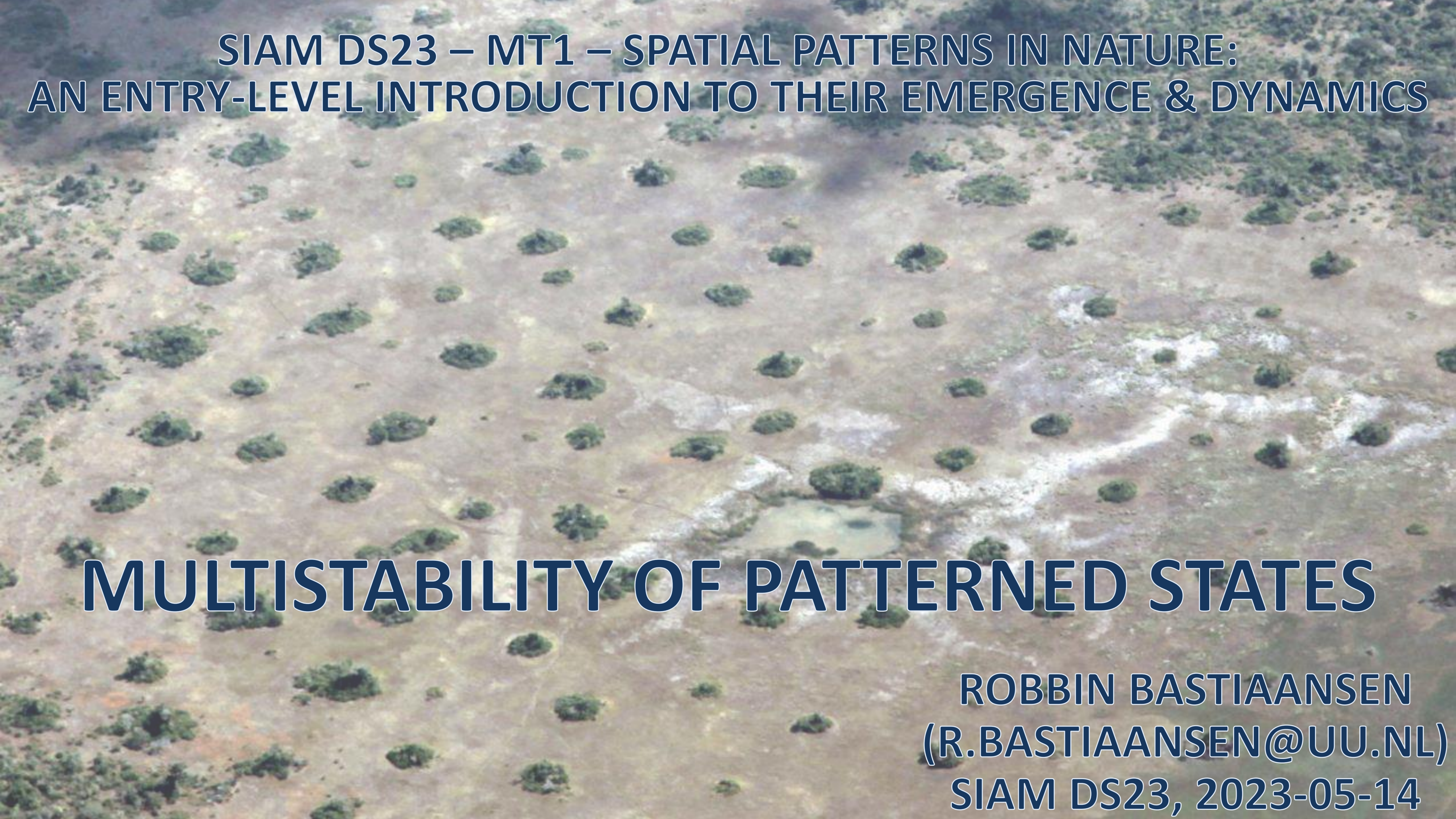


SIAM DS23 – MT1 – SPATIAL PATTERNS IN NATURE: AN ENTRY-LEVEL INTRODUCTION TO THEIR EMERGENCE & DYNAMICS

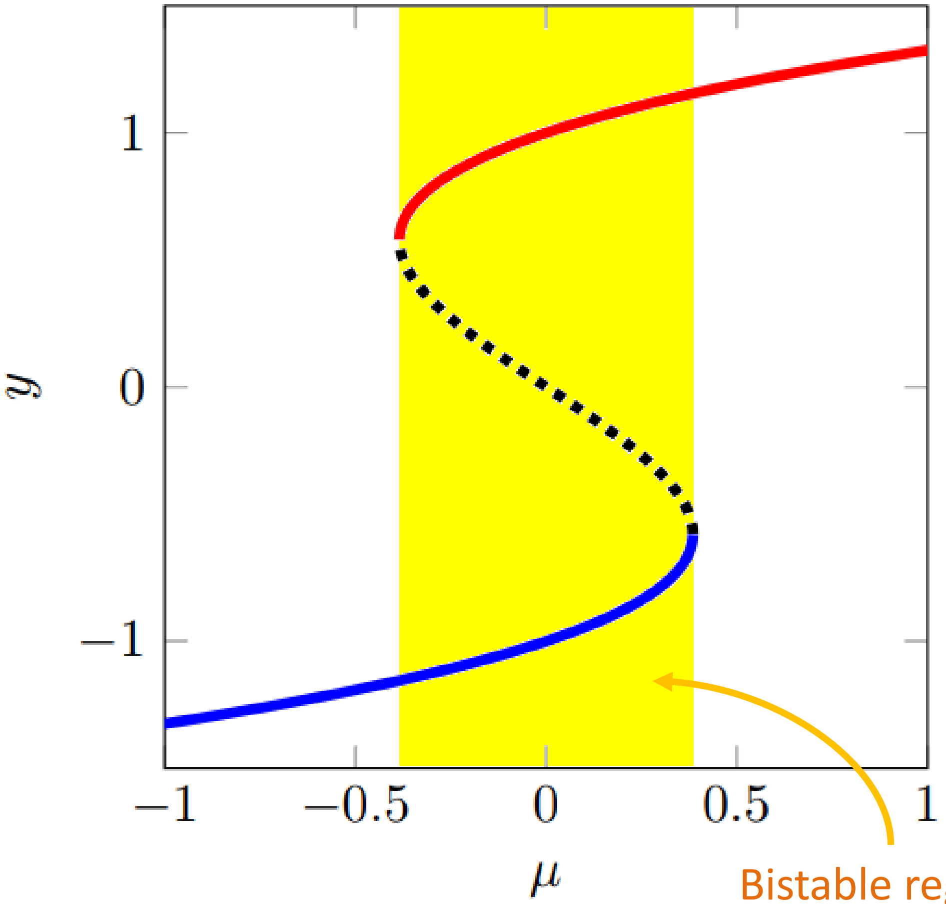
MULTISTABILITY OF PATTERNED STATES

ROBBIN BASTIAANSEN
(R.BASTIAANSEN@UU.NL)
SIAM DS23, 2023-05-14

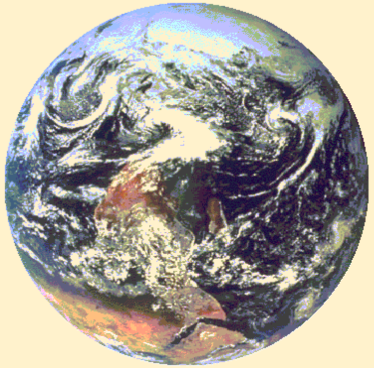
Bistability in dynamical systems

Canonical example:

$$\frac{dy}{dt} = y(1 - y^2) + \mu$$



Bistable regime



Planetary transitions

Ecosystem shifts

Bistability

2 different stable states

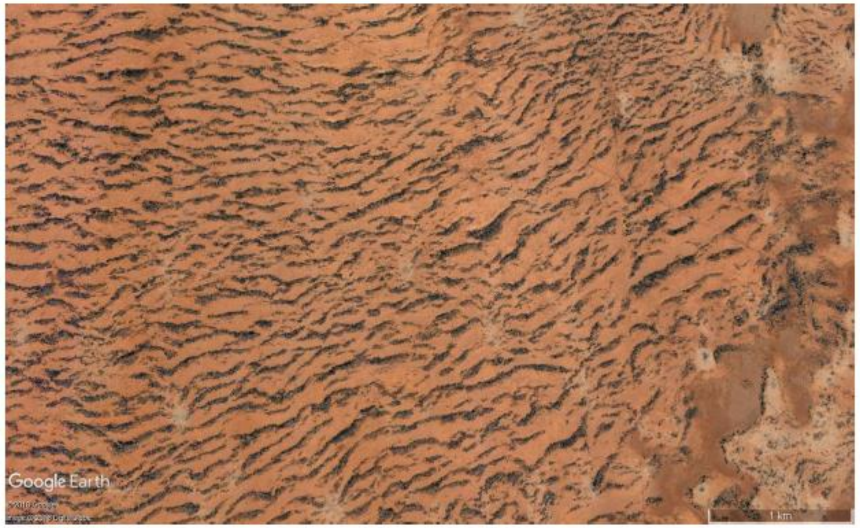
Each has different properties and characterstics

Tipping Points

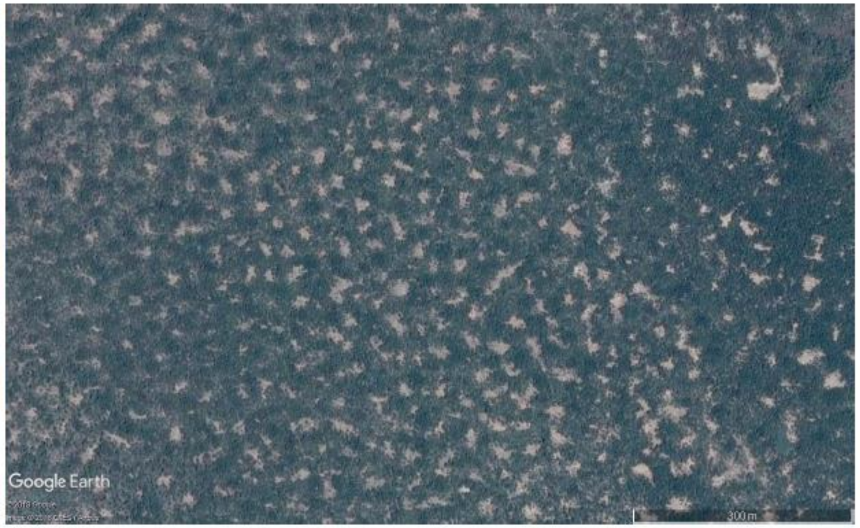
Transitions between alternative states

Bistability is found in many systems!

Example system: dryland ecosystems

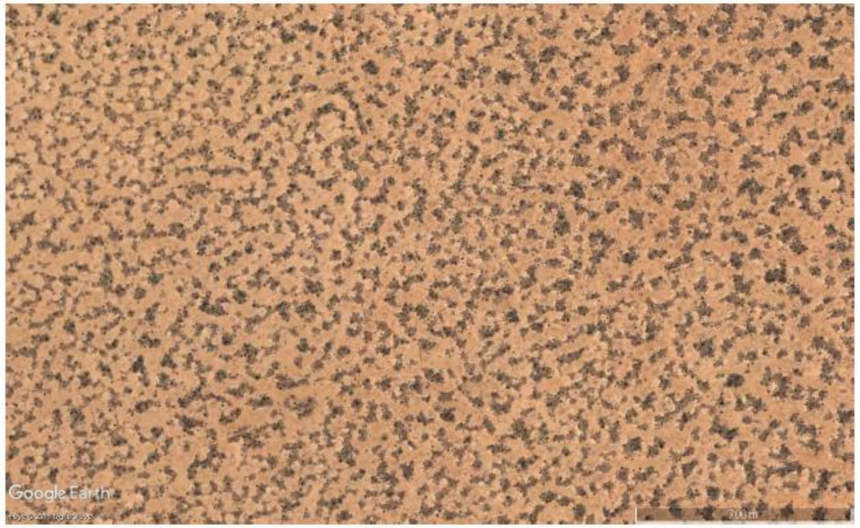


(a) Bands in Somalia



(b) Gaps in Niger

(c) Spots in Zambia



(d) Maze in Sudan

Example system: dryland ecosystems

Extended-Klausmeier model

$$w_t = w_{xx} + (h_x w)_x - w + a - wv^2$$
$$v_t = D^2 v_{xx} - mv + wv^2$$

w : water

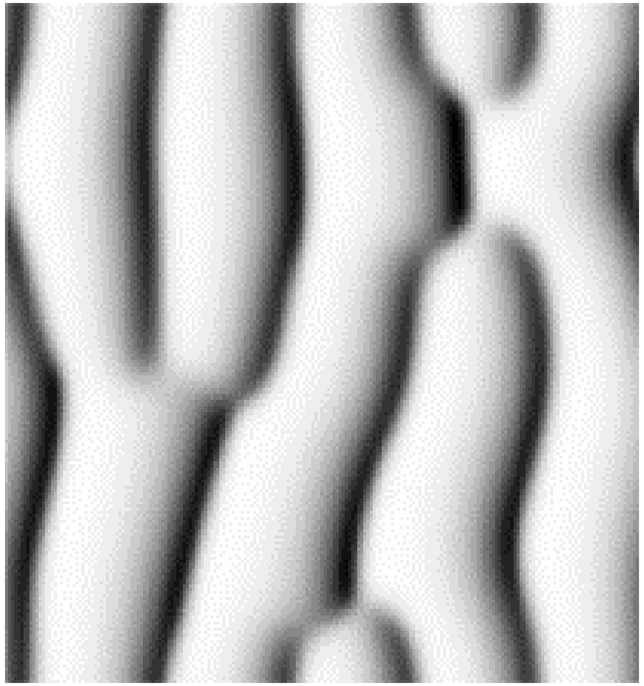
v : vegetation

h : height

D : ratio of diffusion

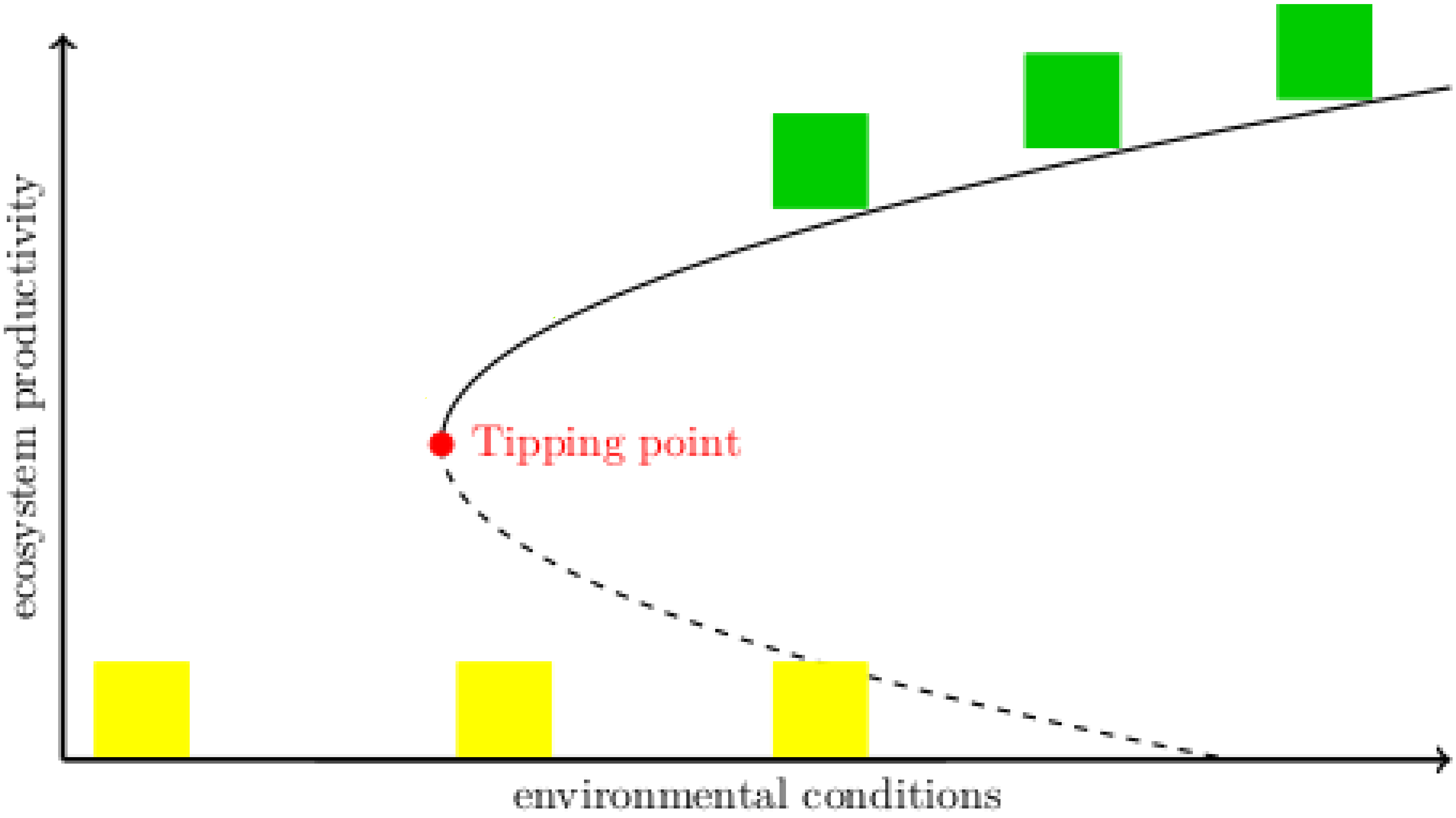
a : rainfall

m : mortality

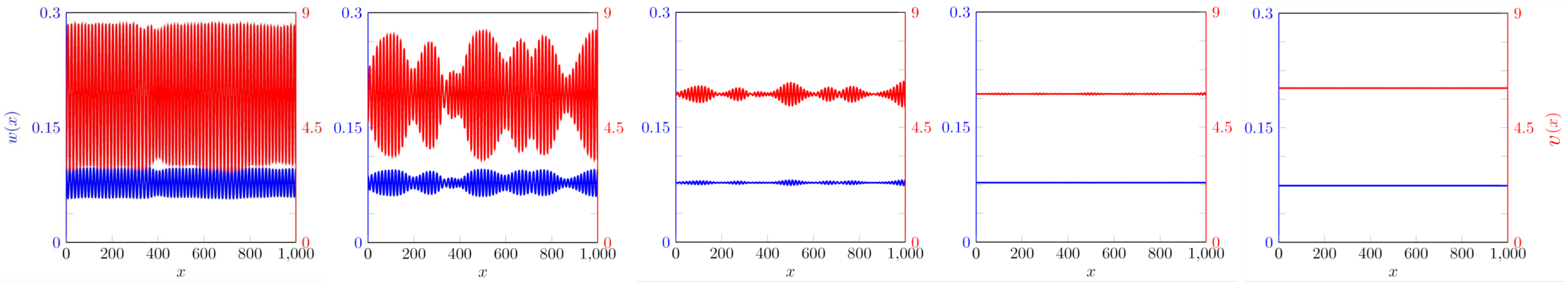


[Klausmeier, 1999]

Bifurcation diagram of non-spatial model



The orgin of patterns in extended-Klausmeier model



Low rainfall

Critical rainfall
Onset of patterns

High rainfall

Turing Patterns [Turing, 1952]

Found in most reaction-diffusion equations

Patterns after Turing bifurcation

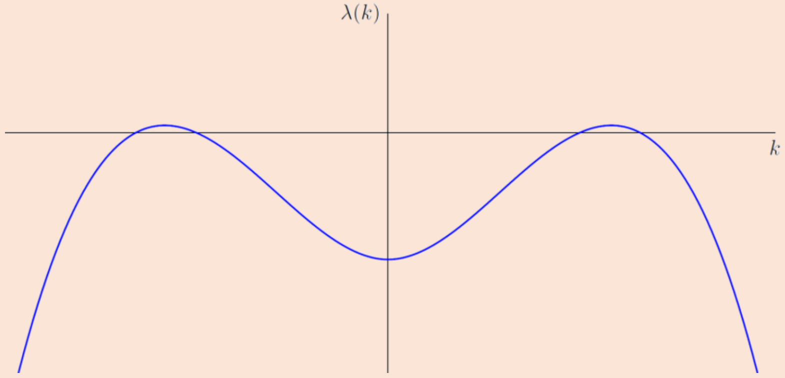
Emerging patterns

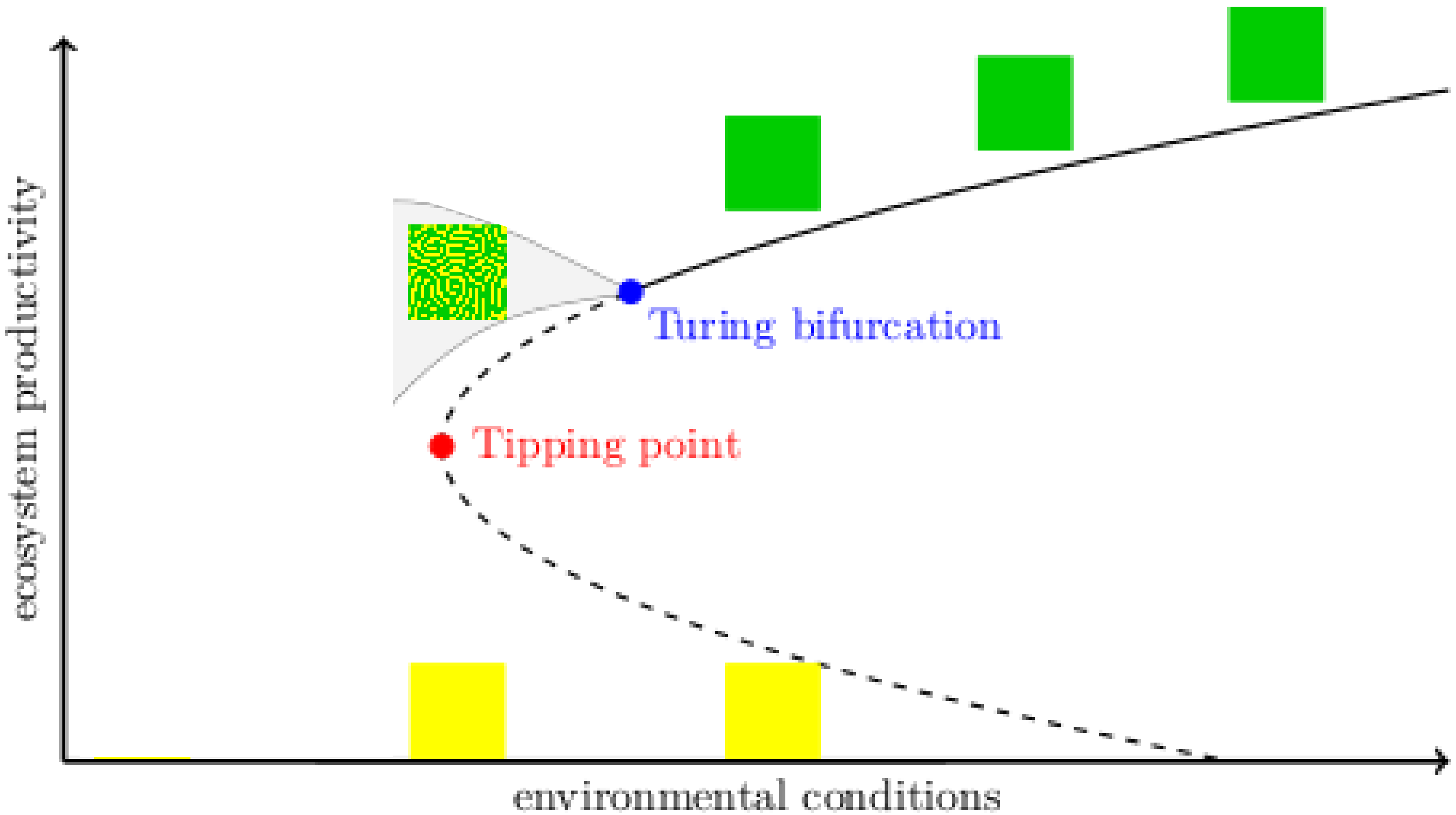
Instability to non-uniform perturbations

$$\binom{u}{v} = \binom{u_*}{v_*} + e^{\lambda t} e^{ikx} \binom{\bar{u}}{\bar{v}}$$

→ Dispersion relation

$$\lambda(k) = \cdots$$





Weakly non-linear analysis

Ginzburg-Landau equation / Amplitude Equation

& Eckhaus/Benjamin-Feir-Newel criterion

[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Busse balloon

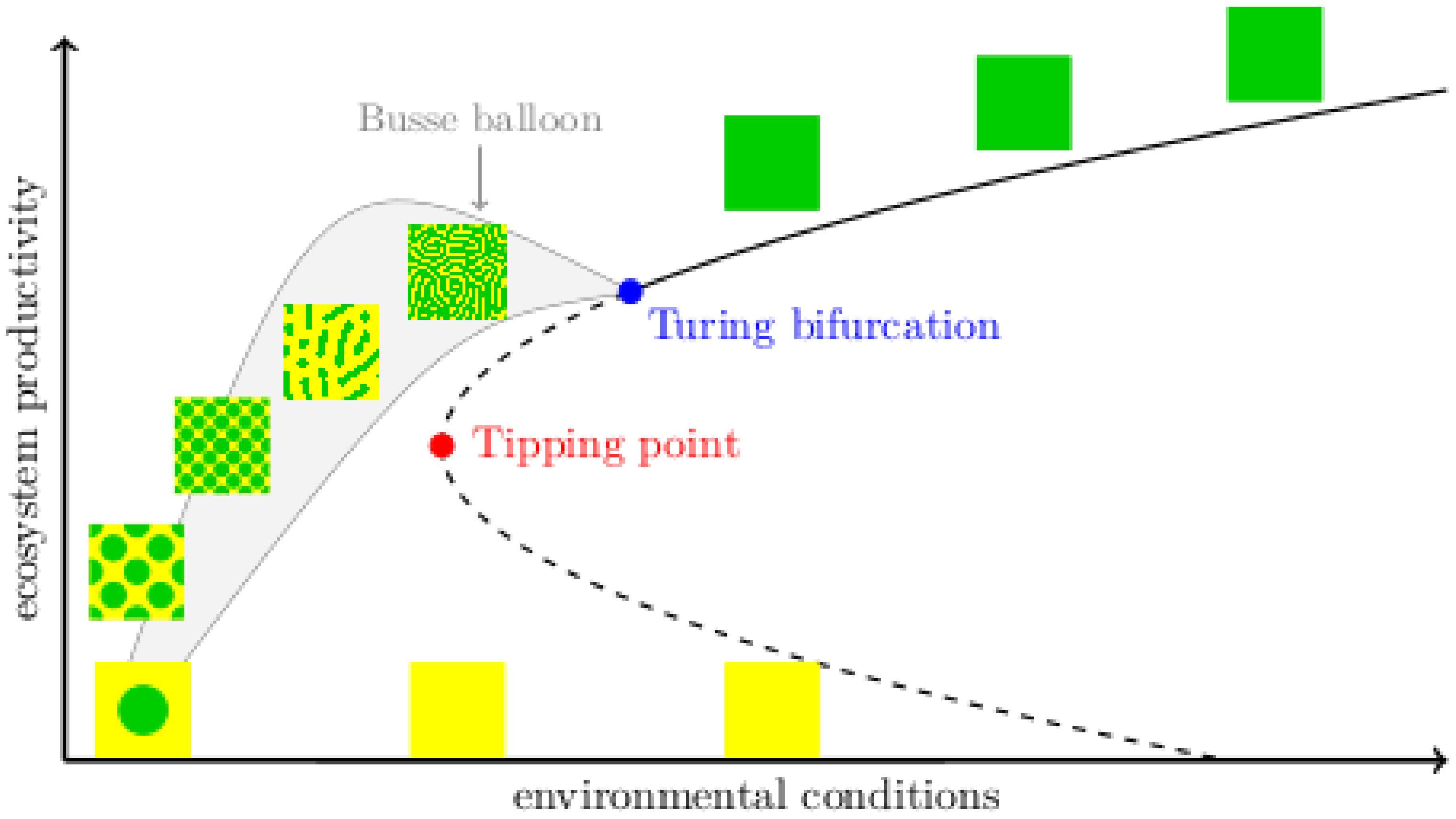
Busse balloon

A model-dependent shape in *(parameter, observable)* space that indicates all stable patterned solutions to the PDE.

Construction Busse balloon

Via numerical continuation

few general results on the shape of Busse balloon



Busse balloon

Idea originates from thermal convection

[Busse, 1978]

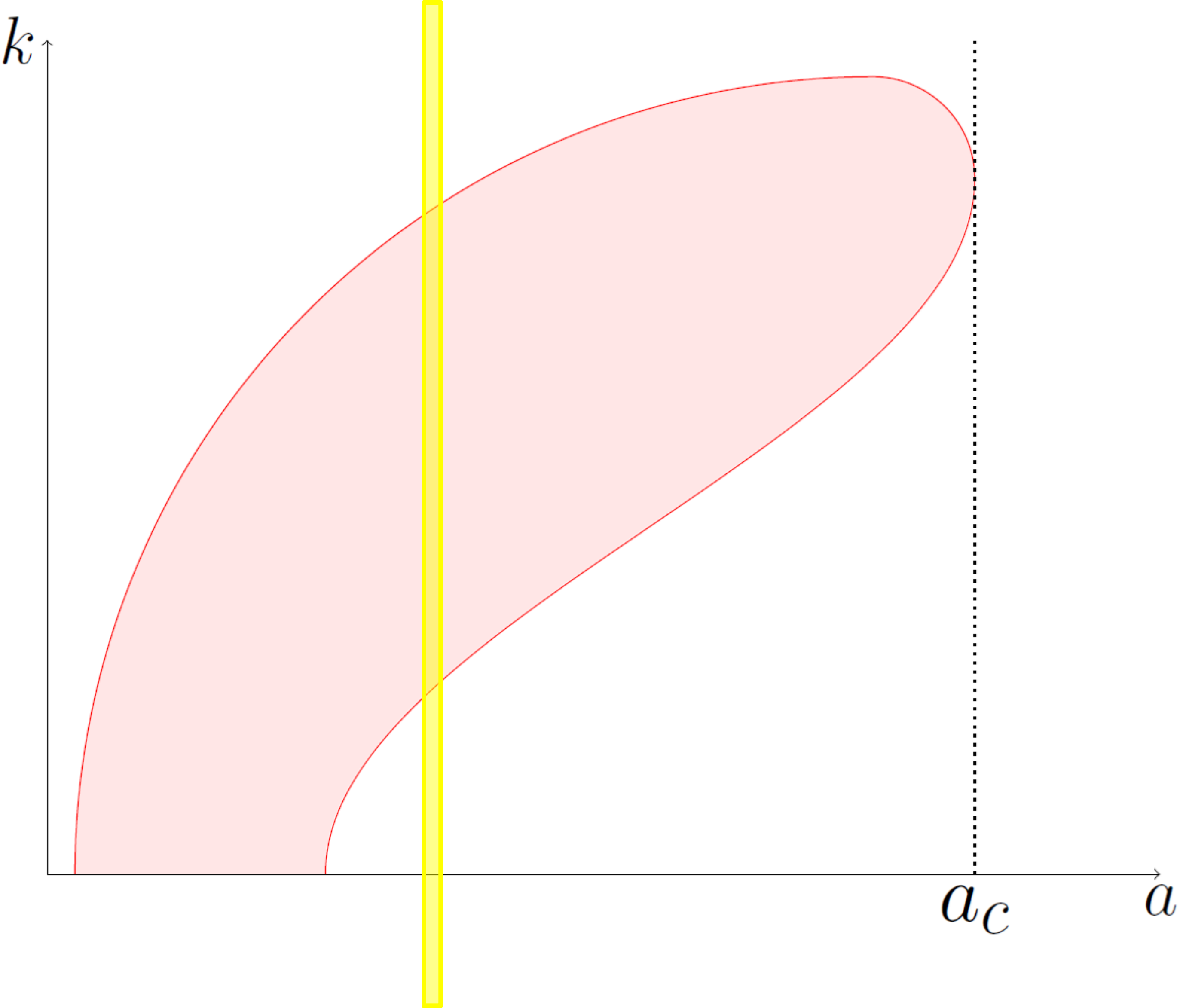
Rayleigh Bénard thermal convection

Busse balloon [Busse, 1978]

A *Busse balloon* is a model-dependent shape in (*parameter*, *wavenumber*)-space that indicates all combinations of parameter and wavenumber that represent stable solutions of the model.

Video source: wikiRigaou (wikimedia commons)

Multistability in the Busse balloon



<u>Observation 1:</u>

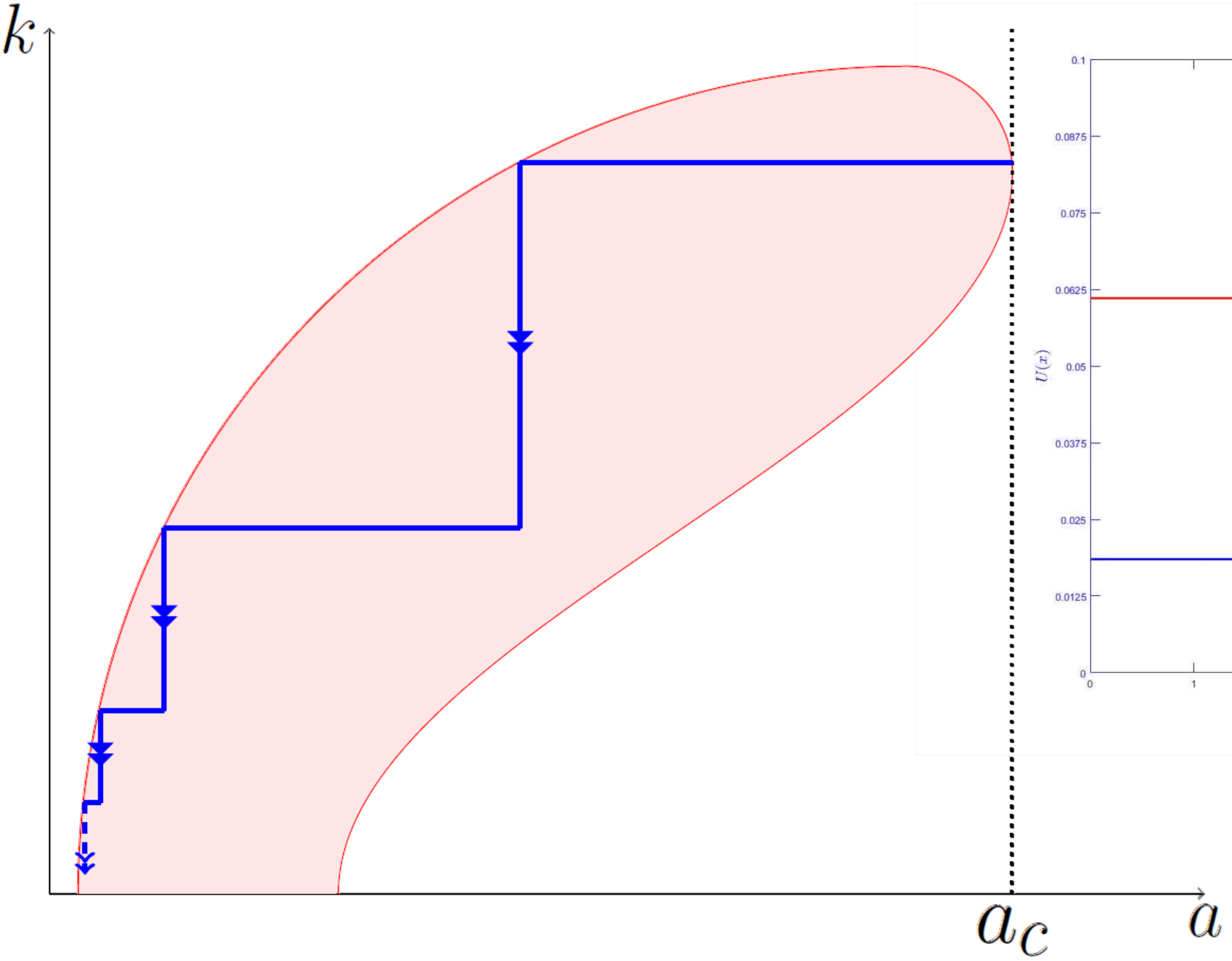
For a fixed parameter value, there is a **continuous** range of wavenumbers possible.

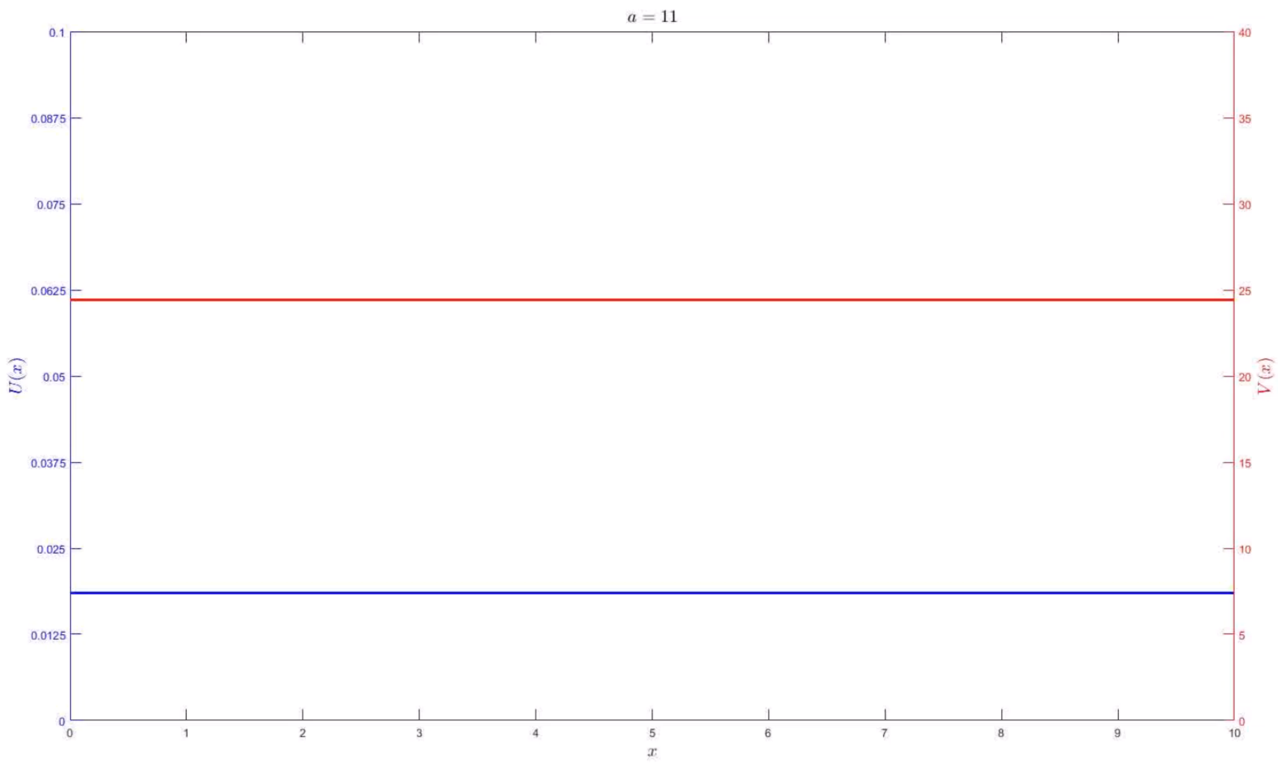
That is, there is a large **multistability** of stable pattern states to the PDE

<u>Consequence 1:</u>

Specifying only parameter values is ambiguous, as it does not correspond to only one patterned state.

A Walk through the Busse balloon



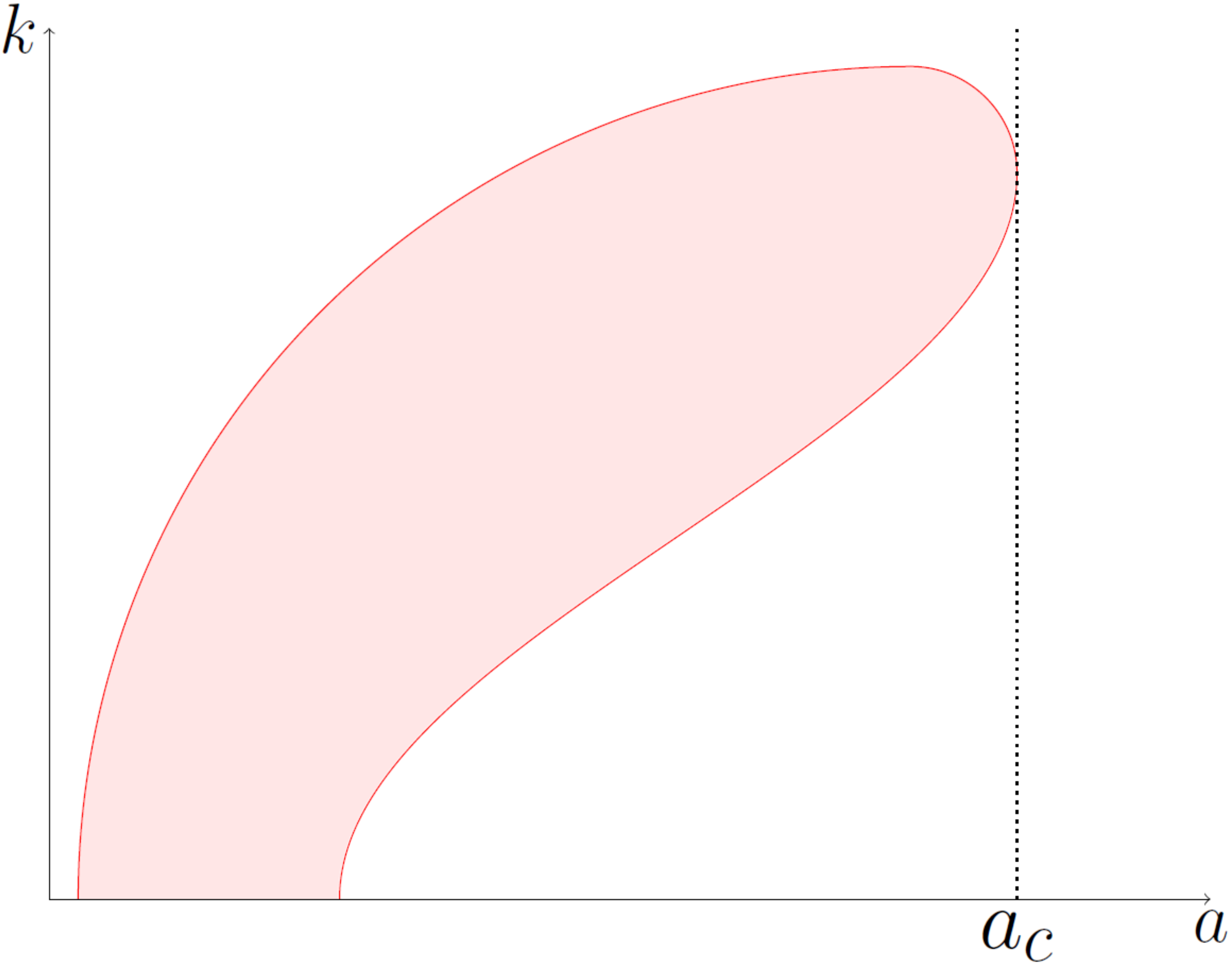


More in Part 2 of this minitutorial

<u>Consequence 2:</u>

In patterned shifts, you see smaller transitions from one patterned states to another

The shape of Busse balloon

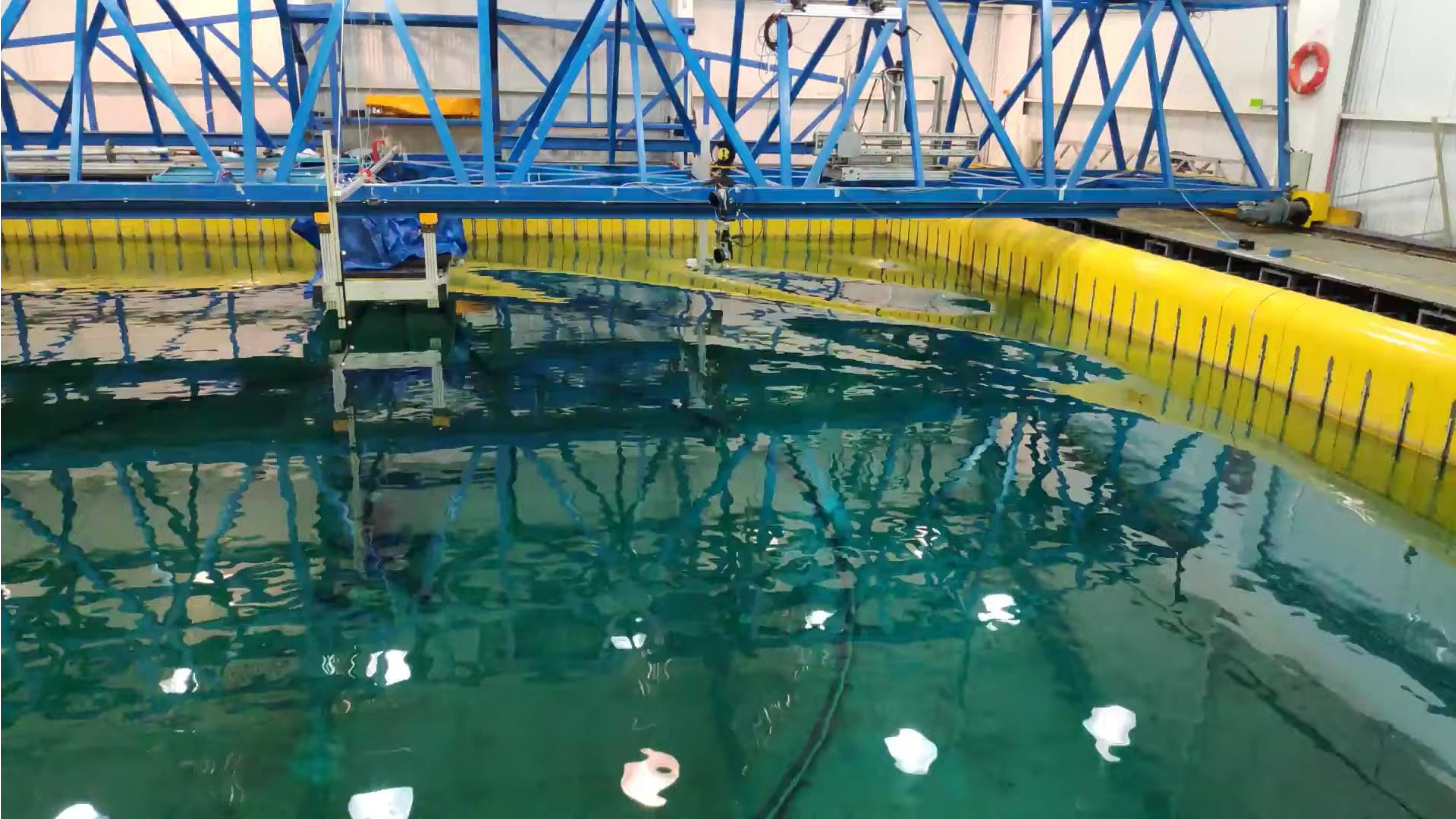


Only few generic results

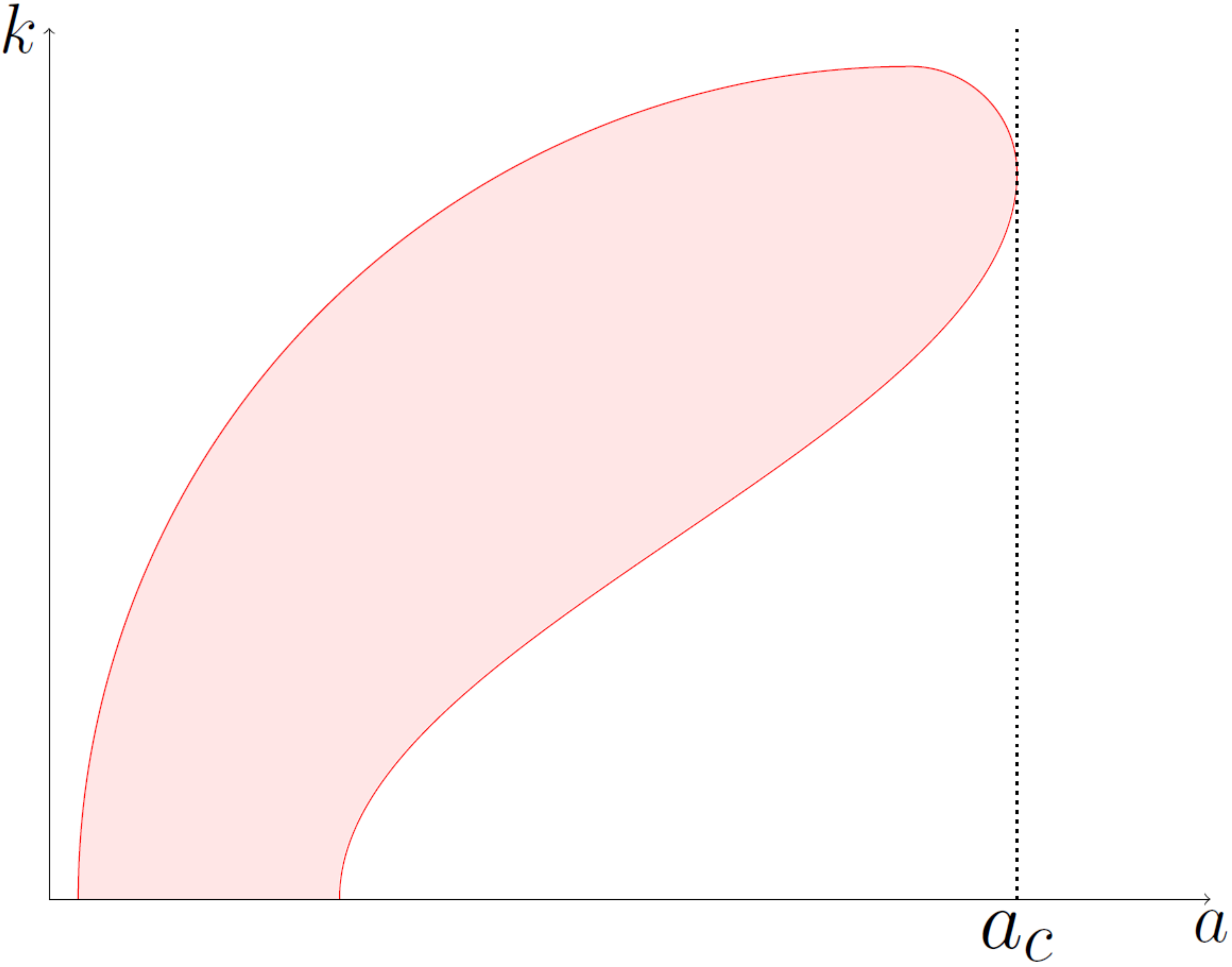
Shape is **model**-dependent

Shape is **domain**-dependent

Models on finite domains:
Shape breaks up into finitely many lines



The shape of Busse balloon



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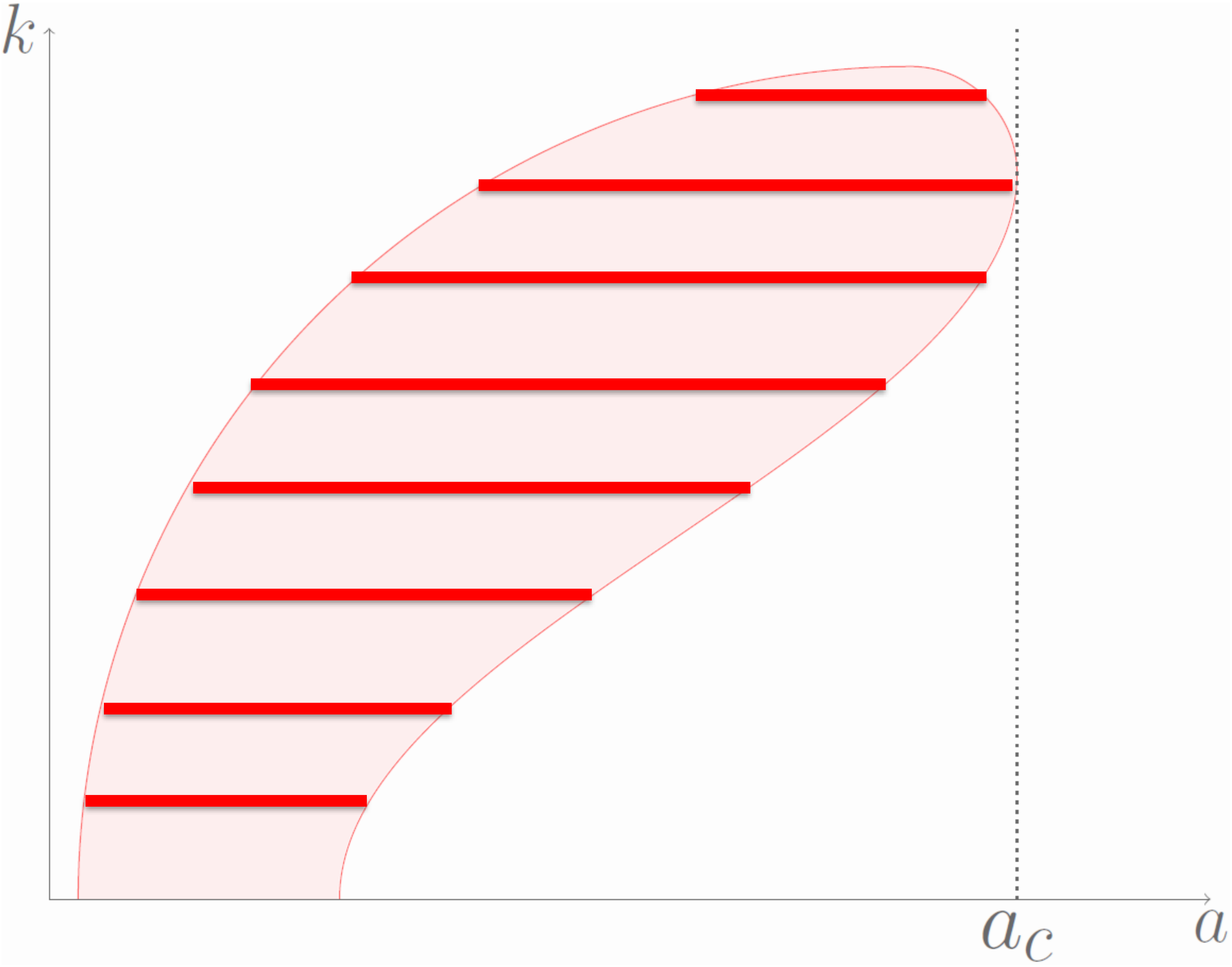


Only few generic results

Shape is **model**-dependent

Shape is **domain**-dependent

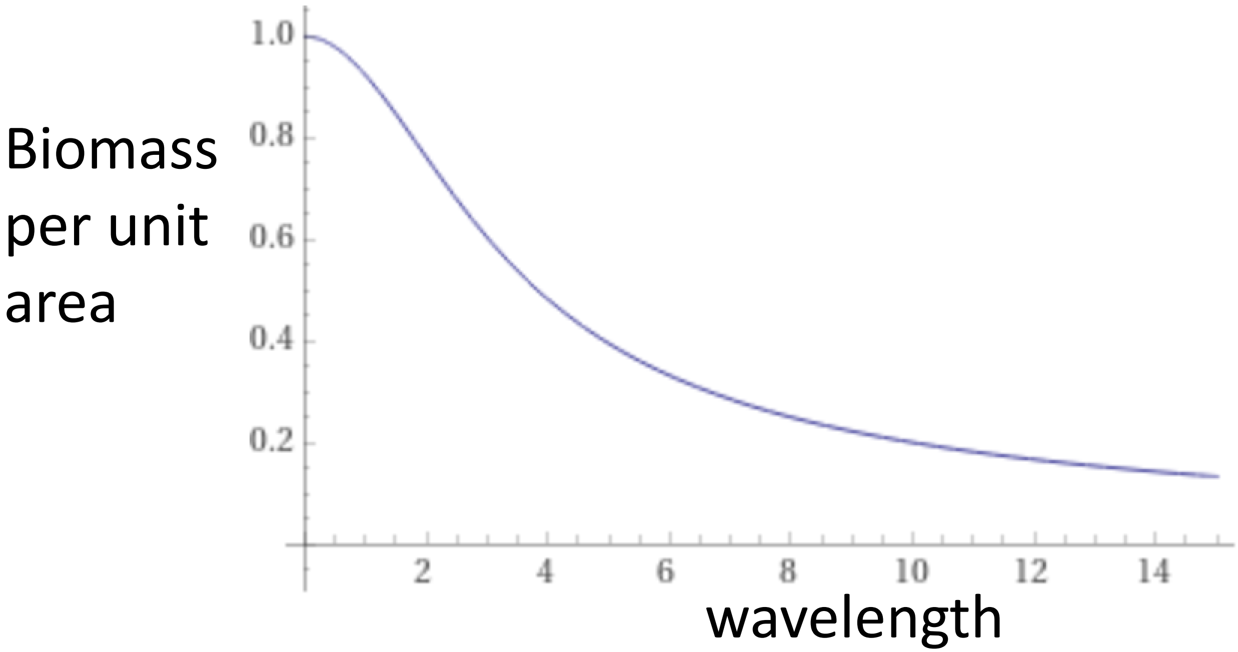
Models on finite domains:
Shape breaks up into finitely many lines

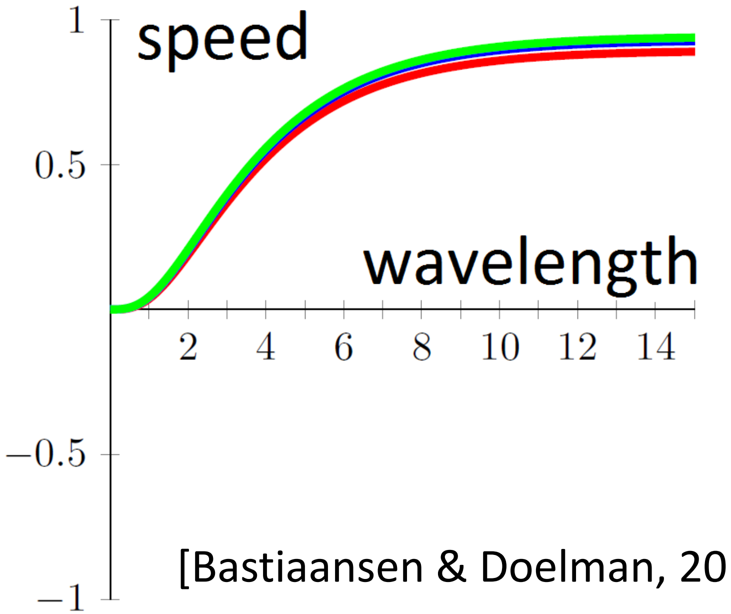
"Quantization of Busse balloon"

Observables in multistable systems

Observation 2:

The value of an observable depends on the parameters and the precise patterned state





[Bastiaansen & Doelman, 2019]

Figures can be made using the same techniques as will be explained by Peter next

Multistability in real patterned systems?

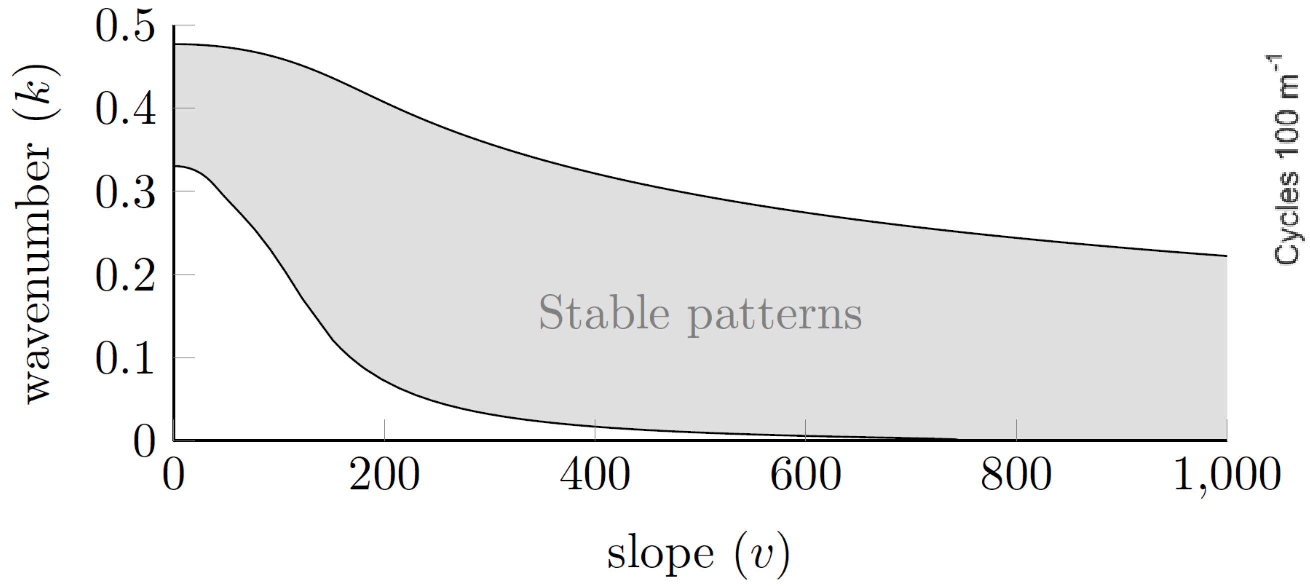
Comparison with sites in Horn of Africa

- Environmental conditions constant within site
- Topography main environmental variation

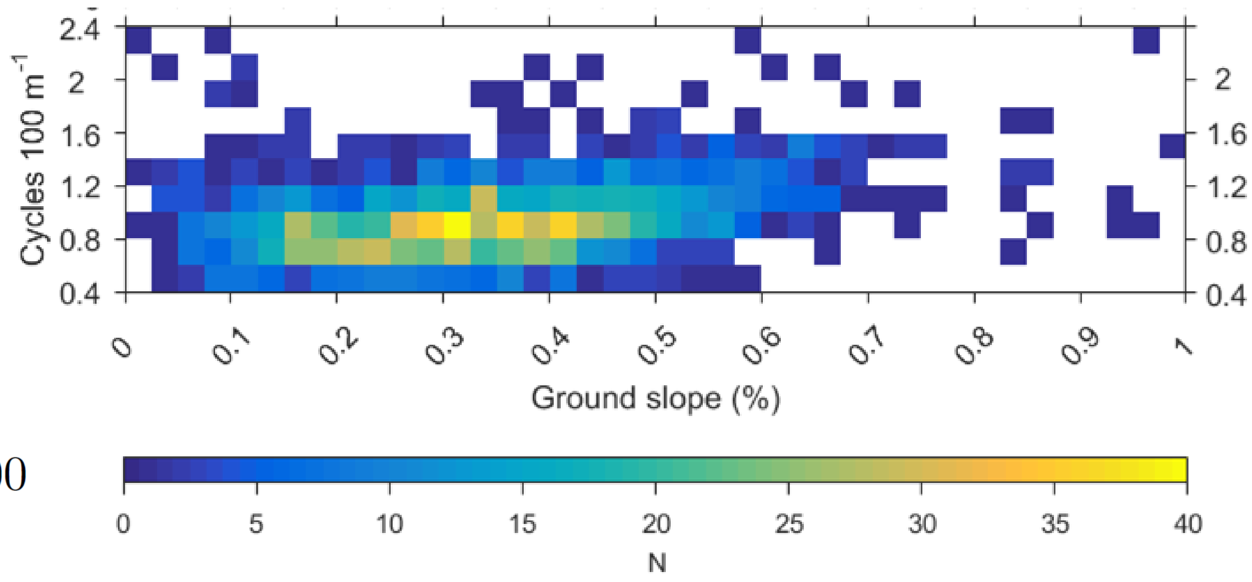
Method

- Divide site into smaller windows
- Per window, find wavenumber, biomass and migration speed

Busse balloon in dryland ecosystems



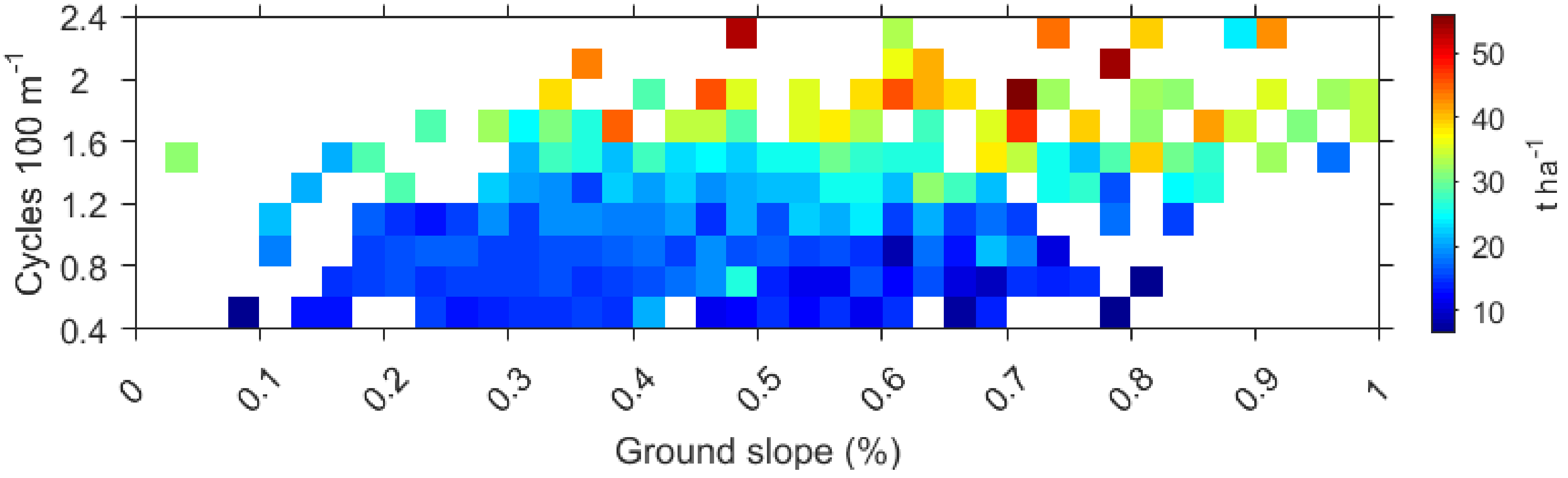
extended-Klausmeier model



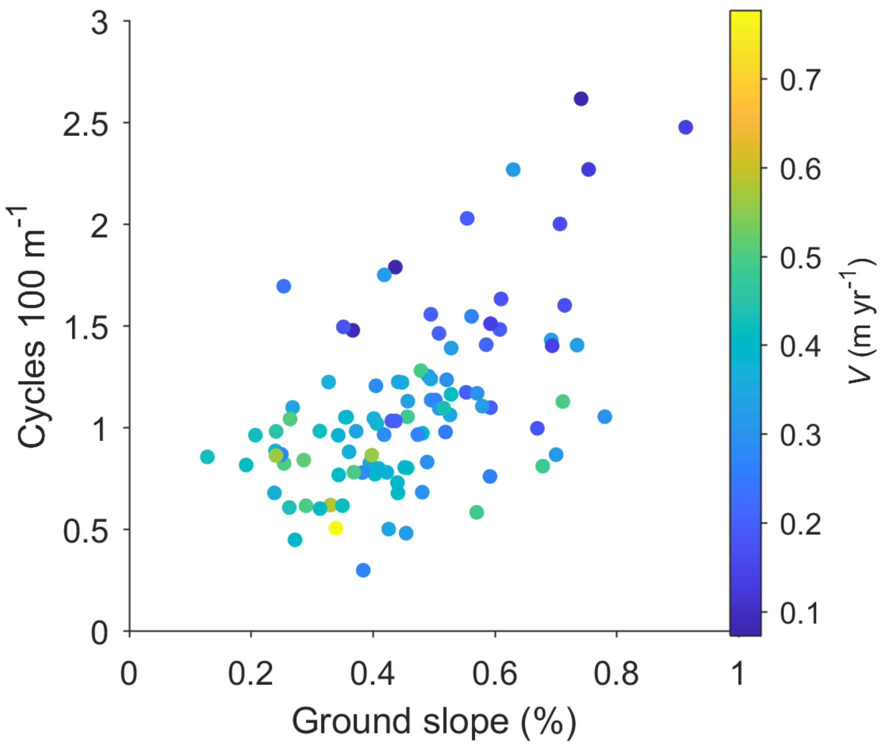
Somalia data

Wide wavenumber spread in both

Biomass and migration speed



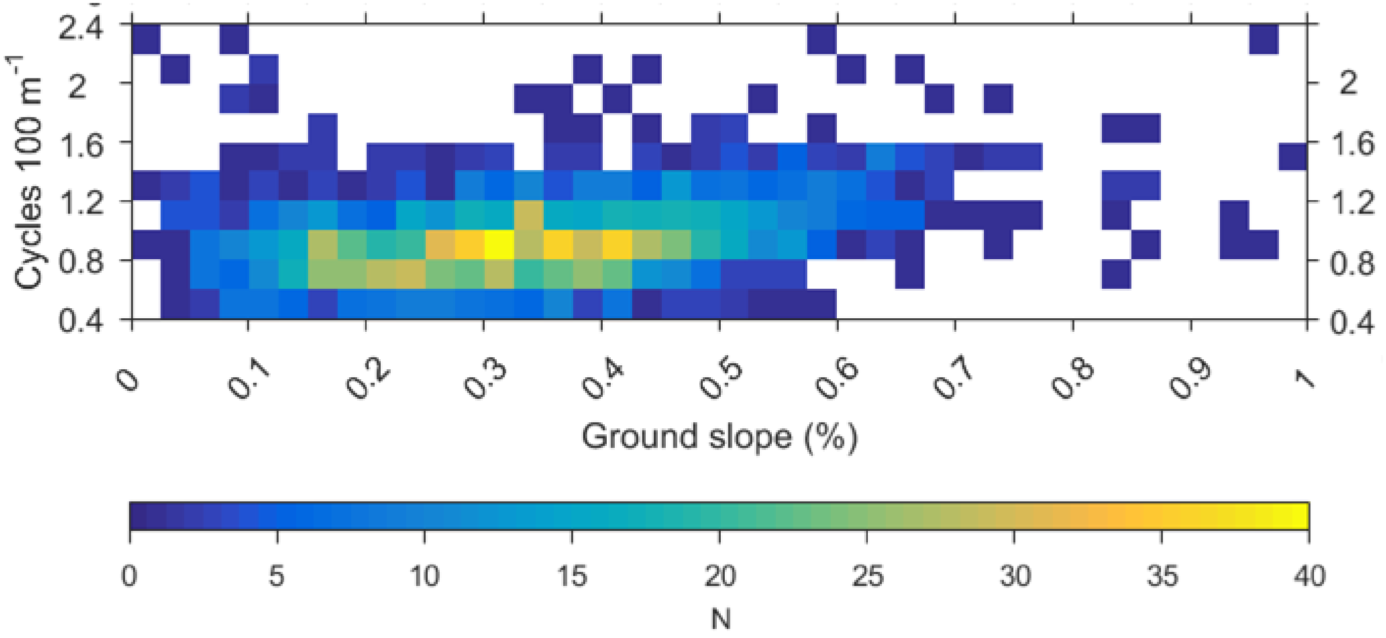
Biomass data



Migration speed data

Biomass and migration speed change with wavenumber

BUT: why not just averaging?



Looks like there is just a preferred patterned for each slope value

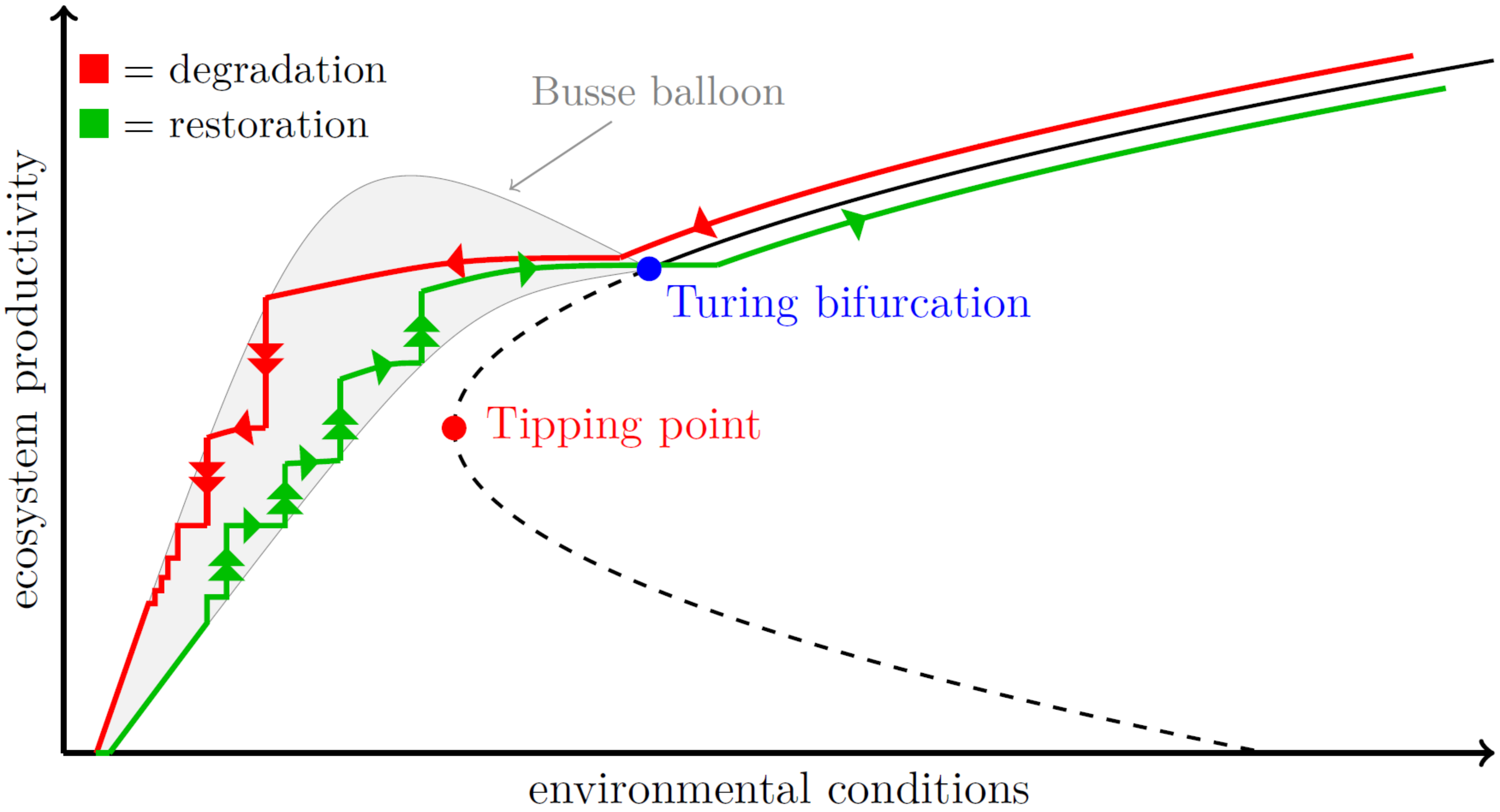
Two reasons:

1. You lose information on the pattern-dependence
2. You do <u>NOT</u> gain information about <u>THE</u> preferred pattern for the given parameters

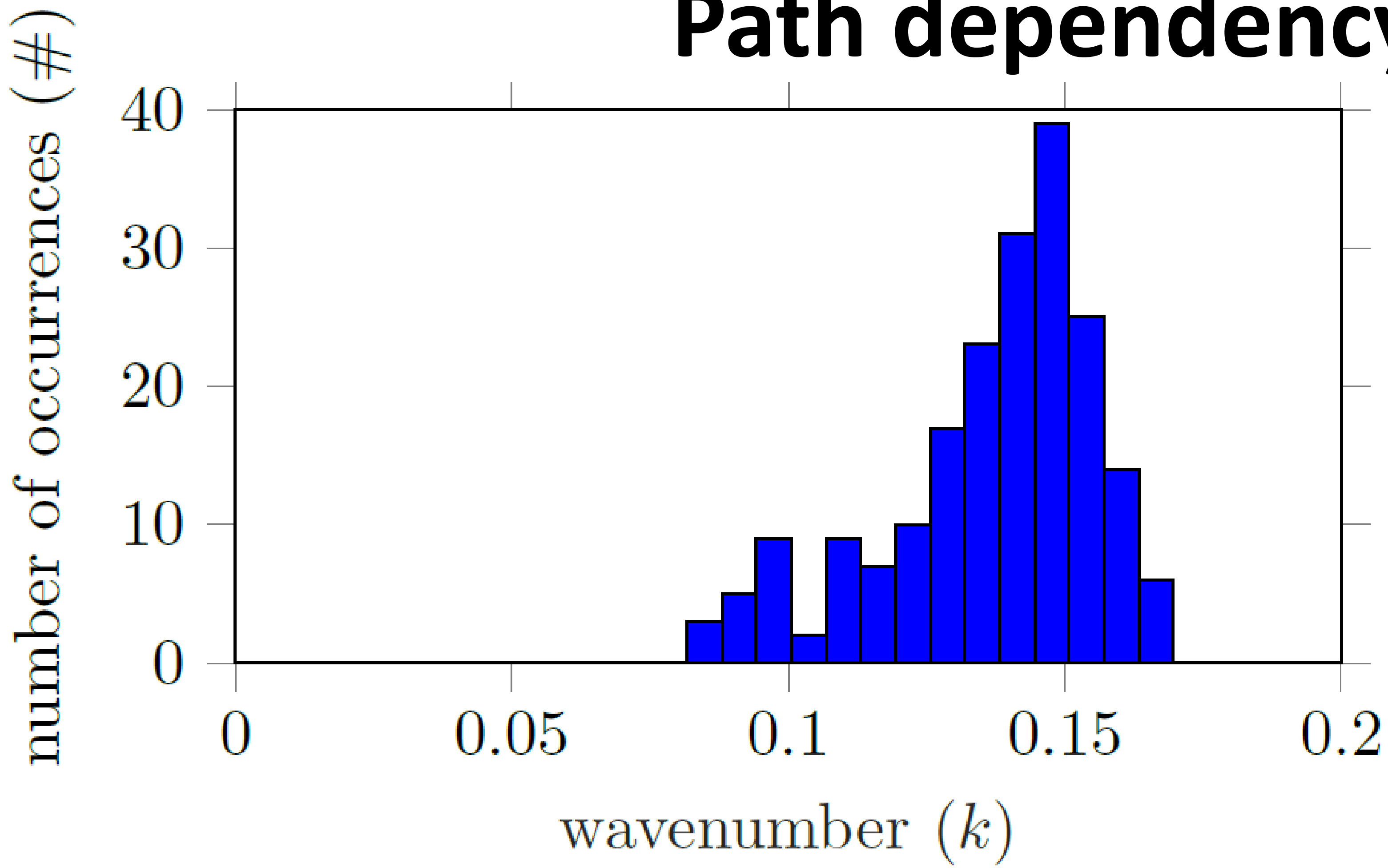
<u>Recommendation 1:</u>

Do NOT average over all observed states without thinking!

Path dependency



Path dependency



Example of wavenumber spread in model in a decreasing rainfall scenario

Recommendation 2:
Think about creation and history of patterns

Consequence 3:
Pattern distribution depends on both the current parameters and the path taken to get there ('history dependency')

In theory, you should be able to get insight into the past by looking at the current distribution.
[Sheratt (2014), PNAS]

Summary - Multistability of patterned states

Observations:

1. In models and real patterned systems there is a large **multistability** of stable pattern states
2. The value of an observable depends on the parameters and the precise patterned state

Consequences:

1. Specifying only parameter values is ambiguous, as it does not correspond to only one patterned state
2. In patterned shifts, you see smaller transitions from one patterned states to another
3. Pattern distribution depends on both the current parameters and the path taken to get there (‘history dependency’)

Recommendations:

1. Do NOT average over all observed states without thinking!
2. Think about creation and history of patterns

Slides available at:
bastiaansen.github.io/
MTpatterns/patternMT
.html