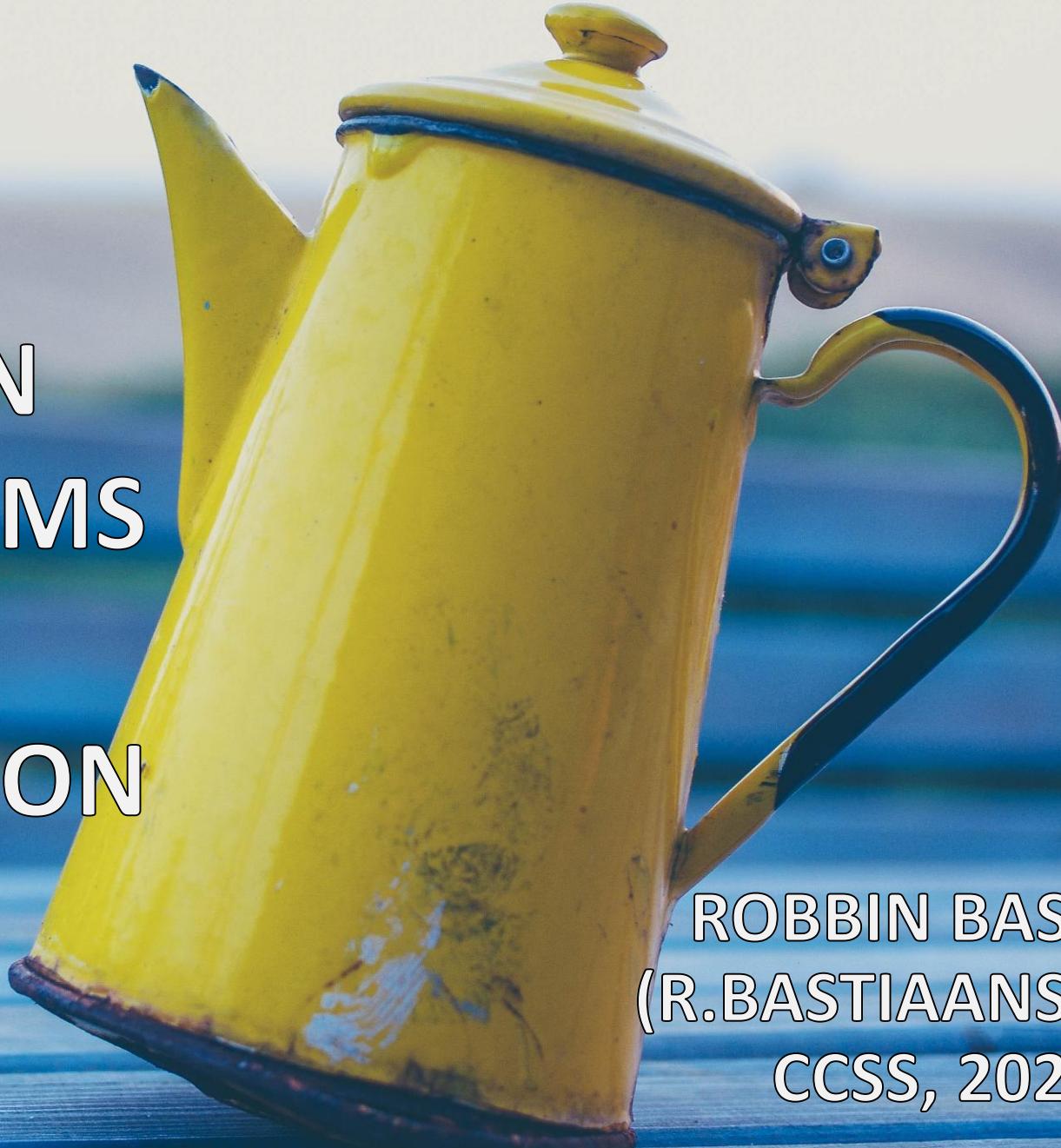


TIPPING BEHAVIOUR IN COMPLEX SYSTEMS

—

AN INTRODUCTION

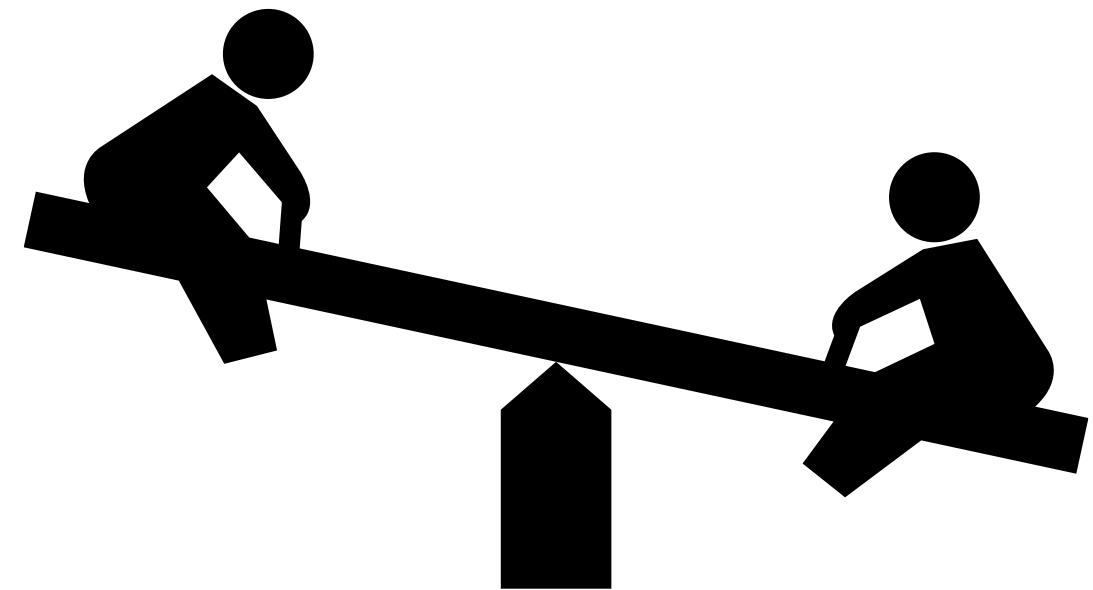
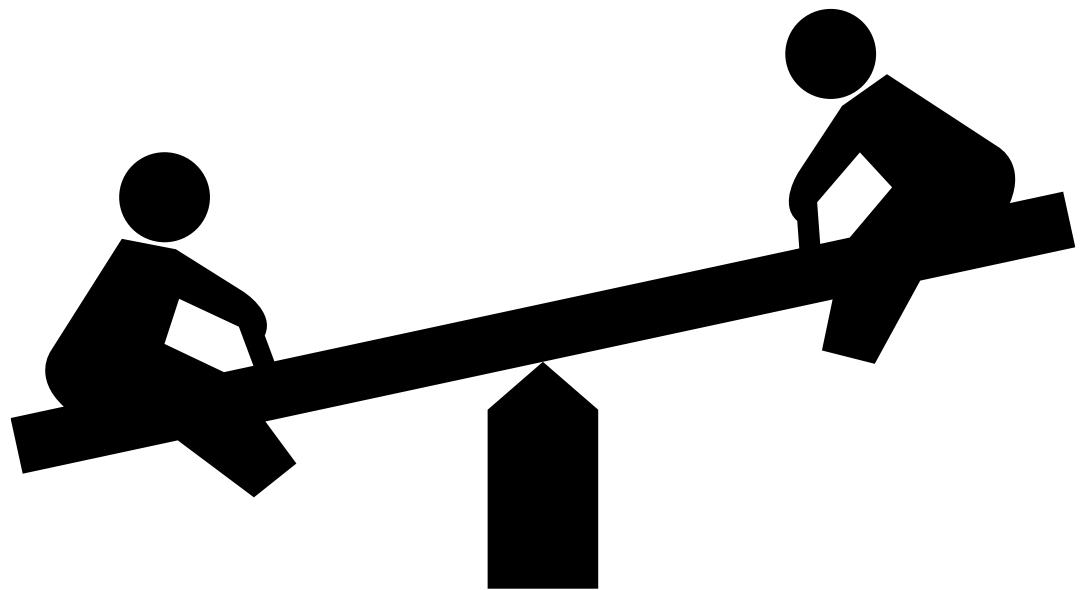


ROBBIN BASTIAANSEN
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CCSS, 2025-09-11

Tipping Points

IPCC AR6 (2021) : “a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly”





more weight on left



more weight on right



more weight on left

more weight on right

THE POCKET TELEPHONE: WHEN IT WILL RING!



The latest modern horror in the way of inventions is supposed to be the pocket telephone. We can imagine the moments this instrument will choose for action!
—(By W. K. Haselden)

The diffusion of innovation



critical mass achieved: it's everywhere

The diffusion of innovation

THE POCKET TELEPHONE: WHEN IT WILL RING!



The latest modern horror in the way of inventions is supposed to be the pocket telephone. We can imagine the moments this instrument will choose for action!
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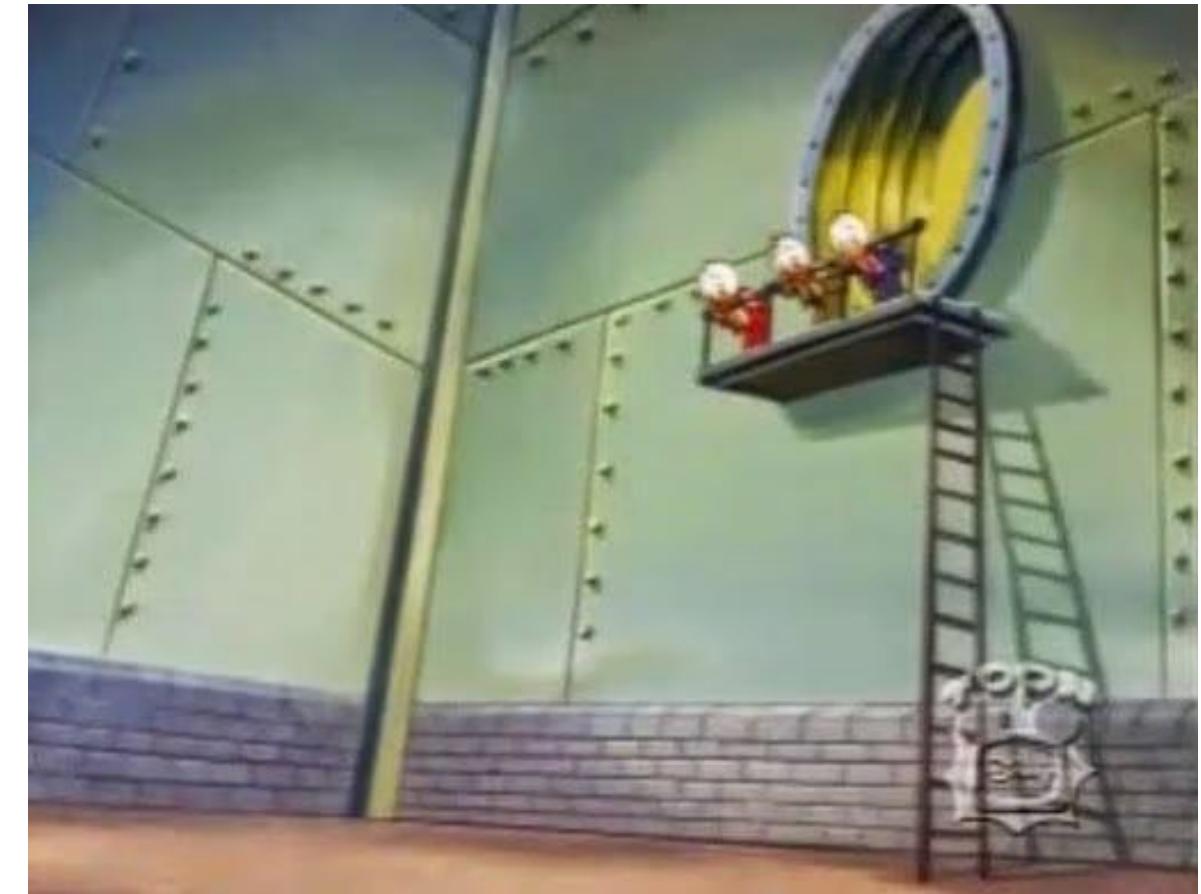
only early adaptors

critical mass achieved: it's everywhere

“How did you go bankrupt? Two ways: Gradually, then suddenly.”
[The Sun Also Rises by Ernest Hemingway]



few bad decisions



few too many

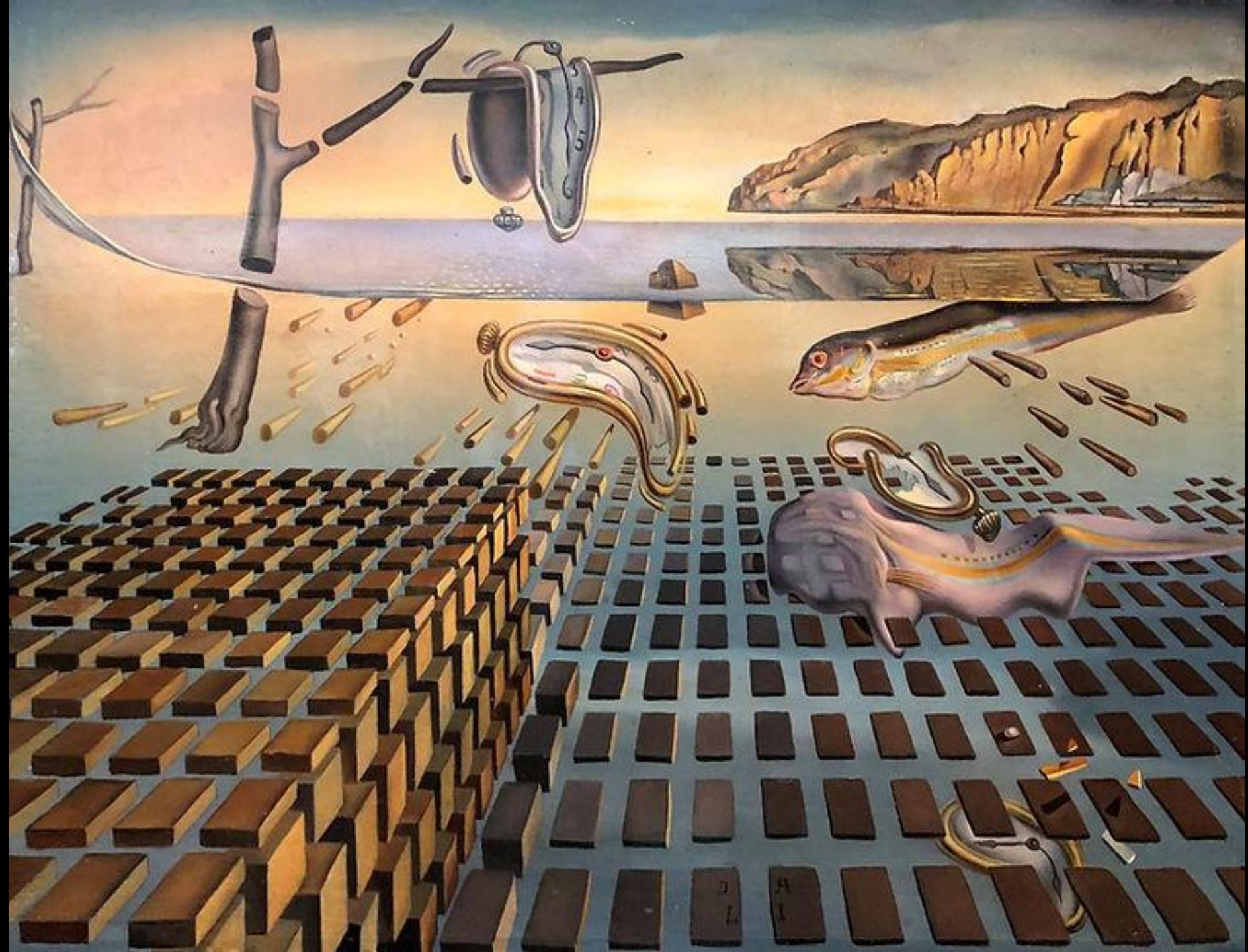
'Intelligent, articulate, thought-provoking'
OBSERVER

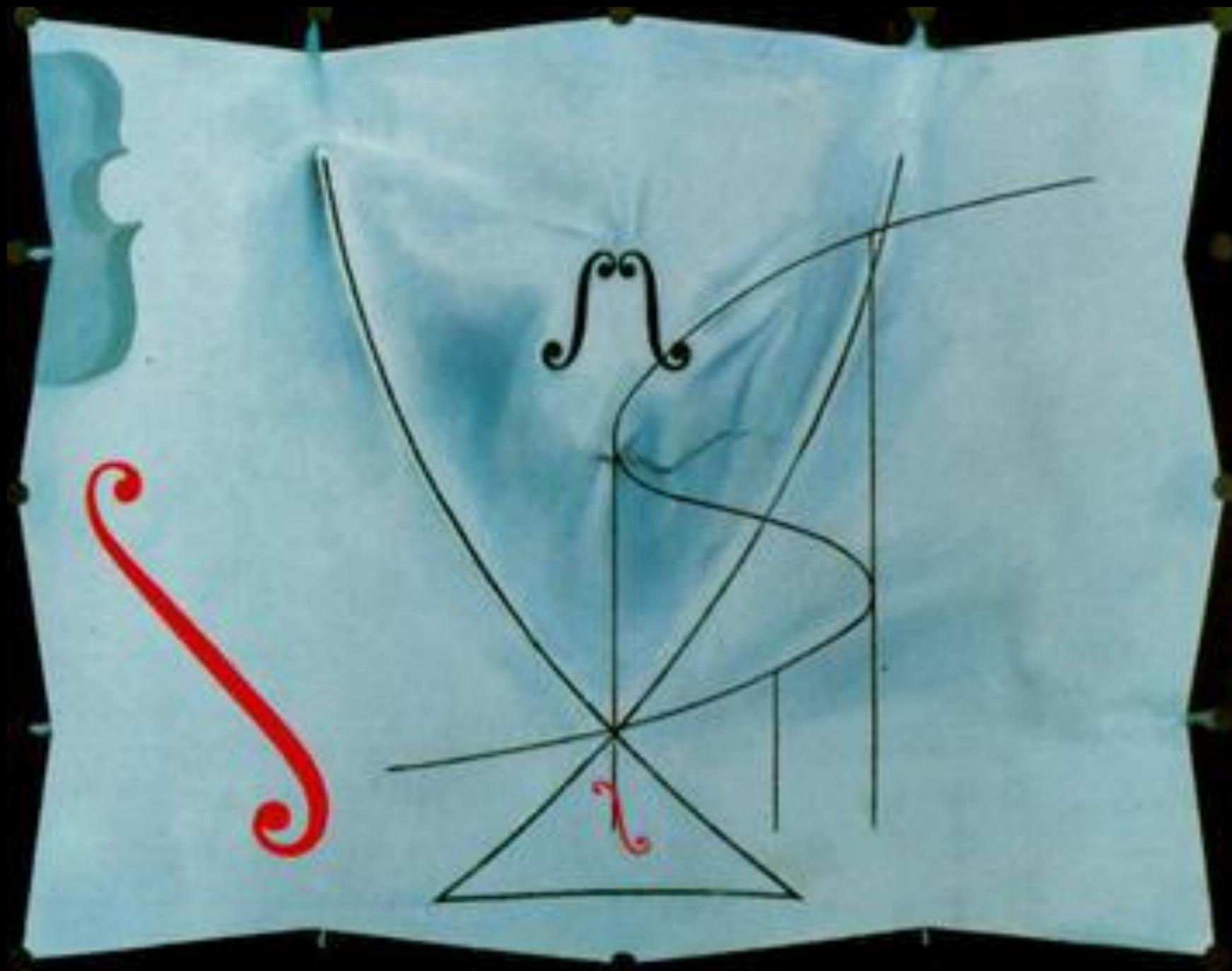


**THE
TIPPING
POINT
MALCOLM
GLADWELL**

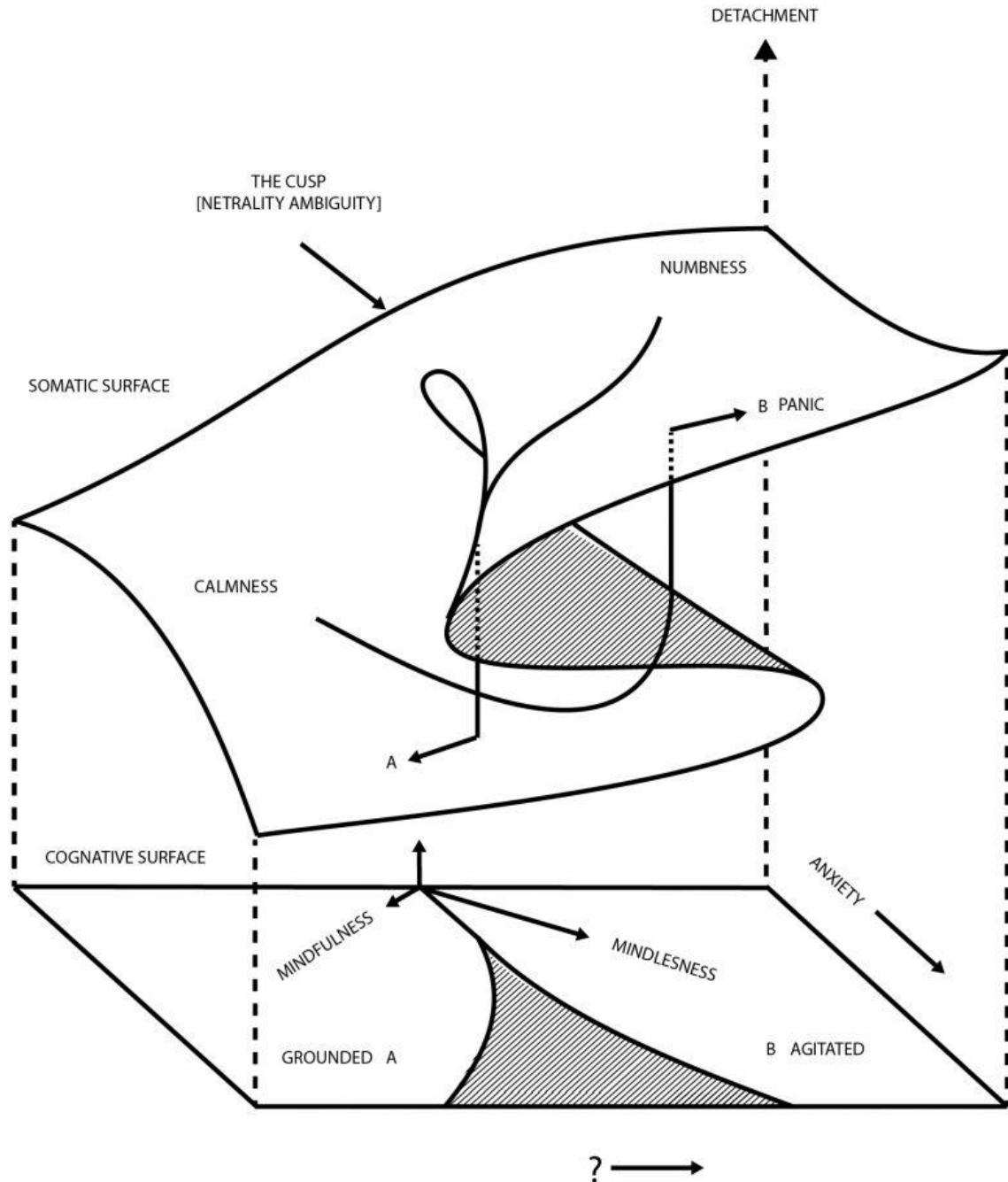
HOW LITTLE THINGS CAN
MAKE A BIG DIFFERENCE

The International Number One Bestseller





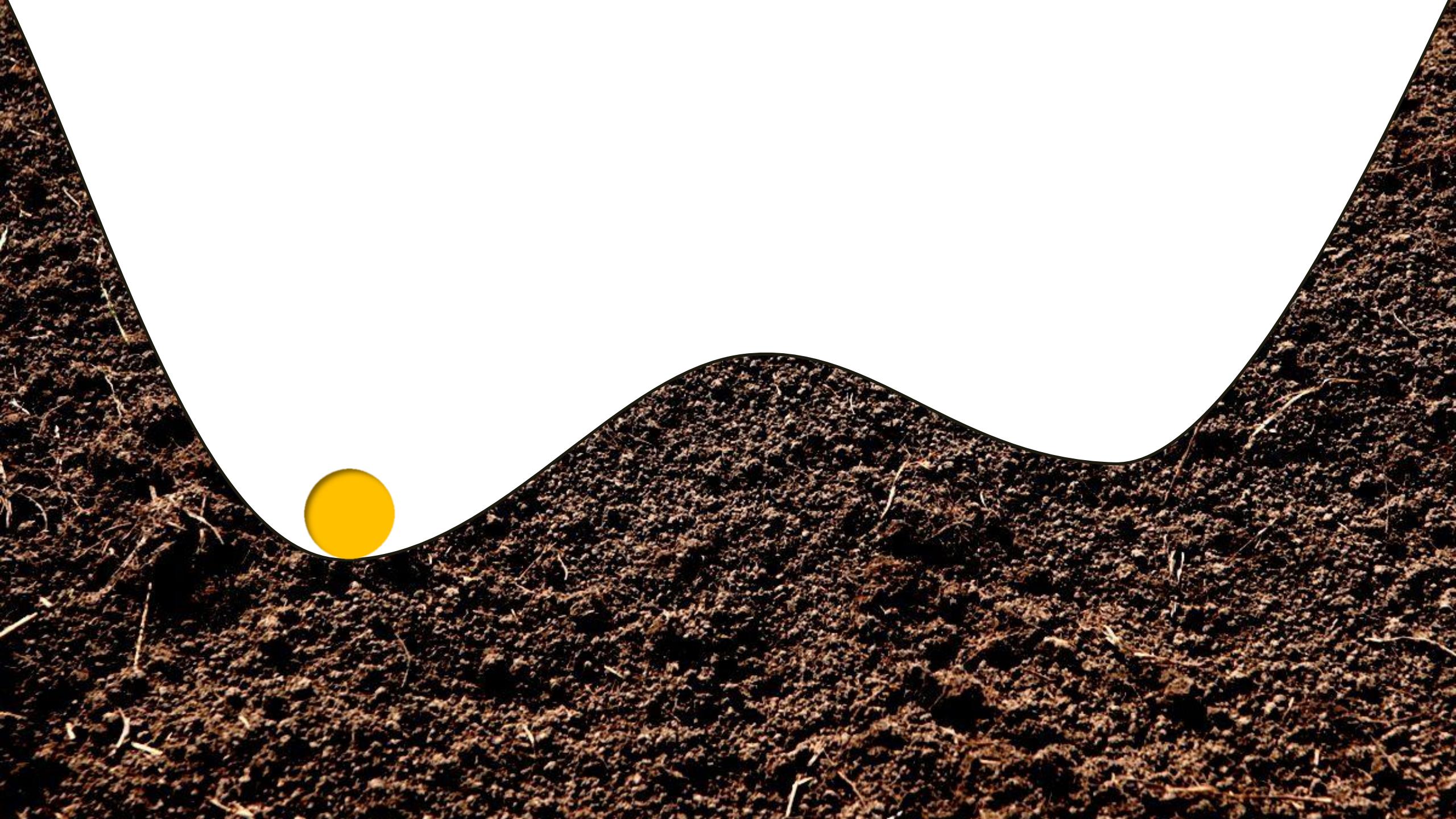
Catastrophe theory

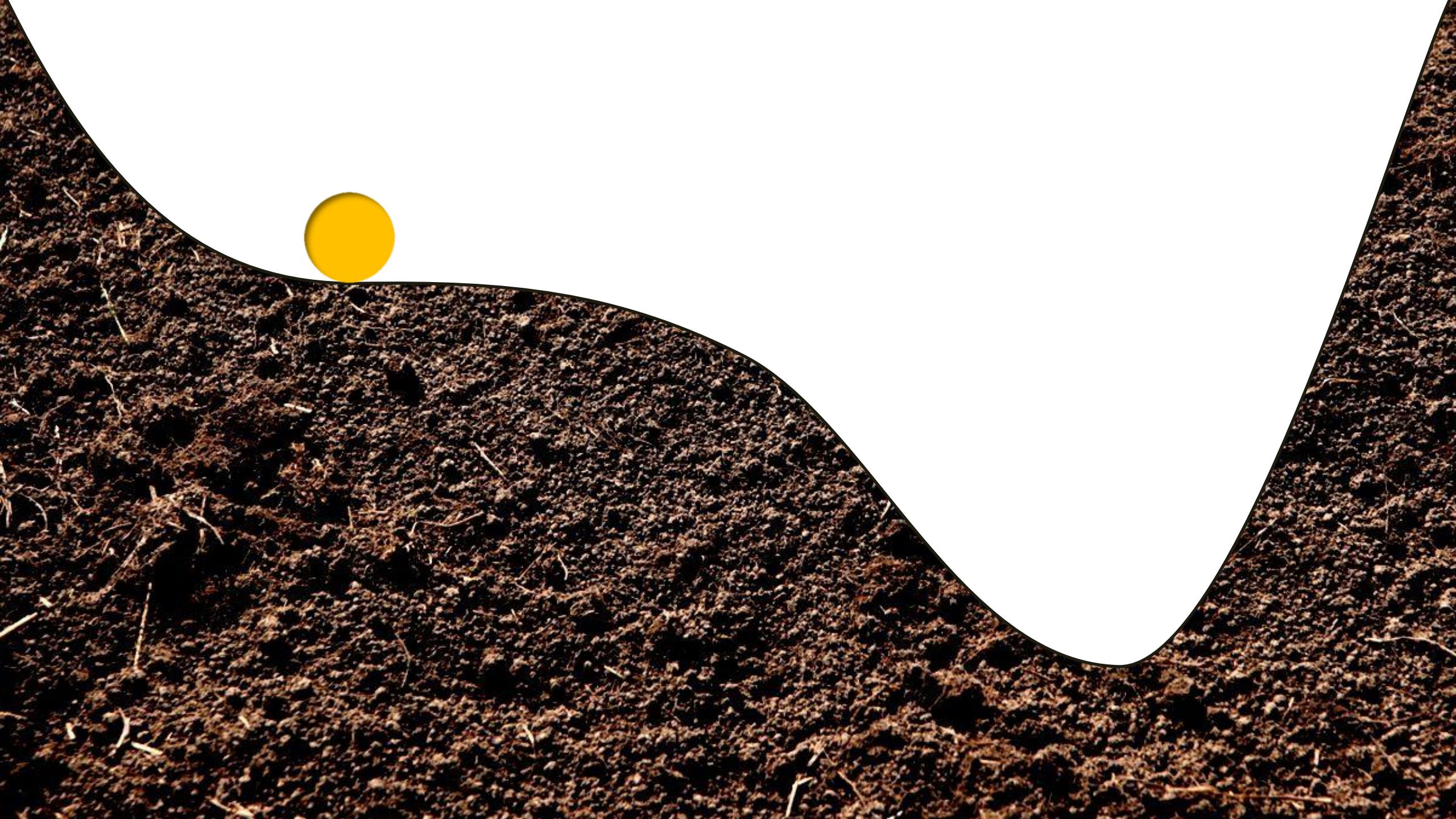


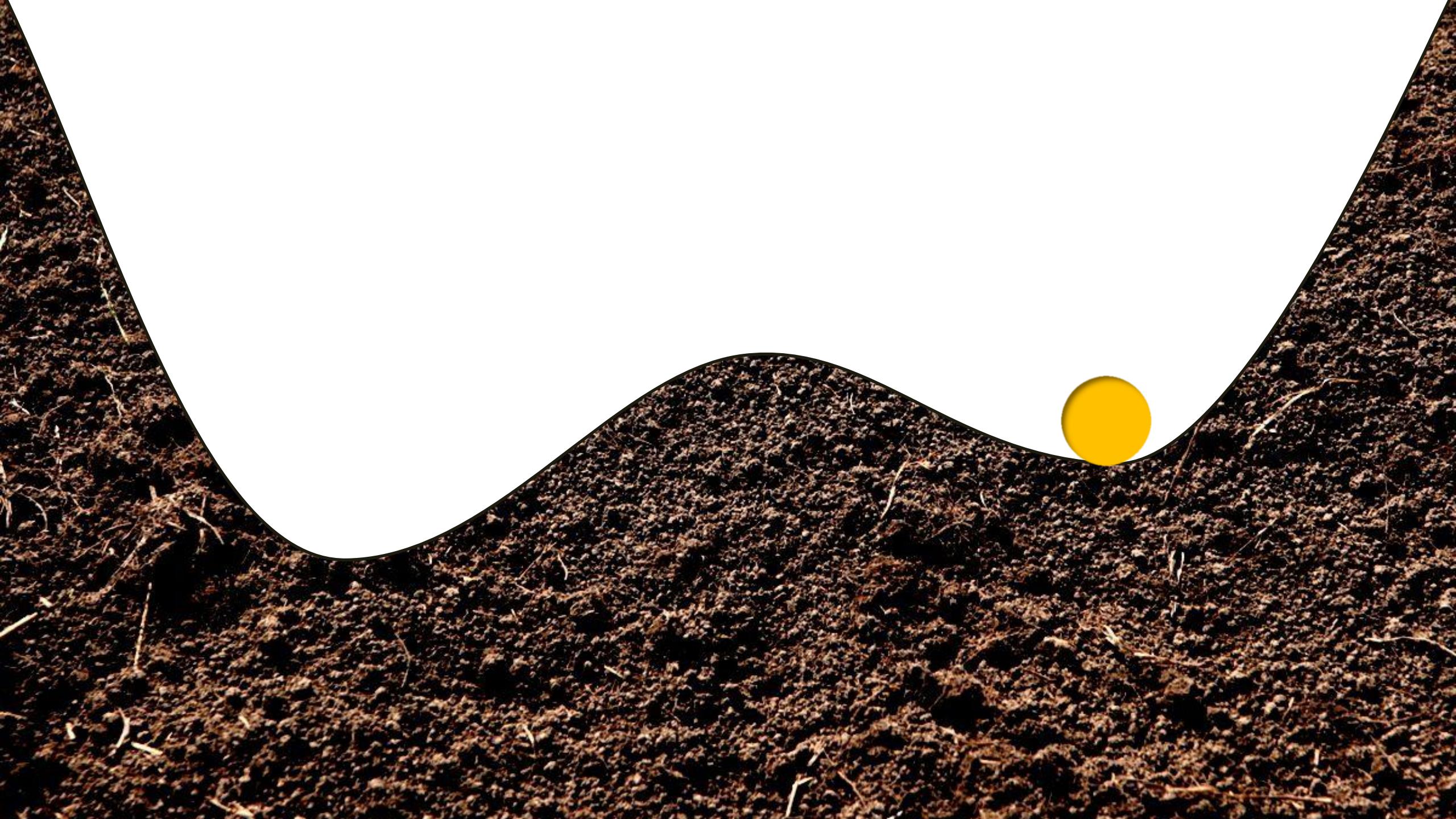
Studies structural changes in the potential function, Energy function or Lyapunov function of a dynamical system

$$\frac{dy}{dt} = -\frac{\delta V(y)}{\delta y}$$





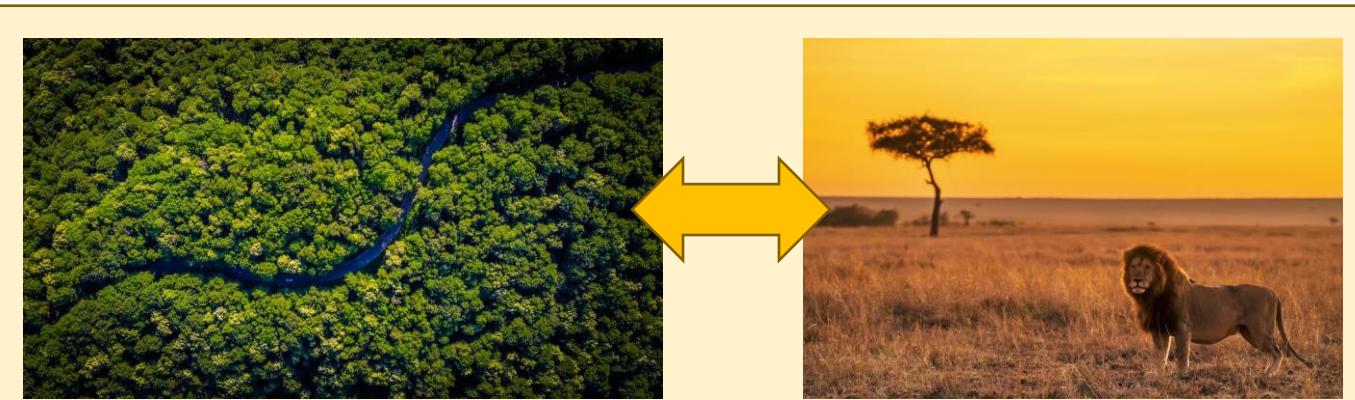
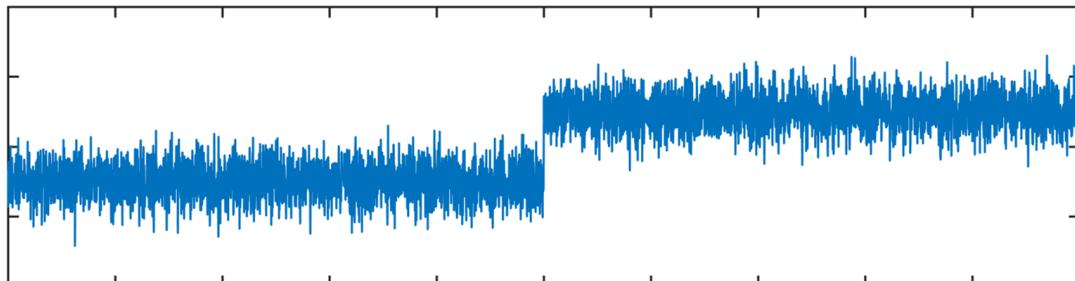




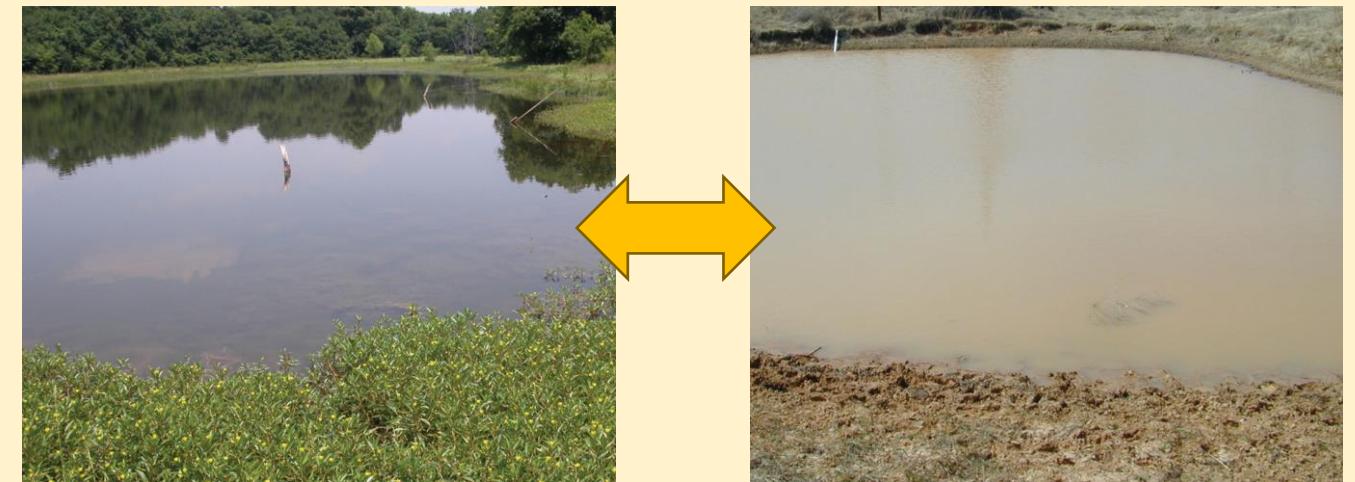


Critical shifts

Classic Literature
[Holling, 1973]
[Noy-Meier, 1975]
[May, 1977]

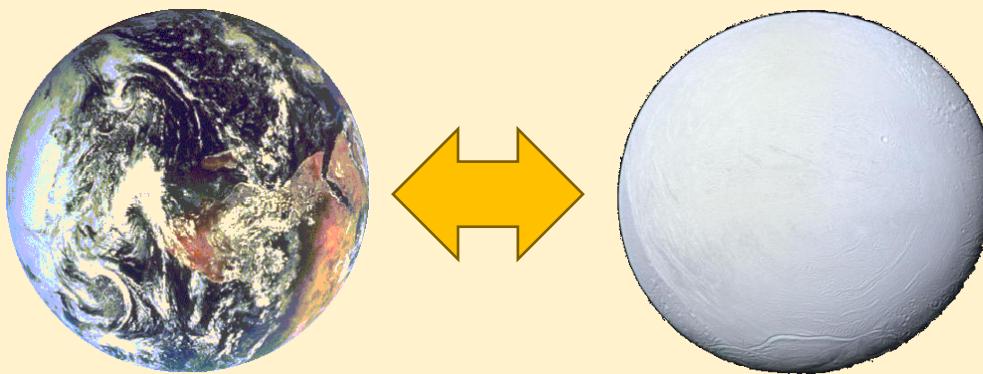


Ecosystem shifts



Critical shifts

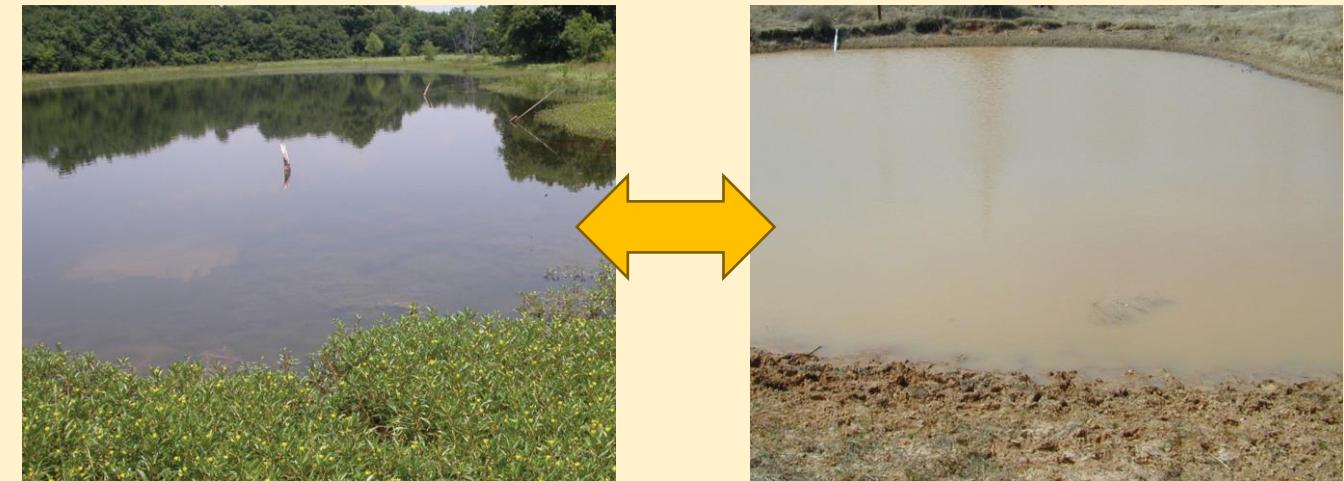
Classic Literature
[Holling, 1973]
[Noy-Meier, 1975]
[May, 1977]



Planetary transitions

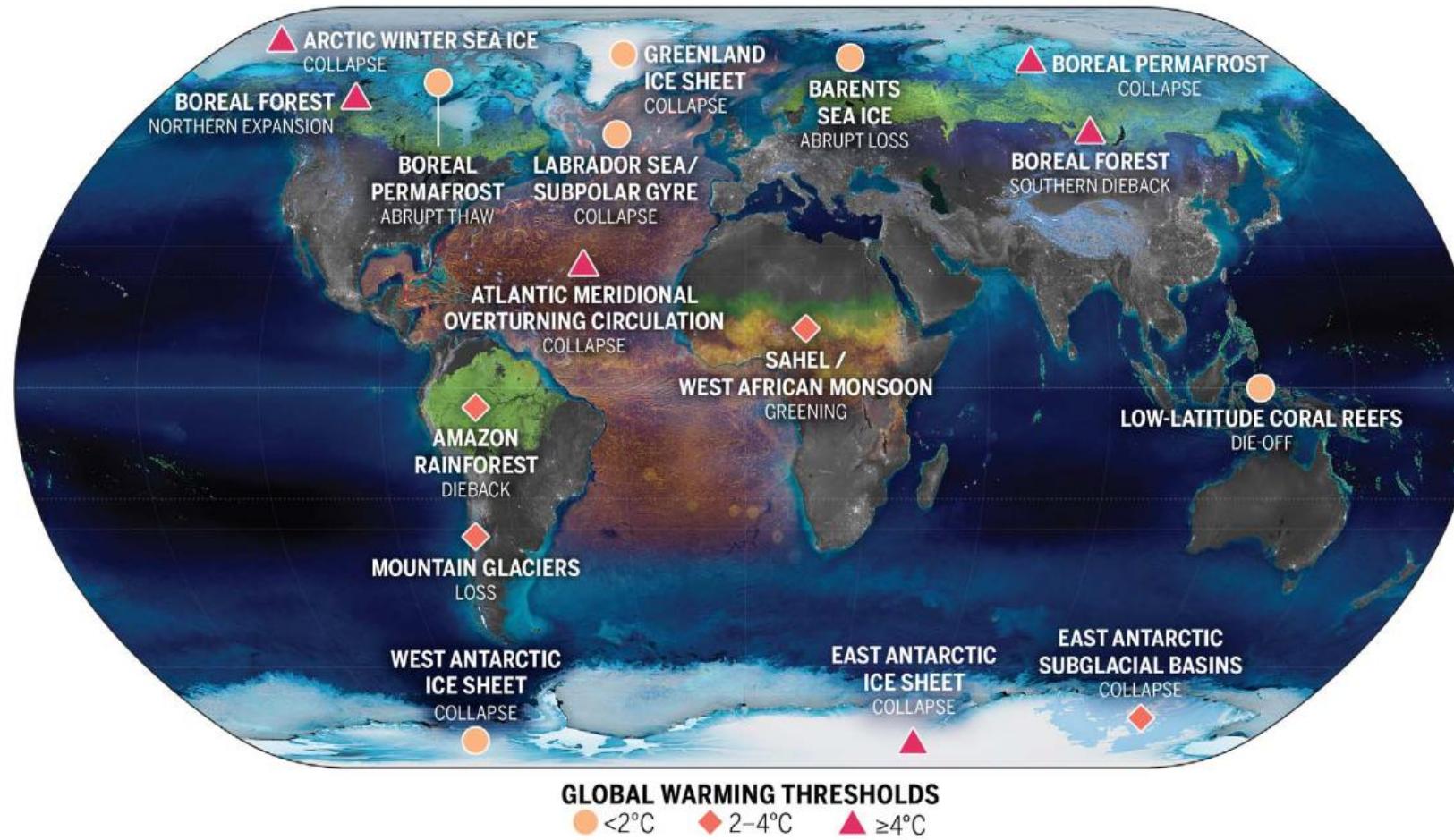


Ecosystem shifts



Tipping Points

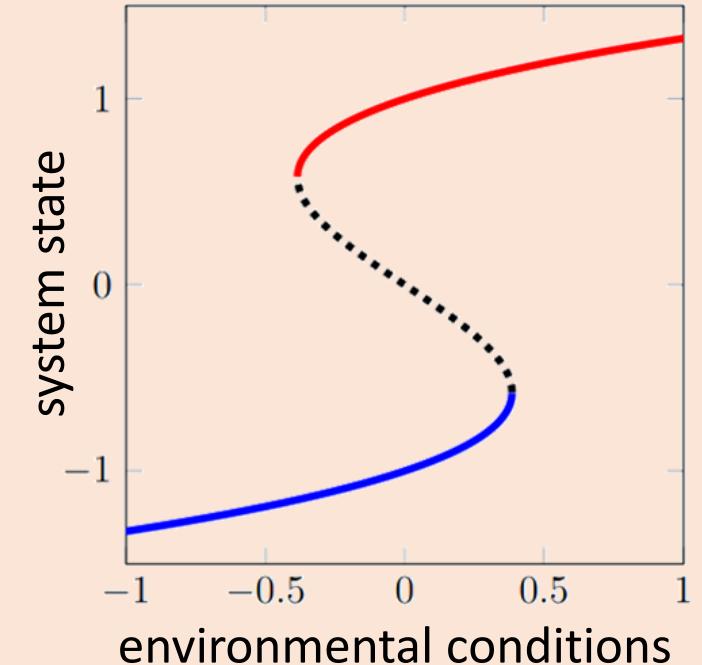
IPCC AR6 (2021) : “a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly”



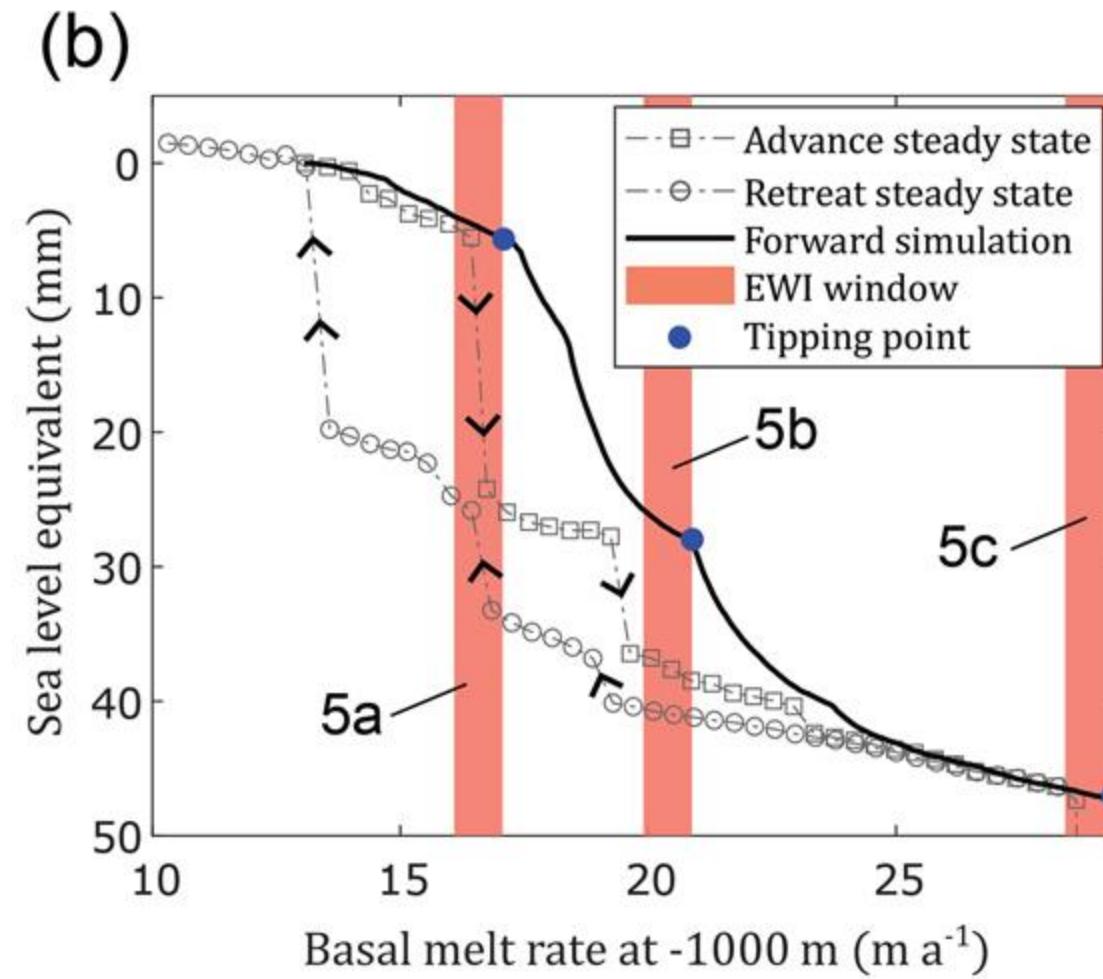
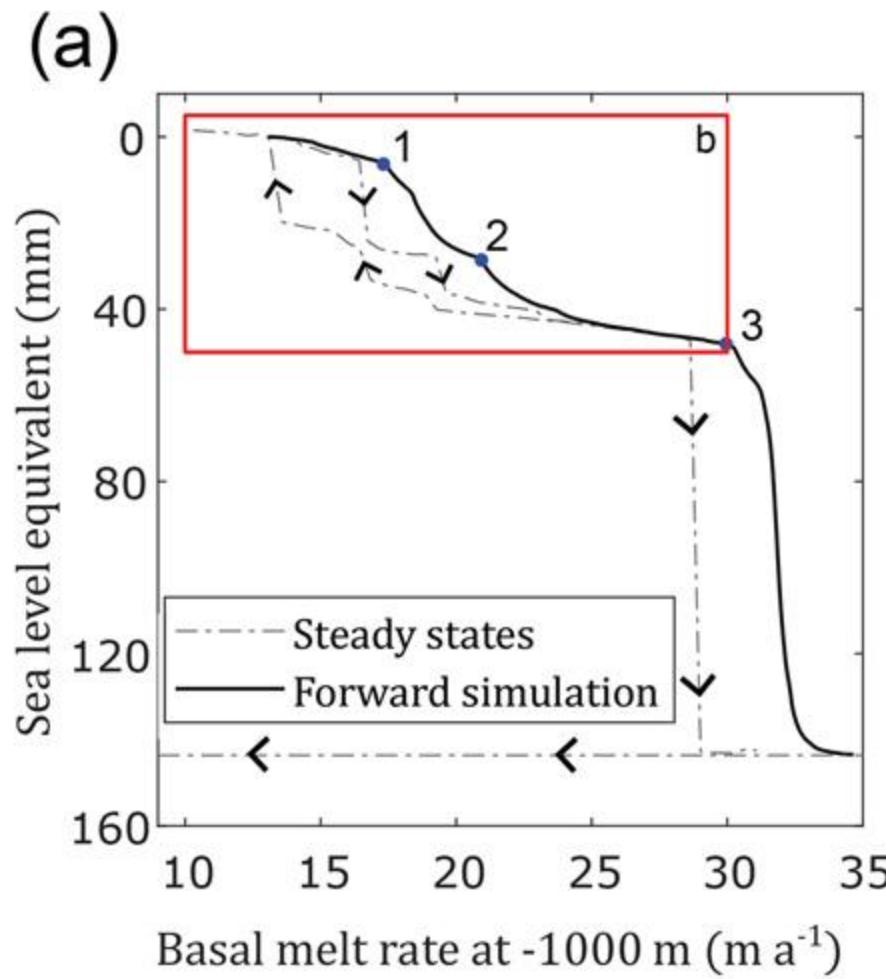
Mathematics

Tipping points \leftrightarrow Bifurcations

$$\frac{dy}{dt} = f(y, \mu)$$



Tipping - Pine Island Glacier, West Antarctica

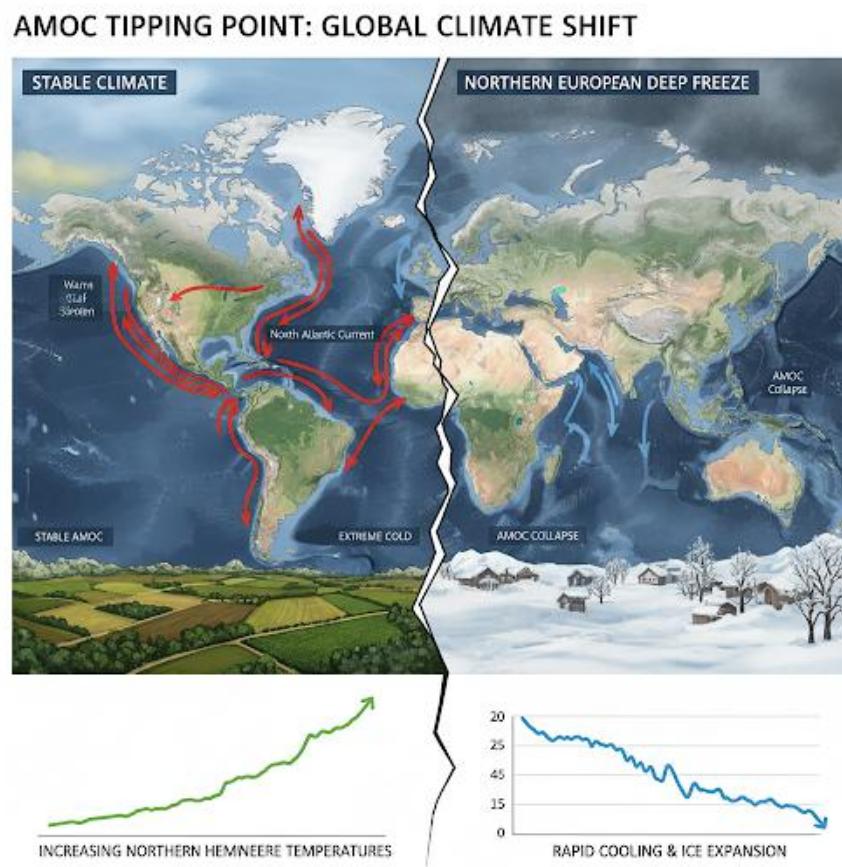


[Rosier et al, 2021]

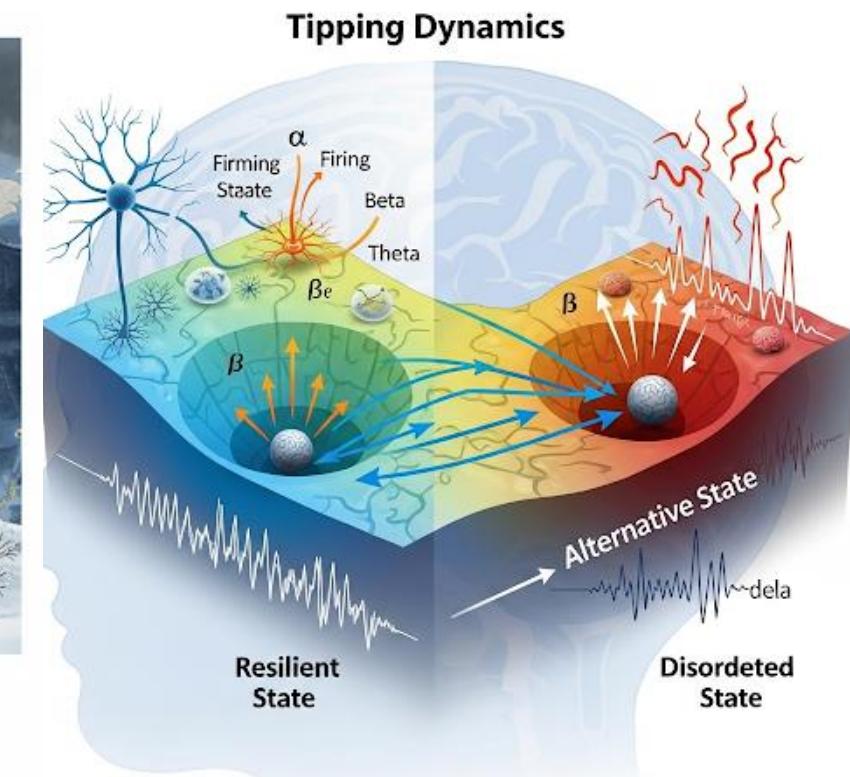
Tipping in other systems



Power grid



Ocean circulation



Brain dynamics

Images by Gemini for visualisation. They do not have scientific meaning. I guess the tipping point for AI singularity is still far away...

Classic Theory of Tipping

$$\frac{d\vec{y}}{dt} = f(\vec{y}; \mu)$$

Canonical example:

$$\frac{dy}{dt} = y(1 - y^2) + \mu$$

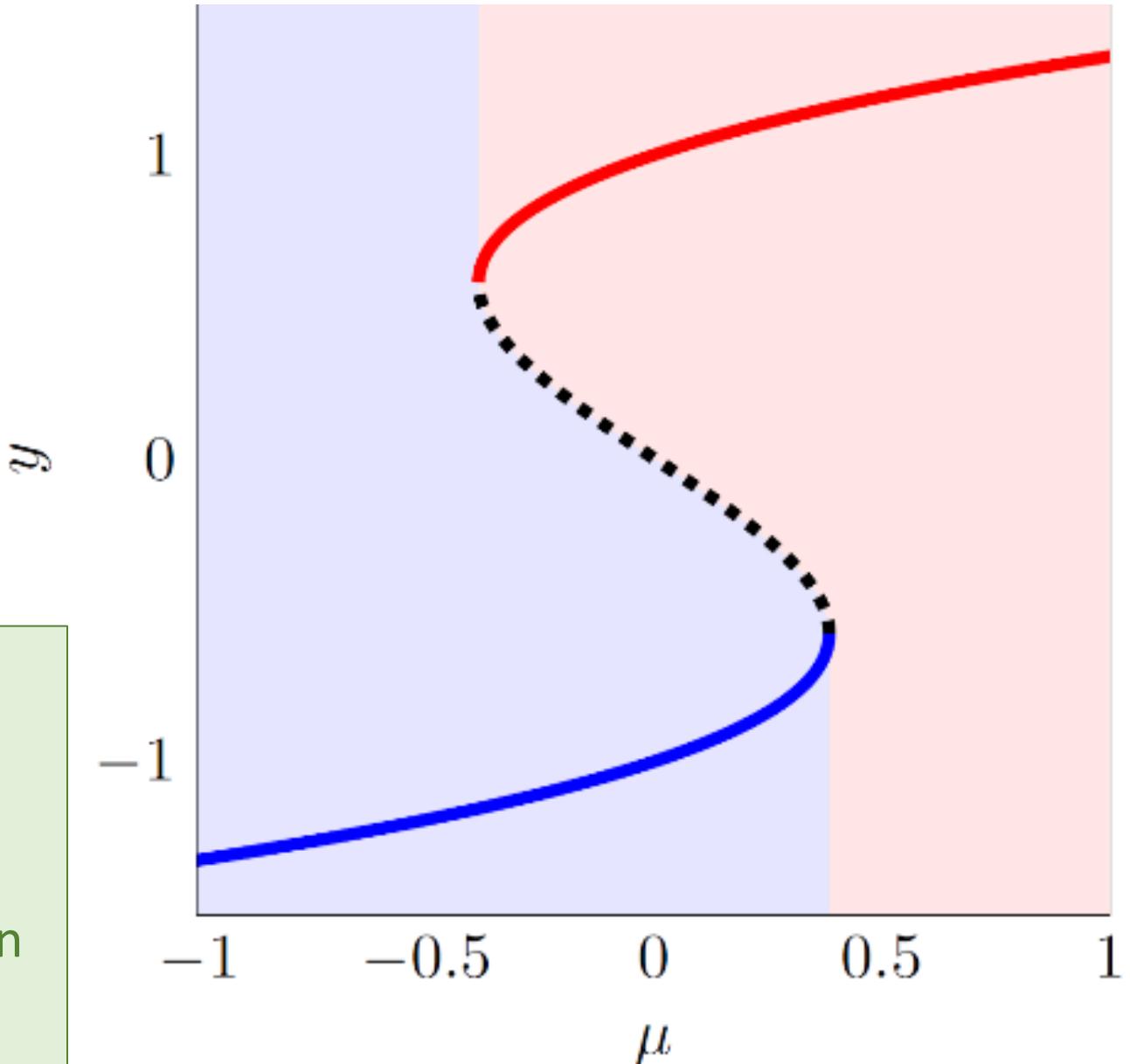
Tipping

[Ashwin et al, 2012]

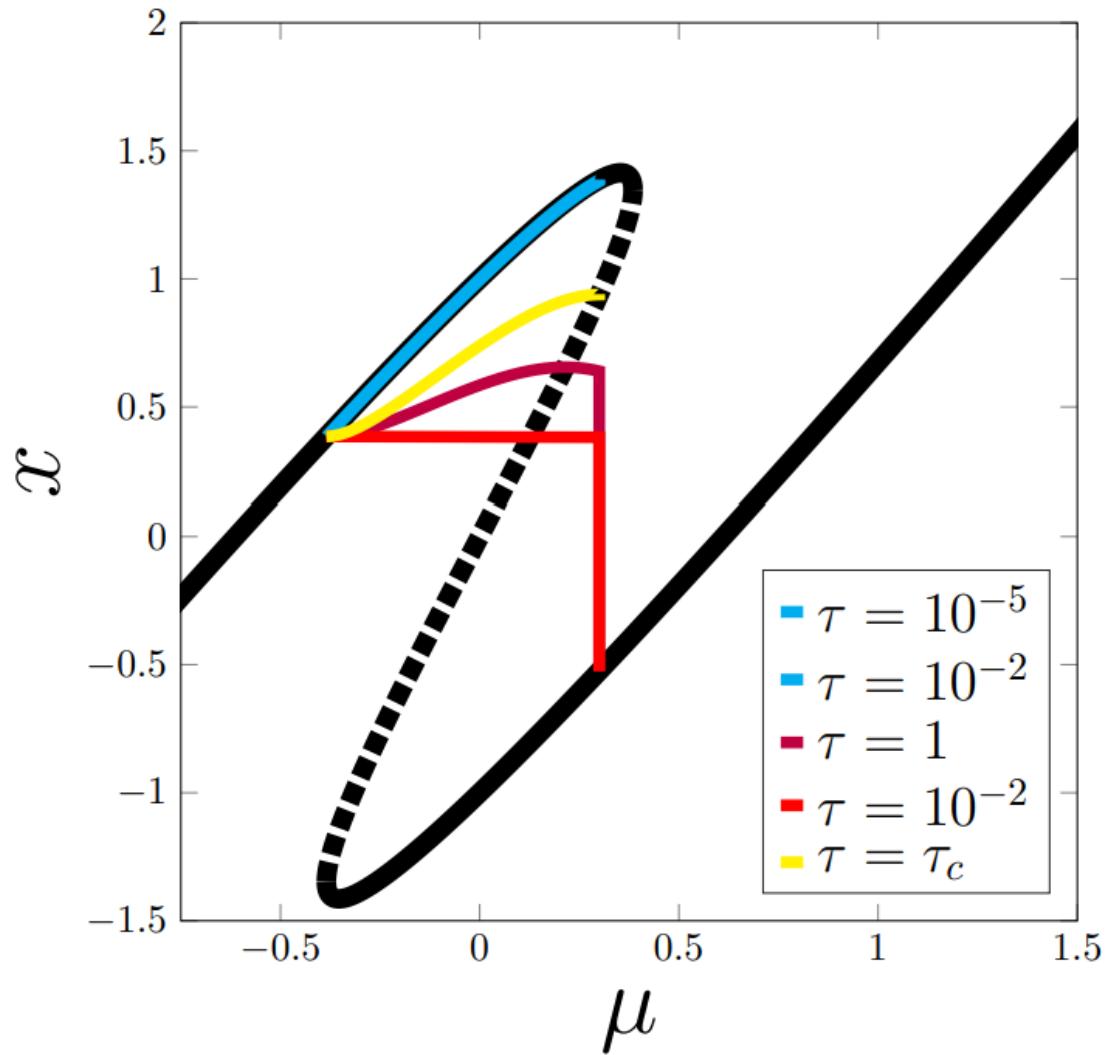
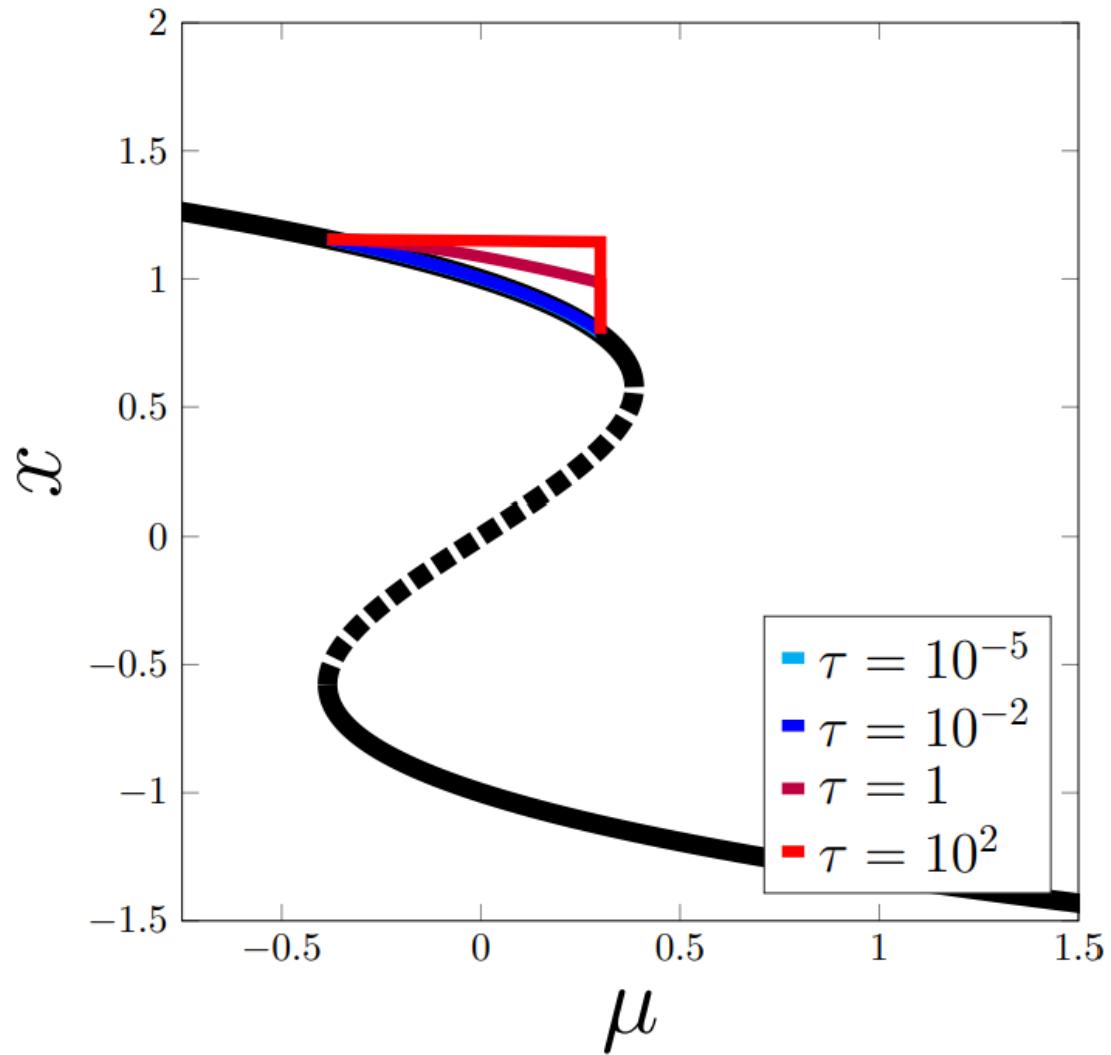
Bifurcation-Tipping : Basin disappears

Noise-Tipping : Forced outside Basin

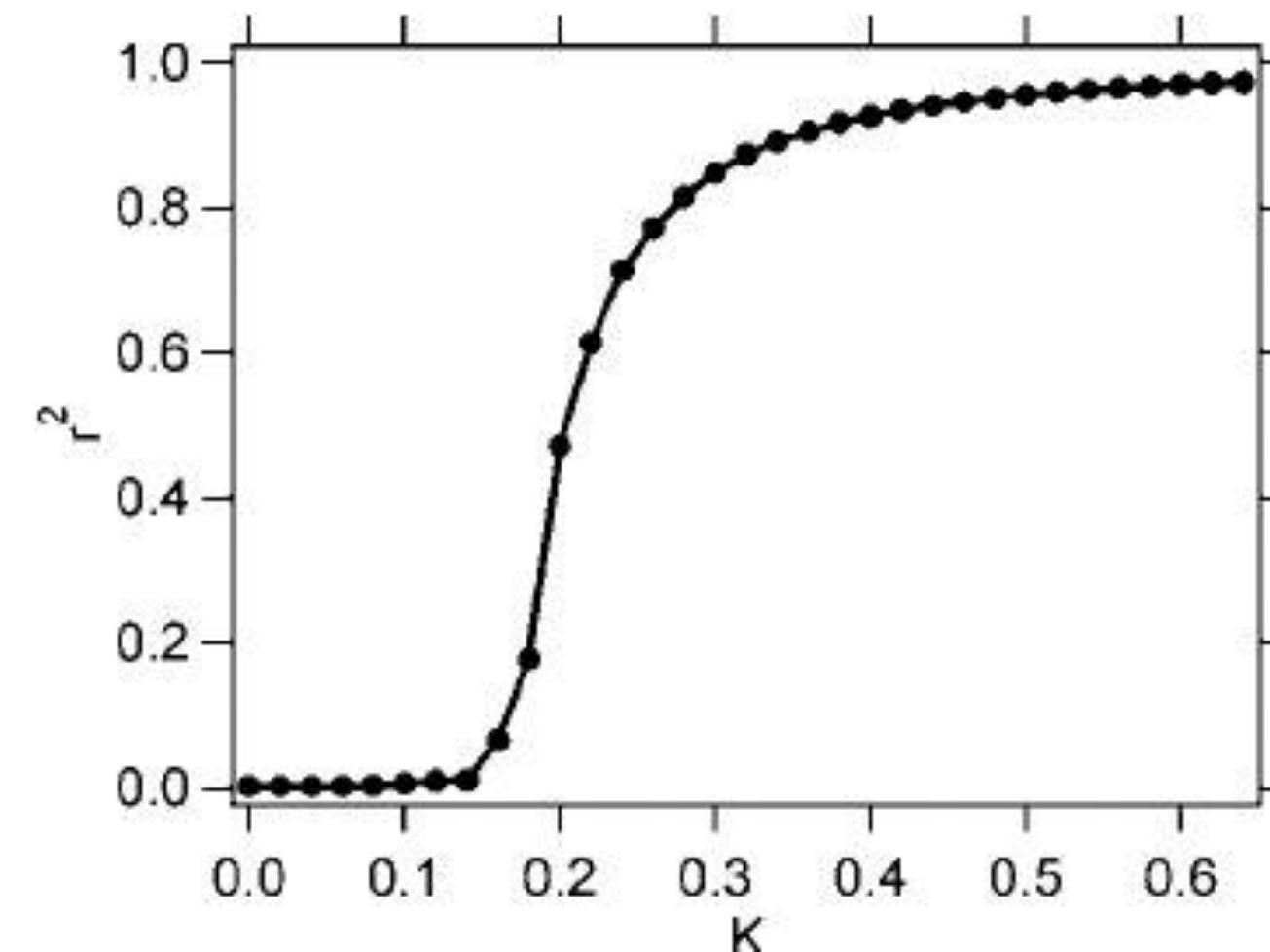
Rate-Tipping : *(more complicated)*



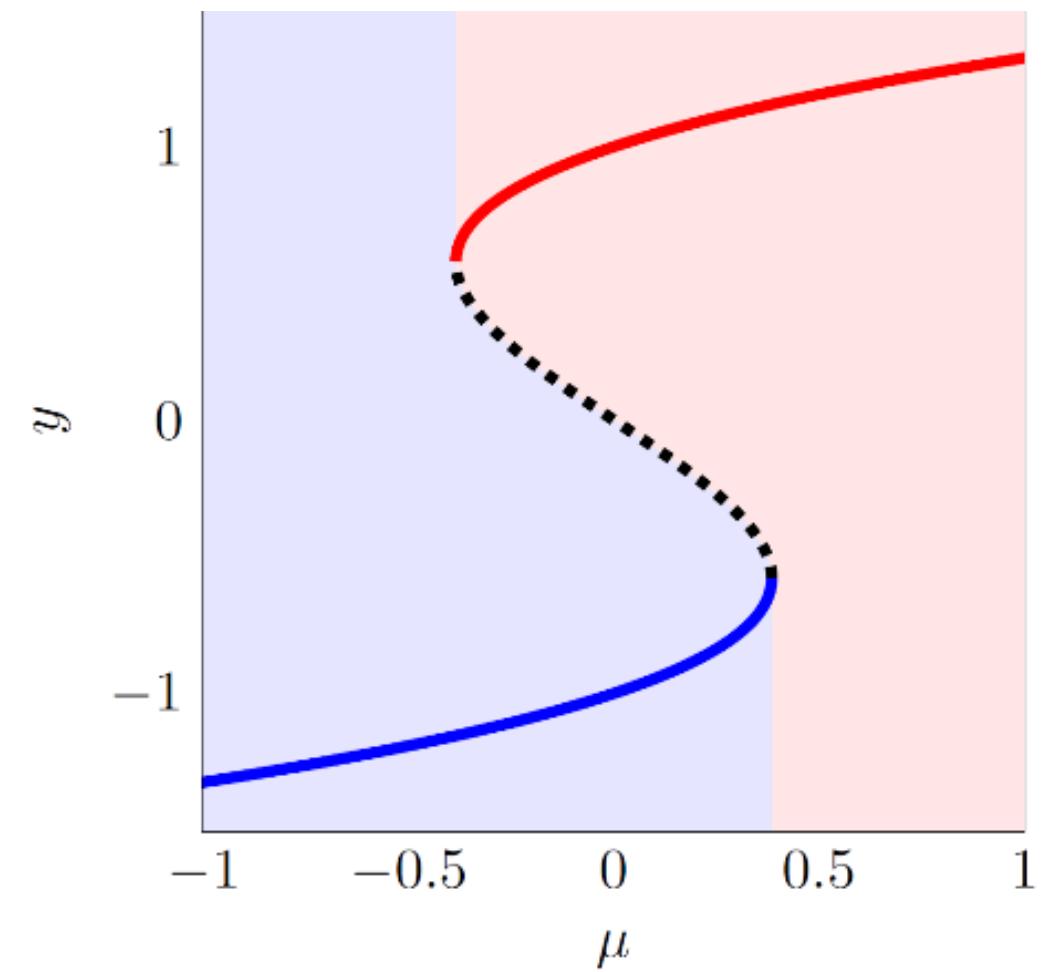
Rate-Tipping



Phase transitions



Kuramoto model [English, 2007]





Timescales



Systems with multiple time scales

$$\frac{dx}{dt} = f(x; \mu)$$

$$\frac{d\mu}{dt} = \tau g(\tau t)$$



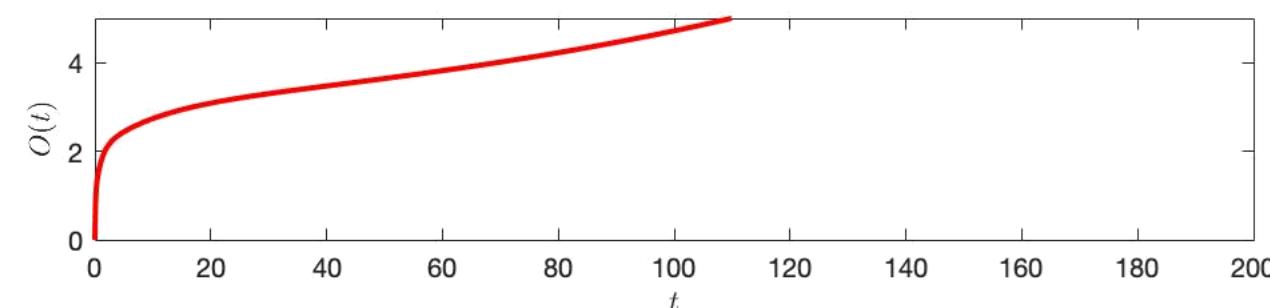
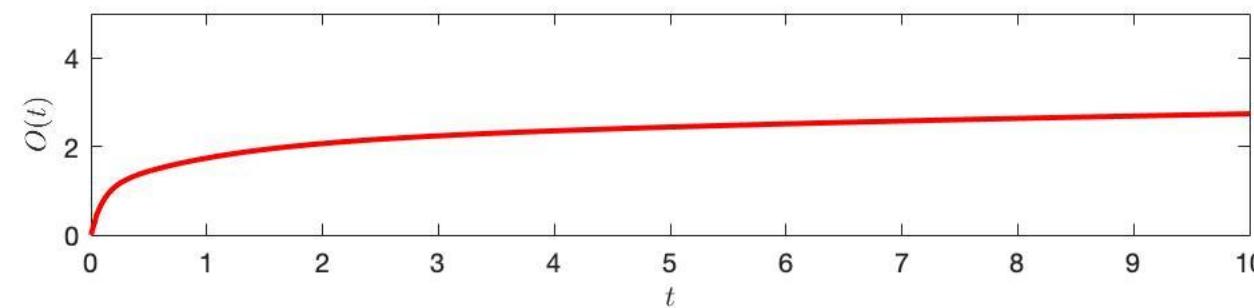
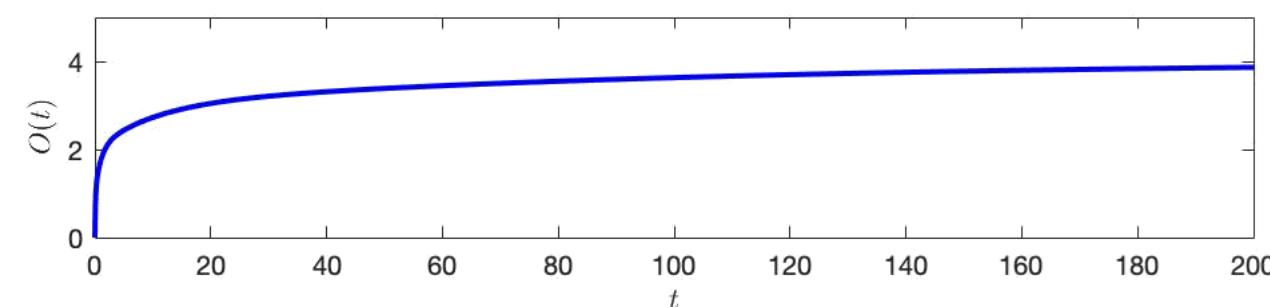
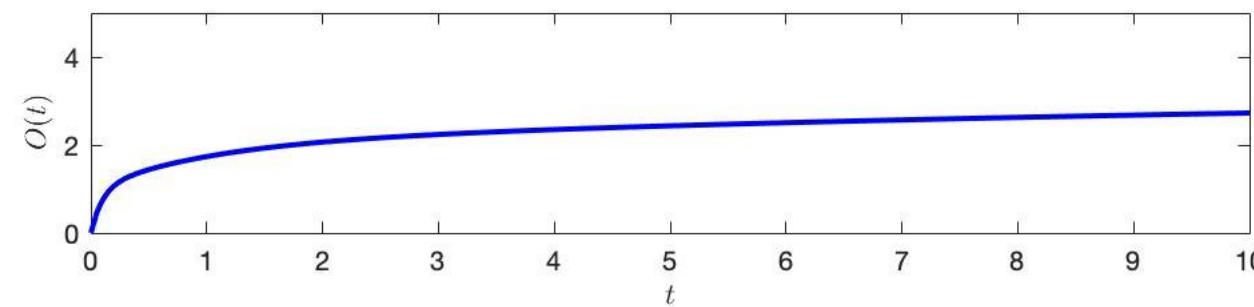
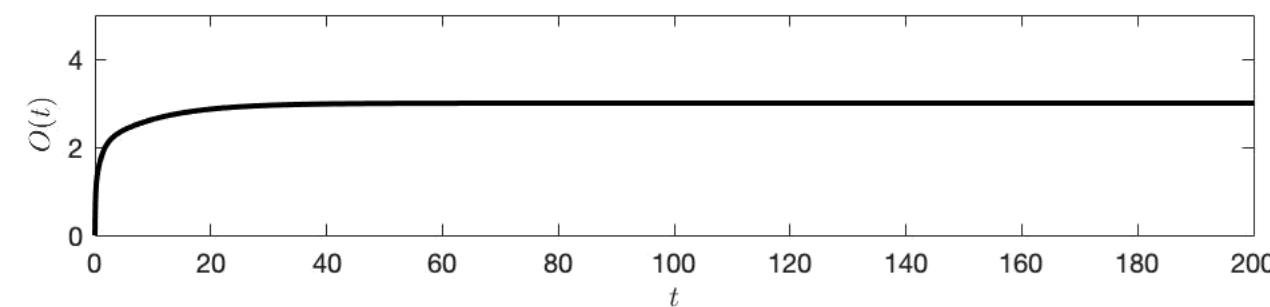
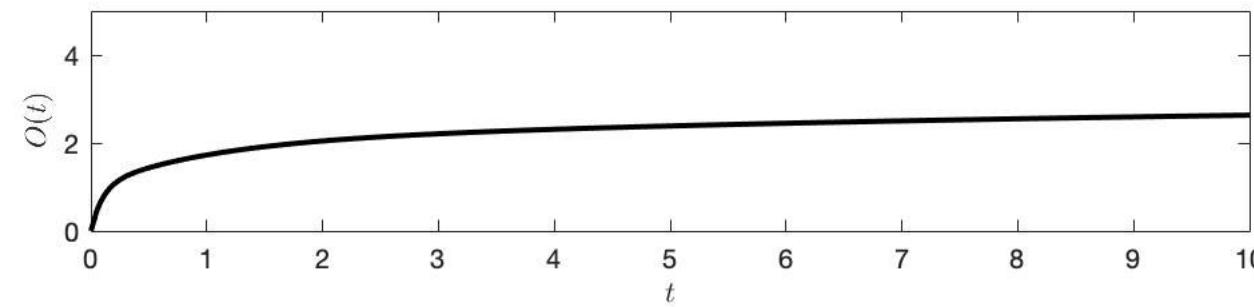
Time Scale Separation

$\tau \ll 1$: forcing slow compared to system dynamics \rightarrow B-tipping

$\tau \gg 1$: forcing fast compared to system dynamics \rightarrow S-tipping

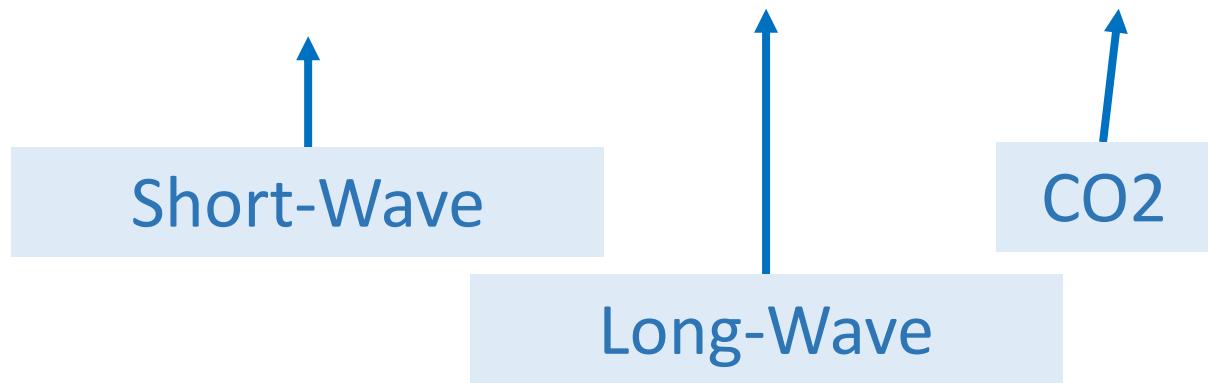
$\tau = \mathcal{O}(1)$: forcing comparable to system dynamics \rightarrow R-tipping

Importance of timescales



EXAMPLE 1: Multiscale Global Energy Balance Model

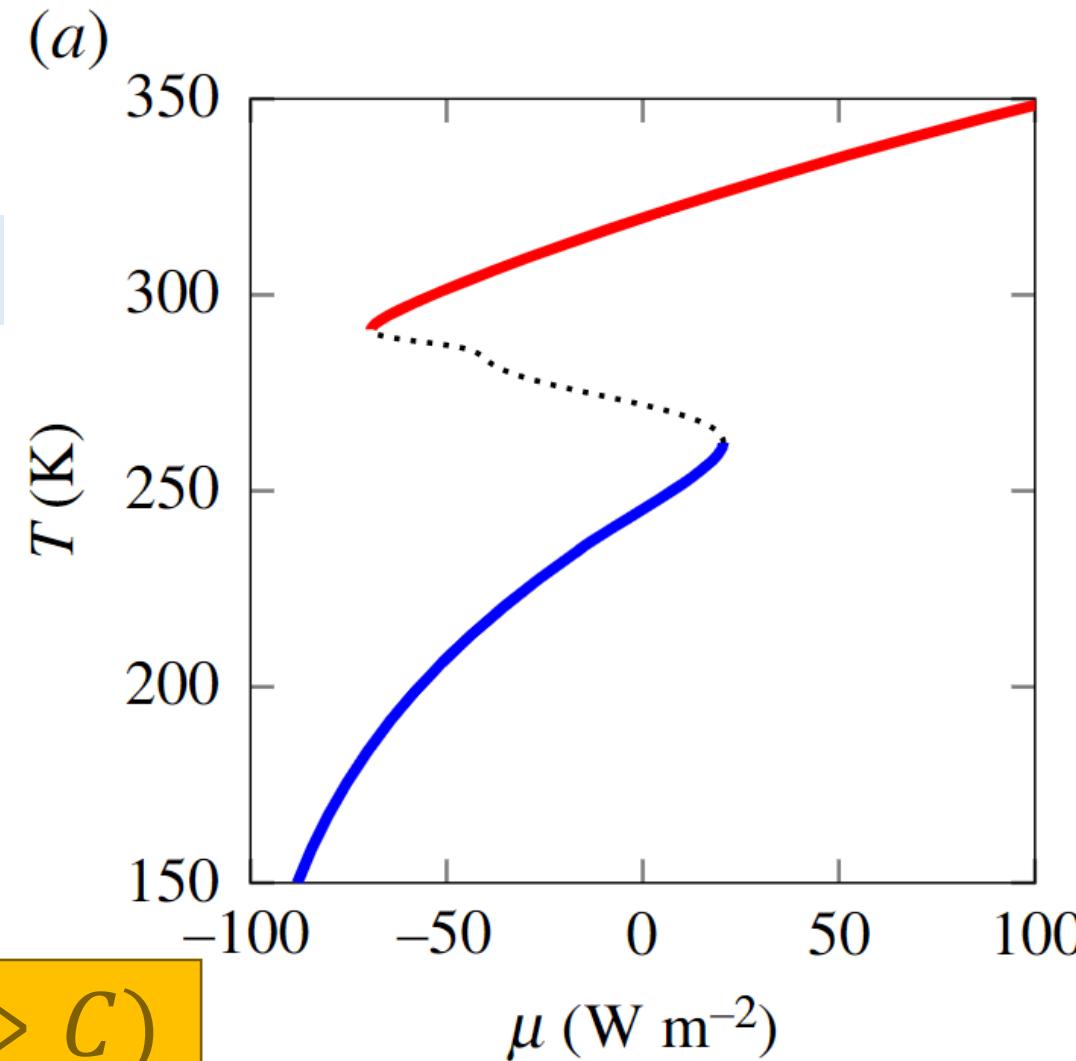
$$C \frac{dT}{dt} = Q_0(1 - \alpha) - \epsilon(T)\sigma T^4 + \mu$$



$$\tau_\alpha \frac{d\alpha}{dt} = \alpha_0(T) - \alpha$$

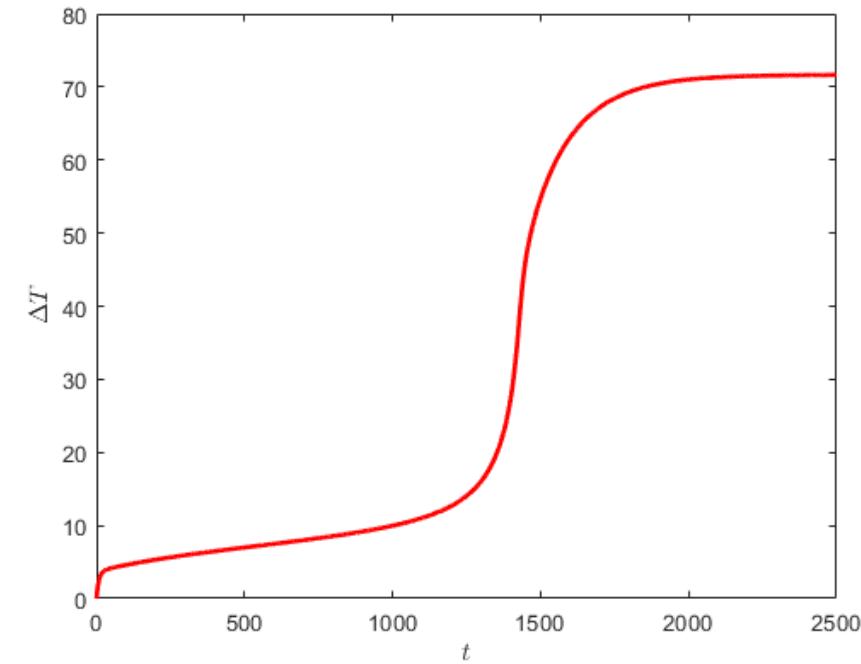
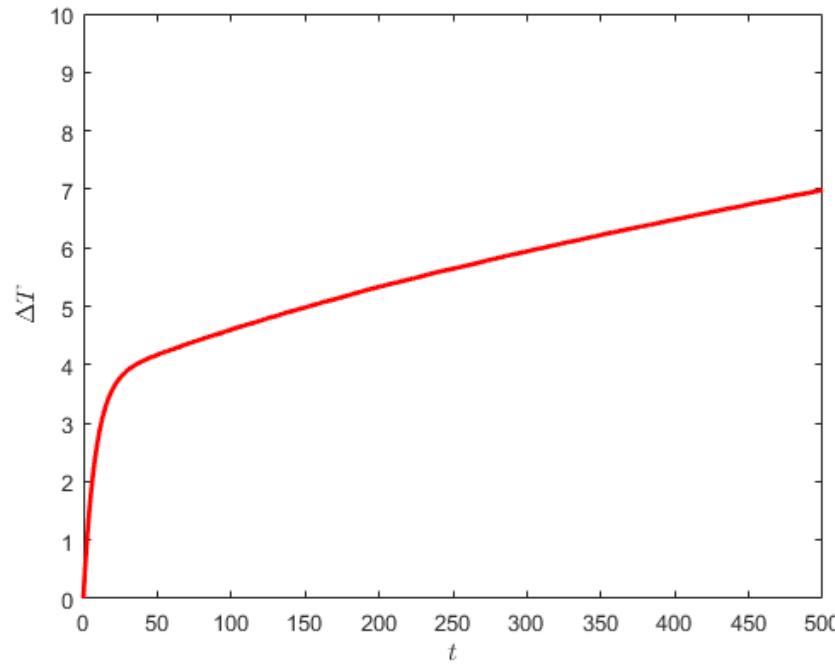
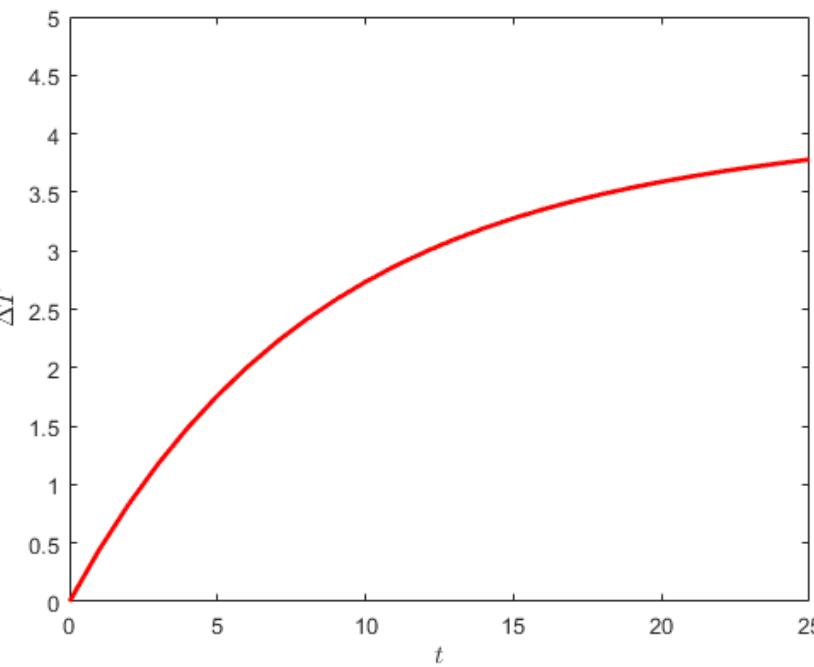
Dynamic albedo

Internal Time Scale Separation ($\tau_\alpha \gg C$)



Abrupt 4xCO₂ forcing experiment

- Initialize for μ_0 (initial CO₂-levels)
 - Change to μ_1 (4xCO₂ levels)
- Look at dynamics

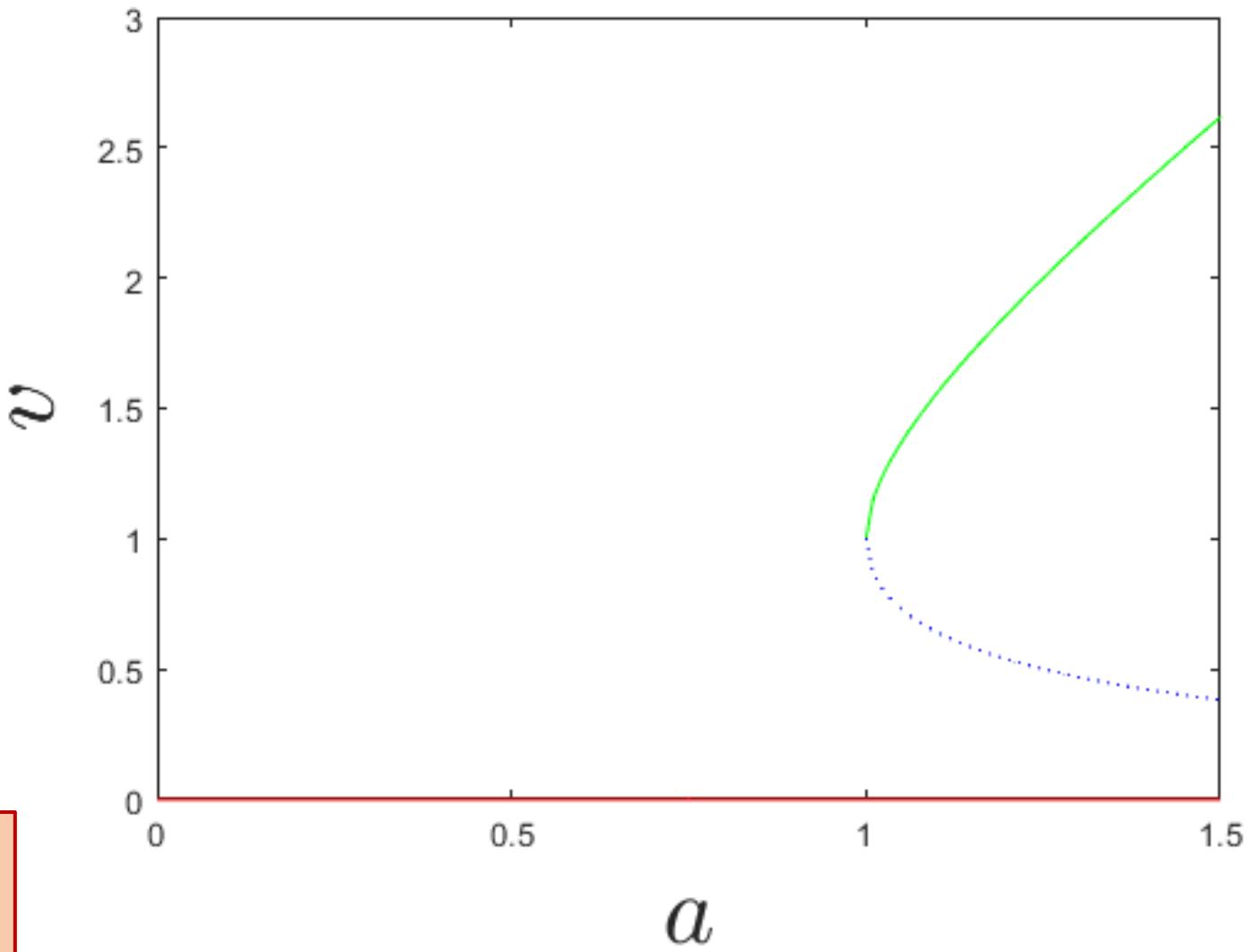


EXAMPLE 2: Time scale of feedback

$$\begin{aligned}\frac{du}{dt} &= a - u - uv^2 \\ \frac{dv}{dt} &= uv^2 - mv\end{aligned}$$

Parameters:

$$m = 0.5$$



EXAMPLE 2: Time scale of feedback

$$\frac{du}{dt} = a - u - uvs$$

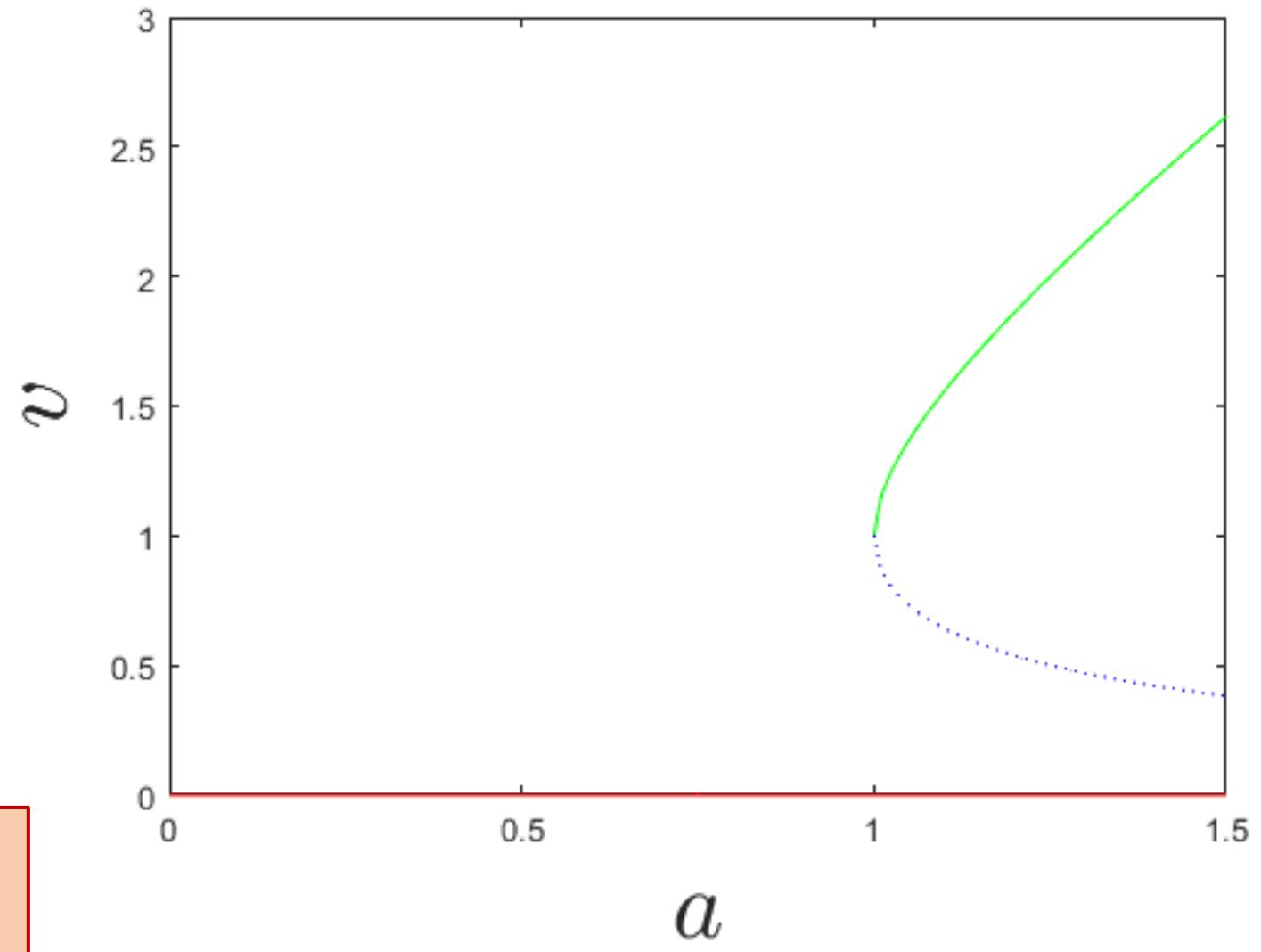
$$\frac{dv}{dt} = uvs - mv$$

$$\frac{ds}{dt} = \tau_{INT} (v - s)$$

Parameters:

$$m = 0.5$$

$$\tau_{INT} = 10^{-5}$$



EXAMPLE 2: Time scale of feedback

$$\frac{du}{dt} = a - u - uvs$$

$$\frac{dv}{dt} = uvs - mv$$

$$\frac{ds}{dt}$$

$$\frac{ds}{dt} = \tau_{INT} (v - s)$$

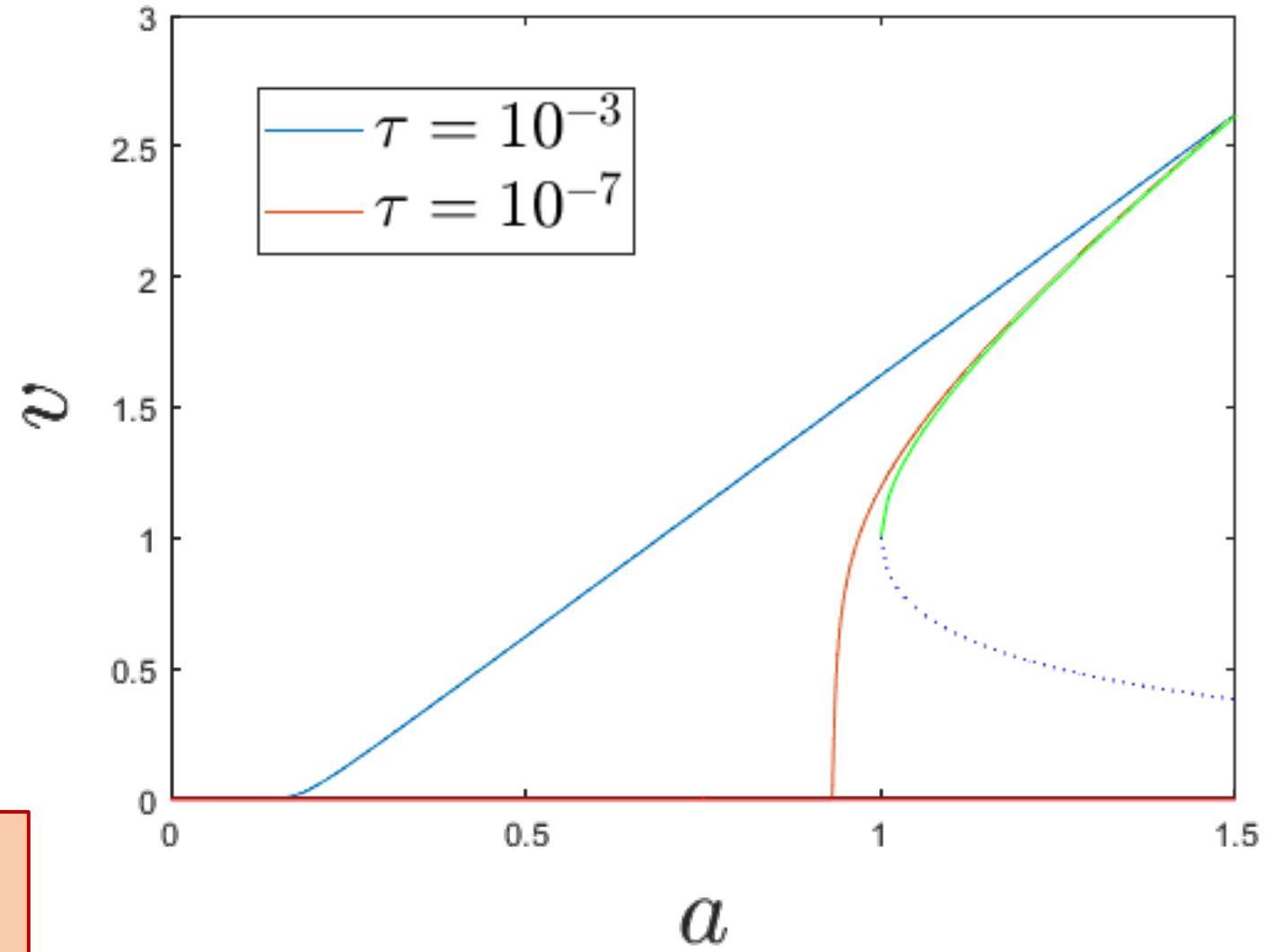
$$\frac{da}{dt}$$

$$\frac{da}{dt} = -\tau$$

Parameters:

$$m = 0.5$$

$$\tau_{INT} = 10^{-5}$$



Spatial effects



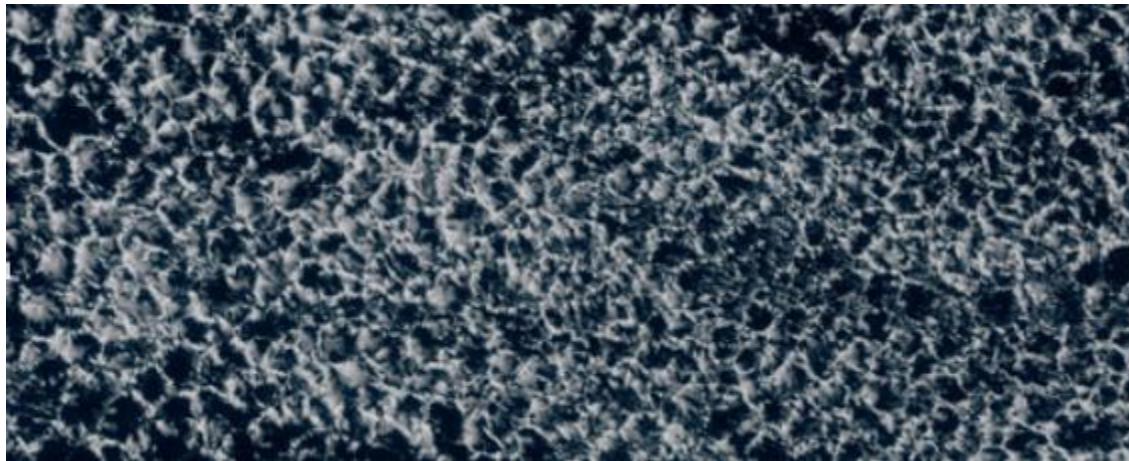
Spatially extended systems



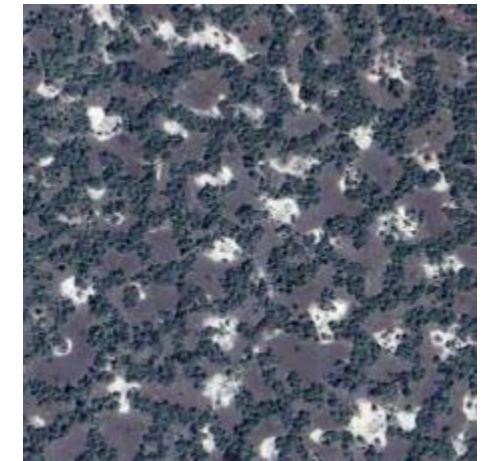
Examples of spatial patterning – regular patterns



mussel beds



clouds



savannas



melt ponds



drylands

Examples of spatial patterning – spatial interfaces

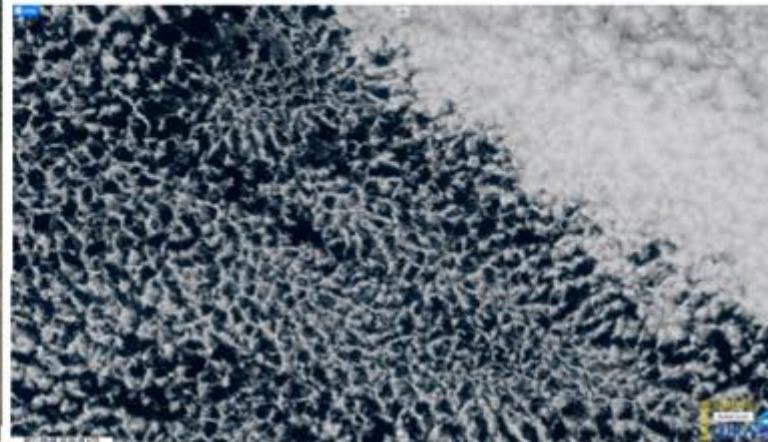
tropical forest
& savanna
ecosystems

[Google Earth]



types of
stratocumulus
clouds

[RAMMB/CIRA SLIDER]



sea-ice & water
at Eltanin Bay

[NASA's Earth observatory]

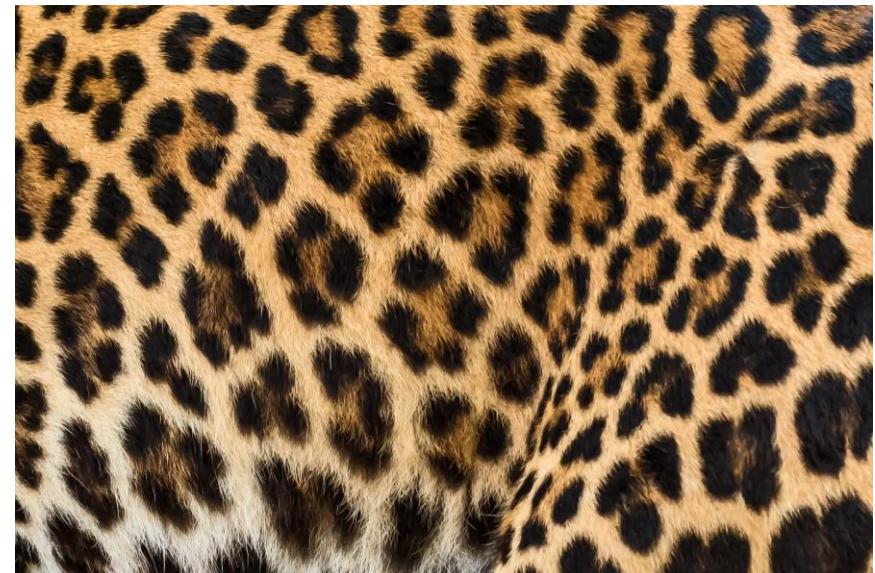


algae bloom
in Lake St. Clair

[NASA's Earth observatory]



Examples of spatial patterning – animals



Examples of spatial patterning – sociology

The population of the United States is not distributed evenly. Instead, we tend to bunch up in communities, leaving the spaces in between more sparsely inhabited. Most Americans live in or near cities; today 53 percent live in the 20 largest cities. 75 percent of all Americans live in metropolitan areas.

This map shows population density. The relative height of each major city reflects its population in 1990.
Source: U.S. Census Bureau

Go West. Nevada is the fastest growing state, followed by Arizona, Idaho, Colorado, and Utah.

Wyoming has the lowest population density of all states in the lower 48 with an average of five people per square mile.

What happens in the empty spaces? Some of it is farming country. More than one quarter of America's crop land is used to grow corn. One third of what is produced is exported to other countries.

Chicago, the country's third largest city, has a population of about three million people. There are 21 states with populations smaller than this city.

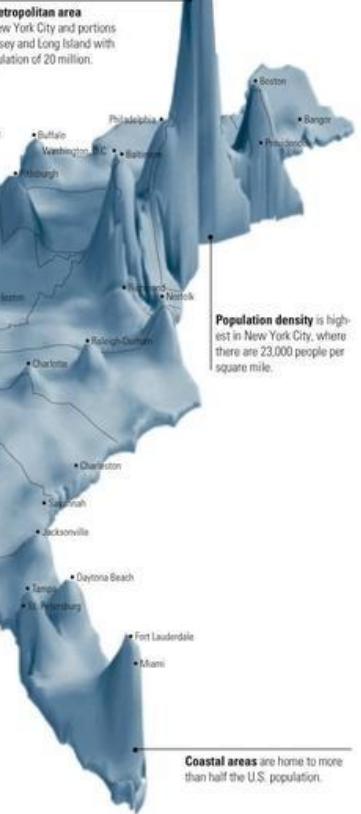
Largest metropolitan area includes New York City and portions of New Jersey and Long Island with a total population of 20 million.

Population Distribution

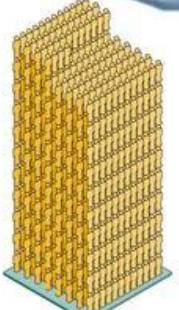
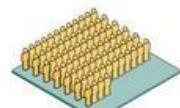
Where do we live?
Where don't we live?



Population density is highest in New York City, where there are 23,000 people per square mile.



Approximately one in nine Americans lives in the nation's most populous state—California. More than 15 million people live in the Los Angeles, Riverside, and Orange County metropolitan area.



Distributing our population evenly would put an average of 76 people per square mile.

New Jersey is the most densely populated state with an average of more than 1,000 people per square mile.

Alaska is a sparsely populated state with an average of one person per square mile.

Examples of spatial patterning – physics

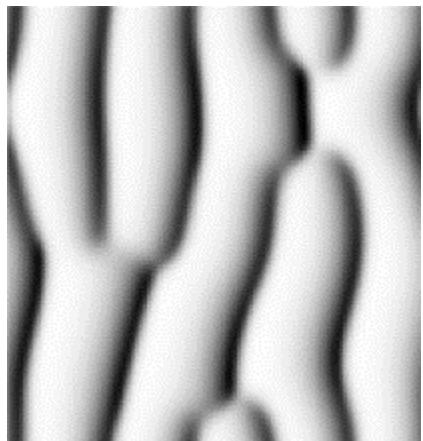
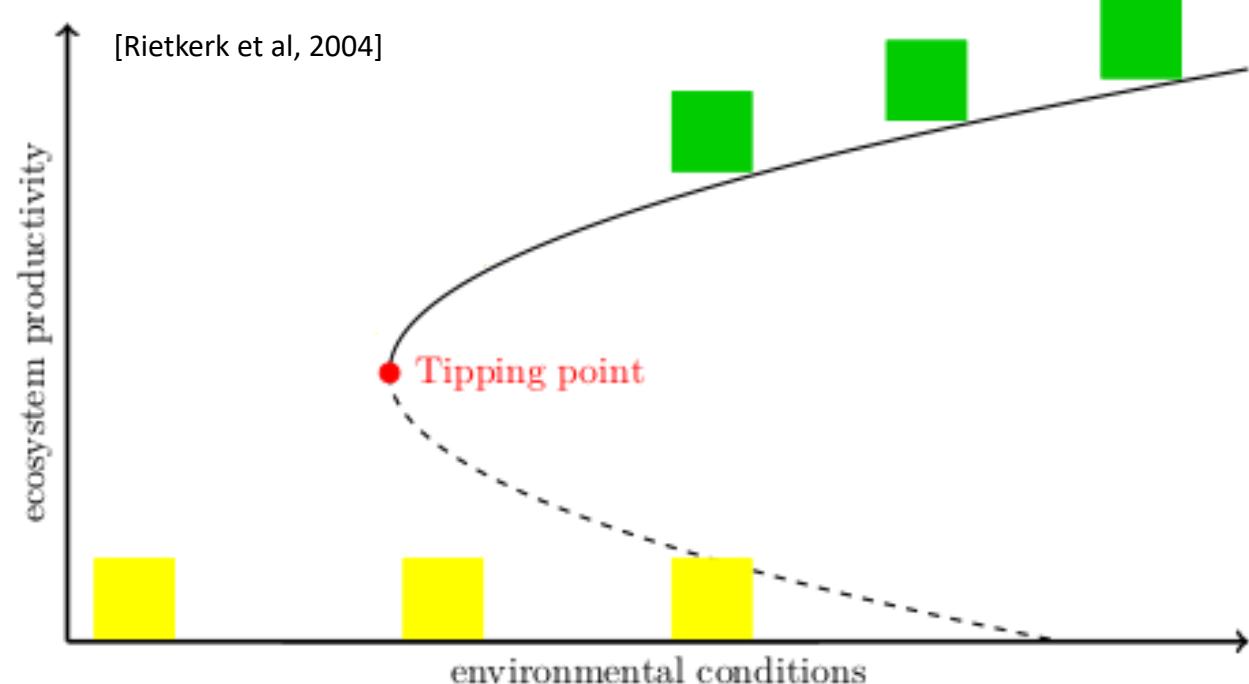


Patterns in models

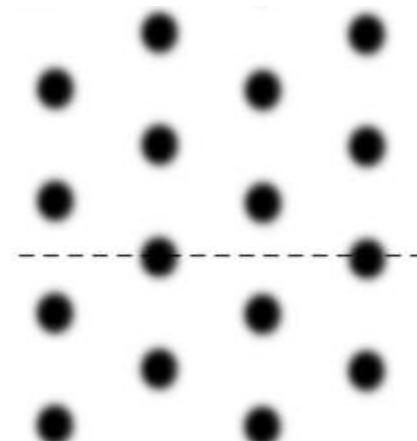
Add spatial transport:

Reaction-Diffusion equations:

$$\begin{cases} \frac{du}{dt} = f(u, v) + D_u \Delta u \\ \frac{dv}{dt} = g(u, v) + D_v \Delta v \end{cases}$$



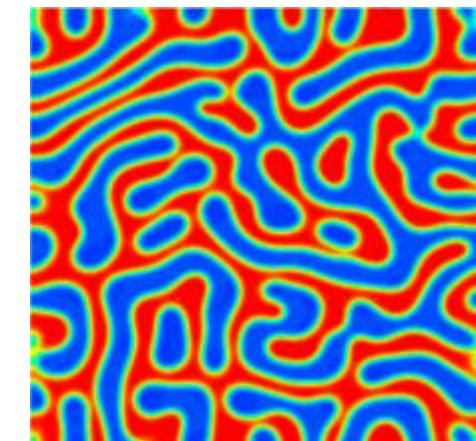
[Klausmeier, 1999]



[Gilad et al, 2004]



[Rietkerk et al, 2002]



[Liu et al, 2013]

Turing patterns

$$\begin{cases} \frac{du}{dt} = f(u, v) + D_u \Delta u \\ \frac{dv}{dt} = g(u, v) + D_v \Delta v \end{cases}$$

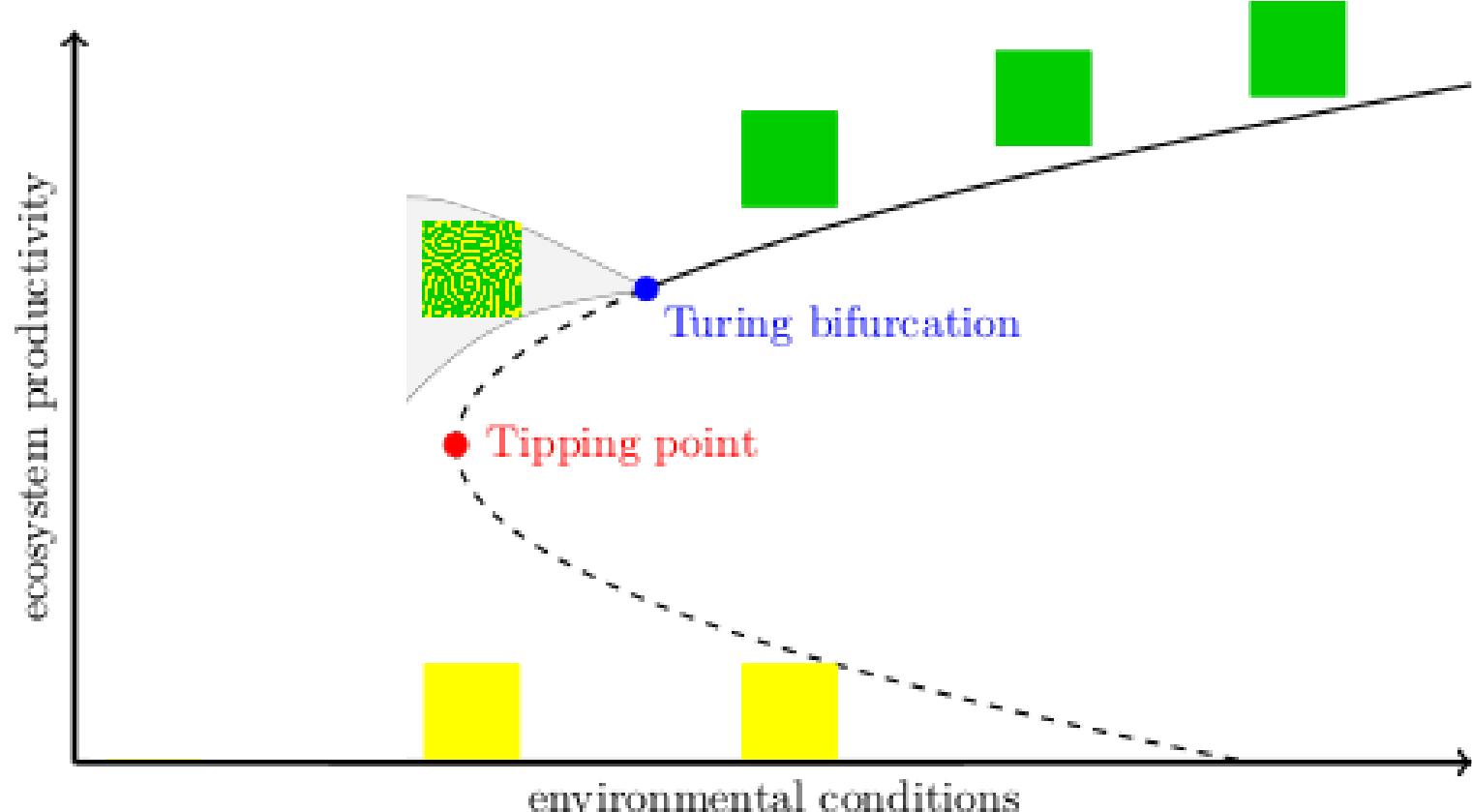
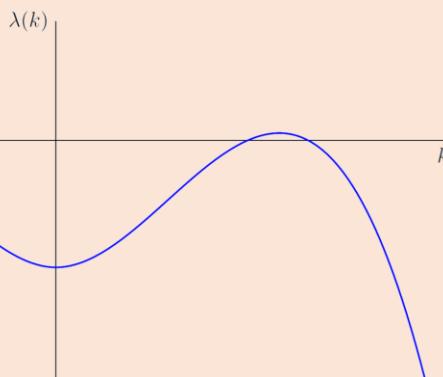
Turing bifurcation

Instability to non-uniform perturbations

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t} e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

→ Dispersion relation

$$\lambda(k) = \dots$$



Weakly non-linear analysis
Ginzburg-Landau equation / Amplitude Equation
& Eckhaus/Benjamin-Feir-Newell criterion
[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Busse balloon

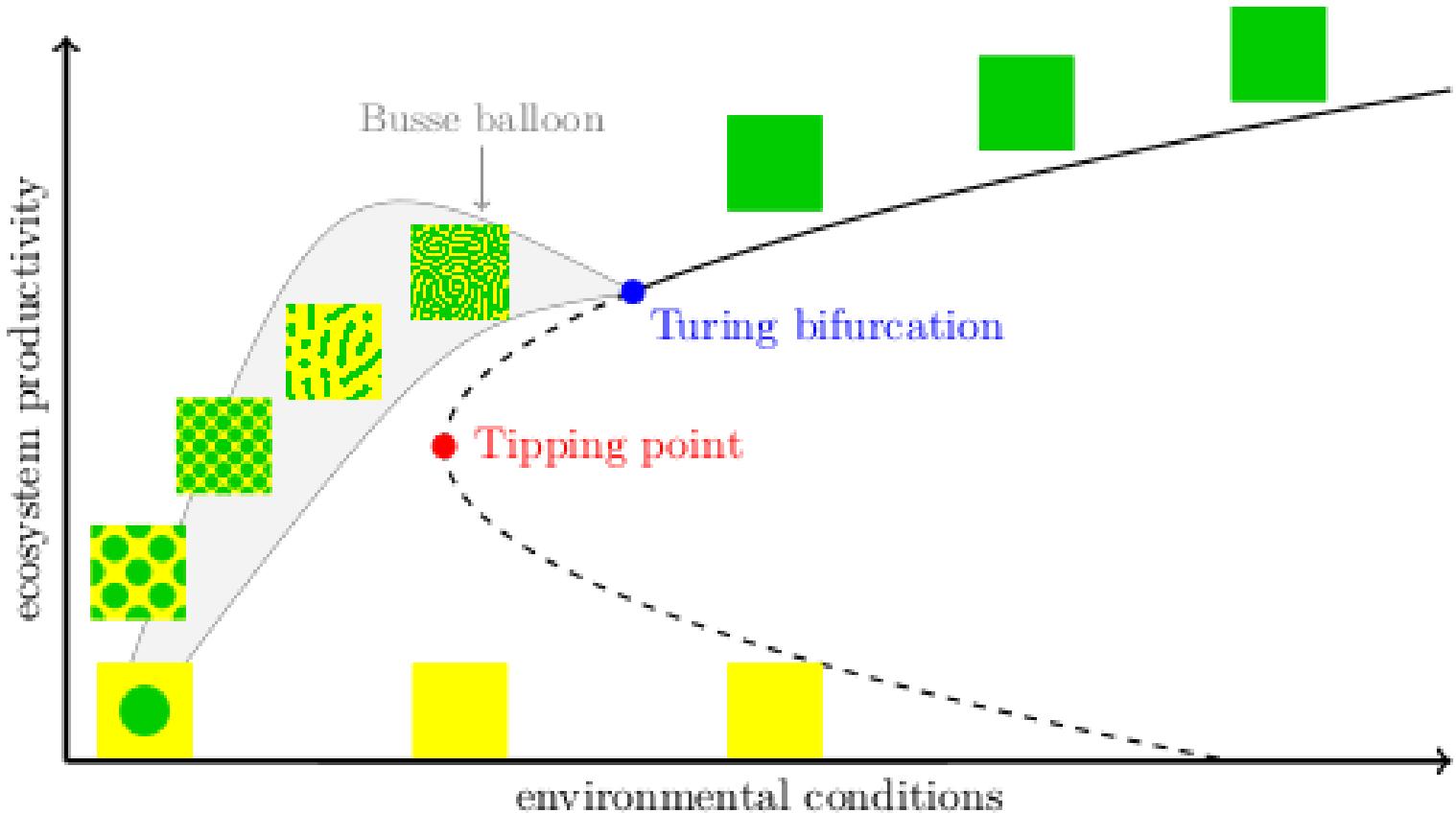
Busse balloon

A model-dependent shape in (*parameter, observable*) space that indicates all stable patterned solutions to the PDE.

Construction Busse balloon

Via numerical continuation
few general results on the
shape of Busse balloon

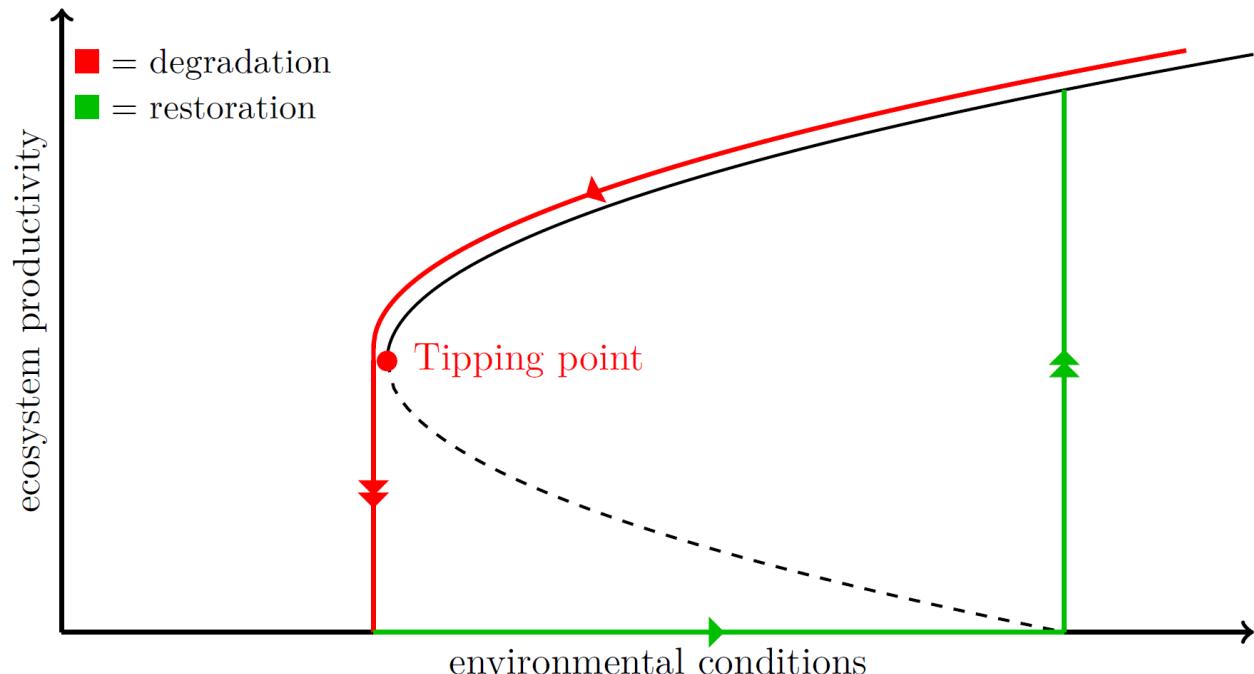
$$\begin{cases} \frac{du}{dt} = f(u, v) + D_u \Delta u \\ \frac{dv}{dt} = g(u, v) + D_v \Delta v \end{cases}$$



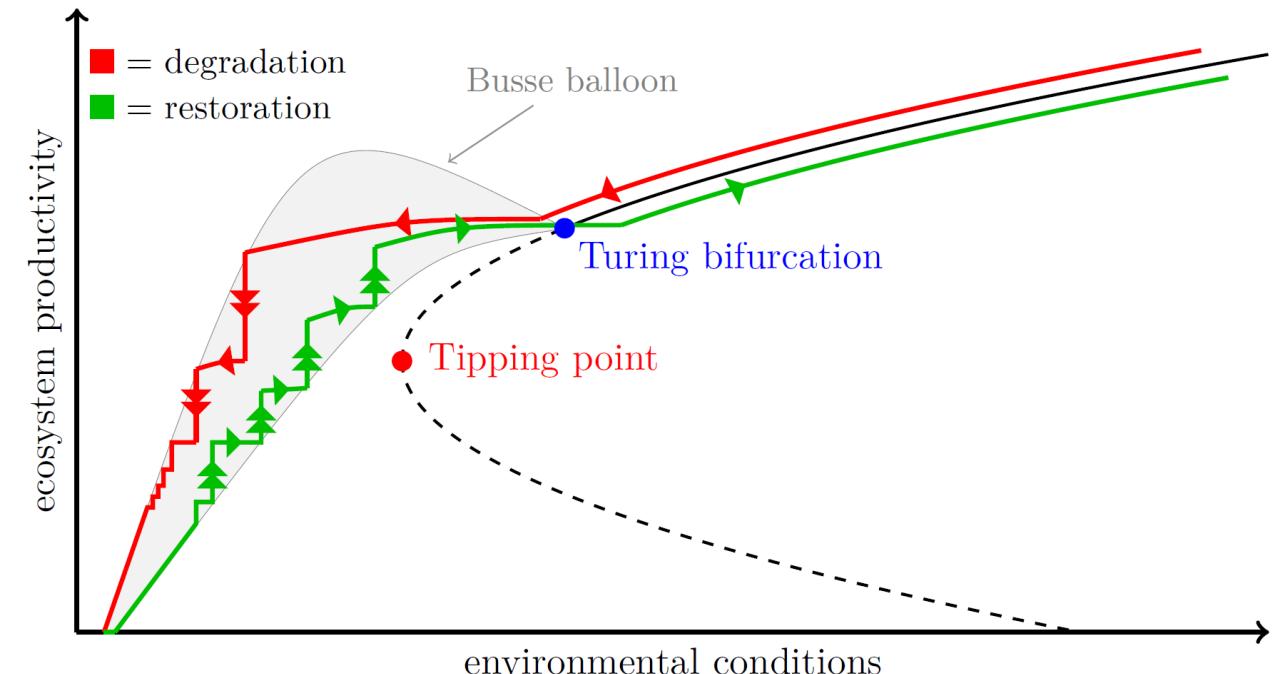
Busse balloon

Idea originates from thermal convection
[Busse, 1978]

Tipping of spatial patterns



Classic tipping



Tipping of patterns

System heterogeneity



Heterogeneous systems

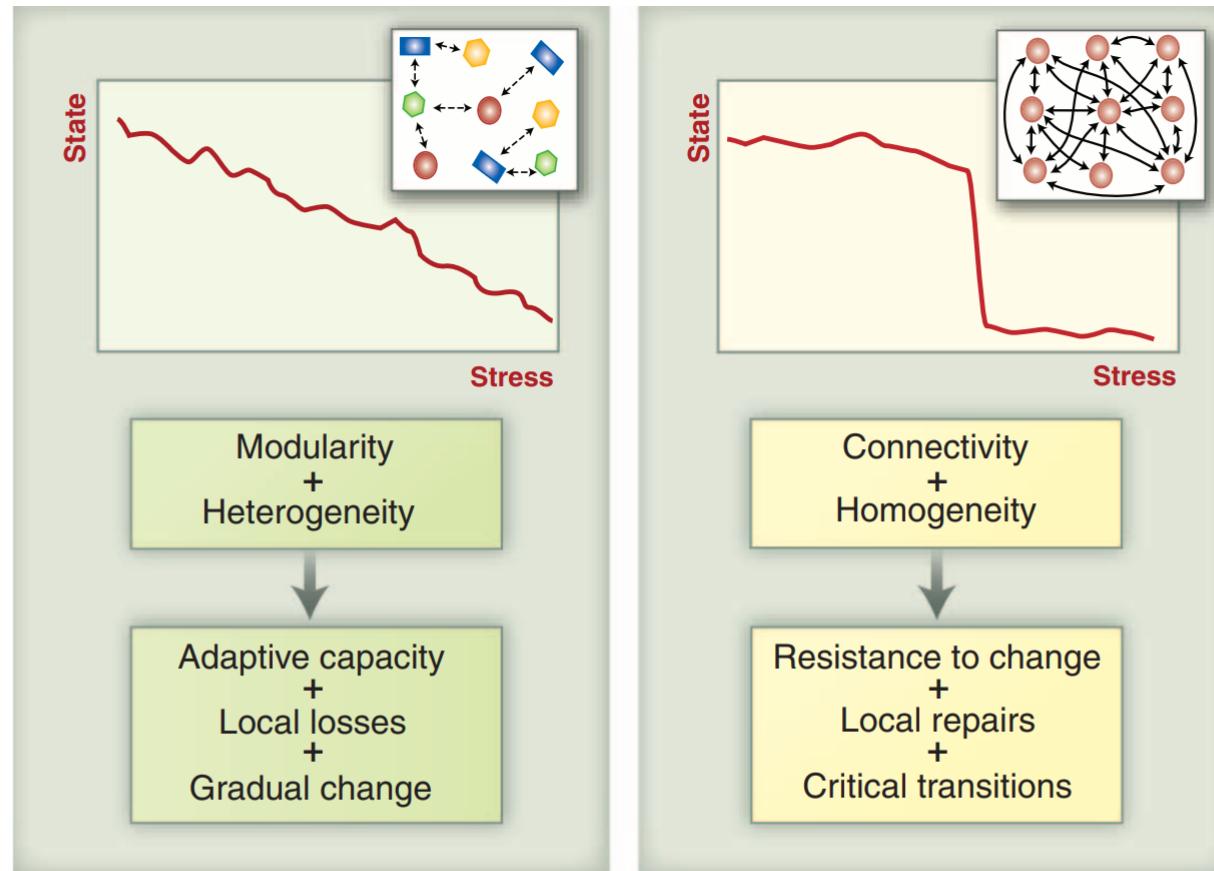
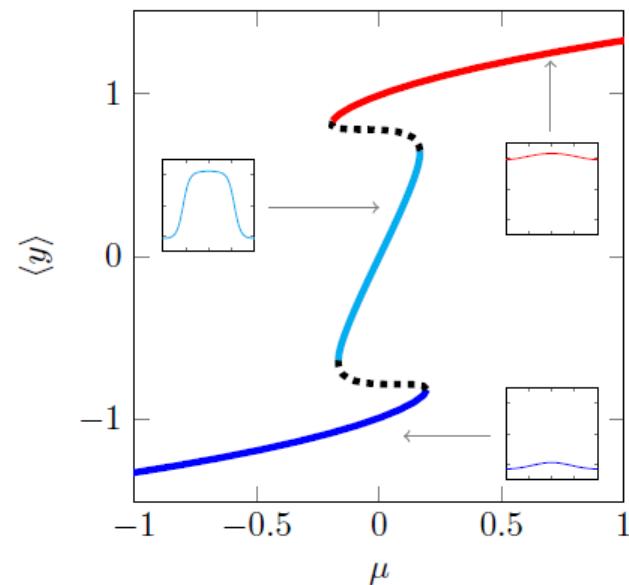
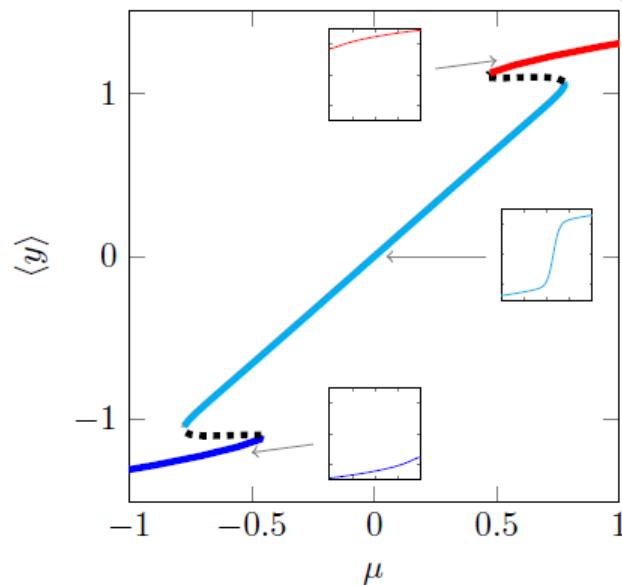


Fig. 1. The connectivity and homogeneity of the units affect the way in which distributed systems with local alternative states respond to changing conditions. Networks in which the components differ (are heterogeneous) and where incomplete connectivity causes modularity tend to have adaptive capacity in that they adjust gradually to change. By contrast, in highly connected networks, local losses tend to be “repaired” by subsidiary inputs from linked units until at a critical stress level the system collapses. The particular structure of connections also has important consequences for the robustness of networks, depending on the kind of interactions between the nodes of the network.

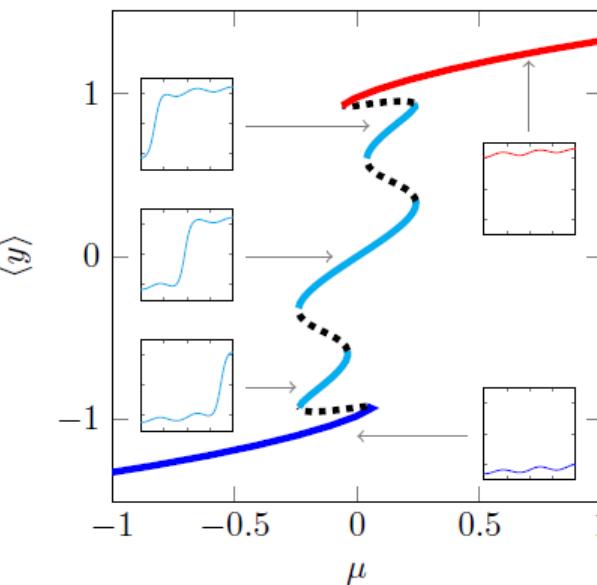
Other Spatial Heterogeneities



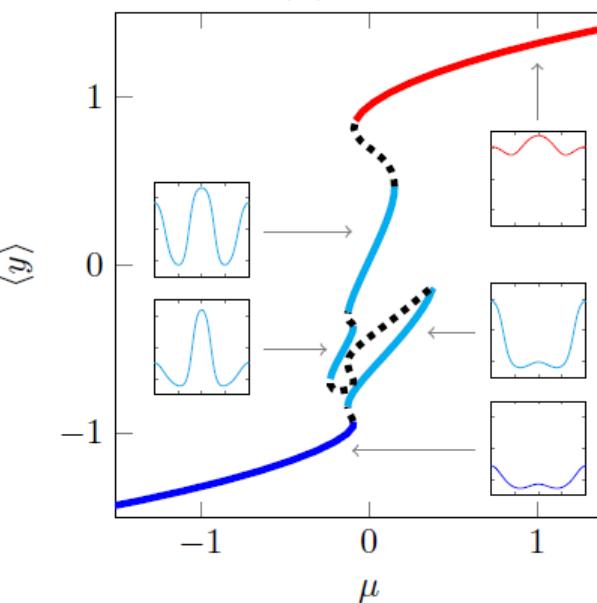
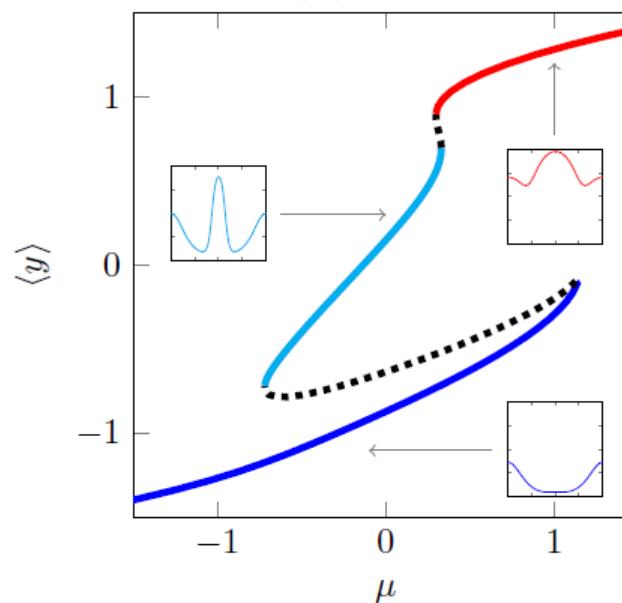
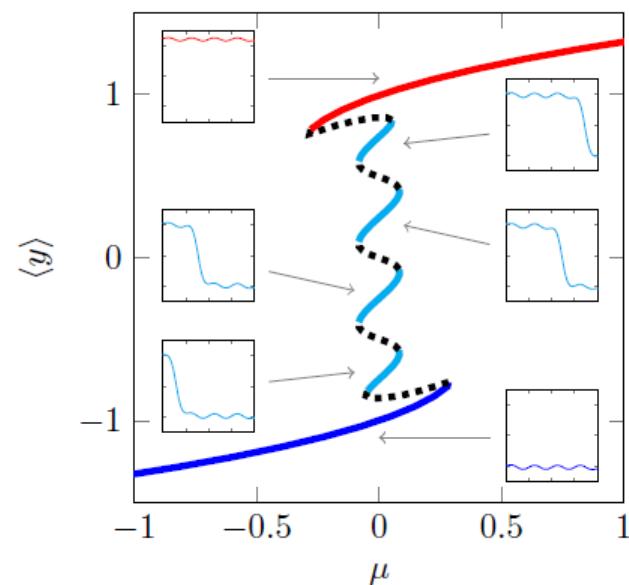
(a)



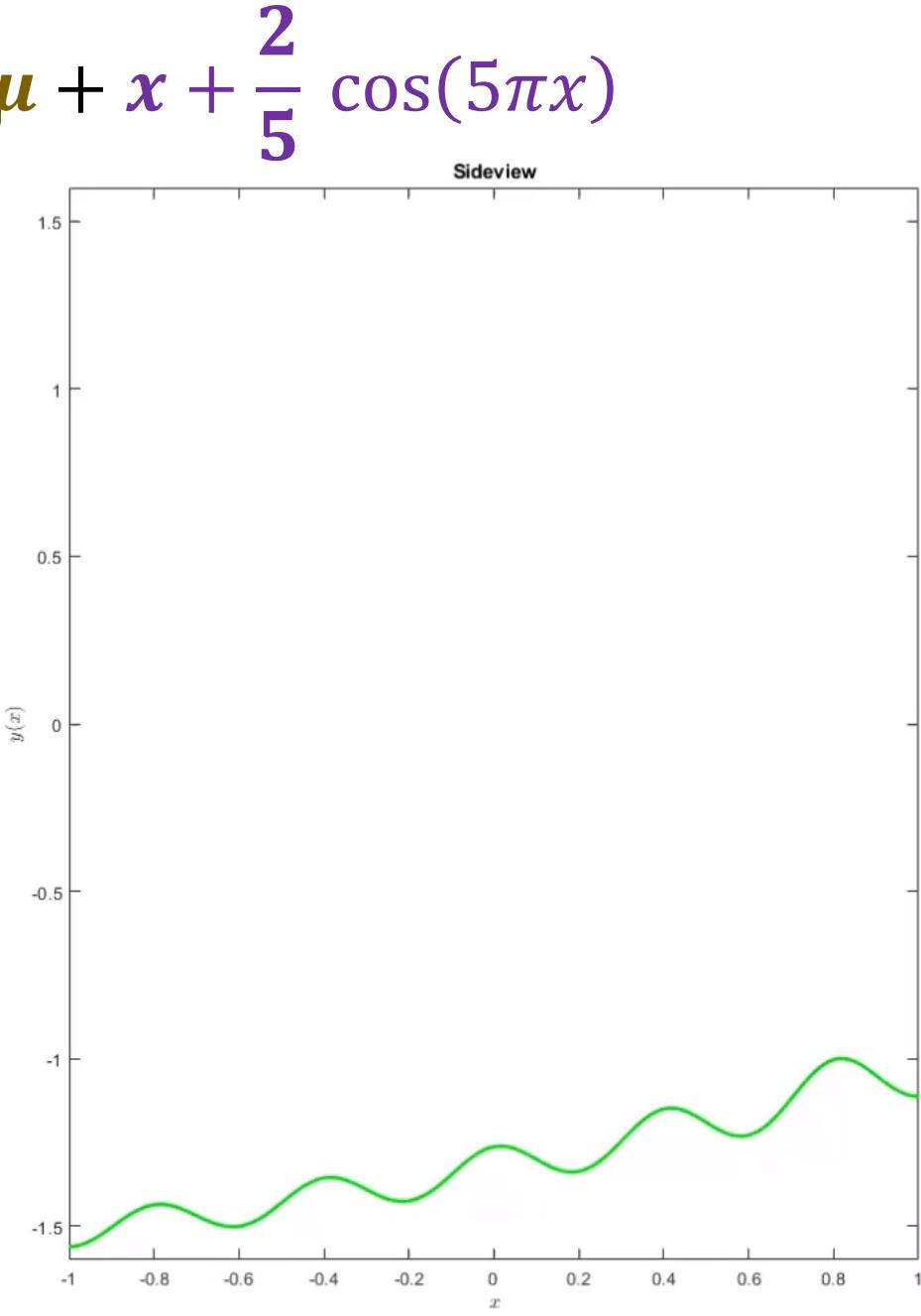
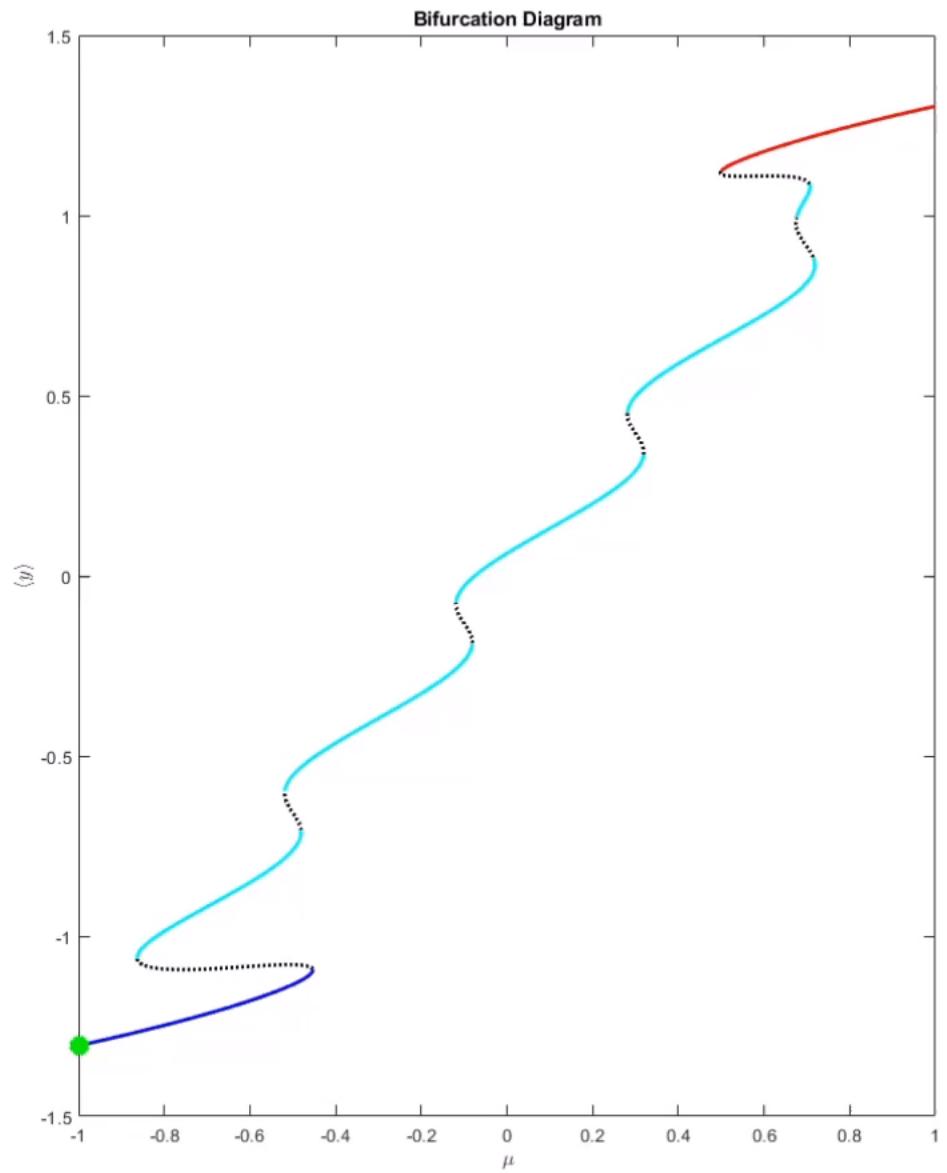
(b)



(c)



$$y_t = D y_{xx} + y(1 - y^2) + \mu + x + \frac{2}{5} \cos(5\pi x)$$



TIPPING BEHAVIOUR IN COMPLEX SYSTEMS

—

AN INTRODUCTION



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CCSS, 2025-09-11

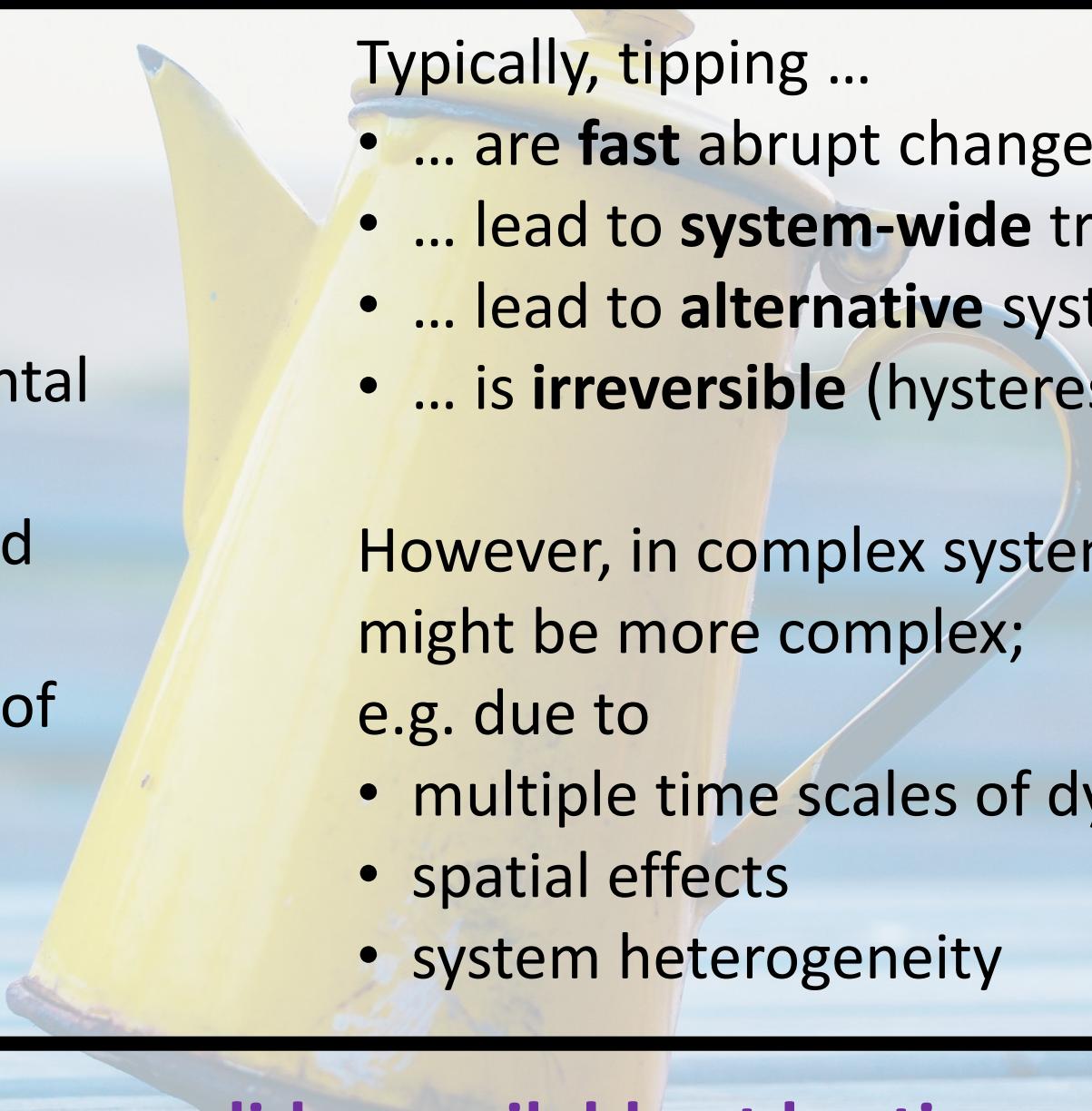


Summary

Tipping can occur in many complex systems

Types of tipping:

- B-tipping: due to incremental parameter shift
- N-tipping: due to noise and state change
- R-tipping: due to fastness of change
- ...



Typically, tipping ...

- ... are **fast** abrupt changes
- ... lead to **system-wide** transition
- ... lead to **alternative** system state
- ... is **irreversible** (hysteresis)

However, in complex systems things might be more complex;

e.g. due to

- multiple time scales of dynamics
- spatial effects
- system heterogeneity

