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Some Combinatorial Properties of the Initial Segment of $C(\mathfrak{a}\mathfrak{a})$

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Abstract:
In the thesis, we look at an inner model of ZFC called $C(\mathfrak{aa})$. It was first constructed and investigated by Juliette Kennedy, Menachem Magidor and Jouko Väänänen in their article *Inner Models From Extended Logics: Part 2*. It is similar to Gödel's constructible universe, but instead of using definability in first-order logic in the construction, it uses definability in stationary logic.

Stationary logic is an extension of first-order logic with a second-order quantifier $\mathfrak{aa}$ ("almost all"). Essentially, this quantifier tells you whether a definable subset of the set of countable subsets of a model contains a club. Stationary logic has many similar properties as first-order logic – such as completenss and compactness – making it a very natural logic to study.

In order to investigate $C(\mathfrak{aa})$, Kennedy, Magidor and Väänänen proved (under large cardinal asssumptions) that its levels satisfy something called club determinacy. Under club determinacy, all regular cardinals of V are measurable in $C(\mathfrak{aa})$. They also developed the theory of $\mathfrak{aa}$-mice and $\mathfrak{aa}$-iterations, which yield a Theorem similar to Gödel's condensation Lemma in L. We will do this here as well. However, many of our definitions are slightly different. Most notably, Kennedy, Magidor and Väänänen use a definition of $C(\mathfrak{aa})$ that is reminiscent of the Jensen hierarchy, since due to a technicality, the levels of the hierarchy need to be closed under pairing. However, following Gabriel Goldberg and John Steel in their paper *The structure of* $C(aa)$, we circumvent this issue by instead using a more complicated pairing function. Thus, our definition of $C(\mathfrak{aa})$ is more in line with the traditional construction of Gödel's constructible universe.

In the last section, we use the techniques developed earlier to prove several combinatorial properties of the initial segment of $C(\mathfrak{aa})$. First, we extend Kennedy's, Magidor's and Väänänen's proof of diamond in $C(\mathfrak{aa})$ to diamonds for all regular cardinals below ω_1^V. Then, we look at a reflection principle from the book *Generalized Descriptive Set Theory and Classification Theory* by Friedman, Hyttinen, and Weinstein (Kulikov). Lastly, we show that every regular, not ineffable, cardinal $\kappa < \omega_1^V$ in $C(\mathfrak{aa})$ has a $\diamondsuit_\kappa^+$-sequence in $C(\mathfrak{aa})$. In all of these Theorems, we assume club determinacy.

Keywords: constructible universe, diamond, inner model, stationary logic, stationary tower forcing

Contents

Chapter 1

Introduction

An inner model of ZF is a transitive class that contains all ordinals and satisfies the axioms of ZF. Gödel's constructible universe L is the most well-known example. It was introduced by Gödel in 1939 in order to establish the consistency (relative to ZF) of axiom of choice, as well as the generalised continuum hypothesis. Since then, the study of inner models has evolved into an independent area of research.

The standard definition of the constructible universe is as follows:

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= \operatorname{def}(L_\alpha) \\ L_\beta &= \bigcup_{\alpha<\beta} L_\alpha, \text{ if } \beta \text{ is a limit ordinal} \\ L &= \bigcup_{\alpha\in \mathrm{Ord}} L_\alpha \end{aligned}$$

Here, $\operatorname{def}(L_\alpha)$ is the set of all subsets of L_α that are definable (from parameters) in first-order logic. In the articles *Inner Models from Extended Logics: Part 1* [8], and *Part 2* [9], Juliette Kennedy, Menachem Magidor and Jouko Väänänen have explored what happens if definability in first-order logic is replaced by definability in some extension of first-order logic. In their second article, they introduce the inner model $C(\mathfrak{aa})$, which arises from stationary logic. They prove, assuming large cardinal assumptions, that every regular cardinal of V is a measurable cardinal in $C(\mathfrak{aa})$. This makes this inner model very different from L, since the existence of measurable cardinals is inconsistent with the axiom $V = L$. Furthermore, one of the main goals of inner model theory is the construction of inner models with similar properties as L, but allowing large cardinals that are incompatible with L. Kennedy's, Magidor's and Väänänen's approach is quite different from the usual approach involving so called extender models.

Stationary logic was first studied extensively in the article *Stationary logic* by Jon Barwise, Matt Kaufmann and Michael Makkai. It is an extension of first-order logic with a second-order quantifier "aa", expressing that "for almost all countable subsets". This is formalised using a generalisation of closed unbounded sets (club sets). In countable languages, stationary logic satisfies both completeness and compactness. There's also a downward Löwenheim-Skolem Theorem, saying that every consistent theory in a countable language has a model of cardinality ω_1. Thus, stationary logic has many of the desirable properties of first-order logic, motivating its study.

This thesis has two goals. The first is to construct $C(\mathsf{aa})$ in detail and develop the tools that are needed to understand its properties. Here we are closely following the article *Inner Models from Extended Logics: Part 2*. However, our definition of $C(\mathsf{aa})$ is different. The most straightforward definition of $C(\mathsf{aa})$ is the following:

$$\begin{aligned} C_0 &:= \emptyset \\ C_{\alpha+1} &:= \mathrm{def}_{\mathsf{aa}}(C_\alpha) \\ C_\beta &:= \bigcup_{\alpha<\beta} C_\alpha, \text{ if } \beta \text{ is a limit ordinal} \\ C(\mathsf{aa}) &:= \bigcup_{\alpha\in\mathrm{Ord}} C_\alpha, \end{aligned}$$

where $\mathrm{def}_{\mathsf{aa}}(C_\alpha)$ is the set of all subsets of C_α definable in stationary logic. However, it is an open question whether this construction satisfies axiom of choice. In the canonical proof of axiom of choice in L, one uses the fact that $\{L_\beta \mid \beta < \alpha\}$ is a definable subset of L_α, whenever α is a limit ordinal. It is not clear whether this is true for the sets C_α. In the proof, one also uses the fact that the first-order satisfaction relation is absolute. This is not the case for stationary logic. However, it is possible to circumvent these issues by adding a predicate to each level C_α. The predicate codes facts about the earlier levels. To be able to do this, we need that C_α is closed under ordered pairs. This is not true if we define ordered pairs in the standard way. Kennedy, Magidor and Väänänen solved this issue by instead defining $C(\mathsf{aa})$ in an entirely different way, using so called rudimentary functions. We will not be doing this. Instead, we will introduce a more complicated pairing function, and define $C(\mathsf{aa})$ in a similar way as above. This is the definition of $C(\mathsf{aa})$ that Gabriel Goldberg and John Steel use in their article *The structure of* $C(\mathsf{aa})$ [4].

The second goal of this thesis is to show in detail how the techniques developed by Kennedy, Magidor and Väänänen can be used to investigate the combinatorial structure of $C(\mathsf{aa})$. Most notably, we prove that every regular, not ineffable cardinal $\kappa < \omega_1^V$ has a $\Diamond_\kappa^+$-sequence.

Throughout the thesis, the reader is assumed to be familiar with basic notions of set theory. In particular, club and stationary sets on ordinals, the constructible universe L and basic definitions related to forcing. Some experience with large cardinals might also be useful. For an introduction to club and stationary sets, we refer the reader to chapter 8 in Jech's book *Set Theory* [6]. For the constructible universe, we refer the reader to chapter 13 in Jech. For forcing, we refer the reader to chapter IV in Kunen's book *Set Theory* [10].

Some comments on notation: We write $\overline{a}, \overline{b}, ...$ for tuples. The length of a tuple is denoted $|\overline{a}|$. If f is a function and $\overline{a} = (a_0, ..., a_{n-1})$, then by $f(\overline{a})$ we mean the tuple $(f(a_0), ..., f(a_{n-1}))$. With $f[A]$, we mean the image of A under f, i.e. the set $\{f(a) \mid a \in A\}$. If κ is a cardinal, then $\mathcal{P}_\kappa(A)$ is the set of all subsets of A of cardinality strictly less than κ. We use letters $\mathcal{M}, \mathcal{N}, \mathcal{H}$,... for structures. It is implicit that their corresponding domains/universes are M, N, H,... For formulas φ, we write $\varphi(\overline{x})$ to indicate that all free variables of φ are among $\overline{x} = (x_0, ..., x_{n-1})$. A formula with parameters $\varphi(\overline{a})$ is identified with the tuple $(\varphi, \overline{a})$. We denote our set-theoretical universe with V. In almost every case, the formulas we talk about are elements of V, i.e. they are not meta-mathematical formulas. When the vocabulary is countable, the formulas are identified with natural numbers.

There's a generalisation of L that we occasionally will be using in some arguments. Let A and M be sets and $\mathrm{def}_A(M)$ the set of all $X \subseteq M$ such that X is definable in first-order logic over the structure $(M, \in, A \cap M)$. We define the class $L[A]$ as follows:

$$\begin{aligned}
L_0[A] &:= \emptyset \\
L_{\alpha+1}[A] &:= \mathrm{def}_A(L_\alpha[A]) \\
L_\beta[A] &:= \bigcup_{\alpha<\beta} L_\alpha[A], \text{ if } \beta \text{ is a limit ordinal} \\
L[A] &:= \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha[A]
\end{aligned}$$

This is an inner model of ZFC. Furthermore, if M is any inner model of ZFC such that $A \cap M \in M$, then $L[A] \subseteq M$ (Theorem 13.22 in Jech). There's also another common generalisation of L, denoted $L(A)$:

$$\begin{aligned}
L_0(A) &:= \text{the transitive closure of } \{A\} \\
L_{\alpha+1}(A) &:= \mathrm{def}(L_\alpha) \\
L_\beta(A) &:= \bigcup_{\alpha<\beta} L_\alpha(A), \text{ if } \beta \text{ is a limit ordinal} \\
L(A) &:= \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha(A)
\end{aligned}$$

This is the least inner model of ZF containing A. It is worth noting that $L(A)$ need not satisfy axiom of choice.

The inner model $C(\mathfrak{aa})$ is particularly interesting since it allows the existence of measurable cardinals (contrary to L). A cardinal κ is *measurable* if there's a non-principal, κ-complete ultrafilter on κ. An ultrafilter is called κ-complete if it is closed under intersections of size strictly less than κ. Measurable cardinals are inaccessible, so their existence cannot be proved from ZFC alone. Furthermore, κ is measurable iff there's an elementary embedding $j : V \to M \subseteq V$, such that M is an inner model and κ is the critical point of j, i.e. κ is the least ordinal such that $j(\kappa) > \kappa$. For more information about measurable cardinals, see chapter 10 and 17 in Jech.

Lastly, a remark on club and stationary sets. These are often only defined for regular cardinals, or ordinals of uncountable cofinality. However, especially in Chapter 5, we will need these notions for all limit ordinals. Thus, the definition we will be using is the following:

Definition 1.0.1 (Club and stationary sets)**.** Let $\alpha \geq \omega$ be a limit ordinal. A subset $C \subseteq \alpha$ is a *club* in α if the following two conditions hold:

1. C is unbounded in α, i.e. for every $\beta < \alpha$ there's $\gamma > \beta$ such that $\gamma \in C$.
2. If $\beta < \alpha$ is a limit ordinal such that $\sup(C \cap \beta) = \beta$, then $\beta \in C$.

A subset $S \subseteq \alpha$ is *stationary*, if for all club $C \subseteq \alpha$, $S \cap C \neq \emptyset$.

It is clear that this definition also works in the case that $\text{cf}\ \alpha = \omega$.

Chapter 2

Stationary logic and $C(\mathfrak{aa})$

2.1 Syntax and semantics of stationary logic

A vocabulary τ for stationary logic is a set of constant symbols and relation symbols. To define the syntax of stationary logic, we will need both first-order variables $x, y, z, ...$, and second-order variables $X, Y, Z, ...$. We assume that there's only countable many variables of each kind.

Definition 2.1.1 (Stationary logic formulas)**.** Let τ be a vocabulary for stationary logic. τ-formulas are defined inductively as follows:

1. Atomic formulas: If x, y are first-order variables (or constants), then "$x = y$" is a formula. If X is a second-order variable and x is a first-order variable (or a constant), then "$X(x)$" is a formula. If $R \in \tau$ is an n-ary relation symbol and $(x_0, ..., x_{n-1})$ is an n-tuple of variables (or constants), then "$R(x_0, ..., x_{n-1})$" is a formula.

2. If φ, ψ are formulas and x is a first-order variable, then "$(\varphi \wedge \psi)$", "$\neg\varphi$" and "$\exists x\varphi$" are formulas.

3. If φ is a formula and X is a second-order variable, then "$\mathfrak{aa}X\varphi$" and "$\mathfrak{stat}X\varphi$" are formulas.

We will also use standard abbreviations such as $\forall$, $\vee$, $\rightarrow$ and $\leftrightarrow$. The set of all τ-formulas of stationary logic is denoted $\mathcal{L}_{\mathfrak{aa}}(\tau)$. This is the least set closed under the above operations. Given a τ-structure $\mathcal{M}$, we occasionally write $\mathcal{L}_{\mathfrak{aa}}(\mathcal{M})$ for $\mathcal{L}_{\mathfrak{aa}}(\tau)$. If τ is countable, $\mathcal{L}_{\mathfrak{aa}}(\tau)$ is identified with a subset of ω.

When talking about formulas, we will exclusively use lowercase letters for first-order variables and uppercase letters for second-order variables. The same will be done for parameters.

The following definition is a natural generalisation of club and stationary sets, and it will serve as the basis for the semantics of stationary logic.

Definition 2.1.2. Let κ be a regular cardinal and M a set. Then $C \subseteq \mathcal{P}_\kappa(M)$ is a *club* in $\mathcal{P}_\kappa(M)$ if the following holds:

1. C is closed: For all $\alpha < \kappa$, if $(A_i)_{i<\alpha}$ is an increasing sequence of sets in C, i.e. $A_i \subseteq A_j$ whenever $i < j < \alpha$ and $A_i \in C$ for every $i < \alpha$, then $\cup_{i<\alpha} A_i \in C$.

2. C is unbounded: For every $A \in \mathcal{P}_\kappa(M)$, there's $B \in C$ such that $B \supseteq A$.

$S \subseteq \mathcal{P}_\kappa(M)$ is said to be *stationary* in $\mathcal{P}_\kappa(M)$ if $S \cap C \neq \emptyset$ for every club $C \subseteq \mathcal{P}_\kappa(M)$.

Definition 2.1.3. Let $\mathcal{M}$ be a τ-structure with domain M and $\varphi(X, \overline{x}, \overline{Y}) \in \mathcal{L}_{\mathfrak{aa}}(\tau)$. Then for all $\overline{a} \in M^{|\overline{x}|}$ and $\overline{B} \in \mathcal{P}_{\omega_1}(M)^{|\overline{Y}|}$,

$$\begin{aligned} \mathcal{M} \vDash \mathfrak{aa}X\varphi(X, \overline{a}, \overline{B}) &\iff \{A \in \mathcal{P}_{\omega_1}(M) \mid \mathcal{M} \vDash \varphi(A, \overline{a}, \overline{B})\} \\ &\qquad\qquad \text{contains a club in } \mathcal{P}_{\omega_1}(M) \\ \mathcal{M} \vDash \mathfrak{stat}X\varphi(X, \overline{a}, \overline{B}) &\iff \{A \in \mathcal{P}_{\omega_1}(M) \mid \mathcal{M} \vDash \varphi(A, \overline{a}, \overline{B})\} \\ &\qquad\qquad \text{is stationary in } \mathcal{P}_{\omega_1}(M) \end{aligned}$$

For $a \in M$ and $A \in \mathcal{P}_{\omega_1}(M)$, we put $\mathcal{M} \vDash A(a) \iff a \in A$. The interpretation of first-order connectives and quantifiers is defined in the standard way.

When interpreting a formula in a model $\mathcal{M}$, first-order variables are assigned elements of M, whereas second-order variables are assigned *countable* subsets of M.

Example 2.1.4. The quantifiers $\mathfrak{aa}$ and $\mathfrak{stat}$ have a similar relationship as the existential and universal quantifier in first-order logic. Let $\mathcal{M}$ be any τ-structure, $\varphi(X, \overline{x}, \overline{Y}) \in \mathcal{L}_{\mathfrak{aa}}(\tau)$, $\overline{a} \in M^{|\overline{x}|}$ and $\overline{B} \in \mathcal{P}_{\omega_1}(M)^{|\overline{Y}|}$. Then,

$$\begin{aligned} \mathcal{M} \vDash \neg\mathfrak{aa}X\neg\varphi(X, \overline{a}, \overline{B}) &\iff \{A \in \mathcal{P}_{\omega_1}(M) \mid \mathcal{M} \vDash \neg\varphi(A, \overline{a}, \overline{B})\} \\ &\qquad\qquad \text{does not contain a club in } \mathcal{P}_{\omega_1}(M) \\ &\iff \{A \in \mathcal{P}_{\omega_1}(M) \mid \mathcal{M} \vDash \varphi(A, \overline{a}, \overline{B})\} \\ &\qquad\qquad \text{intersects every club in } \mathcal{P}_{\omega_1}(M) \\ &\iff \mathcal{M} \vDash \mathfrak{stat}X\varphi(X, \overline{a}, \overline{B})\} \end{aligned}$$

So $\mathfrak{stat}$ is definable using $\mathfrak{aa}$. In this thesis, we will mostly be using $\mathfrak{aa}$.

Example 2.1.5. Having cofinality ω is expressible in stationary logic, as witnessed by

$$\psi(x) := \mathfrak{aa}X\forall y(y \in x \to \exists z(y \in z \land z \in x \land X(z)))$$

That is, if α is an ordinal and M is a transitive set containing α, then

$$(M, \in) \vDash \psi(\alpha) \Longleftrightarrow \text{cf } \alpha = \omega \text{ (in } V).$$

To see this, first suppose that $(M, \in) \vDash \psi(\alpha)$. Then there's a club $C \subseteq \mathcal{P}_{\omega_1}(M)$ such that for all $A \in C$, A is cofinal in α. In particular, C is non-empty and A is countable. Thus, α has cofinality ω.

Now assume that cf $\alpha = \omega$. Let $B \in \mathcal{P}_{\omega_1}(M)$ be such that B is cofinal in α and let $C := \{D \in \mathcal{P}_{\omega_1}(M) \mid D \supseteq B\}$. Then C is a club and,

$$C \subseteq \{A \in \mathcal{P}_{\omega_1}(M) \mid (M, \in) \vDash \forall y(y \in \alpha \to \exists z(y \in z \wedge z \in \alpha \wedge A(z)))\}$$

Thus, $(M, \in) \vDash \psi(\alpha)$.

Definition 2.1.6 (μ-club)**.** Suppose that κ and μ are regular cardinals with $\mu < \kappa$. A subset $c \subseteq \kappa$ is called a *μ-club* if the following holds:

1. c is unbounded in κ.
2. If $\alpha < \kappa$ has cofinality μ and $\sup(c \cap \alpha) = \alpha$, then $\alpha \in c$.

Example 2.1.7. Given a regular cardinal $\kappa > \omega$, consider the following formula:

$$\psi(x) := \mathfrak{aa}X(\sup(X \cap \kappa) \in x)$$

Note that we abuse notation here. X is a second order variable whereas κ is a first order parameter. However, we can express that x is in "$X \cap \kappa$" using the formula "$X(x) \wedge x \in \kappa$". Defining supremum of this set is also straightforward, so we omit the details. The point is that the formula expresses that x contains an ω-club of κ. That is, given $a \subseteq \kappa$ and a transitive set M such that $\kappa \in M$, we have that

$$(M, \in) \vDash \psi(a) \Longleftrightarrow a \text{ contains an } \omega\text{-club (in } V)$$

Let's prove this statement. For "$\Rightarrow$", suppose that $(M, \in) \vDash \psi(a)$. Then there's a club C in $\mathcal{P}_{\omega_1}(M)$ such that

$$C \subseteq \{A \in \mathcal{P}_{\omega_1}(M) \mid \sup(A \cap \kappa) \in a\}$$

We claim that $c := \{\sup(A \cap \kappa) \mid A \in C\} \subseteq a$, is an ω-club in κ. Given $\alpha < \kappa$, there's $A \in C$ such that $\alpha \in A$. Hence, $\alpha < \sup(A \cap \kappa) \in c$ and c is unbounded. Now assume that $\alpha_i \in c$, $i < \omega$, is an increasing sequence. Then we have corresponding $A_i \in C$ such that $\alpha_i = \sup(A_i \cap \kappa)$ for all $i < \omega$. Furthermore, $\cup_{i<\omega} A_i \in C$ and $\cup_{i<\omega} \alpha_i = \sup((\cup_{i<\omega} A_i) \cap \kappa)$. Thus, $\cup_{i<\omega} \alpha_i \in c$ and c is closed under increasing ω-sequences.

For the other direction, assume that $c \subseteq a$ is an ω-club. We show that $C := \{A \in \mathcal{P}_{\omega_1}(M) \mid \sup(A \cap \kappa) \in c\}$ is a club in $\mathcal{P}_{\omega_1}(M)$. Given $B \in \mathcal{P}_{\omega_1}(M)$, there's $\alpha > \sup(B \cap \kappa)$ such that $\alpha \in c$. Take any $A \in \mathcal{P}_{\omega_1}(M)$ with $A \supseteq B$ and $\sup(A \cap \kappa) = \alpha$. Then $A \in C$, so C is unbounded. Now suppose that $(A_i)_{i<\omega}$, is an increasing sequence of sets in C. Then $\cup_{i<\omega} A_i \in C$ since $\sup((\cup_{i<\omega} A_i) \cap \kappa) \in c$. Therefore, C is a club.

In Chapter 4, we will need a concept of relativisation for $\mathfrak{aa}$-formulas.

Definition 2.1.8. Let τ and σ be vocabularies such that $\sigma \subseteq \tau$. Let $\mathcal{M}$ be a τ-structure and $\mathcal{N}$ a σ-structure and suppose that $\mathcal{N}$ is a submodel of $\mathcal{M}$ (restricted to σ) with $\mathcal{N} \in \mathcal{M}$. Furthermore, suppose that σ contains $\in$, with its natural interpretation. Let $\varphi \in \mathcal{L}_{\mathfrak{aa}}(\sigma)$. We define the *relativisation* of φ to $\mathcal{N}$, $\varphi^{\mathcal{N}}$, as follows:

1. $(x = y)^{\mathcal{N}} := (x = y)$.
2. If $R \in \sigma$ is a relation symbol, then $(R(\overline{x}))^{\mathcal{N}} := R(\overline{x})$.
3. If X is a second-order variable and x is first-order, then $(X(x))^{\mathcal{N}} := X(x) \wedge x \in N$.
4. $(\varphi \wedge \psi)^{\mathcal{N}} := \varphi^{\mathcal{N}} \wedge \psi^{\mathcal{N}}$.
5. $(\neg\varphi)^{\mathcal{N}} := \neg\varphi^{\mathcal{N}}$.
6. $(\exists x\varphi(x))^{\mathcal{N}} := \exists x(x \in N \wedge \varphi^{\mathcal{N}}(x))$.
7. $(\mathfrak{aa}X\varphi(X))^{\mathcal{N}} := \mathfrak{aa}X(\varphi^{\mathcal{N}}(X))$.

Lemma 2.1.9. *Let $\mathcal{M}$ and $\mathcal{N}$ be as in definition 2.1.8. Let $\varphi(\overline{x}, \overline{Y}) \in \mathcal{L}_{\mathfrak{aa}}(\sigma)$, $\overline{a} \in N^{|\overline{x}|}$ and $\overline{A} \in \mathcal{P}_{\omega_1}(M)^{|\overline{Y}|}$. Then,*

$$\mathcal{M} \vDash \varphi^{\mathcal{N}}(\overline{a}, \overline{A}) \Longleftrightarrow \mathcal{N} \vDash \varphi(\overline{a}, \overline{A \cap N})$$

Proof. The proof is by induction on the structure of φ. If $\varphi(x) = X(x)$, then for all $A \in \mathcal{P}_{\omega_1}(M)$ and $a \in N$,

$$\mathcal{M} \vDash (A(a))^{\mathcal{N}} \Longleftrightarrow \mathcal{M} \vDash A(a) \wedge a \in N \Longleftrightarrow \mathcal{N} \vDash (A \cap N)(a)$$

Here, "$(A \cap N)(a)$" means that $a \in A \cap N$, but we want to distinguish between first-order and second-order parameters. The other atomic cases are straightforward. Now suppose that $\varphi = (\psi \wedge \chi)$, $\overline{a} \in N^{<\omega}$ and $\overline{A} \in \mathcal{P}_{\omega_1}(M)^{<\omega}$. Then,

$$\begin{aligned}
\mathcal{M} \vDash \varphi^{\mathcal{N}}(\overline{a}, \overline{A}) &\Longleftrightarrow \mathcal{M} \vDash \psi^{\mathcal{N}}(\overline{a}, \overline{A}) \wedge \chi^{\mathcal{N}}(\overline{a}, \overline{A}) \\
&\Longleftrightarrow \mathcal{M} \vDash \psi^{\mathcal{N}}(\overline{a}, \overline{A}) \text{ and } \mathcal{M} \vDash \chi^{\mathcal{N}}(\overline{a}, \overline{A}) \\
&\Longleftrightarrow \mathcal{N} \vDash \psi(\overline{a}, \overline{A \cap N}) \text{ and } \mathcal{N} \vDash \chi(\overline{a}, \overline{A \cap N}) \qquad \text{(I.A.)} \\
&\Longleftrightarrow \mathcal{N} \vDash \varphi(\overline{a}, \overline{A \cap N})
\end{aligned}$$

The induction-step for negation is identical. Thus, we look at the induction-step for the existential quantifier. Assume that $\varphi = \exists y\psi(y)$, $\overline{a} \in N^{<\omega}$ and

$\overline{A} \in \mathcal{P}_{\omega_1}(M)^{<\omega}$. Then,

$$\begin{aligned}
\mathcal{M} \models \varphi^{\mathcal{N}}(\overline{a}, \overline{A}) &\iff \mathcal{M} \models \exists y (y \in N \wedge \psi^{\mathcal{N}}(y, \overline{a}, \overline{A})) \\
&\iff \text{there's } b \in N \text{ such that } \mathcal{M} \models \psi^{\mathcal{N}}(b, \overline{a}, \overline{A}) \\
&\iff \text{there's } b \in N \text{ such that } \mathcal{N} \models \psi(b, \overline{a}, \overline{A \cap N}) \qquad \text{(I.A.)} \\
&\iff \mathcal{N} \models \exists y \psi(y, \overline{a}, \overline{A \cap N}) \\
&\iff \mathcal{N} \models \varphi(\overline{a}, \overline{A \cap N})
\end{aligned}$$

Lastly, we do the induction-step for the $\mathfrak{aa}$-quantifier. Suppose that $\varphi = \mathfrak{aa} X \psi(X)$, $\overline{a} \in N^{<\omega}$ and $\overline{A} \in \mathcal{P}_{\omega_1}(M)^{<\omega}$. Now,

$$\begin{aligned}
\mathcal{M} \models \varphi^{\mathcal{N}}(\overline{a}, \overline{A}) &\iff \mathcal{M} \models \mathfrak{aa} X (\psi^{\mathcal{N}}(X, \overline{a}, \overline{A})) \\
&\iff \{B \in \mathcal{P}_{\omega_1}(M) \mid \mathcal{M} \models \psi^{\mathcal{N}}(B, \overline{a}, \overline{A})\} \\
&\qquad\quad \text{contains a club in } \mathcal{P}_{\omega_1}(M) \\
&\iff \{B \in \mathcal{P}_{\omega_1}(M) \mid \mathcal{N} \models \psi(B \cap N, \overline{a}, \overline{A \cap N})\} \\
&\qquad\quad \text{contains a club in } \mathcal{P}_{\omega_1}(M) \qquad \text{(I.A.)} \\
&\iff \{B \in \mathcal{P}_{\omega_1}(N) \mid \mathcal{N} \models \psi(B, \overline{a}, \overline{A \cap N})\} \\
&\qquad\quad \text{contains a club in } \mathcal{P}_{\omega_1}(N) \qquad (\star, \text{ see below.}) \\
&\iff \mathcal{N} \models \mathfrak{aa} X \psi(X, \overline{a}, \overline{A \cap N}) \\
&\iff \mathcal{N} \models \varphi(\overline{a}, \overline{A \cap N})
\end{aligned}$$

The step $(\star)$ needs some justification. For "$\Rightarrow$", suppose that $C \subseteq \mathcal{P}_{\omega_1}(M)$ is a club of sets B such that $\mathcal{N} \models \psi(B \cap N, \overline{a}, \overline{A \cap N})$. Let $D = \{B \cap N \mid B \in C\}$. Clearly, for every $B' \in D$, $\mathcal{N} \models \psi(B', \overline{a}, \overline{A \cap N})$ holds. It is also easy to see that D is a club in $\mathcal{P}_{\omega_1}(N)$.

For "$\Leftarrow$", suppose that $C \subseteq \mathcal{P}_{\omega_1}(N)$ is a club of sets B such that $\mathcal{N} \models \psi(B, \overline{a}, \overline{A \cap N})$ for all $B \in C$. Let $D = \{B' \in \mathcal{P}_{\omega_1}(M) \mid B' \cap N \in C\}$. Then for all $B' \in D$, $\mathcal{N} \models \psi(B' \cap N, \overline{a}, \overline{A \cap N})$. Again, it is straightforward to check that D is a club in $\mathcal{P}_{\omega_1}(M)$. □

One can extend a Hilbert-style deductive system for first-order logic with the following axioms:

A0. $\mathfrak{aa} X \varphi(X) \leftrightarrow \mathfrak{aa} Y \varphi(Y)$

A1. $\neg \mathfrak{aa} X \bot$

A2. $\mathfrak{aa} X (X(x))$, $\mathfrak{aa} X \forall x (Y(x) \to X(x))$

A3. $(\mathfrak{aa} X \varphi(X) \wedge \mathfrak{aa} X \psi(X)) \to \mathfrak{aa} X (\varphi(X) \wedge \psi(X))$

A4. $\mathfrak{aa} X (\varphi(X) \to \psi(X)) \to (\mathfrak{aa} X \varphi(X) \to \mathfrak{aa} X \psi(X))$

A5. $\forall x \mathfrak{aa} X \varphi(X, x) \to \mathfrak{aa} X \forall x (X(x) \to \varphi(X, x))$

The result is a sound and complete proof system for stationary logic, see [1] for details. These axioms will be needed in chapter 4, where we are dealing with theories in stationary logic.

2.2 The Quine-Rosser pairing function

The object $\varphi(\overline{a}) = \varphi(a_0, ..., a_{n-1})$ is identified with the tuple $(\varphi, a_0, ..., a_{n-1})$. If we use the Kuratowski definition of ordered pairs, then this object will have a higher rank in the cumulative hierarchy than all the coordinates of $\overline{a}$. This will be a problem when constructing $C(\mathfrak{aa})$. Kennedy, Magidor and Väänänen solved this issue in [9] by using the Jensen hierarchy, whose levels are closed under the Kuratowski ordered pairs. However, we will be using a different approach, following Goldberg and Steel [4]. The idea is to use a more complicated pairing function, one that doesn't change the rank of objects with rank $\geq \omega$. The tradeoff is that the construction of $C(\mathfrak{aa})$ becomes more intuitive. Thus, we introduce the *Quine-Rosser pairing function.*[1]

First, for each $n < \omega$ and $x \in V$, let

$$\sigma_n(x) := \begin{cases} x + n & \text{if } x \in \omega \\ x & \text{otherwise} \end{cases}$$

Then, for $a_0, ..., a_{n-1} \in V$, define

$$\mathcal{QR}(a_0, ..., a_{n-1}) := \bigcup_{m<n} \{\sigma_n[X] \cup \{m\} \mid X \in a_m\}$$

Lemma 2.2.1. *For $a_0, ..., a_{n-1}, b_0, ..., b_{n-1} \in V$,*

$$\mathcal{QR}(a_0, ..., a_{n-1}) = \mathcal{QR}(b_0, ..., b_{n-1}) \Longleftrightarrow a_m = b_m \textit{ for all } m < n.$$

Proof. The direction from left to right is the nontrivial case, so assume that $\mathcal{QR}(a_0, ..., a_{n-1}) = \mathcal{QR}(b_0, ..., b_{n-1})$. We show that $a_m = b_m$ for all $m < n$. Given $X \in a_m$, we have that $\sigma_n[X] \cup \{m\} \in \mathcal{QR}(b_0, ..., b_{n-1})$. Now, $\sigma_n[X] \cup \{m\} = \sigma_n[X'] \cup \{m\}$ for some $X' \in b_m$. But σ_n is injective, so $X = X'$. Thus, $X \in b_m$ and $a_m \subseteq b_m$. Similarly, we get that $b_m \subseteq a_m$. $\square$

Lemma 2.2.2. *If $\alpha \geq \omega$, $a \subseteq V_\alpha$ and $m < n$, then $\{\sigma_n[X] \cup \{m\} \mid X \in a\} \subseteq V_\alpha$.*

Proof. If $X \in a$, then $X \subseteq V_\beta$ for some $\beta < \alpha$. If β is infinite, then $\sigma_n[X] \cup \{m\} \subseteq V_\beta$. Otherwise if $\beta < \omega$, then $\sigma_n[X] \cup \{m\} \subseteq V_N$ for some $N < \omega$, so $\sigma_n[X] \cup \{m\} \in V_\omega$. In either case, $\{\sigma_n[X] \cup \{m\} \mid X \in a\} \subseteq V_\alpha$. $\square$

[1] The construction presented here is a generalisation of the pairing functions introduced in [12] and [13].

Corollary 2.2.3. *If $\alpha \geq \omega$ and $a_0, ..., a_{n-1} \in V_\alpha$, then $\mathcal{QR}(a_0, ..., a_{n-1}) \in V_\alpha$.*

Proof. Let $\beta < \alpha$ be the least β such that $a_0, ..., a_{n-1} \subseteq V_\beta$. Then by Lemma 2.2.2, $\mathcal{QR}(a_0, ..., a_{n-1}) \subseteq V_\beta$. Thus, $\mathcal{QR}(a_0, ..., a_{n-1}) \in V_\alpha$. $\square$

From now on, we identify the n-tuple $(a_0, ..., a_{n-1})$ with $\mathcal{QR}(a_0, ..., a_{n-1})$. Furthermore, if M is closed under $\mathcal{QR}$, then by M^n we mean the first-order definable subset of M consisting precisely of the n-tuples of M.

2.3 Constructing $C(\mathfrak{aa})$

With $\mathrm{def}_{\mathfrak{aa}}(\mathcal{M})$, we mean the set of all $X \subseteq M$, such that there's $\varphi(x, \overline{y}) \in \mathcal{L}_{\mathfrak{aa}}(\mathcal{M})$ and $\overline{a} \in M^{<\omega}$ with $X = \{b \in M \mid \mathcal{M} \models \varphi(b, \overline{a})\}$. A subset $X \subseteq M$ is said to be $\mathfrak{aa}$-*definable* or *definable in stationary logic*, if $X \in \mathrm{def}_{\mathfrak{aa}}(\mathcal{M})$. Clearly, if X is definable in first-order logic (from parameters), then it is $\mathfrak{aa}$-definable.

Definition 2.3.1. The $\mathfrak{aa}$-constructible hierarchy is defined by recursion as follows:

$$\begin{aligned}
&C_\omega := V_\omega \\
&C_{\alpha+1} := \mathrm{def}_{\mathfrak{aa}}(\mathfrak{C}_\alpha), \text{ if } \alpha \text{ is an infinite ordinal} \\
&C_\gamma := \bigcup_{\alpha<\gamma} C_\alpha, \text{ if } \alpha \text{ is a limit ordinal} > \omega \\
&\mathfrak{C}_\alpha := (C_\alpha, \in, T_\alpha),\ \alpha \geq \omega \\
&T_\alpha := \{(\beta, \varphi(\overline{a})) \mid \beta < \alpha,\ \overline{a} \in C_\beta^{<\omega},\ \varphi \in \mathcal{L}_{\mathfrak{aa}}(\in, T) \text{ and } \mathfrak{C}_\beta \models \varphi(\overline{a})\}
\end{aligned}$$

Then $C(\mathfrak{aa}) := \bigcup_{\alpha\geq\omega} C_\alpha$. The *levels* $\langle \mathfrak{C}_\alpha \mid \alpha \geq \omega \rangle$ of $C(\mathfrak{aa})$ are structures in vocabulary $\{\in, T\}$, where T is a binary relation symbol and for each $\alpha \geq \omega$, $T^{\mathfrak{C}_\alpha} = T_\alpha$.

Note that $T_\omega = \emptyset$ since C_α is only defined for $\alpha \geq \omega$. If $C(\mathfrak{aa})$ is well-defined, then it is a class by the Recursion Theorem. For this, we need to check that T_α actually is a predicate on C_α, i.e. that $(\beta, \varphi(\bar{a})) \in C_\alpha$ whenever $\beta < \alpha$, $\varphi \in \mathcal{L}_{\mathfrak{aa}}(\in, T)$ and $\overline{a} \in C_\alpha^{<\omega}$.

Lemma 2.3.2. *For each $\alpha \geq \omega$, the definition of $\mathfrak{C}_\alpha$ is well-defined, in the sense that $\mathfrak{C}_\alpha$ is closed under $\mathcal{QR}$. That is, $C_\alpha^{<\omega} \subseteq C_\alpha$.*

Proof. For $\alpha = \omega$, the statement follows from Corollary 2.2.3. Now, suppose that $C_\beta^{<\omega} \subseteq C_\beta$ for all $\beta < \alpha$. Then C_α is well-defined and we want to prove that $C_\alpha^{<\omega} \subseteq C_\alpha$. If α is a limit ordinal, then it is immediate by the induction assumption. Thus, assume that $\alpha = \beta + 1$ for some ordinal β. We need to show that if $a_0, ..., a_{n-1} \subseteq C_\beta$ are $\mathfrak{aa}$-definable, then $\mathcal{QR}(a_0, ..., a_{n-1})$ is a subset of C_β and is $\mathfrak{aa}$-definable. It is sufficient to show that if $a \subseteq C_\beta$ is $\mathfrak{aa}$-definable, then $\{\sigma_n[X] \cup \{m\} \mid X \in a\}$ is an $\mathfrak{aa}$-definable subset of C_β for any $m < n$. It is easy to see that *if* this is a subset of C_β, then it is $\mathfrak{aa}$-definable provided that a is $\mathfrak{aa}$-definable.

Thus, everything boils down to proving this: If $X \in C_\beta$, then $\sigma_n[X] \cup \{m\} \in C_\beta$. If $\beta = \omega$, then this is trivial. If β is a successor ordinal, say $\beta = \gamma + 1$, then $X \subseteq C_\gamma$ and X is $\mathfrak{aa}$-definable. Since $\omega \subseteq C_\gamma$, it follows that $\sigma_n[X] \cup \{m\}$ is a subset of C_γ and is $\mathfrak{aa}$-definable. Thus, $\sigma_n[X] \cup \{m\} \in C_\beta$.

If β is a limit ordinal $> \omega$, pick $\gamma < \beta$ such that $X \subseteq C_\gamma$ and X is $\mathfrak{aa}$-definable. Again, since $\omega \subseteq C_\gamma$, it follows that $\sigma_n[X] \cup \{m\}$ is a subset of C_γ and is $\mathfrak{aa}$-definable. Therefore, $\sigma_n[X] \cup \{m\} \in C_{\gamma+1}$, and so $\sigma_n[X] \cup \{m\} \in C_\beta$. $\square$

The following properties can be proved almost verbatim as for L (See e.g. Theorem 13.3 in Jech [6]):

Theorem 2.3.3. *Let $\beta \leq \alpha$ be infinite ordinals. Then,*

1. *C_α is transitive, $C_\beta \in C_\alpha$ and $C_\beta \subseteq C_\alpha$.*
2. *$\alpha \subseteq C_\alpha$ and $|C_\alpha| = |\alpha|$.*
3. *$C(\mathfrak{aa})$ is an inner model of* ZF.

2.4 The canonical well-ordering of $C(\mathfrak{aa})$

Our goal is to construct a well-ordering of $C(\mathfrak{aa})$. We also want that it is definable in a proper class of $\mathfrak{C}_\alpha$:s. Let $\mathfrak{C}_\beta \in C_\alpha$, $\beta < \alpha$. If we define the $\mathfrak{aa}$-satisfaction predicate within $\mathfrak{C}_\alpha$, then the formulas that are true in $\mathfrak{C}_\beta$ according to $\mathfrak{C}_\alpha$ may differ from the formulas that are true in $\mathfrak{C}_\beta$ according to V. However, using the predicate T_α, $\mathfrak{C}_\alpha$ has an "oracle" for the $\mathfrak{aa}$-satisfaction relation in V. That is, for each $\varphi \in \mathcal{L}_{\mathfrak{aa}}(\in, T)$ and $\overline{a} \in C_\beta$, we have that:

$$\mathfrak{C}_\beta \models \varphi(\overline{a}) \iff \mathfrak{C}_\alpha \models T_\alpha(\beta, \varphi(\overline{a}))$$

Lemma 2.4.1. *For all ordinals $\alpha \geq \omega$, the set $\{(\beta, \mathfrak{C}_\beta) \mid \beta < \alpha\}$ is definable over $\mathfrak{C}_\alpha$ and the same as in V.*

Proof. We prove this by induction on α. For $\alpha = \omega$ it is clear, since then the set is empty. Suppose that α is a successor ordinal, say $\alpha = \beta + 1$. Then by the induction assumption, $\{(\gamma, \mathfrak{C}_\gamma) \mid \gamma < \beta\}$ is definable over $\mathfrak{C}_\beta$. Thus, $\{(\gamma, \mathfrak{C}_\gamma) \mid \gamma < \beta\} \in C_\alpha$. Furthermore, $C_\beta \in C_\alpha$ and since $T_\beta \subseteq C_\beta$ is definable over $\mathfrak{C}_\beta$, it follows that $T_\beta \in C_\alpha$. Thus, $(\beta, \mathfrak{C}_\beta) \in C_\alpha$. Therefore,

$$\{(\gamma, \mathfrak{C}_\gamma) \mid \gamma < \alpha\} = \{(\gamma, \mathfrak{C}_\gamma) \mid \gamma < \beta\} \cup \{(\beta, \mathfrak{C}_\beta)\} \subseteq C_\alpha$$

is definable over $\mathfrak{C}_\alpha$. Now assume that α is a limit ordinal. Given $\beta < \alpha$, we can construct a first-order[2] formula $\psi(y, z)$, such that for all $c \in C_\alpha$, $\mathfrak{C}_\alpha \models \psi(c, C_\beta)$ iff $c = C_{\beta+1}$:

$$\begin{aligned}\psi(y,z) := \forall x[x \in y \leftrightarrow [x \subseteq z \wedge \exists \varphi \in \mathcal{L}_{\mathfrak{aa}}(\in, T) \\ \exists \overline{b} \in z^{<\omega} \forall a(a \in x \leftrightarrow T_\alpha(\beta, \varphi(a, \overline{b}))))]]\end{aligned}$$

Let $A = \{(\beta, \mathfrak{C}_\beta) \mid \beta < \alpha\} \subseteq C_\alpha$. We want to show that A is $\mathfrak{aa}$-definable over C_α. For a pair $(\gamma, c) \in C_\alpha$, $(\gamma, c) \in A$ iff $\mathfrak{C}_\alpha$ believes there's a function F with the following properties:

1. The domain of F is $\delta \setminus \omega$ for some ordinal $\delta > \gamma$.
2. $F(\omega) = V_\omega$.
3. For all $\beta \in \delta \setminus \omega$, $F(\beta + 1)$ is the unique d such that $\psi(d, F(\beta))$ holds.
4. If $\beta \in \delta \setminus \omega$ is a limit ordinal, then $F(\beta) = \bigcup_{\nu<\beta} F(\nu)$.

This is clearly expressible in first-order logic. By the induction assumption, for every ordinal $\gamma < \alpha$, we can find such a function within C_α. Thus, A is definable over $\mathfrak{C}_\alpha$. $\square$

Let $<^*$ be a well-ordering of $\mathcal{L}_{\mathfrak{aa}}(\in, T)$. Recall that we identify $\mathcal{L}_{\mathfrak{aa}}(\in, T)$ with a subset of ω, so we can just use the natural ordering of ω to be our $<^*$. For each $X \in C(\mathfrak{aa})$, let

$$\mathrm{rk}(X) := \text{ the least ordinal } \alpha \text{ such that } X \in C_\alpha$$

Note that if $\mathrm{rk}(X) > \omega$, then it is a successor ordinal. Furthermore, let

$$\begin{aligned}\mathrm{fm}(X) := \text{the } <^*\text{-least formula } \varphi(x, \overline{y}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T) \text{ such that for some} \\ \overline{a} \in C^{<\omega}_{\mathrm{rk}(X)-1},\ X = \{b \in C_{\mathrm{rk}(X)-1} \mid \mathfrak{C}_{\mathrm{rk}(X)-1} \models \varphi(b, \overline{a})\}\end{aligned}$$

If $\gamma > \omega$ is a limit ordinal, then rk is well-defined and absolute over $\mathfrak{C}_\gamma$. By the above remarks, if we replace "$\mathfrak{C}_{\mathrm{rk}(X)-1} \models \varphi(b, \overline{a})$" with $T_\gamma(\mathrm{rk}(X) - 1, \varphi(b, \overline{a}))$ in the definition of fm, then also fm is absolute over $\mathfrak{C}_\gamma$.

Now, define a well-ordering $<_{\mathfrak{aa}}$ of $C(\mathfrak{aa})$ as follows: For $X, Y \in C(\mathfrak{aa})$, let $X <_{\mathfrak{aa}} Y$ iff

1. $X, Y \in V_\omega$ and $X <_L Y$, or[3]
2. $\mathrm{rk}(X) < \mathrm{rk}(Y)$, or
3. $\mathrm{rk}(X) = \mathrm{rk}(Y)$ and $\mathrm{fm}(X) <^* \mathrm{fm}(Y)$, or

[2]The formula has defined notions but clearly there's a "real" first-order formula expressing the same thing.

[3]$<_L$ being the canonical well-ordering of L.

4. $\mathrm{rk}(X) = \mathrm{rk}(Y)$, $\mathrm{fm}(X) = \mathrm{fm}(Y) = \varphi$, and there's $\overline{a} \in C^{<\omega}_{\mathrm{rk}(X)-1}$ such that:
   (a) $X = \{b \in C_{\mathrm{rk}(X)-1} \mid \mathfrak{C}_{\mathrm{rk}(X)-1} \vDash \varphi(b, \overline{a})\}$,
   (b) and for all $\overline{c} \in C^{<\omega}_{\mathrm{rk}(X)-1}$ such that $Y = \{b \in C_{\mathrm{rk}(X)-1} \mid \mathfrak{C}_{\mathrm{rk}(X)-1} \vDash \varphi(b, \overline{c})\}$, we have that $\overline{a} <_{\mathfrak{aa}} \overline{c}$.

This is a well-ordering since $<_L$ and $<^*$ are well-orderings, and the image of both fm and rk are well-ordered. The following Theorem is immediate by the construction:

Theorem 2.4.2. *Let $\gamma > \omega$ be a limit ordinal, $X, Y \in C(\mathfrak{aa})$ and $Y \in C_\gamma$. Then $X <_{\mathfrak{aa}} Y \iff X \in C_\gamma$ and $\mathfrak{C}_\gamma \vDash X <_{\mathfrak{aa}} Y$.*

Theorem 2.4.3. $C(\mathfrak{aa})$ *satisfies axiom of choice.*

Proof. Given $a \in C(\mathfrak{aa})$, let α be a limit ordinal such that $a \in C_\alpha$. Since C_α is transitive, a is also a subset of C_α. Now, $<_{\mathfrak{aa}}$ is definable over C_α, so $<_{\mathfrak{aa}}\restriction a$ is a definable subset of C_α. Hence, $<_{\mathfrak{aa}}\restriction a \in C_{\alpha+1}$, and so

$$\mathfrak{C}_{\alpha+1} \vDash \text{“There's a well-ordering of } a\text{.”}$$

It follows that every set in $C(\mathfrak{aa})$ can be well-ordered within $C(\mathfrak{aa})$. Thus, $C(\mathfrak{aa})$ satisfies axiom of choice. □

In the next section, we will need the following Lemma:

Lemma 2.4.4. *For each $\alpha \geq \omega$, there's a bijection $f : C_\alpha \to \alpha$ such that $f \in L(\mathfrak{C}_\alpha)$.*

Proof. Recall that $L(\mathfrak{C}_\alpha)$ is the least inner model of ZF containing $\mathfrak{C}_\alpha$. Note that if $\omega \leq \alpha < \beta$, then $L(\mathfrak{C}_\alpha) \subseteq L(\mathfrak{C}_\beta)$. We prove the claim by induction on α. For $\alpha = \omega$, it is clear, since $L(\mathfrak{C}_\omega)$ contains a bijection $f : V_\omega \to \omega$.

Suppose that $\alpha = \beta + 1$. Then by the induction assumption there's $g \in L(\mathfrak{C}_\beta)$ such that $g : C_\beta \to \beta$ is a bijection. Hence, it is sufficient to find a bijection $f : C_\alpha \to C_\beta$ with $f \in L(\mathfrak{C}_\alpha)$. The Cantor-Bernstein Theorem does not require axiom of choice, so it is enough to show that

$$L(\mathfrak{C}_\alpha) \vDash |C_\beta| \leq |C_\alpha| \wedge |C_\alpha| \leq |C_\beta|.$$

The inclusion map witnesses that $L(\mathfrak{C}_\alpha) \vDash |C_\beta| \leq |C_\alpha|$. Define $f : C_\alpha \to C_\beta$ as follows: Let $f(b)$ be the g-least pair $(\varphi, \overline{a})$ such that for all $c \in C_\alpha$, $c \in b$ iff $\mathfrak{C}_\alpha \vDash T_\alpha(\beta, \varphi(c, \overline{a}))$. Here, “$g$-least” means least in the well-ordering induced by $g : C_\beta \to \beta$. Then, f is a well-defined injection and $f \in L(\mathfrak{C}_\alpha)$.

Now assume that α is a limit ordinal. We want to find a bijection from C_α onto α in $L(\mathfrak{C}_\alpha)$. It is sufficient to find an injection $f : C_\alpha \to \alpha \times \alpha$ in $L(\mathfrak{C}_\alpha)$, since the axiom of choice is not needed to prove that $|\alpha \times \alpha| = |\alpha|$.

For each $\beta < \alpha$, by the induction assumption $L(\mathfrak{C}_\beta)$ contains a bijection $f : C_\beta \to \beta$. This bijection can be used to construct a well-ordering of the transitive closure of $\{\mathfrak{C}_\beta\}$. Such a well-ordering can be extended to a well-ordering of $L(\mathfrak{C}_\beta)$ (so $L(\mathfrak{C}_\beta)$ does in fact satisfy axiom of choice), and we may assume that there's a canonical well-ordering $<_\beta$ of $L(\mathfrak{C}_\beta)$.

Define $f : C_\alpha \to \alpha \times \alpha$ as follows: Let $f(b) = (\beta, f_\beta(b))$, where β is the least ordinal such that $b \in C_\beta$ and f_β is the $<_\beta$-least bijection from C_β onto β. It is easy to see that f is an injection and that $f \in L(\mathfrak{C}_\alpha)$. $\square$

Chapter 3

Club determinacy

3.1 Basic properties of $C(\mathfrak{aa})$

Definition 3.1.1. A τ structure $\mathcal{M}$ is said to be *club determined* if for all $\varphi(X,\overline{x},\overline{Y}) \in \mathcal{L}_{\mathfrak{aa}}(\tau)$, $\overline{a} \in M^{|\overline{x}|}$ and $\overline{B} \in \mathcal{P}_{\omega_1}(M)^{|\overline{Y}|}$, we have that

$$\mathcal{M} \models \mathfrak{aa}X\varphi(X,\overline{a},\overline{B}) \text{ or } \mathcal{M} \models \mathfrak{aa}X\neg\varphi(X,\overline{a},\overline{B})$$

If for every infinite ordinal α, $\mathfrak{C}_\alpha$ is club determined, then we say that $C(\mathfrak{aa})$ is *club determined*, or satisfies *club determinacy.*

Notice that if $\mathcal{M}$ is club determined, then $\mathcal{M} \models \neg\mathfrak{aa}X\varphi(X,\overline{a},\overline{B})$ iff $\mathcal{M} \models \mathfrak{aa}X\neg\varphi(X,\overline{a},\overline{B})$.

Example 3.1.2. Suppose that $\mathcal{M}$ is a club determined structure. Then for all $\varphi(X,\overline{x},\overline{Y}) \in \mathcal{L}_{\mathfrak{aa}}(\tau)$, $\overline{a} \in M^{|\overline{x}|}$ and $\overline{B} \in \mathcal{P}_{\omega_1}(M)^{|\overline{Y}|}$,

$$\begin{aligned} \mathcal{M} \models \mathfrak{stat}X\varphi(X,\overline{a},\overline{B}) &\iff \mathcal{M} \models \neg\mathfrak{aa}X\neg\varphi(X,\overline{a},\overline{B}) \\ &\iff \mathcal{M} \models \mathfrak{aa}X\neg\neg\varphi(X,\overline{a},\overline{B}) \qquad \text{(club determinacy)} \\ &\iff \mathcal{M} \models \mathfrak{aa}X\varphi(X,\overline{a},\overline{B}) \end{aligned}$$

So, the quantifiers $\mathfrak{aa}$ and $\mathfrak{stat}$ coincide on club determined structures.

Theorem 3.1.3 (Kennedy, Magidor and Väänänen, [9])**.** *If $C(\mathfrak{aa})$ is club determined, then every regular $\kappa \geq \omega_1$ in V is a measurable cardinal in $C(\mathfrak{aa})$.*

Proof. Let $\kappa > \omega$ be regular and choose γ such that $\mathcal{P}(\kappa)^{C(\mathfrak{aa})} \in \mathfrak{C}_\gamma$. Consider the following filter on κ:

$$\begin{aligned} \mathcal{F} &= \{a \subseteq \kappa \mid a \in C(\mathfrak{aa}) \text{ and } a \text{ contains an } \omega\text{-club in } V\} \\ &= \{a \subseteq \kappa \mid a \in \mathfrak{C}_\gamma \wedge \mathfrak{C}_\gamma \models \mathfrak{aa}X(\sup(X \cap \kappa) \in a)\} \end{aligned}$$

This filter is κ-complete. Let $\lambda < \kappa$ and suppose that $a_i \in \mathcal{F}$ for each $i < \lambda$. Then there's an ω-club $c_i \subseteq a_i$ for every $i < \lambda$. Now, $\cap_{i<\lambda} c_i$ is still an ω-club and $\cap_{i<\lambda} c_i \subseteq \cap_{i<\lambda} a_i$. Thus, $\cap_{i<\lambda} a_i \in \mathcal{F}$.

Club determinacy implies that this filter is an ultrafilter in $C(\mathfrak{aa})$. To see this, suppose that $a \subseteq \kappa$, $a \in C(\mathfrak{aa})$ and $a \notin \mathcal{F}$. Then,

$$\mathfrak{C}_\gamma \models \neg \mathfrak{aa} X(\sup(X \cap \kappa) \in a)$$

So by club determinacy,

$$\mathfrak{C}_\gamma \models \mathfrak{aa} X(\sup(X \cap \kappa) \notin a)$$

Hence, $\kappa \setminus a \in \mathcal{F}$. □

Corollary 3.1.4. *If $V = L$, then $C(\mathfrak{aa})$ is not club determined.*

Proof. If $V = L$, then $C(\mathfrak{aa}) = L$ as L is the smallest inner model. Thus, $C(\mathfrak{aa})$ does not have a measurable cardinal, so it cannot be club determined by the previous Theorem. □

Suppose that $\mathbb{P}$ is a forcing notion and G is a $\mathbb{P}$-generic filter over V. When we write something like $C(\mathfrak{aa})^V = C(\mathfrak{aa})^{V[G]}$, what we mean is that the definition of $C(\mathfrak{aa})$ inside $V[G]$ coincide with the definition of $C(\mathfrak{aa})$ inside $V \subseteq V[G]$.[1] For the proof of Theorem 3.1.6, we will need the following Lemma:

Lemma 3.1.5 ([6], Lemma 31.3.)**.** *Let λ be an uncountable cardinal, $\mathbb{P}$ be a countably closed forcing notion, G a $\mathbb{P}$-generic filter over V and $S \subseteq \mathcal{P}_{\omega_1}(\lambda)$ stationary. Then S is stationary also in $V[G]$.*

Note that this also holds if λ is countable, since then S is stationary iff $\lambda \in S$. Furthermore, note that we may replace λ with any set M, and the statement of the Lemma still holds. We can just fix a bijection $f : M \to |M|$, and this bijection will preserve stationarity. That is, $S \subseteq \mathcal{P}_{\omega_1}(M)$ is stationary iff $\{f[A] \mid A \in S\} \subseteq \mathcal{P}_{\omega_1}(|M|)$ is stationary. Lastly, note that the converse of the Lemma is also true, since $V \subseteq V[G]$.

Theorem 3.1.6 (Kennedy, Magidor and Väänänen, [9])**.** *Let $\mathbb{P}$ be a countably closed forcing notion, and let G be $\mathbb{P}$-generic over V. Then $C(\mathfrak{aa})^V = C(\mathfrak{aa})^{V[G]}$.*

[1] Formally this means the following: Let $\psi(x, y)$ be a formula of set theory that defines the class function $\{(\alpha, C_\alpha) \mid \alpha \geq \omega\}$. Furthermore, let $p \in G$ be a condition such that $p \Vdash \exists x \exists y(\psi(x, y) \wedge \tau \in y)$ for some $\mathbb{P}$-name τ. Then there's $q \in G$ and $a \in V$ such that $q \Vdash \tau = \check{a}$.

Proof. Let $\mathbb{P}$ and G be as in the Theorem. By induction on $\alpha \geq \omega$, we will prove that $\mathfrak{C}_\alpha^V = \mathfrak{C}_\alpha^{V[G]}$. If $\alpha = \omega$, then trivially $V_\omega^V = V_\omega^{V[G]}$. Now, suppose that $\mathfrak{C}_\alpha^V = \mathfrak{C}_\alpha^{V[G]}$, we will show that

$$C_{\alpha+1}^V = C_{\alpha+1}^{V[G]} \tag{3.1}$$

$$T_{\alpha+1}^V = T_{\alpha+1}^{V[G]} \tag{3.2}$$

By definition, $C_{\alpha+1}^V = \mathrm{def}_{\mathfrak{aa}}(\mathfrak{C}_\alpha)^V$ and $C_{\alpha+1}^{V[G]} = \mathrm{def}_{\mathfrak{aa}}(\mathfrak{C}_\alpha)^{V[G]}$, and so (3.1) follows from (3.2). Thus we show that for all $\varphi(\overline{x}, \overline{X}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T)$, $\overline{a} \in C_\alpha^{<\omega}$ and $\overline{A} \in \mathcal{P}_{\omega_1}(C_\alpha)^{<\omega}$, the following holds:

$$\left(\mathfrak{C}_\alpha \models \varphi(\overline{a}, \overline{A})\right)^V \iff \left(\mathfrak{C}_\alpha \models \varphi(\overline{a}, \overline{A})\right)^{V[G]}$$

Note that $\mathcal{P}_{\omega_1}(C_\alpha)^V = \mathcal{P}_{\omega_1}(C_\alpha)^{V[G]}$, since the forcing is countably closed. We prove the claim by induction on the structure of φ. The atomic case for $\in$ is trivial and the atomic case for T follows by the induction assumption (on α, not φ). The induction-step for all first-order connectives and quantifiers is a straightforward application of the induction assumption. Thus, we will look at the induction-step for the quantifier $\mathfrak{stat}$, and then we have proved it for $\mathfrak{aa}$ as well.

$$\begin{aligned}
(\mathfrak{C}_\alpha \models \mathfrak{stat} X \varphi(X, \overline{a}, \overline{A}))^V &\iff \{B \in \mathcal{P}_{\omega_1}(C_\alpha) \mid \mathfrak{C}_\alpha \models \varphi(B, \overline{a}, \overline{A})\} \\
&\quad \text{is stationary in } V \\
&\iff \{B \in \mathcal{P}_{\omega_1}(C_\alpha) \mid \mathfrak{C}_\alpha \models \varphi(B, \overline{a}, \overline{A})\} \\
&\quad \text{is stationary in } V[G] \qquad\qquad \text{(Lemma 3.1.5)} \\
&\iff (\mathfrak{C}_\alpha \models \mathfrak{stat} X \varphi(X, \overline{a}, \overline{A}))^{V[G]}
\end{aligned}$$

□

Remark 3.1.7. Note that it follows from Theorem 3.1.6 that we can collapse any regular cardinal $\kappa > \omega_1$ to ω_1 without changing $C(\mathfrak{aa})$ or any of its levels. Let $\mathbb{P}$ be the set of all functions $f : \alpha \to \kappa$, $\alpha < \omega_1$, ordered by reverse inclusion. Then if G is $\mathbb{P}$-generic over V, $F = \cup G$ will be a surjection from ω_1^V to κ in $V[G]$. Furthermore, this forcing is countably closed, so it does not change $C(\mathfrak{aa})$. This observation will be needed when we prove club determinacy, since any failure of club determinacy of $\mathfrak{C}_\alpha$ may w.l.o.g. be assumed to happen when $|\alpha| = \omega_1$.

3.2 A short guide to stationary tower forcing

Stationary towers is a forcing notion that, among other things, give rise to natural generic embeddings with low critical points. A *generic embedding* is an elementary embedding of V into some forcing extension $V[G]$ of V.

We will use them to prove club determinacy in $C(\mathfrak{aa})$ under large cardinal assumptions. This section is a short summary of the facts we will utilise. The standard reference for stationary towers is [11], where most of the proofs can be found. The proof of the last Theorem can be found in [9].

There's a generalised notion of stationarity which is due to Woodin. Let X be a set. Then, $a \subseteq \mathcal{P}(X)$ is said to be *stationary* in $\mathcal{P}(X)$ if for every $f : X^{<\omega} \to X$, there's some $Z \in a$ such that $f[Z^{<\omega}] \subseteq Z$. If a is stationary in $\mathcal{P}(X)$, then $\cup a = X$, so we say that a is *stationary* if a is stationary in $\mathcal{P}(\cup a)$.

Definition 3.2.1 (Stationary tower)**.** Let κ be strongly inaccessible. The *full stationary tower* at κ is:

$$\mathbb{P}_{<\kappa} := \{a \in V_\kappa \mid a \text{ is stationary}\}$$

and the *countable tower* is

$$\begin{aligned} \mathbb{Q}_{<\kappa} : &= \{a \in \mathbb{P}_{<\kappa} \mid a \subseteq \mathcal{P}_{\omega_1}(\cup a)\} \\ &= \{a \in V_\kappa \mid a \text{ is stationary in } \mathcal{P}_{\omega_1}(\cup a)\}, \end{aligned}$$

where on the last line we talk about stationarity in our previous sense. For a and b in $\mathbb{P}_{<\kappa}$ or $\mathbb{Q}_{<\kappa}$, we let $a \leq b$ if $\cup a \supseteq \cup b$, and for every $Z \in a$, $Z \cap (\cup b) \in b$.

Lemma 3.2.2 ([11], Fact 2.2.3.)**.** *Let G be a generic filter over $\mathbb{P}_{<\kappa}$ or $\mathbb{Q}_{<\kappa}$ and $X \in V_\kappa$. Then, $U_X = \{a \in G \mid \cup a = X\} \subseteq \mathcal{P}(\mathcal{P}(X))$ is an ultrafilter on $\mathcal{P}(X)$.*

Of course, this ultrafilter may only exist in a generic extension of V. Given $X \in V_\kappa$, let (M_X, E_X) denote the ultrapower of V over U_X and $j_X : (V, \in) \to (M_X, E_X)$ the canonical embedding. That is, given $f, g : \mathcal{P}(X) \to V$, we let $f \sim_{U_X} g$ iff $\{Z \subseteq X \mid f(Z) = g(Z)\} \in U_X$ and $[f]_{U_X} E_X [g]_{U_X}$ iff $\{Z \subseteq X \mid f(Z) \in g(Z)\} \in U_X$. Furthermore, given $a \in V$, $j_X(a) = [c_a]_{U_X}$, where c_a is the constant function mapping every element to a.

If $X \subseteq Y$, we can define a canonical embedding $j_{XY} : (M_X, E_X) \to (M_Y, E_Y)$ as follows: For each $f : \mathcal{P}(X) \to V$, let $f_Y : \mathcal{P}(Y) \to V$ be such that $f_Y(Z) = f(Z \cap X)$. Then $j_{XY}([f]_{U_X}) = [f_Y]_{U_Y}$, which is a well-defined elementary embedding. This gives rise to a directed system of models $\langle (M_X, E_X) \mid X \in V_\kappa \rangle$, which has a limit model (M, E). There's no guarentee that the model (M, E) is well-founded, but if κ is a *Woodin cardinal*, then it is. The definition of a Woodin cardinal is not important, but we include it for completeness.

Definition 3.2.3 (Woodin cardinal)**.** An inaccessible cardinal κ is called *Woodin*, if for every $f : \kappa \to \kappa$, there's $\lambda < \kappa$, such that $f[\lambda] \subseteq \lambda$ and λ is the critical point of some elementary embedding $j : V \to M \subseteq V$, having the additional property that $V_{j(f)(\lambda)} \subseteq M$.

Notice that if κ is a Woodin cardinal, then there's a measurable cardinal λ below κ.

Theorem 3.2.4 ([11], Theorem 2.5.8.)**.** *Let κ be a Woodin cardinal. For every generic $G \subseteq \mathbb{P}_{<\kappa}$, the limit model (M, E) is well-founded and closed under sequences of length $< \kappa$ in $V[G]$.*

The same holds for $\mathbb{Q}_{<\kappa}$ ([11], Theorem 2.7.7.). Whenever this is true, we identify M with its transitive collapse. From the embeddings j_X, j_{XY}, $X \subseteq Y \in V_\kappa$, we obtain a limit embedding $j : V \to M \subseteq V[G]$. This embedding is elementary and satisfies the following properties:

Theorem 3.2.5 ([11])**.** *For every generic G over $\mathbb{P}_{<\kappa}$ or $\mathbb{Q}_{<\kappa}$, the following holds:*

1. *$U_X = \{a \subseteq \mathcal{P}(X) \mid j[X] E j(a)\}$*
2. *$a \in G \iff j[\cup a] E j(a)$.*
3. *If $\gamma \in G$ and γ is an ordinal, then the critical point of j is γ.*

Theorem 3.2.6 ([11], Theorem 2.7.8.)**.** *Let κ be a Woodin cardinal and $j : V \to M \subseteq V[G]$ the generic embedding associated with $\mathbb{Q}_{<\kappa}$. Then the critical point of j is ω_1 and $j(\omega_1) = \kappa$.*

By the elementarity of j, $j(\omega_1) = \kappa$ means that κ is the first uncountable ordinal in M. Furthermore, M is an inner model of $V[G]$, so we must have that $\kappa \leq \omega_1^{V[G]}$. In particular, this means that forcing with $\mathbb{Q}_{<\kappa}$ collapses every cardinal $\lambda < \kappa$, so that λ is countable in $V[G]$.

Theorem 3.2.7 ([9], Proposition 4.9.)**.** *Let κ be a Woodin cardinal and let G be $\mathbb{Q}_{<\kappa}$-generic over V. If $A \in V$ and $S \subseteq \mathcal{P}_\kappa(A)$ is stationary in V, then S is stationary also in $V[G]$.*

3.3 Obtaining club determinacy

Lemma 3.3.1. *Suppose that M is a set with cardinality ω_1 and $f : M \to \omega_1$ is a bijection. Then $S \subseteq \mathcal{P}_{\omega_1}(M)$ is stationary in $\mathcal{P}_{\omega_1}(M)$ iff*

$$S^* = \{\beta < \omega_1 \mid f^{-1}[\beta] \in S\}$$

is stationary in ω_1.

Proof. "$\Rightarrow$" Assume, for the sake of contradiction, that S is stationary but S^* is not. Pick a club $C^* \subseteq \omega_1$ such that $C^* \cap S^* = \emptyset$. Define $C := \{A \in \mathcal{P}_{\omega_1}(M) \mid f[A] \in C^*\}$. We claim that C is a club in $\mathcal{P}_{\omega_1}(M)$.

Take any $B \in \mathcal{P}_{\omega_1}(M)$. We can find $\beta > \sup f[B]$ such that $\beta \in C^*$. Then $f^{-1}[\beta] \supseteq B$ and $f^{-1}[\beta] \in C$. Now, suppose that $A_i \in C$, $i < \omega$, is an increasing sequence. Then $f[A_i] \in C^*$ for all $i < \omega$. Thus, $f[\cup_{i<\omega} A_i] = \cup_{i<\omega} f[A_i] \in C^*$. Therefore, $\cup_{i<\omega} A_i \in C$. This shows that C is a club.

Pick $A \in C \cap S$. Then $f[A] \in C^*$. Furthermore, $f^{-1}[f[A]] = A \in S$, so in particular, $f[A] \in S^*$. This is a contradiction. It follows that S^* is stationary in ω_1.

"$\Leftarrow$" Assume, for the sake of contradiction, that S^* is stationary but S is not. Pick a club $C \subseteq \mathcal{P}_{\omega_1}(M)$ such that $S \cap C = \emptyset$. Define $C^* := \{\beta < \omega_1 \mid f^{-1}[\beta] \in C\}$. As before, we claim that C^* is a club in ω_1.

To show that C^* is unbounded, pick $\beta_0 < \omega_1$ and $B_0 \in C$ such that $B_0 \supseteq f^{-1}[\beta_0]$. For each $n < \omega$, let $\beta_{n+1} = \sup f[B_n]$ and $B_{n+1} \in C$ be such that $B_{n+1} \supseteq f^{-1}[\beta_{n+1}]$. Let $B := \cup_{n<\omega} B_n$ and $\beta := \cup_{n<\omega} \beta_n$. Then $B \in C$ is a witness for $\beta \in C^*$ and $\beta > \beta_0$.

Now assume that β_i, $i < \omega$, is a increasing sequence in C^*. Then $(f^{-1}[\beta_i])_{i<\omega}$ is an increasing sequence in C, so $\cup_{i<\omega} f^{-1}[\beta_i] \in C$. Since, $\cup_{i<\omega} f^{-1}[\beta_i] = f^{-1}[\cup_{i<\omega} \beta_i]$, we have that $\cup_{i<\omega} \beta_i \in C^*$.

Thus, C^* is a club. Pick $\beta \in C^* \cap S^*$. Then $f^{-1}[\beta] \in S \cap C$, a contradiction. $\square$

The set of I-positive sets of an ideal $I \subseteq \mathcal{P}(X)$, is the set $I^+ = \mathcal{P}(X) \setminus I$. The *non-stationary ideal on* ω_1, denoted NS_{ω_1}, is the ideal on ω_1 with positive sets being precisely the stationary subsets of ω_1. An ideal I is called κ-saturated if every antichain in I^+ is of size $< \kappa$. We won't need these definitions directly, but they help make the assumptions of the following Lemma clearer:

Lemma 3.3.2. *If there's a measurable cardinal and NS_{ω_1} is ω_2-saturated, then*

1. *For every α such that $|\alpha| = \omega_1$, there's $x \subseteq \omega$ and $f \in L[x]$ such that $f : \alpha \to \omega_1^V$ is a bijection.*
2. *Let δ be a Woodin cardinal and $j : V \to M \subseteq V[G]$ the generic embedding we get from forcing with $\mathbb{Q}_{<\delta}$. Then $j \restriction \omega_2 \in V$. Furthermore, if $a \subseteq \omega_2$ is countable, then there's $b \in V$ such that $\Vdash_{\mathbb{Q}_{<\delta}} j(\check{a}) = \check{b}$.*

Proof. This follows from a Theorem by Woodin (Theorem 38.3. in Jech [6]), combined with Lemma 4.10. and 4.11. in [9]. $\square$

Remark 3.3.3. If there's a Woodin cardinal, then we can force the ω_2-saturation of the non-stationary ideal on ω_1 with a semi-proper forcing (Theorem 38.1. in Jech [6]). Semi-proper forcing preserves stationary sets of ω_1 (Theorem 34.4. in Jech [6]). Suppose that $|\alpha| = \omega_1$ and $S \subseteq \mathcal{P}_{\omega_1}(C_\alpha)$ is stationary. Since $|C_\alpha| = \omega_1$, we can fix a bijection $f : C_\alpha \to \omega_1$. Then, $S^* = \{\beta < \omega_1 \mid f^{-1}[\beta] \in S\}$ is a stationary subset of ω_1 by Lemma 3.3.1. Thus, it remains stationary in a generic extension. It follows that also S remains stationary. Looking at the proof of Theorem 3.1.6, this means that semi-proper forcing does not change the levels $\mathfrak{C}_\alpha$ for $\alpha < \omega_2$.

Given a regular cardinal λ, we introduce a new quantifer aa_λ. It is similar to aa, but expresses the existence of a club set in $\mathcal{P}_\lambda(M)$ rather than $\mathcal{P}_{\omega_1}(M)$, for some set M. That is, given $\varphi(X, \overline{a}, \overline{Y}) \in \mathcal{L}_{\mathsf{aa}_\lambda}(\tau)$, a τ-model $\mathcal{M}$, $\overline{a} \in M^{<\omega}$ and $\overline{B} \in \mathcal{P}_\lambda(M)^{<\omega}$, we define

$$\mathcal{M} \models \mathsf{aa}_\lambda X \varphi(X, \overline{a}, \overline{B}) \iff \{A \in \mathcal{P}_\lambda(M) \mid \mathcal{M} \models \varphi(A, \overline{a}, \overline{B})\} \text{ contains a club in } \mathcal{P}_\lambda(M)$$

We can construct an auxiliary hierarchy $\langle \mathfrak{C}^\lambda_\alpha \mid \alpha \geq \omega \rangle$ by using definability with aa_λ-formulas instead of aa-formulas in Definition 2.3.1. Similarly, we can define club determinacy in terms of aa_λ-formulas:

Definition 3.3.4. A τ structure $\mathcal{M}$ is said to be *λ-club determined* if for all $\varphi(X, \overline{x}, \overline{Y}) \in \mathcal{L}_{\mathsf{aa}_\lambda}(\tau)$, $\overline{a} \in M^{|\overline{x}|}$ and $\overline{B} \in \mathcal{P}_\lambda(M)^{|\overline{Y}|}$, we have that

$$\mathcal{M} \models \mathsf{aa}_\lambda X \varphi(X, \overline{a}, \overline{B}) \text{ or } \mathcal{M} \models \mathsf{aa}_\lambda X \neg\varphi(X, \overline{a}, \overline{B})$$

In the following proof, we will occasionally need to interpret aa-formulas as aa_λ-formulas. Whenever we do this, we write "$\models_\lambda$" instead of "$\models$", for the satisfaction relation. The idea of the proof is the following: We suppose that club determinacy fails at a level $\mathfrak{C}_\alpha$, and assume w.l.o.g. that $|\alpha| = \omega_1$. We obtain a stationary subset $S^* \subseteq \omega_1$ such that also its complement is stationary. We force with $\mathbb{Q}_{<\delta}$ and obtain a generic embedding $j : V \to M \subseteq V[G]$. By induction we prove that $(\mathfrak{C}^\delta_{j(\alpha)})^V = \mathfrak{C}^M_{j(\alpha)}$. We use this fact to show that $j(S^*)$ is in V and is independent[2] of the generic filter G. We can choose G such that either $S^* \in G$ or $\omega_1 \setminus S^* \in G$. Now, by Theorem 3.2.5, this means that both $\omega_1 \in j(S^*)$ and $\omega_1 \notin j(S^*)$, a contradiction.

Theorem 3.3.5 (Kennedy, Magidor and Väänänen, [9]). *If there's a proper class of Woodin cardinals, then $C(\mathsf{aa})$ is club determined.*

Proof. Suppose that club determinacy fails at a level $\mathfrak{C}_\alpha$ and that α is least such that this happens. We can collapse $|\alpha|$ to ω_1 in accordance with Remark 3.1.7. This forcing preserves all Woodin cardinals above α. Thus, we may assume that $|\alpha| = \omega_1$. By Remark 3.3.3, we may assume that the non-stationary ideal on ω_1 is ω_2-saturated. Since we have a Woodin cardinal, we also have a measurable cardinal, so we can apply Lemma 3.3.2 to find $x \subseteq \omega$ such that $L[x]$ contains a bijection $f : \alpha \to \omega_1^V$. Furthermore, we may take f to be the $<_{L[x]}$-least such bijection.

Let δ be a Woodin cardinal and $j : V \to M \subseteq V[G]$ the generic embedding obtained from forcing with $\mathbb{Q}_{<\delta}$. Let $\gamma = j(\alpha)$. By Theorem 3.2.6, ω_1 is the critical point of j and $j(\omega_1) = \delta$. Thus, by the elementarity of j, $j(f) : \gamma \to \delta$ is a bijection and least in the canonical well-ordering of $L[x]^M$. Note that $\delta \leq \gamma$. Let $g := j(f)$. Since $L[x]^V = L[x]^M$, we have that $g \in V$. So the situation looks like this:

[2] This means that if G_1 and G_2 are $\mathbb{Q}_{<\delta}$-generic over V and $j_1 : V \to M_1 \subseteq V[G_1]$, $j_2 : V \to M_2 \subseteq V[G_2]$ are the associated embeddings, then $j_1(S^*) = j_2(S^*) \in V$.

1. Club determinacy fails in V at $\mathfrak{C}_\alpha$, and we have a bijection $f : \alpha \to \omega_1^V$.
2. Club determinacy fails in $M \subseteq V[G]$ at $\mathfrak{C}_\gamma^M$, and we have a bijection $g : \gamma \to \delta$. Furthermore, $g \in V$ and g is independent of the generic filter G.

We will need these functions later. Now, assume that $\varphi(X, \overline{a}, \overline{B})$ witnesses the failure of club determinacy at $\mathfrak{C}_\alpha$, in the sense that both

$$S := \{A \in \mathcal{P}_{\omega_1}(C_\alpha) \mid \mathfrak{C}_\alpha \models \varphi(A, \overline{a}, \overline{B})\},$$

and its complement are stationary in $\mathcal{P}_{\omega_1}(C_\alpha)$. We may further assume that no subformula of φ witnesses the failure of club determinacy. Now, since j is elementary, we have that both

$$j(S) = \{A \in (\mathcal{P}_\delta(C_\gamma^M))^M \mid (\mathfrak{C}_\gamma^M \models \varphi(A, j(\overline{a}), j(\overline{B})))^M\},$$

and its complement are stationary in $(\mathcal{P}_\delta(C_\alpha^M))^M$. Furthermore, the elementarity of j also implies that γ is minimal such that club determinacy fails in $C(\mathfrak{aa})^M$, and no subformula of φ witnesses the failure of club determinacy in $\mathfrak{C}_\gamma^M$.

Claim. For all $\beta < \gamma$, $\mathfrak{C}_\beta^M = \mathfrak{C}_\beta^\delta$ and $\mathfrak{C}_\beta^\delta$ satisfies δ-club determinacy for all formulas with parameters[3] in $V \cap M$.

Proof. We prove this by induction on β. Clearly, $\mathfrak{C}_\omega^M = \mathfrak{C}_\omega^\delta$. Suppose the claim is true for β, we prove it for $\beta + 1$. Now, by definition,

$$C_{\beta+1}^M = \mathrm{def}_{\mathfrak{aa}}^M(\mathfrak{C}_\beta^M)$$
$$C_{\beta+1}^\delta = \mathrm{def}_{\mathfrak{aa}}^\delta(\mathfrak{C}_\beta^\delta)$$

These sets are equal if the following holds: For all $\psi(\overline{x}, \overline{Y}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T)$, $\overline{a} \in C_\beta^M$ and $\overline{B} \in (\mathcal{P}_\delta(C_\beta^\delta))^V$, we have that

$$(\mathfrak{C}_\beta^\delta \models_\delta \psi(\overline{a}, \overline{B}))^V \iff (\mathfrak{C}_\beta^M \models \psi(\overline{a}, \overline{B}))^M \tag{3.3}$$

If 3.3 holds, then also $T_{\beta+1}^M = T_{\beta+1}^\delta$. Thus, we prove 3.3 by induction on the structure of ψ. The atomic case for $\in$ is trivial and the atomic case for T follows by the induction assumption on β. The induction-step for all first-order connectives and quantifiers is a straightforward application of the induction assumption. Thus, we restrict ourselves to the $\mathfrak{aa}$-quantifier.

Suppose first that $(\mathfrak{C}_\beta^\delta \models_\delta \mathfrak{aa}X\psi(X, \overline{a}, \overline{B}))^V$. We want to show that $(\mathfrak{C}_\beta^M \models \mathfrak{aa}X\psi(X, \overline{a}, \overline{B}))^M$. By the induction assumption, the equivalence 3.3 holds

[3] This is relevant only for second-order parameters. Also note that $(\mathcal{P}_\delta(C_\beta^\delta))^V \subseteq (\mathcal{P}_{\omega_1}(C_\beta^M))^M$ since M is closed under sequences of length $< \delta$ in $V[G]$, i.e. sequences of length ω in $V[G]$. Thus, $(\mathcal{P}_\delta(C_\beta^\delta))^V = (\mathcal{P}_\delta(C_\beta^\delta))^V \cap M$.

for ψ. Let C be a club of sets in $(\mathcal{P}_\delta(C^\delta_\beta))^V$ such that for all $A \in C$, $(\mathfrak{C}^\delta_\beta \models_\delta \psi(A, \overline{a}, \overline{B}))^V$. By Theorem 3.2.7, C is stationary in $V[G]$. By the minimality of γ, $\mathfrak{C}^M_\beta$ is club determined. Suppose for a contradiction that $(\mathfrak{C}^M_\beta \models \mathsf{aa}X\neg\psi(X, \overline{a}, \overline{B}))^M$. Let D be a club in $(\mathcal{P}_{\omega_1}(C^M_\beta))^M$ such that for all $A \in D$, $(\mathfrak{C}^M_\beta \not\models \psi(A, \overline{a}, \overline{B}))^M$. Pick $A \in C \cap D$. Then by the induction assumption, $(\mathfrak{C}^\delta_\beta \not\models_\delta \psi(A, \overline{a}, \overline{B}))^V$, a contradiction.

Now assume that $(\mathfrak{C}^M_\beta \models \mathsf{aa}X\psi(X, \overline{a}, \overline{B}))^M$. We want to show that $(\mathfrak{C}^\delta_\beta \models_\delta \mathsf{aa}X\psi(X, \overline{a}, \overline{B}))^V$. By the induction assumption, the equivalence 3.3 holds for ψ. Let D be a club of sets in $(\mathcal{P}_{\omega_1}(C^M_\beta))^M$ such that $(\mathfrak{C}^M_\beta \models \psi(A, \overline{a}, \overline{B}))^M$ for all $A \in D$. M is closed under ω-sequences so D is a club also in $V[G]$. Suppose for a contradiction that $(\mathfrak{C}^\delta_\beta \not\models \mathsf{aa}X\psi(X, \overline{a}, \overline{B}))^V$. By assumption, $\mathfrak{C}^\delta_\beta$ satisfies δ-club determinacy for formulas with parameters in $V \cap M$. Thus, there's a club C in $(\mathcal{P}_\delta(C^\delta_\beta))^V$ such that for all $A \in C$, $(\mathfrak{C}^\delta_\beta \not\models \psi(A, \overline{a}, \overline{B}))^V$. By Theorem 3.2.7, C is stationary in $V[G]$. Pick $A \in C \cap D$. Then $(\mathfrak{C}^M_\beta \not\models \psi(A, \overline{a}, \overline{B}))^M$, a contradiction.

This proves equivalence 3.3. We still need to show that $\mathfrak{C}^\delta_{\beta+1}$ is δ-club determined for formulas with parameters in $V \cap M$. Assume that δ-club determinacy fails for $\mathfrak{C}^\delta_{\beta+1}$ and that $\chi(X, \overline{a}, \overline{B})$ is a minimal witness for this, with $\overline{a} \in C^\delta_{\beta+1}$ and $\overline{B} \in (\mathcal{P}_\delta(C^\delta_{\beta+1}))^V \subseteq (\mathcal{P}_{\omega_1}(C^M_{\beta+1}))^M$. That is,

$$E := \{A \in (\mathcal{P}_\delta(C^\delta_{\beta+1}))^V \mid \mathfrak{C}^\delta_{\beta+1} \models \chi(A, \overline{a}, \overline{B})\}, \text{ and}$$
$$E^C := \{A \in (\mathcal{P}_\delta(C^\delta_{\beta+1}))^V \mid \mathfrak{C}^\delta_{\beta+1} \not\models \chi(A, \overline{a}, \overline{B})\}$$

are both stationary in $(\mathcal{P}_\delta(C^\delta_{\beta+1}))^V$. By Theorem 3.2.7, E and E^C are stationary in $V[G]$. Since χ is the minimal failure of δ-club determinacy with parameters in $V \cap M$, we can go through the proof of 3.3 up until 3.4 to obtain:

$$(\mathfrak{C}^\delta_{\beta+1} \models_\delta \chi(A, \overline{a}, \overline{B}))^V \iff (\mathfrak{C}^M_{\beta+1} \models \chi(A, \overline{a}, \overline{B}))^M \tag{3.4}$$

for any $A \in (\mathcal{P}_\delta(C^\delta_{\beta+1}))^V$. It follows that

$$E \subseteq H := \{A \in (\mathcal{P}_{\omega_1}(C^M_{\beta+1}))^M \mid \mathfrak{C}^\delta_{\beta+1} \models \chi(A, \overline{a}, \overline{B})\}, \text{ and}$$
$$E^C \subseteq H^C := \{A \in (\mathcal{P}_{\omega_1}(C^M_{\beta+1}))^M \mid \mathfrak{C}^\delta_{\beta+1} \not\models \chi(A, \overline{a}, \overline{B})\}$$

are both stationary in M, contradicting that $\mathfrak{C}^M_{\beta+1}$ is club determined.

Lastly, we have to do the induction-step for limit β. Since $\mathfrak{C}^\delta_\beta = \bigcup_{\xi<\beta} \mathfrak{C}^\delta_\xi$ and $\mathfrak{C}^M_\beta = \bigcup_{\xi<\beta} \mathfrak{C}^M_\xi$, it follows directly from the induction assumption that $\mathfrak{C}^\delta_\beta = \mathfrak{C}^M_\beta$. The proof that $\mathfrak{C}^\delta_\beta$ satisfies δ-club determinacy is the same as above. □

Claim. $\mathfrak{C}^M_\gamma = \mathfrak{C}^\delta_\gamma$, and for all $\overline{a} \in C^\delta_\gamma$ and $A, \overline{B}$ in $(\mathcal{P}_\delta(C^\delta_\gamma))^V \cap M$, we have that

$$(\mathfrak{C}^\delta_\gamma \models_\delta \varphi(A, \overline{a}, \overline{B}))^V \iff (\mathfrak{C}^M_\gamma \models \varphi(A, \overline{a}, \overline{B}))^M$$

Proof. The proof is precisely the same as above, we just stop the induction on formulas when we reach φ. $\square$

Now, use Lemma 2.4.4 to pick a bijection $f' : C_\alpha \to \alpha$ such that $f' \in L(\mathfrak{C}_\alpha)$. Then, $g' = j(f') : C_\gamma^M \to \gamma$ is a bijection and $g' \in L(\mathfrak{C}_\gamma^M)$. Furthermore, define $\tilde{f} = f \circ f'$ and $\tilde{g} = g \circ g'$. Then $\tilde{f} : C_\alpha \to \omega_1^V$ and $\tilde{g} : C_\gamma^M \to \delta$ are bijections. Again, by the elementarity of j, we have that $\tilde{g} = j(\tilde{f})$. Since $\mathfrak{C}_\gamma^M = \mathfrak{C}_\gamma^\delta$, it follows that $\tilde{g} \in V$. Note that g is independent of which generic filter G we use, so also $\tilde{g}$ is independent of G. By the claim above, we know that

$$j(S) \cap V = \{A \in (\mathcal{P}_\delta(C_\gamma^\delta))^V \mid \mathfrak{C}_\gamma^\delta \vDash_\delta \varphi(A, j(\overline{a}), j(\overline{B}))\}$$

Given $\overline{a}$ in C_α and $\overline{B}$ in $(\mathcal{P}_{\omega_1}(C_\alpha))^V$, we have that $\tilde{f}(\overline{a})$ is a tuple of elements of ω_1^V and $\tilde{f}[\overline{B}]$ is a tuple of countable subsets of ω_1^V in V. By Lemma 3.3.2, $j(\tilde{f}(\overline{a}))$ and $j(\tilde{f}[\overline{B}])$ are in V. Thus, $j(\overline{a}) = \tilde{g}^{-1}(j(\tilde{f}(\overline{a})))$ and $j(\overline{B}) = \tilde{g}^{-1}[j(\tilde{f}[\overline{B}])]$ are in V. Therefore, $j(S) \cap V \in V$.

By Lemma 3.3.1, $S^* := \{\beta < \omega_1^V \mid \tilde{f}^{-1}[\beta] \in S\}$ is a stationary subset of ω_1 such that also $\omega_1 \setminus S^*$ is stationary. Now, $j(S^*)$ is in V, since $\tilde{g}$ is in V:

$$\begin{aligned} j(S^*) &= \{\beta < \delta \mid \tilde{g}^{-1}[\beta] \in j(S)\} \\ &= \{\beta < \delta \mid \tilde{g}^{-1}[\beta] \in j(S) \cap V\} \end{aligned}$$

Furthermore, note that $j(S^*)$ is independent of the generic filter G. This means that we can choose generic filters G_1 and G_2 such that $S^* \in G_1$ and $\omega_1 \setminus S^* \in G_2$. Note that both of these are conditions in $\mathbb{Q}_{<\delta}$. Let j_1 and j_2 be the associated embeddings. By Theorem 3.2.5, we get that $j_1[\cup S^*] \in j_1(S^*)$ in the first case and $j_2[\cup(\omega_1 \setminus S^*)] \in j_2(\omega_1 \setminus S^*)$ in the second case. Now, $j_1[\cup S^*] = j_1[\omega_1] = \omega_1$, and similarly, $j_2[\cup(\omega_1 \setminus S^*)] = \omega_1$. Thus, we have that $\omega_1 \in j_1(S^*)$ in the first case but $\omega_1 \notin j_2(S^*)$ in the second case. However, $j(S^*)$ was independent of the generic filter, so $j_1(S^*) = j_2(S^*)$, and both of these are in V. This is a contradiction. $\square$

From now on, we will assume that $C(\mathfrak{aa})$ is club determined.

Chapter 4

Developing the theory of $\mathfrak{aa}$-mice

The canonical proofs of many combinatorial properties of L uses the condensation principle in L: If $\mathcal{M} \preccurlyeq L_\gamma$ is an elementary submodel, where γ is a limit ordinal, then $\mathcal{M}$ collapses to L_α for some $\alpha \leq \gamma$. However, this is not true in $C(\mathfrak{aa})$, as the following example illustrates:

Example 4.0.1. Let $\mathcal{M} = (M, E, T^{\mathcal{M}}) \preccurlyeq \mathfrak{C}_{\omega_2}$ be countable such that $\omega_1 \in M$. Note that

$$\mathfrak{C}_{\omega_2} \vDash T_{\omega_2}(\omega_1, \mathfrak{aa}X\exists x \neg X(x))$$

Then if $\pi : \mathcal{M} \to \mathcal{N} = (N, \in, T^{\mathcal{N}})$ is the transitive collapse of $\mathcal{M}$, we have that

$$\mathcal{N} \vDash T^{\mathcal{N}}(\pi(\omega_1), \mathfrak{aa}X\exists x \neg X(x))$$

However, $\pi(\omega_1)$ is countable, so $\mathfrak{C}_{\pi(\omega_1)} \vDash \neg\mathfrak{aa}X\exists x\neg X(x)$. Thus, $\mathcal{N}$ is not equal to $\mathfrak{C}_\alpha$ for any α.

It is still possible to achieve a principle in $C(\mathfrak{aa})$ that is similar to the condensation principle in L. After we've taken an elementary submodel of some level of $C(\mathfrak{aa})$, the trick is to add – in ω_1 many steps – a club that witnesses the truth of $\mathfrak{aa}$-formulas. Then the result will be a level $\mathfrak{C}_\gamma$ for some γ. This is Theorem 4.3.6, which will be our main tool in Chapter 5. However, the route there is somewhat technical.

4.1 $\mathfrak{aa}$-premice

Definition 4.1.1. Let $\xi \geq 0$ be an ordinal. $\mathcal{M} = (M, \in, T^{\mathcal{M}}, S^{\mathcal{M}}, (P_i^{\mathcal{M}})_{i<\xi})$ is a *potential* $\mathfrak{aa}$*-premouse*, if

1. M is transitive and $V_\omega \subseteq M$.
2. $T^{\mathcal{M}} \subseteq M \times M$

3. $S^{\mathcal{M}}$ is a complete consistent theory in stationary logic extending the first-order elementary diagram of $(M, \in, T^{\mathcal{M}}, (P_i^{\mathcal{M}})_{i<\xi})$.

4. $S^{\mathcal{M}}$ contains the club determinacy schema, i.e.

$$\mathfrak{aa}\overline{Y}(\mathfrak{aa}X\varphi(X, \overline{a}, \overline{Y}) \vee \mathfrak{aa}X\neg\varphi(X, \overline{a}, \overline{Y})) \in S^{\mathcal{M}},$$

for each $\varphi(X, \overline{x}, \overline{Y}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi})$ and $\overline{a} \in M^{|\overline{x}|}$.

5. If $\exists x\varphi(x) \in S^{\mathcal{M}}$, then $\varphi(a) \in S^{\mathcal{M}}$ for some $a \in M$.

6. For every $i < \xi$, $P_i^{\mathcal{M}}$ is a unary relation on M such that

$$\mathfrak{aa}X\forall x(P_i(x) \to X(x)) \in S^{\mathcal{M}}.$$

7. $\mathfrak{aa}X\exists x(\neg X(x)) \in S^{\mathcal{M}}$.

Note that $S^{\mathcal{M}}$ is a theory with constants for all $a \in M$. For convenience, we will identify the formulas in $S^{\mathcal{M}}$ with elements of $\mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi}) \times M^{<\omega}$. If $\varphi(\overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi})$ and $\overline{a} \in M^{|\overline{x}|}$, then we will write $\mathcal{M} \vDash_S \varphi(\overline{a})$ instead of $\varphi(\overline{a}) \in S^{\mathcal{M}}$. This is to indicate that the formula is true in $\mathcal{M}$ according to $S^{\mathcal{M}}$, but it need not actually be true. Also note that ξ may be 0, in which case we have no predicates P_i.

Let $\mathcal{N} \in \mathcal{M}$ be a structure in vocabulary $\{\in, T\}$. Using $S^{\mathcal{M}}$, we can look at the subsets of $\mathcal{N}$ which are definable in stationary logic, not in reality, but according to $S^{\mathcal{M}}$. Thus, let $\mathrm{def}^S_{\mathfrak{aa}}(\mathcal{N})$ be the set of all $X \subseteq N$, such that there's $\varphi(x, \overline{y}) \in \mathcal{L}_{\mathfrak{aa}}(\mathcal{N})$ and $\overline{a} \in N^{<\omega}$ with $X = \{b \in N \mid \mathcal{M} \vDash_S \varphi^{\mathcal{N}}(b, \overline{a})\}$. Now, we can define a sequence $\langle \mathfrak{C}_\alpha^{\mathcal{M}} \mid \alpha < \gamma\rangle$ as in definition 2.3.1, but using $\mathrm{def}^S_{\mathfrak{aa}}$ instead of $\mathrm{def}_{\mathfrak{aa}}$. Here, γ is the least ordinal such that $\mathfrak{C}_\gamma^{\mathcal{M}} \notin M$. Note that $\gamma \leq o(\mathcal{M})$, where $o(\mathcal{M})$ is the least ordinal not in M. It is now possible to define what an $\mathfrak{aa}$-premouse is.

Definition 4.1.2 ($\mathfrak{aa}$-premouse)**.** A potential $\mathfrak{aa}$-premouse $\mathcal{M}$ is an $\mathfrak{aa}$-*premouse* if

1. $\gamma = o(\mathcal{M})$ is a limit ordinal,

2. $M = \bigcup_{\alpha<\gamma} C_\alpha^{\mathcal{M}}$,

3. $T^{\mathcal{M}} = \{(\alpha, \varphi(\overline{a})) \mid \alpha < \gamma, \overline{a} \in (C_\alpha^{\mathcal{M}})^{<\omega}, \varphi \in \mathcal{L}_{\mathfrak{aa}}(\in, T), \mathcal{M} \vDash_S \varphi^{\mathfrak{C}_\alpha^{\mathcal{M}}}(\overline{a})\}$,

4. If $\mathcal{M} \vDash_S \mathfrak{aa}\overline{X}\exists x\varphi(\overline{X}, x, \overline{a})$ for some $\varphi(\overline{X}, x, \overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi})$, then $\mathcal{M} \vDash_S \mathfrak{aa}\overline{X}\exists x(\varphi(\overline{X}, x, \overline{a}) \wedge \forall y(y <_{\mathfrak{aa}} x \to \neg\varphi(\overline{X}, y, \overline{a})))$. Here, "$<_{\mathfrak{aa}}$" is defined using the same formula as in section 2.4. See the remark below for justification.

Remark 4.1.3. Suppose that $\mathcal{M}$ is a potential $\mathfrak{aa}$-premouse that satisfies the criteria (1) – (3) in the above definition. Then given $\mathfrak{C}_\alpha^{\mathcal{M}} \in \mathcal{M}$, $C_{\alpha+1}^{\mathcal{M}}$ is definable over $(M, \in, T^{\mathcal{M}})$. This is witnessed by the following formula, expressing that "$y = \mathrm{def}^S_{\mathfrak{aa}}(\mathfrak{C}_\alpha^{\mathcal{M}})$":

$$\forall x[x \in y \leftrightarrow [x \subseteq C_\alpha \wedge \exists \varphi \in \mathcal{L}_{\mathfrak{aa}}(\in, T)$$
$$\exists \overline{b} \in C_\alpha^{<\omega} \forall a(a \in x \leftrightarrow T(\alpha, \varphi(a, \overline{b})))]]$$

Therefore, the sequence $\langle \mathfrak{C}_\alpha^{\mathcal{M}} \mid \alpha < o(\mathcal{M}) \rangle$ is definable over $\mathcal{M}$ by the same argument that $\langle \mathfrak{C}_\alpha \mid \alpha < \beta \rangle$ is definable over $\mathfrak{C}_\beta$ (lemma 2.4.1).

Since we required that $o(\mathcal{M})$ is a limit ordinal, it is possible to define $<_{\mathfrak{aa}}$ within every $\mathfrak{aa}$-premouse $\mathcal{M}$ in the same manner as in $\mathfrak{C}_\alpha$, if α is a limit ordinal. This yields a definable well-ordering $<_{\mathfrak{aa}}^{\mathcal{M}}$ over $\mathcal{M}$, and if for some first-order formula φ, $\mathcal{M} \vDash \exists x \varphi(x)$, then also,

$$\mathcal{M} \vDash \exists x(\varphi(x) \wedge \forall y(y <_{\mathfrak{aa}} x \rightarrow \neg\varphi(y)))$$

That is,

$$\mathcal{M} \vDash \text{“There's a } <_{\mathfrak{aa}}\text{-least } x \text{ such that } \varphi(x).\text{”}$$

The fourth criteria of being an $\mathfrak{aa}$-premouse is needed to make sure that this is true when "$\vDash$" is replaced by "$\vDash_S$", the formula is an $\mathfrak{aa}$-formula, and when we are within a block of $\mathfrak{aa}$-quantifiers.

Example 4.1.4. For every ordinal $\alpha \geq \omega$, let S_α be the set of $\mathfrak{aa}$-formulas true in $\mathfrak{C}_\alpha$. That is, $\varphi(\overline{a}) \in S_\alpha$ iff $(\alpha, \varphi(\overline{a})) \in T_{\alpha+1}$. Then, for every limit ordinal $\alpha \geq \omega_1^V$, $(C_\alpha, \in, T_\alpha, S_\alpha)$ is an $\mathfrak{aa}$-premouse since we assume that $C(\mathfrak{aa})$ is club determined. Furthermore, let $\mathcal{M}' = (M', E, T^{\mathcal{M}'}) \preccurlyeq \mathfrak{C}_\alpha$ be an elementary submodel and $\pi : \mathcal{M}' \rightarrow (M, \in, T^{\mathcal{M}})$ its transitive collapse. Let

$$S^{\mathcal{M}} = \{\varphi(\overline{a}) \mid \overline{a} \in M^{<\omega}, \varphi(\overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T) \text{ and } \mathfrak{C}_\alpha \vDash \varphi(\pi^{-1}(\overline{a}))\}$$

Then $\mathcal{M} = (M, \in, T^{\mathcal{M}}, S^{\mathcal{M}})$ is an $\mathfrak{aa}$-premouse. Note that in this case, $\xi = 0$.

4.2 $\mathfrak{aa}$-ultrapowers

Throughout this section, $\mathcal{M}$ is an $\mathfrak{aa}$-premouse. For each $\varphi(X, x, \overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi})$ and $\overline{a} \in M$ such that $\mathcal{M} \vDash_S \mathfrak{aa}X \exists x \varphi(X, x, \overline{a})$, we extend our vocabulary with an auxiliary notation $f_{\varphi(X,x,\overline{a})}(X)$, to denote the element that according to $S^{\mathcal{M}}$ is $<_{\mathfrak{aa}}$-least satisfying $\varphi(X, x, \overline{a})$, if such exists. That is,

$$\mathcal{M} \vDash_S \mathfrak{aa}X \exists x \varphi(X, x, \overline{a}) \rightarrow \mathfrak{aa}X[\varphi(X, f_{\varphi(X,x,\overline{a})}(X), \overline{a}) \wedge$$
$$\forall y(y <_{\mathfrak{aa}} f_{\varphi(X,x,\overline{a})}(X) \rightarrow \neg\varphi(X, y, \overline{a}))]$$

By criteria (4) in definition 4.1.2,

$\mathcal{M} \vDash_S \mathfrak{aa}X\exists x\varphi(X,x,\overline{a}) \to \mathfrak{aa}X\exists x[\varphi(X,x,\overline{a}) \wedge \forall y(y <_{\mathfrak{aa}} x \to \neg\varphi(X,y,\overline{a}))]$,

so this extension is conservative (and consistent), and we may treat $f_{\varphi(X,x,\overline{a})}(X)$ as any other term in the language.

Let K be the set of all formulas $\varphi(X,x,\overline{a})$ with $\varphi(X,x,\overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi})$ and $\overline{a} \in M^{|\overline{x}|}$ such that $\mathcal{M} \vDash_S \mathfrak{aa}X\exists x\varphi(X,x,\overline{a})$. We define an equivalence relation $\sim$ on K as follows: Given $\varphi(X,x,\overline{a}), \psi(X,x,\overline{b}) \in K$, let

$$\varphi(X,x,\overline{a}) \sim \psi(X,x,\overline{b}) \Longleftrightarrow \mathcal{M} \vDash_S \mathfrak{aa}X \left(f_{\varphi(X,x,\overline{a})}(X) = f_{\psi(X,x,\overline{b})}(X)\right)$$

When the variables or parameters of a formula are not directly relevant for the context, we will simplify our notation and write $f_\varphi(X)$ instead of e.g. $f_{\varphi(X,x,\overline{a})}(X)$.

Definition 4.2.1 ($\mathfrak{aa}$-ultrapower)**.** The $\mathfrak{aa}$-ultrapower $\mathcal{M}^*$ of $\mathcal{M}$ is the structure $(M^*, E, T^{\mathcal{M}^*}, S^{\mathcal{M}^*}, (P_i^{\mathcal{M}^*})_{i<\xi+1})$, where:

1. $M^* = K/\sim$ is the set of equivalence classes of $\sim$.
2. $[\varphi]E[\psi] \Longleftrightarrow \mathcal{M} \vDash_S \mathfrak{aa}X\,(f_\varphi(X) \in f_\psi(X))$.
3. $([\varphi],[\psi]) \in T^{\mathcal{M}^*} \Longleftrightarrow \mathcal{M} \vDash_S \mathfrak{aa}XT\,(f_\varphi(X), f_\psi(X))$.
4. For each $i < \xi$, $P_i^{\mathcal{M}^*} = \{[\varphi] \in M^* \mid \mathcal{M} \vDash_S \mathfrak{aa}XP_i\,(f_\varphi(X))\}$.
5. $P_\xi^{\mathcal{M}^*} = \{[x = a] \mid a \in M\}$.
6. $$\begin{aligned} S^{\mathcal{M}^*} = \{\psi(P_\xi, [\varphi_0], ..., [\varphi_{n-1}]) \mid\ & \psi(X,\overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi}) \text{ and} \\ & \mathcal{M} \vDash_S \mathfrak{aa}X\psi(X, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))\}. \end{aligned}$$

There's a canonical mapping $j : \mathcal{M} \to \mathcal{M}^*$ given by $j(a) = [x = a]$ for $a \in M$. We will subsequently show that if $\mathcal{M}^*$ is well-founded, then its transitive collapse is an $\mathfrak{aa}$-premouse and j composed with the collapsing function is a *weak* $\mathfrak{aa}$*-elementary embedding.*

Definition 4.2.2 (Weak $\mathfrak{aa}$-elementary embedding)**.** Let $\mathcal{M} = (M, \in, T^{\mathcal{M}}, S^{\mathcal{M}}, (P_i^{\mathcal{M}})_{i<\xi})$ and $\mathcal{N} = (N, \in, T^{\mathcal{N}}, S^{\mathcal{N}}, (P_i^{\mathcal{N}})_{i<\xi'})$ be potential $\mathfrak{aa}$-premice such that $\xi \leq \xi'$. Then $\theta : \mathcal{M} \to \mathcal{N}$ is a weak $\mathfrak{aa}$-elementary embedding if for all $\varphi(\overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi})$ and $\overline{a} \in M^{|\overline{x}|}$, we have that

$$\mathcal{M} \vDash_S \varphi(\overline{a}) \Longleftrightarrow \mathcal{N} \vDash_S \varphi(\theta(\overline{a}))$$

Recall the axioms A0 – A5 from Chapter 2. We will need them in the proof of the following Lemma:

Lemma 4.2.3. *Let $\mathcal{M}^*$ be the $\mathfrak{aa}$-ultrapower of $\mathcal{M}$. If $\psi(X, \overline{x})$ is a first-order formula in vocabulary $\{\in, T, (P_i)_{i<\xi}\}$, then*

$$\mathcal{M}^* \vDash \psi(P_\xi, [\varphi_0], ..., [\varphi_{n-1}]) \iff \mathcal{M} \vDash_S \mathfrak{aa} X \psi(X, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))$$

for all $[\varphi_0], ..., [\varphi_{n-1}] \in M^*$.

Proof. The proof is by induction on the structure of ψ. For atomic cases, only the case when $\psi(x) = P_\xi(x)$ is non-trivial. So given $[\varphi(X, x, \overline{a})] \in M^*$, we need to show that $\mathcal{M}^* \vDash P_\xi([\varphi(X, x, \overline{a})]) \iff \mathcal{M} \vDash_S \mathfrak{aa} X(X(f_\varphi(X)))$. Note that $\mathcal{M}^* \vDash P_\xi([\varphi(X, x, \overline{a})])$ iff there's $b \in M$ such that $\mathcal{M} \vDash_S \mathfrak{aa} X(f_\varphi(X) = b)$.

Claim. The following are equivalent:

1. There's $b \in M$ such that $\mathcal{M} \vDash_S \mathfrak{aa} X(f_\varphi(X) = b)$.
2. $\mathcal{M} \vDash_S \mathfrak{aa} X(X(f_\varphi(X)))$.

Proof. First suppose that there's $b \in M$ such that $\mathcal{M} \vDash_S \mathfrak{aa} X(f_\varphi(X) = b)$. By A2, $\mathcal{M} \vDash_S \forall x \mathfrak{aa} X(X(x))$, so in particular, $\mathcal{M} \vDash_S \mathfrak{aa} X(X(b))$. Then by A3 it follows that $\mathcal{M} \vDash_S \mathfrak{aa} X(X(f_\varphi(X)))$.

Now assume that $\mathcal{M} \vDash_S \mathfrak{aa} X(X(f_\varphi(X)))$. By A5,

$$\mathcal{M} \vDash_S \forall y \mathfrak{aa} X(f_\varphi(X) \neq y) \to \mathfrak{aa} X \forall y(X(y) \to f_\varphi(X) \neq y)$$

and so,

$$\mathcal{M} \vDash_S \forall y \mathfrak{aa} X(f_\varphi(X) \neq y) \to \mathfrak{aa} X \forall y(f_\varphi(X) = y \to \neg X(y))$$

By substitution,

$$\mathcal{M} \vDash_S \forall y \mathfrak{aa} X(f_\varphi(X) \neq y) \to \mathfrak{aa} X(\neg X(f_\varphi(X)))$$

By contraposition,

$$\mathcal{M} \vDash_S \neg \mathfrak{aa} X(\neg X(f_\varphi(X))) \to \exists y \neg \mathfrak{aa} X(f_\varphi(X) \neq y)$$

By club determinacy,

$$\mathcal{M} \vDash_S \mathfrak{aa} X(X(f_\varphi(X))) \to \exists y \mathfrak{aa} X(f_\varphi(X) = y)$$

Hence, $\mathcal{M} \vDash_S \exists y \mathfrak{aa} X(f_\varphi(X) = y)$. By the definition of a potential $\mathfrak{aa}$-premouse, there's $b \in M$ such that $\mathcal{M} \vDash_S \mathfrak{aa} X(f_\varphi(X) = b)$. $\square$

This completes the atomic cases. The induction-step for $\wedge$ is an easy application of the induction assumption (and A3). In the induction-step for negation we need club determinacy. We omit the details and focus on the induction-step for the existential quantifier. First suppose that $\mathcal{M}^* \vDash$

$\exists x\psi(P_\xi, x, [\varphi_0], ..., [\varphi_{n-1}])$. Then there's $[\chi] \in M^*$ such that $\mathcal{M}^* \vDash \psi(P_\xi, [\chi], [\varphi_0], ..., [\varphi_{n-1}])$. By the induction assumption,

$$\mathcal{M} \vDash_S \mathfrak{aa}X\psi(X, f_\chi(X), f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))$$

Hence,

$$\mathcal{M} \vDash_S \mathfrak{aa}X\exists x\psi(X, x, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))$$

This proves "$\Rightarrow$". Now assume that,

$$\mathcal{M} \vDash_S \mathfrak{aa}X\exists x\psi(X, x, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))$$

Let χ be the formula $\psi(X, x, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))$. Then,

$$\mathcal{M} \vDash_S \mathfrak{aa}X\psi(X, f_\chi(X), f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))$$

So by the induction assumption, $\mathcal{M}^* \vDash \psi(P_\xi, [\chi], [\varphi_0], ..., [\varphi_{n-1}])$. Therefore, $\mathcal{M}^* \vDash \exists x\psi(P_\xi, x, [\varphi_0], ..., [\varphi_{n-1}])$. This proves "$\Leftarrow$". □

Lemma 4.2.4. *Let $\mathcal{M}^*$ be the $\mathfrak{aa}$-ultrapower of $\mathcal{M}$. Then $S^{\mathcal{M}^*}$ is consistent.*

Proof. For the sake of a contradiction, assume that $S^{\mathcal{M}^*}$ is not consistent. Let $\psi_i(P_\xi, [\varphi_0], ..., [\varphi_n])$, $i < m$, be a finite sequence of formulas in $S^{\mathcal{M}^*}$ such that

$$\bigwedge_{i<m} \psi_i(P_\xi, [\varphi_0], ..., [\varphi_{n-1}]) \vdash \bot$$

Then,

$$\mathcal{M} \vDash_S \bigwedge_{i<m} \mathfrak{aa}X\psi_i(X, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))$$

By A3 we get that

$$\mathcal{M} \vDash_S \mathfrak{aa}X \bigwedge_{i<m} \psi_i(X, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))$$

By induction on the length of $\mathcal{L}_{\mathfrak{aa}}$-proofs it follows that $\mathcal{M} \vDash_S \mathfrak{aa}X\bot$. This contradicts A1 and the consistency of $S^{\mathcal{M}}$. □

Theorem 4.2.5. *Let $\mathcal{M}^*$ be the $\mathfrak{aa}$-ultrapower of $\mathcal{M}$ and $\psi(X, \overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi})$. Then*

$$\mathcal{M}^* \vDash_S \psi(P_\xi, [\varphi_0], ..., [\varphi_{n-1}]) \Longleftrightarrow \mathcal{M} \vDash_S \mathfrak{aa}X\psi(X, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))$$

for all $[\varphi_0], ..., [\varphi_{n-1}] \in M^*$.

Proof. The direction from right to left is the definition of $S^{\mathcal{M}^*}$. For the direction from left to right, suppose that $\mathcal{M}^* \vDash_S \psi(P_\xi, [\varphi_0], ..., [\varphi_{n-1}])$ holds but $\mathcal{M} \nvDash_S \mathfrak{aa}X\psi(X, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))$. By club determinacy,

$$\mathcal{M} \vDash_S \mathfrak{aa}X\neg\psi(X, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))$$

Thus, $\mathcal{M}^* \vDash_S \neg\psi(P_\xi, [\varphi_0], ..., [\varphi_{n-1}])$. This contradicts the consistency of $S^{\mathcal{M}^*}$. □

Theorem 4.2.6. *Let $\mathcal{M}^*$ be the $\mathfrak{aa}$-ultrapower of $\mathcal{M}$ and suppose that $\mathcal{M}^*$ is well-founded. Let $\pi : \mathcal{M}^* \to \mathcal{N}$ be its transitive collapse. Then $\mathcal{N}$ is a potential $\mathfrak{aa}$-premouse and $\pi \circ j$ is a weak $\mathfrak{aa}$-elementary embedding.*

Proof. First we show that $\mathcal{N}$ is a potential $\mathfrak{aa}$-premouse:

1. $\mathcal{M}$ is transitive so $\mathcal{M}$ satisfies the axiom of extensionality. By Lemma 4.2.3, $\mathcal{M}^*$ satisfies the axiom of extensionality. Thus, M^* collapses to a transitive set N. If $a \in V_\omega$, then

$$\mathcal{M} \vDash \forall x(x \in a \leftrightarrow \bigvee_{b \in a} x = b)$$

and this is a first-order formula. By Lemma 4.2.3 and induction, $\pi([x = a]) = a$. Thus, $V_\omega \subseteq N$.

2. By construction.

3. By Lemma 4.2.3, $S^{\mathcal{M}^*}$ extends the first-order elementary diagram of $(M^*, \in, T^{\mathcal{M}^*}, (P_i^{\mathcal{M}^*})_{i<\xi+1})$. Consistency was proved in Lemma 4.2.4. As for completeness, it follows from the club determinacy schema in $S^{\mathcal{M}}$. If

$$\mathcal{M}^* \nvDash_S \psi(P_\xi, [\varphi_0], ..., [\varphi_{n-1}]),$$

then by Theorem 4.2.5,

$$\mathcal{M} \nvDash_S \mathfrak{aa}X\psi(X, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X)).$$

Thus, by club determinacy,

$$\mathcal{M} \vDash_S \mathfrak{aa}X\neg\psi(X, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X)).$$

Hence, $\mathcal{M}^* \vDash_S \neg\psi(P_\xi, [\varphi_0], ..., [\varphi_{n-1}])$.

4. Suppose that $\psi(P_\xi, \overline{x}, \overline{Y}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi+1})$ and $[\varphi_0], ..., [\varphi_{n-1}] \in M^*$. Then,

$$\begin{aligned}\mathcal{M} \vDash_S \mathfrak{aa}Z\mathfrak{aa}\overline{Y}[&\mathfrak{aa}X\psi(X, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X), \overline{Y})\vee\\ &\mathfrak{aa}X\neg\psi(X, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X), \overline{Y})]\end{aligned}$$

So by Theorem 4.2.5,

$$\begin{aligned}\mathcal{M}^* \vDash_S \mathfrak{aa}\overline{Y}[&\mathfrak{aa}X\psi(P_\xi, [\varphi_0], ..., [\varphi_{n-1}], \overline{Y})\vee\\ &\mathfrak{aa}X\neg\psi(P_\xi, [\varphi_0], ..., [\varphi_{n-1}], \overline{Y})]\end{aligned}$$

5. Suppose that $\psi(P_\xi, x, \overline{y}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi+1})$ and $[\varphi_0], ..., [\varphi_{n-1}] \in M^*$. Then,

$$\begin{aligned}
&\mathcal{M}^* \vDash_S \exists x \psi(P_\xi, x, [\varphi_0], ..., [\varphi_{n-1}]) \Rightarrow \\
&\mathcal{M} \vDash_S \mathfrak{aa} X \exists x \psi(X, x, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X)) \Rightarrow \\
&\mathcal{M} \vDash_S \mathfrak{aa} X \psi(X, f_\psi(X), f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X)) \Rightarrow \\
&\mathcal{M}^* \vDash_S \psi(P_\xi, [\psi], [\varphi_0], ..., [\varphi_{n-1}])
\end{aligned}$$

Thus, $[\psi] = [\psi(X, x, f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))] \in M^*$ is a witness of $\mathcal{M}^* \vDash_S \exists x \psi(P_\xi, x, [\varphi_0], ..., [\varphi_{n-1}])$.

6. For $i < \xi$ this follows from Theorem 4.2.5. For $i = \xi$, we have that

$$\mathcal{M}^* \vDash_S \mathfrak{aa} X \forall x (P_\xi(x) \to X(x)) \iff \mathcal{M} \vDash_S \mathfrak{aa} Y \mathfrak{aa} X \forall x (Y(x) \to X(x))$$

So the claim follows from A2.

7. Follows directly from Theorem 4.2.5.

Note that we proved (2)-(7) for $\mathcal{M}^*$ instead of $\mathcal{N}$. Since $\mathcal{M}^* \cong \mathcal{N}$, it follows that $\mathcal{N}$ is a potential $\mathfrak{aa}$-premouse. Lastly, we show that $\pi \circ j$ is a weak $\mathfrak{aa}$-elementary embedding. Let $\varphi(\overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi})$ and $a_0, ..., a_{n-1} \in M$. Then,

$$\begin{aligned}
&\mathcal{N} \vDash_S \varphi(\pi(j(a_0)), ..., \pi(j(a_{n-1}))) \iff \\
&\mathcal{M}^* \vDash_S \varphi([x = a_0], ..., [x = a_{n-1}]) \iff \\
&\mathcal{M} \vDash_S \mathfrak{aa} X \varphi(f_{x=a_0}(X), ..., f_{x=a_{n-1}}(X)) \iff \\
&\mathcal{M} \vDash_S \varphi(a_0, ..., a_{n-1})
\end{aligned}$$

□

Theorem 4.2.7. *Let $\mathcal{M}^*$ be the $\mathfrak{aa}$-ultrapower of $\mathcal{M}$ and suppose that $\mathcal{M}^*$ is well-founded. Let $\pi : \mathcal{M}^* \to \mathcal{N}$ be its transitive collapse. Then $\mathcal{N}$ is an $\mathfrak{aa}$-premouse.*

Proof. We already know that $\mathcal{N}$ is a potential $\mathfrak{aa}$-premouse. We need to show that $\mathcal{N}$ satisfies criteria (1) – (4) in Definition 4.1.2. By remark 4.1.3, there's a first-order formula $\phi(x, y)$ such that for all $a, b \in M$, $\mathcal{M} \vDash \phi(a, b)$ iff for some ordinal $\alpha < o(\mathcal{M})$, $a = \alpha$ and $b = C_\alpha^{\mathcal{M}}$. Then since $\mathcal{M}$ satisfies criteria (1) and (2), $\mathcal{M} \vDash \forall x \exists y \exists z (\phi(y, z) \wedge x \in z)$. By Lemma 4.2.3, $\phi(x, y)$ defines a function F over $\mathcal{N}$ and $\mathcal{N} \vDash \forall x \exists y \exists z (\phi(y, z) \wedge x \in z)$. Thus, criteria (1) and (2) follows from (3). We show that criteria (3) holds. By Lemma 4.2.3, it follows that:

$$\mathcal{N} \vDash \forall x \forall y \left(T(x, y) \to \left(x \in \mathrm{Ord} \wedge y \in \left(\mathcal{L}_{\mathfrak{aa}}(\in, T) \times F(x)^{<\omega} \right) \right) \right)$$

Therefore,

$$T^{\mathcal{N}} \subseteq \{(\alpha, \varphi(\overline{a})) \mid \alpha < o(\mathcal{N}), \overline{a} \in F(\alpha)^{<\omega}, \varphi \in \mathcal{L}_{\mathfrak{aa}}(\in, T)\}$$

Hence, we only need to show that $(\alpha, \varphi(\overline{a})) \in T^{\mathcal{N}}$ iff $\mathcal{N} \vDash_S \varphi^{\mathfrak{C}_\alpha^{\mathcal{N}}}(\overline{a})$, for all $\alpha < o(\mathcal{N})$, $\varphi \in \mathcal{L}_{\mathfrak{aa}}(\in, T)$ and $\overline{a} \in C_\alpha^{\mathcal{N}}$. By induction on $\alpha \geq \omega$, $\alpha < o(\mathcal{N})$, we show the following two statements:

$$F(\alpha) = C_\alpha^{\mathcal{N}} \tag{a}$$

$$\text{For all } \varphi \in \mathcal{L}_{\mathfrak{aa}}(\in, T) \text{ and } \overline{a} \in C_\alpha^{<\omega},\ (\alpha, \varphi(\overline{a})) \in T^{\mathcal{N}} \iff \mathcal{N} \vDash_S \varphi^{\mathfrak{C}_\alpha^{\mathcal{N}}}(\overline{a}) \tag{b}$$

W.l.o.g. $o(\mathcal{N}) > \omega$. Then by the definition of F, $F(\omega) = V_\omega$. We will show that if (a) holds for α, then also (b) holds for α. Suppose for a contradiction that there's $\varphi \in \mathcal{L}_{\mathfrak{aa}}(\in, T)$ and $[\psi], [\varphi_0], ..., [\varphi_{n-1}] \in M^*$ such that

$$\mathcal{M}^* \vDash_S T([\psi], \varphi([\varphi_0], ..., [\varphi_{n-1}])) \wedge \neg\varphi^{F([\psi])}([\varphi_0], ..., [\varphi_{n-1}])$$

Then,

$$\begin{aligned}\mathcal{M} \vDash_S \mathfrak{aa}X[T(f_\psi(X), \varphi(f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X)))\wedge \\ \neg\varphi^{F(f_\psi(X))}(f_{\varphi_0}(X), ..., f_{\varphi_{n-1}}(X))]\end{aligned}$$

So,

$$\mathcal{M} \vDash_S \mathfrak{aa}X \exists\alpha\exists\overline{x}(T(\alpha, \varphi(\overline{x})) \wedge \neg\varphi^{F(\alpha)}(\overline{x}))$$

Thus, since X does not appear in the right hand side of the formula, $\mathcal{M} \vDash \exists\alpha\exists\overline{x}(T(\alpha, \varphi(\overline{x})) \wedge \neg\varphi^{F(\alpha)}(\overline{x}))$. This contradicts that $\mathcal{M}$ is an $\mathfrak{aa}$-premouse.

Similarly, suppose for a contradiction that there's $\varphi \in \mathcal{L}_{\mathfrak{aa}}(\in, T)$ and $[\psi], [\varphi_0], ..., [\varphi_{n-1}] \in M^*$ such that

$$\mathcal{M}^* \vDash_S \neg T([\psi], \varphi([\varphi_0], ..., [\varphi_{n-1}])) \wedge \varphi^{F([\psi])}([\varphi_0], ..., [\varphi_{n-1}])$$

From this we obtain a contradiction in the same way as above. This show that (a) implies (b).

Now by the definition of F, (b) implies that $F(\alpha+1) = C_{\alpha+1}^{\mathcal{N}}$. Hence, the induction-step for successor ordinals is clear. At limit α, (a) is immediate and (b) follows from (a) as before.

Lastly, criteria (4) follows from Theorem 4.2.5 and club determinacy. $\square$

Remark 4.2.8. In the above proof, when we write

$$\mathcal{M}^* \vDash_S T([\psi], \varphi([\varphi_0], ..., [\varphi_{n-1}])) \wedge \neg\varphi^{F([\psi])}([\varphi_0], ..., [\varphi_{n-1}]),$$

note that this statement only makes sense for every particular instance of a formula φ. That is, when we say "Suppose there's φ such that...", this quantification is not done within $S^{\mathcal{M}}$, but on the "outside".

Lemma 4.2.9. $j[M] \neq M^*$.

Proof. By the definition of a potential $\mathfrak{aa}$-premouse, $\mathcal{M} \vDash_S \mathfrak{aa}X\exists x(\neg X(x))$. Thus, $[\neg X(x)] \in M^*$. Suppose for the sake of contradiction that there's $a \in M$ such that $j(a) = [\neg X(x)]$. Then,

$$\mathcal{M} \vDash_S \mathfrak{aa}X(f_{\neg X(x)}(X) = a)$$

so in particular, $\mathcal{M} \vDash_S \mathfrak{aa}X(\neg X(a))$. This contradicts A2. $\square$

Lemma 4.2.10. *The least ordinal γ such that $\mathcal{M} \vDash_S \neg\mathfrak{aa}X\forall x(x \in \gamma \to X(x))$ is the critical point of j.*

Proof. First suppose that $\alpha < \gamma$. Then, $\mathcal{M} \vDash_S \mathfrak{aa}X\forall x(x \in \alpha \to X(x))$. If $[\varphi]E[x = \alpha]$, then $\mathcal{M} \vDash_S \mathfrak{aa}X(f_\varphi(X) \in \alpha)$. By combining the formulas, it follows that $\mathcal{M} \vDash_S \mathfrak{aa}X(X(f_\varphi(X)))$. By the proof of Lemma 4.2.3 we can find $\beta \in M$ such that $\mathcal{M} \vDash_S \mathfrak{aa}X(f_\varphi(X) = \beta)$. Hence, $[\varphi] = [x = \beta]$. Therefore, $[x = \alpha]$ collapses to α, so α is below the critical point.

Next, consider the element $[x \in \gamma \wedge \neg X(x)] \in \mathcal{M}^*$. We claim that this element witnesses that γ is the critical point of j. We have that $[x \in \gamma \wedge \neg X(x)]Ej(\gamma)$. Suppose there's $\alpha < \gamma$ such that $[x \in \gamma \wedge \neg X(x)] = [x = \alpha]$. Then,

$$\mathcal{M} \vDash_S \mathfrak{aa}X(f_{x\in\gamma\wedge\neg X(x)}(X) = \alpha)$$

Hence, $\mathcal{M} \vDash_S \mathfrak{aa}X(\neg X(\alpha))$. This contradicts A2. Therefore, $[x \in \gamma \wedge \neg X(x)]$ is according to $\mathcal{M}^*$ an ordinal larger than all ordinals of the form $j(\beta)$ for $\beta < \gamma$, but smaller than $j(\gamma)$. Thus, γ is the critical point of j. $\square$

Lemma 4.2.11. *Let $\mathcal{K}$ be an $\mathfrak{aa}$-premouse of the form $(C_\alpha, \in, T_\alpha, S'_\alpha, (P_i^{\mathcal{K}})_{i<\xi})$, $\alpha \geq \omega_1^V$, where S'_α is the set of true formulas (with parameters) in the structure $(C_\alpha, \in, T_\alpha, (P_i^{\mathcal{K}})_{i<\xi})$. Suppose that $\mathcal{M} = (M, \in, T^{\mathcal{M}}, S^{\mathcal{M}}, (P_i^{\mathcal{M}})_{i<\xi})$ is a countable*[1] *$\mathfrak{aa}$-premouse and $\theta : \mathcal{M} \to \mathcal{K}$ is a weak $\mathfrak{aa}$-elementary embedding. Let $\mathcal{M}^*$ be the $\mathfrak{aa}$-ultrapower of $\mathcal{M}$. Then there is:*

1. $P_\xi^{\mathcal{K}'} \subseteq C_\alpha$,
2. *a structure $\mathcal{K}' = (C_\alpha, \in, T_\alpha, S''_\alpha, (P_i^{\mathcal{K}})_{i<\xi}, P_\xi^{\mathcal{K}'})$ such that*

$$S''_\alpha = \{\varphi(\overline{a}) \mid \varphi(\overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi+1}), \overline{a} \in C_\alpha \textit{ and } (C_\alpha, \in, T_\alpha, (P_i^{\mathcal{K}'})_{i<\xi+1}) \vDash \varphi(\overline{a})\},$$

3. *and a weak $\mathfrak{aa}$-elementary embedding $\theta^* : \mathcal{M}^* \to \mathcal{K}'$.*

[1] Countable in V and $\xi < \omega_1^V$.

Proof. Given $[\varphi(X,x,\overline{a})] \in M^*$, we have $\mathcal{M} \vDash_S \mathfrak{aa}X\exists x\varphi(X,x,\overline{a})$. By the weak $\mathfrak{aa}$-elementarity of θ, $\mathcal{K} \vDash_S \mathfrak{aa}X\exists x\varphi(X,x,\theta(\overline{a}))$. Since we required that this actually coincide with the satisfaction in $\mathcal{K}$, we get $\mathcal{K} \vDash \mathfrak{aa}X\exists x\varphi(X,x,\theta(\overline{a}))$. Let $D_{\overline{a},\varphi} \subseteq P_{\omega_1}(C_\alpha)$ be a club of sets such that if $A \in D_{\overline{a},\varphi}$, then $\mathcal{K} \vDash \exists x\varphi(A,x,\theta(\overline{a}))$. Furthermore, let

$$Q := \bigcap \{D_{\overline{a},\varphi} \mid [\varphi(X,x,\overline{a})] \in M^*\}$$

Then since $\mathcal{M}$ is countable in V, it follows by the ω_1-completeness of the club filter that Q is non-empty. Fix $A^* \in Q$. Define $P_\xi^{\mathcal{K}'} = A^*$ and $\theta^* : M^* \to C_\alpha$ according to

$$\theta^*([\varphi(X,x,\overline{a})]) = \text{ the } <_{\mathfrak{aa}} -\text{least } b \in C_\alpha \text{ such that } \mathcal{K} \vDash \varphi(A^*,b,\theta(\overline{a}))$$

Let S''_α be the $\mathfrak{aa}$-theory of the structure $(C_\alpha, \in, T_\alpha, (P_i^{\mathcal{K}})_{i<_{\xi+1}})$. Then given $a \in M$, $\theta^*(j(a)) = \theta^*([x = a]) = \theta(a)$. We show that θ^* is a weak $\mathfrak{aa}$-elementary embedding. Let $\psi(P_\xi, \overline{y}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi+1})$ and $[\varphi_0(X,x,\overline{a}_0)]$, ..., $[\varphi_{n-1}(X,x,\overline{a}_{n-1})] \in M^*$. Then,

$$\begin{aligned}
&\mathcal{M}^* \vDash_S \psi(P_\xi, [\varphi_0(X,x,\overline{a}_0)], ..., [\varphi_{n-1}(X,x,\overline{a}_{n-1})])\\
\Rightarrow &\mathcal{M} \vDash_S \mathfrak{aa}X\psi(X, f_{\varphi_0(X,x,\overline{a}_0)}(X), ..., f_{\varphi_{n-1}(X,x,\overline{a}_{n-1})}(X))\\
\Rightarrow &\mathcal{K} \vDash \mathfrak{aa}X\psi(X, f_{\varphi_0(X,x,\theta(\overline{a}_0))}(X), ..., f_{\varphi_{n-1}(X,x,\theta(\overline{a}_{n-1}))}(X))
\end{aligned}$$

By the definition of A^*, it follows that

$$\begin{aligned}
&\mathcal{K} \vDash \psi(P_\xi, f_{\varphi_0(X,x,\theta(\overline{a}_0))}(A^*), ..., f_{\varphi_{n-1}(X,x,\theta(\overline{a}_{n-1}))}(A^*))\\
\Rightarrow &\mathcal{K} \vDash \psi(P_\xi, \theta^*([\varphi_0(X,x,\overline{a}_0)]), ..., \theta^*([\varphi_{n-1}(X,x,\overline{a}_{n-1})]))\\
\Rightarrow &\mathcal{K}' \vDash_S \psi(P_\xi, \theta^*([\varphi_0(X,x,\overline{a}_0)]), ..., \theta^*([\varphi_{n-1}(X,x,\overline{a}_{n-1})]))
\end{aligned}$$

The other direction follows from club determinacy, since if

$$\mathcal{M}^* \nvDash_S \psi(P_\xi, [\varphi_0(X,x,\overline{a}_0)], ..., [\varphi_{n-1}(X,x,\overline{a}_{n-1})])$$

Then by club determinacy,

$$\mathcal{M} \vDash_S \mathfrak{aa}X\neg\psi(X, f_{\varphi_0(X,x,\overline{a}_0)}(X), ..., f_{\varphi_{n-1}(X,x,\overline{a}_{n-1})}(X)),$$

and we can proceed as above. □

Lemma 4.2.11 implies that if we have a countable $\mathfrak{aa}$-premouse $\mathcal{M}$ and a weak $\mathfrak{aa}$-elementary embedding of $\mathcal{M}$ into a level $\mathfrak{C}_\alpha$ of $C(\mathfrak{aa})$, then its $\mathfrak{aa}$-ultrapower is well-founded. Then, $\mathcal{M}^*$ is in fact an $\mathfrak{aa}$-premouse, and we identify $\mathcal{M}^*$ with its transitive collapse.

4.3 Iterating $\mathfrak{aa}$-ultrapowers

Definition 4.3.1 ($\mathfrak{aa}$-iteration and $\mathfrak{aa}$-mouse)**.** Let $\mathcal{M}_0$ be an $\mathfrak{aa}$-premouse. The $\mathfrak{aa}$-*iteration* starting from $\mathcal{M}_0$ is the sequence of models

$$\langle \mathcal{M}_\alpha \mid 0 \leq \alpha \leq \gamma \rangle, \text{ such that}$$

1. For each $\alpha < \gamma$, $\mathcal{M}_{\alpha+1}$ is the $\mathfrak{aa}$-ultrapower of $\mathcal{M}_\alpha$.
2. If $\alpha \leq \gamma$ is a limit ordinal, then $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$ (the direct limit).
3. $\gamma \leq \omega_1$ is the least ordinal such that $\mathcal{M}_{\gamma+1}$ is not well-founded, or ω_1 if no such exists.

If $\gamma = \omega_1$, then $\mathcal{M}_0$ is called an $\mathfrak{aa}$-*mouse*.

Some comments on the limit steps might be required. First, note that the composition of two weak $\mathfrak{aa}$-elementary embeddings is a weak $\mathfrak{aa}$-elementary embedding. So if we have $\langle \mathcal{M}_\alpha \mid 0 \leq \alpha \leq \gamma \rangle$ and for every $\alpha \leq \gamma$, $\mathcal{M}_\alpha$ is an $\mathfrak{aa}$-premouse, then we also have weak $\mathfrak{aa}$-elementary embeddings $j_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ for all $\alpha < \beta \leq \gamma$.

Suppose that $\delta \leq \omega_1^V$ is a limit ordinal and for all $\alpha < \beta < \delta$, $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are $\mathfrak{aa}$-premice and $j_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is a weak $\mathfrak{aa}$-elementary embedding. Then in order to construct $\mathcal{M}_\delta$, we first take the direct limit of all $(M_\alpha, \in, T^{\mathcal{M}_\alpha}, (P_i)_{i<\xi(\alpha)})$, for $\alpha < \delta$. Then, since the embeddings $j_{\alpha,\beta}$ restricted to this structure are first-order elementary embeddings, we know that we have a well-defined limit model $(M_\delta, \in, T^{\mathcal{M}_\delta}, (P_i)_{i<\xi(\delta)})$, with $\xi(\delta) = \sup_{\alpha<\delta} \xi(\alpha)$. Also, we have a first-order elementary embedding $j_{\alpha,\delta} : M_\alpha \to M_\delta$ for all $\alpha < \delta$. Now, we can set

$$S^{\mathcal{M}_\delta} := \{\psi(j_{\alpha,\delta}(\overline{a})) \mid \alpha < \delta,\ \psi(\overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\xi(\delta)}) \text{ and } \mathcal{M}_\alpha \models_S \psi(\overline{a})\}$$

Then, using similar arguments as in the previous section, one can show that if $\mathcal{M}_\delta$ is well-founded, then it is an $\mathfrak{aa}$-premouse and $j_{\alpha,\delta}$ is a weak $\mathfrak{aa}$-elementary embedding. For the following Theorem, recall that by S_α, we mean the set of $\mathfrak{aa}$-formulas (with parameters from C_α) that are true in $\mathfrak{C}_\alpha$.

Theorem 4.3.2. *Suppose that $(M, E, T^{\mathcal{M}}) \preccurlyeq \mathfrak{C}_\alpha$ for some limit ordinal $\alpha \geq \omega_1^V$, and that M is countable in V. Let $S^{\mathcal{M}} = S_\alpha \cap M$ and $\mathcal{M}_0 = (M_0, \in, T^{\mathcal{M}_0}, S^{\mathcal{M}_0})$ be the transitive collapse of $\mathcal{M} = (M, E, T^{\mathcal{M}}, S^{\mathcal{M}})$. Then $\mathcal{M}_0$ is an $\mathfrak{aa}$-mouse.*

Proof. Let $\pi : \mathcal{M} \to \mathcal{M}_0$ be the transitive collapse and $\theta_0 = \pi^{-1} : \mathcal{M}_0 \to \mathfrak{C}_\alpha$. Then θ_0 is a weak $\mathfrak{aa}$-elementary embedding. We can apply Lemma 4.2.11 inductively to obtain for each $\beta \leq \omega_1^V$, a weak $\mathfrak{aa}$-elementary embedding

$$\theta_\beta : \mathcal{M}_\beta \to \mathcal{K}_\beta := (C_\alpha, \in, T_\alpha, S_\alpha^\beta, (P_i^{\mathcal{K}_\beta})_{i<\beta})$$

If $\delta \leq \omega_1^V$ is a limit, then we define $\theta_\delta : \mathcal{M}_\delta \to \mathcal{K}_\delta$ as follows: For each $a \in M_\delta$, fix $\beta < \delta$ and $b \in M_\beta$ such that $a = j_{\beta,\delta}(b)$. Then $\theta_\delta(a) := \theta_\beta(b)$. Clearly, θ_δ is a weak $\mathfrak{aa}$-elementary embedding. It follows that $\mathcal{M}_\beta$ is well-founded for each $\beta \leq \omega_1^V$. □

Lemma 4.3.3. *Let $\mathcal{M} = (M, \in, T^{\mathcal{M}}, S^{\mathcal{M}}, (P_i^{\mathcal{M}})_{i<\xi})$ be an $\mathfrak{aa}$-premouse and $\mathcal{M}^*$ its $\mathfrak{aa}$-ultrapower. Then for each $i < \xi$, $j[P_i^{\mathcal{M}}] = P_i^{\mathcal{M}^*}$.*

Proof. Recall that $P_i^{\mathcal{M}^*} = \{[\varphi] \in M^* \mid \mathcal{M} \vDash_S \mathfrak{aa}X P_i(f_\varphi(X))\}$. Trivially, $j[P_i^{\mathcal{M}}] \subseteq P_i^{\mathcal{M}^*}$. For the other direction, suppose that $[\varphi] \in P_i^{\mathcal{M}^*}$. Then by criteria (6) of being a potential $\mathfrak{aa}$-premouse, it follows that $\mathcal{M} \vDash_S \mathfrak{aa}X(X(f_\varphi(X)))$. By the proof of Lemma 4.2.3, there's $b \in M$ such that $\mathcal{M} \vDash_S \mathfrak{aa}X(f_\varphi(X) = b)$. Thus, $[\varphi] = [x = b]$. □

Lemma 4.3.4. *Suppose that $\mathcal{M}_0$ is a countable (in V) $\mathfrak{aa}$-mouse with no predicates P_i. Let $\langle \mathcal{M}_\alpha \mid 0 \leq \alpha \leq \omega_1^V \rangle$ be its $\mathfrak{aa}$-iteration. Then $C = \{P_i^{\mathcal{M}_{\omega_1}} \mid i < \omega_1^V\}$ is a club in $\mathcal{P}_{\omega_1}(M_{\omega_1})$.*

Proof. Note that by Lemma 4.3.3, for each $i < \omega_1^V$, $P_i^{\mathcal{M}_{\omega_1}} = j_{i,\omega_1}[M_i]$. Since at limit steps in the $\mathfrak{aa}$-iteration, we take unions, it follows that C is closed under increasing ω-sequences. By Lemma 4.2.9, the sequence $\langle P_i^{\mathcal{M}_{\omega_1}} \mid i < \omega_1 \rangle$ is continuously increasing. Now, suppose that $A \in \mathcal{P}_{\omega_1}(M_{\omega_1})$. Since ω_1^V is regular, we can find $\alpha < \omega_1^V$ such that $A \subseteq j_{\alpha,\omega_1}[M_\alpha]$. Thus, there's $B \in C$ such that $B \supseteq A$. □

Lemma 4.3.5. *Let $\mathcal{M}$ be an $\mathfrak{aa}$-mouse with no predicates P_i and $\mathcal{M}_{\omega_1}$ its ω_1-th iteration. For each $\varphi(\overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T, (P_i)_{i<\omega_1})$ and $\overline{a} \in M_{\omega_1}^{<\omega}$, we have that:*

$$\mathcal{M}_{\omega_1} \vDash_S \varphi(\overline{a}) \iff \mathcal{M}_{\omega_1} \vDash \varphi(\overline{a})$$

Proof. The proof is by induction on the structure of φ. If φ is atomic, then the statement follows from Lemma 4.2.3 since $\mathcal{M}_{\omega_1}$ is an $\mathfrak{aa}$-premouse. The induction-steps for conjunction and negation are straightforward. In the induction-step for the existential quantifier, we use the critera (5) of being a potential $\mathfrak{aa}$-premouse. We look at the induction-step for the $\mathfrak{aa}$-quantifier in detail.

Suppose that $\varphi(\overline{a}) = \mathfrak{aa}X\psi(X, \overline{a})$, $\mathcal{M}_{\omega_1} \vDash_S \mathfrak{aa}X\psi(X, \overline{a})$ and that the equivalence holds for ψ. Fix $\beta < \omega_1^V$ such that $\overline{a} = j_{\beta,\omega_1}(\overline{b})$ for some $\overline{b} \in M_\beta^{<\omega}$, and for all P_i occuring in φ, $i < \beta$. Then by the weak $\mathfrak{aa}$-elementarity of j_{β,ω_1}, $\mathcal{M}_\beta \vDash_S \mathfrak{aa}X\psi(X, \overline{b})$. Then also, $\mathcal{M}_\alpha \vDash_S \mathfrak{aa}X\psi(X, j_{\beta,\alpha}(\overline{b}))$ for all $\beta \leq \alpha < \omega_1^V$. Hence, $\mathcal{M}_{\alpha+1} \vDash_S \psi(P_\alpha, j_{\beta,\alpha+1}(\overline{b}))$ for all $\beta \leq \alpha < \omega_1^V$. Thus, $\mathcal{M}_{\omega_1} \vDash_S \psi(P_i, \overline{a})$ for all $\beta \leq i < \omega_1^V$. By the induction assumption, $\mathcal{M}_{\omega_1} \vDash \psi(P_i, \overline{a})$ for all $\beta \leq i < \omega_1^V$. From Lemma 4.3.4, it follows that $\mathcal{M}_{\omega_1} \vDash \mathfrak{aa}X\psi(X, \overline{a})$, as desired.

For the other direction, suppose that $\mathcal{M}_{\omega_1} \nvDash_S \mathfrak{aa}X\psi(X, \overline{a})$. Then by club determinacy, $\mathcal{M}_{\omega_1} \vDash_S \mathfrak{aa}X\neg\psi(X, \overline{a})$. Thus, we can argue as above. □

Theorem 4.3.6 (Kennedy, Magidor and Väänänen, [9]). *Let $\mathcal{M}$ be an $\mathfrak{aa}$-mouse and $\mathcal{M}_{\omega_1}$ its ω_1-th iteration. For each $\alpha \leq o(\mathcal{M}_{\omega_1})$,*

1. $T^{\mathcal{M}_{\omega_1}} \restriction \alpha = T_\alpha$
2. $C_\alpha^{\mathcal{M}_{\omega_1}} = C_\alpha$

In particular, there's an ordinal $\gamma = o(\mathcal{M}_{\omega_1})$ such that $(M_{\omega_1}, \in, T^{\mathcal{M}_{\omega_1}}) = \mathfrak{C}_\gamma$.

Proof. We prove (1) and (2) simultaneously by induction on α. The only non-trivial case is the successor-step. So suppose that (1) and (2) holds for $\alpha < o(\mathcal{M}_{\omega_1})$. Now by definition,

$$\begin{aligned} C_{\alpha+1} &= \mathrm{def}_{\mathfrak{aa}}(\mathfrak{C}_\alpha) \\ C_{\alpha+1}^{\mathcal{M}_{\omega_1}} &= \mathrm{def}_{\mathfrak{aa}}^{S^{\mathcal{M}_{\omega_1}}}(\mathfrak{C}_\alpha^{\mathcal{M}_{\omega_1}}) \end{aligned}$$

By our induction assumption, $\mathfrak{C}_\alpha = \mathfrak{C}_\alpha^{\mathcal{M}_{\omega_1}}$, so the above sets are equal if $T^{\mathcal{M}_{\omega_1}} \restriction (\alpha+1) = T_{\alpha+1}$. Given $\varphi(\overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T)$ and $\overline{a} \in C_\alpha^{|\overline{x}|}$, we have that

$$\begin{aligned} (\alpha, \varphi(\overline{a})) \in T^{\mathcal{M}_{\omega_1}} \restriction (\alpha+1) &\Longleftrightarrow \mathcal{M}_{\omega_1} \models_S \varphi^{\mathfrak{C}_\alpha^{\mathcal{M}_{\omega_1}}}(\overline{a}) \\ &\Longleftrightarrow \mathcal{M}_{\omega_1} \models_S \varphi^{\mathfrak{C}_\alpha}(\overline{a}) && \text{(I.A.)} \\ &\Longleftrightarrow \mathcal{M}_{\omega_1} \models \varphi^{\mathfrak{C}_\alpha}(\overline{a}) && \text{(Lemma 4.3.5)} \\ &\Longleftrightarrow \mathfrak{C}_\alpha \models \varphi(\overline{a}) && \text{(Lemma 2.1.9)} \\ &\Longleftrightarrow (\alpha, \varphi(\overline{a})) \in T_{\alpha+1} \end{aligned}$$

□

Remark 4.3.7. Whenever we apply Theorem 4.3.6, we iterate the $\mathfrak{aa}$-ultrapower construction ω_1^V-many times, not $\omega_1^{C(\mathfrak{aa})}$-many times. This is because we want the club in Lemma 4.3.4 to be a club in the sense of V, not in the sense of $C(\mathfrak{aa})$. So from the perspective of $C(\mathfrak{aa})$, the iterations are done a measurable number of times. Furthermore, since each application of the $\mathfrak{aa}$-ultrapower construction adds new elements, Theorem 4.3.6 always gives us a level $\mathfrak{C}_\gamma$ for some $\gamma \geq \omega_1^V$.

Chapter 5

Combinatorial properties of $C(\mathfrak{aa})$ below ω_1^V

Let $\varphi(x, \overline{y})$ be a first-order formula in vocabulary $\{\in, T\}$ and α a limit ordinal. Since we have a definable well-ordering $<_{\mathfrak{aa}}$ of $\mathfrak{C}_\alpha$, we also have definable Skolem functions $F_\varphi : C_\alpha^{|\overline{y}|} \to C_\alpha$:

$$F_\varphi(\overline{a}) = \begin{cases} <_{\mathfrak{aa}}\text{-least } b \text{ such that } \mathfrak{C}_\alpha \models \varphi(b, \overline{a}), & \text{if such exists} \\ \emptyset \text{ otherwise} \end{cases}$$

Given $A \subseteq C_\alpha$, let $\mathrm{Sk}(A)^{\mathfrak{C}_\alpha}$ be the Skolem-closure of A under all F_φ. So, $\mathrm{Sk}(A)^{\mathfrak{C}_\alpha} \preccurlyeq \mathfrak{C}_\alpha$ and $\left|\mathrm{Sk}(A)^{\mathfrak{C}_\alpha}\right| = |A| + \omega$. Suppose that $\mathrm{Sk}(A)^{\mathfrak{C}_\alpha} = (M', E, T^{\mathcal{M}'})$ and define

$$S^{\mathcal{M}'} := \{\psi(\overline{a}) \mid \psi(\overline{x}) \in \mathcal{L}_{\mathfrak{aa}}(\in, T), \overline{a} \in \mathrm{Sk}(A)^{\mathfrak{C}_\alpha} \text{ and } \mathfrak{C}_\alpha \models \psi(\overline{a})\}.$$

Let $\mathcal{M}$ be the transitive collapse of $(M', E, T^{\mathcal{M}'}, S^{\mathcal{M}'})$. Then $\mathcal{M}$ is an $\mathfrak{aa}$-premouse. In the following section, when we talk about the transitive collapse of $\mathrm{Sk}(A)^{\mathfrak{C}_\alpha}$, we are refering to $\mathcal{M}$, which is an $\mathfrak{aa}$-premouse. Note that if $A \in C(\mathfrak{aa})$, then $\mathrm{Sk}(A)^{\mathfrak{C}_\alpha} \in C(\mathfrak{aa})$, and also $\mathcal{M} \in C(\mathfrak{aa})$.

Lemma 5.0.1. *Let $\kappa \in \mathfrak{C}_\gamma$ be regular in $C(\mathfrak{aa})$ and γ a limit ordinal. Let $A \subseteq \mathfrak{C}_\gamma$ be finite and for each $\alpha < \kappa$, let $\mathcal{H}_\alpha = \mathrm{Sk}(\alpha \cup A)^{\mathfrak{C}_\gamma}$. Then, $D = \{\alpha < \kappa \mid \mathcal{H}_\alpha \cap \kappa = \alpha\}$ is a club in κ.*

Proof. **Claim.** If $\gamma < \kappa$ is a limit ordinal, then $\mathcal{H}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{H}_\alpha$.

Proof of the claim: Note that if $\overline{a} \in \left(\bigcup_{\alpha<\gamma} \mathcal{H}_\alpha\right)^{<\omega}$, then for every φ, $F_\varphi(\overline{a}) \in \bigcup_{\alpha<\gamma} \mathcal{H}_\alpha$. So, $\bigcup_{\alpha<\gamma} \mathcal{H}_\alpha$ contains $\gamma \cup A$ and is closed under every F_φ. Thus, $\mathcal{H}_\gamma \subseteq \bigcup_{\alpha<\gamma} \mathcal{H}_\alpha$. The converse is immediate. $\square$

To show that D is unbounded, let $\alpha_0 < \kappa$ and for every $n < \omega$, let $\alpha_{n+1} = \cup(\mathcal{H}_{\alpha_n} \cap \kappa)$, and $\alpha = \cup_{n<\omega} \alpha_n$. Then $\alpha \geq \alpha_0$. We will show that $\alpha \in D$.

For the sake of contradiction, suppose not. Then there's $\beta \in \mathcal{H}_\alpha$ such that $\alpha \leq \beta < \kappa$. By the claim, $\beta \in \mathcal{H}_{\alpha_n}$ for some $n < \omega$. Then $\alpha_{n+1} > \beta$, which is a contradiction.

That D contains all its limit points follows immediately from the claim. Thus, D is a club in κ. □

5.1 Diamonds

Remark 5.1.1. Let $\mathcal{M}_0$ be an $\mathfrak{aa}$-mouse and $\langle \mathcal{M}_i \mid i \leq \omega_1^V \rangle$ its $\mathfrak{aa}$-iteration. If $\alpha \in M_0$ is an ordinal such that α is below the critical point of $j_{0,1} : \mathcal{M}_0 \to \mathcal{M}_1$, then $j_{0,\omega_1}(\alpha) = \alpha$. Recall that the critical point of $j_{0,1}$ is the least γ such that $\mathcal{M}_0 \models_S \neg\mathfrak{aa}X\forall x(x \in \gamma \to X(x))$ (Lemma 4.2.10). So α is below the critical point if for all $\beta \leq \alpha$, $\mathcal{M}_0 \models_S \mathfrak{aa}X\forall x(x \in \beta \to X(x))$. If this holds, then for all $i \leq \omega_1^V$, we have $\mathcal{M}_i \models_S \mathfrak{aa}X\forall x(x \in \beta \to X(x))$.

The combinatorial principle $\diamondsuit$ was originally introduced by Jensen in 1971 to show the existence of Suslin trees in L [7].

Definition 5.1.2. Let κ be regular and $R \subseteq \kappa$ stationary. A sequence $\langle X_\alpha \mid \alpha \in R \rangle$, $X_\alpha \subseteq \alpha$, is a $\diamondsuit_\kappa(R)$-sequence if for all $X \subseteq \kappa$, the set $\{\alpha \in R \mid X \cap \alpha = X_\alpha\}$ is stationary. If such a sequence exists, then we say that $\diamondsuit_\kappa(R)$ holds.

Theorem 5.1.3. *Let $\kappa < \omega_1^V$ be regular in $C(\mathfrak{aa})$ and $R \subseteq \kappa$ stationary. Then $\diamondsuit_\kappa(R)$ holds.*

Proof. We define a sequence $\langle (X_\alpha, E_\alpha) \mid \alpha \in R \rangle$ by recursion as follows: If $\alpha \in R$ is a limit ordinal, let (X_α, E_α) be the $<_{\mathfrak{aa}}$-least pair such that $X_\alpha, E_\alpha \subseteq \alpha$, E_α is club in α and for all $\beta \in E_\alpha \cap R$, $X_\alpha \cap \beta \neq X_\beta$. If no such pair exists, or if α is 0 or a successor ordinal, let $X_\alpha = E_\alpha = \emptyset$. We claim that $\langle X_\alpha \mid \alpha \in R \rangle$ is a $\diamondsuit_\kappa(R)$-sequence.

Suppose not. Then there's a $<_{\mathfrak{aa}}$-least pair (X, E) such that $X, E \subseteq \kappa$, E is club in κ and for all $\alpha \in E \cap R$, $X \cap \alpha \neq X_\alpha$. Let $\gamma \geq \omega_1^V$ be a limit ordinal such that $\mathcal{P}(\kappa)^{C(\mathfrak{aa})} \in \mathfrak{C}_\gamma$ and

$$A = \{\kappa, R, X, E, \langle (X_\alpha, E_\alpha) \mid \alpha \in R \rangle\} \subseteq \mathfrak{C}_\gamma$$

Now, given D as in Lemma 5.0.1, we can find $\lambda \in D \cap R$. Let $\pi : \mathcal{H}_\lambda \to \mathcal{M}_0$ be the transitive collapse of $\mathcal{H}_\lambda$, where $\mathcal{H}_\lambda = \mathrm{Sk}(\lambda \cup A)^{\mathfrak{C}_\gamma}$. Then π satisfies the following properties:

1. $\pi \restriction \lambda = \mathrm{id}$, and $\pi(\kappa) = \lambda$.
2. $\pi(X) = X \cap \lambda$, $\pi(E) = E \cap \lambda$ and $\pi(R) = R \cap \lambda$.
3. $\pi(\langle (X_\alpha, E_\alpha) \mid \alpha \in R \rangle) = \langle (X_\alpha, E_\alpha) \mid \alpha \in R \cap \lambda \rangle$.

Since $|\mathcal{M}_0| = \lambda$ is countable in V, $\mathcal{M}_0$ is an $\mathfrak{aa}$-mouse by Theorem 4.3.2. Thus, the $\mathfrak{aa}$-iteration $\langle \mathcal{M}_\alpha \mid \alpha \leq \omega_1 \rangle$ is well-defined. By Theorem 4.3.6, there's an ordinal δ such that $(M_{\omega_1}, \in, T^{\mathcal{M}_{\omega_1}}) = \mathfrak{C}_\delta$. Furthermore, λ is below the critical point of $j : \mathcal{M}_0 \to \mathcal{M}_1$, since for all $\beta \leq \lambda$,

$$\mathfrak{C}_\gamma \models \mathfrak{aa} X \forall x (x \in \beta \to X(x)).$$

Thus,

$$\begin{aligned}\mathfrak{C}_\delta \models \text{“}&(X \cap \lambda, E \cap \lambda) \text{ is the } <_{\mathfrak{aa}}\text{-least pair such that } E \cap \lambda \text{ is a club in } \lambda \\ &\text{and for all } \alpha \in (E \cap \lambda) \cap (R \cap \lambda),\ (X \cap \lambda) \cap \alpha \neq X_\alpha.\text{”}\end{aligned}$$

By Theorem 2.4.2, this is true in $\mathfrak{C}_\gamma$. Hence, by definition, $(X_\lambda, E_\lambda) = (X \cap \lambda, E \cap \lambda)$. Furthermore, since $E \cap \lambda$ is unbounded in λ, we get that $\lambda \in E$. So we have $\lambda \in E \cap R$ such that $X \cap \lambda = X_\lambda$. This is a contradiction, since our assumption was that for all $\alpha \in E \cap R$, $X \cap \alpha \neq X_\alpha$. □

5.2 A reflection principle regarding Σ_1-formulas

The following reflection principle appeared first in *Generalized Descriptive Set Theory and Classification Theory* by Friedman, Hyttinen and Weinstein (Kulikov) [3]. There, it was used in Theorem 18 to show that in L, the κ-Borel* sets are precisely the $\Sigma^1_1(\kappa)$ sets. The proof in $C(\mathfrak{aa})$ is similar to the proof in [5], but some adjustments are needed. With ZF^- we mean all axioms of ZFC except the power set axiom. If $\mu < \kappa$ are regular cardinals, then with E^κ_μ we mean the set $\{\alpha < \kappa \mid \operatorname{cf} \alpha = \mu\}$.

Theorem 5.2.1. *Suppose that $\mu < \kappa < \omega_1^V$ are a regular cardinals in $C(\mathfrak{aa})$. Let $\psi(x, y)$ be a first-order Σ_1-formula of set theory. Then for all $\eta, \xi \in 2^\kappa$, $C(\mathfrak{aa}) \models \psi(\eta, \xi)$ if and only if the set*

$$A = \{\alpha < \kappa \mid \exists \beta > \alpha \left(\mathfrak{C}_\beta \models \mathrm{ZF}^- \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha) \wedge \text{“}\alpha \textit{ is regular”}\right)\}$$

contains a μ-club.

Proof. “$\Rightarrow$” Let θ be large enough such that

$$\mathfrak{C}_\theta \models \mathrm{ZF}^- \wedge \psi(\eta, \xi) \wedge \text{“}\kappa \text{ is regular”}.$$

For every $\alpha < \kappa$, let $\mathcal{H}_\alpha = \mathrm{Sk}(\alpha \cup \{\kappa, \eta, \xi\})^{\mathfrak{C}_\theta}$ and $D = \{\alpha < \kappa \mid \mathcal{H}_\alpha \cap \kappa = \alpha\}$. By Lemma 5.0.1, D is a club in κ, so in particular it is a μ-club. We claim that $D \subseteq A$.

Suppose that $\alpha \in D$. Let $\mathcal{M}_0$ be the transitive collapse of $\mathcal{H}_\alpha$. Since $\mathcal{M}_0$ is countable in V, $\mathcal{M}_0$ is an $\mathfrak{aa}$-mouse. Thus, its ω_1-th iteration $\mathcal{M}_{\omega_1}$ is $\mathfrak{C}_\beta$ for some $\beta > \alpha$. Furthermore, α is below the critical point of $j : \mathcal{M}_0 \to \mathcal{M}_1$, so we get that

$$\mathfrak{C}_\beta \models \mathrm{ZF}^- \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha) \wedge \text{“}\alpha \text{ is regular”}.$$

Thus, $\alpha \in A$.
"$\Leftarrow$" Suppose that $\psi(\eta, \xi)$ does not hold. Given a μ-club C, let θ be such that $C \in C_\theta$ and

$$\mathfrak{C}_\theta \models \mathrm{ZF}^- \wedge \neg\psi(\eta, \xi) \wedge \text{“}\kappa \text{ is regular”}.$$

Furthermore, we may assume that $\mathfrak{C}_\theta$ calculates all cofinalities below κ correctly. Similarly as before, define $\mathcal{H}_\alpha = \mathrm{Sk}(\alpha \cup \{\kappa, C, \eta, \xi\})^{\mathfrak{C}_\theta}$ and $D = \{\alpha \in E^\kappa_\mu \mid \mathcal{H}_\alpha \cap \kappa = \alpha\}$. By Lemma 5.0.1, D is a μ-club. Let α_0 be the least μ-cofinal limit point of D. Then C is unbounded in α_0, so $\alpha_0 \in C$. We show that $\alpha_0 \notin A$.

Suppose for a contradiction that $\alpha_0 \in A$. Then there's $\beta > \alpha_0$ such that

$$\mathfrak{C}_\beta \models \mathrm{ZF}^- \wedge \psi(\eta \restriction \alpha_0, \xi \restriction \alpha_0) \wedge \text{“}\alpha_0 \text{ is regular”}.$$

Let $\mathfrak{C}_\lambda$ be the ω_1-th $\mathfrak{aa}$-iteration of the transitive collapse of $\mathcal{H}_{\alpha_0}$. Since ψ is Σ_1, $\beta > \lambda$. Now, for $\alpha < \alpha_0$, define

$$\begin{aligned}
\mathcal{H}'_\alpha &= \mathrm{Sk}(\alpha \cup \{\alpha_0, C \cap \alpha_0, \eta \restriction \alpha_0, \xi \restriction \alpha_0\})^{\mathfrak{C}_\lambda}, \text{ and} \\
D' &= \{\alpha \in (E^{\alpha_0}_\mu)^{\mathfrak{C}_\theta} \mid \mathcal{H}'_\alpha \cap \alpha_0 = \alpha\}
\end{aligned}$$

Note that $E^{\alpha_0}_\mu = (E^{\alpha_0}_\mu)^{\mathfrak{C}_\theta} \in \mathfrak{C}_\lambda$, so $D', \mathcal{H}'_\alpha \in C_\beta$. Furthermore, for each $\alpha < \alpha_0$, note that

$$\mathcal{H}_\alpha \cong \mathcal{H}'_\alpha,$$

so in particular, $\mathcal{H}_\alpha \cap \kappa = \alpha \iff \mathcal{H}'_\alpha \cap \alpha_0 = \alpha$. Thus, $D' = D \cap \alpha_0 \in \mathfrak{C}_\beta$. Now since $\mathfrak{C}_\beta \models \mathrm{ZF}^-$,

$$\mathfrak{C}_\beta \models \text{“There's } \gamma \leq \alpha_0 \text{ and an order-preserving bijection from } \gamma \text{ to } D \cap \alpha_0.\text{”}$$

However, the only possibility for γ is μ, so $\mathfrak{C}_\beta$ must believe that α_0 is singular. This is a contradiction. $\square$

5.3 More diamonds

In this section we look at a combinatorial principle known as $\diamondsuit^+_\kappa$. It is stronger than $\diamondsuit_\kappa$, and was originally used in the construction of κ-Kurepa trees in L. Before looking at this principle in $C(\mathfrak{aa})$, we need one more fact about $\mathfrak{aa}$-mice.

Lemma 5.3.1 (Kennedy, Magidor and Väänänen, [9]). *Suppose that $\mathcal{M}_0$ is an $\mathfrak{aa}$-mouse, $\kappa < \omega_1^V$, $\kappa \in M_0$, $\mathcal{M}_{\omega_1} = \mathfrak{C}_\alpha$, and κ is below the critical point of $j : \mathcal{M}_0 \to \mathcal{M}_1$. Then, $\mathcal{P}(\kappa) \cap M_0 = \mathcal{P}(\kappa) \cap C_\alpha$.*

Proof. Suppose not. Then there's $r \subseteq \kappa$ such that $r \in C_\alpha$, $r = j_{\xi+1,\omega_1}(r^*)$ for some $r^* \in M_{\xi+1}$ and $\xi < \omega_1^V$, but $r \neq j_{\xi,\omega_1}(s)$ for all $s \in M_\xi$. Suppose that $r^* = [\varphi(X, x, \overline{a})]$ for some $\overline{a} \in M_\xi$. Now,

$$\mathfrak{C}_\alpha \models \exists x(x \subseteq \kappa \wedge \forall y(y \in j_{\xi+1,\omega_1}([\varphi(X, x, \overline{a})]) \leftrightarrow y \in x))$$

Thus,

$$\mathfrak{C}_\alpha \models \mathfrak{aa}X\exists x(x \subseteq \kappa \wedge \forall y(y \in f_{\varphi(X,x,j_{\xi,\omega_1}(\overline{a}))}(X) \leftrightarrow y \in x))$$

Let $D \subseteq \mathcal{P}_{\omega_1}(M_\alpha)$ be a club of sets A such that $\mathfrak{C}_\alpha \models \exists x(x \subseteq \kappa \wedge \forall y(y \in f_\varphi(A) \leftrightarrow y \in x))$. Given $t \subseteq \kappa$ with $t \in C_\alpha$, let

$$D_t := \{A \in \mathcal{P}_{\omega_1}(C_\alpha) \mid \mathfrak{C}_\alpha \models t \subseteq \kappa \wedge \forall y(y \in f_{\varphi(X,x,j_{\xi,\omega_1}(\overline{a}))}(A) \leftrightarrow y \in t)\}$$

Then, $D = \bigcup_{t \subseteq \kappa, t \in C_\alpha} D_t$. Since ω_1^V is a strong limit in $C(\mathfrak{aa})$, it follows that C_α only has countably many subsets of κ from the perspective of V. Thus, this is a countable union, so by the ω_1-completeness of the club filter, there's $t \subseteq \kappa$ such that D_t is stationary. By club determinacy of $\mathfrak{C}_\alpha$, it follows that D_t contains a club. Thus,

$$\mathfrak{C}_\alpha \models \exists x\mathfrak{aa}X(x \subseteq \kappa \wedge \forall y(y \in f_{\varphi(X,x,j_{\xi,\omega_1}(\overline{a}))}(X) \leftrightarrow y \in x))$$

Therefore,

$$\mathcal{M}_\xi \models_S \exists x\mathfrak{aa}X(x \subseteq \kappa \wedge \forall y(y \in f_{\varphi(X,x,\overline{a})}(X) \leftrightarrow y \in x))$$

So by criteria (5) of being a potential $\mathfrak{aa}$-premouse, there's $s \in M_\xi$ such that

$$\mathcal{M}_\xi \models_S \mathfrak{aa}X(s \subseteq \kappa \wedge \forall y(y \in f_{\varphi(X,x,\overline{a})}(X) \leftrightarrow y \in s))$$

It follows that $r = j_{\xi,\omega_1}(s)$, which is a contradiction. $\square$

Definition 5.3.2 ($\diamondsuit_\kappa^+$)**.** Let κ be a regular cardinal and $\langle \mathcal{A}_\alpha \mid \alpha < \kappa \rangle$ a sequence such that for all $\alpha < \kappa$, $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$ and $|\mathcal{A}_\alpha| \leq |\alpha|$. Then this sequence is a $\diamondsuit_\kappa^+$-sequence if for all $A \subseteq \kappa$, there's a club $D \subseteq \kappa$ such that if $\alpha \in D$, then $A \cap \alpha \in \mathcal{A}_\alpha$ and $D \cap \alpha \in \mathcal{A}_\alpha$.

At certain large cardinals, so called ineffable cardinals, the principle $\diamondsuit_\kappa^+$ automatically fails:

Definition 5.3.3 (Ineffable cardinal)**.** A regular, uncountable cardinal κ is called *ineffable*, if for all sequences $\langle X_\alpha \mid \alpha < \kappa \rangle$, with $X_\alpha \subseteq \alpha$, there's $X \subseteq \kappa$ such that $\{\alpha < \kappa \mid X_\alpha = X \cap \alpha\}$ is stationary.

Ineffable cardinals are at least weakly compact. In the next Theorem, we show that every regular, not ineffable, cardinal $\kappa < \omega_1^V$ in $C(\mathfrak{aa})$ has a $\diamondsuit_\kappa^+$-sequence in $C(\mathfrak{aa})$. This is similar to the situation in L, where every regular, not ineffable, cardinal κ satisfies $\diamondsuit_\kappa^+$. Since ω_1^V is measurable in $C(\mathfrak{aa})$, the assumption that κ is not ineffable is necessary, as there are always many ineffable cardinals below a measurable.

For more information about ineffable cardinals, the principle $\diamondsuit_\kappa^+$, κ-Kurepa trees and their relationship, we refer the reader to Devlin's book *Constructibility* [2].

Theorem 5.3.4. *Let $\kappa < \omega_1^V$ be regular and not ineffable in $C(\mathfrak{aa})$. Then there's a $\diamondsuit_\kappa^+$-sequence.*

Proof. Let $\langle X_\alpha \mid \alpha < \kappa\rangle$ witness that κ is *not* ineffable. Thus, for all $X \subseteq \kappa$, there's a club E such that whenever $\alpha \in E$, then $X_\alpha \neq X \cap \alpha$. Let $\theta \geq \omega_1^V$ be a large enough limit ordinal such that $\mathcal{P}(\kappa)^{C(\mathfrak{aa})} \in C_\theta$ and $\langle X_\alpha \mid \alpha < \kappa\rangle \in C_\theta$. For each $\alpha < \kappa$, let $\mathcal{B}_\alpha := \mathrm{Sk}(\alpha \cup \{\alpha, \langle X_\alpha \mid \alpha < \kappa\rangle\})^{\mathfrak{C}_\theta}$ and $\mathcal{A}_\alpha := \mathcal{B}_\alpha \cap \mathcal{P}(\alpha)$.

We claim that $\langle \mathcal{A}_\alpha \mid \alpha < \kappa\rangle$ is a $\diamondsuit_\kappa^+$-sequence. Clearly, $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$ and $|\mathcal{A}_\alpha| \leq |\alpha|$. Fix a subset $A \subseteq \kappa$. For each $\alpha < \kappa$, define

$$\mathcal{H}_\alpha := \mathrm{Sk}(\alpha, \{\kappa, A, \langle X_\alpha \mid \alpha < \kappa\rangle\})^{\mathfrak{C}_\theta}$$

By Lemma 5.0.1, $D := \{\alpha < \kappa \mid \mathcal{H}_\alpha \cap \kappa = \alpha\}$ is a club in κ. We claim that D is as wanted. Pick $\alpha \in D$. We want to show that $A \cap \alpha, D \cap \alpha \in \mathcal{A}_\alpha$. Let $\pi : \mathcal{B}_\alpha \to \mathcal{M}_0$ be the transitive collapse of $\mathcal{B}_\alpha$ and $\pi' : \mathcal{H}_\alpha \to \mathcal{N}_0$ the transitive collapse of $\mathcal{H}_\alpha$. By the elementarity of π and π', the following holds:

1. $\pi \restriction \alpha = \mathrm{id}$, $\pi(\alpha) = \alpha$ and $\pi(X_\alpha) = X_\alpha$.
2. $\pi' \restriction \alpha = \mathrm{id}$, $\pi'(\kappa) = \alpha$ and $\pi'(A) = A \cap \alpha$.
3. $\pi'(\langle X_\beta \mid \beta < \kappa\rangle) = \langle X_\beta \mid \beta < \alpha\rangle$.

Since $\mathcal{M}_0$ and $\mathcal{N}_0$ are countable in V, they are $\mathfrak{aa}$-mice. Thus, if $\mathcal{M}_{\omega_1}$ is the ω_1-th $\mathfrak{aa}$-iteration of $\mathcal{M}_0$, then $\mathcal{M}_{\omega_1} = \mathfrak{C}_\gamma$ for some limit ordinal $\gamma \geq \omega_1^V$. Similarly, $\mathcal{N}_{\omega_1} = \mathfrak{C}_\delta$ for some limit ordinal $\delta \geq \omega_1^V$. First, we show that $\delta < \gamma$. Since α is below the critical point of $j : \mathcal{M}_0 \to \mathcal{M}_1$, it follows that $X_\alpha \in C_\gamma$. We show that $X_\alpha \notin C_\delta$, which is sufficient.

For the sake of contradiction, assume that $X_\alpha \in C_\delta$. Since $\mathcal{N}_0 \preccurlyeq \mathfrak{C}_\delta$ and since κ is not ineffable,

$$\mathfrak{C}_\delta \models \text{“There's a club } E' \subseteq \alpha \text{ such that if } \beta \in E', \text{ then } X_\alpha \cap \beta \neq X_\beta.\text{”}$$

By Lemma 5.3.1, $X_\alpha, E' \in \mathcal{N}_0$. Let $X, E \in C_\theta$ be such that $\pi'(X) = X_\alpha$ and $\pi'(E) = E'$. By the elementarity of π',

$$\mathfrak{C}_\theta \models \text{“}E \subseteq \kappa \text{ is a club such that if } \beta \in E, \text{ then } X \cap \beta \neq X_\beta.\text{”}$$

Now, $E' = E \cap \alpha$, so E is unbounded in α. Thus, $\alpha \in E$. However, this means that $X \cap \alpha \neq X_\alpha$, which contradicts that $\pi'(X) = X_\alpha$.

Thus, we have proved that $\delta < \gamma$. Since $A \cap \alpha \in C_\delta$, also $A \cap \alpha \in C_\gamma$. By Lemma 5.3.1, $A \cap \alpha \in \mathcal{M}_0$. Thus, $A \cap \alpha \in \mathcal{A}_\alpha$. Lastly, we need to show that $D \cap \alpha \in A_\alpha$. By the same argument, it is sufficient to show that $D \cap \alpha \in C_\gamma$.

For each $\beta < \alpha$, let $\mathcal{H}'_\beta := \mathrm{Sk}(\beta \cup \{\alpha, A \cap \alpha, \langle X_\beta \mid \beta < \alpha \rangle\})^{\mathfrak{C}_\delta}$, and define

$$D' := \{\beta < \alpha \mid \mathcal{H}'_\beta \cap \alpha = \beta\}$$

Then, for all $\beta < \alpha$, $\mathcal{H}_\beta \cong \mathcal{H}'_\beta$. In particular, $\mathcal{H}_\beta \cap \kappa = \beta \iff \mathcal{H}'_\beta \cap \alpha = \beta$. Hence, $D' = D \cap \alpha$. Since $D' \in C_\gamma$, also $D \cap \alpha \in C_\gamma$. $\square$

5.4 Some final remarks

We have now showed that many properties of L carries over to $C(\mathfrak{aa})$, especially below ω_1^V. Goldberg and Steel have showed that ω_1^V is the first measurable cardinal in $C(\mathfrak{aa})$. Thus, $C(\mathfrak{aa})$ looks very much like L below ω_1^V. In fact, we even have that,

$$\mathfrak{C}_{\omega_1^V} \vDash V = L$$

Proving this is actually trivial, since if α is countable and $\varphi(X, \overline{x})$ is a first-order formula in vocabulary $\{\in, T\}$, then for all $\overline{a} \in C_\alpha$,

$$\mathfrak{C}_\alpha \vDash \mathfrak{aa} X \varphi(X, \overline{a}) \iff \mathfrak{C}_\alpha \vDash \varphi(C_\alpha, \overline{a}) \iff \mathfrak{C}_\alpha \vDash \varphi(\top, \overline{a})$$

By induction on $\mathfrak{aa}$-formulas and countable α, we obtain that $C_\alpha = L_\alpha$ for all $\alpha < \omega_1^V$. Thus, it follows that $C_{\omega_1^V} = L_{\omega_1^V}$. However, this does not at all mean that $(V_{\omega_1^V})^{C(\mathfrak{aa})} = (V_{\omega_1^V})^L$. In general, given $\kappa < \omega_1^V$, many of the elements of $\mathcal{P}(\kappa)^{C(\mathfrak{aa})}$ appears after ω_1^V in the levels $\langle \mathfrak{C}_\alpha \mid \alpha \geq \omega \rangle$. For the reader familiar with $0^\sharp$, we may remark that $\omega_1^{C(\mathfrak{aa})}$ is inaccessible in L. This is because $C(\mathfrak{aa})$ has a measurable cardinal, and hence also $0^\sharp \in C(\mathfrak{aa})$. Thus, a bijection $f : \omega_1^L \to \omega$ exists in $C(\mathfrak{aa})$, but it appears in some level $\mathfrak{C}_\alpha$ for $\alpha > \omega_1^V$.

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