Systems and control theory: a short crash course

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Introduction and background: some systems theory

Systems theory: a dynamical system in interaction with the environment.
Typically, there are variables that can be manipulated (inputs) and variables that are observed (outputs), and a dynamical system that transforms one to the other.

Examples:

$$\begin{cases} x'(t) & = f(x, u, t) \\ y(t) & = g(x, u, t) \end{cases}$$

where $u(t)$ is the input, $y(t)$ is the output and $x(t)$ is the so-called state variable.

Introduction and background: some systems theory

The functions f and g may be very complicated, may involve all kinds of dynamics, and the dynamical system also needs not be a continuous time system. The dynamics could be driven by difference equations, could be driven by discrete event systems, may be infinite dimensional and coefficients may be time varying.

Simplest example: time-invariant linear finite dimensional causal systems. These may be *realized* as follows:

$$\begin{cases} x'(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$$

for some matrices A, B, C, D of appropriate sizes.

Introduction and background: some systems theory

Linear time invariant system:

$$x'(t) = Ax(t) + Bu(t), \qquad x(0) = x_0.$$

Here A is an $n \times n$ matrix, $x \in \mathbb{R}^n$, B is an $n \times m$ matrix, $u \in \mathbb{R}^m$. x is called the *state*, u is called the *input*.

Solution of the system:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,ds.$$

Denote this by $x(t; u, x_0)$.

System is called *stable* if all solutions of $x'(t) = Ax(t)$ tend to zero when $t \to \infty$. Equivalently, all eigenvalues of A are in the open left half plane.

Introduction and background: controllability

Linear system:

$$x'(t) = Ax(t) + Bu(t), \qquad x(0) = x_0.$$

A is an $n \times n$ matrix, $x \in \mathbb{R}^n$, B is an $n \times m$ matrix, $u \in \mathbb{R}^m$.

Problem: given x_0 and x_1, is there a $t_1 > 0$ and some input $u(t)$ for $0 \leq t \leq t_1$ such that $x(t_1; u, x_0) = x_1$? If so the system is called *controllable*.

Theorem *The system is controllable if and only if*

$$\operatorname{Im}\begin{bmatrix} B & AB & \cdots A^{n-1}B \end{bmatrix} = \mathbb{R}^n.$$

Introduction and background: feedback control

Linear system:

$$x'(t) = Ax(t) + Bu(t), \qquad x(0) = x_0.$$

A is an $n \times n$ matrix, $x \in \mathbb{R}^n$, B is an $n \times m$ matrix, $u \in \mathbb{R}^m$.

Problem: choose $u(t)$ so that $x(t)$ has some desired property, e.g., $\lim_{t\to\infty} x(t) = 0$. Such an input is called a *stabilizing* input. If it exists for all choices of x_0 the system is called *stabilizable*.

Idea: use *state feedback*, i.e., let $u(t)$ depend on $x(t)$.

Theorem *If the system is controllable then there exists a matrix F such that $A + BF$ has all its eigenvalues in the open left half plane.*

So, if we define $u(t) = Fx(t)$, then we have a feedback that stabilizes the system:

$$x'(t) = Ax(t) + Bu(t) = (A + BF)x(t).$$

Introduction and background: feedback control

However, mostly $x(t)$ is not available for feedback. The system is

$$\begin{cases} x'(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$$

Even if the matrices are all known, the initial state x_0 may be unknown.

Idea: make a state observer, i.e., a linear time invariant system that produces an approximation $z(t)$ of $x(t)$ based on knowledge of $u(t)$ and $y(t)$. Use $z(t)$ for feedback.

This works perfectly well, provided the system is *observable*. That is, every initial state can in principle be reconstructed from the outputs. Dual concept to controllability.

Example

Automatic control of a Segway (movie).

Introduction and background: LQ optimal control problem

Linear system:

$$x'(t) = Ax(t) + Bu(t), \qquad x(0) = x_0.$$

A is an $n \times n$ matrix, $x \in \mathbb{R}^n$, B is an $n \times m$ matrix, $u \in \mathbb{R}^m$.

Quadratic cost function

$$J(x_0, u) = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt$$

with Q positive semi-definite and R positive definite.

Goal: minimize $J(x_0, u)$ over all *stabilizing* input functions u, that is,over all input functions for which $\lim_{t\to\infty} x(t) = 0$.

Assumption: (A, B) stabilizable and (A^T, Q) also stabilizable.

Introduction and background: LQ optimal control solution

$$x'(t) = Ax(t) + Bu(t), \qquad x(0) = x_0.$$

$$J(x_0, u) = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt$$

Under stabilizability assumptions: solution is known and easy to implement. The minimizing input is given by feedback:

$$u(t) = -R^{-1} B^T X x(t),$$

where X solves

$$XBR^{-1}B^T X - A^T X - XA - Q = 0$$

with the eigenvalues of $A - BR^{-1}B^T X$ in the open left half plane. Minimal cost is

$$\min_u J(x_0, u) = x_0^T X x_0.$$

Conclusion

Theory for finite dimensional linear time-invariant systems rather complete.

Nonlinear systems: lots of theory available.

Dynamics in discrete time.

Discrete event systems.

Hybrid systems.

Interconnected systems, control based on architecture of the interconnections.

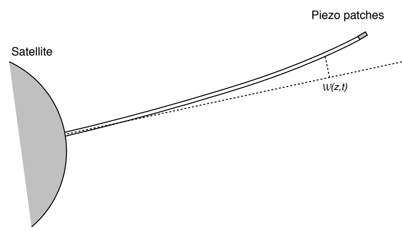
Infinite dimensional systems: control of systems where the dynamics is governed by p.d.e. or delay differential equations. Lots of theory available.

Conclusion

Example: boundary control for a vibrating string

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta}\left(T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right)$$

for ζ in the interval $[a, b]$, with left endpoint fixed: $\frac{\partial w}{\partial t}(a, t) = 0$, and control via a force at the right endpoint: $\frac{\partial w}{\partial t}(b, t) = u(t)$.

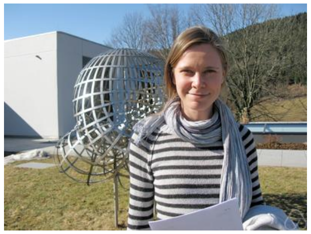


This problem does not fit in the form described above, even if we take A to be the generator of a semigroup, but it does fit in the context of Port-Hamiltonian boundary control systems as described in a recent text by Birgit Jacob and Hans Zwart.

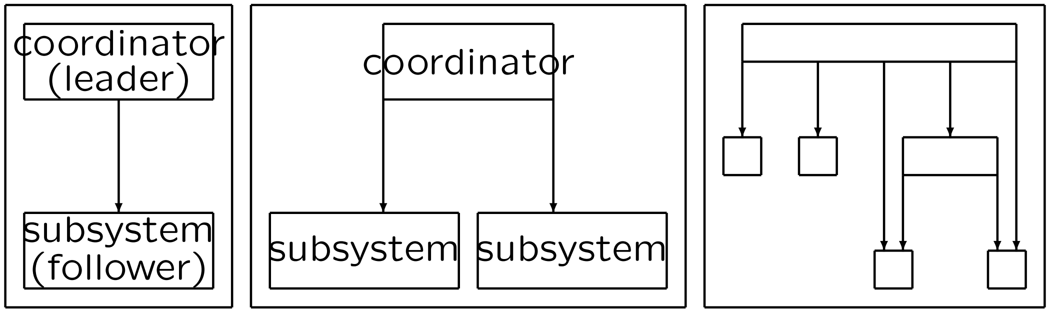
End of crash course on systems theory

Linear quadratic optimal control for coordinated systems

Pia Kempker, André Ran, Jan van Schuppen



Coordinated systems



Different types of systems:
leader-follower system (left),
coordinated system (center),
hierarchical system (right).

All of them have a coordinator (top level) and subsystems. Information flow is downwards, no information goes up.

Simplest types are the two left ones.
These we will discuss, with focus on the first.

Leader-follower

The leader's state is denoted by $x_2(t)$, it's input by $u_2(t)$, the follower's state is denoted by $x_1(t)$, it's input by $u_1(t)$. Assuming linear systems behavior we then have

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

We are interested in feedback control that preserves the structure: if

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

then

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_{11}F_{11} & B_{11}F_{12} + B_{12}F_{22} \\ 0 & B_{22}F_{22} \end{bmatrix}\right)\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Leader-follower

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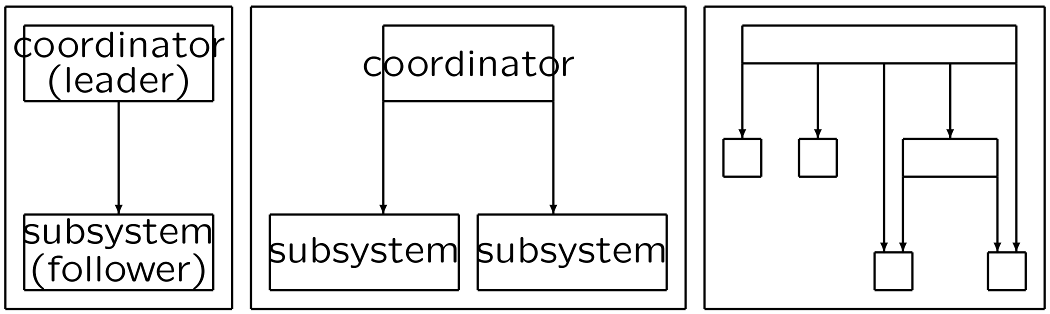
then

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} + B_{11}F_{11} & A_{12} + B_{11}F_{12} + B_{12}F_{22} \\ 0 & A_{22} + B_{22}F_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Coordinated systems



Coordinated systems

The coordinator's state is denoted by $x_c(t)$, it's input by $u_c(t)$, the subsystems' are denoted by $x_1(t)$ and $x_2(t)$, inputs by $u_1(t)$ and $u_2(t)$. There is no interconnection between subsystem 1 and subsystem 2.

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_c(t) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_c(t) \end{bmatrix}.$$

Again, the feedback to keep the structure is of interest:

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_c(t) \end{bmatrix} = \begin{bmatrix} F_{11} & 0 & F_{1c} \\ 0 & F_{22} & F_{2c} \\ 0 & 0 & F_{cc} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_c(t) \end{bmatrix}.$$

LQ problem for leader-follower system

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix},$$

$$J\left(x_0, \begin{bmatrix} u_1(\cdot) \\ u_2(\cdot) \end{bmatrix}\right) = \int_0^\infty \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dt$$

with $Q_{11} \geq 0$, $Q_{22} \geq 0$ and $R_{11} > 0$, $R_{22} > 0$.

Define

$$\mathcal{U}^\Delta = \left\{ \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \mid A + BF \text{ is stable}, F \text{ may depend on } x_0 \right\}.$$

Problem

$$\min_{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathcal{U}^\Delta} J\left(x_0, \begin{bmatrix} u_1(\cdot) \\ u_2(\cdot) \end{bmatrix}\right).$$

Towards a solution

Suppose F_{22} is fixed, so that $u_2(t) = F_{22}x_2(t)$.
Insert in state equations:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12}+B_{12}F_{22} \\ 0 & A_{22}+B_{22}F_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} u_1(t), \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix},$$

Insert in cost function:

$$J\left(x_0, u_1(\cdot)\right) = \int_0^\infty \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22}+F_{22}^T B_{22} F_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u_1^T R_{11} u_1 \, dt$$

This is a standard LQ optimal control problem for the input u_1, then $u_1 = -R^{-1}B^T X x = -R_{11}^{-1}\begin{bmatrix} B_{11}^T & 0 \end{bmatrix} X x$

Algorithm for given F_{22}

Recall Riccati equation necessary for solution

$$XBR^{-1}B^TX - XA - A^TX - Q = 0$$

Write $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$.

The Riccati equation reduces to equations for the entries X_{ij}.

- An algebraic Riccati equation
  $X_{11}B_{11}R_{11}^{-1}B_{11}^TX_{11} - X_{11}A_{11} - A_{11}^TX_{11} - Q_{11} = 0$,
  defining $F_{11} = -R_{11}^{-1}B_{11}^TX_{11}$,
- A matrix Lyapunov equation for X_{12}
  $(A_{11} + B_{11}F_{11})^TX_{12} + X_{12}(A_{22} + B_{22}F_{22}) + X_{11}(A_{12} + B_{12}F_{22}) = 0$,
  defining $F_{12} = -R_{11}^{-1}B_{11}^TX_{12}$,
- A similar Lyapunov equation for X_{22}.

Algorithm for the general problem

The input is $u_1(t) = \begin{bmatrix} F_{11} & F_{12} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$,

the minimal cost *given* F_{22} is $J_{F_{22}}(x_0, u_1) = x_0^T \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} x_0$.

The idea is now to minimize $J_{F_{22}}(x_0, u_1)$ over all stabilizing F_{22}, using Matlab's command `fmincon`.

Algorithm for the general problem, discussion

The idea is now to minimize $J_{F_{22}}(x_0, u_1)$ over all stabilizing F_{22}, using Matlab's command `fmincon`.

Conjecture. There is a unique F_{22} that minimizes the cost (under suitable hypothesis such as stabilizability of the system).

We did not find a counterexample testing randomly generated examples. Proof would be easy if the set of stabilizing feedback matrices for the pair (A_{22}, B_{22}) is convex, but that is not the case with respect to an entry-wise parametrization. However, it might be convex with respect to an alternative parametrizaton.

Example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, R = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

Given f_{22} the solution of the algebraic Riccati equation is

$$X = \begin{bmatrix} \alpha\left(1 + \sqrt{1 + \frac{1}{\alpha}}\right) & \alpha f_{22} \\ \alpha f_{22} & \frac{1}{2}(\alpha + \beta) f_{22}^2 \end{bmatrix}$$

Corresponding cost with initial condition $x(0)$ is $x(0)^T X x(0)$ which is quadratic in f_{22}, and is easily minimized over f_{22}.

The unique minimizer is

$$f_{22} = \frac{2\alpha x_1(0)}{(\alpha + \beta) x_2(0)}.$$

Discussion

Most important observation: the optimal F_{22} (and hence also most of the other matrices involved in the optimal feedback) depends on x_0.

This holds in general, and is in contrast to the standard optimal LQ control.

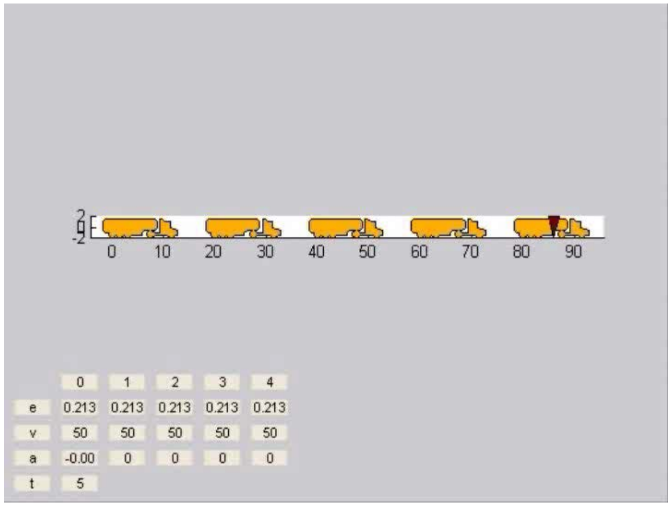
For $\beta \to \infty$ the solutions to the standard optimal LQ control problem and the solution to the problem keeping the structure will approach the same minimal cost.
For $\beta \downarrow 0$ the relative difference in cost between the two goes to infinity.

The general scalar case is easy, and in that case the conjecture holds.

More discussion

Problem for coordinated systems rather than leader-follower can be dealt with in essentially the same way. Same for hierarchical systems.

Applications: vehicle formation systems



LQ control with event-based feedback: set up

The set up of the problem is as before: leader-follower system

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix},$$

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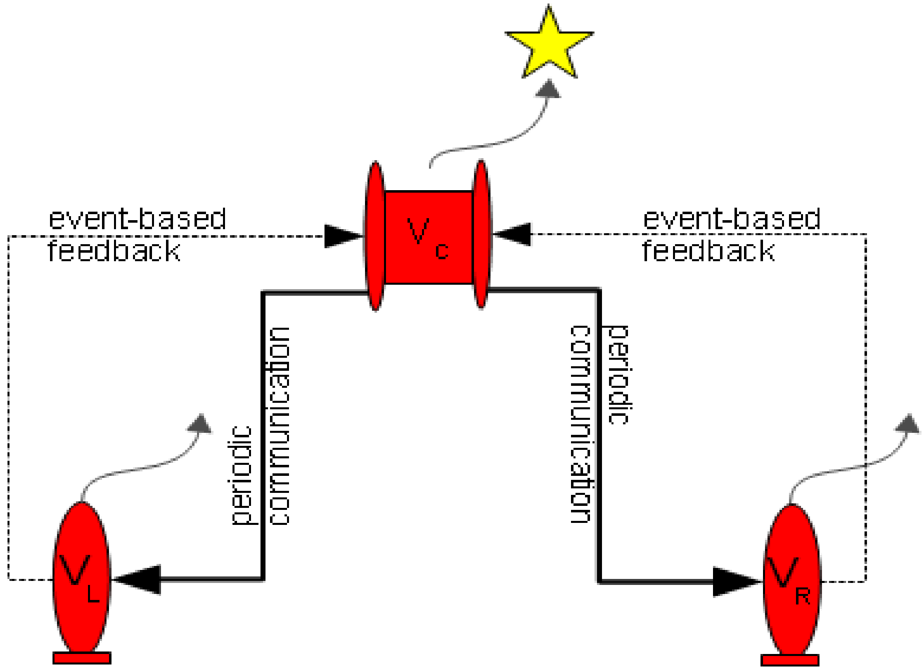
with $Q_{11} \geq 0$, $Q_{22} \geq 0$ and $R_{11} > 0$, $R_{22} > 0$.

However, the idea is to consider another set of feedback rules to minimize J over, *and to allow information of the follower to be available to the leader* .

Unrestricted information flow from follower to leader: that is the standard LQ problem.
Idea is to restrict information from follower to leader to certain time instants, determined by discrete events.

Example for coordinated system (two subsystems)



Back to leader follower model: some tricks

Unrestricted communication. Put $G = -R^{-1}B^T X$, where X solves the Riccati equation for the full system. Decompose

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

Trick one: define rank one matrix

$$T(t) = \frac{1}{\|x_2(t)\|^2} x_1(t) x_2^T(t).$$

Closed loop system $u = Gx$:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

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Closed loop system $u = Gx$:

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because $T(t)x_2(t) = x_1(t)$.

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Closed loop system $u = Gx$:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} + B_{11}G_{11} & A_{12} + B_{11}G_{12} + B_{12}(G_{21}T(t) + G_{22}) \\ 0 & A_{22} + B_{22}(G_{21}T(t) + G_{22}) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

because $T(t)x_2(t) = x_1(t)$.

This has the desired form, but that is deceptive as $T(t)$ still depends on $x_1(t)$ for all t. Idea: replace $T(t)$ by a piecewise constant function of t.

Piecewise constant $T(t)$ and a guard condition

Remember $x_1(t) = T(t)x_2(t)$. Initially set $t_{old} = 0$.
Our guard condition compares $G_{21}x_1(t) = G_{21}T(t)x_2(t)$ with $G_{21}T(t_{old})x_2(t)$, and the follower sends its current state to the leader at those times when the difference is too large, also then replacing t_{old} by the current time.

We have to be more precise concerning "too large":
the *guard condition* is

$$\|G_{21}(x_1(t) - T(t_{old})x_2(t))\| \leq re^{-\beta t},$$

where $r > 0$ and $\beta > 0$ are pre-specified parameters.

Note that the follower has access to both its own state $x_1(t)$ and to the state $x_2(t)$ of the leader, so it has all necessary information for the guard condition.

Some results

Control law

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = g_{r,\beta}(x(t)) = \begin{bmatrix} G_{11}x_1(t) + G_{12}x_2(t) \\ G_{21}T_{j(t)} + G_{22}x_2(t) \end{bmatrix},$$

where

$$T_{j(t^+)} = \begin{cases} T_{j(t^-)} & \text{if guard} \leq re^{-\beta t} \\ \frac{1}{\|x_2(t)\|^2} x_1(t) x_2^T(t) & \text{if guard} > re^{-\beta t} \end{cases}$$

and

$$\text{guard} = \|G_{21}(x_1(t) - T_{j(t^-)}x_2(t)\|.$$

Proposition *For any $r > 0$ and $\beta > 0$ this control law leads to an exponentially stable closed loop system.*

One more result and some conjectures

Proposition *Difference between cost J_0 in the unrestricted information case and the cost $J_{r,\beta}$ coming with this control law is bounded as follows:*

$$J_{r,\beta} - J_0 \leq \|R^{1/2}\|^2 \frac{r^2}{2\beta},$$

for any intial state.

One more result and some conjectures

Proposition *Difference between cost J_0 in the unrestricted information case and the cost $J_{r,\beta}$ coming with this control law is bounded as follows:*

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for any intial state.

Regarding the number of times that information is sent, set those times to $t_1 < t_2 < t_3 \cdots$.

Conjecture *For β small enough there is an $\varepsilon > 0$ so that $t_{j+1} - t_j > \varepsilon$, that is, there are no infinite resets in finite time.*
For all $r > 0$ there is a β_{max} and a T_N such that the guard condition is satisfied for all $\beta < \beta_{max}$ and all $t > T_N$. In other words: there are only finitely many resets.

First example

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

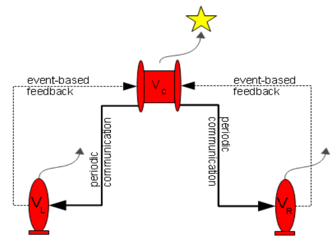
$$J(x_0, u) = \int_0^\infty \|x(t)\|^2 + \|u(t)\|^2 \, dt.$$

We take $\beta = 10^{-4}$, consider $5 * 10^4$ time steps of size 10^{-4} each (for comparison). Effect of r is given in the next table.

r	$J_{r\beta}$	number of resets
0	7.3674	$5 * 10^4$
0.01	7.3674	798
0.1	7.3690	116
1	7.5441	17
10	7.6883	0

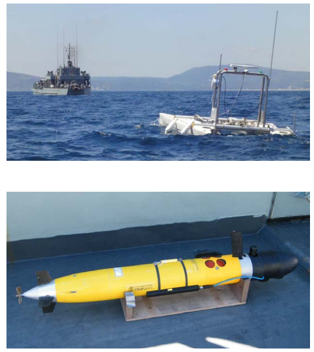
Similar results for different β.

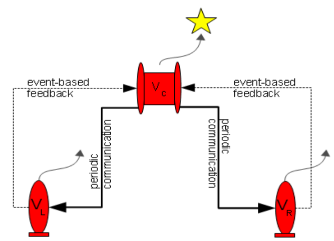
Second example



Wireless communication (sonar), very costly in terms of energy. No problem from surface vessel to submarine, but problematic the other way: cost is about 30% of total available energy with unrestricted feedback (i.e., every time instant).
This limits the underwater-time.

Second example

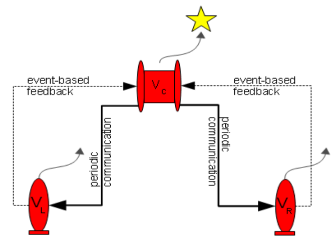




We did simulation with mathematical model for real systems, also incorporating delays in communcation, probability of information packet loss at 10% and disturbances by waves and currents.

Optimal control cost: tracking problem, with cost on deviation from reference trajectory and on acceleration.

Second example



Results: compared to unlimited communication cost increase in cost function is around 11% (over several runs of the simulation), and communication between submarines and surface vessel took place in 110 out of 1000 time steps.

Conclusion: if communication is very costly, it may be cheaper to use our algorithm.

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Supplementary references

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Finally

Thank you for your attention

Questions?