(Geometric Singular) Perturbation Theory

by Arjen Doelman

perturbation: $\exists \varepsilon: \; 0 < \varepsilon << 1$

often in the neighborhood of a bifurcation

natural small quantity

– in fluidodynamics viscosity

ex: air at the boundary layer of a plane wing

Reaction - diffusion equations: $\begin{cases} V_t = \varepsilon^2 V_{xx} + g(u,v) = 0 \\ u_t = u_{xx} + f(u,v) = 0 \end{cases}$

regular setting (ODE)

$$\begin{cases} \dot{x} = f(x,t;\varepsilon) \\ x(0) = a(\varepsilon) \end{cases} \quad \text{in } \mathbb{R}^n$$

assume sufficient smoothness

$\xrightarrow{\text{expand}}$ $f(x,\varepsilon) = f_0(x) + \varepsilon f_1(x) + \dots$

$$x(t) = x_0(t) + \varepsilon x_1(t) + \dots$$

$$a(\varepsilon) = a_0 + \varepsilon a_1 + \dots$$

$$\frac{d}{dt}x = \dot{x}_0 + \varepsilon \dot{x}_1 + \dots = f_0(x) + \varepsilon f_1(x) + \dots$$

$$= f_0(x_0 + \varepsilon x_1 + \dots) + \varepsilon f_1(x_0 + \varepsilon x_1 + \dots) + \dots$$

$$= f_0(x_0) + \frac{\partial f_0}{\partial x}(x_0)\cdot \varepsilon x_1 + \dots + \varepsilon f_1(x_0) + \varepsilon^2 \frac{\partial f_1}{\partial x}(x_0)\cdot x_1 + \dots$$

$\mathcal{O}(1)$: $\begin{cases} \dot{x}_0 = f_0(x_0) \\ x_0(0) = a_0 \end{cases}$ non-linear

$\mathcal{O}(\varepsilon)$: $\begin{cases} \dot{x}_1 = \underbrace{\frac{\partial f_0}{\partial x}(x_0)\cdot x_1}_{\text{linear}} + \underbrace{f_1(x_0)}_{\text{+ inhomogeneous}} \\ x_1(0) = a_1 \end{cases}$

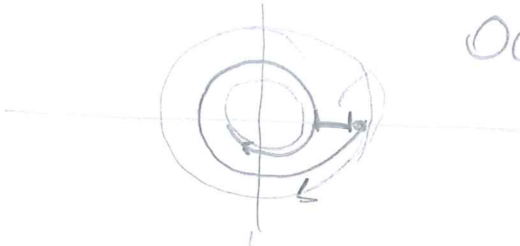
up to ε^N, $N \geq 1$ $\quad x(t) = x_0(t) + \dots + \varepsilon^N x_N(t) + \dots$

Poincaré $\quad \forall T > 0 \; \exists \varepsilon_0 > 0$ and $C > 0$ s.t. $\|x - (x_0 + \dots + \varepsilon^N x_N)\| < C\varepsilon^{N+1} \quad \forall t < T$

Thus 'validity' only up to

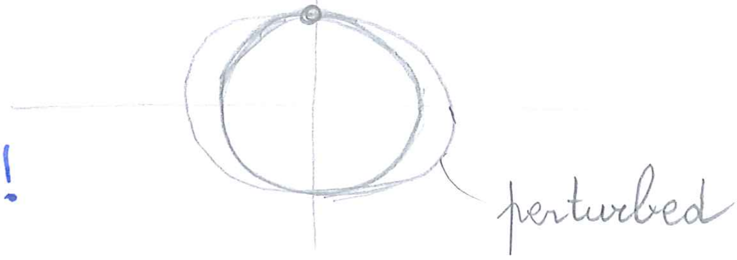
$$T = \mathcal{O}(1) \quad \text{w.r.t. } \varepsilon$$

Ex. • $\ddot{x} + \underbrace{\varepsilon \dot{x}}_{\text{friction}} + x = 0$ $\qquad O(1)$ it $t \sim \frac{1}{\varepsilon}$



• $\ddot{x} + (1-\varepsilon)x = 0$

the perturbed system lags behind!



perturbation theory: this aproach with a higher T

SINGULAR PERTURBATION

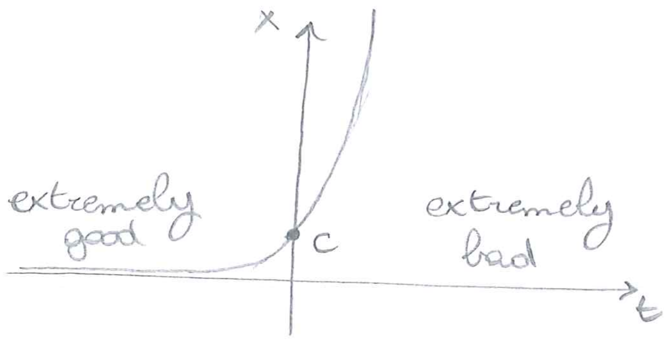
" the $\varepsilon = 0$ limit systems differ significantly from the $\varepsilon \neq 0$ system"

Ex. $\varepsilon \dot{x} = x \xrightarrow{\varepsilon \downarrow 0} 0 = x$

?

RELATION

$x(t) = Ce^{\frac{1}{\varepsilon}t}$ $\qquad$ x-



in the $\dot{x} = f(x,t;\varepsilon)$ setting

SP $\longleftrightarrow$ $f(x,t;\varepsilon)$ is NOT smooth as function of ε, for $\varepsilon \to 0$

$\dot{x} = \frac{1}{\varepsilon} x$ $\qquad f(x,\varepsilon) = \frac{x}{\varepsilon}$ $\qquad$ → that's why it doesn't follow Poincaré

Ex. $\begin{cases} \varepsilon \ddot{u} + \dot{u} - u = 0 \\ u(0) = u(1) = 0 \end{cases}$ $\longleftrightarrow \frac{d}{dx}$ $\qquad$ B.V.P.

WORKS for $\varepsilon \neq 0$

for $\varepsilon \downarrow 0$, what about the B.V.?

$\varepsilon = 0 \qquad \dot{u} = u \qquad u(t) = Ce^{x}$

$u(x) = e^{\lambda x} \qquad \varepsilon\lambda^2 + \lambda - 1 = 0$

$$\lambda_\pm = \frac{-1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon}$$

$\sqrt{1+4\varepsilon}$: $1 + 2\varepsilon + O(\varepsilon^2)$

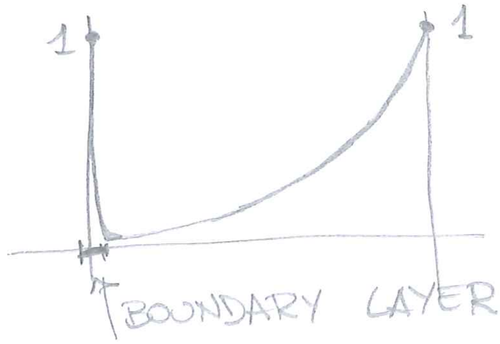
$\lambda_+ = 1 + O(\varepsilon)$

$\lambda_- = -\frac{1}{\varepsilon} + O(\varepsilon)$

at leading order after B.C.

$$u(x) = \frac{e-1}{e} e^{-\frac{1}{\varepsilon}x} + \frac{1}{e} e^{x}$$

only "non-zero" for x close to 0



$$\begin{cases} \dot{u} = v \\ \dot{v} = \frac{1}{\varepsilon}(u - v) \end{cases} \quad x \in [0,1] \quad \rightarrow \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \end{pmatrix}}_{f \text{ NOT smooth}} \begin{pmatrix} u \\ v \end{pmatrix}$$

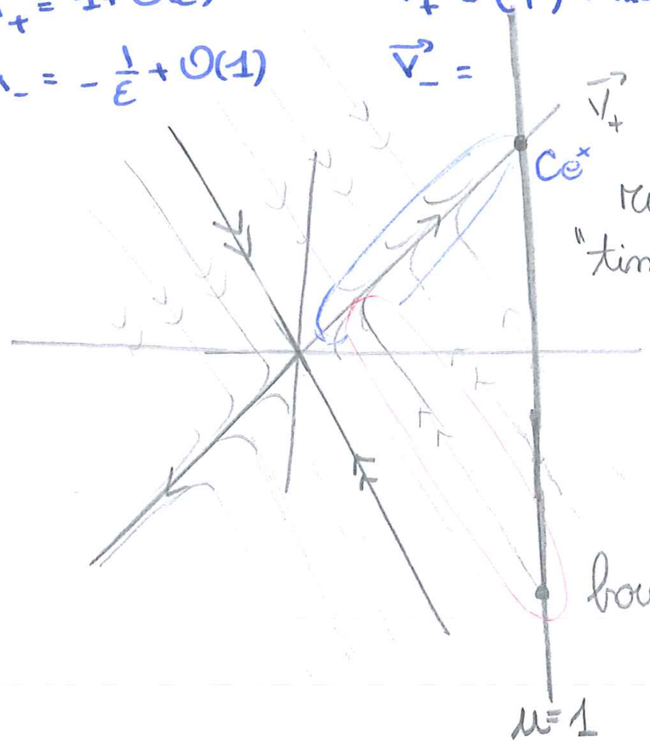
or rescaling time
$$\begin{cases} u' = \varepsilon v \\ v' = (u - v) \end{cases} \quad f \text{ smooth} \quad \text{BUT} \quad x \in [0, \tfrac{1}{\varepsilon}]$$

outside the Poincaré time scale

eigenvalues $\lambda_+ = 1 + O(\varepsilon)$ $\quad \vec{v}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \text{h.o.t.}$

$\lambda_- = -\frac{1}{\varepsilon} + O(1)$ $\quad \vec{v}_- =$

saddle point



restriction: "time of flight" = 1

General (*)

$$\begin{cases} \dot{x} = f(x, y; \varepsilon) \\ \dot{y} = \varepsilon g(x, y; \varepsilon) \end{cases} \quad \begin{matrix} x \in \mathbb{R}^k \\ y \in \mathbb{R}^l \end{matrix} \Big\} \in \mathbb{R}^N = \mathbb{R}^{k+l}$$

FAST SCALING

background
$$\begin{cases} u_{xx} + F(u, v) = 0 \\ \varepsilon^2 v_{xx} + G(u, v) = 0 \end{cases} \rightarrow \begin{cases} u_x = p \\ p_x = -F(u, v) \\ \varepsilon v_x = q \\ \varepsilon q_x = -g(u, v) \end{cases}$$

$x \rightarrow X$ rescaling

$$\begin{cases} u_X = \varepsilon p \\ p_X = -\varepsilon F(u, v) \\ v_X = q \\ q_X = -g(u, v) \end{cases}$$

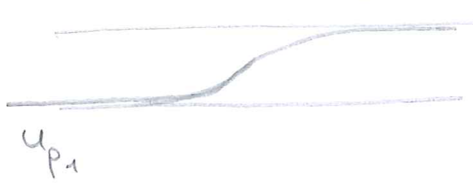
prototype
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 - x_1^2 + \varepsilon h(x_1, x_2, y) \\ \dot{y} = \varepsilon g(x_1, x_2, y) \end{cases}$$

(*) Assume I have critical points P_1, P_2 $(P_1 = P_2)$

$\hookrightarrow f(x,y) = g(x,y) = 0$

can I construct a HOMOCLINIC or HETEROCLINIC orbits?

unbounded domain, no help from Poincaré.



and same for $V_{P_1} \to V_{P_2}$

(*) rescale 'time' $\longrightarrow$ $\begin{cases} \varepsilon x' = f(x,y,\varepsilon) \\ y' = g(x,y,\varepsilon) \end{cases}$ $\iff$ $\begin{cases} \dot{x} = f(x,y,\varepsilon) \\ \dot{y} = g(x,y,\varepsilon) \cdot \varepsilon \end{cases}$

SLOW SCALING | FAST SCALING

$\downarrow \varepsilon \downarrow 0$ | $\downarrow \varepsilon \downarrow 0$

Manifold $\mathcal{M}_0$ $x\ (\exists\ x(y))$ $\to$ $\begin{cases} f(x,y,0) = 0 \\ y' = g(x,y,0) \end{cases}$ SLOW REDUCED LIMIT

$y' = g(x(y), y, 0) \to$

1-dim dynamical system on $\mathcal{M}_0$

$\begin{cases} \dot{x} = f(x,y,0) \\ \dot{y} = 0 \end{cases}$ FAST REDUCED LIMIT

$y = y_0$ parameter and $\dot{x} = f(x, y_0, 0)$

1-parameter dyn. sys.

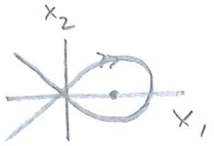
NOTE: $\mathcal{M}_0$ corresponds to the collection of critical points of the fast reduced limit

Example on the prototype case

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 - x_1^2 + \varepsilon h(x_1, x_2, y) \\ \dot{y} = \varepsilon g(x_1, x_2, y) \end{cases} \longrightarrow \begin{cases} \varepsilon x_1' = x_2 \\ \varepsilon x_2' = x_1 - x_1^2 + \varepsilon h(x_1, x_2, y) \\ y' = g(x_1, x_2, y) \end{cases}$$

$\downarrow \varepsilon \downarrow 0$ | $\downarrow \varepsilon \downarrow 0$

FRL $\begin{cases} \ddot{x}_1 = x_1 - x_1^2 \\ \dot{y} = 0 \end{cases}$



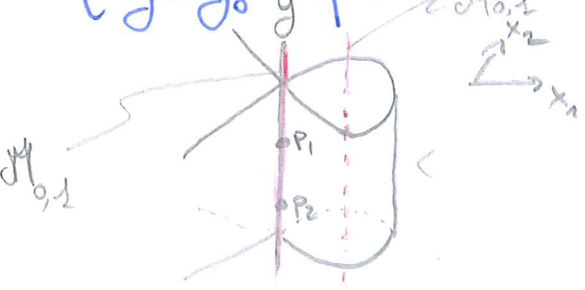
SRL $\begin{cases} 0 = x_2 \\ 0 = x_1 - x_1^2 \\ y' = g(x_1, x_2, y) \end{cases}$

$\mathcal{M}_{0,1}$ and $\mathcal{M}_{0,2}$: $x_2 = 0$, $x_1 = 0, +1$

$y' = \begin{cases} g(0,0,y) \\ g(1,0,y) \end{cases}$

$\begin{cases} \ddot{x}_1 = x_1 - x_1^2 \\ y = y_0\ y \text{ parameter} \end{cases}$

if it has 2 critical points, let's assume



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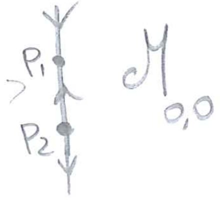
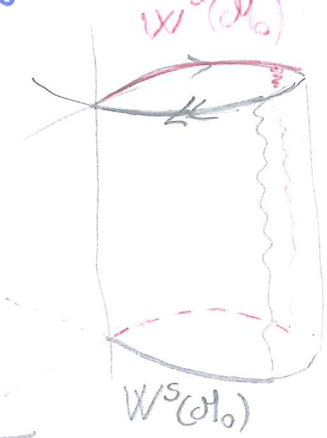


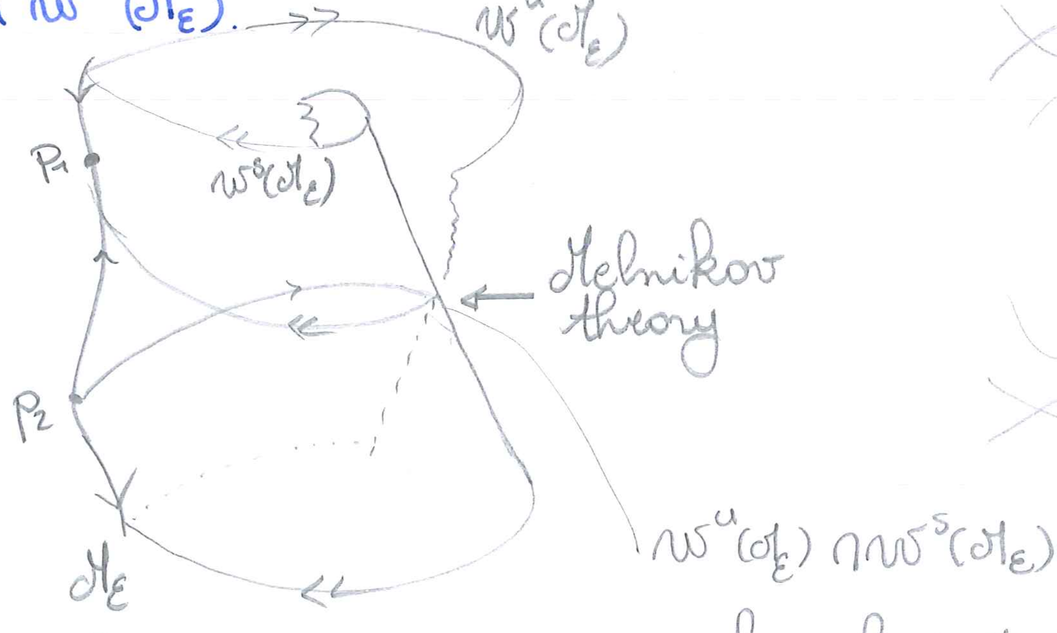
THEOREM Fenichel 1

If M_0 corresponds to critical points of FRL that have eigenvalues λ with $Re(\lambda) \neq 0$, then M_0 persists as invariant manifold M_ε for the full system. at leading order



THEOREM Fenichel 2

M_0 has stable and unstable manifolds $W^s(M_0)$ and $W^u(M_0)$
if M_0 is normaly hyperbolic ($Re(\lambda) \neq 0$) then also $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$.



Heteroclinic $p_1 \to p_2$ ⟶ γ_* homoclinic to M_ε

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Some references:

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