

Spatial Patterns in Nature

An Entry-Level
Introduction to Their
Emergence and
Dynamics

SIAM DS23,
Minitutorial MT1-2

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Minitutorial overview

- ~~Introduction~~
- ~~Multistability and patterns~~
- ~~Explicit construction of front solutions~~
 - ~~– Existence~~
 - ~~– Stability~~
- ~~Dynamics of existing structures~~
- Summary & Outlook

Spatial Patterns:

- 🌀 Turing Patterns
- 🌀 Coexistence States

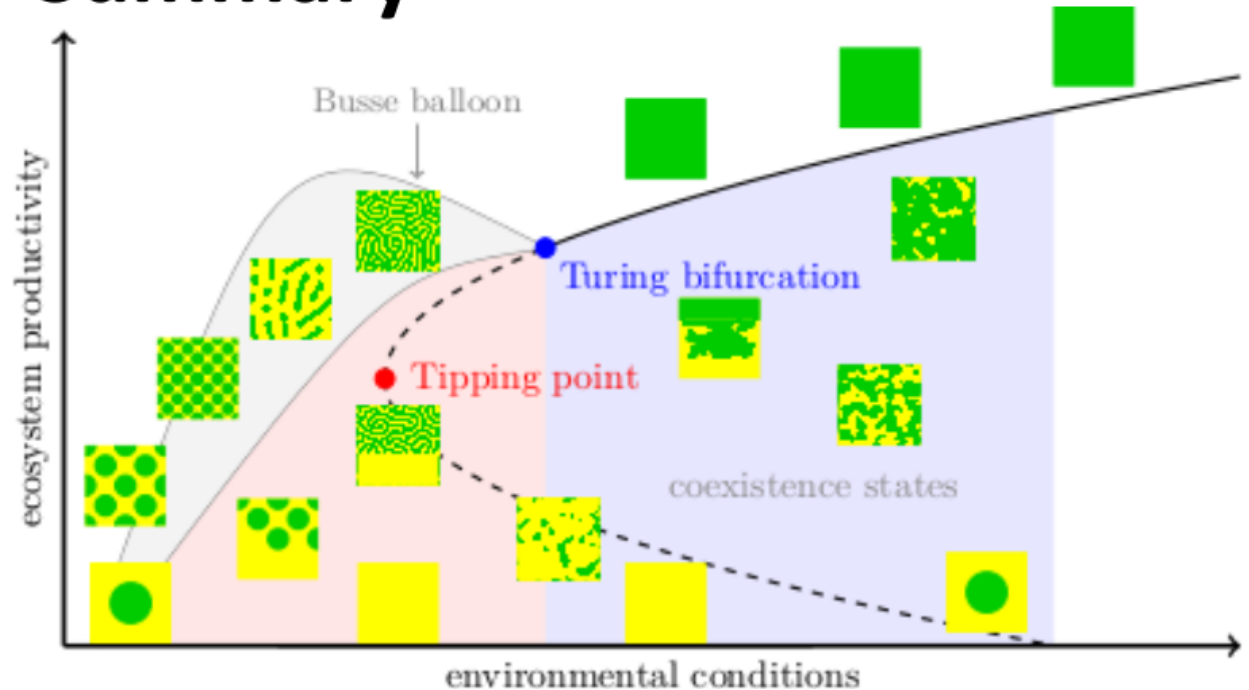
Tipping can be more subtle:

- 📊 Spatial reorganization
- 📊 Fragmented Tipping

Dynamics of Patterns is:

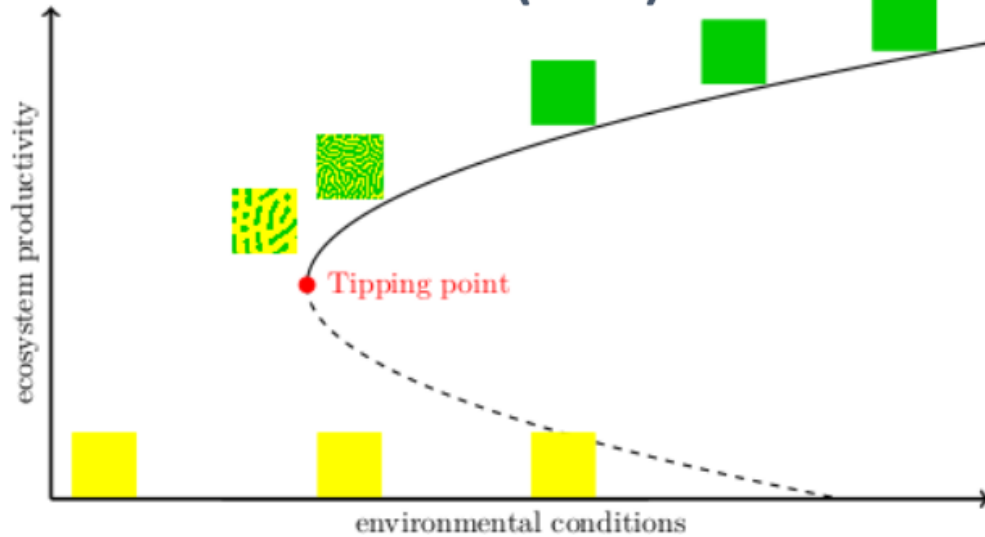
- 🐢 Slow Pattern Adaptation
- 🐇 Fast Pattern Degradation

Summary



What if the system tips?

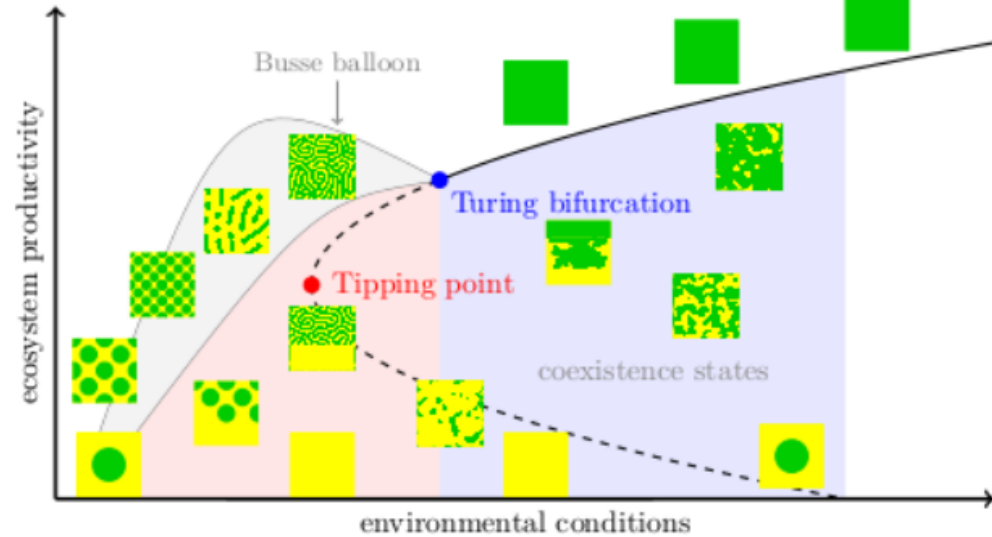
Classic (ODE)



Crossing a Tipping Point:
→ Always full reorganization

Early Warning Signals
signal for WHEN

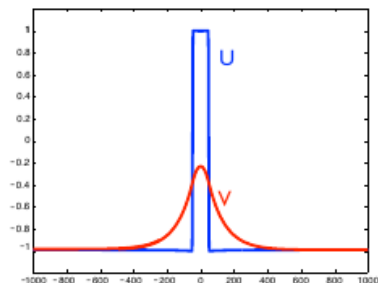
PDE



Crossing a bifurcation:
Now also possible:
→ Spatial reorganization (Turing patterns)
→ Fragmented tipping (coexistence states)

Early Warning Signals
need to signal WHEN & WHAT

Existence of stationary pulse



$$U_t = \varepsilon^2 U_{xx} + U - U^3 - \varepsilon(\alpha V + \gamma)$$

$$\tau V_t = V_{xx} + U - V$$

where $(x, t) \in \mathbb{R} \times \mathbb{R}_+$; $0 < \varepsilon \ll 1$; $0 < \tau, \alpha, \gamma \in \mathbb{R}$

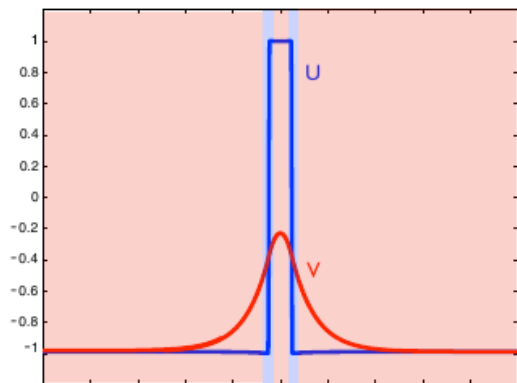
Observations:

- **stationary:** $\partial_t = 0$

$$0 = \varepsilon^2 u_{xx} + u - u^3 - \varepsilon(\alpha v + \gamma)$$

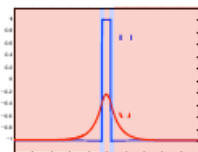
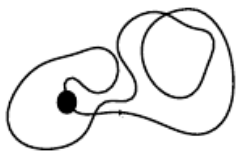
$$0 = v_{xx} + u - v$$

- five distinct spatial regions (due to smallness of ε)



large, U constant (± 1), V changing (outer/slow region)

small, U changing, V constant (inner/fast region)



4D phase portrait

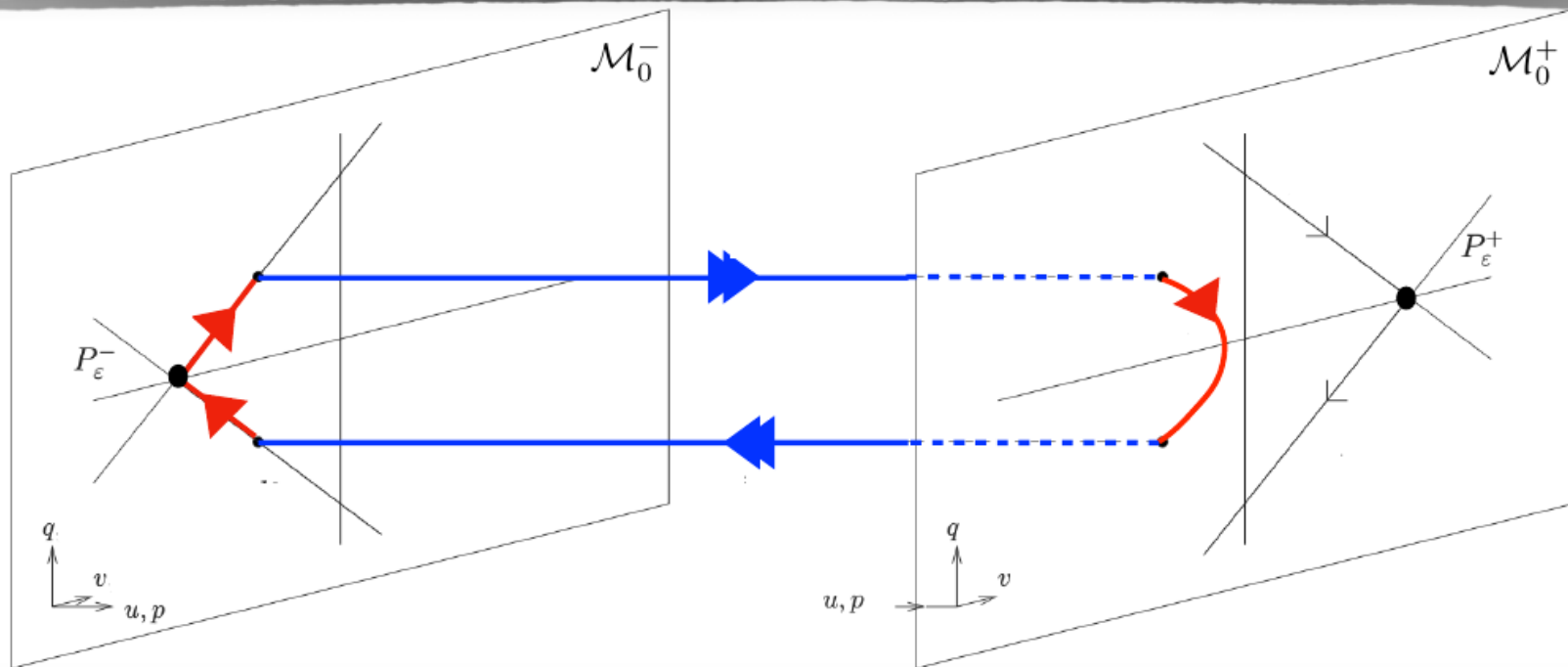
The dynamics of **slow reduced system** is constraint to fixed points of the **fast reduced system**.

Critical Manifold

$$\mathcal{M}_0 := \{(u, p, v, q) \in \mathbb{R}^4 : p = 0, 0 = u - u^3\}$$

$$\mathcal{M}_0 = \mathcal{M}_0^+ \cup \mathcal{M}_0^- \cup \mathcal{M}_0^0$$

$u = 1$ $u = -1$ $u = 0$

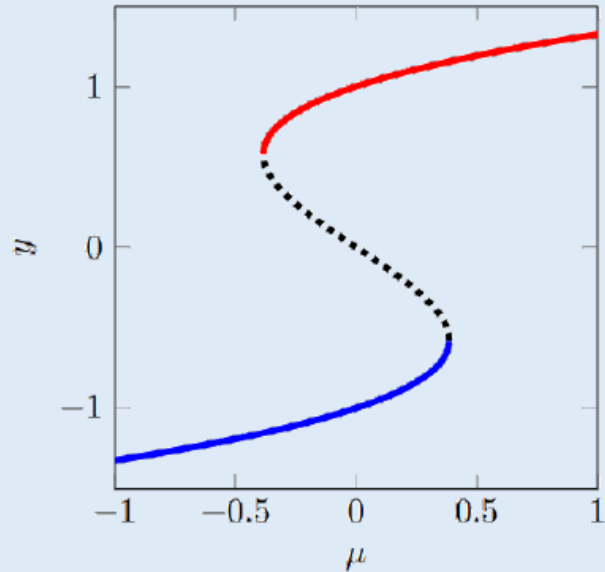


Outlook



A spatially heterogeneous world

Classic Tipping



Example:

$$\frac{dy}{dt} = y(1 - y^2) + \mu$$

Tipping in Spatially Heterogeneous Systems

Spatial Transport

Spatial Variation in
Environmental Conditions

Example:

$$\frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2} + y(1 - y^2) + \mu + \frac{1}{2} \cos(\pi x)$$

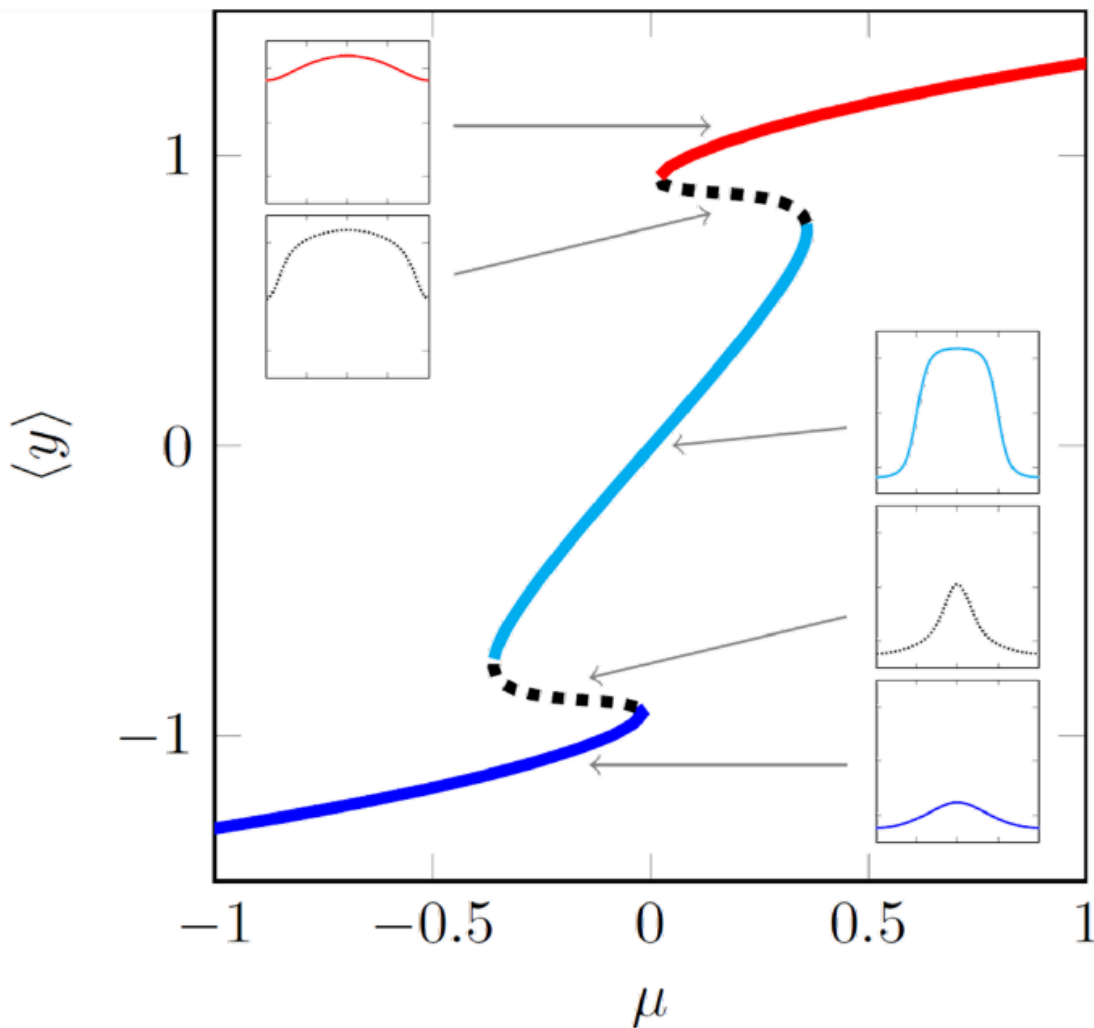
Adding Spatial Heterogeneity

$$\frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2} + f(y, \mathbf{x}; \mu)$$

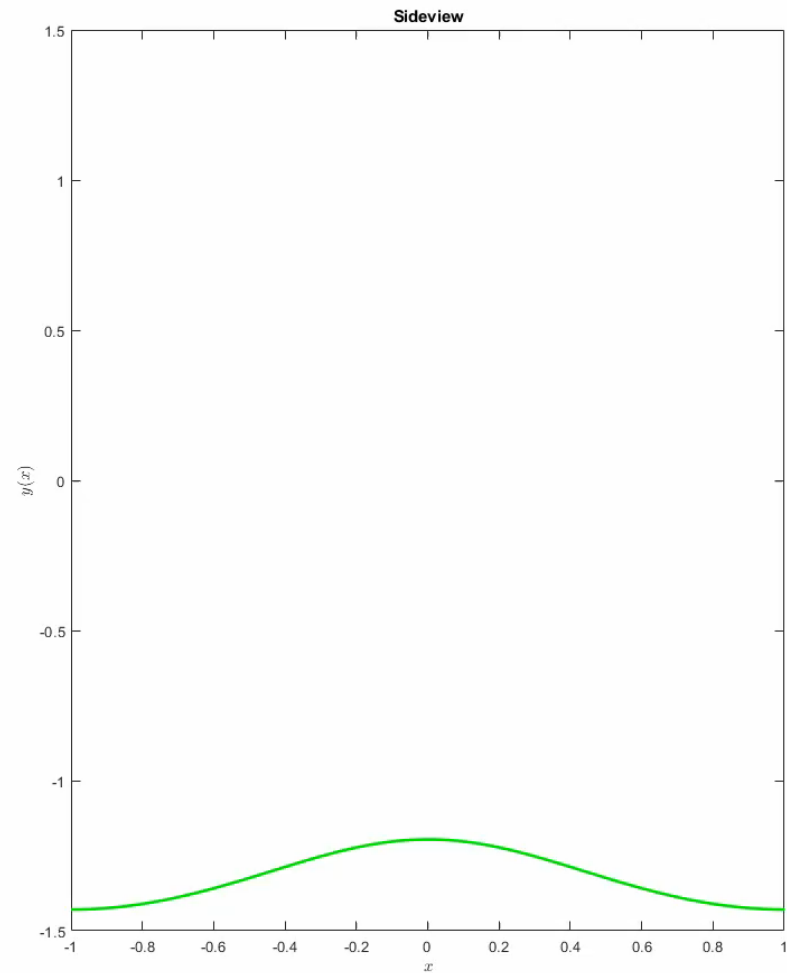
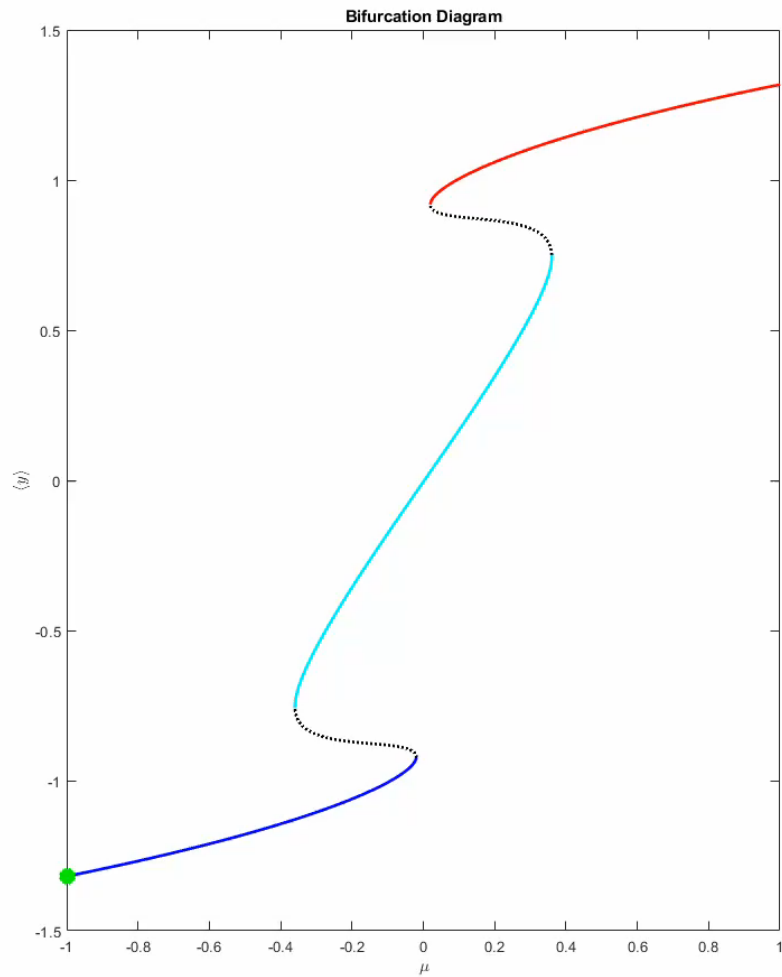
Now, the **local** difference in potentials determines the front movement

New behaviour:

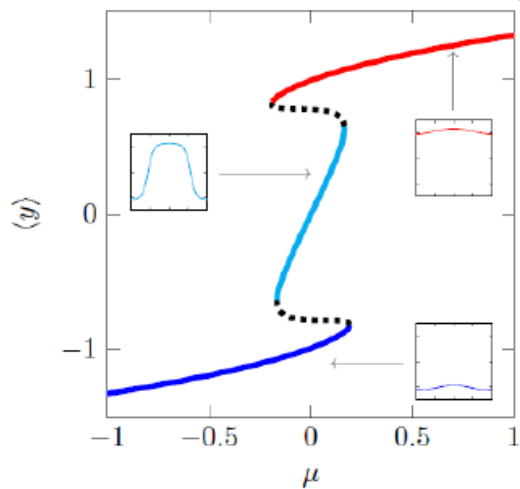
- Multi-fronts can be stationary
- Maxwell point is smeared out



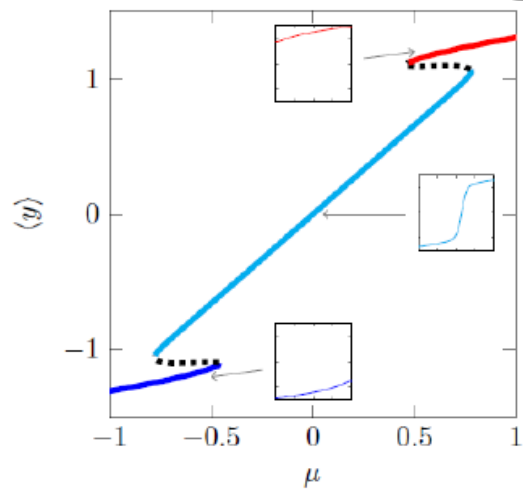
Fragmented Tipping



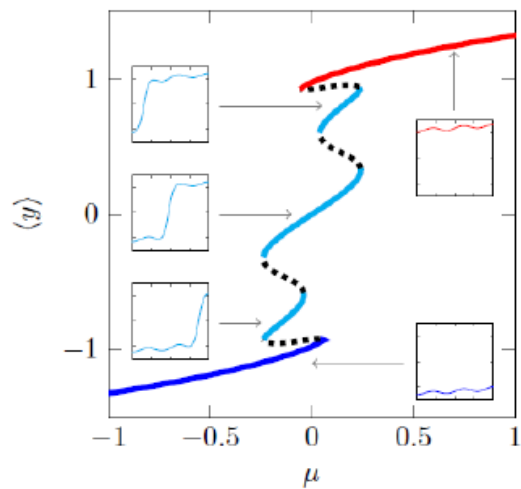
Other Spatial Heterogeneities



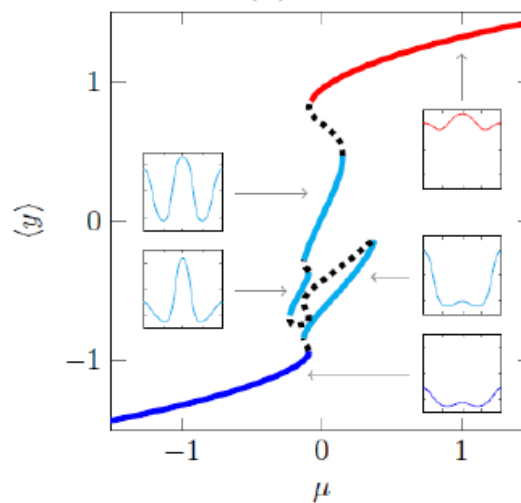
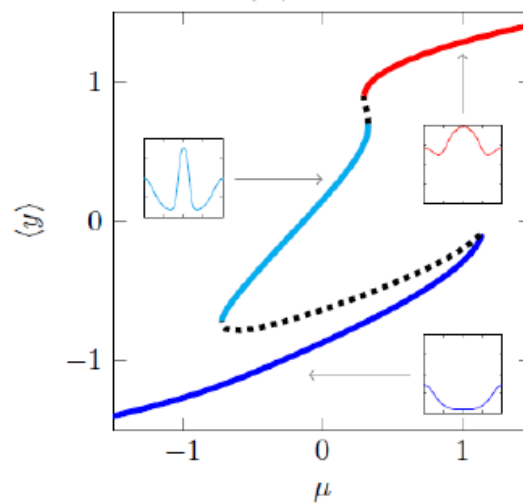
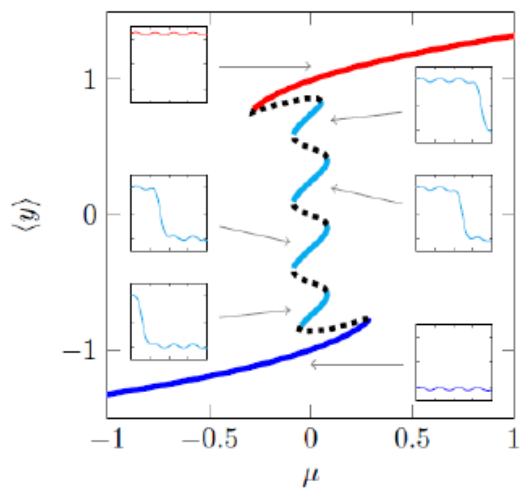
(a)



(b)

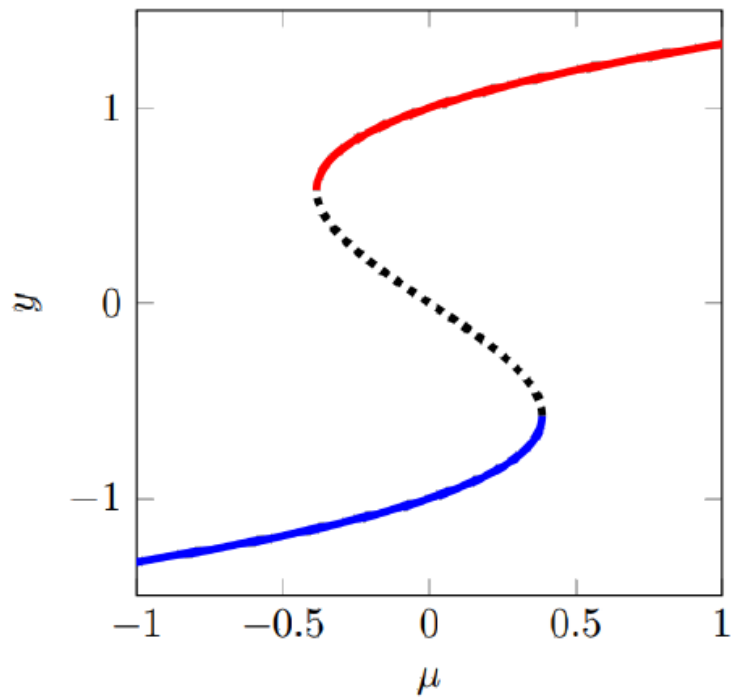


(c)



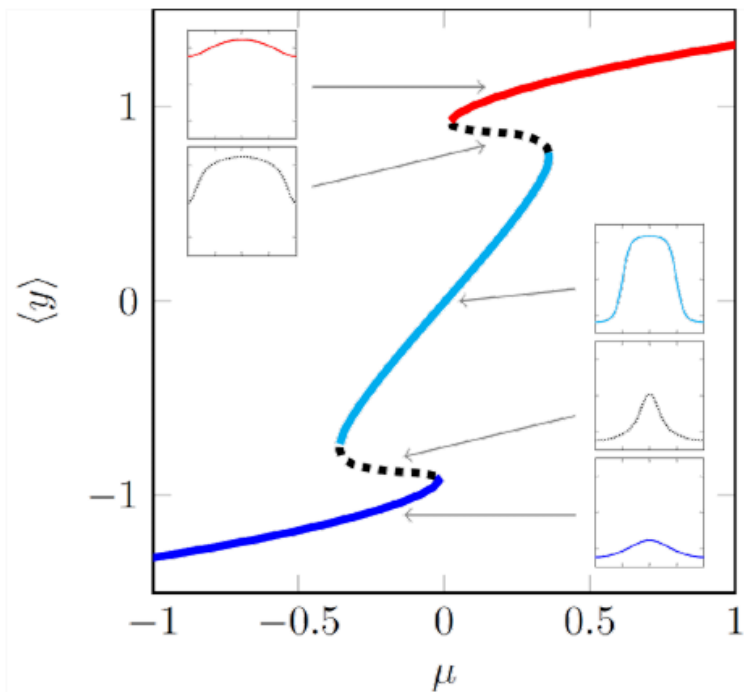
Fragmented Tipping

Classic tipping



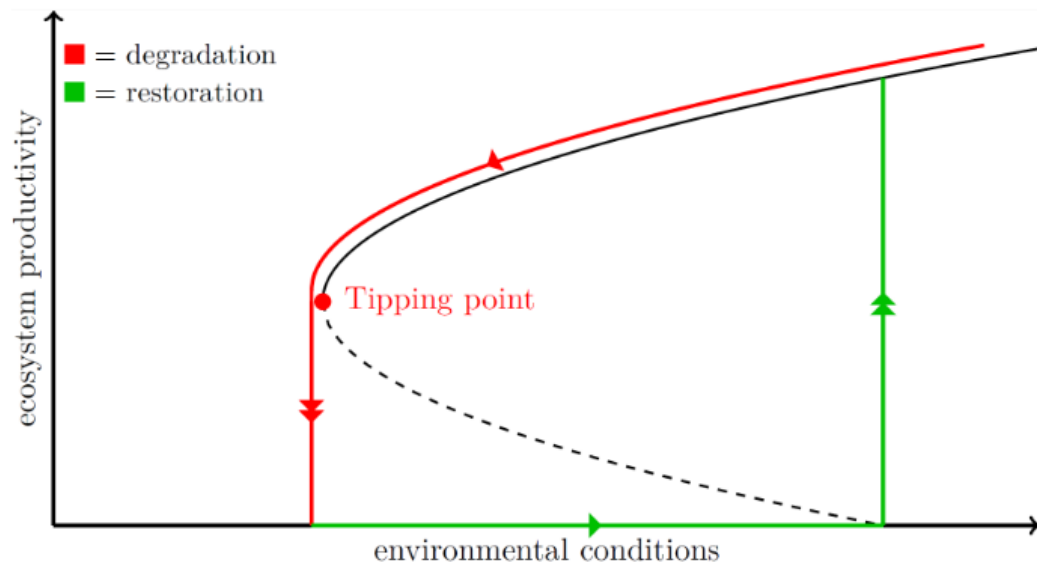
Tipping leads to full reorganisation

Tipping in a heterogeneous world

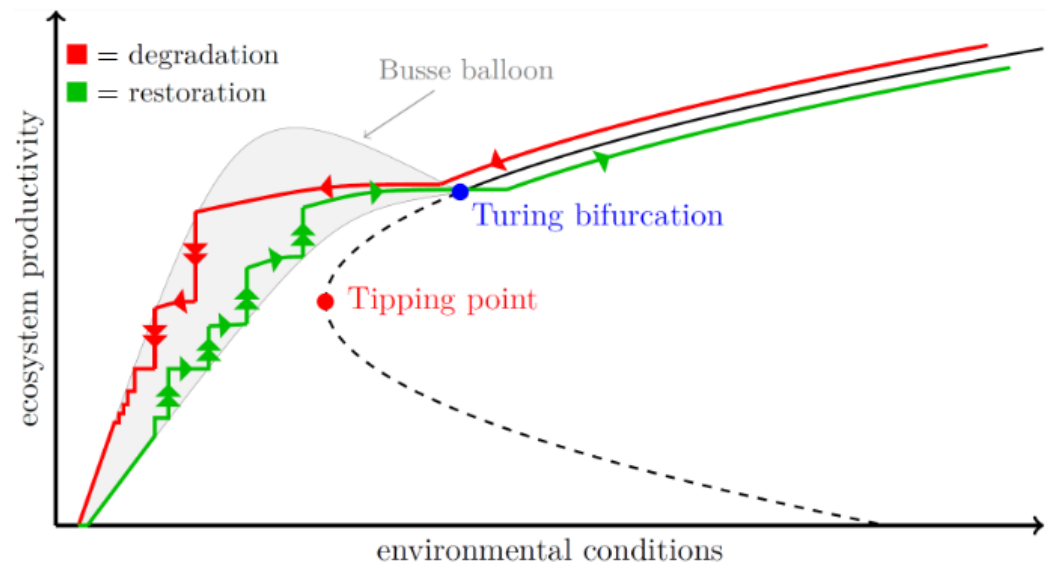


Fragmented tipping possible:
Only part of the domain reorganises

Tipping of (Turing) patterns

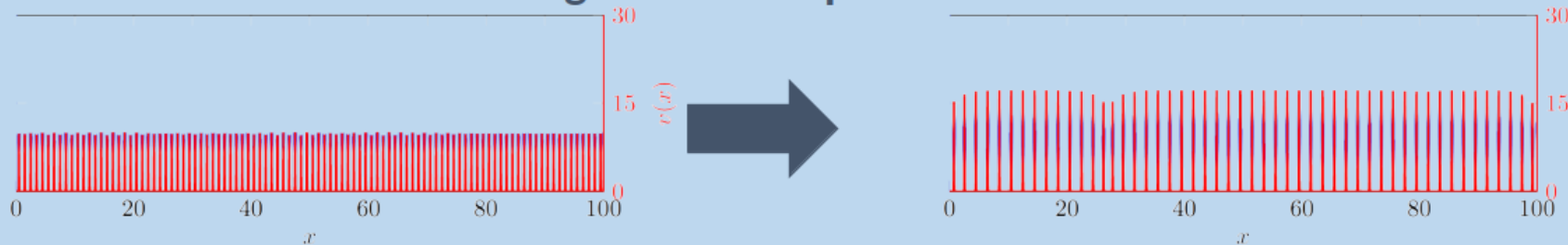


Classic tipping



Tipping of patterns

Degradation of patterns



More questions

- What determines the shape of the Busse balloon?
- How are transient patterns selected? (path through Busse balloon)
- What is the influence of spatial heterogeneities? (terrain features, roads, etc.)
- What can happen in 2D? Is this a fundamentally different context?
- Is a PDE always the best model to describe patterns? When is an 'effective' ODE pulse/front interaction model more useful?
- ...

Interesting minisymposia:

- MS 29, *Tipping Points in Natural Systems*
- MS 32 & 47, *Patterns in Nonlinear PDEs*
- MS 36 & 50, *Singular Perturbation Methods for Multi-Scale Infinite-Dimensional Systems*
- MS 62 & 75, *Pattern Formation in Nature: from Busse Balloons to Homoclinic Snaking*
- MS 73 & 86, *Dynamical Systems Methods in Climate Modeling*
- MS 88 & 103, *Branching Out: a New Generation's Perspective on Spatial Localisation in Higher Dimensions*
- MS 99 & 100, *Rate-Induced Tipping*
- MS 114 & 128, *Patterns in Earth's Climate System*
- MS 130 & 144, *Modeling and Data-Driven Methods for Collective Behaviour and Pattern Formation*
- MS 138, 158 & 173, *Front Propagation and Invasion Phenomena*