition 6** (Planar rooted tree, PR-tree). A tree is a *planar rooted tree* if one vertex is marked and the tree is embedded in the plane. In this paper we add the additional restriction that no vertex are allowed to have *degree* two. The degree of a vertex is the number of edges which contains that vertex. The marked vertex is called the *root* and should have degree one. All other vertices with degree one are called *leaves*. The vertices with degree one are called *outer vertices* and all other vertices are called *inner vertices*.



**Definition 7** (Planar rooted binary tree, PRB-tree). A *planar rooted binary tree* is a planar rooted tree with no vertex of degree greater than three.

## 2.3 Polytopes and Their Combinatorial Interpretation

A *polytope* is a generalization of a polygon or a polyhedron to any dimension. Basically, an  $n$ -polytope can be seen as a shape in  $n$ -dimensional space with "flat" sides. This intuitive notion is made mathematical by the notion of the *convex hull* of a set of non-coplanar points in  $\mathbb{R}^n$ .

A  $k$ -cell is a  $k$ -polytope which is a part of an  $n$ -polytope, e.g. the 0-cells of a cube are its vertices, the 1-cells of a cube are its edges and the 2-cells of a cube are the squares making up the faces of the cube. The 3-cell is the cube in itself.

**Example 3.** The simplest kind of polytope is a *simplex*. A simplex of dimension  $n$  is the convex hull of  $(n + 1)$  non-coplanar points in  $\mathbb{R}^n$ . The intuitive notion of a simplex is that it is an  $n$ -dimensional triangle. A simplex of dimension 2 is any triangle and a simplex of dimension 3 is any tetrahedron.

The combinatorial structure of a polytope is the way different cells are connected to each other, i.e. the size and proportions are irrelevant. The combinatorial structure of a polytope can be described by something called a *face poset*. In this article we define polytopes in terms of their combinatorial features and by a *realization* we mean a way to describe the polytope geometrically.

**Definition 8** (Partially ordered set). A partially ordered set, also called *poset*, is a set,  $S$ , together with a relation,  $\preceq$ , such that the relation is defined between certain elements  $a, b \in S$  but not necessarily between all elements. The relation should satisfy the following: for any element  $a \in S$ ,  $a \preceq a$ . If  $a \preceq b$  and  $b \preceq a$  then  $a = b$ . Lastly, if  $a \preceq b$  and  $b \preceq c$  then  $a \preceq c$ .

**Example 4.** A partially ordered set can be visualized with a directed graph called a *Hasse diagram*. This is done in Figure 3. In a Hasse diagram, for an element  $a$  to be

placed on a higher level than another element  $b$ , means that  $b \preceq a$ . The edges in a Hasse diagram illustrate which vertices the order relation is defined between.

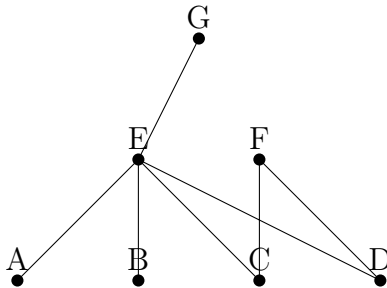


Figure 3: A poset visualized with a Hasse diagram.  $A \preceq E \preceq G$  is true but  $F \preceq G$  is not defined.

**Definition 9** (Face poset). A face poset of a polytope is the poset of all cells ordered such that  $A \preceq B$  if  $A$  is contained in  $B$ , i.e.  $A$  is a cell and a subset of  $B$ .

**Example 5.** Consider a cube with labeled vertices as in Figure 4. Then the face poset is visualized with its Hasse diagram in Figure 5.

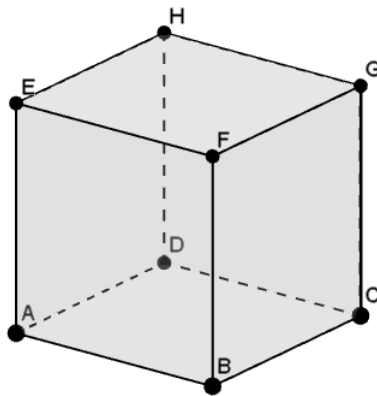


Figure 4: A cube with labeled vertices. The other cells are labeled by what vertices they contain. The face poset consists of all cells ordered such that any cell is less than another if that cell is contained in the other, e.g.  $A \preceq AB \preceq ABEF \preceq ABCDEFGH$ .

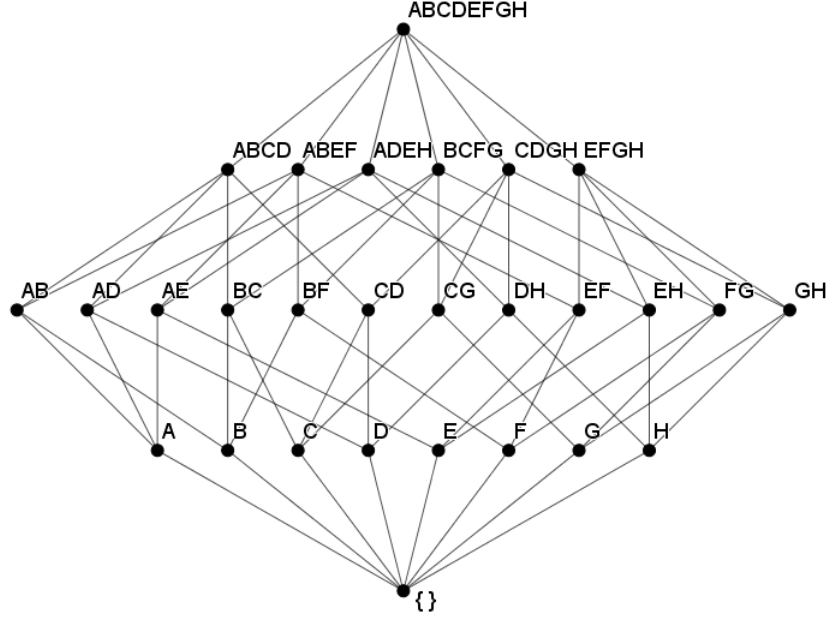


Figure 5: Hasse diagram representing the face poset of the cube in Figure 4. Since the empty set is a subset of any set, the empty set is sometimes included in the face poset.

Intuitively, two posets are *isomorphic* when they are structurally the same. The formal definition can be found below.

**Definition 10** (Isomorphic posets). Two partially ordered sets  $X$  and  $Y$  with order relation  $\preceq$  and  $\trianglelefteq$  respectively, are considered to be isomorphic if there is a one-to-one correspondence  $f : X \rightarrow Y$  such that  $x_1, x_2 \in X$  and  $x_1 \preceq x_2$  if and only if  $f(x_1) \trianglelefteq f(x_2)$ .

The map  $f$  is said to be an *isomorphism*.

### 3 The Associahedra

The Associahedra are a family of polytopes of which there exists one in every dimension. The polytope is defined entirely by its combinatorial structure. However, the simplest way to define what associahedra is requires an understanding of something called *diagonalized polygons*.

### 3.1 Diagonalized Polygons

A regular polygon in which 0 or more diagonals are drawn between any two non-consecutive vertices in such a way that no diagonal cross another diagonal is called a diagonalized polygon. In this article, we denote the set of all diagonalized  $n$ -sided polygons with  $d$  diagonals  $\mathcal{Q}_{n,d}$ . The poset of all diagonalized  $n$ -sided polygons, ordered such that one diagonalized polygon  $P_1$ , satisfies  $P_1 \preceq P_2$  if  $P_1$  can be obtained from  $P_2$  by adding diagonals, is denoted by  $\mathcal{Q}_n$ .

### 3.2 Defining Associahedra

We are now in position to give a definition of the central structure studied in this article.

**Definition 11** (Associahedron). The Associahedron of dimension  $n$ , denoted  $K_{n+2}$ , is the polytope whose face poset is isomorphic to  $\mathcal{Q}_{n+3}$ .

This means that all 0-cells (vertices) of  $K_n$  correspond to all fully diagonalized  $(n+1)$ -sided polygons, i.e. diagonalized polygons with  $n-2$  diagonals. All 1-cells correspond to diagonalized polygons with  $n-3$  diagonals, 2-cells correspond to diagonalized polygons with  $n-4$  diagonals and so on.

What determines whether or not two vertices should be connected in  $K_n$ ? From the definition of  $K_n$  it follows that two vertices are connected if the diagonalized polygons corresponding to the vertices differs from each other by only one diagonal. This can easily be realized when considering the Hasse diagram of  $\mathcal{Q}_{n+1}$  which by definition is isomorphic to the face poset of  $K_n$ .

The more general question at this point could be: what determines which cells are contained in another in  $K_n$ ? Again, by the definition of  $K_n$ , a  $k$ -cell is contained in a  $(k+1)$ -cell if and only if all diagonals in the diagonalized polygon corresponding to the  $(k+1)$ -cell are shared by the diagonalized polygon corresponding to the  $k$ -cell.

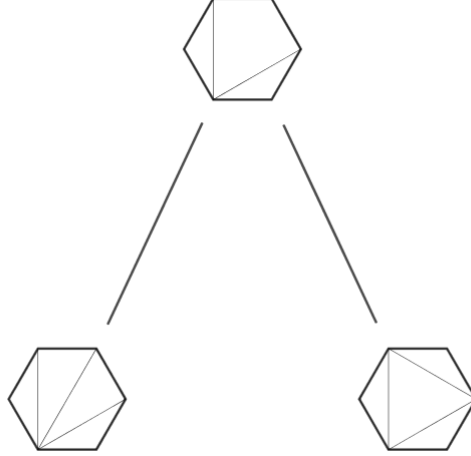


Figure 6: A part of the Hasse diagram of  $\mathcal{Q}_6$ , showing how a diagonalization is less than another diagonalization. The vertices of  $K_5$  which correspond to the two polygons with three diagonals are connected with an edge in  $K_5$  since they both are "less" than the diagonalized polygon with two diagonals corresponding to an edge.

### 3.3 The Correspondence Between PR-Trees and Diagonalized Polygons

There is a direct correspondence between PR-trees and the diagonalized polygons described above. Let  $\mathcal{T}_n$  be the poset of all PR-trees with  $n$  leaves. Then  $\mathcal{T}_n$  is isomorphic to  $\mathcal{Q}_{n+1}$ . This can be realized by letting every side in the polygon correspond to an outer vertex of the PR-tree and draw one inner vertex in every region of the diagonalized polygon and drawing one edge from that vertex to every side of that region. This procedure can be seen in Figure 7.

From this it follows that the associahedra  $K_n$  can also be defined as the polytope whose face poset is isomorphic to  $\mathcal{T}_n$ . In fact, the diagonalized polygons are mostly used to make it easier to understand the group action on the PR-trees. Before we proceed to the next section, we give one more definition. Let  $\mathcal{T}_{n,m}$  be the set of all PR-trees which correspond to  $m$ -cells of  $K_n$ .  $\mathcal{T}_{n,0}$  is then all PRB-trees corresponding to the vertices of  $K_n$ . As a consequence of what is mentioned above,  $\mathcal{T}_{n,m}$  is isomorphic to  $\mathcal{Q}_{n+1,n-2-m}$ .

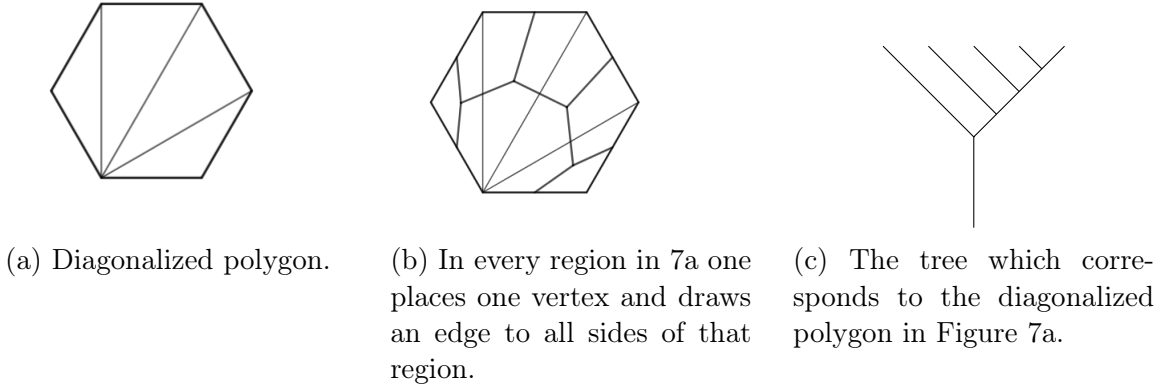


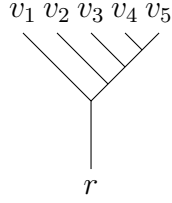
Figure 7: How diagonalized polygons correspond to PR-trees.

## 4 Group Actions on PR-Trees and Diagonalized Polygons

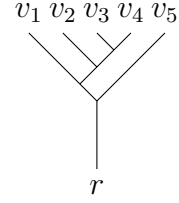
Consider the dihedral group of order  $n$  acting upon the diagonalized  $n$ -sided polygons. Clearly, this action can also be interpreted as an action on the PR-trees with  $n + 2$  leaves. All elements in  $D_n$  can be generated by rotations and reflections. The rotation partitions the set of all PR-trees into orbits in one way and the reflection in an other way.

The rotation  $\rho : \mathcal{T}_n \rightarrow \mathcal{T}_n$ , easily understood on the polygons,  $\rho(\text{⬠}) = \text{⬠}$ , can be interpreted on the PR-trees as moving every outer vertex to the place of the outer vertex which is a neighbour to the original vertex in a counter-clockwise direction such that all connections between the vertices are preserved, i.e. consider the PRB-tree in Figure 8a, then a rotation as dictated by  $\rho$  maps  $v_1 \mapsto r$ ,  $v_2 \mapsto v_1$ ,  $v_3 \mapsto v_2$ ,  $v_4 \mapsto v_3$ ,  $v_5 \mapsto v_4$  and  $r \mapsto v_5$ , yielding the tree in Figure 8b.

For  $K_5$ , there are 4 orbits of the vertices under the action of rotation of lengths 6, 3, 3 and 2, respectively. These orbits, the elements they correspond to, and the labels and colours they are given in this paper can be seen in Table 1. In Table 2 one can find the orbits of the diagonalized polygons under the action of  $D_6$  corresponding to edges of  $K_5$ . The same for faces can be found in Table 3.



(a) PRB-tree with labeled vertices.



(b) Rotation of the tree in Figure 8a.

Figure 8: The tree to the right is a rotation of the tree to the left. The rotation can intuitively be understood either by transforming the tree to a diagonalized polygon, rotate the polygon and transfer back to a tree or by just considering the map  $v_1 \mapsto r$ ,  $v_2 \mapsto v_1$ ,  $v_3 \mapsto v_2$ ,  $v_4 \mapsto v_3$ ,  $v_5 \mapsto v_4$  and  $r \mapsto v_5$ , which should leave all connections between the vertices preserved.

Table 1: The orbits of the vertices of  $K_5$  under the action of rotation, the elements they correspond to and the labels and colours they are given in this paper.

PRB-trees	Fully diagonalized hexagons	Labels	Colour
		$\{r_1, r_2\}$	<span style="color: red;">●</span>
		$\{y_1, y_2, y_3\}$	<span style="color: yellow;">●</span>
		$\{b_1, b_2, b_3\}$	<span style="color: blue;">●</span>
		$\{g_1, g_2, g_3, g_4, g_5, g_6\}$	<span style="color: green;">●</span>



Table 2: The orbits of the PR-trees under the action of rotation corresponding to the edges of  $K_5$ . In the left column one element of the orbit is visualized. All other elements in the orbit are rotations of the elements in the left column. The number of such elements can be found in the column in the middle. In the right column each orbit is labeled by a specific colour. Note that the colours are chosen arbitrarily and that there is no connection to the colours in table 1.

Diagonalized hexagon	Rotations of the hexagon	Colour
	6	<span style="color: red;">●</span>
	6	<span style="color: yellow;">●</span>
	6	<span style="color: blue;">●</span>
	3	<span style="color: green;">●</span>

## 5 Symmetries of $K_5$

Before we proceed to investigating the symmetries of  $K_5$ , we define a *rigid motion*. Note that the definition of a rigid motion may be different in other articles.

Table 3: The orbits of all diagonalizations of a hexagon consisting of one diagonal under the action of  $D_6$ . These correspond to faces of  $K_5$ .

Diagonalized hexagon	Rotations of the hexagon
	6
	3

**Definition 12** (Rigid motion of  $\mathbb{R}^n$ ). A rigid motion of  $\mathbb{R}^n$  is a composition of reflections and rotations, fixing the origin, of  $\mathbb{R}^n$ .

A rigid motion of  $\mathbb{R}^n$  can be seen as rotations and reflections of an  $n$ -dimensional sphere with center at the origin. A rigid motion of a polytope is a rigid motion of  $\mathbb{R}^n$  such that all  $k$ -cells are mapped to another  $k$ -cell. We also define a rigid motion of the set of all  $k$ -cells of a polytope. This is a rigid motion of  $\mathbb{R}^n$  such that every  $k$ -cell is mapped to another  $k$ -cell.

## 5.1 $K_5$ embedded in a Sphere

Since the associahedron is defined through PR-trees, the  $D_6$  action on the PR-trees as described in section 4 must have a geometric interpretation. We propose that the group action on the PR-trees with  $n$  leaves corresponds to a rigid motion of at least one realization of  $K_n$ . This can easily be shown to be the case for  $K_4$ . We will now analyze the symmetries of  $K_5$  and show that this is the case for  $K_5$ .

How different vertices should be connected in order to obtain  $K_5$  was explained in section 3.2. What colour and label each vertex should get when considering dihedral group actions are described in table 1. Now, let these vertices be mapped to a sphere in such a way that the two red vertices are the north and south pole respectively. Place the blue and yellow vertices on the equator such that the blue and yellow vertices alternate and that the distance between consecutive vertices is the same. Three green vertices are then placed on the northern hemisphere in a symmetric way and the same is done for the



remaining three green vertices on the southern hemisphere<sup>2</sup>. The result one should get is something like Figure 9.

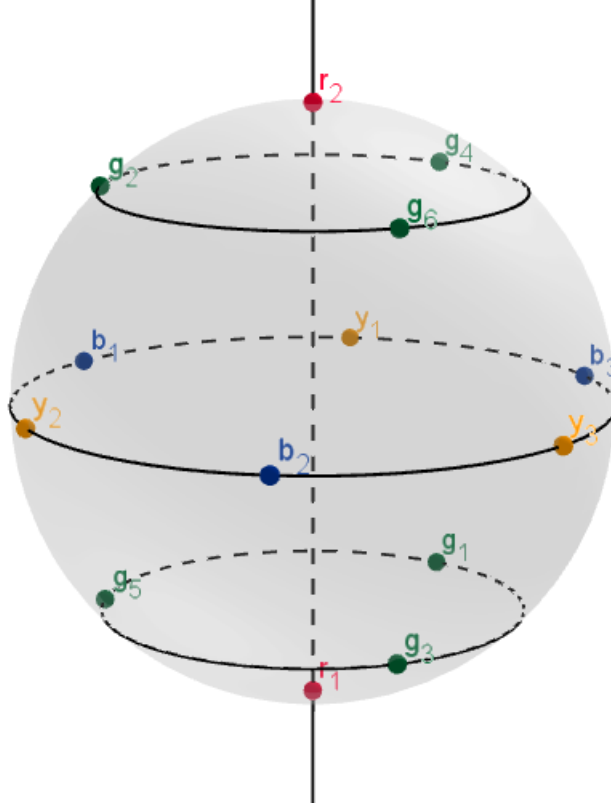


Figure 9: The coloured vertices placed on a sphere. The colours tell in which orbit the corresponding PRB-tree lies in when  $D_6$  acts upon all PRB-trees with 5 leaves. Note that the lines in this picture do not represent the way vertices should be connected in order to obtain  $K_5$  but is used to illustrate the symmetric way in which vertices are placed on the sphere.

From this point is it not hard to connect the vertices with edges in the correct way as described earlier. The result is  $K_5$  since the combinatorial structure is inherent in the connections of the vertices. We have now obtained a symmetric version of  $K_5$  embedded in a sphere. This can be seen in Figure 10. In Figure 11 the same realization is done but here the edges are coloured instead of the vertices according to Table 2.

<sup>2</sup>The PRB-trees corresponding to the green vertices forms an orbit when they are under the action of rotation. However, these "green" PRB-trees can also be separated in two different orbits if one considers the action of two rotations. Then these two orbits consist of three elements each, corresponding to the three elements that are positioned on the north and south pole respectively.

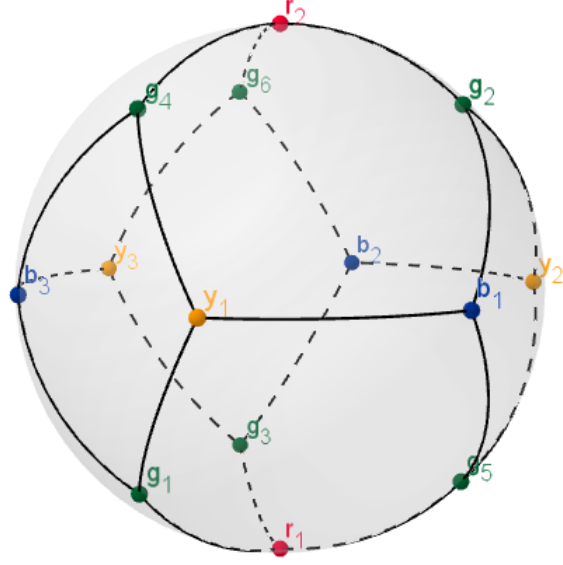


Figure 10:  $K_5$  embedded in a sphere with vertices labeled according to which orbit their corresponding PRB-tree lies in under action of  $D_6$ .

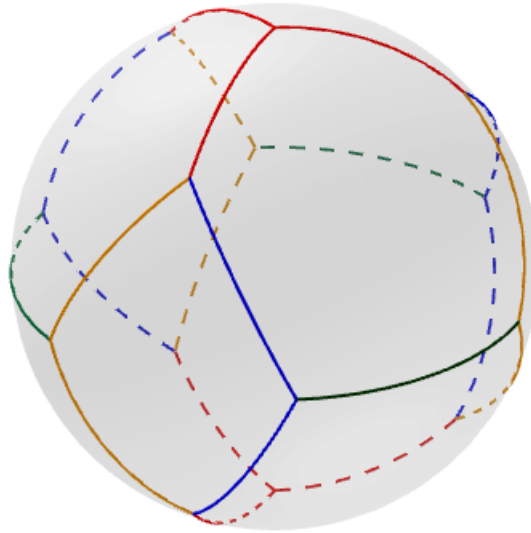


Figure 11: The edges of  $K_5$  coloured as in Table 2.

## 5.2 Rigid Motions of $K_5$

We will now move on to investigate some rigid motions of  $K_5$  as it is realized in this paper. Consider a rotation of  $2\pi/3$  radians in anti-clockwise direction with respect to Figure 10 of  $K_5$  which fixes  $r_1$  and  $r_2$ . This is a rigid motion  $\phi_{2\pi/3} : S \rightarrow S$  whose mapping can be seen on the vertices like:

$$\begin{aligned} & (r_1, r_2, y_1, y_2, y_3, b_1, b_2, b_3, g_1, g_2, g_3, g_4, g_5, g_6) \\ & \mapsto (r_1, r_2, y_2, y_3, y_1, b_2, b_3, b_1, g_5, g_6, g_1, g_2, g_3, g_4) \end{aligned} \tag{1}$$

The reflection through the equatorial plane forms a rigid motion  $\Phi_{\text{eq}} : S \rightarrow S$  which maps the vertices in the following way:

$$\begin{aligned} & (r_1, r_2, y_1, y_2, y_3, b_1, b_2, b_3, g_1, g_2, g_3, g_4, g_5, g_6) \\ & \mapsto (r_2, r_1, y_1, y_2, y_3, b_1, b_2, b_3, g_4, g_5, g_6, g_1, g_2, g_3) \end{aligned} \tag{2}$$

Lastly, consider the reflection through the plane crossing  $r_1$ ,  $r_2$ ,  $g_3$  and  $g_6$ . This plane is perpendicular to the equatorial plane. The reflection is a rigid motion  $\Phi_{\text{orth}} : S \rightarrow S$  which maps the vertices according to:

$$\begin{aligned} & (r_1, r_2, y_1, y_2, y_3, b_1, b_2, b_3, g_1, g_2, g_3, g_4, g_5, g_6) \\ & \mapsto (r_1, r_2, b_1, b_3, b_2, y_1, y_3, y_2, g_5, g_4, g_3, g_2, g_1, g_6) \end{aligned} \tag{3}$$

## 5.3 Symmetries of $K_5$ as Described by Groups

We would like to show that the action of the dihedral group on the PR-trees could be interpreted as a rigid motion of some realization of the associahedron. In Lemma 1 we show that this is the same as just considering the dihedral group action on the PRB-trees and the way in which this affects the associahedron.

**Lemma 1.** *Let  $C(K_n)$  be a realization of  $K_n$  as a convex hull of the set of vertices  $V(K_n)$  interpreted as points in  $C(K_n)$ , then the action of  $D_{n+1}$  on  $\mathcal{T}_n$  corresponds to a*

rigid motion of  $C(K_n)$  if and only if the action of  $D_{n+1}$  on  $\mathcal{T}_{n,0}$  corresponds to a rigid motion of  $V(K_n)$ .

*Proof.* Let  $\rho$  be the action of rotation and  $\xi$  be the action of reflection. Then all elements of  $D_{n+1}$  can be generated by  $\rho$  and  $\xi$  and it is sufficient to investigate the action of these two elements. Let  $C_k$  be a  $k$ -cell of  $C(K_n)$  and  $D(C_k)$  the corresponding diagonalized polygon. From the definition of  $K_n$ ,  $D(C_k)$  can be seen as all diagonals shared by those  $D(C_{k-1})$  such that  $C_{k-1} \subset C_k$ . This means that the order relation is preserved in the poset  $\mathcal{Q}_{n+1}$  under the action of  $D_{n+1}$ , i.e.  $\rho(D(C_{k-1})) \preceq \rho(D(C_k))$  and  $\xi(D(C_{k-1})) \preceq \xi(D(C_k))$ . Hence, all cells in  $K_n$  are connected in the same way as before the group action.

Since the position of a  $k$ -cell can be reduced to the positions of all vertices  $V(K_n)$ , a rigid motion of the vertices as dictated by the action of  $D_n$  must also be a rigid motion of all other cells if the rigid motion is defined separately for all  $k$ -cells. Since the combinatorial structure of  $K_n$  is preserved under the action of the dihedral group, Lemma 1 follows.  $\square$

Lemma 1 says that the group action on the PR-trees leaves the ordering unchanged, and hence,  $D_{n+1}$  action on  $\mathcal{T}_n$  leaves the combinatorial structure of  $K_n$  preserved. Furthermore, since the positions of all cells in a realization of  $K_n$  as a convex hull of a set of points is determined only by the vertices, the dihedral group action on the vertices determines the way other cells are effected by the group action.

*Remark.* Lemma 1 also applies to the realization of  $K_5$  embedded in a sphere as described in section 5.1. This is true because the way different cells are built up in the realization are defined solely by the vertices.

We are now in position to present the main result of this article.

**Theorem 1.** *Let  $\rho$  be the action of rotation and  $\xi$  the action of reflection on  $\mathcal{T}_5$  as dictated by the dihedral group. Let  $C(K_5)$  be the realization of  $K_5$  as described in section 5.1 and  $V(K_5)$  the vertices in this realization. Let  $\psi : \mathcal{T}_5 \rightarrow C(K_5)$  be a map which maps every*

PR-tree to the corresponding cell of  $K_5$ . The action of  $D_6$  on  $\mathcal{T}_5$  is then equivalent to rigid motions of  $C(K_5)$ , more precisely:

$$\begin{array}{ccc} \mathcal{T}_5 & \xrightarrow{\psi} & C(K_5) \\ \rho \downarrow & & \downarrow \Phi_{\text{eq}} \circ \phi_{2\pi/3} \\ \mathcal{T}_5 & \xrightarrow{\psi} & C(K_5) \end{array} \quad \begin{array}{ccc} \mathcal{T}_5 & \xrightarrow{\psi} & C(K_5) \\ \xi \downarrow & & \downarrow \Phi_{\text{orth}} \circ \Phi_{\text{eq}} \\ \mathcal{T}_5 & \xrightarrow{\psi} & C(K_5) \end{array}$$

*Proof.* According to Lemma 1 the theorem follows if one can show that the rotation  $\rho$  and reflection  $\xi$  maps each element  $j$  in  $\mathcal{Q}_{6,3}$  to another element  $i$  in  $\mathcal{Q}_{6,3}$  such that the corresponding vertex to  $j$  in  $C(K_5)$  is mapped by the rigid motion to the corresponding vertex of  $i$ . By function composition of the rigid motions defined in section 5.2 this can easily be checked to be true. The function composition of  $\Phi_{\text{eq}}$  and  $\phi_{2\pi/3}$  is seen on the vertices like:

$$\begin{aligned} & (r_1, r_2, y_1, y_2, y_3, b_1, b_2, b_3, g_1, g_2, g_3, g_4, g_5, g_6) \\ & \mapsto (r_1, r_2, y_2, y_3, y_1, b_2, b_3, b_1, g_5, g_6, g_1, g_2, g_3, g_4) \\ & \mapsto (r_2, r_1, y_2, y_3, y_1, b_2, b_3, b_1, g_2, g_3, g_4, g_5, g_6, g_1) \end{aligned}$$

Which is equivalent to the way  $\rho$  maps the corresponding PRB-trees or fully diagonalized polygons. This equivalence can be realized if one understands Table 1. Now, consider the function composition of  $\Phi_{\text{orth}}$  and  $\Phi_{\text{eq}}$ , this one is seen on the vertices like:

$$\begin{aligned} & (r_1, r_2, y_1, y_2, y_3, b_1, b_2, b_3, g_1, g_2, g_3, g_4, g_5, g_6) \\ & \mapsto (r_1, r_2, b_1, b_3, b_2, y_1, y_3, y_2, g_5, g_4, g_3, g_2, g_1, g_6) \\ & \mapsto (r_2, r_1, b_1, b_3, b_2, y_1, y_3, y_2, g_2, g_1, g_6, g_5, g_4, g_3) \end{aligned}$$

Which is equivalent to reflecting the PRB-trees or fully diagonalized polygons.  $\square$

A question of interest at this point is what type of rigid motion of  $K_5$  the action of different elements of  $D_6$  correspond to. In Table 4 this information can be found. Note that all rigid motions can be explained by compositions of some other rigid motions. Furthermore, note that every rigid motion is a composition of three different rigid motions,

each chosen from the three sets:  $\{\Phi_{\text{orth}}^0, \Phi_{\text{orth}}\}$ ,  $\{\Phi_{\text{eq}}^0, \Phi_{\text{eq}}\}$  and  $\{\phi_{2\pi/3}^0, \phi_{2\pi/3}, \phi_{2\pi/3}^2\}$ , where  $\Phi_{\text{orth}}^0$ ,  $\Phi_{\text{eq}}^0$  and  $\phi_{2\pi/3}^0$  correspond to not doing anything and therefore these are not spelled out. Since each element is chosen from all three "categories", there is a total of  $2 \cdot 2 \cdot 3 = 12$  rigid motions corresponding to each of the twelve elements of  $D_6$ .

Table 4: In the left column the different elements of  $D_6$  are listed. In the right column we show the effect this element have on  $K_5$  when the element acts on  $\mathcal{T}_5$ , i.e. the poset which is isomorphic to the face poset of  $K_5$ .

Element in $D_6$	Corresponding rigid motion of $K_5$
$e$	No effect on $K_5$
$\rho$	$\Phi_{\text{eq}} \circ \phi_{2\pi/3}$
$\rho^2$	$\phi_{2\pi/3}^2 = \phi_{4\pi/3}$
$\rho^3$	$\Phi_{\text{eq}}$
$\rho^4$	$\phi_{2\pi/3}$
$\rho^5$	$\Phi_{\text{eq}} \circ \phi_{2\pi/3}^2 = \Phi_{\text{eq}} \circ \phi_{4\pi/3}$
$\xi$	$\Phi_{\text{orth}} \circ \Phi_{\text{eq}}$
$\xi\rho$	$\Phi_{\text{orth}} \circ \phi_{2\pi/3}$
$\xi\rho^2$	$\Phi_{\text{orth}} \circ \Phi_{\text{eq}} \circ \phi_{4\pi/3}$
$\xi\rho^3$	$\Phi_{\text{orth}}$
$\xi\rho^4$	$\Phi_{\text{orth}} \circ \Phi_{\text{eq}} \circ \phi_{2\pi/3}$
$\xi\rho^5$	$\Phi_{\text{orth}} \circ \phi_{4\pi/3}$

## References

- [1] Müller-Hoissen F, Marcel Pallo J, Stasheff J. *Associahedra, Tamari Lattices and Related Structures*. Basel: Birkhäuser; 2012.
- [2] Stasheff J. (1963). *Homotopy Associativity of H-spaces*. In: *Transactions of the American Mathematical Society*.
- [3] Lee, C. W. (1989). *The associahedron and triangulations of the n-gon*. *European Journal of Combinatorics*, 10(6), 551-560.
- [4] Loday, J. L. (2005). *The multiple facets of the associahedron*. *preprint*.
- [5] Devadoss, S. L., Forcey, S., Reisdorf, S., Showers, P. (2015). *Convex polytopes from nested posets*. *European Journal of Combinatorics*, 43, 229-248.
- [6] Devadoss, S. L. (2009). *A realization of graph associahedra*. *Discrete Mathematics*, 309(1), 271-276.
- [7] Benedetti B., Delucchi E., Moci L. (2015). *Combinatorial Methods in Topology and Algebra*. Paris: Springer.

## A Exact Coordinates for the Sphere-embedding of $K_5$

In Table 5 one can find the exact coordinates of the vertices of  $K_5$  embedded in a sphere as explained in section 5.1.

Table 5: Spherical coordinates for the realization of  $K_5$  as described in section 5.1.

Vertex	Spherical coordinates
$r_1$	$(1, 0, -\frac{\pi}{2})$
$r_2$	$(1, 0, \frac{\pi}{2})$
$y_1$	$(1, \frac{\pi}{6}, 0)$
$y_2$	$(1, \frac{5\pi}{6}, 0)$
$y_3$	$(1, \frac{3\pi}{2}, 0)$
$b_1$	$(1, \frac{\pi}{2}, 0)$
$b_2$	$(1, \frac{7\pi}{6}, 0)$
$b_3$	$(1, \frac{11\pi}{6}, 0)$
$g_1$	$(1, 0, \frac{\pi}{4})$
$g_2$	$(1, \frac{2\pi}{3}, \frac{\pi}{4})$
$g_3$	$(1, \frac{4\pi}{3}, -\frac{\pi}{4})$
$g_4$	$(1, 0, \frac{\pi}{4})$
$g_5$	$(1, \frac{2\pi}{3}, -\frac{\pi}{4})$
$g_6$	$(1, \frac{4\pi}{3}, \frac{\pi}{4})$