



Co-financed by the Connecting Europe Facility of the European Union

# Tipping in Spatially Extended Systems

2022-07-20, Tipping Seminar, Munich  
Robbin Bastiaansen ([r.bastiaansen@uu.nl](mailto:r.bastiaansen@uu.nl))



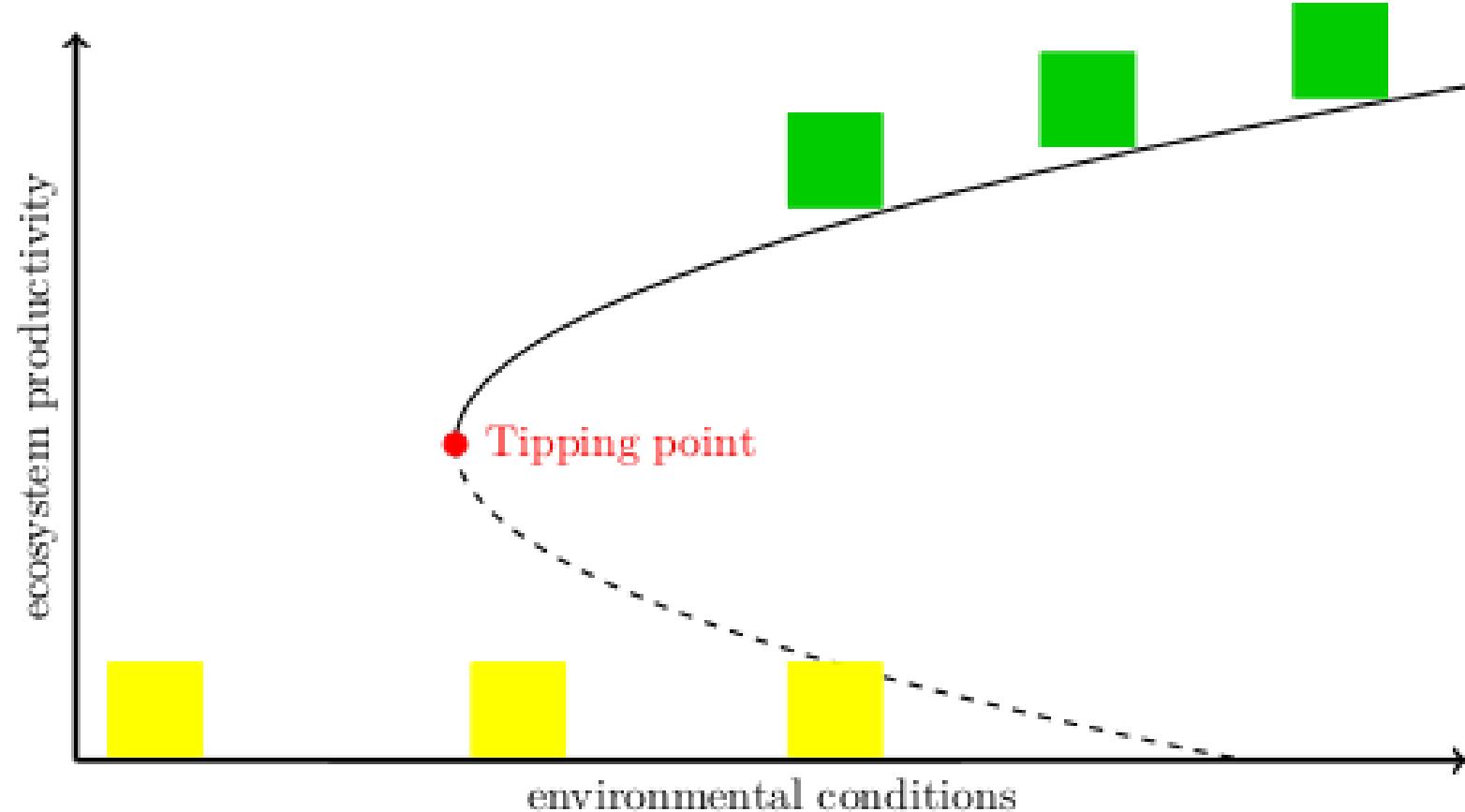
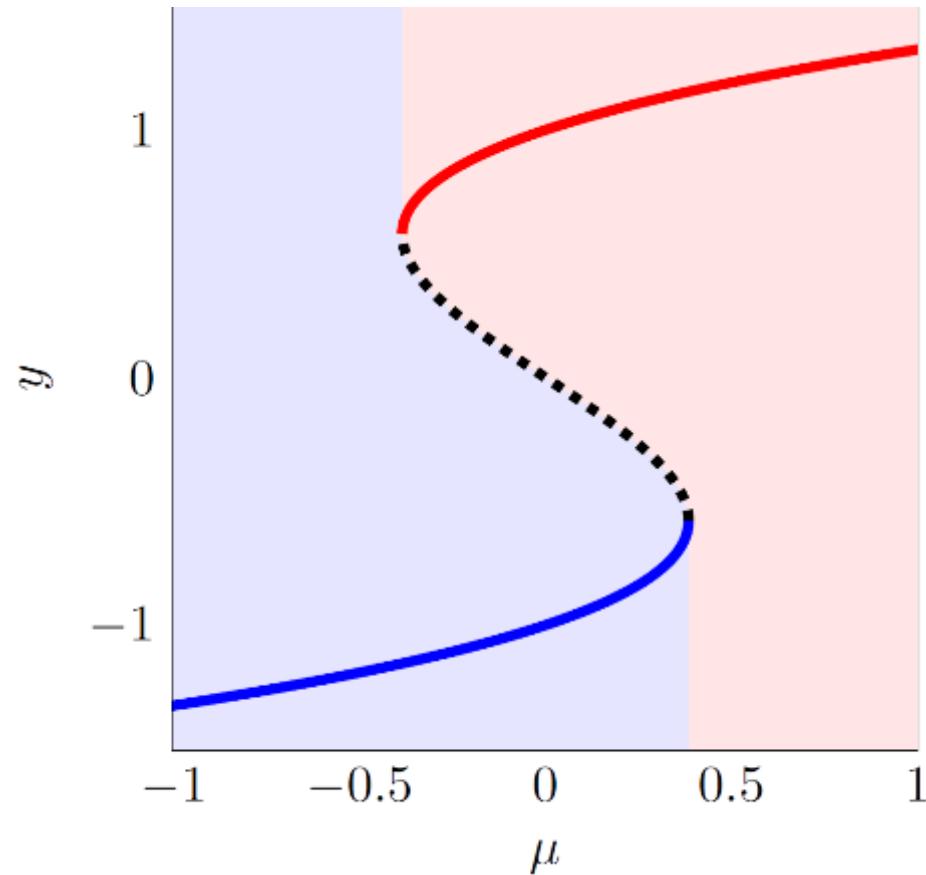
UNIVERSITY OF  
COPENHAGEN



$u^b$



# Classic Theory of Tipping



**Canonical example:**

$$\frac{dy}{dt} = y(1 - y^2) + \mu$$

$$\frac{d\vec{y}}{dt} = f(\vec{y}; \mu)$$

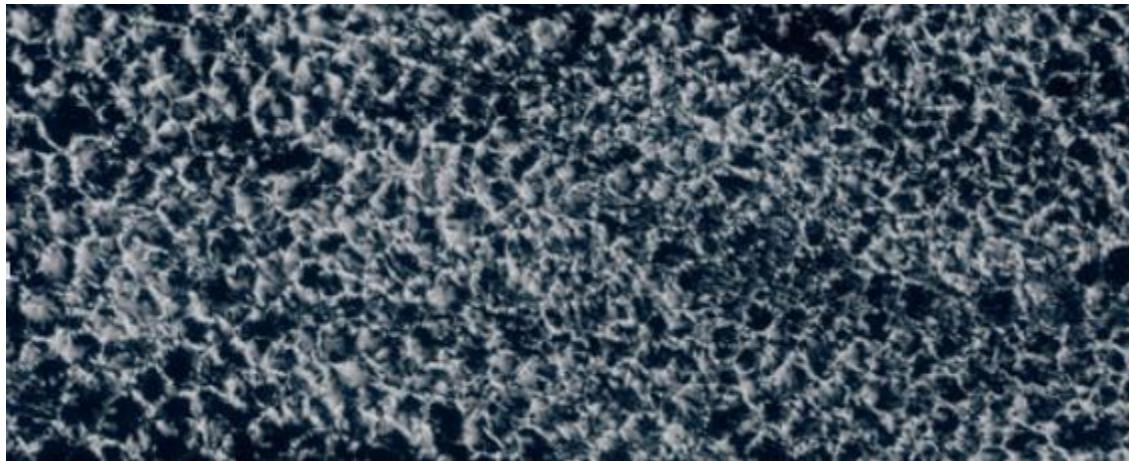
$$\begin{cases} \frac{du}{dt} = f(u, v; \mu) \\ \frac{dv}{dt} = g(u, v; \mu) \end{cases}$$



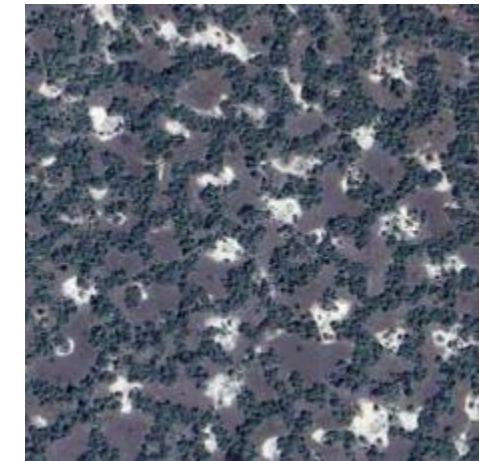
# Examples of spatial patterning – regular patterns



mussel beds



clouds



savannas



melt ponds

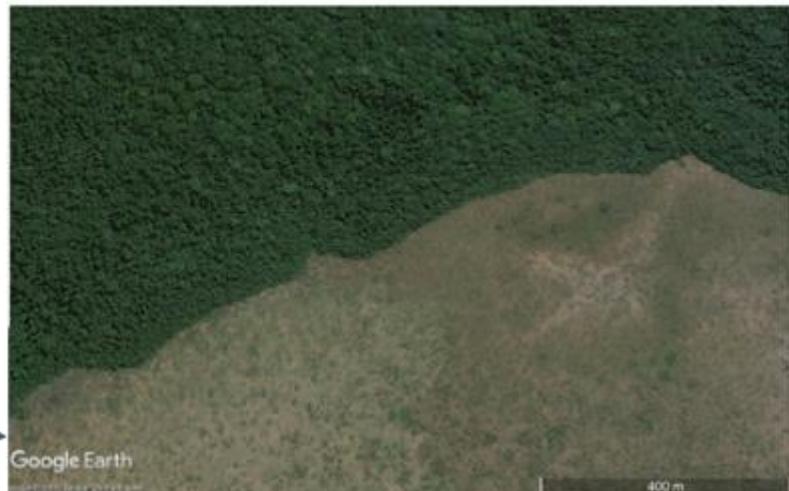


drylands

# Examples of spatial patterning – spatial interfaces

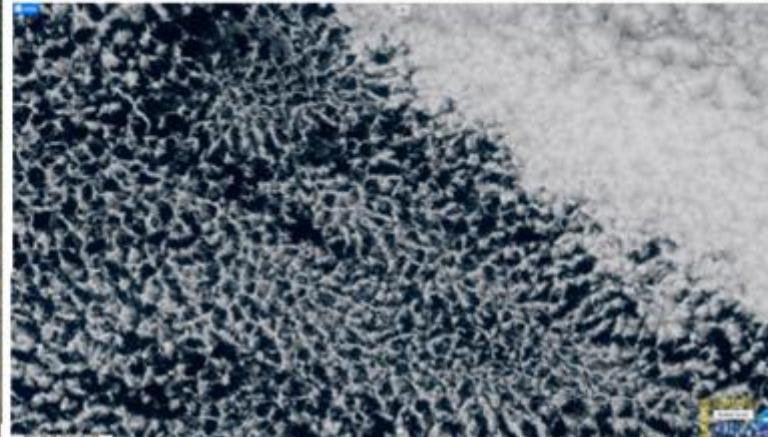
tropical forest  
& savanna  
ecosystems

[Google Earth]



types of  
stratocumulus  
clouds

[RAMMB/CIRA SLIDER]



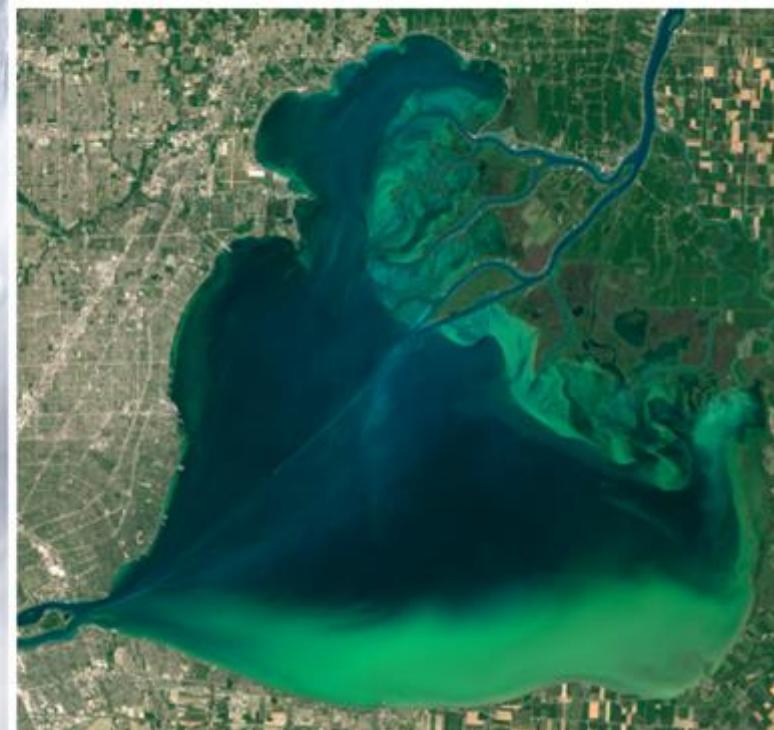
sea-ice & water  
at Eltanin Bay

[NASA's Earth observatory]



algae bloom  
in Lake St. Clair

[NASA's Earth observatory]



An aerial photograph of a vast, open landscape, likely a savanna or coastal plain. The terrain is covered in a light brown or tan color, with numerous small, dark green, rounded shrubs scattered across it. These shrubs are arranged in a roughly rectangular grid, creating a distinct pattern. In the upper right corner, there is a denser concentration of greenery, possibly a forest or a different type of vegetation. The overall scene is a mix of natural patterns and some man-made elements like a few thin lines on the ground.

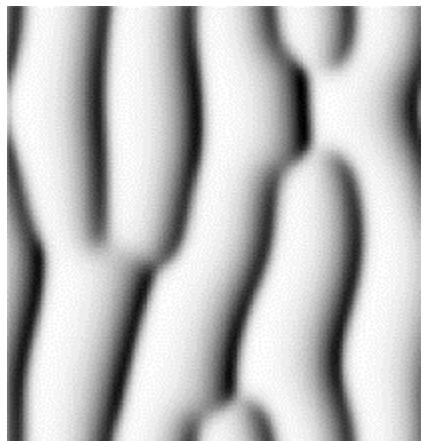
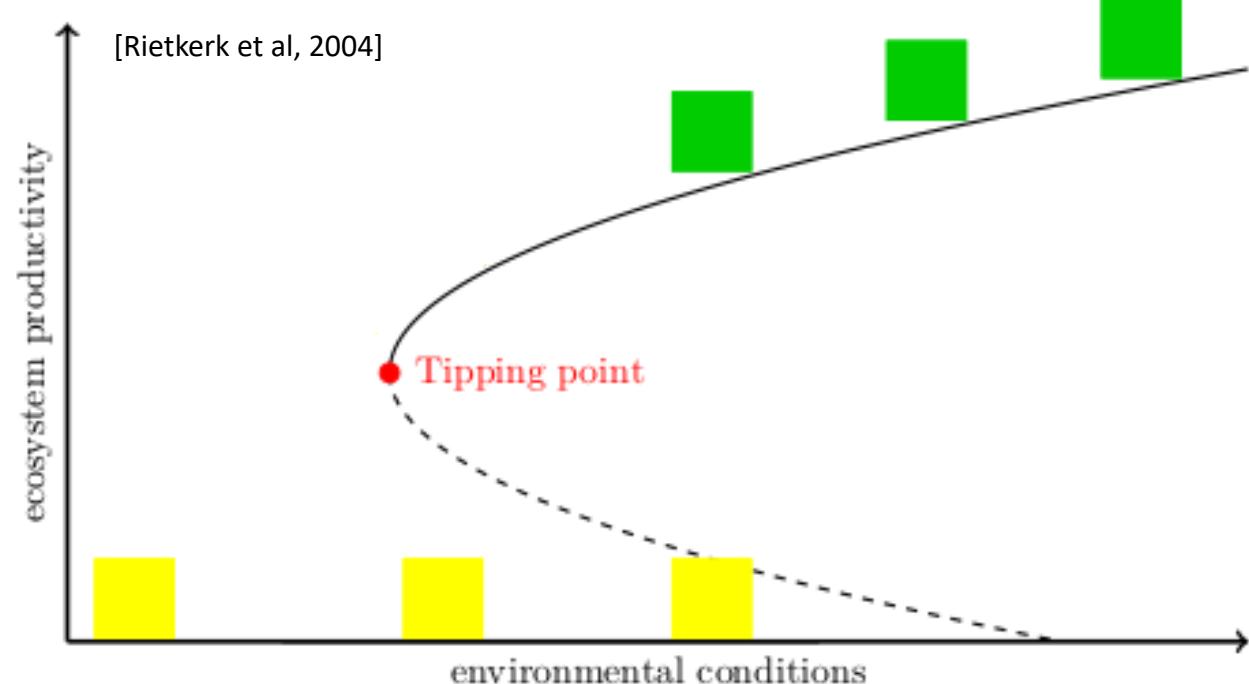
# Part 1: Turing Patterns

# Patterns in models

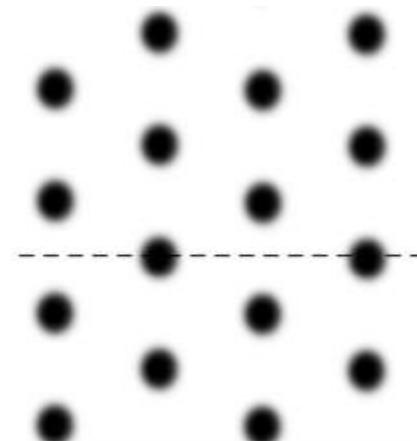
Add spatial transport:

Reaction-Diffusion equations:

$$\begin{cases} \frac{du}{dt} = f(u, v) + D_u \Delta u \\ \frac{dv}{dt} = g(u, v) + D_v \Delta v \end{cases}$$



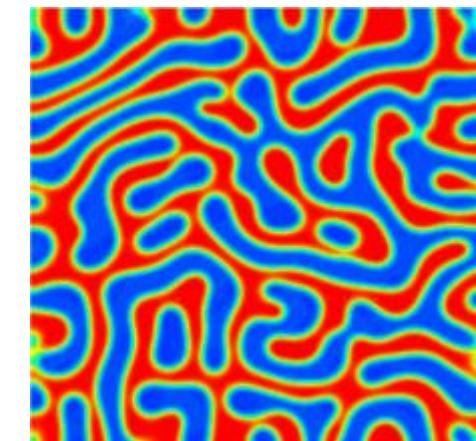
[Klausmeier, 1999]



[Gilad et al, 2004]

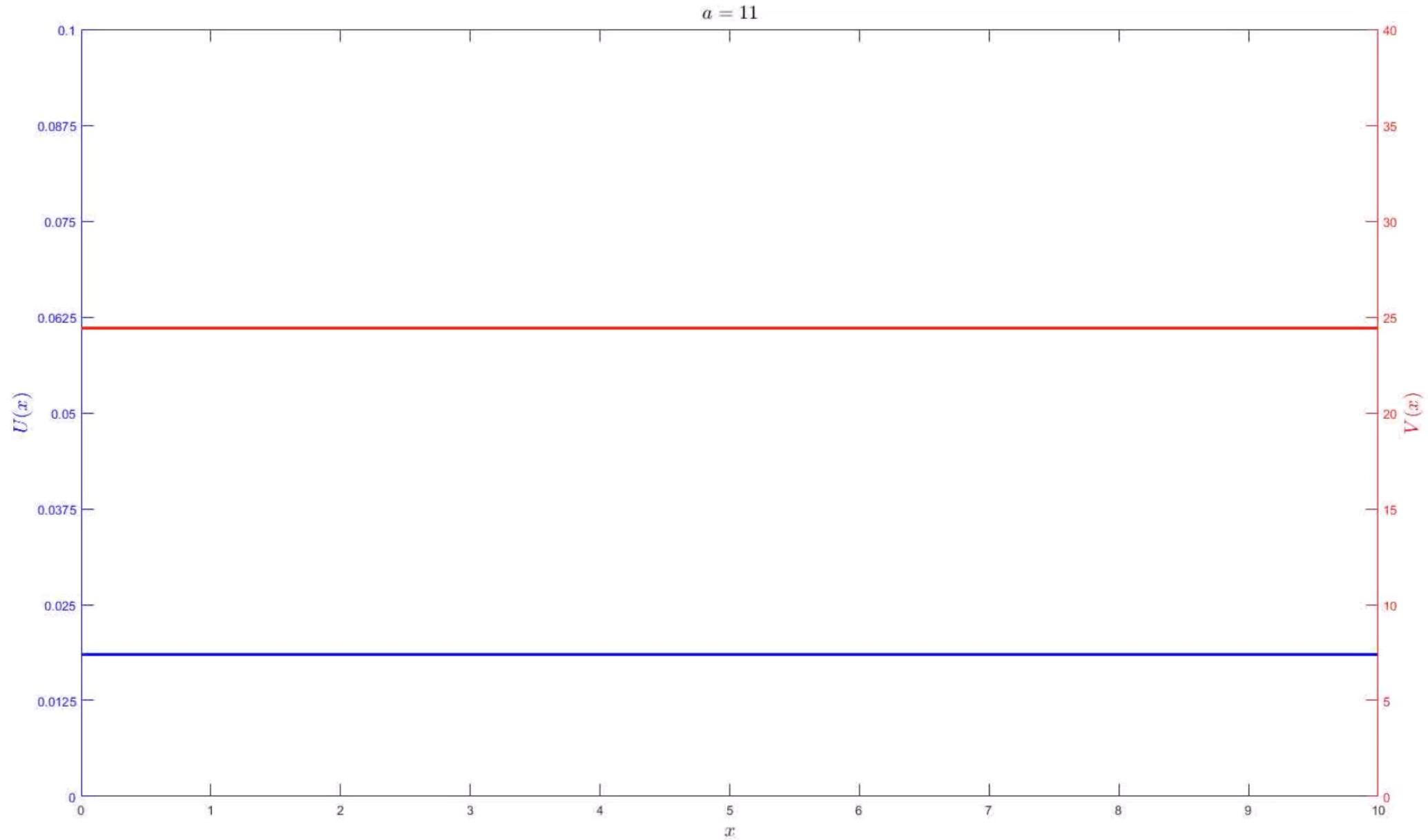


[Rietkerk et al, 2002]



[Liu et al, 2013]

# Behaviour of PDEs



# Turing patterns

$$\begin{cases} \frac{du}{dt} = f(u, v) + D_u \Delta u \\ \frac{dv}{dt} = g(u, v) + D_v \Delta v \end{cases}$$

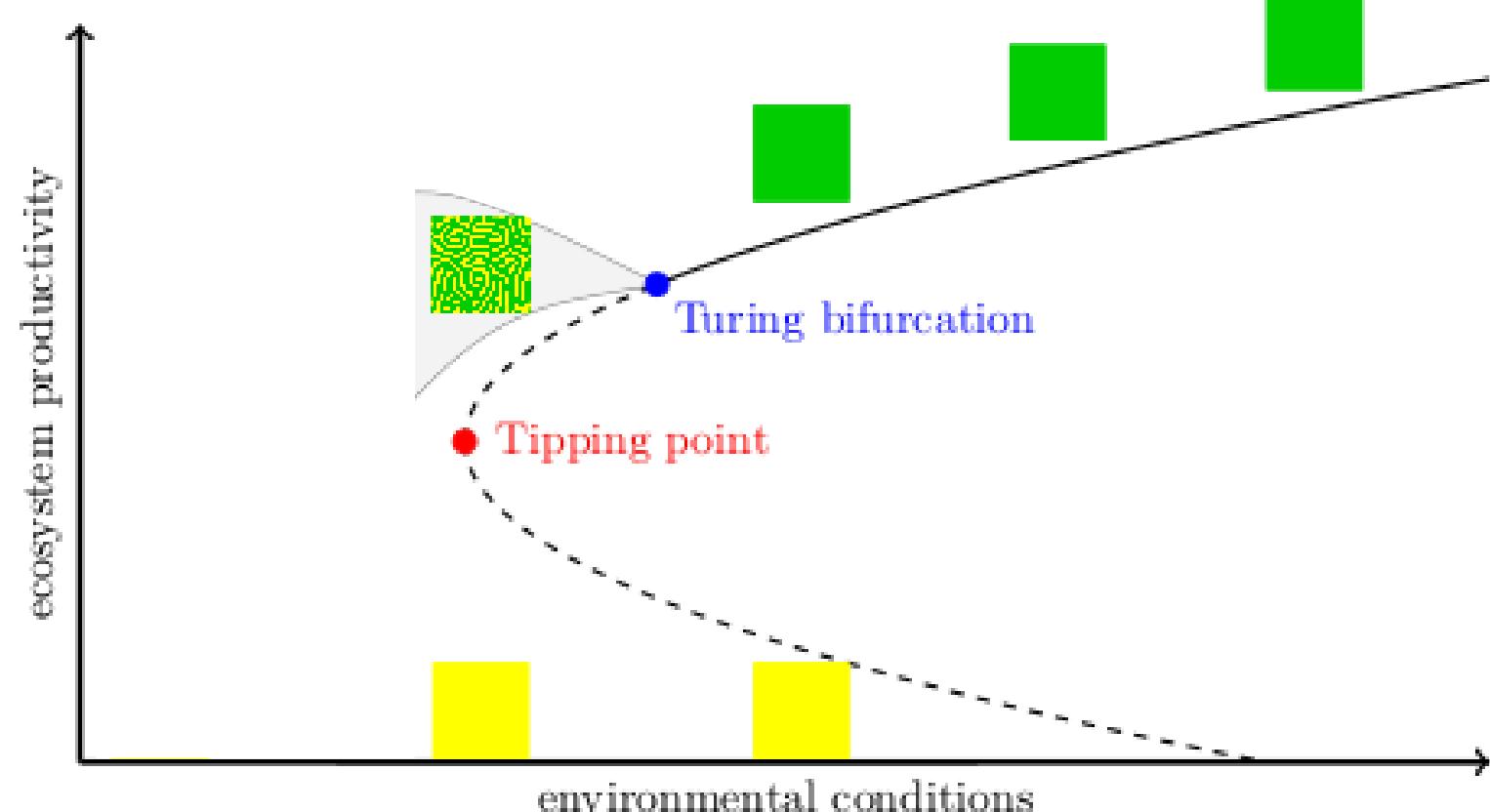
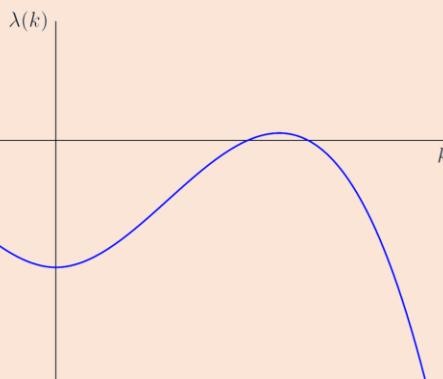
## Turing bifurcation

Instability to non-uniform perturbations

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t} e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

→ Dispersion relation

$$\lambda(k) = \dots$$



**Weakly non-linear analysis**  
Ginzburg-Landau equation / Amplitude Equation  
& Eckhaus/Benjamin-Feir-Newell criterion  
[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

# Busse balloon

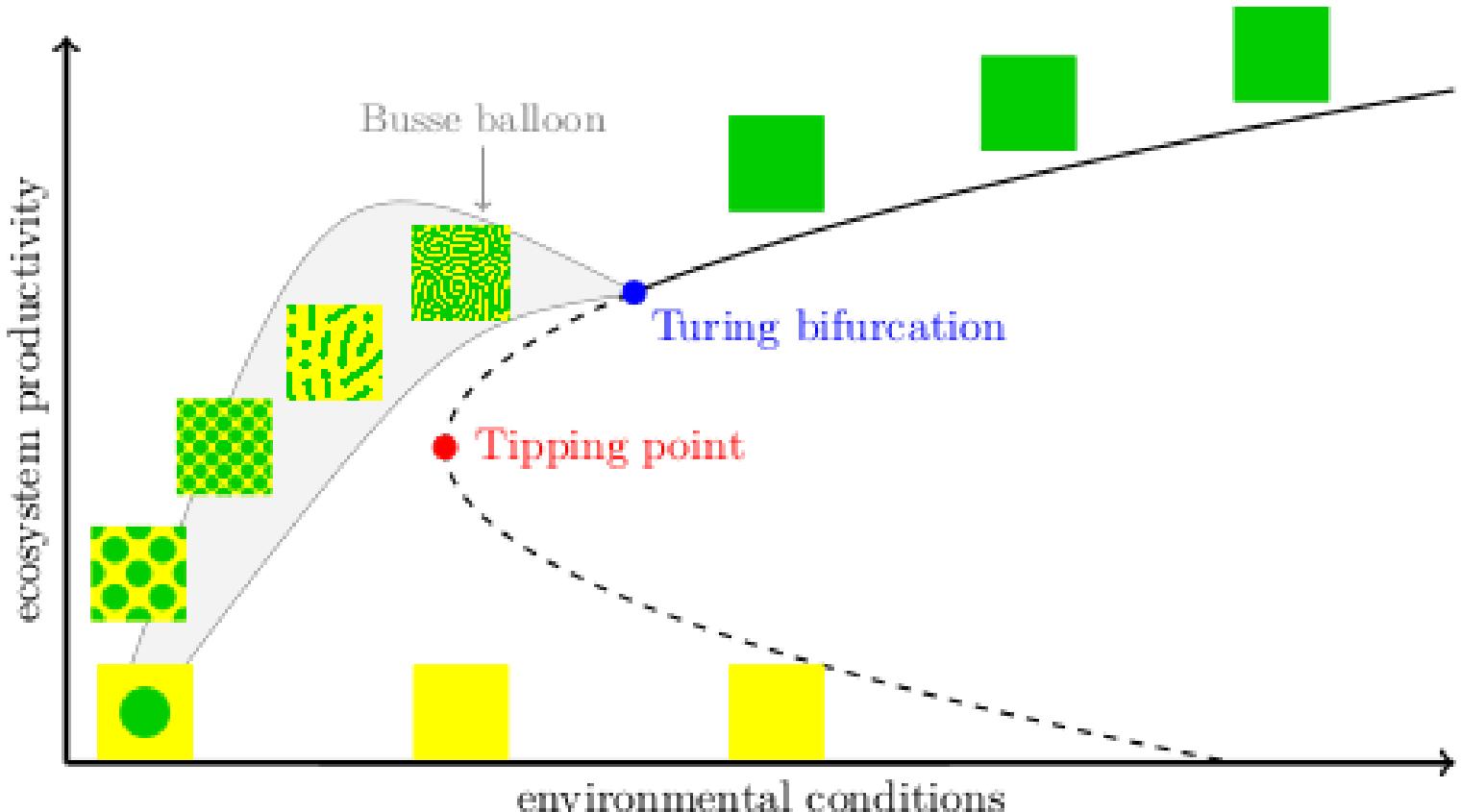
## Busse balloon

A model-dependent shape in (*parameter, observable*) space that indicates all stable patterned solutions to the PDE.

### Construction Busse balloon

Via numerical continuation  
few general results on the  
shape of Busse balloon

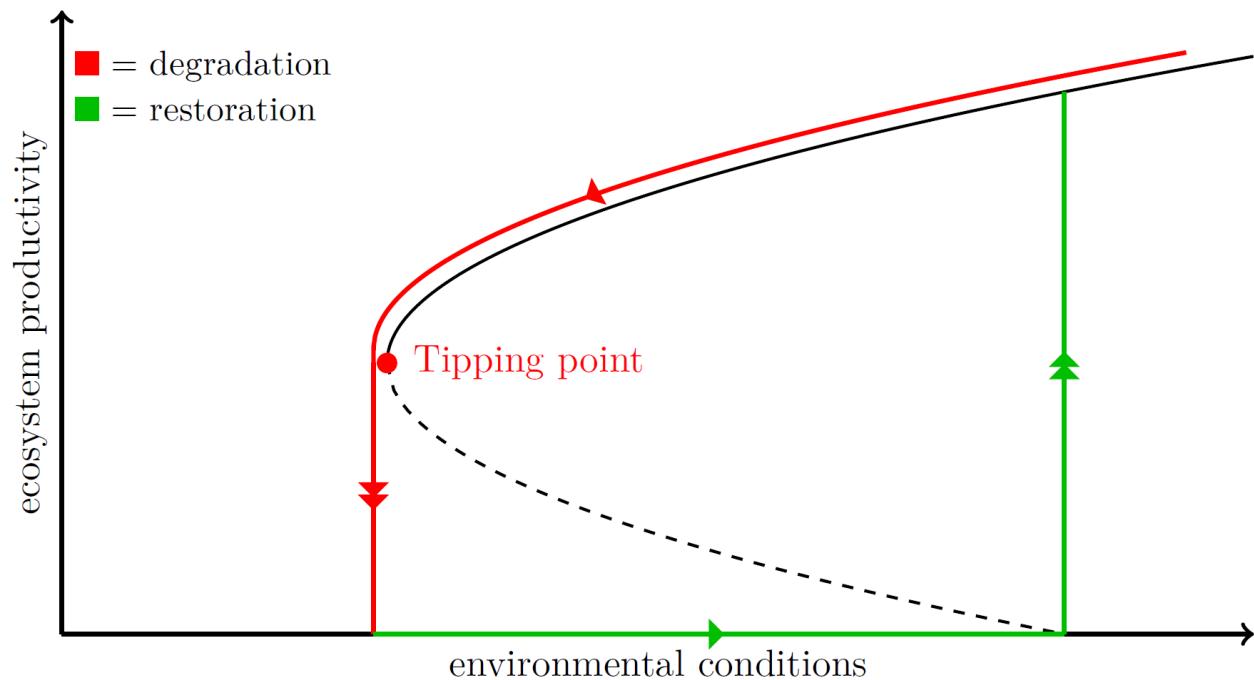
$$\begin{cases} \frac{du}{dt} = f(u, v) + D_u \Delta u \\ \frac{dv}{dt} = g(u, v) + D_v \Delta v \end{cases}$$



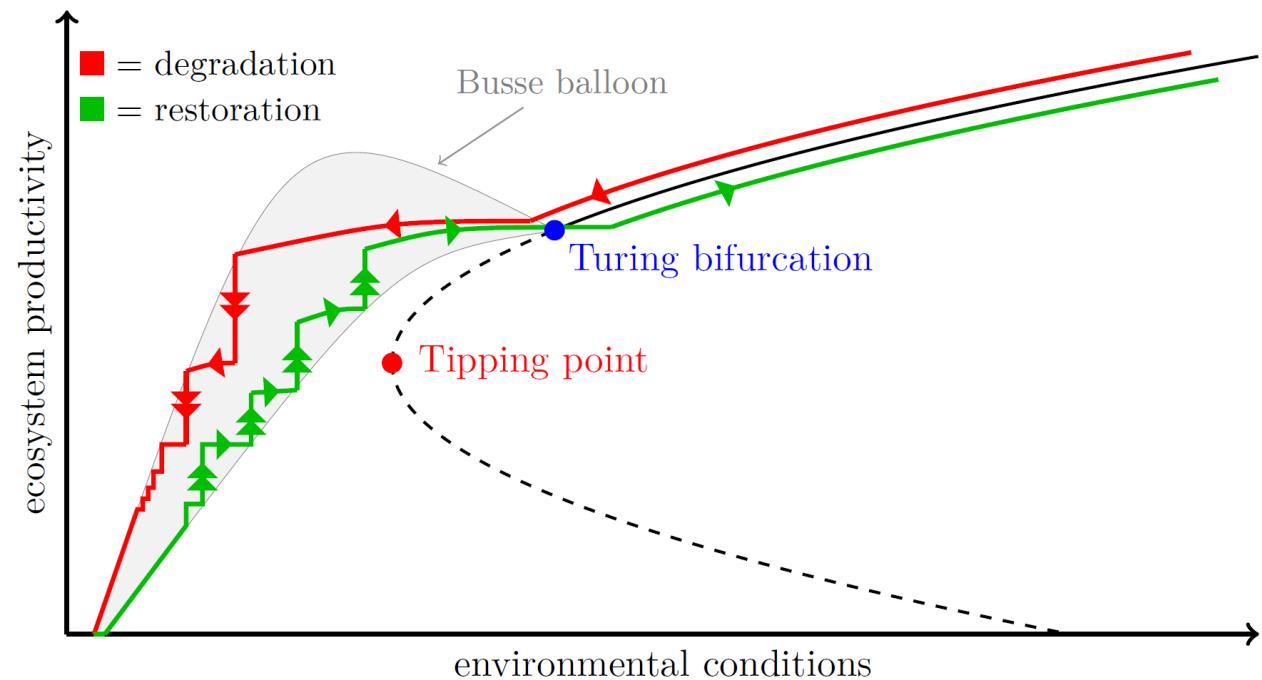
## Busse balloon

Idea originates from thermal convection  
[Busse, 1978]

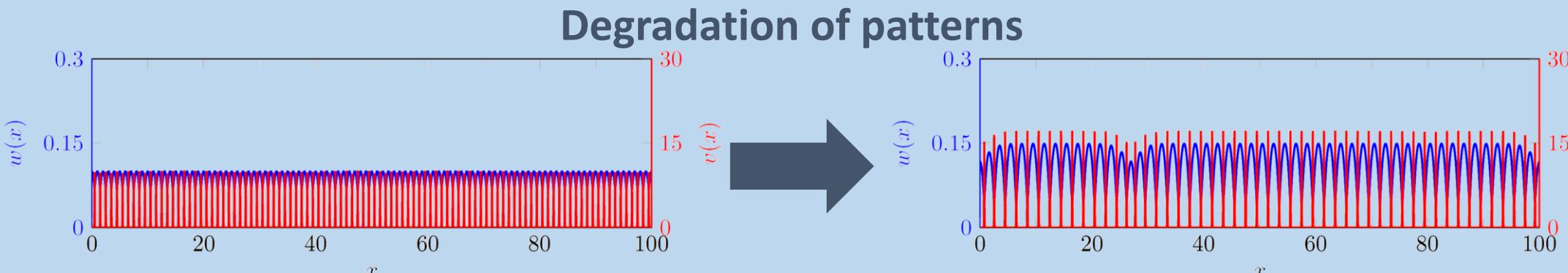
# Tipping of (Turing) patterns



Classic tipping



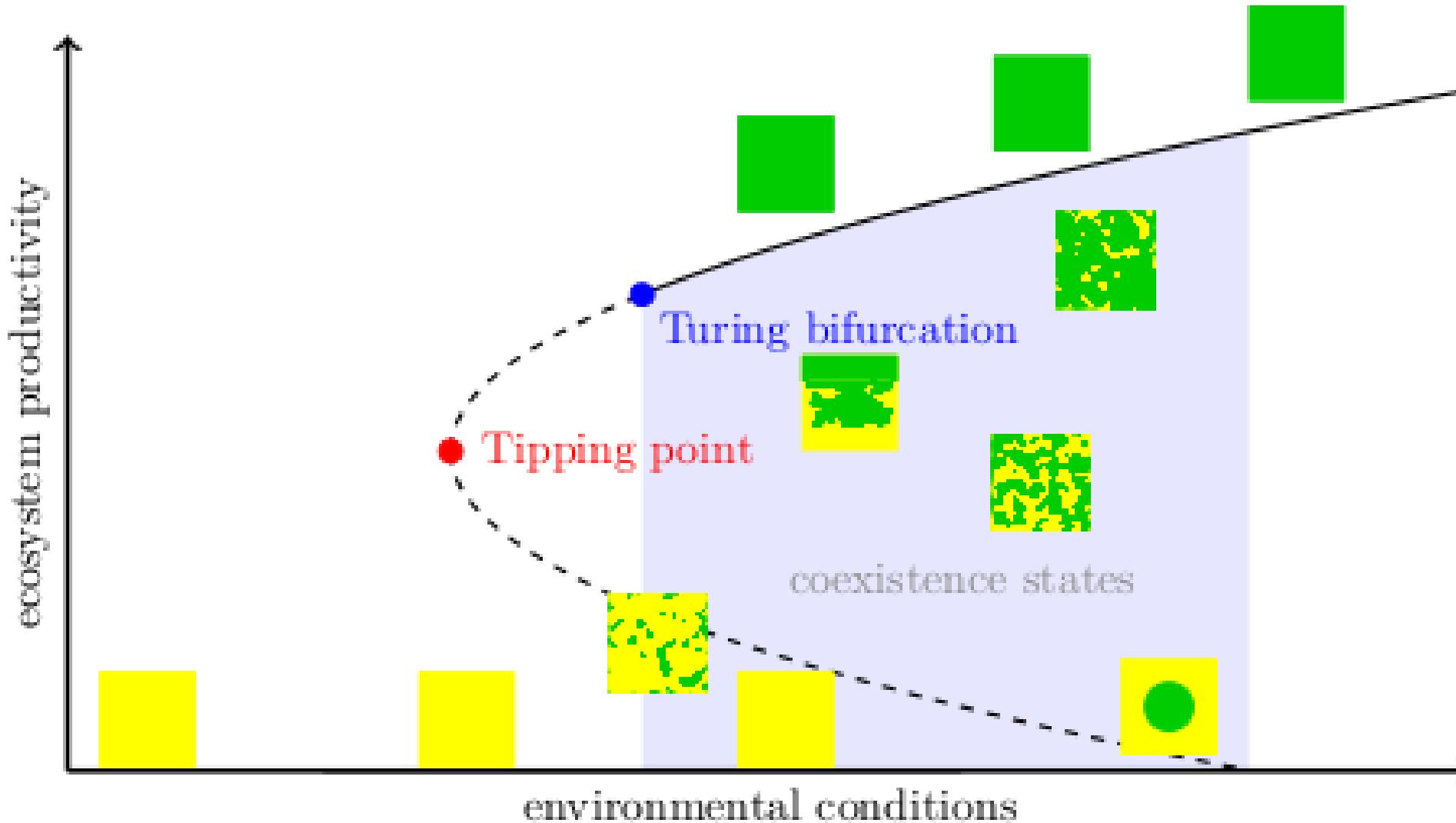
Tipping of patterns





# Part 2: Coexistence States and spatial heterogeneities

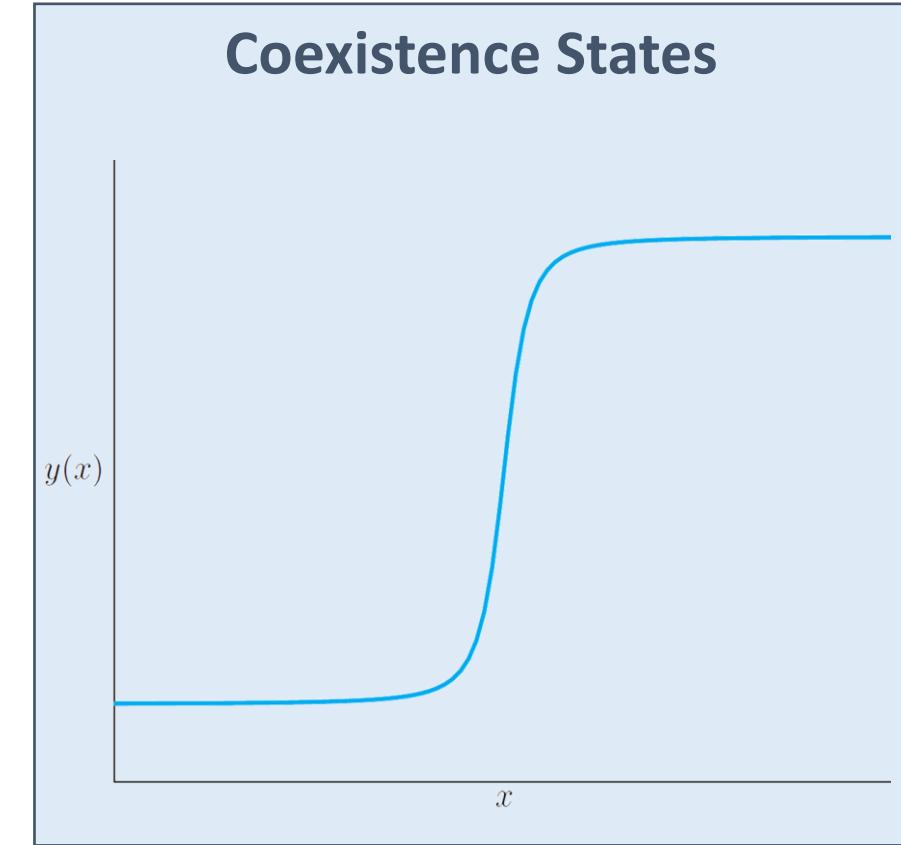
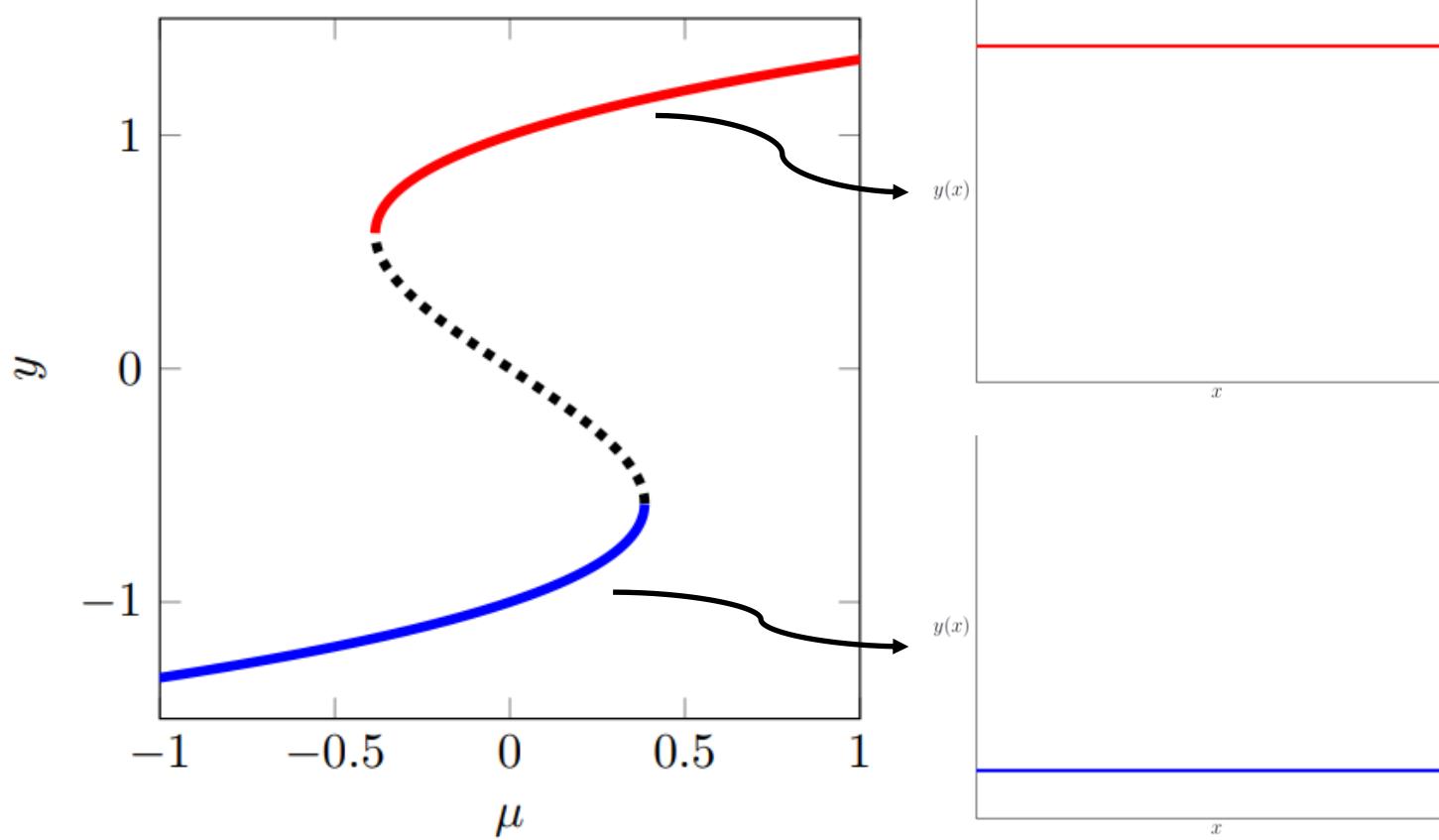
# Coexistence states in bifurcation diagram



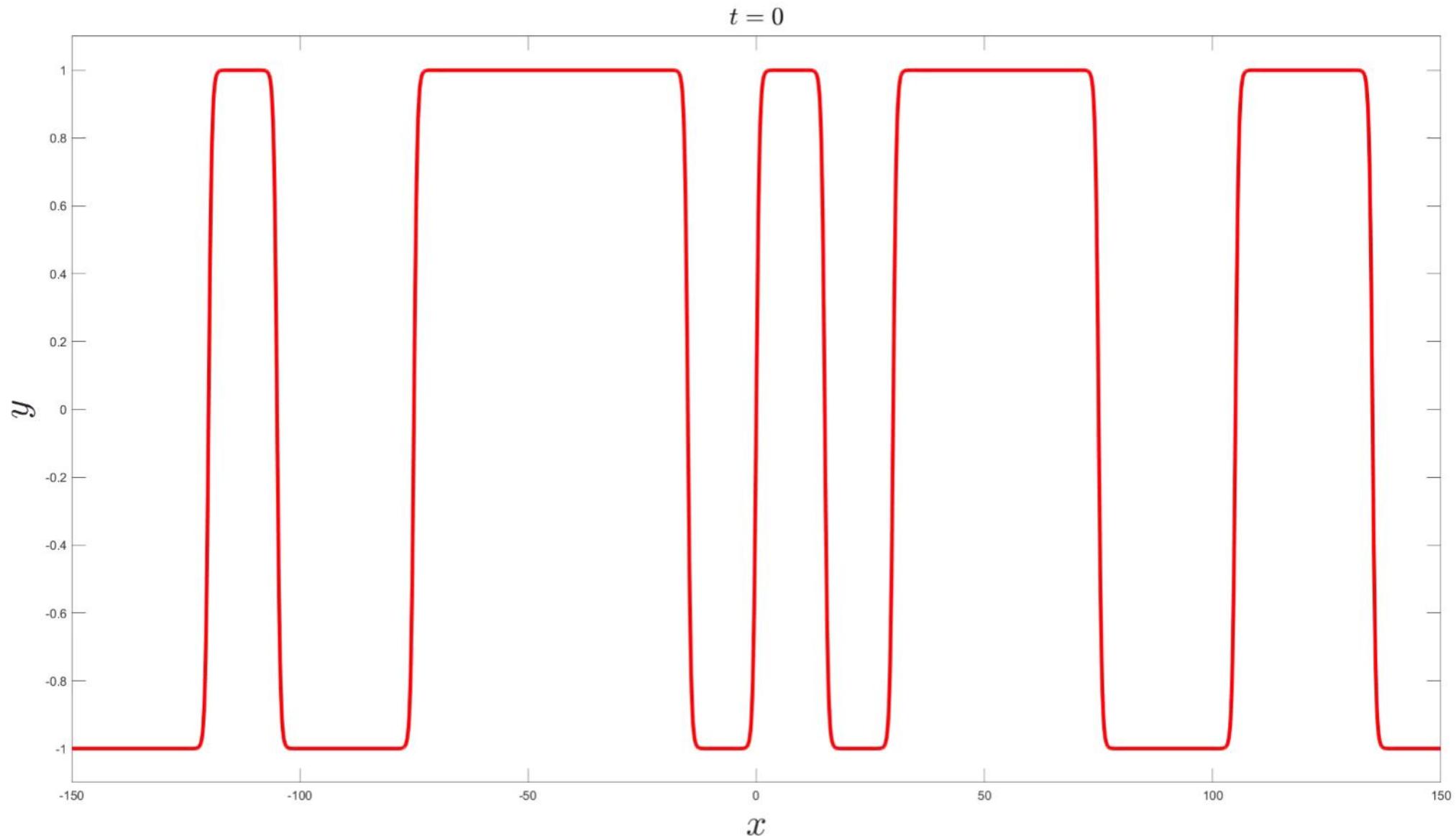
# Coexistence states

Bistable (Allen-Cahn/Nagumo) equation:

$$\frac{\partial y}{\partial t} = y(1 - y^2) + \mu + D \frac{\partial^2 y}{\partial x^2}$$



**Dynamics of  $\frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2} + y(1 - y^2) + \mu$**

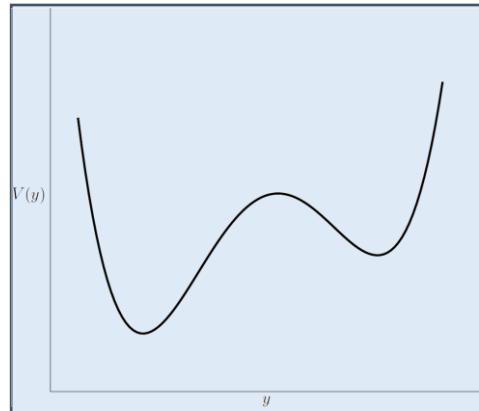


# Front Dynamics

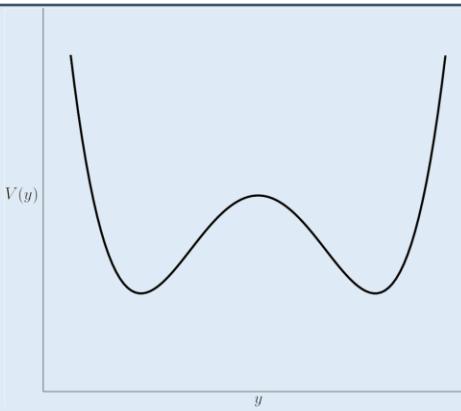
$$\frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2} + f(y; \mu)$$

Potential function  $V(y; \mu)$ :

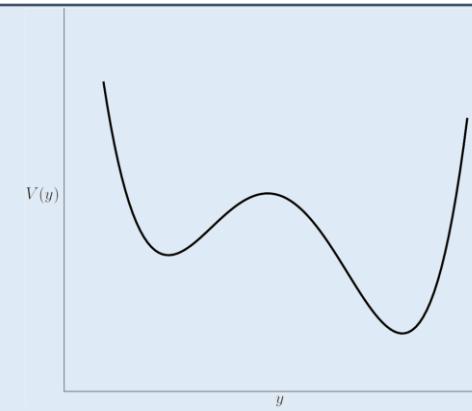
$$\frac{\partial V}{\partial y}(y; \mu) = -f(y; \mu)$$



moves right

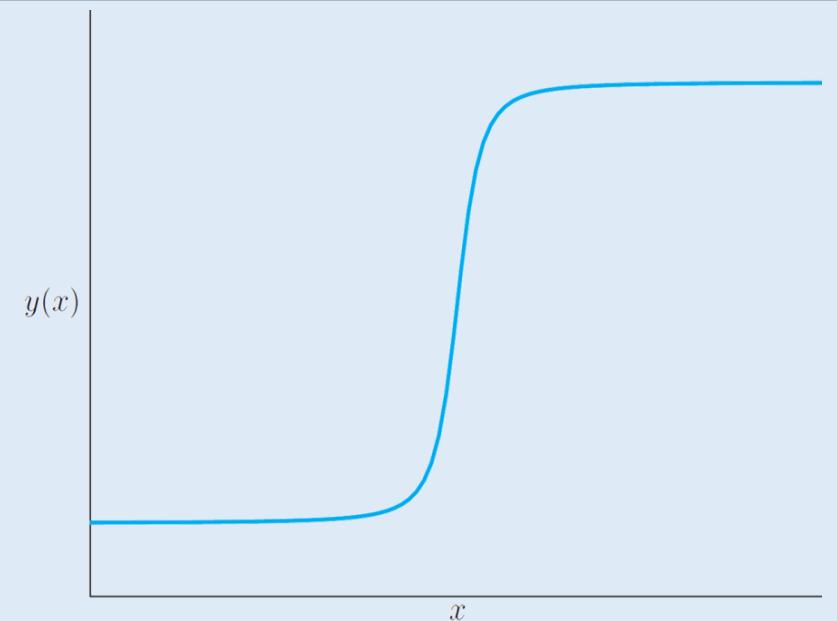
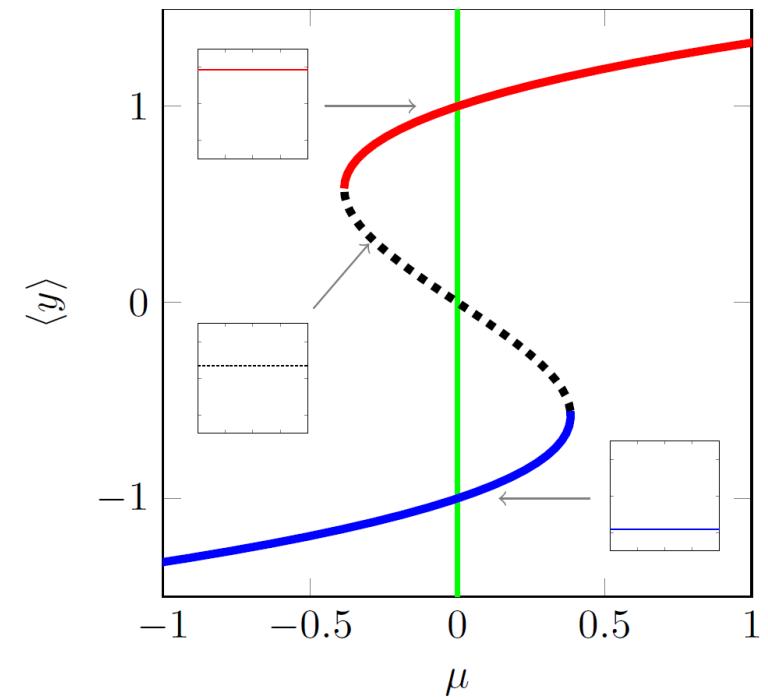


stationary



moves left

**Maxwell Point  $\mu_{maxwell}$**

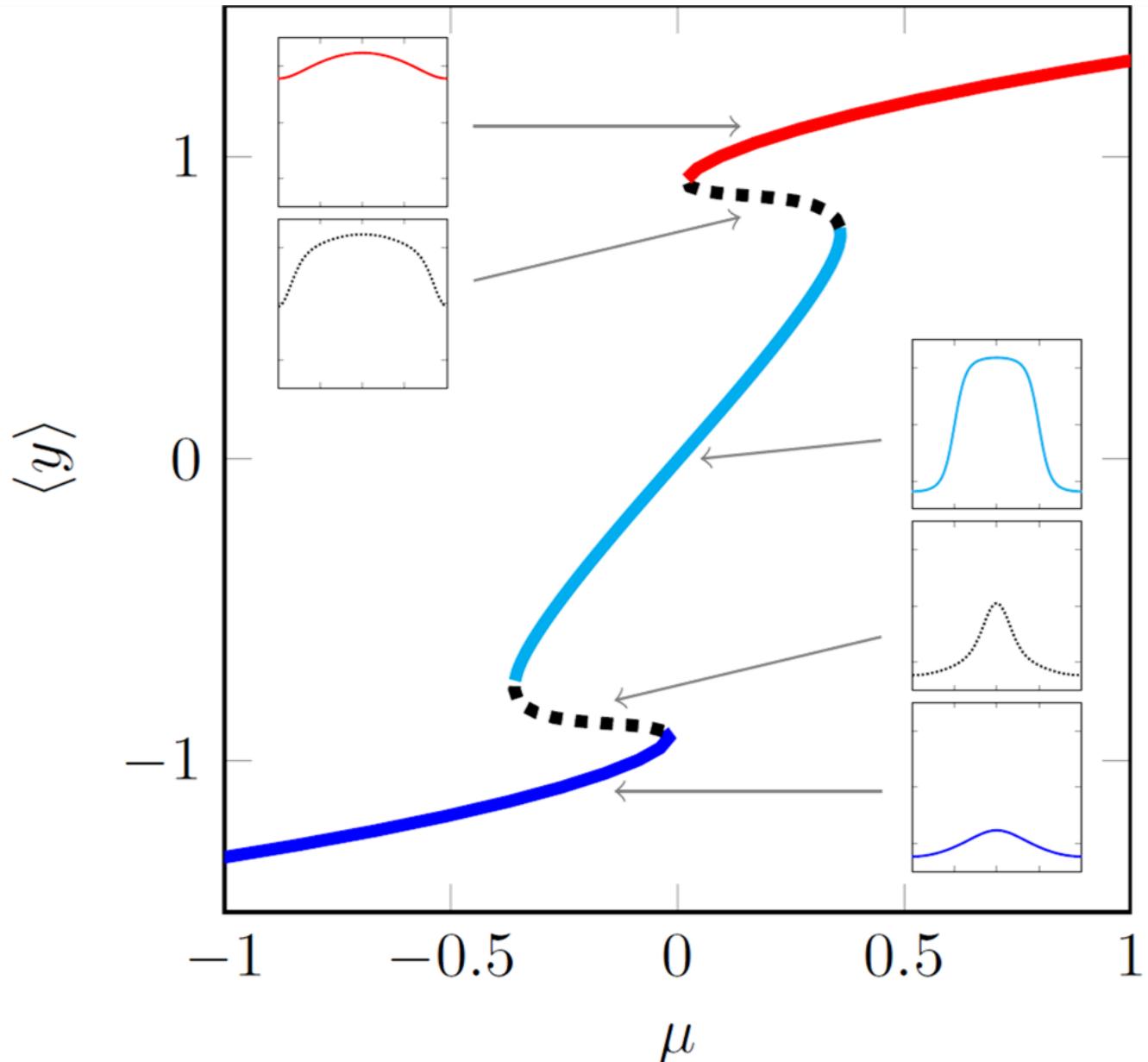


# Adding Spatial Heterogeneity

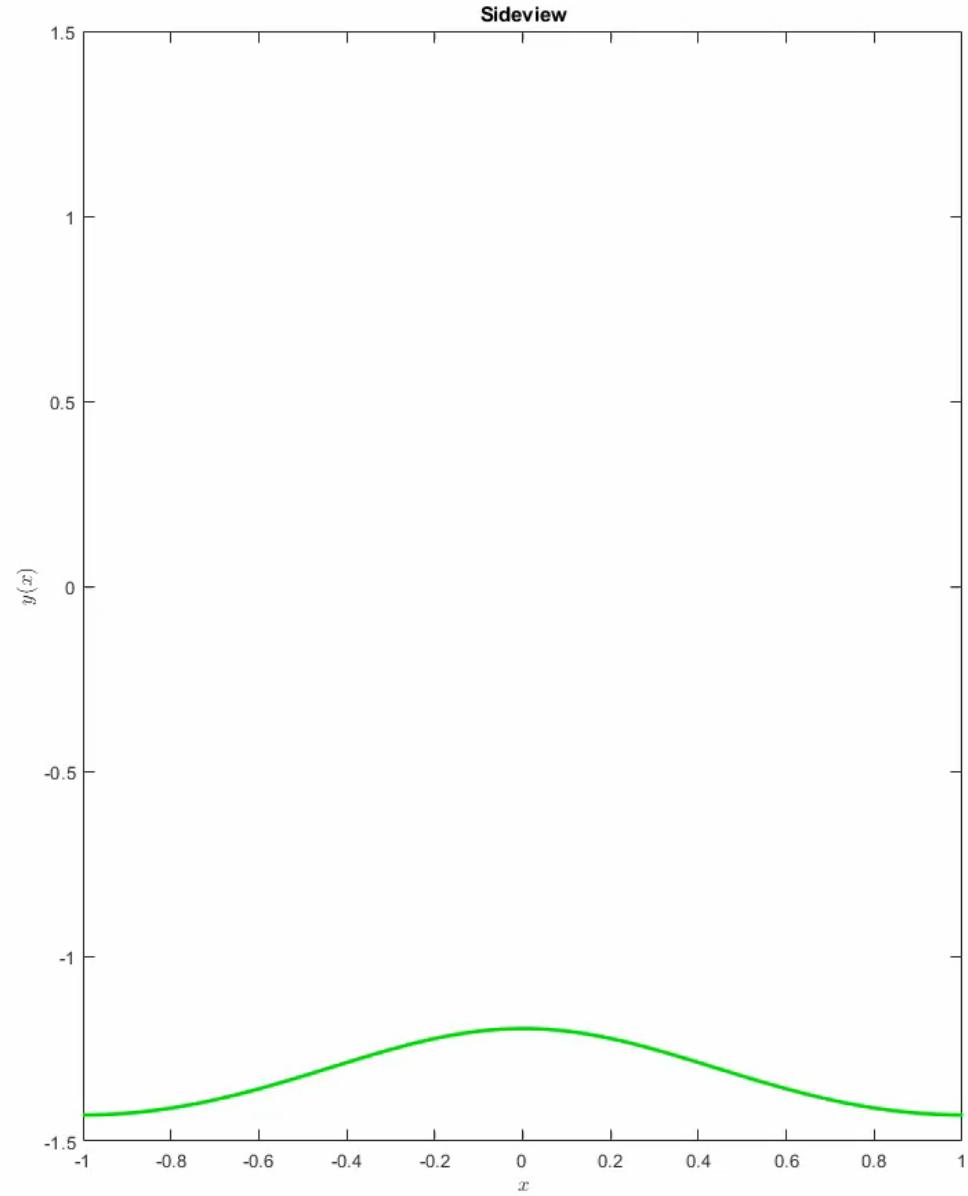
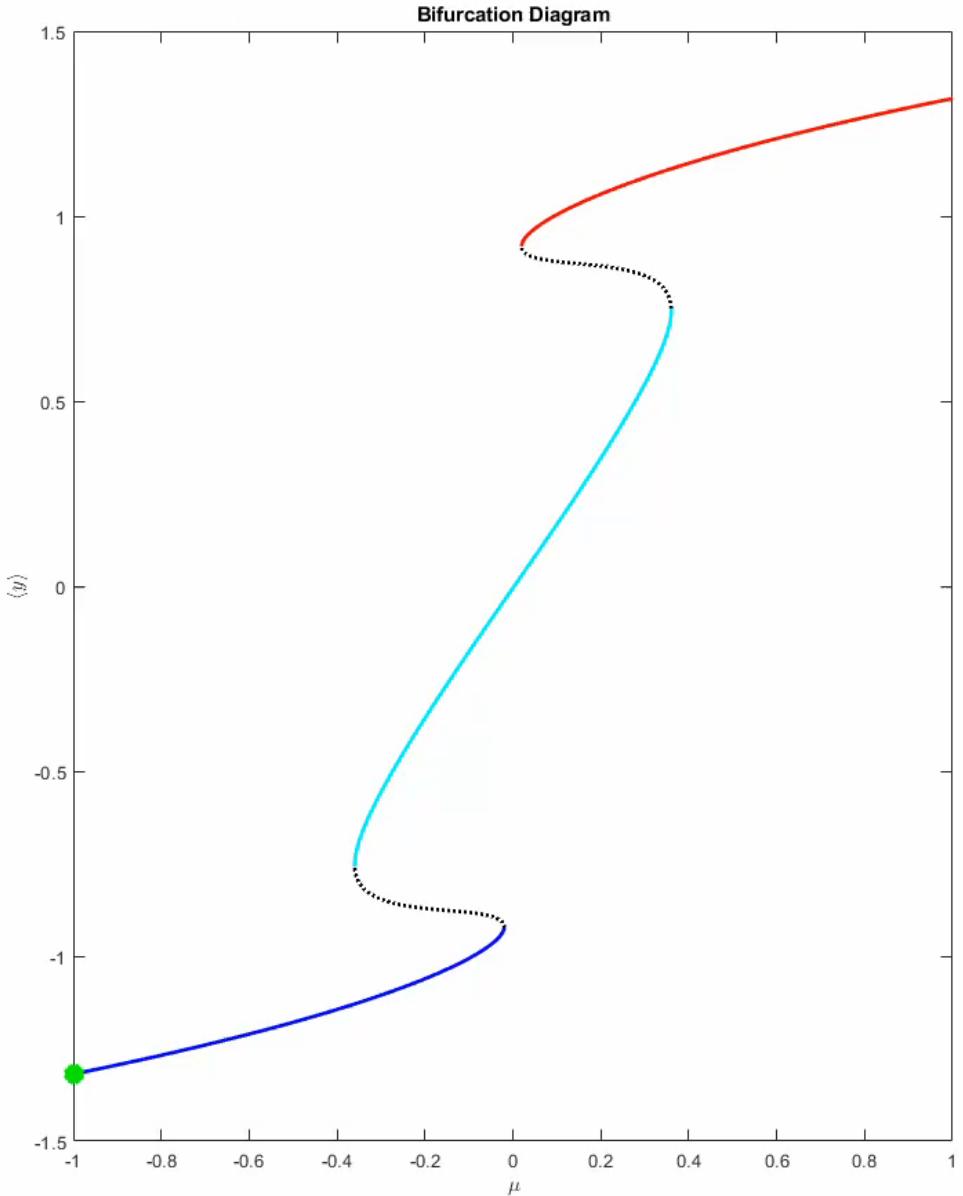
$$\frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2} + f(y, x; \mu)$$

Now, the **local** difference in potentials determines the front movement

- New behaviour:
- Multi-fronts can be stationary
  - Maxwell point is smeared out



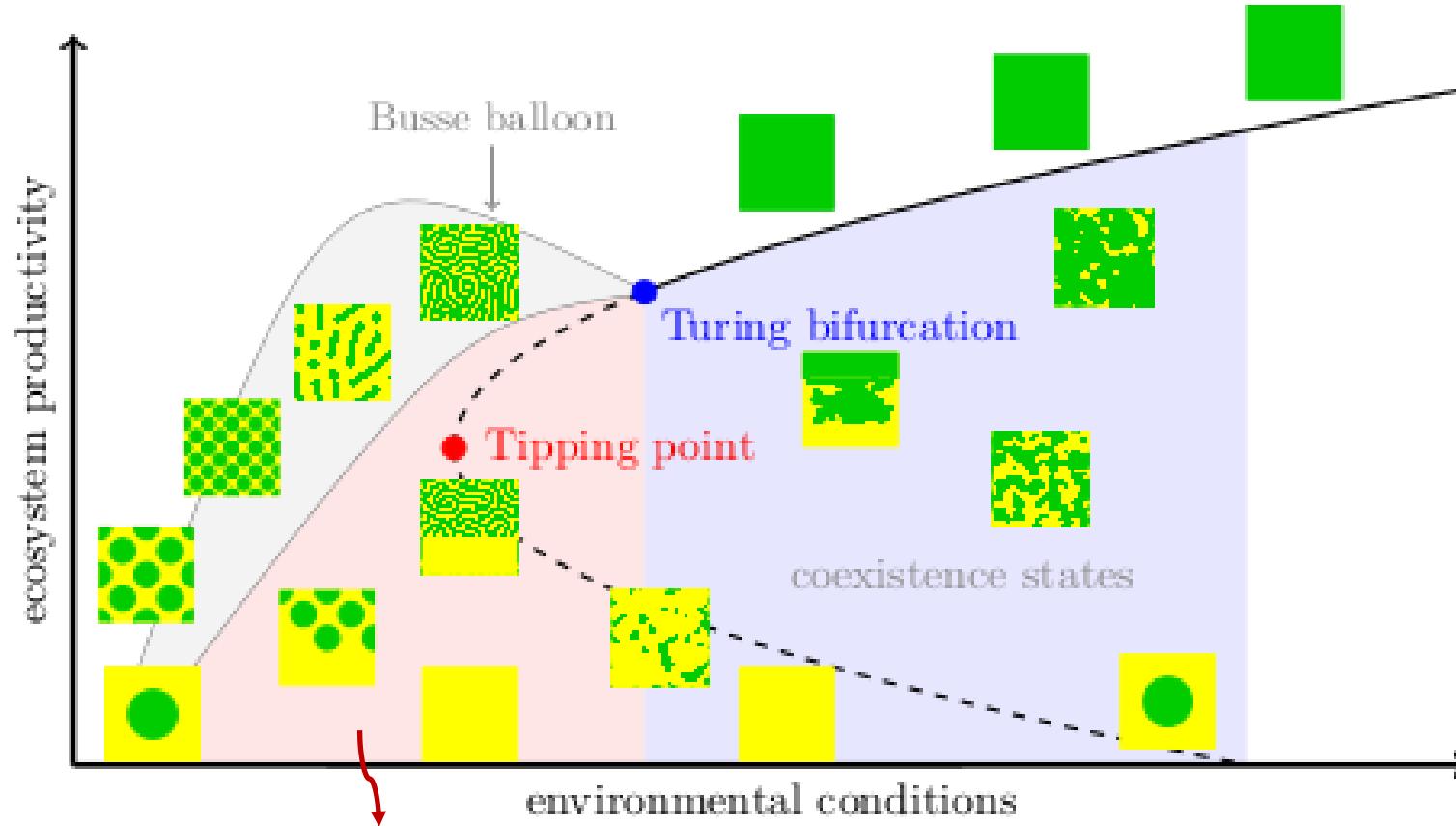
# Fragmented Tipping



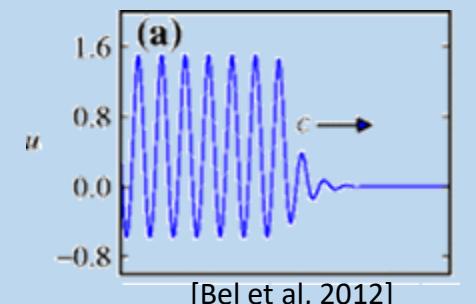
An aerial photograph of a grassland fire on a hillside. The fire is visible as a bright orange and red line of flames moving across the dry, yellowish-brown grass. A large plume of dark smoke is visible in the lower-left foreground, billowing upwards and to the right. The hillside slopes upwards from left to right, with the fire following the contours of the land.

# Part 3: Tipping in Spatially Extended Systems?

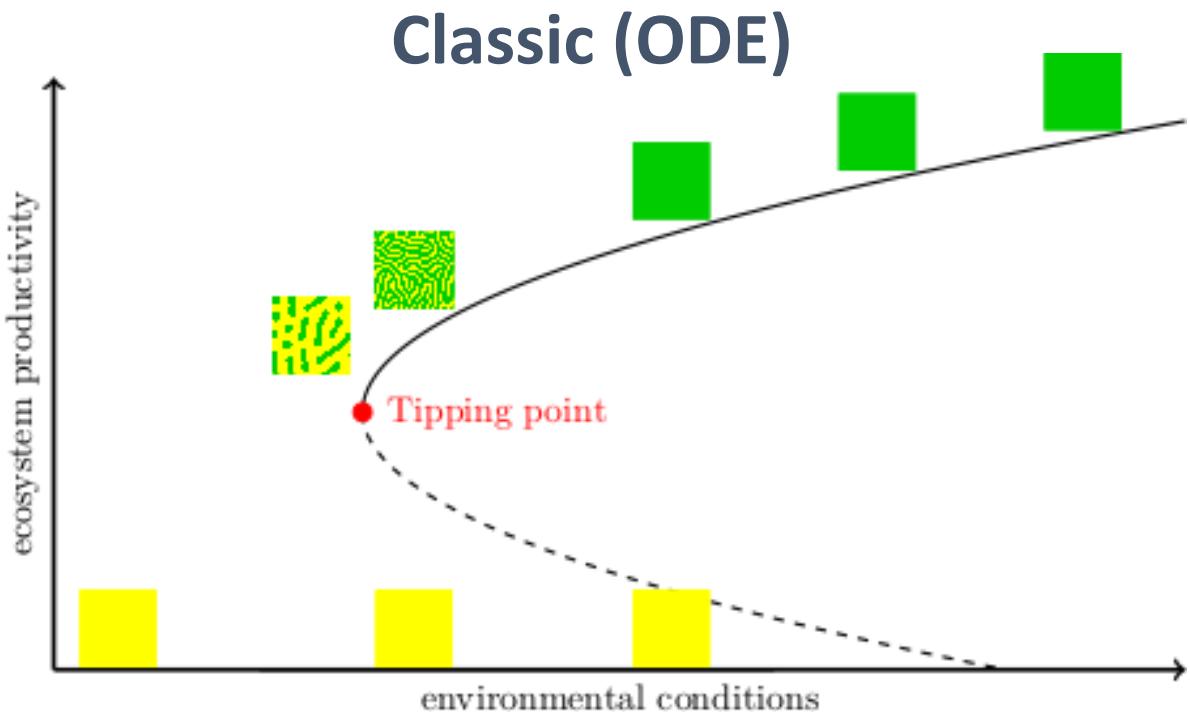
# “Bifurcation Diagram” for spatially extended systems



Coexistence states  
between patterned and  
uniform states also exist

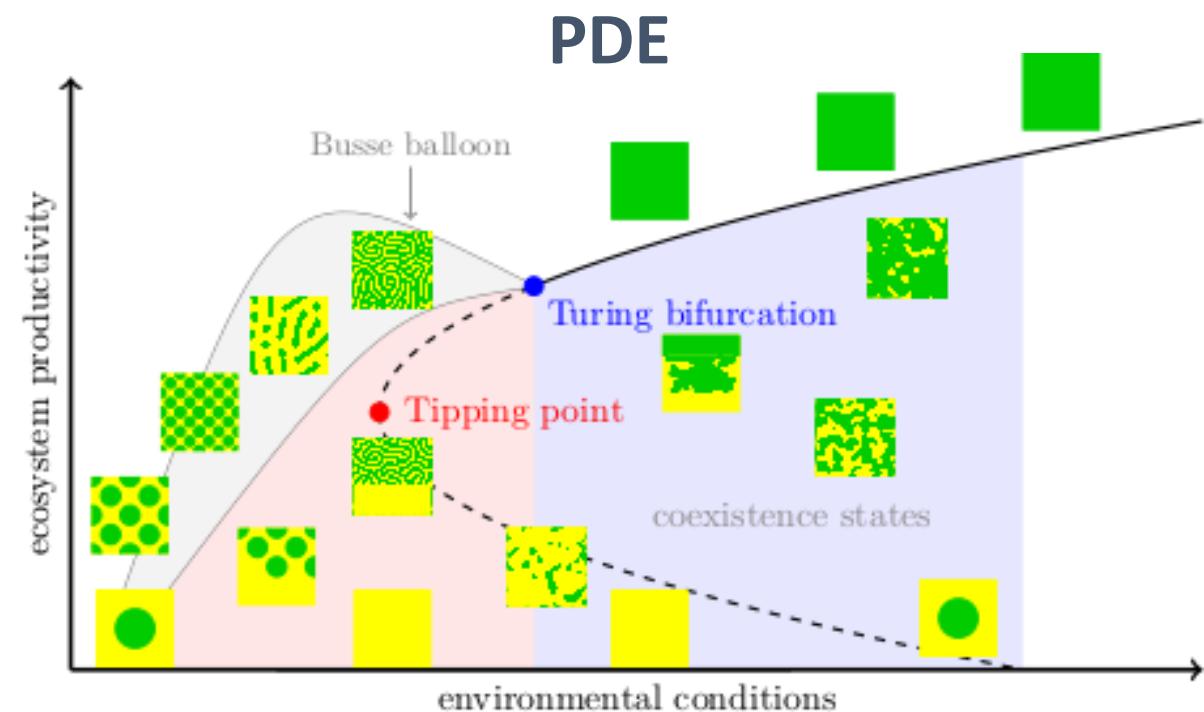


# What if the system tips?



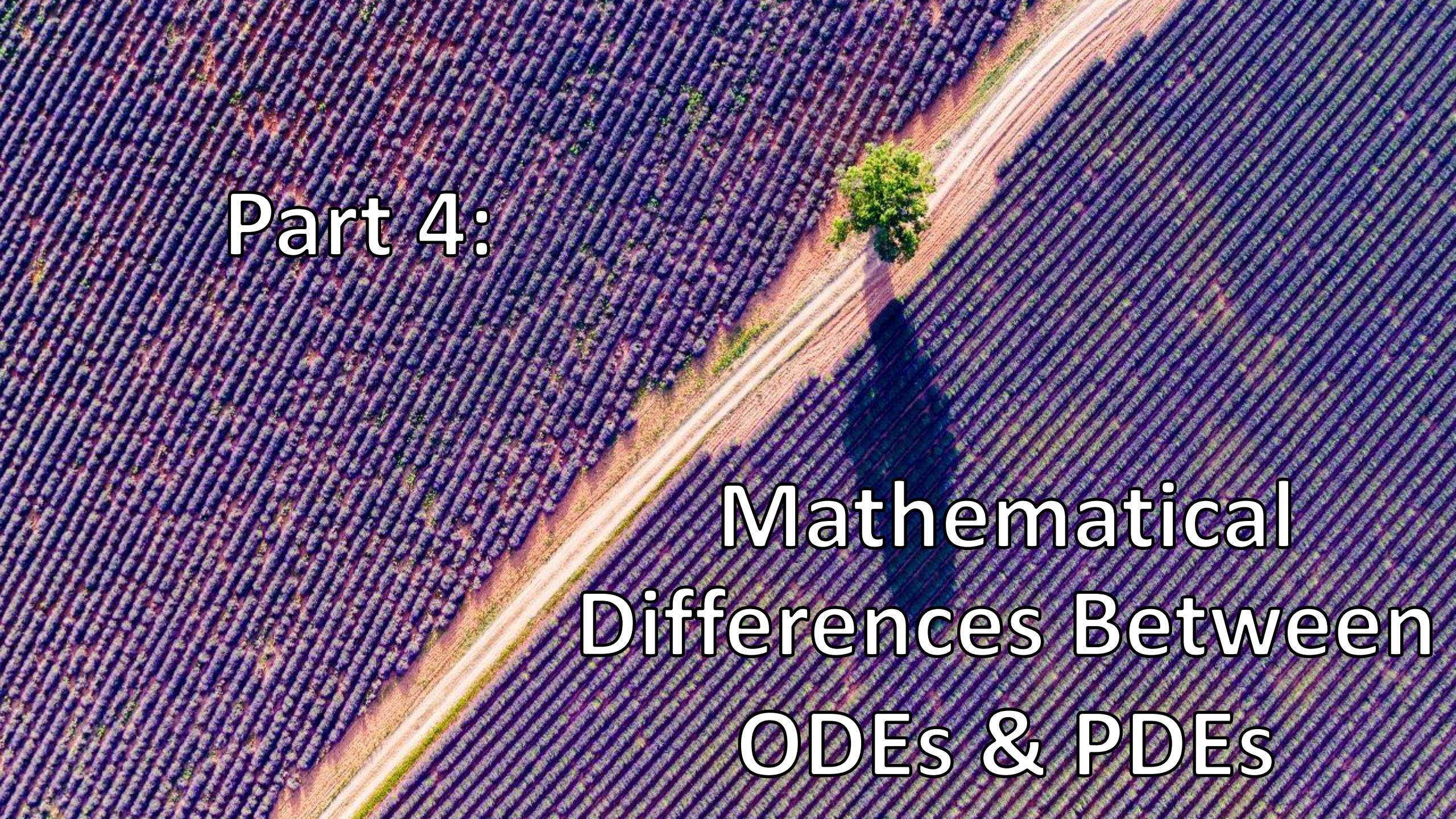
Crossing a Tipping Point:  
→ Always full reorganization

**Early Warning Signals**  
signal for WHEN



Crossing a bifurcation:  
Now also possible:  
→ Spatial reorganization (Turing patterns)  
→ Fragmented tipping (coexistence states)

**Early Warning Signals**  
need to signal WHEN & WHAT

An aerial photograph showing a vast, sprawling lavender field. The field is organized into numerous long, narrow, dark purple rows that stretch across the frame. A single, small, bright green tree stands prominently in the center of a dirt road that cuts through the fields. The overall pattern is highly geometric and repetitive.

**Part 4:**

**Mathematical  
Differences Between  
ODEs & PDEs**

# Differences between ODEs and PDEs

	<u>ODE</u>	<u>PDE</u>
Stationary States	$0 = f(y^*; \mu)$	$0 = y_{xx}^* + f(y^*; \mu)$
Linear Stability	$\lambda \bar{y} = f_y(y^*; \mu) \bar{y}$	$\lambda \bar{y} = \bar{y}_{xx} + f_y(y^*(x); \mu) \bar{y}$

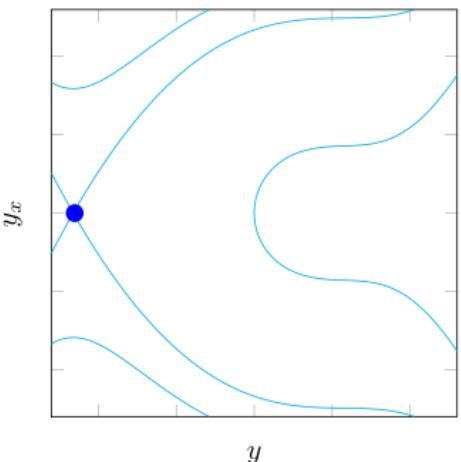
# Stationary States

$$y_t = y_{xx} + f(y; \mu)$$

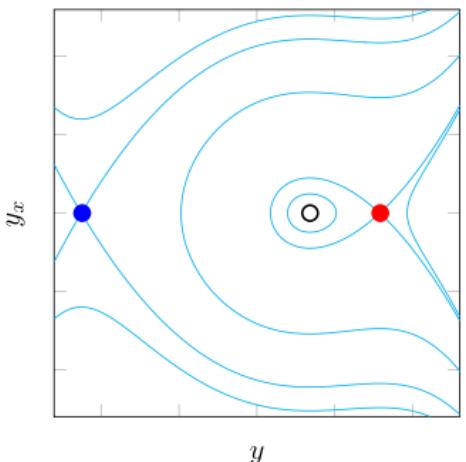
**Stationary states**

$$0 = y_{xx} + f(y; \mu)$$

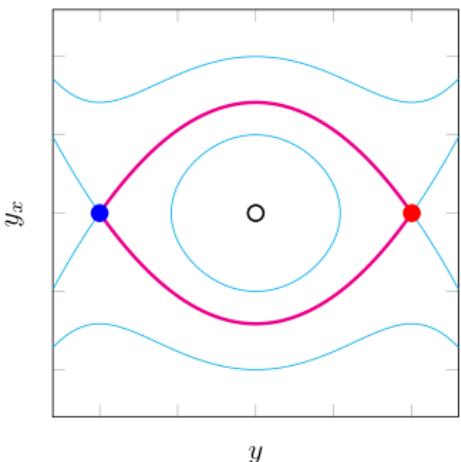
$$\begin{cases} y_x = p \\ p_x = -f(y; \mu) \end{cases}$$



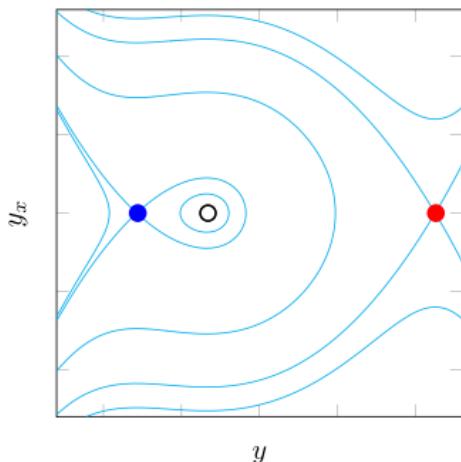
(a)  $\mu < \mu_B$



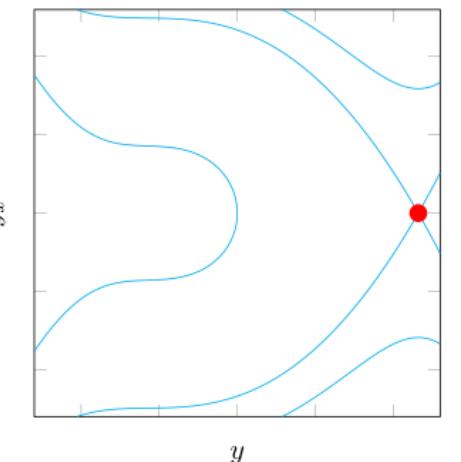
(b)  $\mu_B < \mu < \mu_M$



(c)  $\mu = \mu_M$



(d)  $\mu_M < \mu < \mu_A$

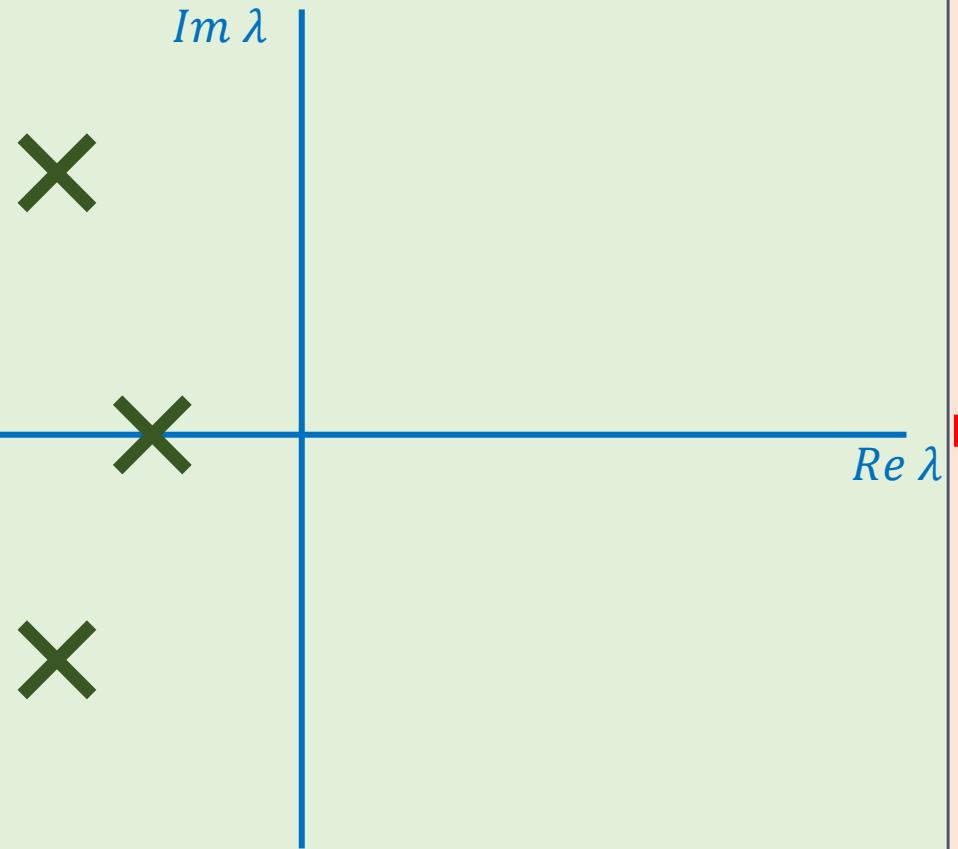


(e)  $\mu > \mu_A$

# Stability of Stationary States

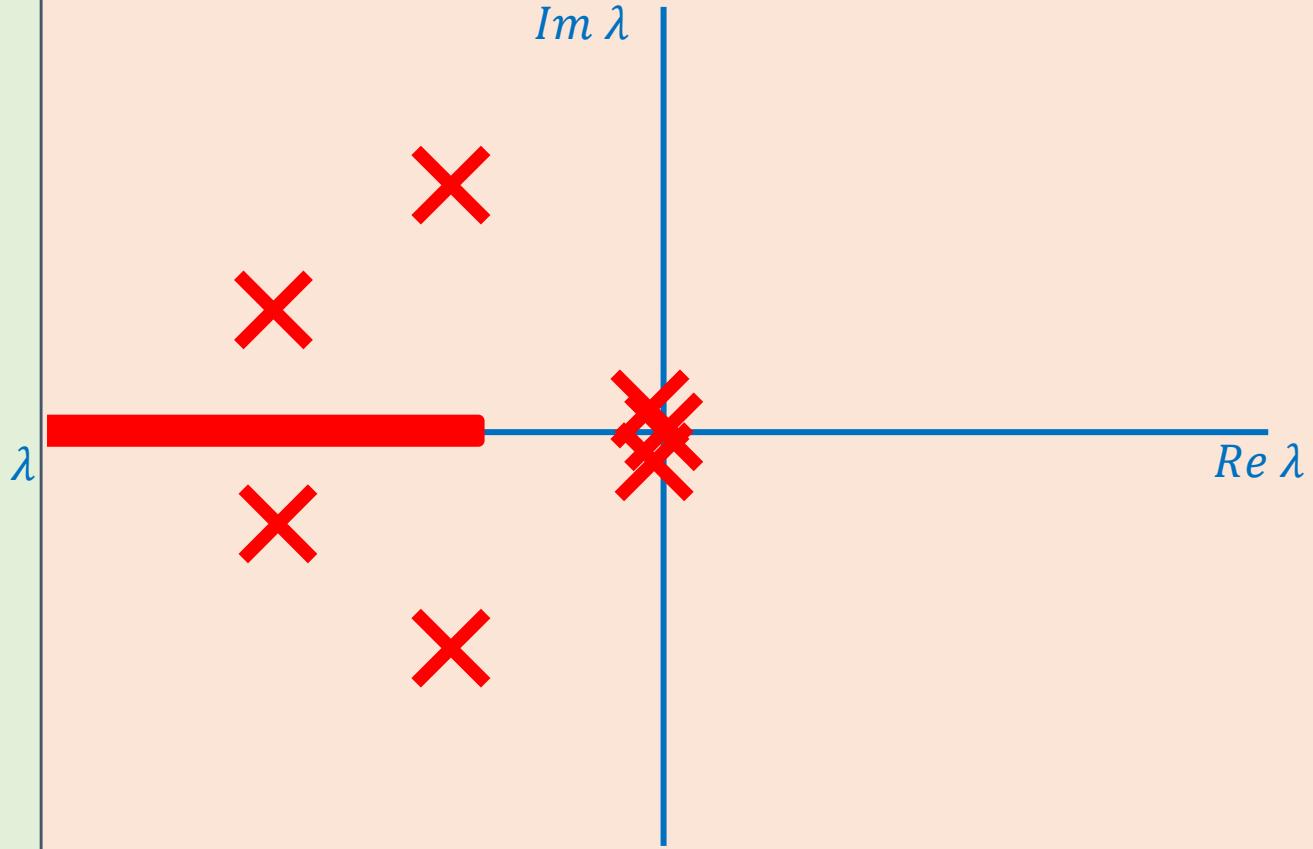
ODE

$$\lambda \bar{y} = f_y(y^*; \mu) \bar{y}$$



PDE

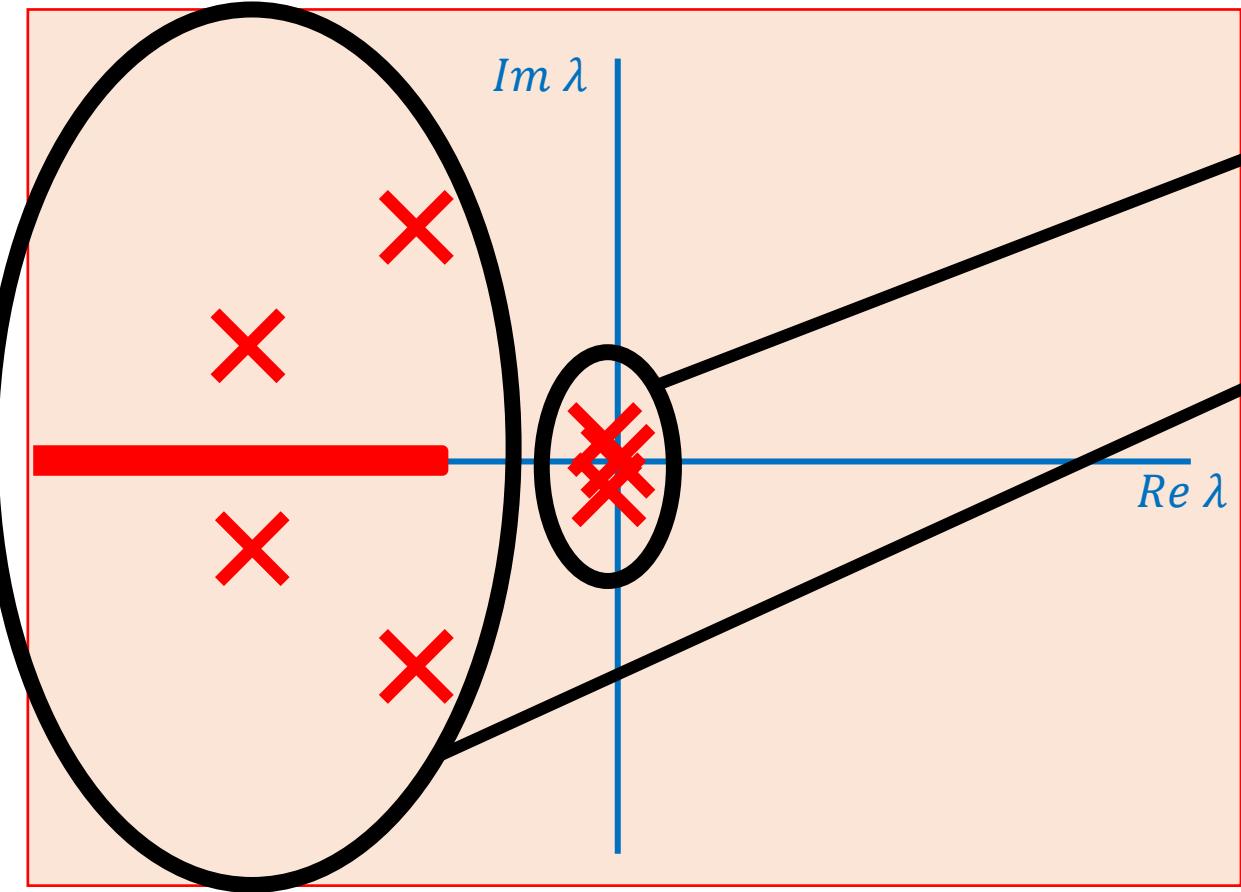
$$\lambda \bar{y} = \bar{y}_{xx} + f_y(y^*(x); \mu) \bar{y}$$



A massive, jagged glacier wall with deep blue ice at its base, set against a dark, rocky mountain background and a calm, light blue lake.

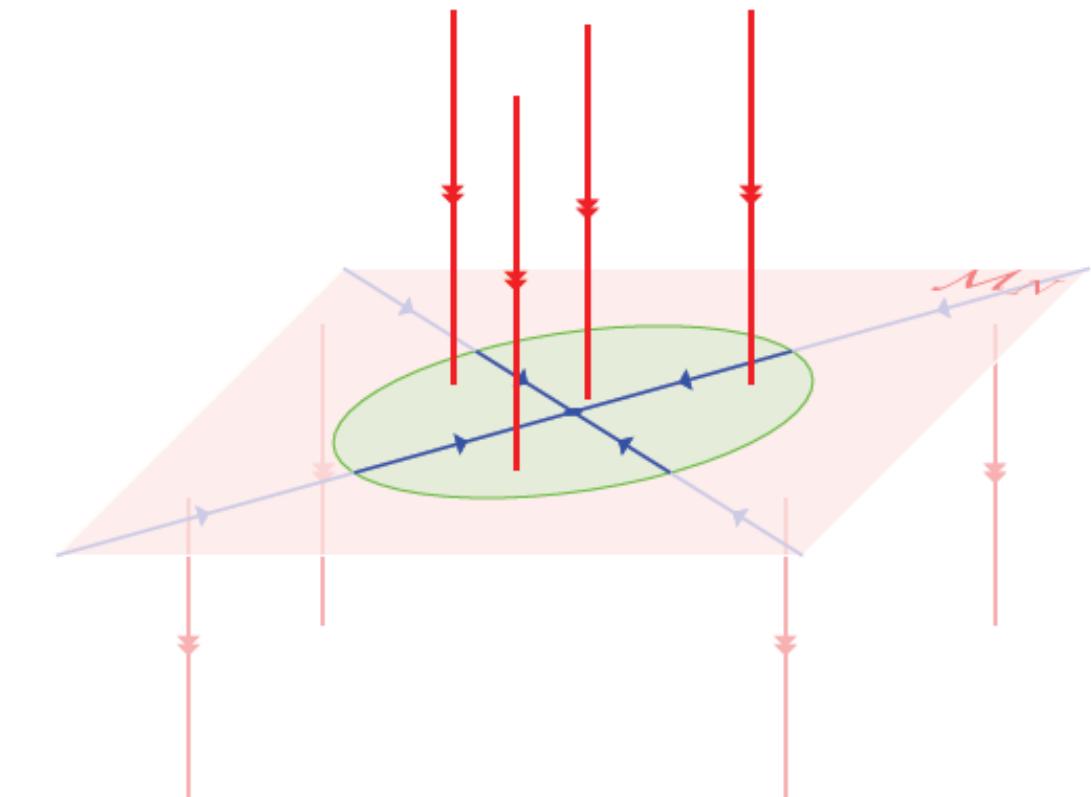
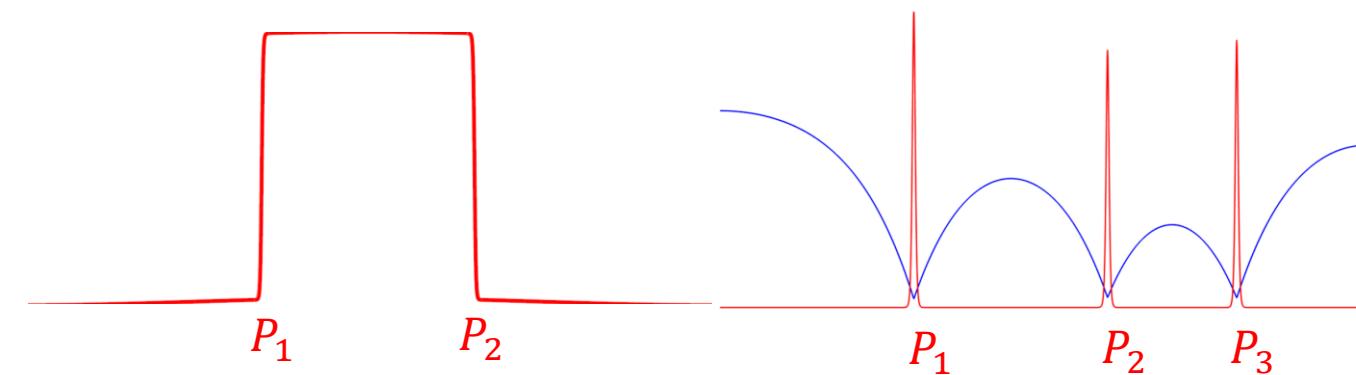
# Part 5: Dynamics & Bifurcations of Patterned States

# Dynamics of Patterned States



1. SLOW Pattern Adaptation

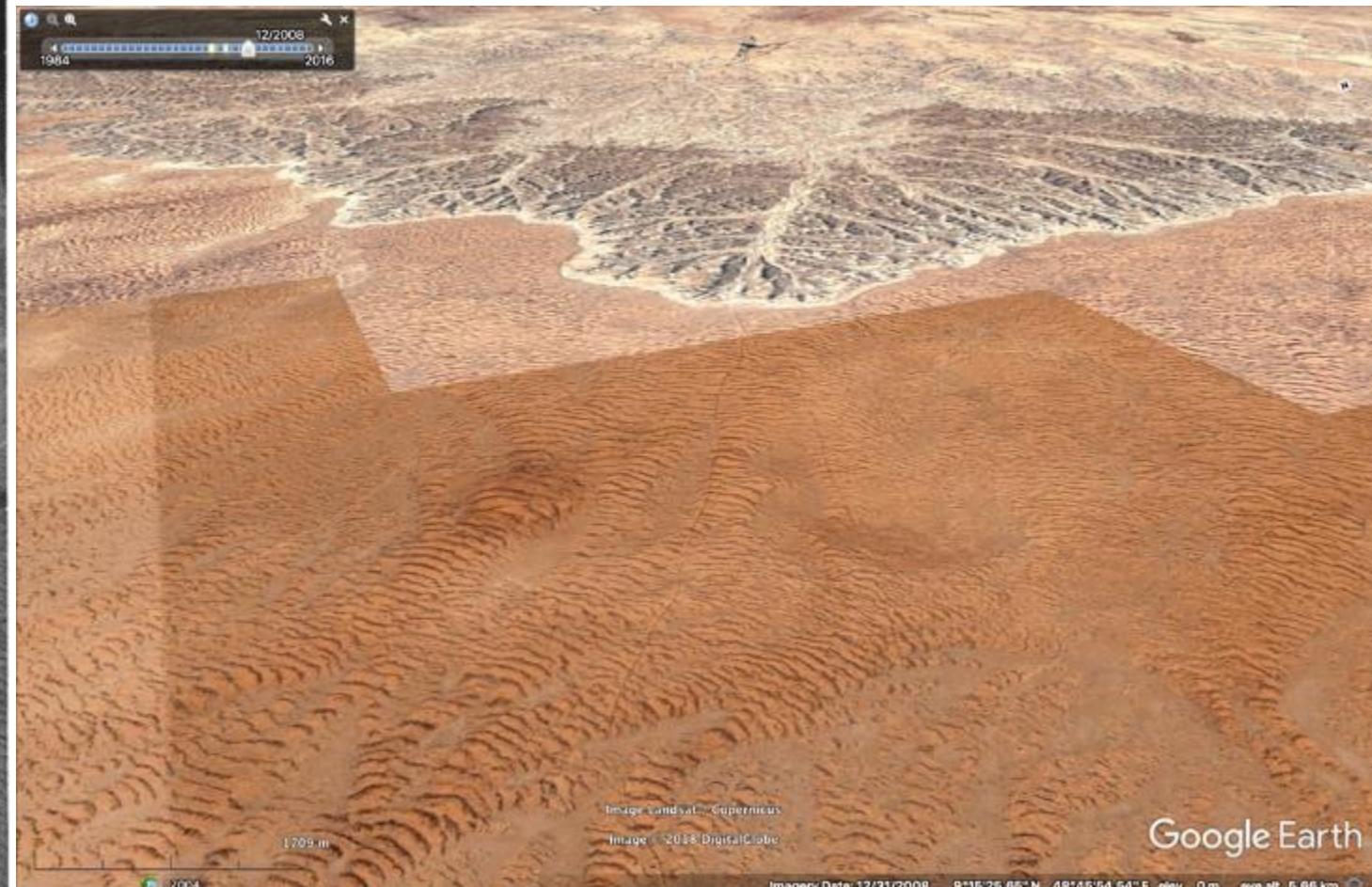
2. FAST Pattern Degradation



# 1. SLOW pattern adaptation

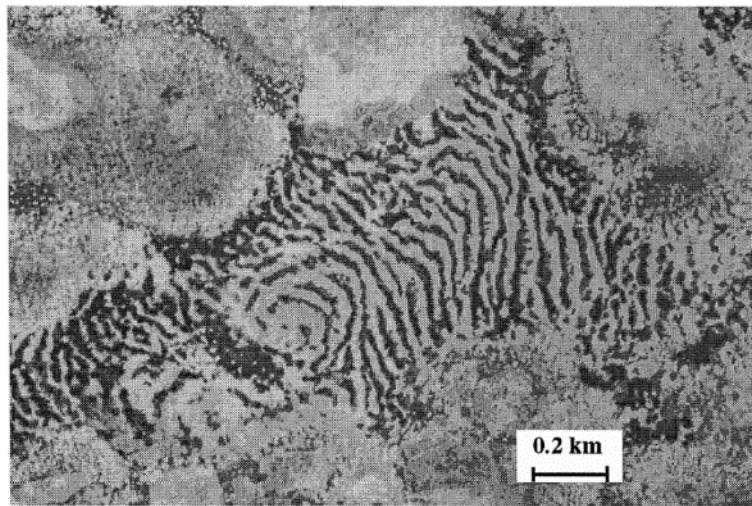


Somaliland, 1948 [Macfadyen, 1950]



Somaliland, 2008

## 2. FAST Pattern Degradation



Niger, 1950 [Valentin, 1999]



Niger, 2008



Niger, 2010



Niger, 2011

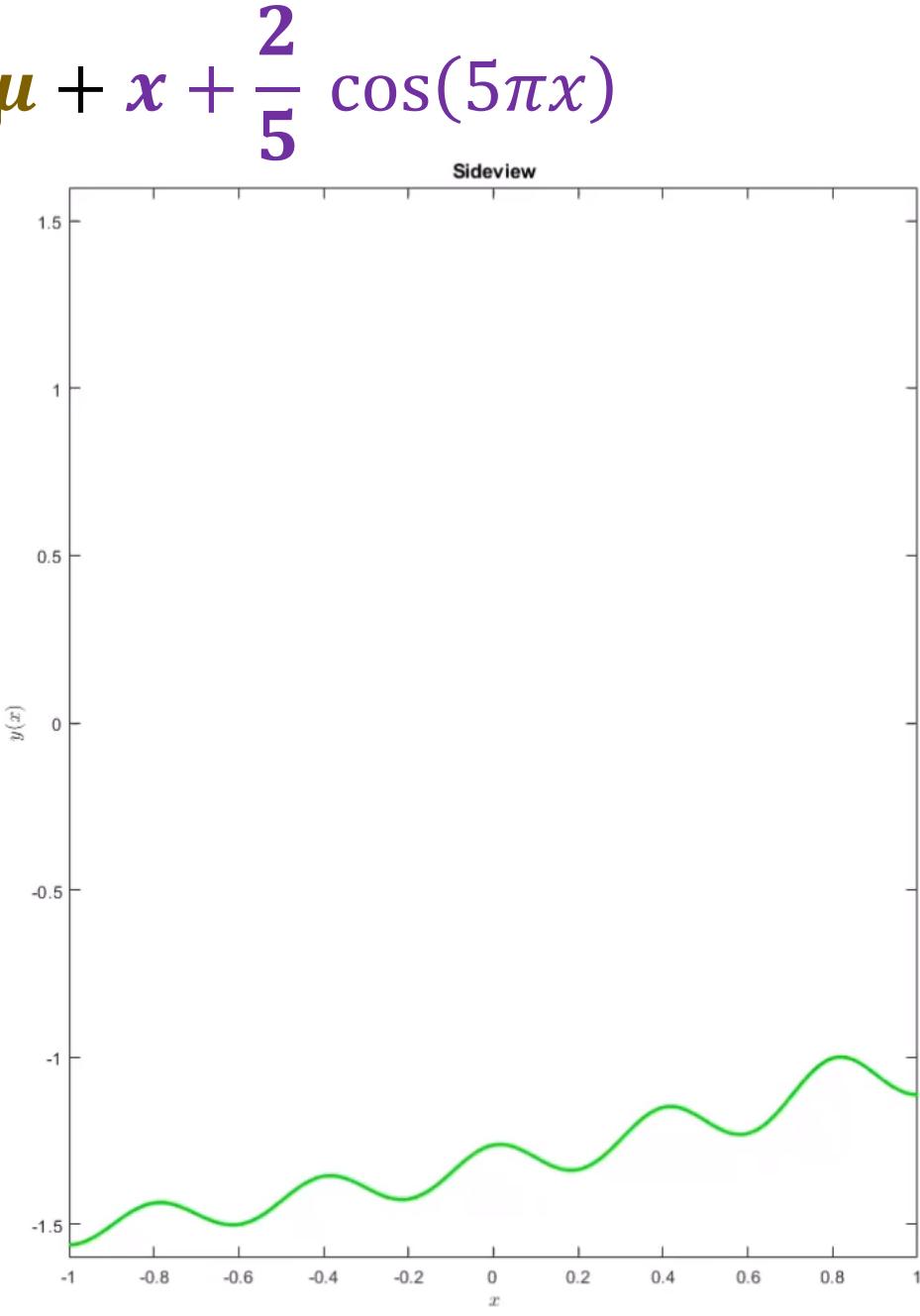
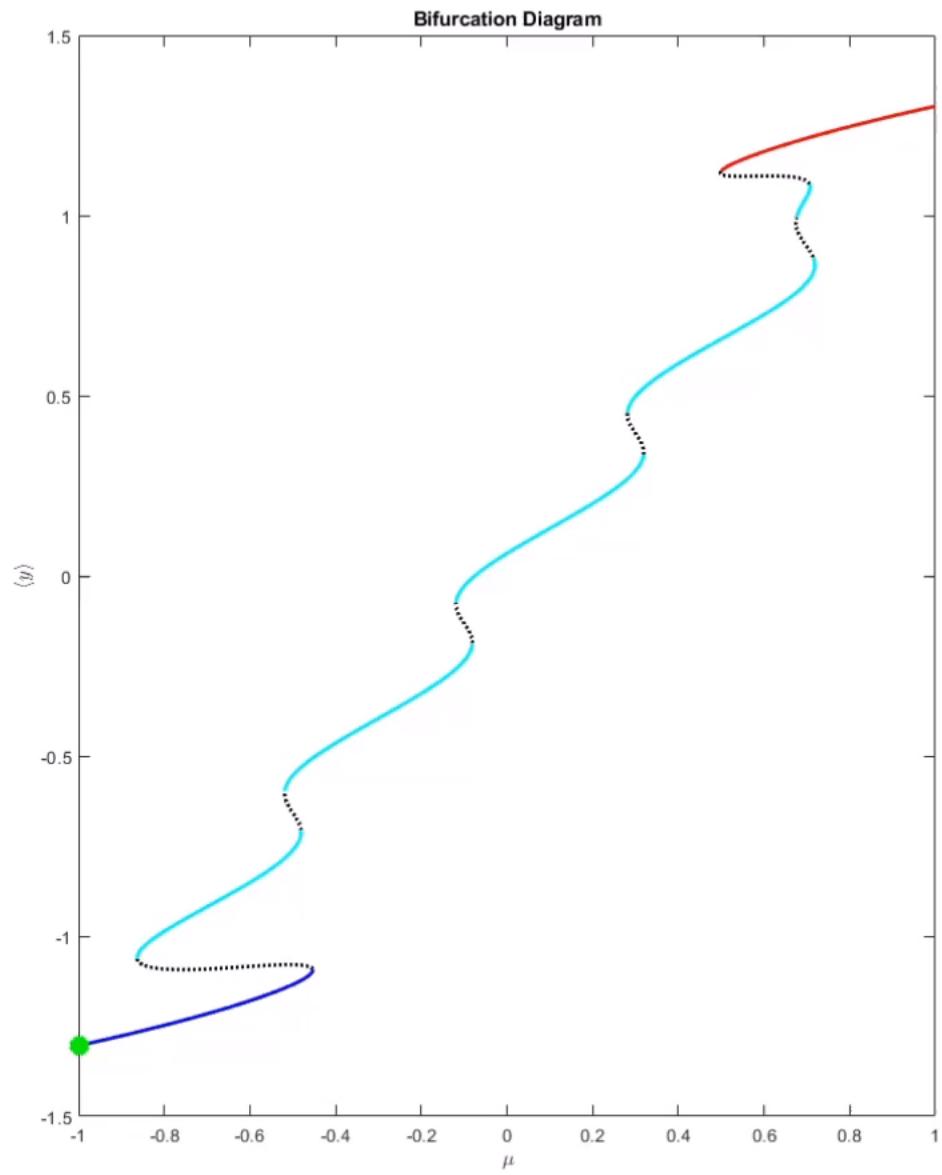


Niger, 2014

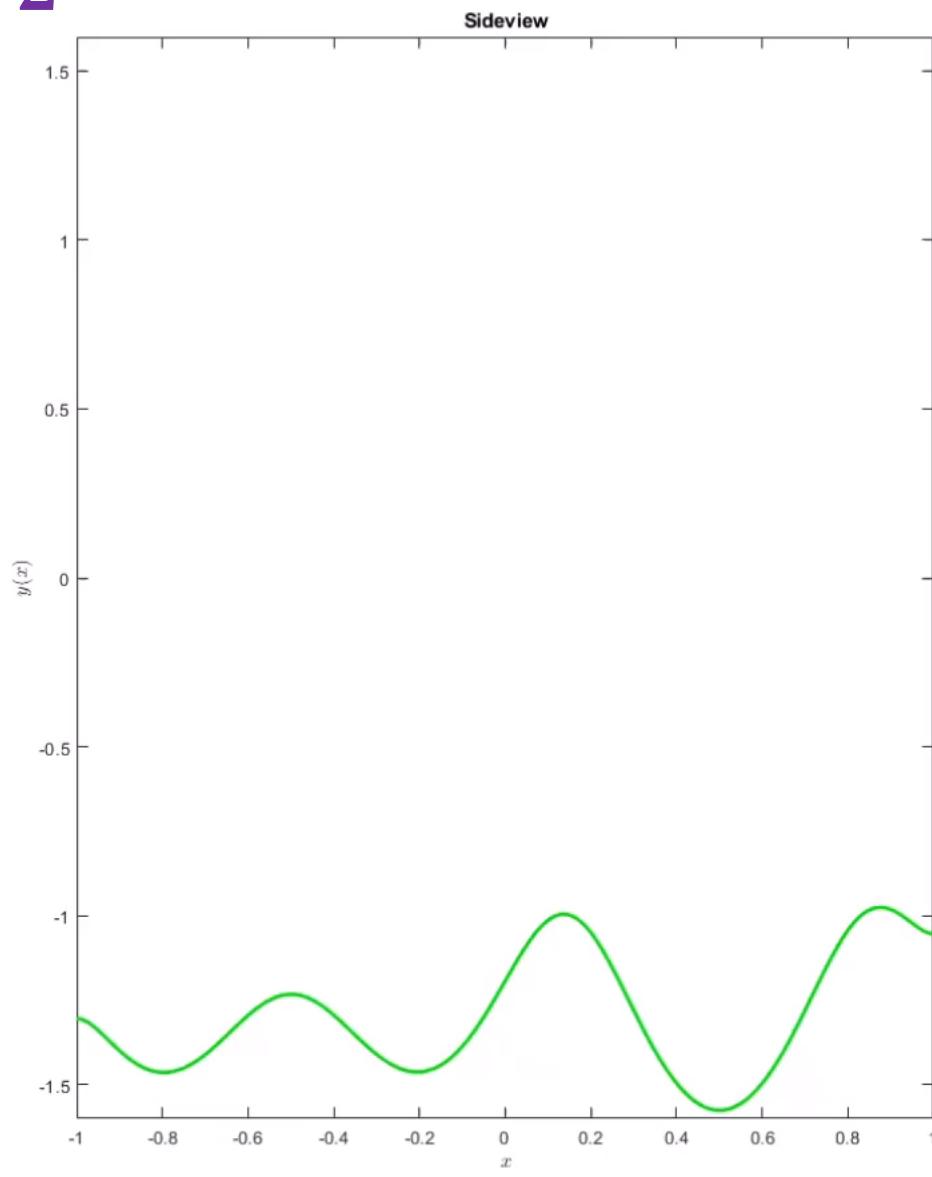
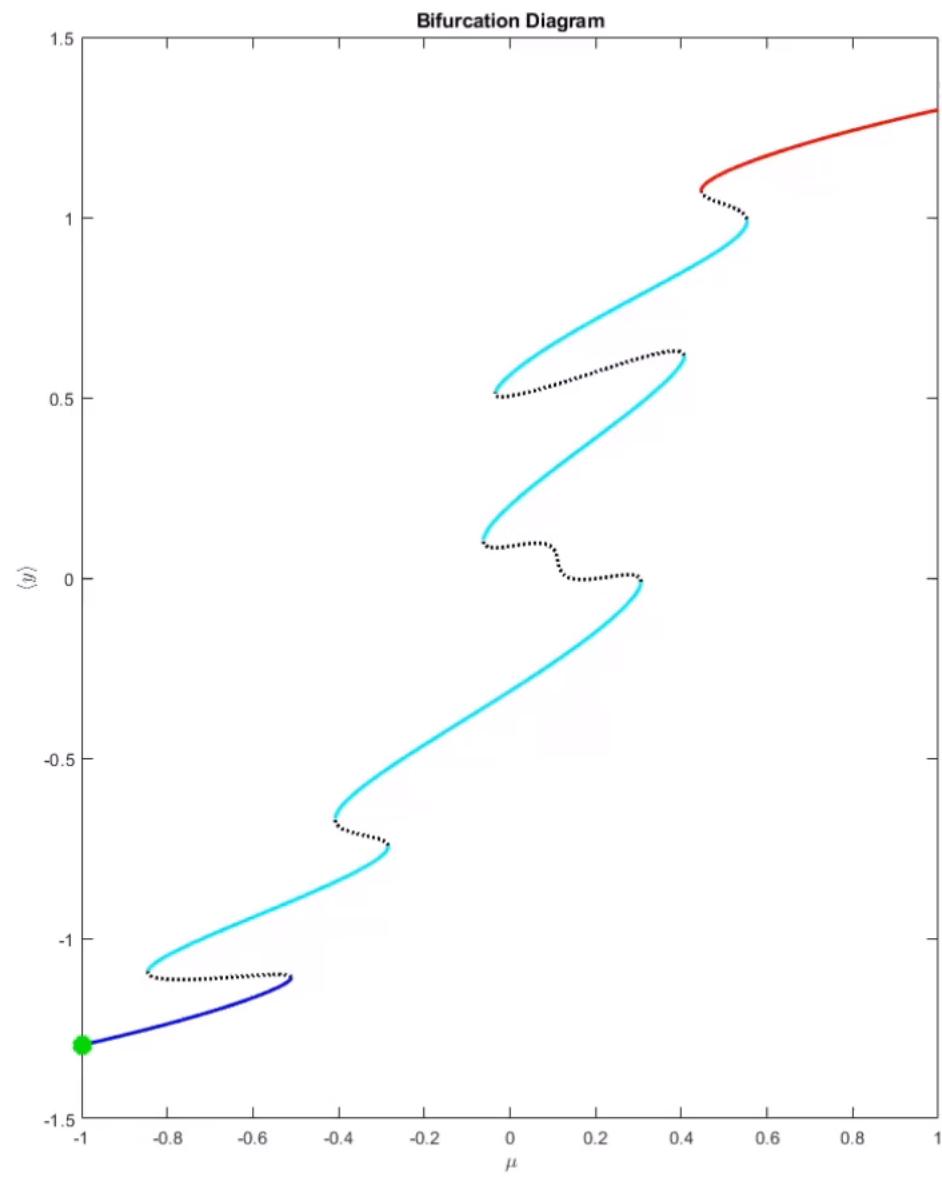


Niger, 2016

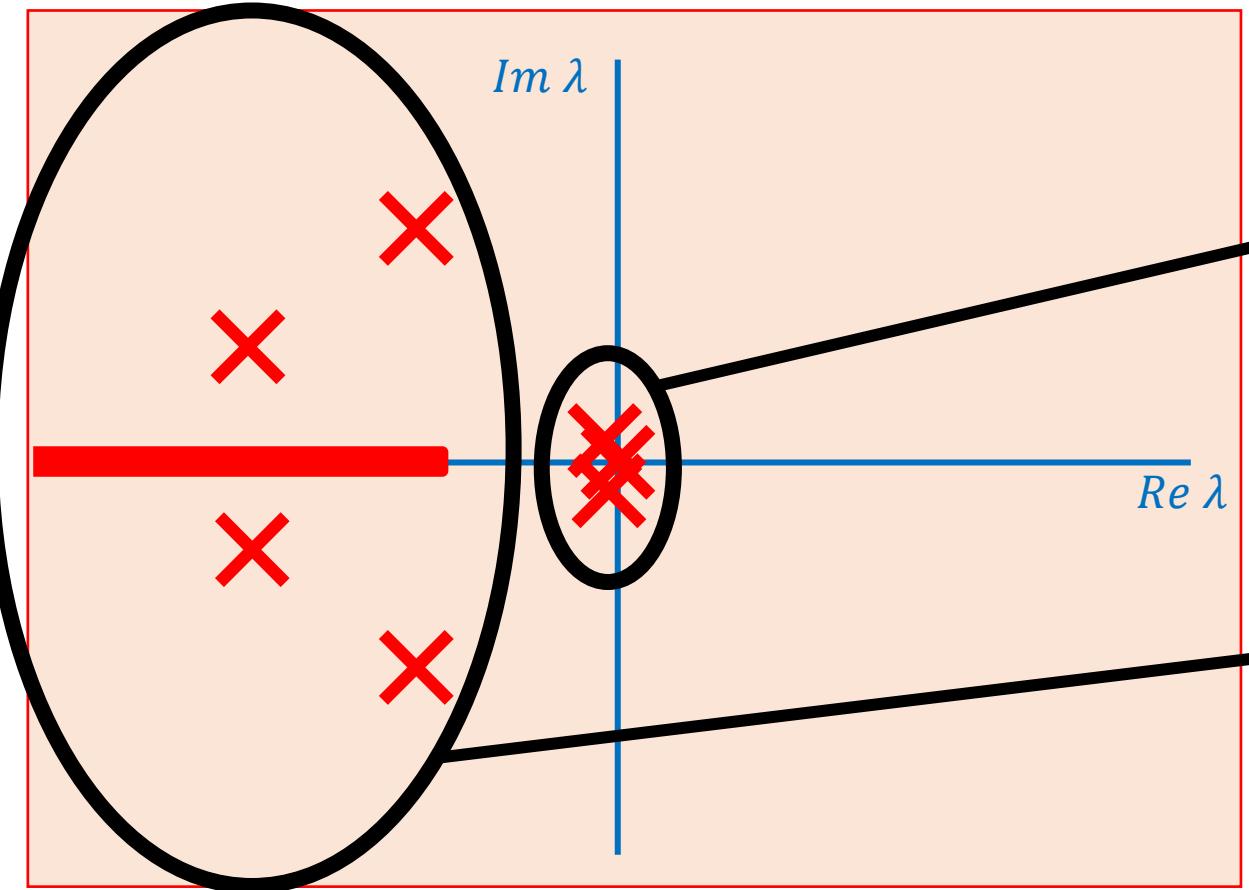
$$y_t = D y_{xx} + y(1 - y^2) + \mu + x + \frac{2}{5} \cos(5\pi x)$$



$$y_t = D y_{xx} + y(1 - y^2) + \mu + \frac{1}{2} \cos(2\pi x) + \sin(3\pi x)$$



# Bifurcations



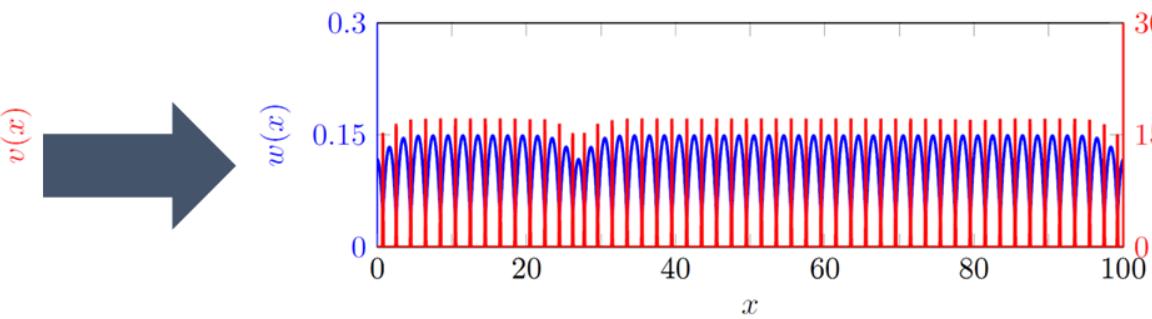
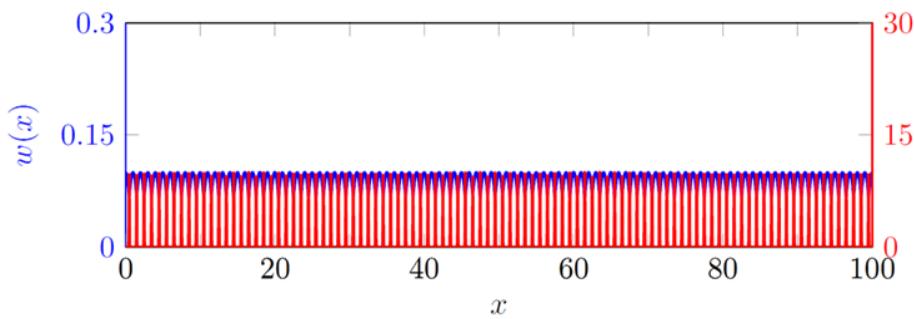
What happens at bifurcation?

## 1. SLOW Pattern Adaptation

At bifurcation:  
→ Location of structure changes

## 2. FAST Pattern Degradation

At bifurcation:  
→ Structures created or destroyed

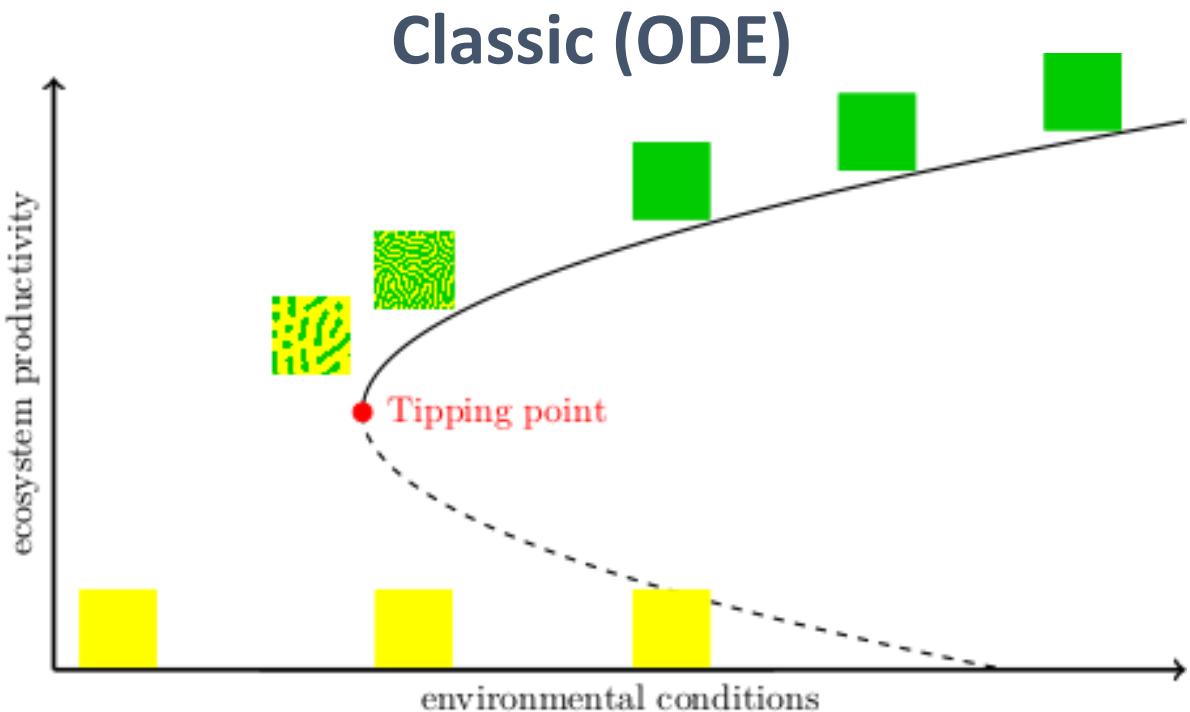


The background image shows a steep hillside where a large portion of the forest has been cleared or killed, leaving a brown, lifeless landscape. This area meets a dense, healthy forest of green coniferous trees on the right side of the slope. The text is overlaid on this image.

# Summary

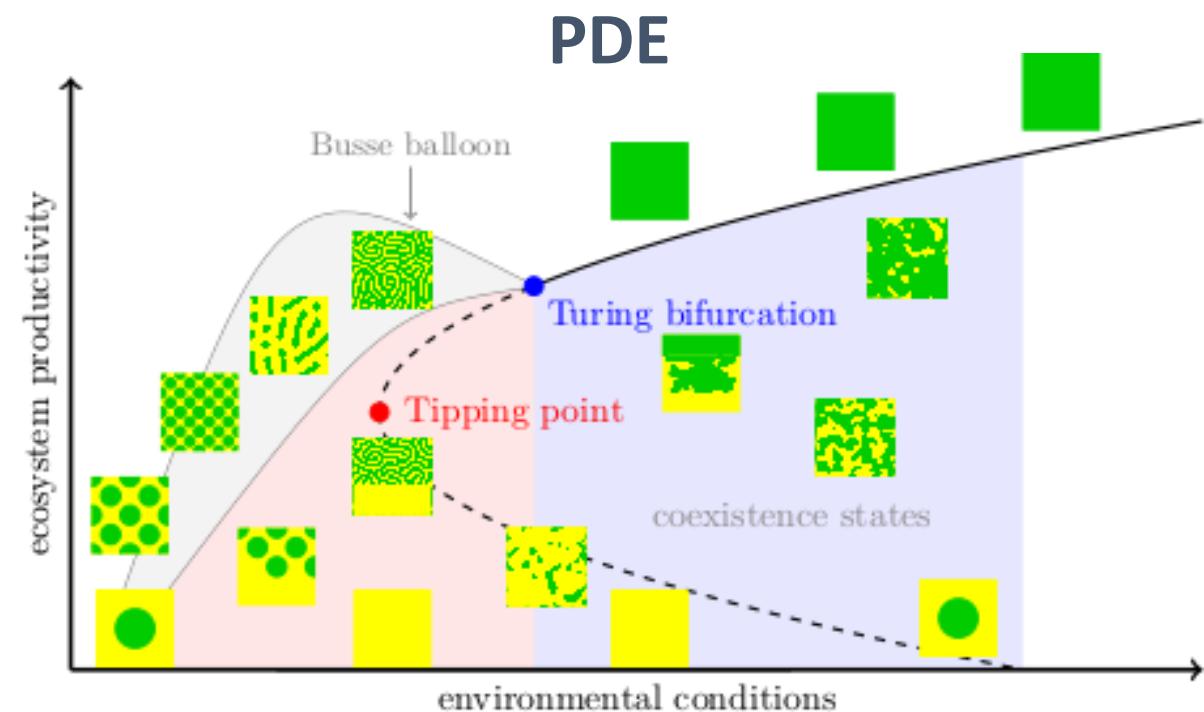
## Tipping in Spatially Extended Systems

# What if the system tips?



Crossing a Tipping Point:  
→ Always full reorganization

**Early Warning Signals**  
signal for WHEN



Crossing a bifurcation:  
Now also possible:  
→ Spatial reorganization (Turing patterns)  
→ Fragmented tipping (coexistence states)

**Early Warning Signals**  
need to signal WHEN & WHAT

# Do systems always behave like this?

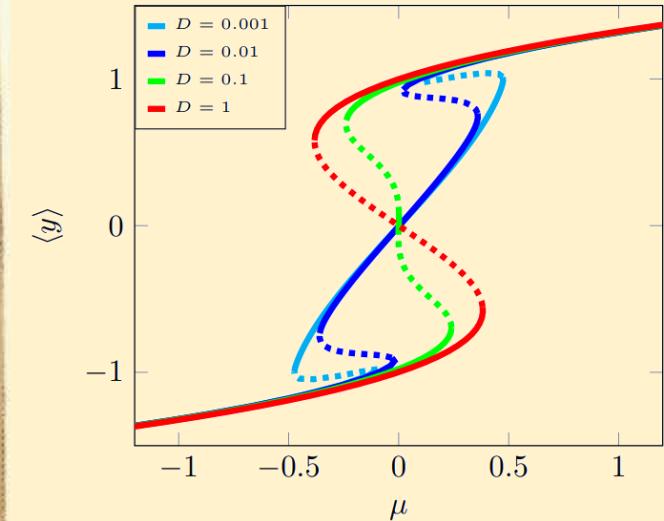
(a.k.a. the small print)

No.

Well-mixed systems



Spatially confined systems



→ Such systems (again) behave like ODEs ←

But even in other systems terms & conditions apply:  
System-specific knowledge is required!

# Spatial Patterns:

- ❖ Turing Patterns

- ❖ Coexistence States

# Tipping can be more subtle:

- ❖ Spatial reorganization

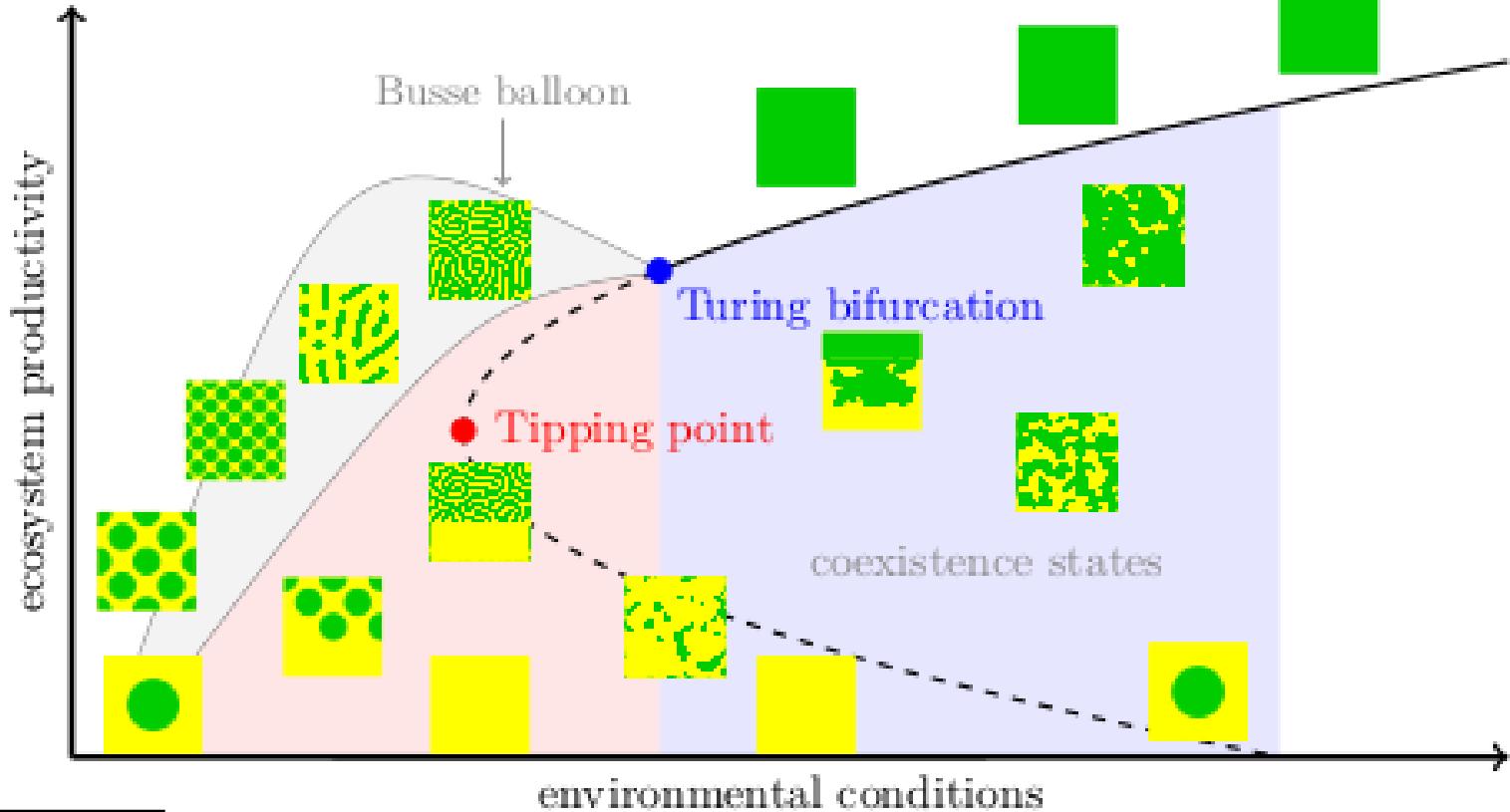
- ❖ Fragmented Tipping

# Dynamics of Patterns is:

- ❖ Slow Pattern Adaptation

- ❖ Fast Pattern Degradation

# Summary



## THANKS TO:

Swarnendu Banerjee

Martina Chirilus-Bruckner

Henk Dijkstra

Olfa Jaïbi

Max Rietkerk

Mara Baudena

Vincent Deblauwe

Maarten Eppinga

Johan van de Koppel

Eric Siero

Alexandre Bouvet

Arjen Doelman

Anna von der Heydt

Stéphane Mermoz

Koen Siteur

Rietkerk, M., Bastiaansen, R., Banerjee, S., van de Koppel, J., Baudena, M., & Doelman, A. (2021). Evasion of tipping in complex systems through spatial pattern formation. *Science*, 374(6564), eabj0359.



Bastiaansen, R., Dijkstra, H. A., & von der Heydt, A. S. (2021). Fragmented Tipping in a spatially heterogeneous world. *Environmental Research Letters*, 17, 045006





