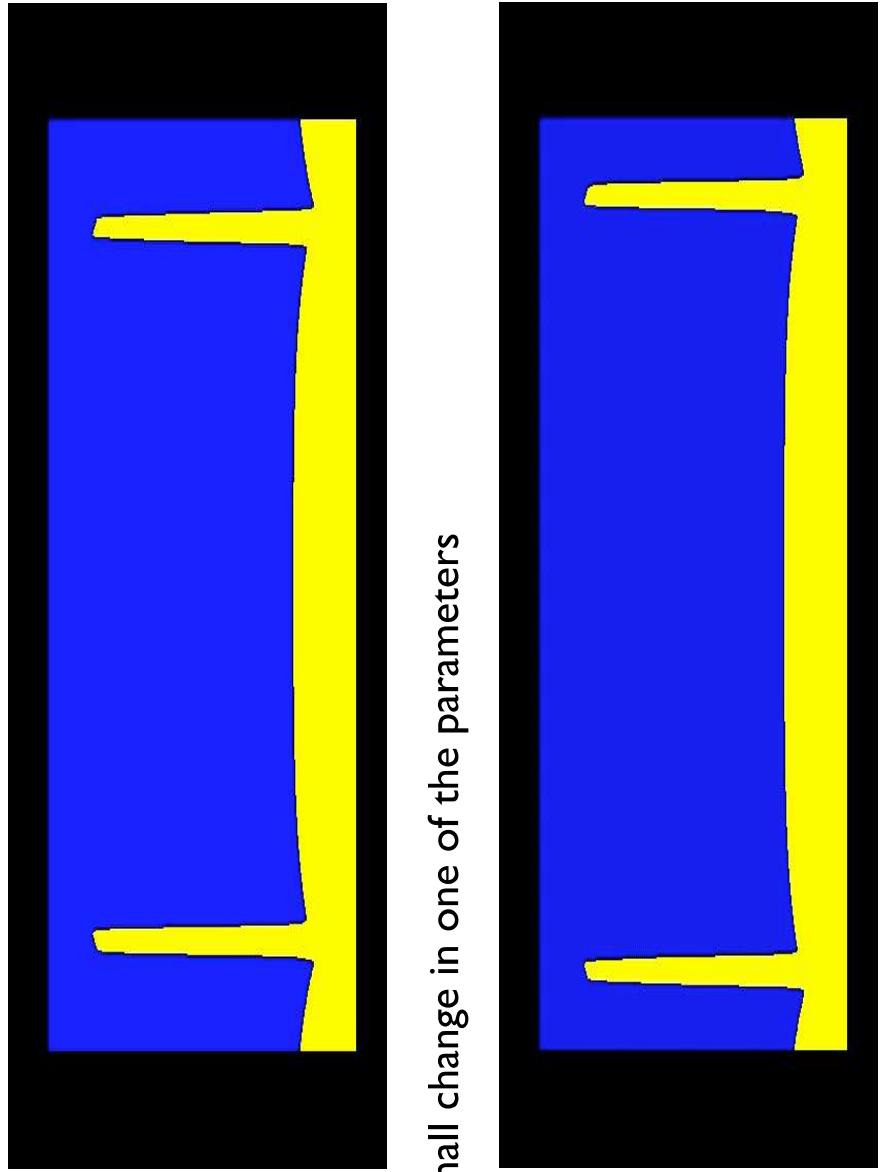
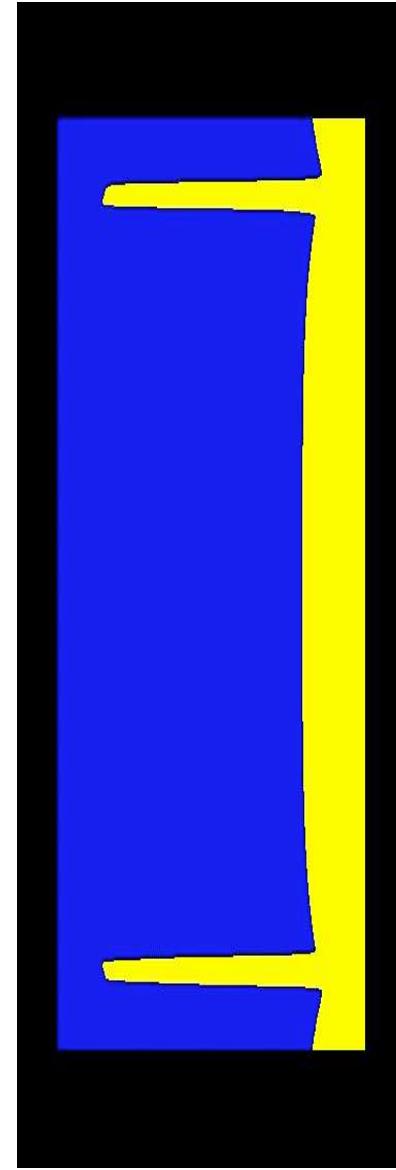
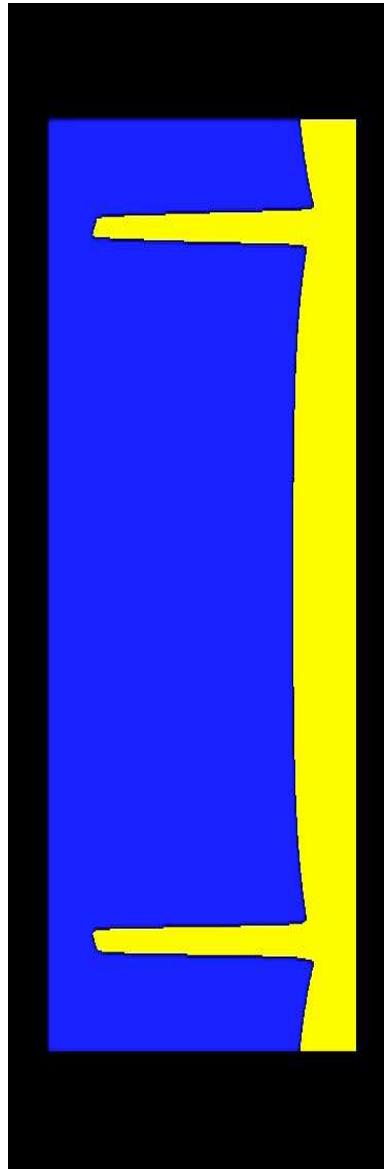


The start of the story ...



Courtesy of Y. Nishiura

The start of the story ...

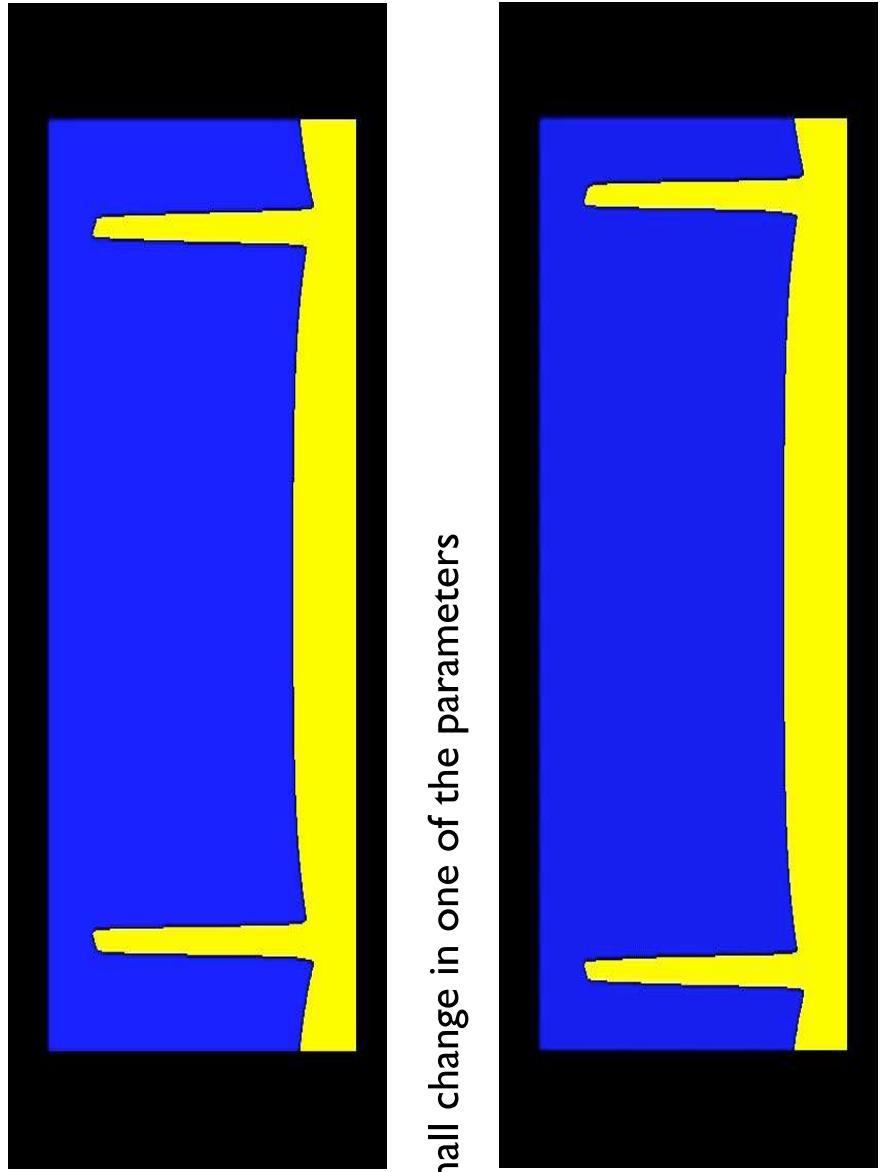


small change in one of the parameters



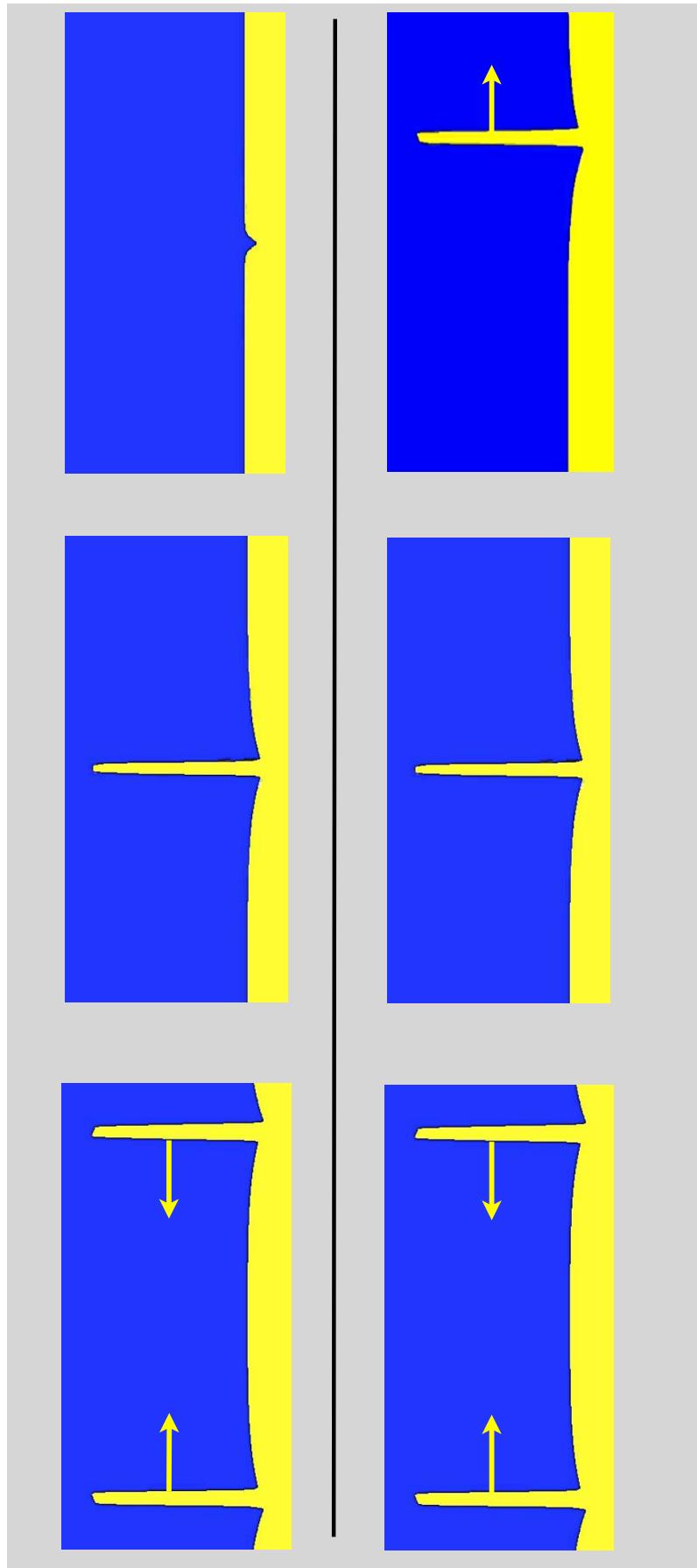
Courtesy of Y. Nishiura

The start of the story ...



Courtesy of Y. Nishiura

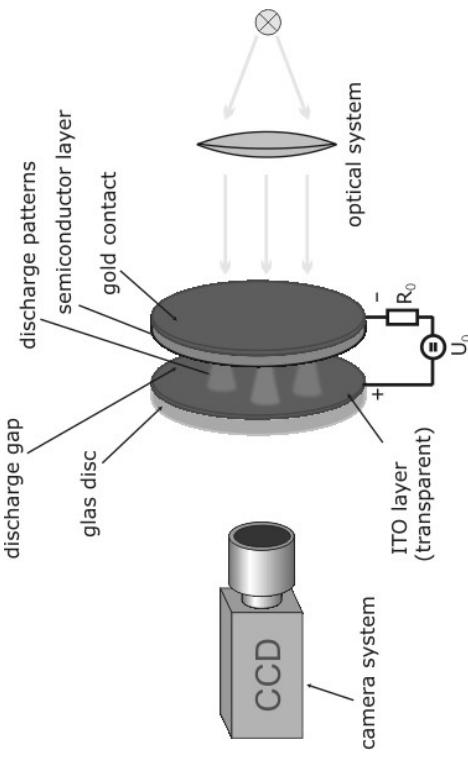
What's going on??



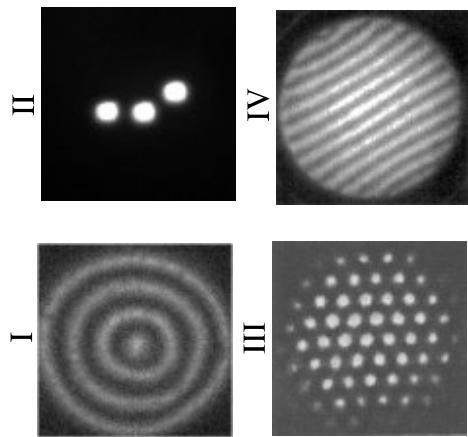
Slightly different parameters (change in the 6th digit of the parameter)... gives completely different dynamics

Model: gas-discharge system

Set up:



Observed 2D patterns:

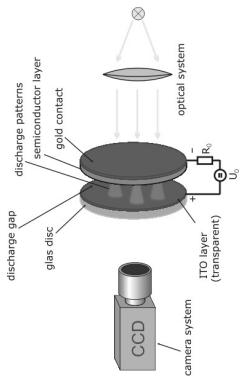


<http://www.uni-muenster.de/Physik.AP/Purwins/Gruppe/Purwins-en.html>

Model

$$\begin{aligned} U_t &= D_U \Delta U + f(U) - \kappa_3 V - \kappa_4 W + \kappa_1 \\ \tau V_t &= D_V \Delta V + U - V \\ \theta W_t &= D_W \Delta W + U - W \end{aligned}$$

U: current density
V: voltage drop
W: surface charge

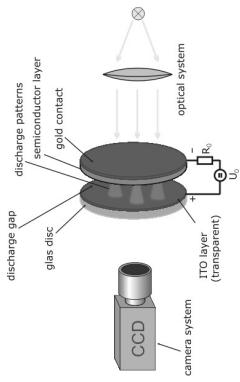


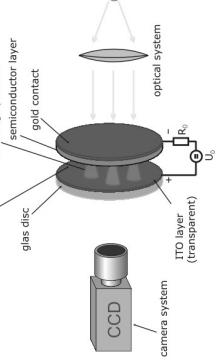
Model

$$f(U) = 2U - U^3$$

$$\begin{aligned}U_t &= D_U \Delta U + f(U) - \kappa_3 V - \kappa_4 W + \kappa_1 \\ \tau V_t &= D_V \Delta V + U - V \\ \theta W_t &= D_W \Delta W + U - W\end{aligned}$$

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Model

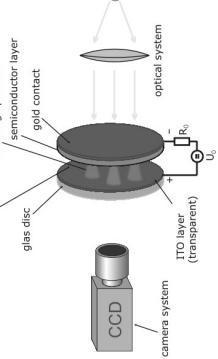
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U: current density
V: voltage drop
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typical parameter values:

$$\begin{aligned} D_U &= 5 \times 10^{-6}, D_V = 5 \times 10^{-5} \\ D_W &= 10^{-2}, \quad \kappa_1 = -7 \\ \kappa_3 &= 1, \quad \kappa_4 = 8.5 \\ \tau &= 20, \quad \theta = 1 \end{aligned}$$



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↓ Nondimensionalise

$$U_t = \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma)$$

$$\tau V_t = \Delta V + U - V$$

$$\theta W_t = D^2 \Delta W + U - W$$

ε small

Model

$$U_t = \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma)$$

$$\tau V_t = \Delta V + U - V$$

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where $0 < \varepsilon \ll 1; D > 1; 0 < \tau, \theta; \alpha, \beta, \gamma \in \mathbb{R}$

Model

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=

fast Allen-Cahn (AC) eqn.

$$U_t = \varepsilon^2 \Delta U + U - U^3 - \varepsilon\gamma$$

- **U**: fast component / activator
- small diffusion
- AC: bistable ($U \approx \pm 1$)
- AC: supports slowly travelling waves
- AC: phase separation
- nonlinear: U^3
- coupling to V, W is small

Model

$$U_t = \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma)$$

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=

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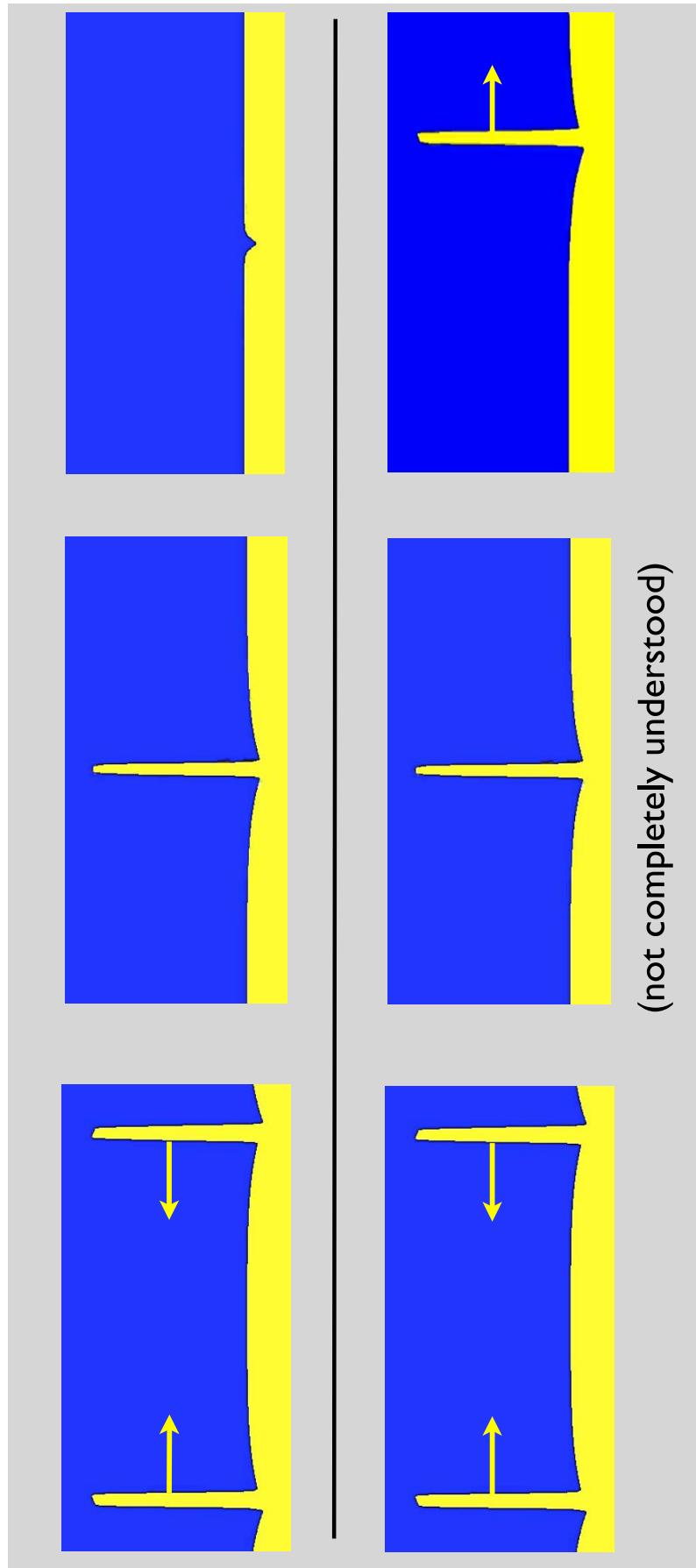
- U : fast component / activator
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- AC: supports slowly travelling waves
- AC: phase separation
- nonlinear: U^3
- coupling to V, W is small

weakly linearly coupled to **2 slow linear equations**

- V, W : slow components / inhibitors
- $O(1)$ diffusion: spatially evolving on a different scale
- linear
- only coupled directly to U and not to each other

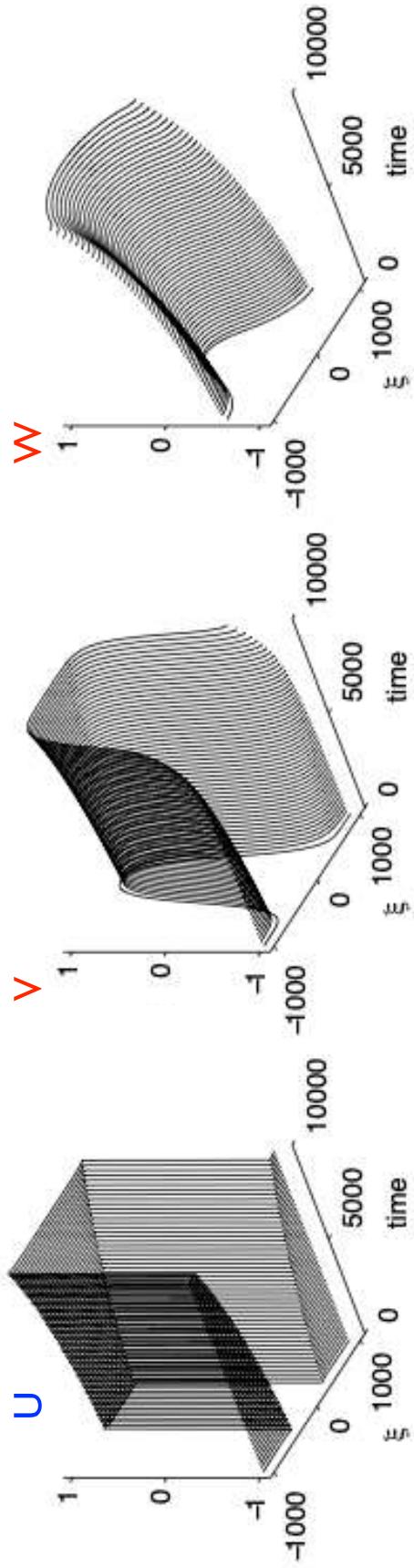
Lucky?

Maybe the equations are “*simple enough*” (singular perturbed, linear slow components, nonlinear term not too complicated and well-studied) to do **rigorous mathematical analysis**, but are “*complex enough*” to support “interesting dynamics”.



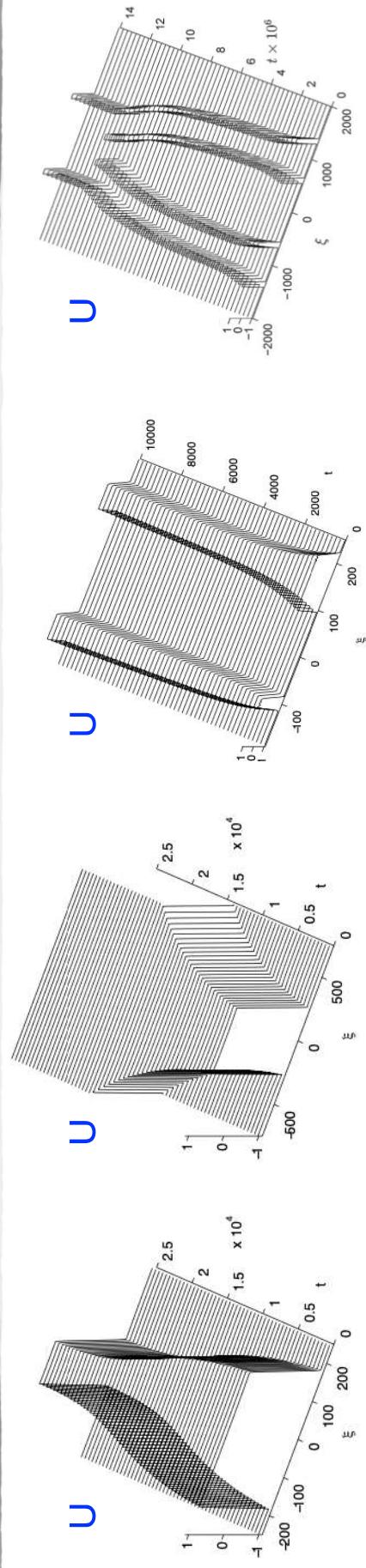
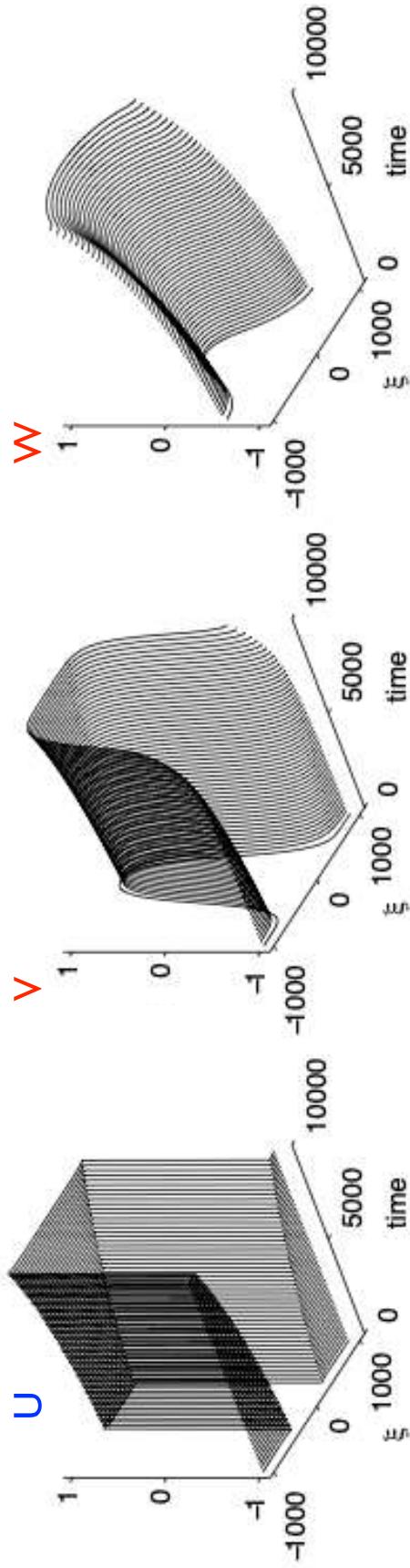
$$\begin{aligned}\textcolor{blue}{U}_t &= \varepsilon^2 \Delta \textcolor{blue}{U} + \textcolor{blue}{U} - \textcolor{blue}{U}^3 - \varepsilon(\alpha \textcolor{red}{V} + \beta \textcolor{red}{W} + \gamma) \\ \tau \textcolor{red}{V}_t &= \Delta \textcolor{red}{V} + \textcolor{blue}{U} - \textcolor{red}{V} \\ \theta \textcolor{red}{W}_t &= D^2 \Delta \textcolor{red}{W} + \textcolor{blue}{U} - \textcolor{red}{W}\end{aligned}$$

But first ... numerics (1D):

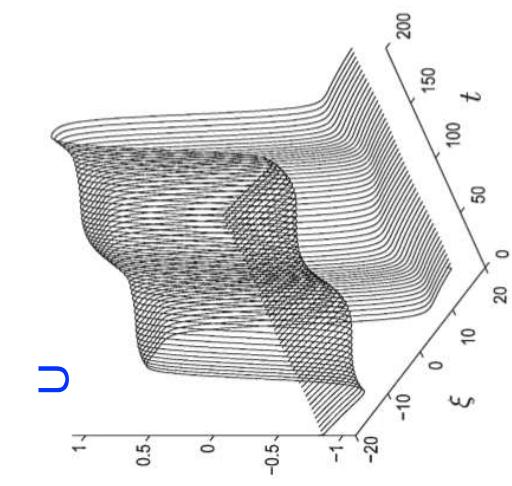


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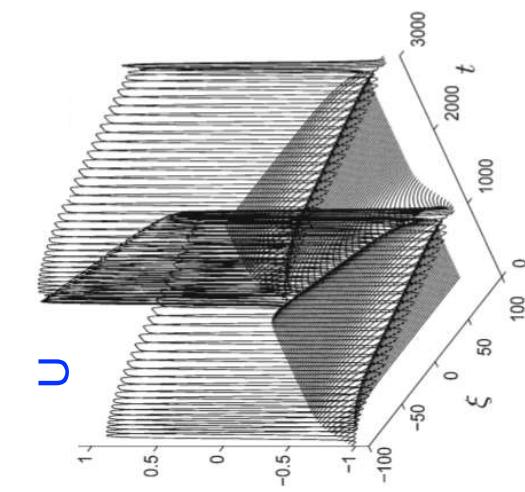
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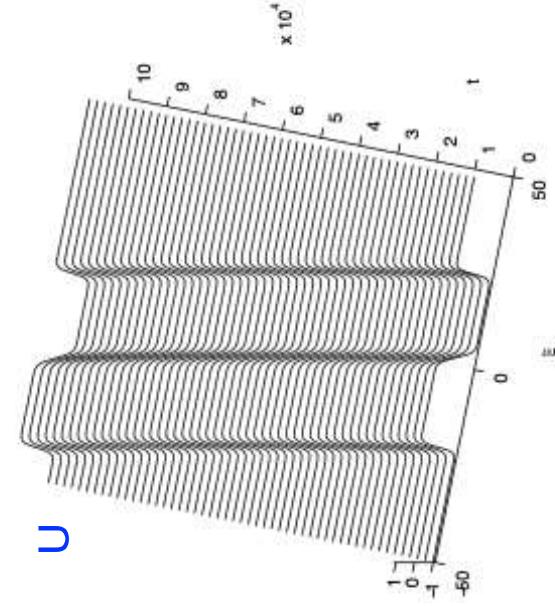
But first ... numerics (1D):



breathing pulse



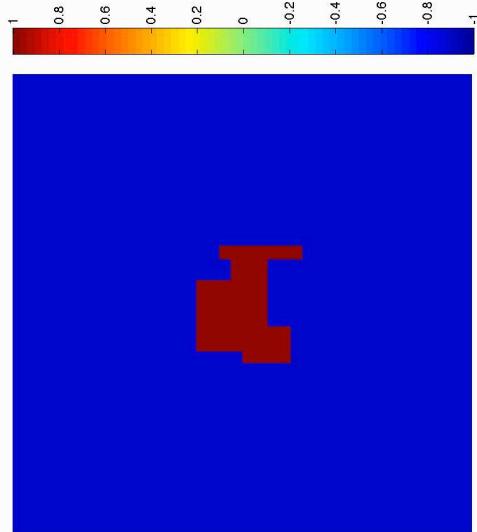
travelling pulse



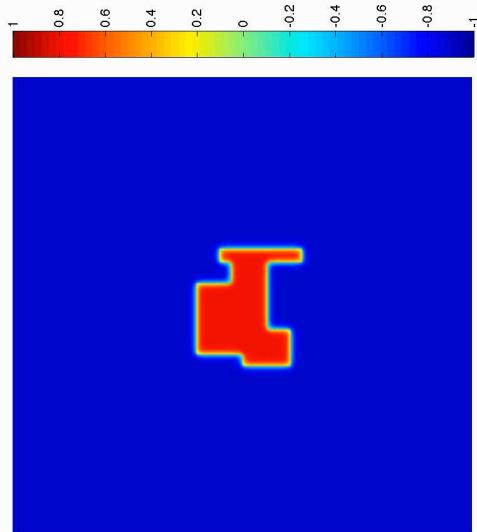
travelling 3-front

But first ... numerics (2D):

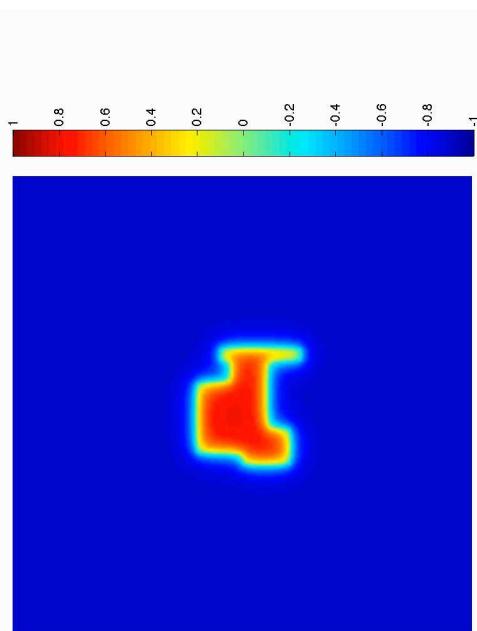
U-component



V-component

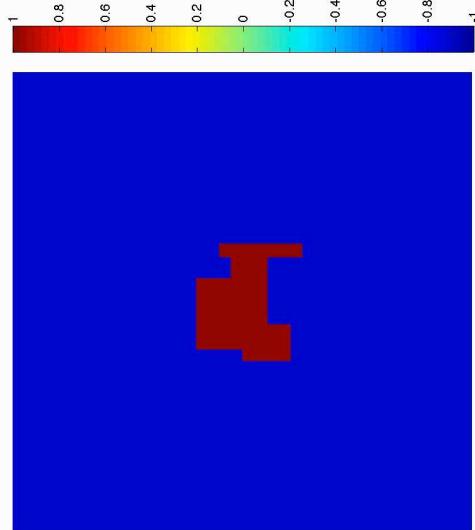


W-component

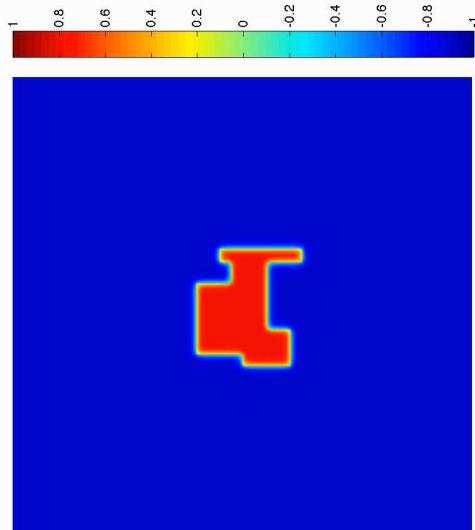


But first ... numerics (2D):

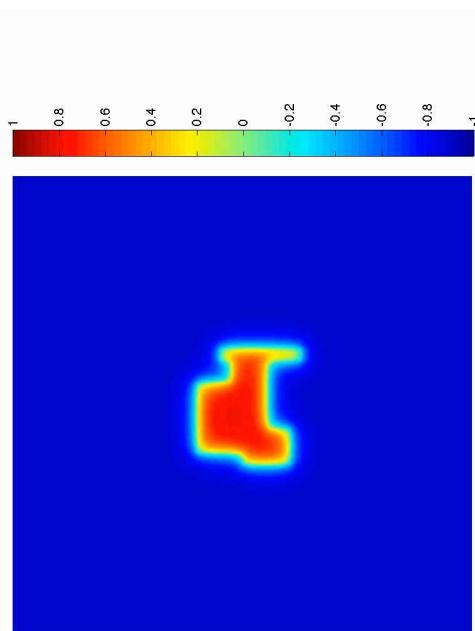
U-component



V-component

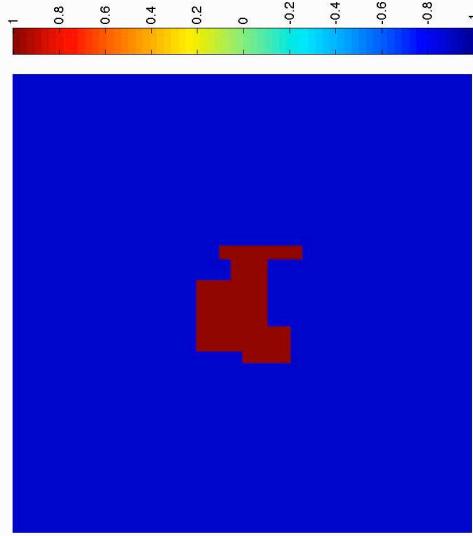


W-component

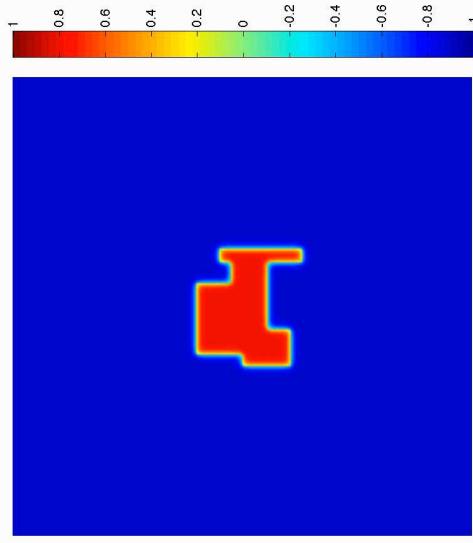


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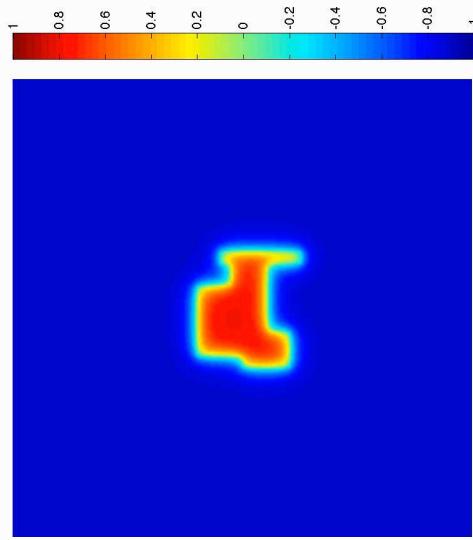
U-component



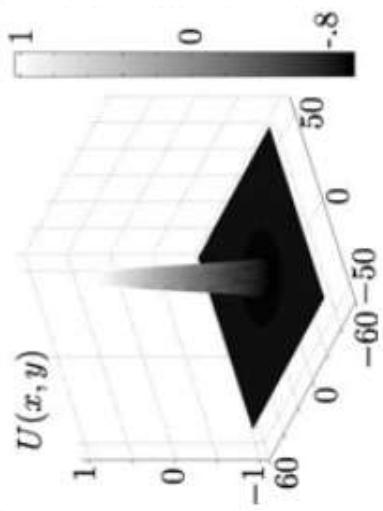
V-component



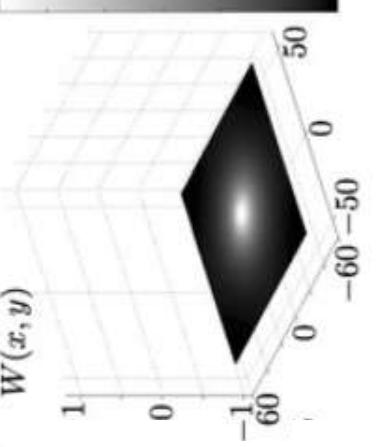
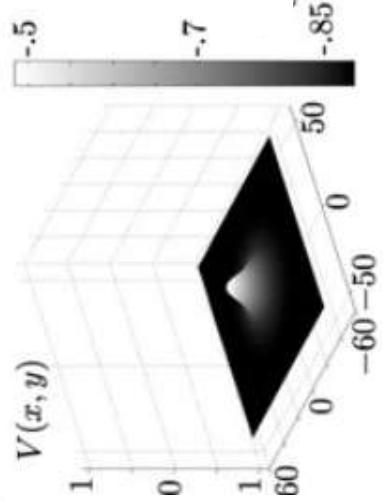
W-component



$U(x, y)$

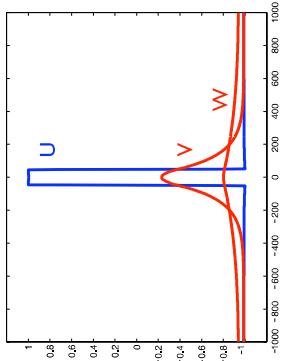


$V(x, y)$



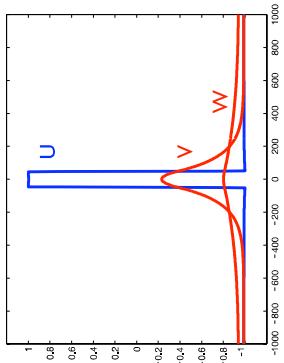
Questions?

- Can we construct **localized structures** (stationary/traveling pulses)? (*existence*)



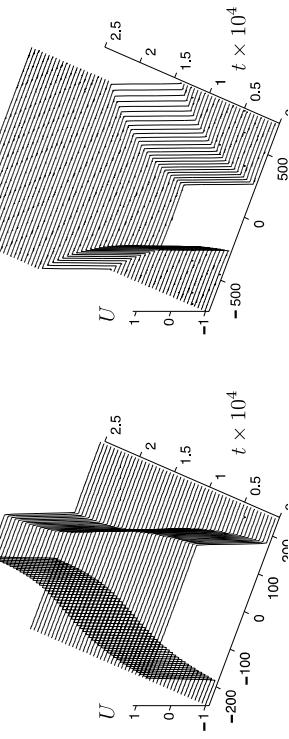
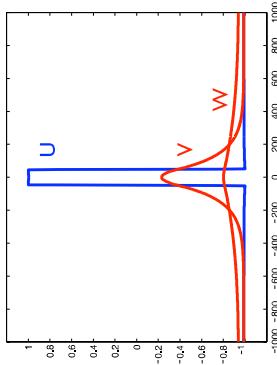
Questions?

- Can we construct **localized structures** (stationary/traveling pulses)? (*existence*)
- Can we determine their **stability**?

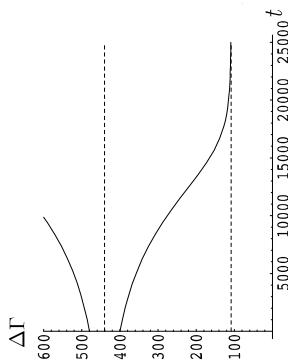


Questions?

- Can we construct **localized structures** (stationary/traveling pulses)? (*existence*)
- Can we determine their **stability**?
- Can we study the *interaction* of several of these structures?

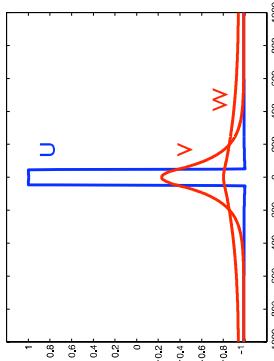


Derive ODE for the distance
between the interfaces ($\Delta\Gamma$)?

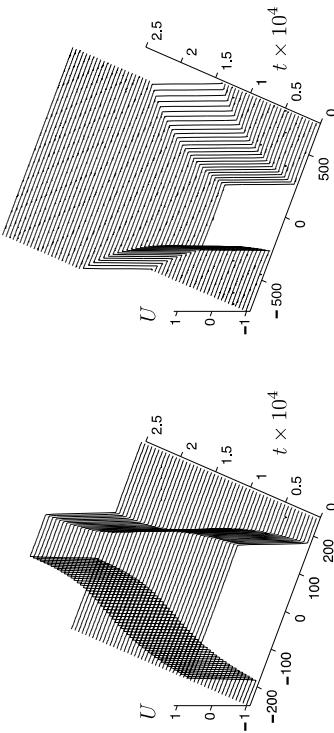
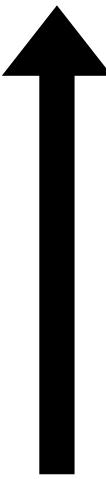


Questions?

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Derive ODE for the distance between the interfaces ($\Delta\Gamma$)?

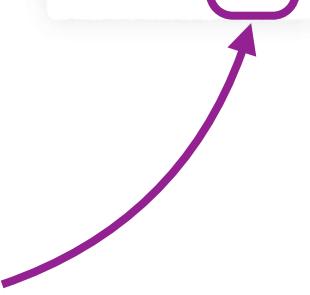


- What is the influence of the **second inhibitor**?

$$U_t = \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma)$$

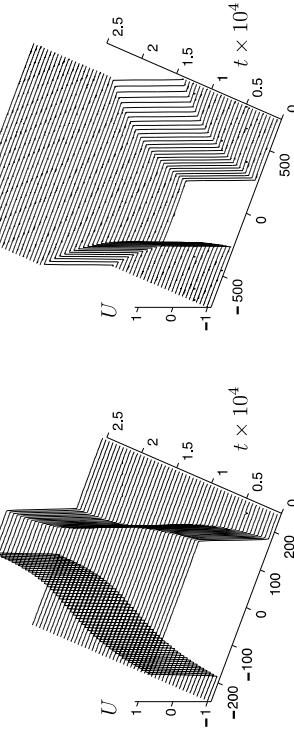
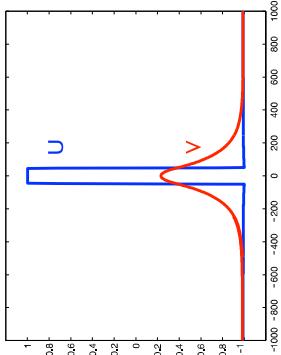
$$\tau V_t = \Delta V + U - V$$

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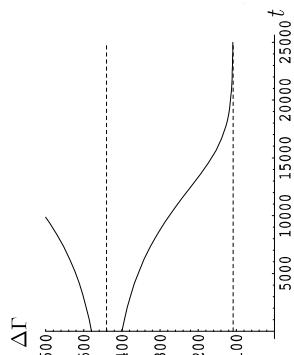
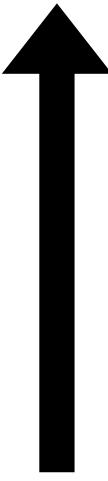


Questions?

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Derive ODE for the distance
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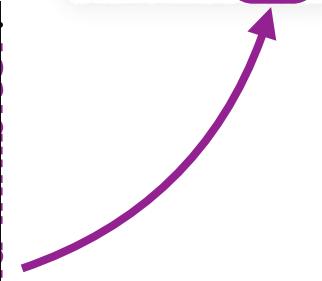
$$\tau V_t = \Delta V + V - V$$

$$\theta W_t - D^2 \Delta W + U = V$$

$$U_t = \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma)$$

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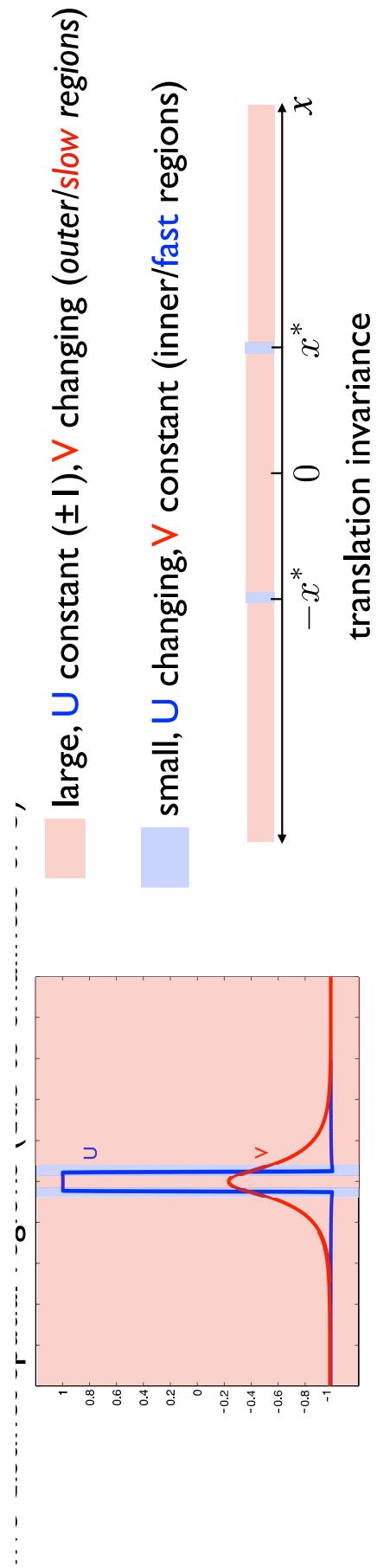


Existence of stationary pulse

$$U_t = \varepsilon^2 U_{xx} + U - U^3 - \varepsilon(\alpha V + \gamma)$$

$$\tau V_t = V_{xx} + U - V$$

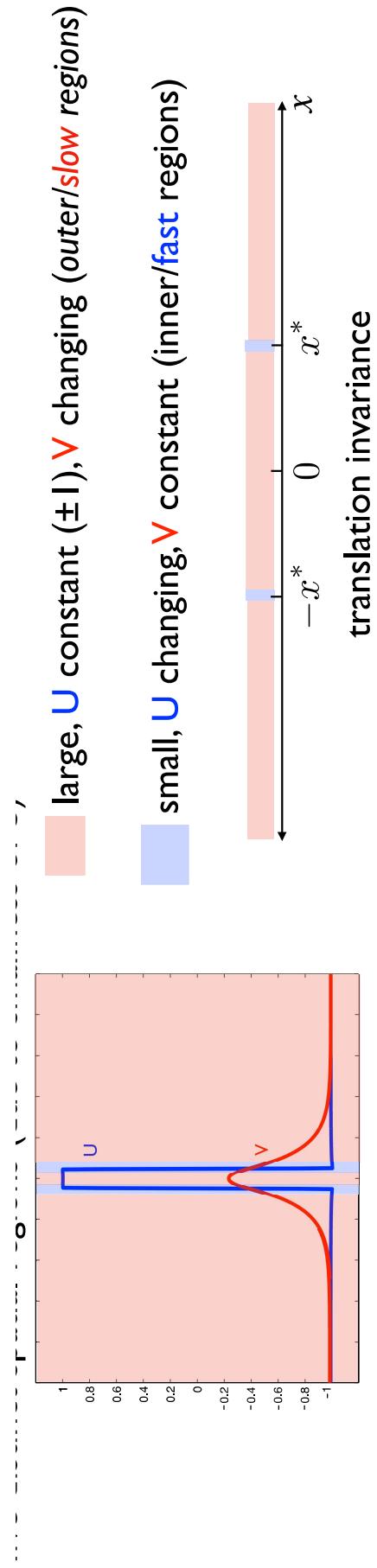
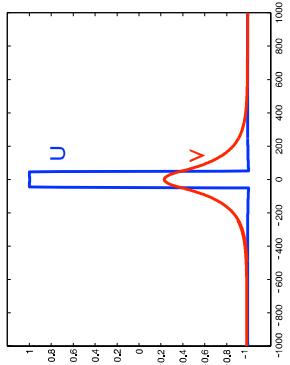
where $(x, t) \in \mathbb{R} \times \mathbb{R}_+; 0 < \varepsilon \ll 1; 0 < \tau, \alpha, \gamma \in \mathbb{R}$



Existence of stationary pulse

$$\begin{aligned} U_t &= \varepsilon^2 U_{xx} + U - U^3 - \varepsilon(\alpha V + \gamma) \\ \tau V_t &= V_{xx} + U - V \end{aligned}$$

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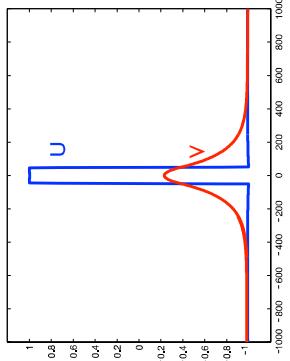


Existence of stationary pulse

$$U_t = \varepsilon^2 U_{xx} + U - U^3 - \varepsilon(\alpha V + \gamma)$$

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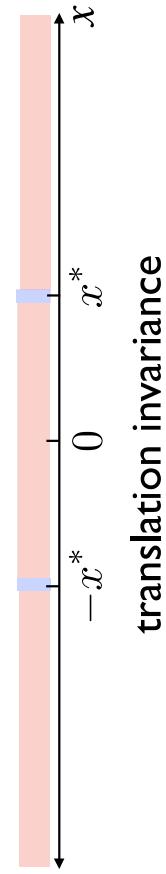
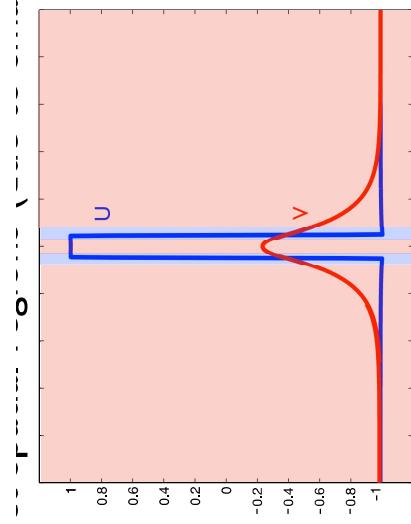
Observations:

- stationary: $\partial_t = 0$

$$0 = \varepsilon^2 u_{xx} + u - u^3 - \varepsilon(\alpha v + \gamma)$$

$$0 = v_{xx} + u - v$$

,
■ large, U constant (± 1), V changing (outer/**slow** regions)
■ small, U changing, V constant (inner/**fast** regions)



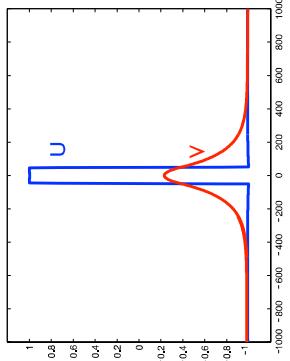
translation invariance

Existence of stationary pulse

$$U_t = \varepsilon^2 U_{xx} + U - U^3 - \varepsilon(\alpha V + \gamma)$$

$$\tau V_t = V_{xx} + U - V$$

where $(x, t) \in \mathbb{R} \times \mathbb{R}_+; 0 < \varepsilon \ll 1; 0 < \tau, \alpha, \gamma \in \mathbb{R}$



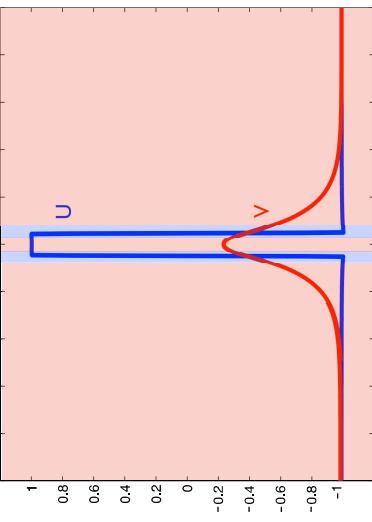
Observations:

- **stationary:** $\partial_t U = 0$

$$0 = \varepsilon^2 u_{xx} + u - u^3 - \varepsilon(\alpha v + \gamma)$$

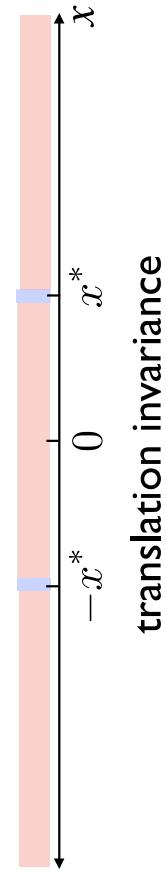
$$0 = v_{xx} + u - v$$

- five distinct spatial regions (due to smallness of ε)



large, **U** constant (± 1), **V** changing (outer/**slow** regions)

small, **U** changing, **V** constant (inner/**fast** regions)

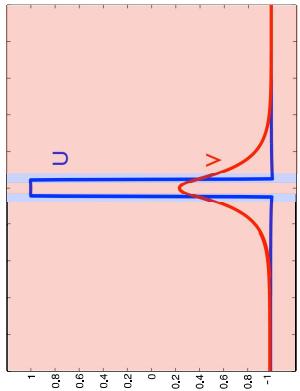


translation invariance

Spatial dynamics

$$0 = \varepsilon^2 \underline{u}_{xx} + \underline{u} - \underline{u}^3 - \varepsilon(\alpha \underline{v} + \gamma)$$

$$0 = \underline{v}_{xx} + \underline{u} - \underline{v}$$

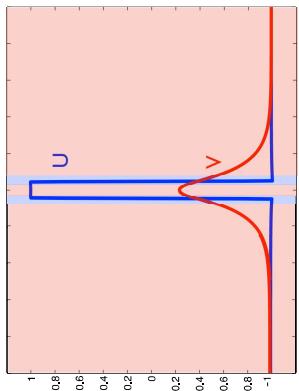


Spatial dynamics

$$\begin{aligned}0 &= \varepsilon^2 \textcolor{blue}{u}_{xx} + \textcolor{blue}{u} - \textcolor{blue}{u}^3 - \varepsilon(\alpha \textcolor{red}{v} + \gamma) \\0 &= \textcolor{red}{v}_{xx} + \textcolor{blue}{u} - \textcolor{red}{v}\end{aligned}$$

introduce $(p, q) = (\varepsilon u_x, v_x)$

$$\begin{aligned}\varepsilon u_x &= p \\ \varepsilon p_x &= -u + u^3 + \varepsilon(\alpha v + \gamma) \\ v_x &= q \\ q_x &= v - u\end{aligned}$$

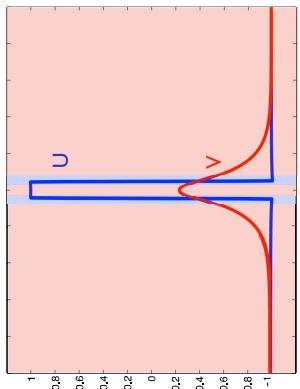


Spatial dynamics

$$\begin{aligned} 0 &= \varepsilon^2 \textcolor{blue}{u}_{xx} + \textcolor{blue}{u} - \textcolor{blue}{u}^3 - \varepsilon(\alpha \textcolor{red}{v} + \gamma) \\ 0 &= \textcolor{red}{v}_{xx} + \textcolor{blue}{u} - \textcolor{red}{v} \end{aligned}$$

introduce $(p, q) = (\varepsilon u_x, v_x)$

$$\begin{aligned} \varepsilon u_x &= p \\ \varepsilon p_x &= -u + u^3 + \varepsilon(\alpha v + \gamma) \\ v_x &= q \\ q_x &= v - u \end{aligned}$$



• Symmetries

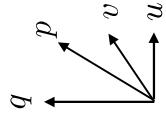
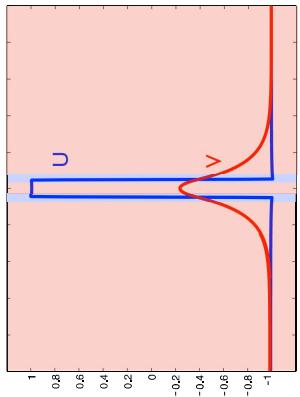
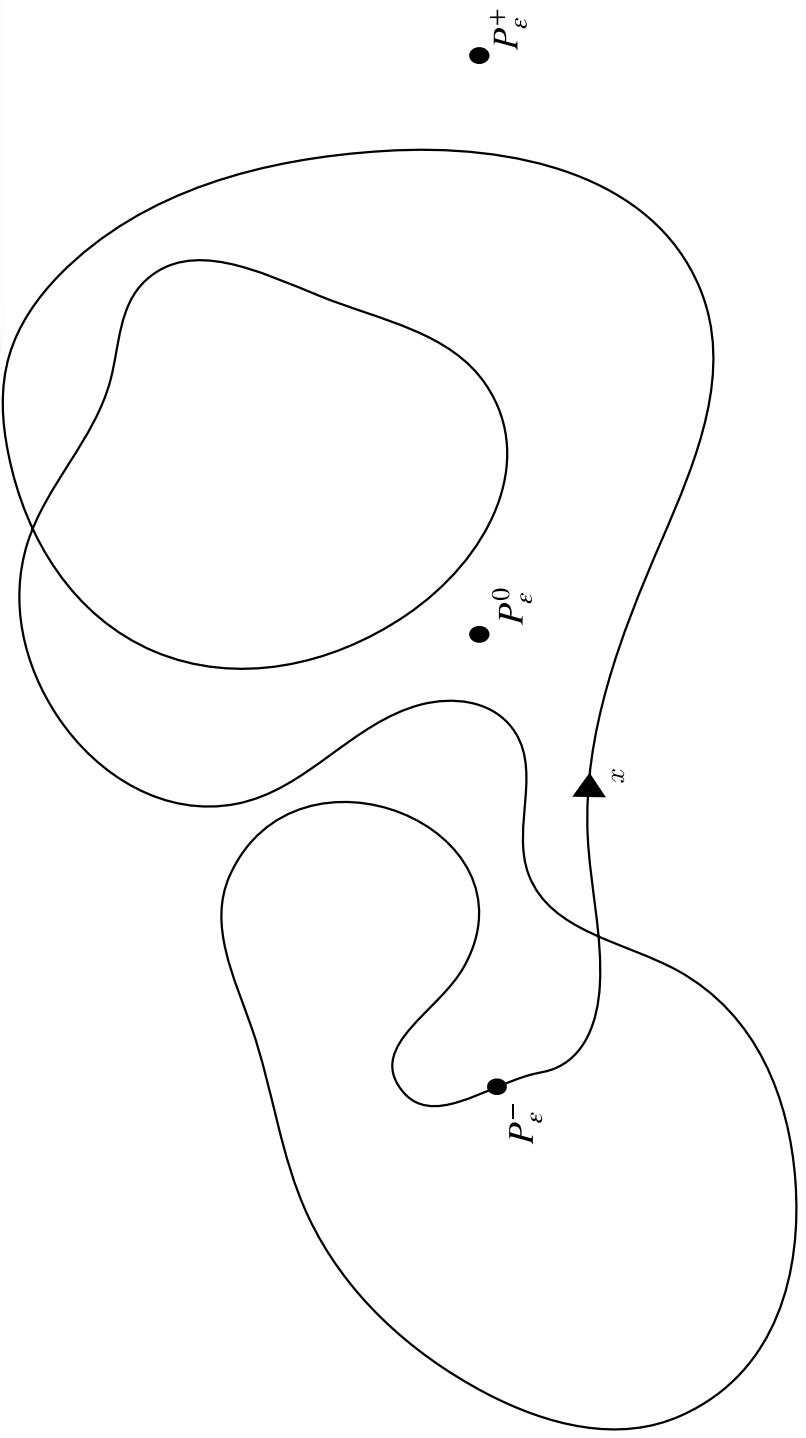
- Three fixed points: $P_\varepsilon^\pm = (u_\varepsilon^\pm, 0, u_\varepsilon^\pm, 0)$ and $P_\varepsilon^0 = (u_\varepsilon^0, 0, u_\varepsilon^0, 0)$, with $u_\varepsilon^\pm = \pm 1 \mp \frac{1}{2}\varepsilon(\alpha \pm \gamma) + \mathcal{O}(\varepsilon^2)$ and $u_\varepsilon^0 = \varepsilon\gamma + \mathcal{O}(\varepsilon^2)$.

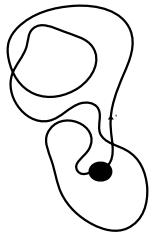
- $P_\varepsilon^\pm = (u_\varepsilon^\pm, 0, u_\varepsilon^\pm, 0)$ are *stable*; $P_\varepsilon^0 = (u_\varepsilon^0, 0, u_\varepsilon^0, 0)$ is *unstable*.
- **Goal:** look for a homoclinic orbit asymptoting to P_ε^-

abstract 4D phase portrait

$$\begin{aligned}\varepsilon u_x &= p \\ \varepsilon p_x &= -u + u^3 + \varepsilon(\alpha v + \gamma)\end{aligned}$$

$$\begin{aligned}v_x &= q \\ q_x &= v - u\end{aligned}$$





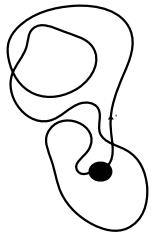
Spatial dynamics

$$\varepsilon u_x = p$$

$$\varepsilon p_x = -u + u^3 + \varepsilon(\alpha v + \gamma)$$

$$v_x = q$$

$$q_x = v - u$$



Spatial dynamics

$$\begin{aligned}\varepsilon u_x &= p \\ \varepsilon p_x &= -u + u^3 + \varepsilon(\alpha v + \gamma) \\ v_x &= q \\ q_x &= v - u\end{aligned}$$

$$\begin{array}{c} \textcolor{red}{x} = \varepsilon \xi \\ \updownarrow \\ \text{equivalent (for } \varepsilon \neq 0\text{)} \end{array}$$

$$\begin{aligned}u_\xi &= p \\ p_\xi &= -u + u^3 + \varepsilon(\alpha v + \gamma) \\ v_\xi &= \varepsilon q \\ q_\xi &= \varepsilon(v - u)\end{aligned}$$



Spatial dynamics

$$\begin{aligned} \varepsilon u_x &= p \\ \varepsilon p_x &= -u + u^3 + \varepsilon(\alpha v + \gamma) \\ v_x &= q \\ q_x &= v - u \end{aligned}$$

$$x = \varepsilon \xi$$



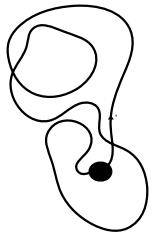
equivalent (for $\varepsilon \neq 0$)

$$\begin{aligned} u_\xi &= p \\ p_\xi &= -u + u^3 + \varepsilon(\alpha v + \gamma) \\ v_\xi &= \varepsilon q \\ q_\xi &= \varepsilon(v - u) \end{aligned}$$



singular limit $\varepsilon \downarrow 0$

$$\begin{aligned} u_\xi &= p \\ p_\xi &= -u + u^3 \\ &+ v, q \text{ constant} \end{aligned}$$



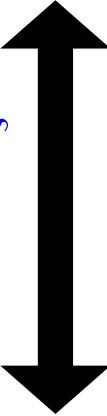
Spatial dynamics

$$\begin{aligned}\varepsilon u_x &= p \\ \varepsilon p_x &= -u + u^3 + \varepsilon(\alpha v + \gamma) \\ v_x &= q \\ q_x &= v - u\end{aligned}$$

$$x = \varepsilon \xi$$

equivalent (for $\varepsilon \neq 0$)

$$\begin{aligned}u_\xi &= p \\ p_\xi &= -u + u^3 + \varepsilon(\alpha v + \gamma) \\ v_\xi &= \varepsilon q \\ q_\xi &= \varepsilon(v - u)\end{aligned}$$



↓ singular limit $\varepsilon \downarrow 0$

↓ singular limit $\varepsilon \downarrow 0$

$$\begin{aligned}v_x &= q \\ q_x &= v - u \\ + p &= 0 \text{ and } u^3 - u = 0\end{aligned}$$

$$\begin{aligned}u_\xi &= p \\ p_\xi &= -u + u^3 \\ + v, q &\text{ constant}\end{aligned}$$



Spatial dynamics

$$\begin{aligned} \varepsilon u_x &= p \\ \varepsilon p_x &= -u + u^3 + \varepsilon(\alpha v + \gamma) \\ v_x &= q \\ q_x &= v - u \end{aligned}$$

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equivalent (for $\varepsilon \neq 0$)

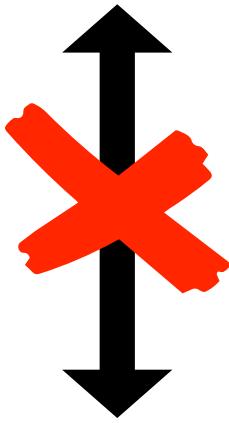
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$$\begin{aligned} v_x &= q \\ q_x &= v - u \\ + p &= 0 \text{ and } u^3 - u = 0 \end{aligned}$$

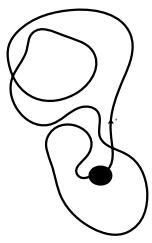
↓ singular limit $\varepsilon \downarrow 0$

$$\begin{aligned} u_\xi &= p \\ p_\xi &= -u + u^3 + \varepsilon(\alpha v + \gamma) \\ v_\xi &= \varepsilon q \\ q_\xi &= \varepsilon(v - u) \end{aligned}$$

$$\begin{aligned} u_\xi &= p \\ p_\xi &= -u + u^3 \\ + v, q &\text{ constant} \end{aligned}$$



no longer equivalent



Spatial dynamics

slow variable/scaling

fast	slow
$\varepsilon u_x = p$	
$\varepsilon p_x = -u + u^3 + \varepsilon(\alpha v + \gamma)$	
$v_x = q$	
$q_x = v - u$	

↓ singular limit $\varepsilon \downarrow 0$

slow
$v_x = q$
$q_x = v - u$
$+ p = 0 \text{ and } u^3 - u = 0$

slow reduced system
(reduced problem)

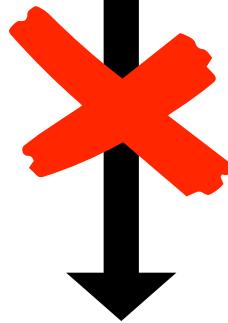
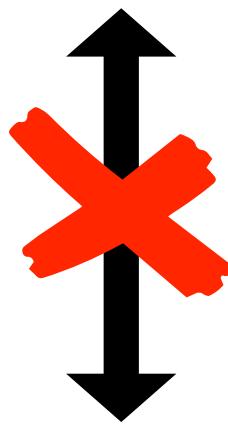
fast variable/scaling

fast	slow
$u_\xi = p$	
$p_\xi = -u + u^3 + \varepsilon(\alpha v + \gamma)$	
$v_\xi = \varepsilon q$	
$q_\xi = \varepsilon(v - u)$	

↓ singular limit $\varepsilon \downarrow 0$

fast
$u_\xi = p$
$p_\xi = -u + u^3$
$+ v, q \text{ constant}$

fast reduced system
(layer problem)



no longer equivalent



Spatial dynamics

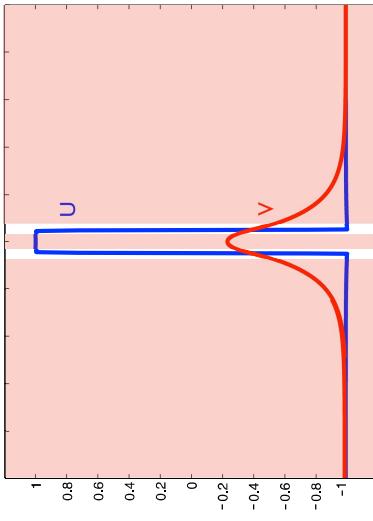
slow reduced system (SRS)

$$v_x = q$$

$$q_x = v - u$$

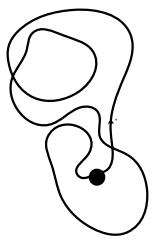
$$+ p = 0 \text{ and } u^3 - u = 0$$

- Describes the dynamics away from the interfaces (slow regions)
- Two algebraic constraints related to the fixed points of the fast reduced system.



u constant (± 1)

v changing

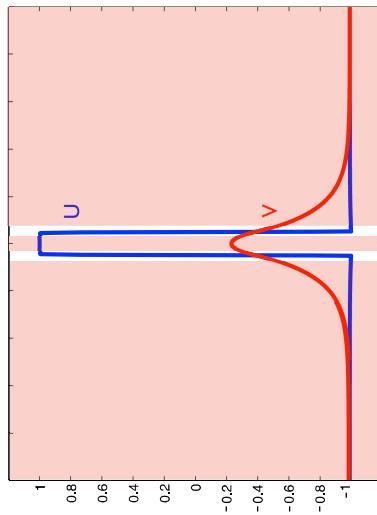


Spatial dynamics

slow reduced system (SRS)

$$\begin{aligned} v_x &= q \\ q_x &= v - u \\ &\quad + p = 0 \text{ and } u^3 - u = 0 \end{aligned}$$

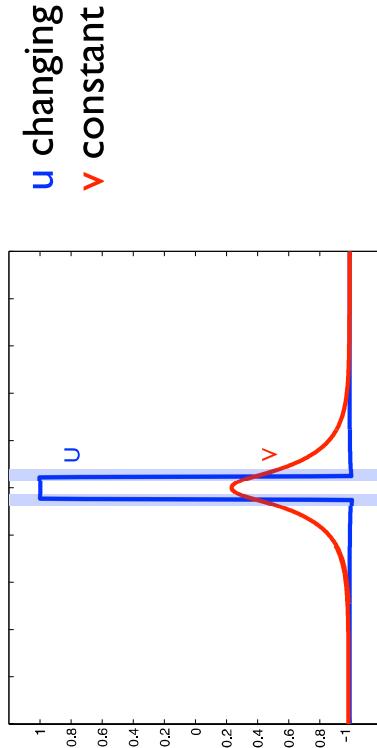
- Describes the dynamics away from the interfaces (**slow** regions)
- Two algebraic constraints related to the fixed points of the **fast reduced system**.



fast reduced system (FRS)

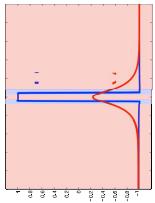
$$\begin{aligned} u_\xi &= p \\ p_\xi &= -u + u^3 \\ &\quad + v, q \text{ constant} \end{aligned}$$

- Describes the dynamics near the the interfaces (**fast** regions)
- **Slow variables** v, q act as parameters



$$u_\xi = p$$
$$p_\xi = -u + u^3$$

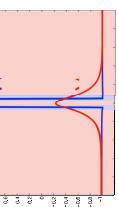
4D phase portrait



The dynamics of **slow reduced system** is constraint to fixed points of the **fast reduced system**.

Critical Manifold

$$\mathcal{M}_0 := \{(u, p, v, q) \in \mathbb{R}^4 : p = 0, 0 = u - u^3\}$$



4D phase portrait

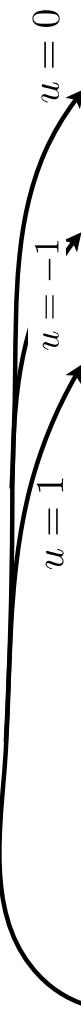
$$\begin{aligned} u_\xi &= p \\ p_\xi &= -u + u^3 \end{aligned}$$

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Critical Manifold

$$\mathcal{M}_0 := \{(u, p, v, q) \in \mathbb{R}^4 : p = 0, 0 = u - u^3\}$$

$$\mathcal{M}_0 = \mathcal{M}_0^+ \cup \mathcal{M}_0^- \cup \mathcal{M}_0^0$$



$$u_\xi = p$$

$$p_\xi = -u + u^3$$

4D phase portrait

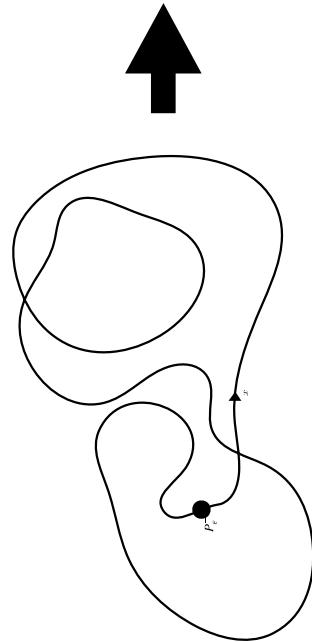
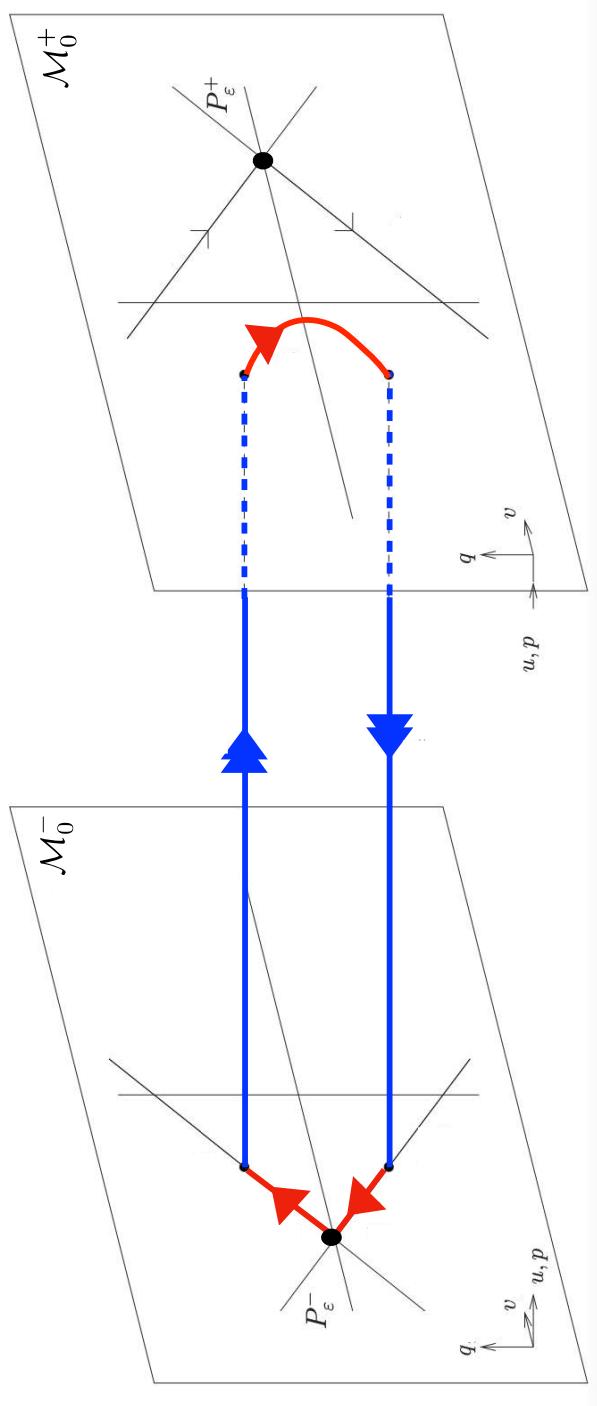
The dynamics of **slow reduced system** is constraint to fixed points of the **fast reduced system**.

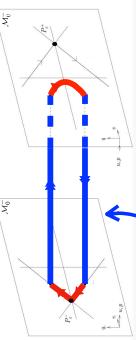
Critical Manifold

$$\mathcal{M}_0 := \{(u, p, v, q) \in \mathbb{R}^4 : p = 0, 0 = u - u^3\}$$

$$\mathcal{M}_0 = \mathcal{M}_0^+ \cup \mathcal{M}_0^- \cup \mathcal{M}_0^0$$

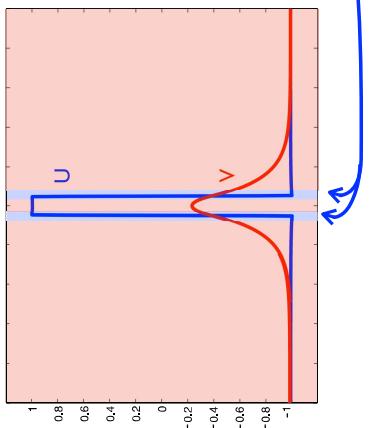
$u = 1$ $u = -1$ $u = 0$

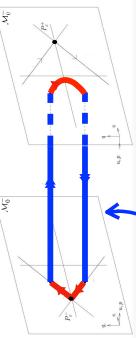




Fast reduced system

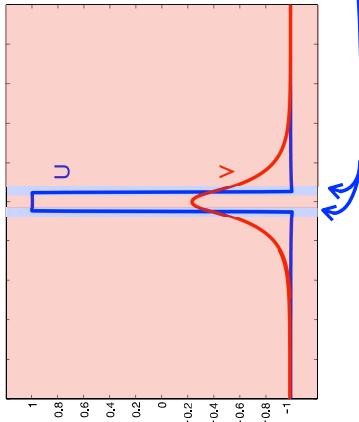
$$\begin{aligned} u_\xi &= p \\ p_\xi &= -u + u^3 \end{aligned}$$





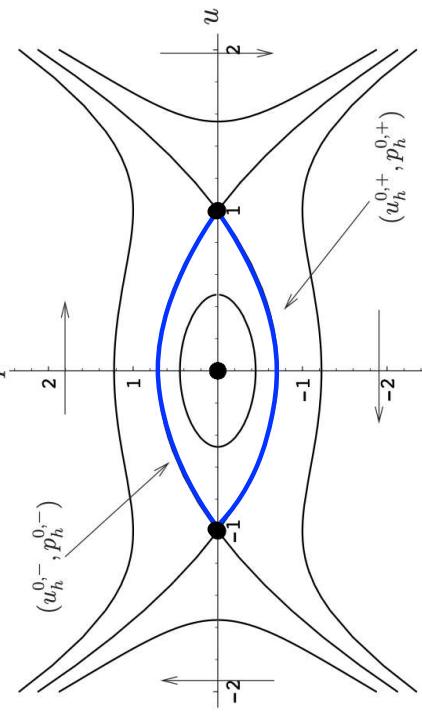
Fast reduced system

$$\begin{aligned} u_\xi &= p \\ p_\xi &= -u + u^3 \end{aligned}$$



• Hamiltonian system

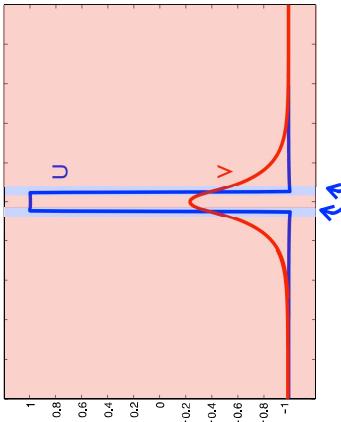
$$\begin{aligned} H(u, p) &= \frac{1}{2}(p^2 + u^2) - \frac{1}{4}(1 + u^4) \quad \blacktriangleup \quad u_\xi = \frac{\partial H}{\partial p} \\ &\qquad\qquad\qquad \blacktriangleup \quad p_\xi = -\frac{\partial H}{\partial u} \quad \blacktriangleright \quad H_\xi = 0 \end{aligned}$$



• Two heteroclinic orbits

$$u_h^{0,\pm} = \mp \tanh \left(\frac{1}{2} \sqrt{2\xi} \right), \quad p_h^{0,\pm} = \mp \frac{1}{2} \sqrt{2} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{2\xi} \right)$$

Fast reduced system



$$\begin{aligned} u_\xi &= p \\ p_\xi &= -u + u^3 \end{aligned}$$

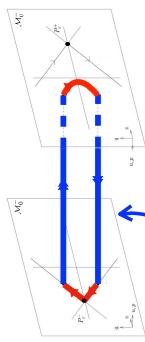
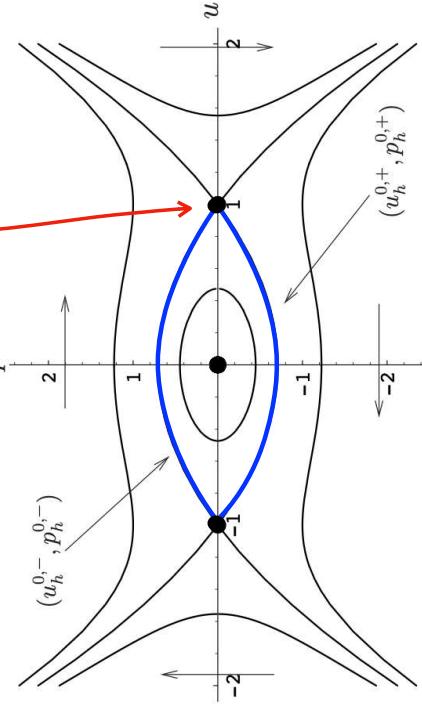
+ v, q constant

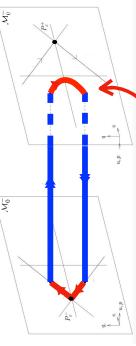
• Hamiltonian system

$$\begin{aligned} H(u, p) &= \frac{1}{2}(p^2 + u^2) - \frac{1}{4}(1 + u^4) \quad \blacktriangleup \quad u_\xi = \frac{\partial H}{\partial p} \\ &\qquad\qquad\qquad \blacktriangleup \quad p_\xi = -\frac{\partial H}{\partial u} \quad \blacktriangleright \quad H_\xi = 0 \end{aligned}$$

• Two heteroclinic orbits

$$u_h^{0,\pm} = \mp \tanh \left(\frac{1}{2}\sqrt{2\xi} \right), \quad p_h^{0,\pm} = \mp \frac{1}{2}\sqrt{2} \operatorname{sech}^2 \left(\frac{1}{2}\sqrt{2\xi} \right)$$





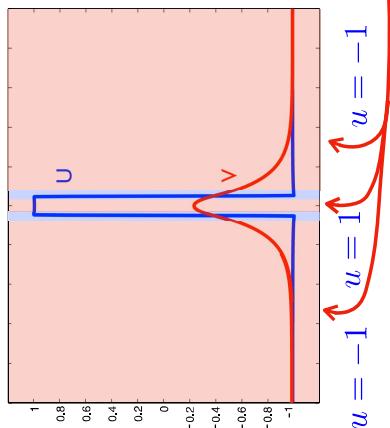
Slow reduced system

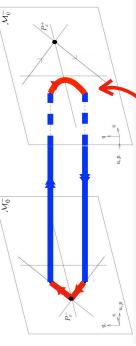
$$v_x = q$$

$$q_x = v - u$$

Constraint to the Critical Manifold

$$\mathcal{M}_0 := \{(u, p, v, q) \in \mathbb{R}^4 : p = 0, 0 = u - u^3\}$$





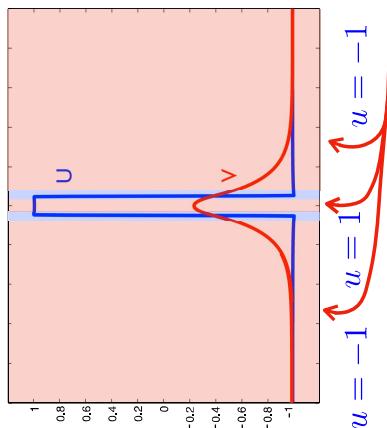
Slow reduced system

$$v_x = q$$

$$q_x = v - u$$

Constraint to the Critical Manifold

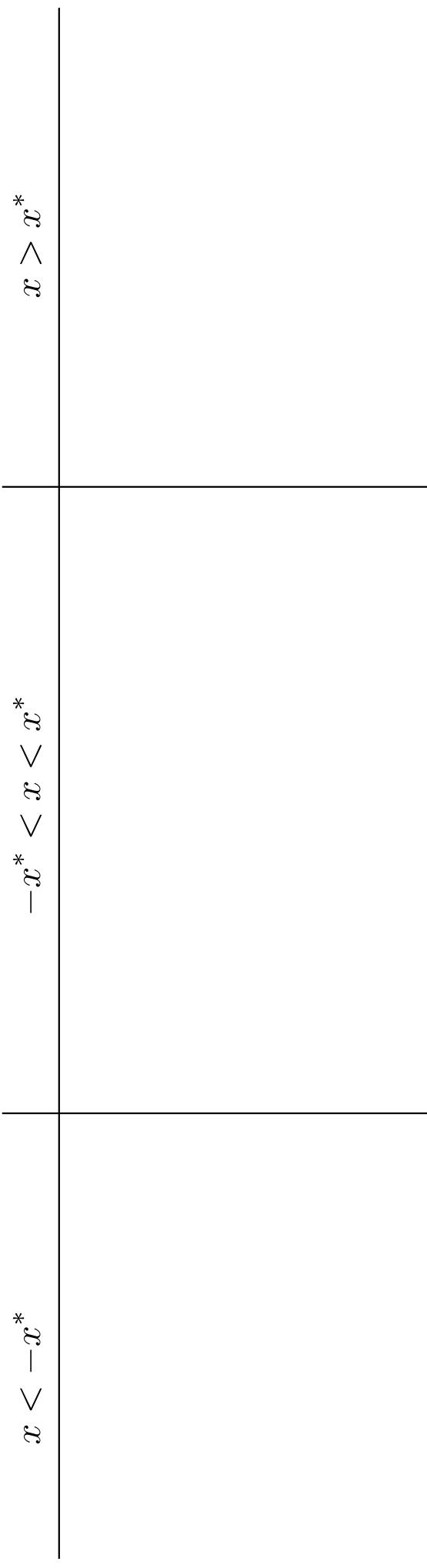
$$\mathcal{M}_0 := \{(u, p, v, q) \in \mathbb{R}^4 : p = 0, 0 = u - u^3\}$$

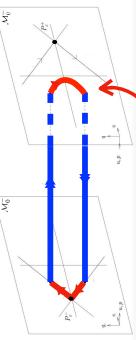


$$x < -x^*$$

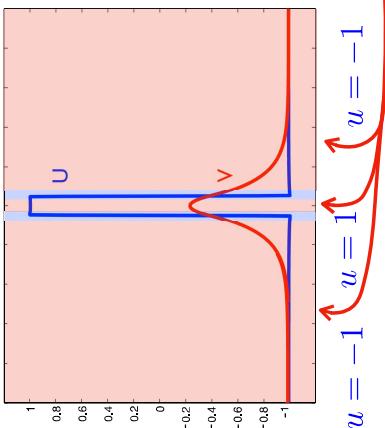
$$-x^* < x < x^*$$

$$x > x^*$$





Slow reduced system



Constraint to the Critical Manifold

$$v_x = q$$

$$q_x = v - u$$

$$\mathcal{M}_0 := \{(u, p, v, q) \in \mathbb{R}^4 : p = 0, 0 = u - u^3\}$$

$x > x^*$

$-x^* < x < x^*$

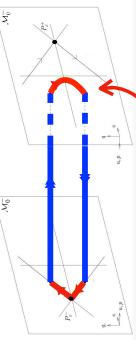
$x < -x^*$

- $u = -1$

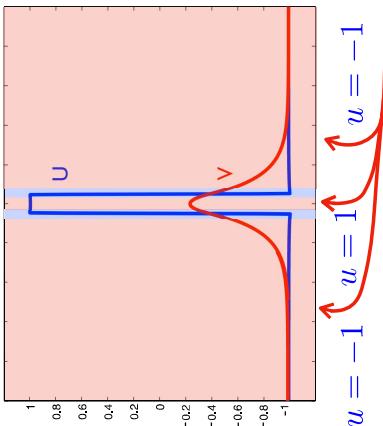
$$\begin{aligned} v_x &= q \\ q_x &= v + 1 \end{aligned}$$

- solution is bounded at $-\infty$

$$v(x) = A_- e^x - 1$$



Slow reduced system



Constraint to the Critical Manifold

$$\mathcal{M}_0 := \{(u, p, v, q) \in \mathbb{R}^4 : p = 0, 0 = u - u^3\}$$

$$v_x = q$$

$$q_x = v - u$$

$$x < -x^* \quad -x^* < x < x^* \quad x > x^*$$

- $u = -1$

$$v_x = q$$

$$q_x = v + 1$$

- solution is bounded at $-\infty$

$$v(x) = A_- e^x - 1$$

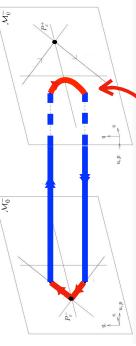
- $u = -1$

$$v_x = q$$

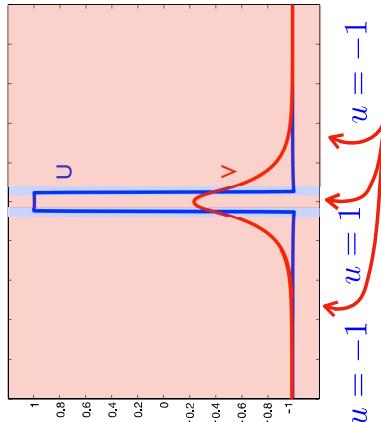
$$q_x = v + 1$$

- solution is bounded at $+\infty$

$$v(x) = A_+ e^{-x} - 1$$



Slow reduced system



Constraint to the Critical Manifold

$$\mathcal{M}_0 := \{(u, p, v, q) \in \mathbb{R}^4 : p = 0, 0 = u - u^3\}$$

$$v_x = q$$

$$q_x = v - u$$

$$x < -x^* \quad -x^* < x < x^* \quad x > x^*$$

- translation invariance

$$\bullet \textcolor{blue}{u = -1}$$

$$\begin{aligned} v_x &= q \\ q_x &= v + 1 \end{aligned}$$

- solution is bounded at $-\infty$

$$v(x) = A_- e^x - 1$$

$$\bullet \textcolor{blue}{u = +1}$$

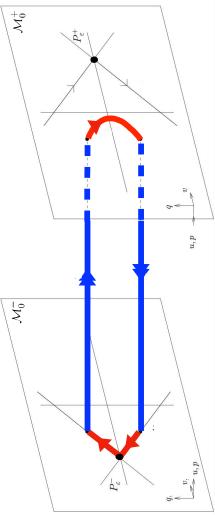
$$\begin{aligned} v_x &= q \\ q_x &= v - 1 \end{aligned}$$

- solution is bounded at $+\infty$

$$v(x) = A_+ e^{-x} - 1$$

$$\bullet \textcolor{blue}{u = -1}$$

$$\begin{aligned} v_x &= q \\ q_x &= v + 1 \end{aligned}$$



Slow reduced system

$$v_x = q$$

$$q_x = v - u$$

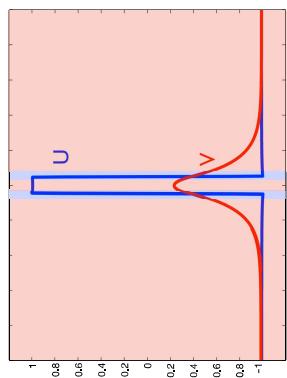
$$-x^* < x < x^*$$

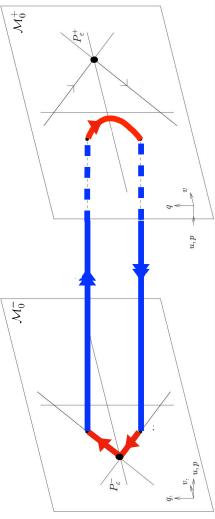
$$v(x) = A_- e^x - 1$$

$$v(x) = A_0^1 e^x + A_0^2 e^{-x} + 1$$

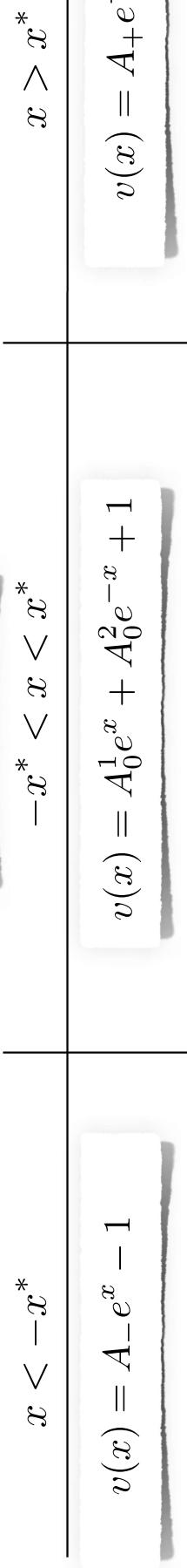
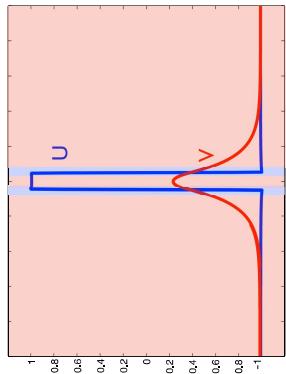
$$x < -x^*$$

$$x > x^*$$





Slow reduced system



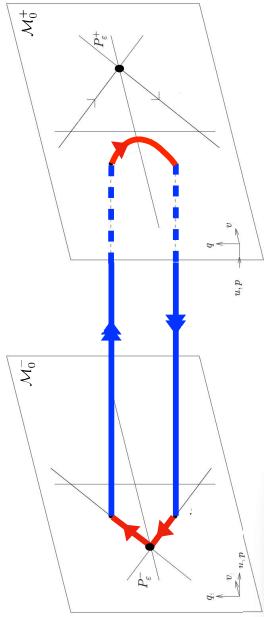
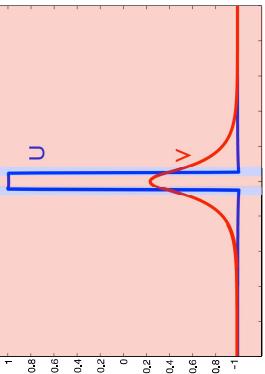
- v, q are constant over the **fast fields**. So solutions (and their derivatives) should match at $\pm x^*$
- 4 eqns with 4 unknowns

$$\begin{aligned}
 A_- e^{-x^*} - 1 &= A_0^1 e^{-x^*} + A_0^2 e^{x^*} + 1 \\
 A_+ e^{-x^*} - 1 &= A_0^1 e^{x^*} + A_0^2 e^{-x^*} + 1 \\
 A_- e^{-x^*} &= A_0^1 e^{-x^*} - A_0^2 e^{x^*} \\
 -A_+ e^{-x^*} &= A_0^1 e^{x^*} - A_0^2 e^{-x^*}
 \end{aligned}$$

$$\begin{aligned}
 A_- &= e^{x^*} - e^{-x^*} = A_+ \\
 A_0^1 &= -e^{-x^*} = A_0^2
 \end{aligned}$$



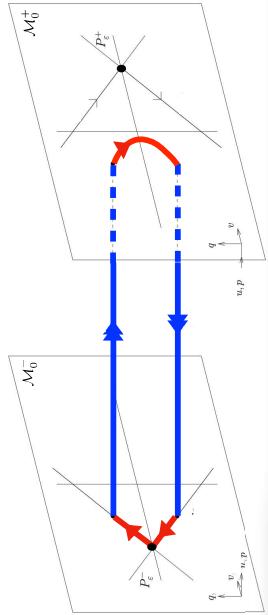
Summarise (so far)



$$u(x) = \begin{cases} \tanh\left(\frac{1}{2\varepsilon}\sqrt{2}(x+x^*)\right), & x \approx -x^* \\ -\tanh\left(\frac{1}{2\varepsilon}\sqrt{2}(x-x^*)\right), & x \approx x^* \end{cases}$$

$$v(x) = \begin{cases} -1, & x \gtrless -x^* \\ e^{x+x^*} - e^{x-x^*} - 1, & x \gtrless -x^* \\ -e^{-2x^*}, & x \approx -x^* \\ -e^{x-x^*} - e^{-x-x^*} + 1, & -x^* \gtrless x \gtrless x^* \\ e^{-x+x^*} - e^{-x-x^*} - 1, & x \gtrless -x^* \end{cases}$$

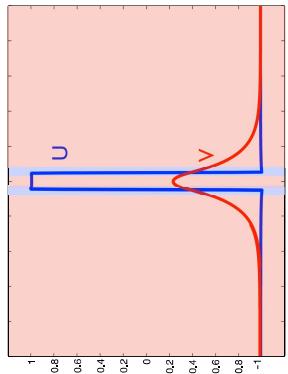
Summarise (so far)



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How to determine x^* ?

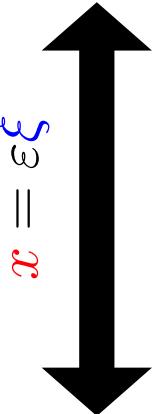


Turning on ε

- Go back to the perturbed system (ε is as small as we want)

$$\begin{aligned}\varepsilon u_x &= p \\ \varepsilon p_x &= -u + u^3 + \varepsilon(\alpha v + \gamma) \\ v_x &= q \\ q_x &= v - u\end{aligned}$$

$$\begin{aligned}u_\xi &= p \\ p_\xi &= -u + u^3 + \varepsilon(\alpha v + \gamma) \\ v_\xi &= \varepsilon q \\ q_\xi &= \varepsilon(v - u)\end{aligned}$$



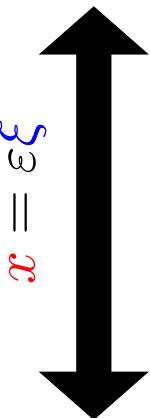
$\textcolor{red}{x} = \varepsilon \xi$

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- No problem for the dynamics in the **slow fields**

- However, in the **fast fields** we cannot guarantee the persistence of the heteroclinic orbits

- Enforcing persistence of the heteroclinics will yield x^*

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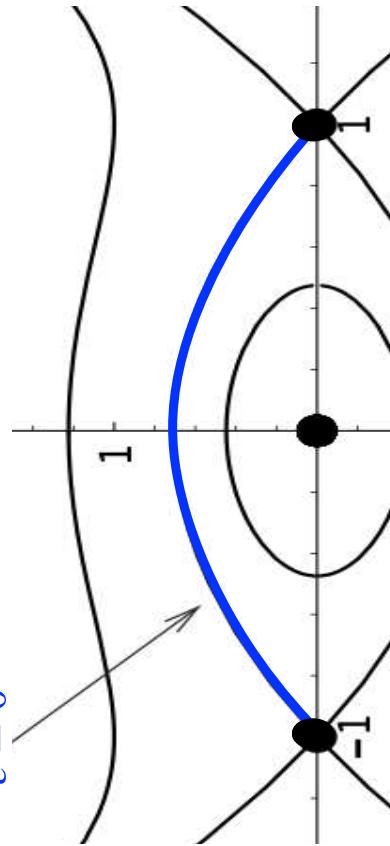
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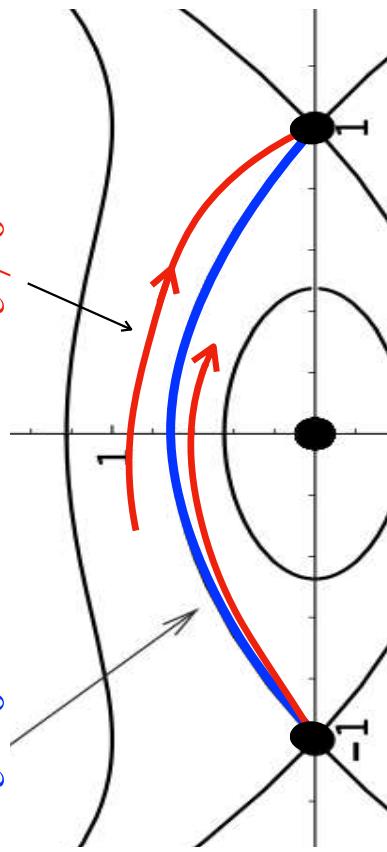
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Determining x^*

$$\begin{aligned}\varepsilon u_x &= p \\ \varepsilon p_x &= -u + u^3 + \varepsilon(\alpha v + \gamma) \\ v_x &= q \\ q_x &= v - u\end{aligned}$$

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- Hamiltonian of the **fast reduced system** is no longer conservative

$$\begin{aligned}H_\xi &= H_u u_\xi + H_p p_\xi \\ &= (u - u^3)p + p(-u + u^3 + \varepsilon(\alpha v + \gamma)) \\ &= \varepsilon p(\alpha v + \gamma)\end{aligned}$$

$$H(u, p) = \frac{1}{2}(p^2 + u^2) - \frac{1}{4}(1 + u^4)$$

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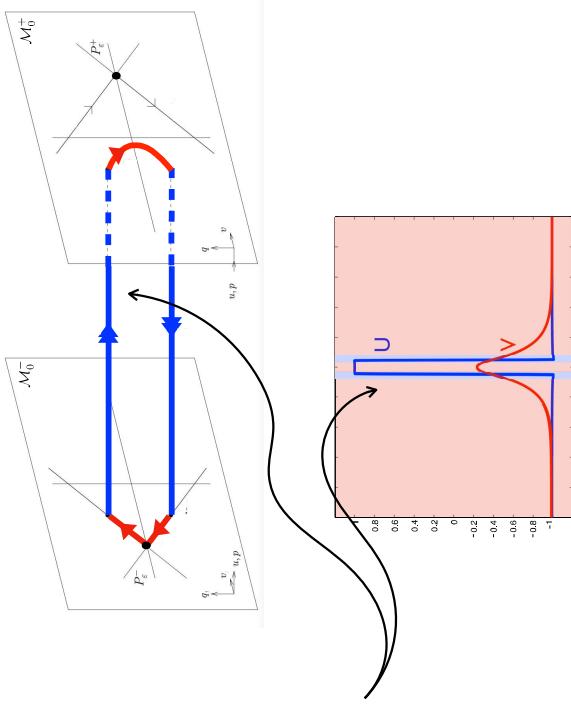
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- Hamiltonian on the critical manifolds $\mathcal{M}_\varepsilon^\pm$ is $\mathcal{O}(\varepsilon^2)$

$$\mathcal{M}_\varepsilon^\pm = \left\{ (u, p, v, q) \in \mathbb{R}^4 : u = \pm 1 - \frac{1}{2}\varepsilon(\alpha v + \gamma) + \mathcal{O}(\varepsilon^2), p = \mathcal{O}(\varepsilon^2) \right\} \quad \blacktriangleup \quad H|_{\mathcal{M}_\varepsilon^\pm} = \mathcal{O}(\varepsilon^2)$$

$u_\xi = p$	$p_\xi = -u + u^3 + \varepsilon(\alpha v + \gamma)$
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Determining x^*



- Two ways to compute ΔH (change of H) when a solution jumps over the **fast field** \mathbf{f}_f from $\mathcal{M}_\varepsilon^-$ to $\mathcal{M}_\varepsilon^+$

- Combining — using the symmetry — gives the persistence/existence

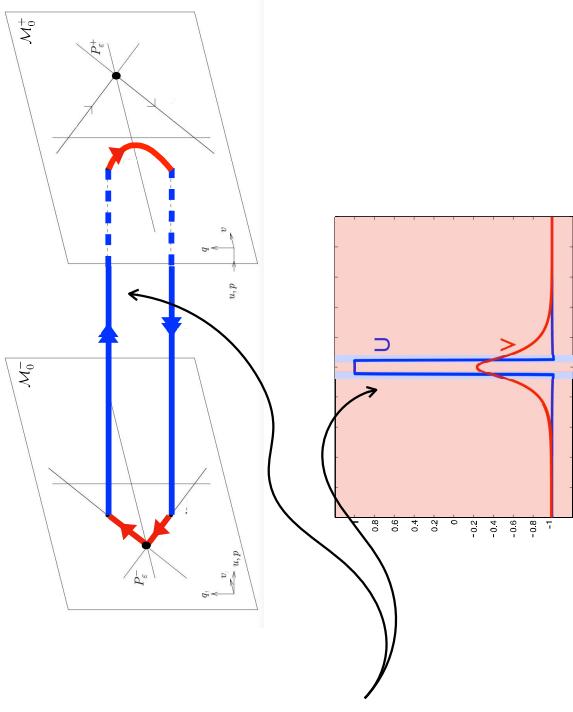
$$-\alpha e^{-2x^*} + \gamma = 0$$

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$(\alpha > \gamma!!)$

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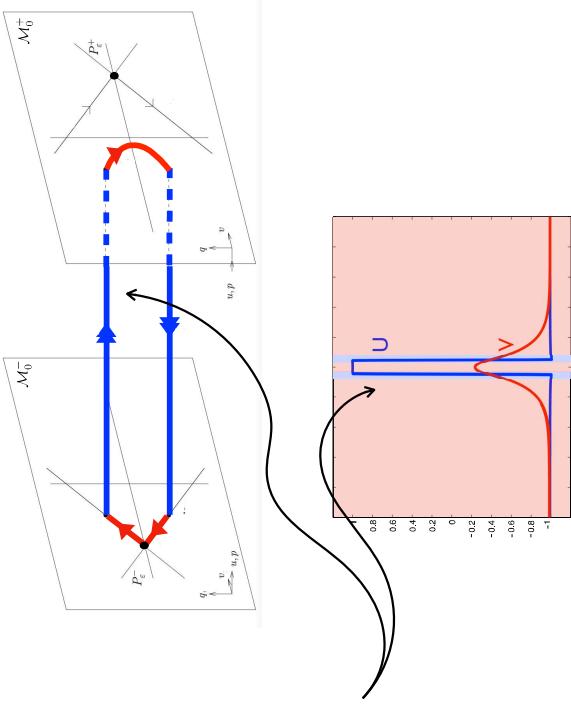
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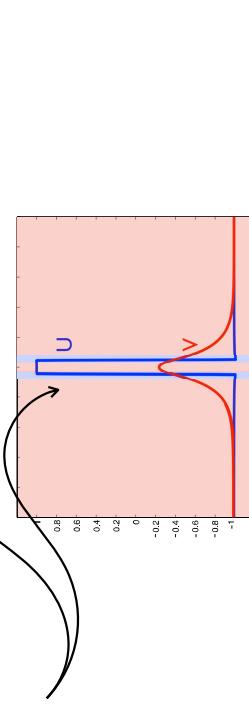
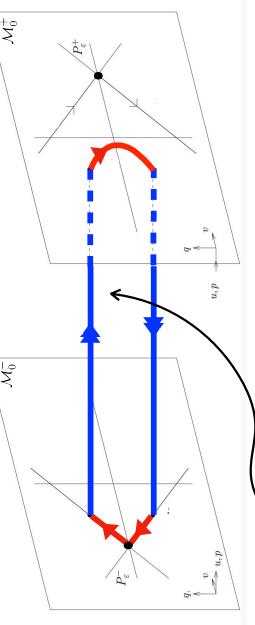
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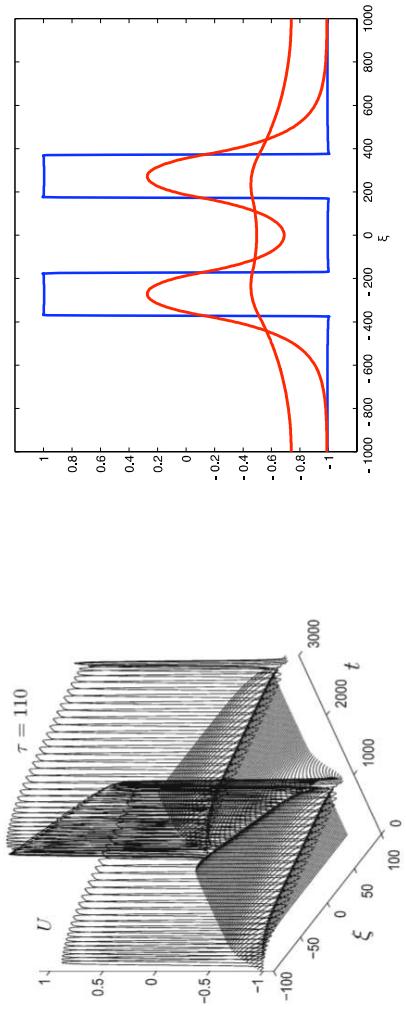
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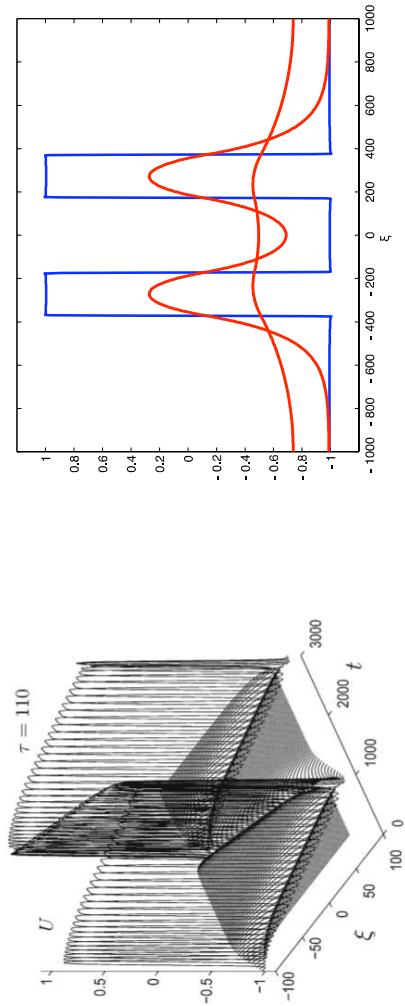
Remarks

- Method can be applied to study stationary 2-pulse solutions and travelling pulse solutions. For the latter τ needs to be large ($\mathcal{O}(1/\varepsilon^2)$)



Remarks

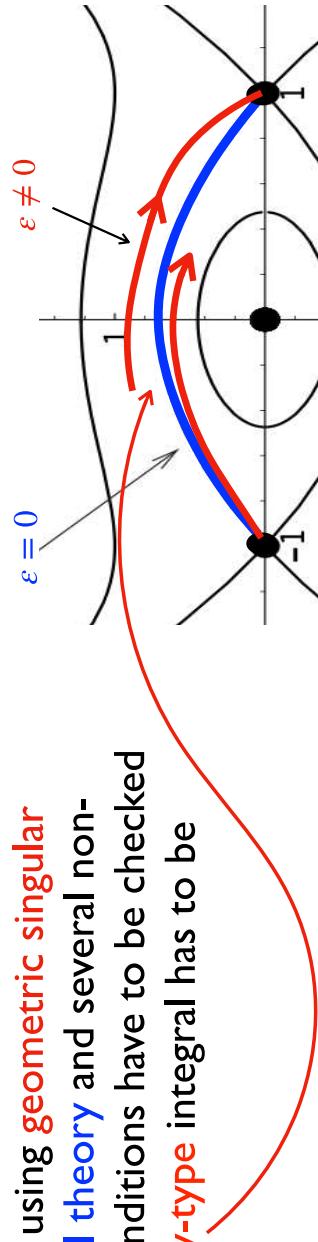
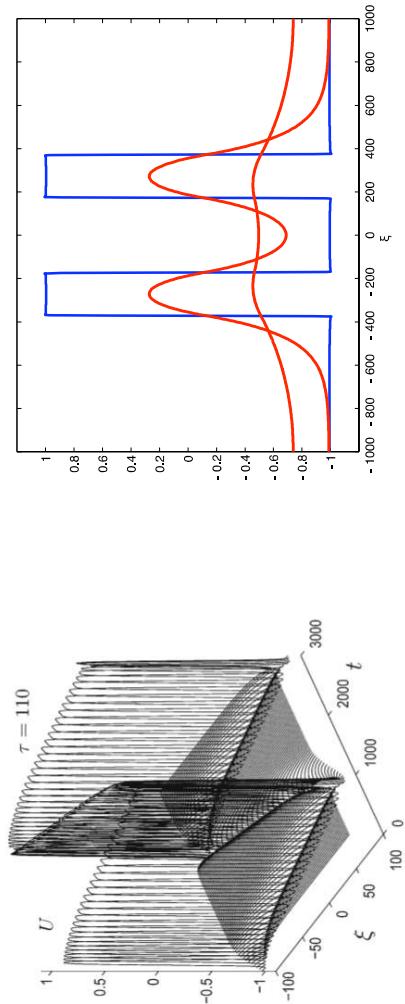
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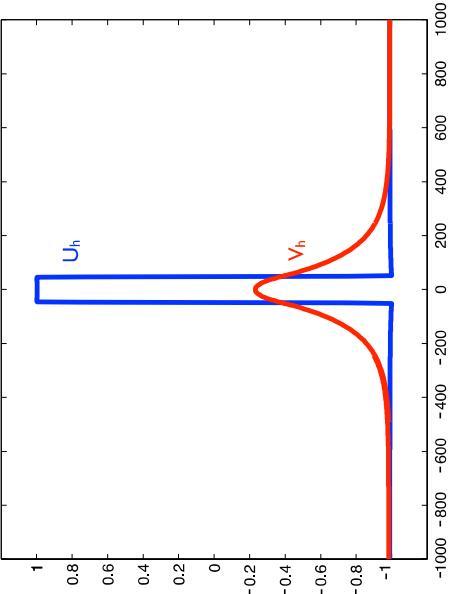
End of Part I

We continue again at 13:30

Are these stationary pulse solutions stable?

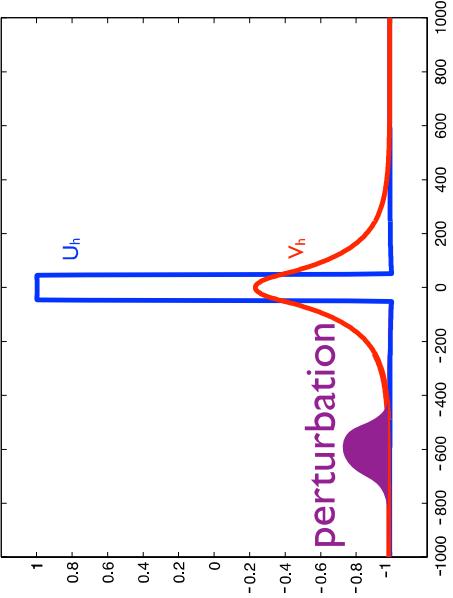
$$\begin{aligned} U_t &= \varepsilon^2 U_{xx} + \textcolor{blue}{U} - \textcolor{blue}{U}^3 - \varepsilon(\alpha \textcolor{red}{V} + \gamma) \\ \tau V_t &= \textcolor{red}{V}_{xx} + \textcolor{blue}{U} - \textcolor{red}{V} \end{aligned}$$

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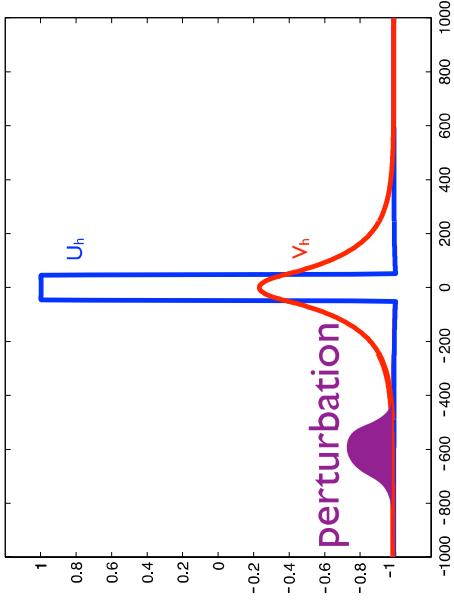
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• Interested in the spectrum of the pulse solution. The spectrum naturally decomposes into two parts: *essential spectrum* and *point spectrum*

- To compute, we substitute

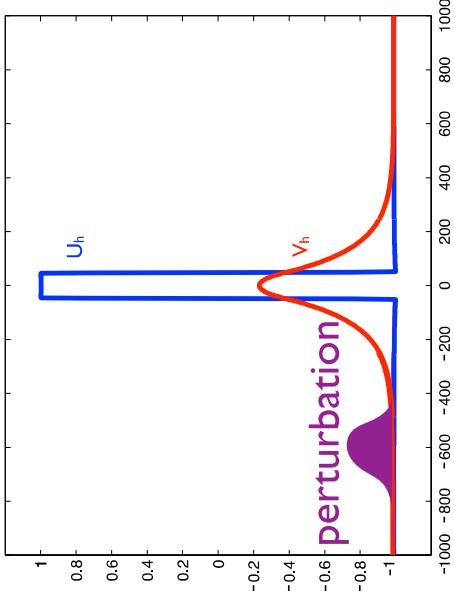
$$(U, V)(x, t) = (\textcolor{blue}{u}_h, \textcolor{red}{v}_h)(x) + (\textcolor{violet}{u}, \textcolor{violet}{v})(x)e^{\lambda t}, \quad \lambda \in \mathbb{C}$$

into the PDE and linearise to obtain the eigenvalue problem

$$\begin{aligned} \varepsilon^2 \textcolor{violet}{u}_{xx} + (1 - 3u_h(x)^2 - \lambda)\textcolor{violet}{u} &= \varepsilon\alpha \textcolor{violet}{v} \\ \textcolor{violet}{v}_{xx} &= (1 + \tau\lambda)\textcolor{violet}{v} - \textcolor{violet}{u} \end{aligned}$$

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Compare to the existence ODE

$$0 = \varepsilon^2 \textcolor{blue}{u}_{h,xx} + \textcolor{blue}{u}_h - \textcolor{blue}{u}_h^3 - \varepsilon(\alpha \textcolor{red}{v}_h + \gamma)$$

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into the PDE and linearise to obtain the eigenvalue problem

$$\varepsilon^2 \textcolor{violet}{u}_{xx} + (1 - 3\textcolor{blue}{u}_h(x)^2 - \lambda)\textcolor{violet}{u} = \varepsilon \alpha \textcolor{violet}{v}$$

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Essential spectrum

$$\varepsilon^2 \mathbf{v}_{xx} + (1 - 3u_h(x)^2 - \lambda) \mathbf{v} = \varepsilon \alpha \mathbf{v}$$

$$\mathbf{v}_{xx} = (1 + \tau \lambda) \mathbf{v} - \mathbf{u}$$

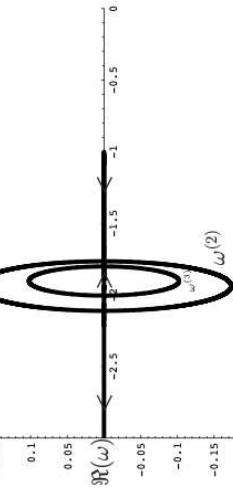
- Essential spectrum is related to instabilities coming from infinity (unbounded domain)

- In general relatively easy to compute

- Write eigenvalue problem as system of 1st order ODEs (in **fast formulation**) and let $\xi \rightarrow \pm \infty$. This gives a 4D matrix equation with matrix

$$\mathsf{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 + \lambda + \mathcal{O}(\varepsilon) & 0 & \varepsilon \alpha & 0 \\ 0 & 0 & 0 & \varepsilon \\ -\varepsilon & 0 & \varepsilon(1 + \tau \lambda) & 0 \end{pmatrix}$$

- The λ for which M has purely imaginary eigenvalues determine the essential spectrum. Here, this gives that the essential spectrum lies (to leading order) in the left half plane given by



$$\Sigma := \{\omega : \Re(\omega) < \chi\}, \quad \max\{-2, -1/\tau\} < \chi < 0$$

- So, no essential instabilities

Point spectrum

$$\varepsilon^2 \textcolor{violet}{u}_{xx} + (1 - 3\textcolor{blue}{u}_h(x)^2 - \lambda) \textcolor{violet}{u} = \varepsilon \alpha \textcolor{violet}{v}$$

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Point spectrum

$\lambda=0$ is eigenvalue related to translation invariance

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It follows that the leading (largest) point eigenvalues are of $\mathcal{O}(\varepsilon^2)$, for which the **slow component** scales as $\mathcal{O}(\varepsilon)$

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$$u_{\xi\xi} + (1 - 3u_h^2)u = \varepsilon^2(\alpha v + \lambda u)$$

$$\varepsilon v_{\xi\xi} = -\varepsilon u + \varepsilon^2 v + \varepsilon^4 \tau \lambda v$$

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$$\varepsilon \downarrow 0$$

fast fields

- u_h known to leading order
- \mathbf{u} can be determined
- \mathbf{q} no longer constant over **fast fields**

$$0 = p$$

$$0 = (3u_h^2 - 1)u$$

$$v_x = q$$

$$q_x = v$$

slow fields

- constraint gives $\mathbf{u}=0$
- (\mathbf{v}, \mathbf{q}) can be solved

Determining λ

- To determine the eigenvalues, we have to look at the $\mathcal{O}(\varepsilon^2)$ terms of the **fast equation** in the **fast fields**

$$u_{\xi\xi} + (1 - 3u_h^2)u = \varepsilon^2(\alpha v + \lambda u)$$

- Expanding $u = u_0 + \varepsilon^2 u_2 + \mathcal{O}(\varepsilon^3)$ and $u_h = u_{h,0} + \varepsilon^2 u_{h,2} + \mathcal{O}(\varepsilon^3)$ gives

$$\mathcal{O}(\varepsilon^2) : \quad Lu_2 := (u_2)_{\xi\xi} + (1 - 3u_{h,0}^2)u_2 = 6u_{h,0}u_{h,2}u_0 + \alpha v^* + \lambda u_0$$

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- Since $Lp_{h,0} = 0$, a solvability condition/Fredholm alternative gives

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$$\int_{I_f} (6u_{h,0}u_{h,2}u_0 + \alpha v^* + \lambda u_0)p_{h,0}d\xi = 0$$

- Only $u_{h,2}$ is unknown. We need higher order information on the stationary pulse solution. This can again be derived from a solvability condition/Fredholm alternative.

Determining λ

- To determine the eigenvalues, we have to look at the $\mathcal{O}(\varepsilon^2)$ terms of the **fast equation** in the **fast fields**

$$u_{\xi\xi} + (1 - 3u_h^2)u = \varepsilon^2(\alpha v + \lambda u)$$

- Expanding $u = u_0 + \varepsilon^2 u_2 + \mathcal{O}(\varepsilon^3)$ and $u_h = u_{h,0} + \varepsilon^2 u_{h,2} + \mathcal{O}(\varepsilon^3)$ gives

$$\mathcal{O}(\varepsilon^2) : \quad Lu_2 := (u_2)_{\xi\xi} + (1 - 3u_{h,0}^2)u_2 = 6u_{h,0}u_{h,2}u_0 + \alpha v^* + \lambda u_0$$

- Since $Lp_{h,0} = 0$, a solvability condition/Fredholm alternative gives

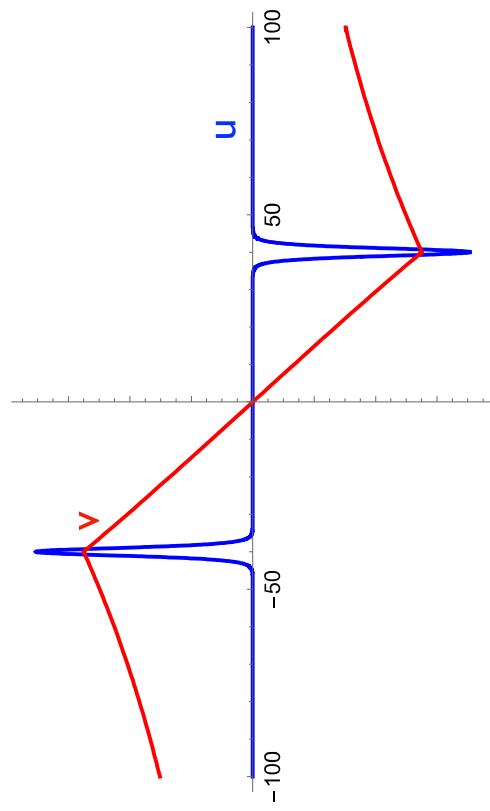
$$\int_{I_f} (6u_{h,0}u_{h,2}u_0 + \alpha v^* + \lambda u_0)p_{h,0}d\xi = 0$$

- Only $u_{h,2}$ is unknown. We need higher order information on the stationary pulse solution. This can again be derived from a solvability condition/Fredholm alternative.

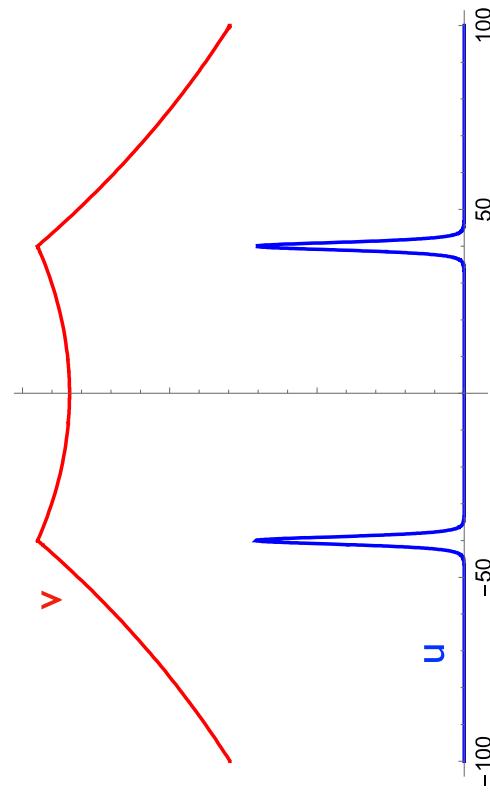
- This gives that the leading (largest) point eigenvalue - next to the translation invariance eigenvalue at 0 - is to leading order given by

$$\lambda = -3\sqrt{2}\alpha\varepsilon^2 e^{-2x^*}$$

Eigenfunctions



translation invariance eigenfunction

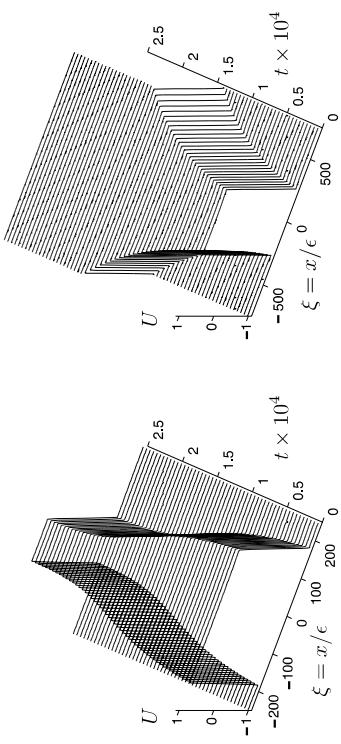


eigenfunction belonging to $\lambda = -3\sqrt{2}\alpha\varepsilon^2 e^{-2x^*}$

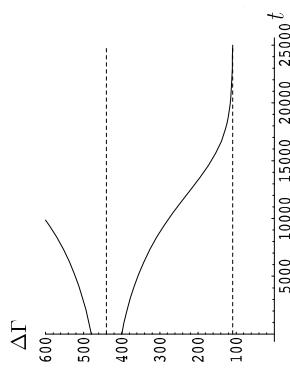
Remarks

- Note that linear stability implies nonlinear stability in this case since the operator associated to the linear problem is sectorial.
- Method can be applied to study the stability of stationary 2-pulse solutions and travelling pulse solutions.
- For the 3-component system the eigenvalue expression becomes more involved
- Results can be made rigorous by using the **Evans function** and **NonLocal Eigenvalue Paradox (NLEP) method**.

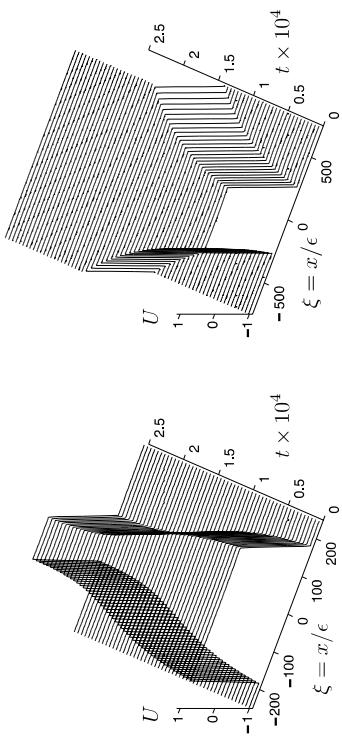
Interactions



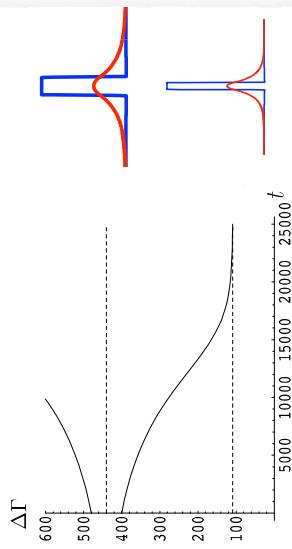
Derive ode for the distance
between the interfaces ($\Delta\Gamma$)?



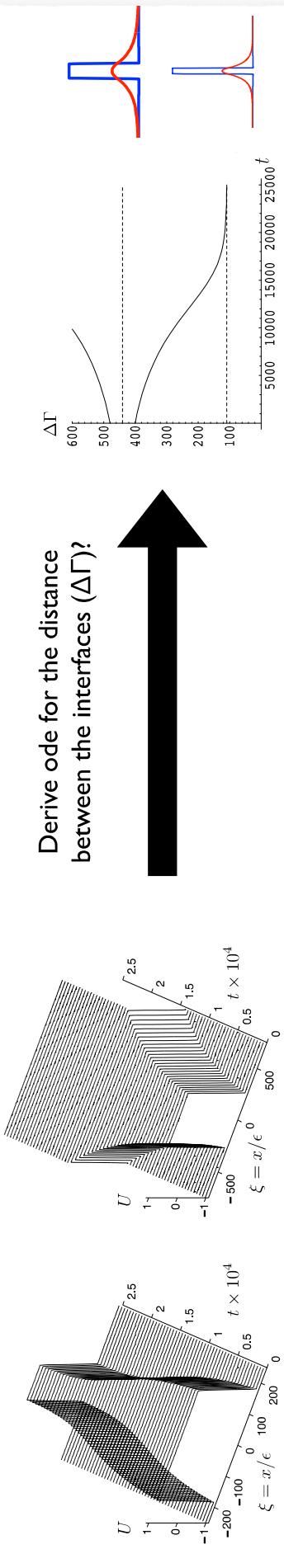
Interactions



Derive ode for the distance
between the interfaces ($\Delta\Gamma$)?

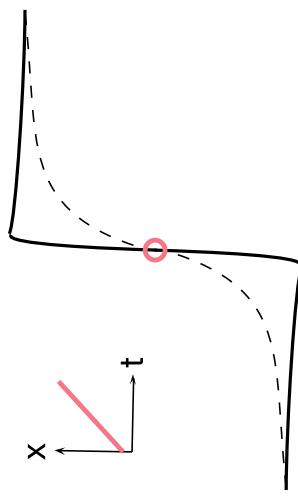


Interactions

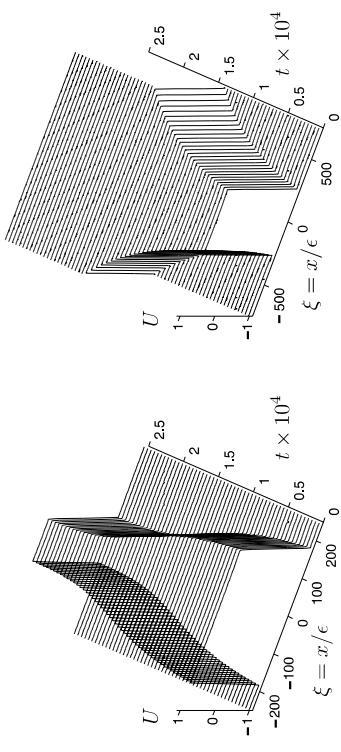


• A I-front solution travels with speed

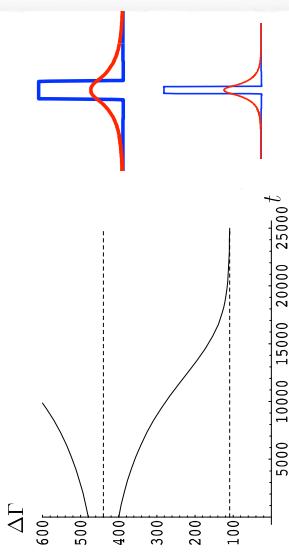
$$c = \frac{3}{\sqrt{2}} \gamma \epsilon^2 \text{ (in the slow scaling)}$$



Interactions



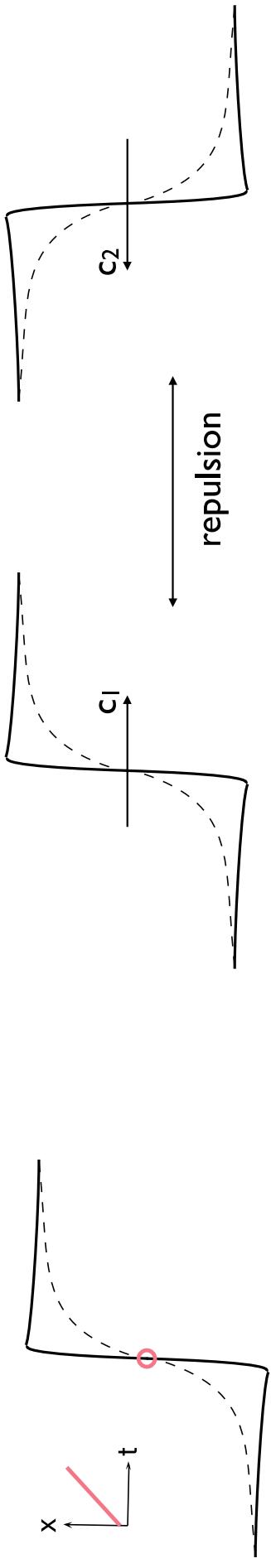
Derive ode for the distance between the interfaces ($\Delta\Gamma$)?



• A I-front solution travels with speed

$$c = \frac{3}{\sqrt{2}} \gamma \epsilon^2 \text{ (in the slow scaling)}$$

- A stationary pulse can be seen as a balance between two I-front solutions wanting to travel in opposite direction while repelling each other



Formal derivation

- Restrict to 2 fronts (I front and I back)
- Fronts naturally travel with a speed of $\mathcal{O}(\varepsilon^2)$
- Denote the location of the i-th front by Γ_i and say moves with speed $\varepsilon^2 c_i(t)$, then

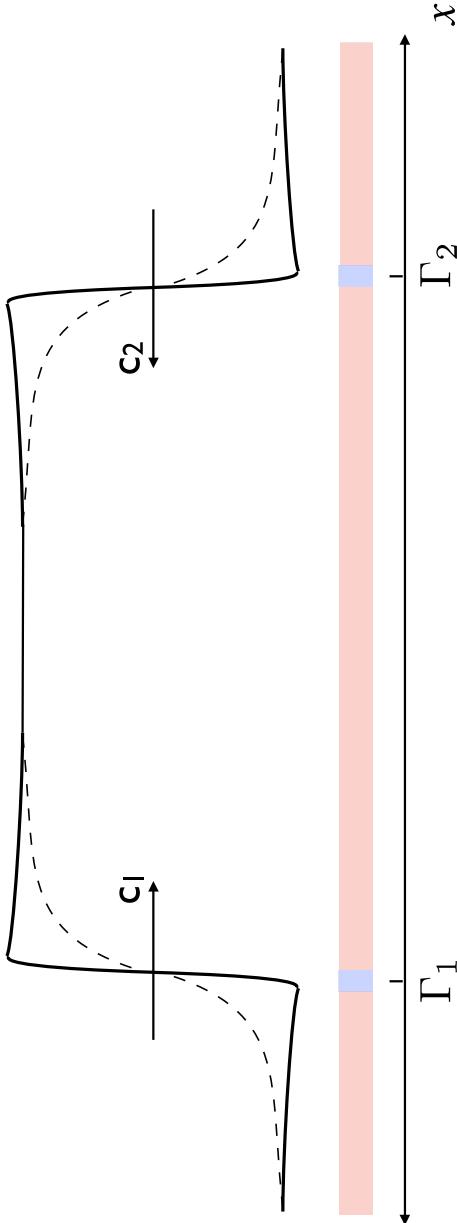
$$\Gamma_i(t) = \Gamma_i(0) + \varepsilon^2 \int_0^t c_i(s) ds \implies \dot{\Gamma}_i(t) = \varepsilon^2 c_i(t)$$

• **Crucially**, the speed has no leading order influence on the **slow equation**!

$$\begin{aligned}\mathcal{U}_t &= \varepsilon^2 \mathcal{U}_{xx} + \mathcal{U} - \mathcal{U}^3 - \varepsilon(\alpha \mathcal{V} + \gamma) \\ \tau \mathcal{V}_t &= \mathcal{V}_{xx} + \mathcal{U} - \mathcal{V}\end{aligned}$$

- Therefore, we can formally introduce two coming frames $x - \varepsilon^2 c_i(t)$

Formal derivation



$$\varepsilon u_x = p$$

$$\varepsilon p_x = -u + u^3 + \varepsilon(\alpha v + \gamma - c_i p)$$

$$v_x = q$$

$$q_x = v - u + \mathcal{O}(\varepsilon^2)$$

$$u_\xi = p$$

$$p_\xi = -u + u^3 + \varepsilon(\alpha v + \gamma - c_i p)$$

$$v_\xi = \varepsilon q$$

$$q_\xi = \varepsilon(v - u) + \mathcal{O}(\varepsilon^3)$$

Formal derivation

- Now, take singular limits $\varepsilon \downarrow 0$ and proceed as before

$$\varepsilon u_x = p$$

$$\varepsilon p_x = -u + u^3 + \varepsilon(\alpha v + \gamma - c_i p)$$

$$v_x = q$$

$$q_x = v - u + \mathcal{O}(\varepsilon^2)$$

$$u_\xi = p$$

$$p_\xi = -u + u^3 + \varepsilon(\alpha v + \gamma - c_i p)$$

$$v_\xi = \varepsilon q$$

$$q_\xi = \varepsilon(v - u) + \mathcal{O}(\varepsilon^3)$$

Formal derivation

- Now, take singular limits $\varepsilon \downarrow 0$ and proceed as before

$$\varepsilon u_x = p$$

$$\varepsilon p_x = -u + u^3 + \varepsilon(\alpha v + \gamma - c_i p)$$

$$v_x = q$$

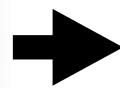
$$q_x = v - u + \mathcal{O}(\varepsilon^2)$$

$$u_\xi = p$$

$$p_\xi = -u + u^3 + \varepsilon(\alpha v + \gamma - c_i p)$$

$$v_\xi = \varepsilon q$$

$$q_\xi = \varepsilon(v - u) + \mathcal{O}(\varepsilon^3)$$



$$u_\xi = p$$

$$p_\xi = -u + u^3$$

+ v, q constant

↓ singular limit $\varepsilon \downarrow 0$

Formal derivation

- Now, take singular limits $\varepsilon \downarrow 0$ and proceed as before

$$\varepsilon u_x = p$$

$$\varepsilon p_x = -u + u^3 + \varepsilon(\alpha v + \gamma - c_i p)$$

$$v_x = q$$

$$q_x = v - u + \mathcal{O}(\varepsilon^2)$$

↓
singular limit $\varepsilon \downarrow 0$

$$v_x = q$$

$$q_x = v - u$$

$$+ p = 0 \text{ and } u^3 - u = 0$$

$$\begin{aligned} u_\xi &= p \\ p_\xi &= -u + u^3 + \varepsilon(\alpha v + \gamma - c_i p) \\ v_\xi &= \varepsilon q \\ q_\xi &= \varepsilon(v - u) + \mathcal{O}(\varepsilon^3) \end{aligned}$$

↓
singular limit $\varepsilon \downarrow 0$

$$u_\xi = p$$

$$p_\xi = -u + u^3$$

+ v, q constant

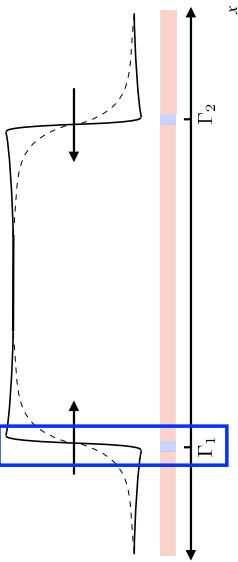
Formal derivation

- Differences appear when we have to turn on ε again; computation for persistence!

- Two ways to compute ΔH (change of Hamiltonian) when a solution jumps over the fast field \mathbf{f}_f from $\mathcal{M}_\varepsilon^-$ to $\mathcal{M}_\varepsilon^+$

$$\Delta H = H|_{\mathcal{M}_\varepsilon^+} - H|_{\mathcal{M}_\varepsilon^-} = \mathcal{O}(\varepsilon^2)$$

and



$$\begin{aligned} \Delta H &= \int_{I_f} H_\xi d\xi = \int_{I_f} H_\xi d\xi = \int_{I_f} \varepsilon p(\alpha v + \gamma) d\xi \\ &= \varepsilon(\alpha v^* + \gamma) \int_{I_f} p_h^{0,-} d\xi + hot \end{aligned}$$

- Combining gives

$$\dot{\Gamma}_1(t) = \varepsilon^2 c_1(t) = \frac{3}{\sqrt{2}} \varepsilon^2 \left(-\alpha e^{-(\Gamma_2 - \Gamma_1)} + \gamma \right)$$

Formal derivation

- Differences appear when we have to turn on ε again; computation for persistence!

- Two ways to compute ΔH (change of Hamiltonian) when a solution jumps over the fast field \mathbf{f}_f from $\mathcal{M}_\varepsilon^-$ to $\mathcal{M}_\varepsilon^+$

$$\Delta H = H|_{\mathcal{M}_\varepsilon^+} - H|_{\mathcal{M}_\varepsilon^-} = \mathcal{O}(\varepsilon^2)$$

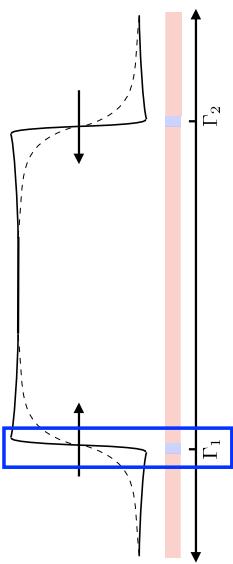
and

$$\begin{aligned} \Delta H &= \int_{I_f} H_\xi d\xi = \int_{I_f} H_\xi d\xi = \int_{I_f} \varepsilon p(\alpha v + \gamma - c_1 p) d\xi \\ &= \varepsilon(\alpha v^* + \gamma) \int_{I_f} p_h^{0,-} d\xi - \varepsilon \int_{I_f} c_1(p_h^{0,-})^2 d\xi + h o t = -\alpha e^{-(\Gamma_2 - \Gamma_1)} + \gamma - \frac{1}{3} \sqrt{2} c_1 \end{aligned}$$

- Combining gives

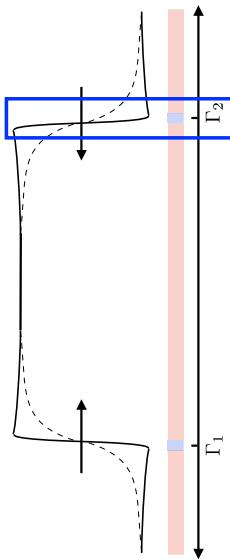
$$\dot{\Gamma}_1(t) = \varepsilon^2 c_1(t) = \frac{3}{\sqrt{2}} \varepsilon^2 \left(-\alpha e^{-(\Gamma_2 - \Gamma_1)} + \gamma \right)$$

$$\begin{aligned} u_\xi &= p \\ p_\xi &= -u + u^3 + \varepsilon(\alpha v + \gamma - c_1 p) \\ v_\xi &= \varepsilon q \\ q_\xi &= \varepsilon(v - u) + \mathcal{O}(\varepsilon^3) \end{aligned}$$



Formal derivation

- No symmetry anymore, so we also have to compute ΔH (change of Hamiltonian) when a solution jumps over the second fast field Γ_2 from $\mathcal{M}_\varepsilon^+$ to $\mathcal{M}_\varepsilon^-$

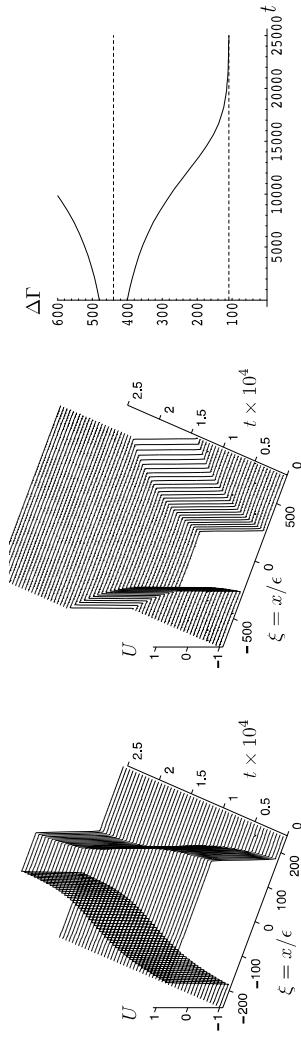


- This gives

$$\dot{\Gamma}_2(t) = \varepsilon^2 c_2(t) = -\frac{3}{\sqrt{2}}\varepsilon^2 \left(-\alpha e^{-(\Gamma_2 - \Gamma_1)} + \gamma \right)$$

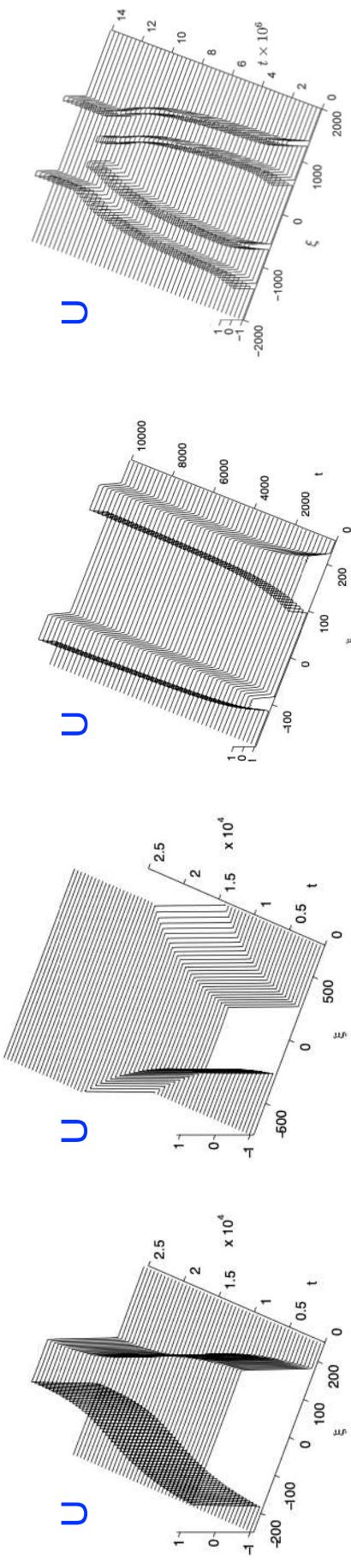
- Thus

$$\begin{aligned} \dot{\Gamma}_1(t) &= \frac{3}{\sqrt{2}}\varepsilon^2 \left(-\alpha e^{-(\Gamma_2 - \Gamma_1)} + \gamma \right) \\ \dot{\Gamma}_2(t) &= -\frac{3}{\sqrt{2}}\varepsilon^2 \left(-\alpha e^{-(\Gamma_2 - \Gamma_1)} + \gamma \right) \end{aligned}$$



Remarks

- Method can be easily extended to N-front solutions



- Results can be made rigorous by using a renormalization group approach. Note that this is a real tour de force.
- Methodology doesn't hold for large τ (travelling pulses) due to the location of the spectrum nor has it been extended to 2D

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