

# Behaviour of Spatial Patterns

2024-05-23, AXIOMA 2024  
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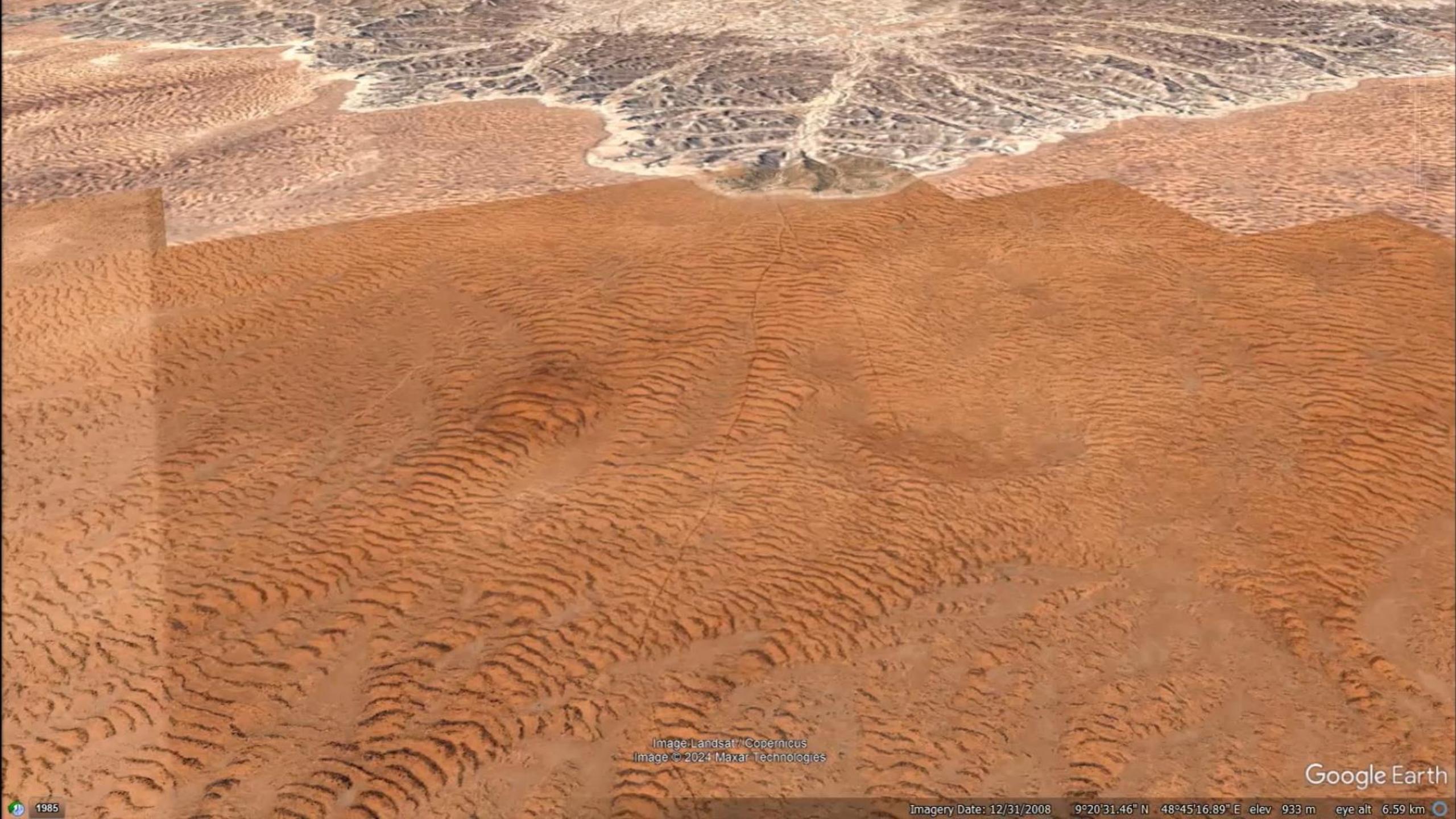


Image:Landsat//Copernicus  
Image © 2024 Maxar Technologies

Google Earth



Image © 2024 Maxar Technologies

Google Earth

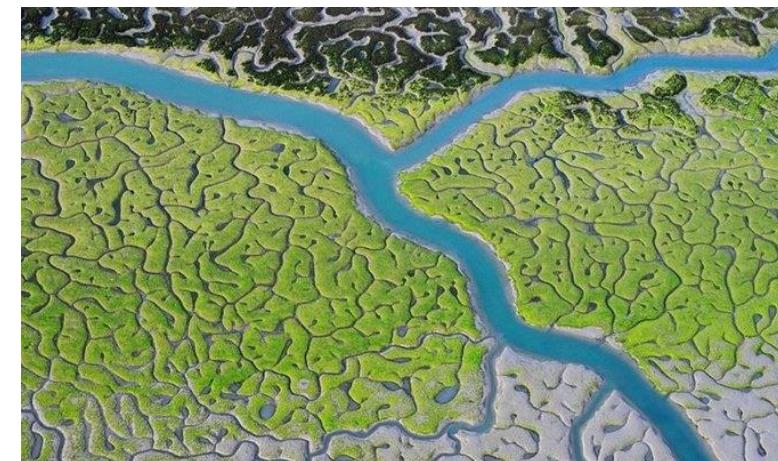
# Examples of spatial patterning – ecosystems



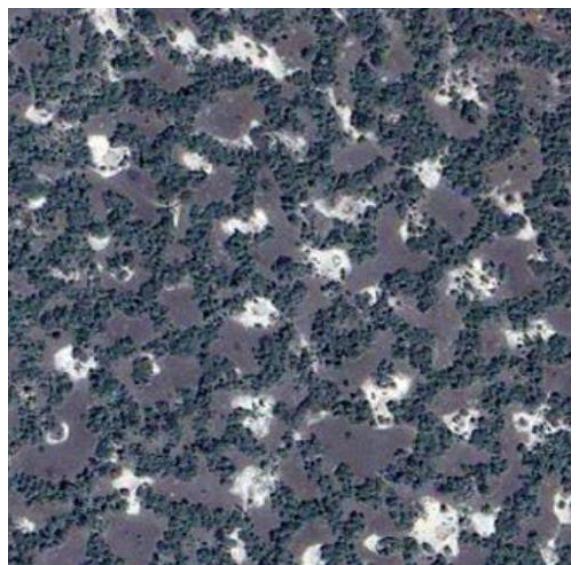
mussel beds



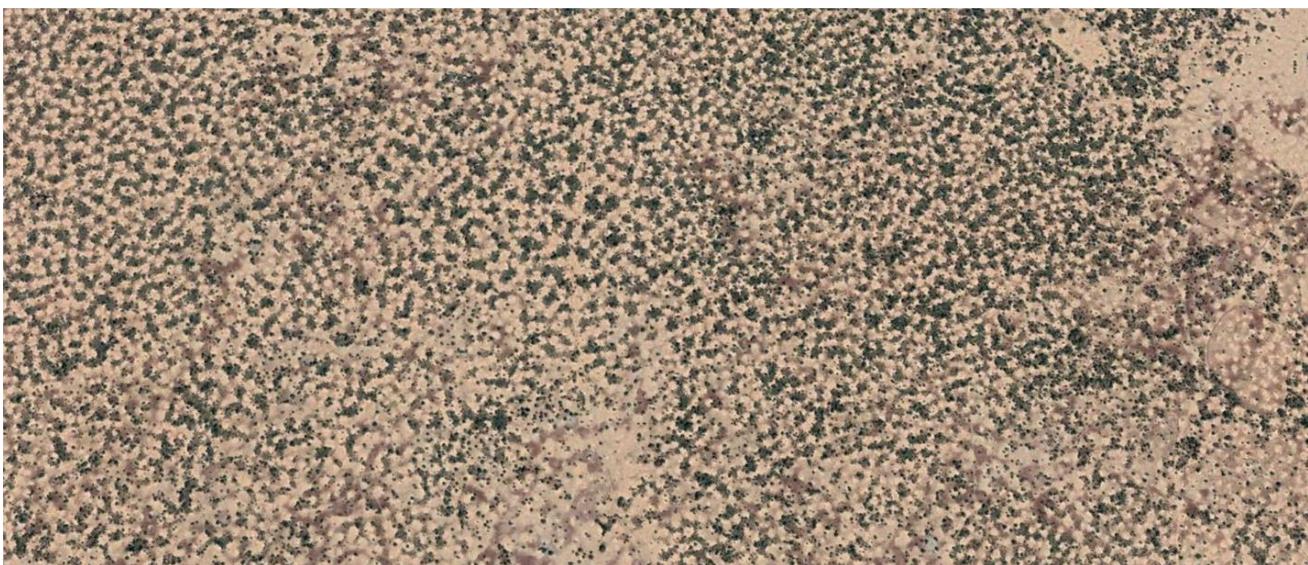
vegetation in coastal systems



marsh formation



savannas

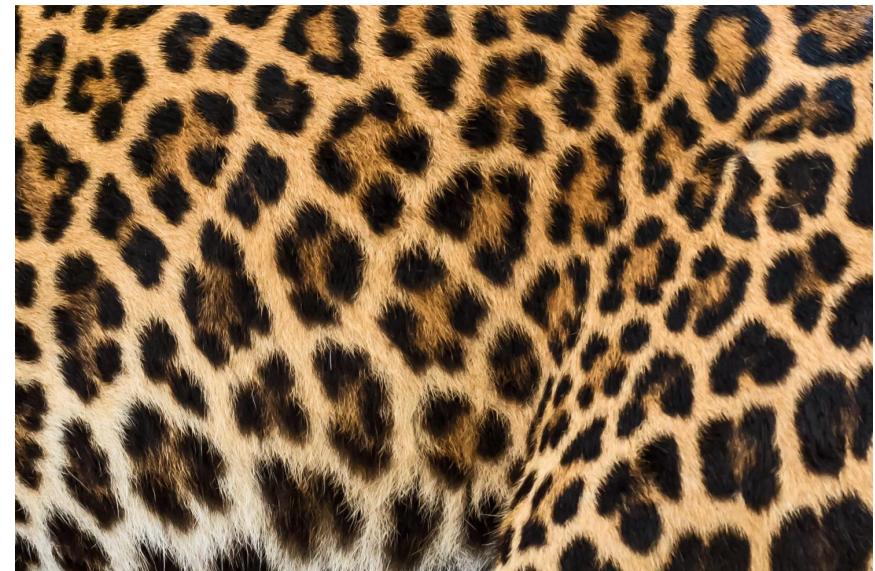


drylands



tropical forests

# Examples of spatial patterning – animals

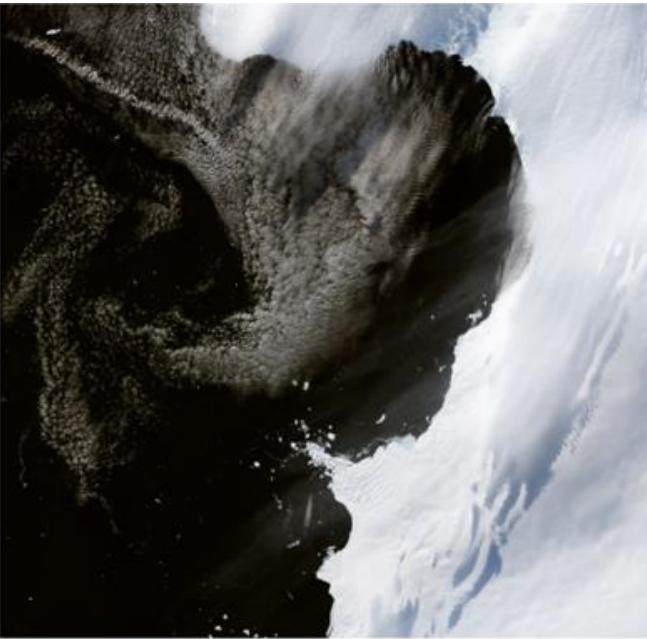


# Examples of spatial patterning – climate



sea-ice & water  
at Eltanin Bay

[NASA's Earth observatory]



melt ponds



sand dunes



clouds

# Examples of spatial patterning – sociology

The population of the United States is not distributed evenly. Instead, we tend to bunch up in communities, leaving the spaces in between more sparsely inhabited. Most Americans live in or near cities; today 53 percent live in the 20 largest cities. 75 percent of all Americans live in metropolitan areas.

This map shows population density. The relative height of each major city reflects its population in 1990.  
Source: U.S. Census Bureau

Go West. Nevada is the fastest growing state, followed by Arizona, Idaho, Colorado, and Utah.

Wyoming has the lowest population density of all states in the lower 48 with an average of five people per square mile.

What happens in the empty spaces? Some of it is farming country. More than one quarter of America's crop land is used to grow corn. One third of what is produced is exported to other countries.

Chicago, the country's third largest city, has a population of about three million people. There are 21 states with populations smaller than this city.

Largest metropolitan area includes New York City and portions of New Jersey and Long Island with a total population of 20 million.

## Population Distribution

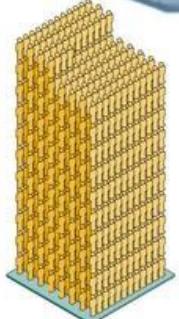
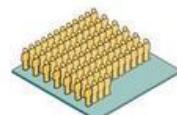
Where do we live?  
Where don't we live?



Population density is highest in New York City, where there are 23,000 people per square mile.

Coastal areas are home to more than half the U.S. population.

Approximately one in nine Americans lives in the nation's most populous state—California. More than 15 million people live in the Los Angeles, Riverside, and Orange County metropolitan area.



Distributing our population evenly would put an average of 76 people per square mile.

New Jersey is the most densely populated state with an average of more than 1,000 people per square mile.

Alaska is a sparsely populated state with an average of one person per square mile.

# Examples of spatial patterning – physics



# Examples of spatial patterning – physics



# Examples of spatial patterning – physics/mussels



# Programme

- 0) Lengthy introduction with patterns in all sort of systems
- 1) How do spatial patterns emerge?
  - Importance for climate tipping points?
- 2) How do spatial patterns behave?
  - Why ice cream does not stay soft?

algal bloom  
in Lake St. Clair  
[NASA's Earth observatory]



# A bit about myself

Since 2022: Assistant professor @ Utrecht University

Joint appointment:

- Mathematical Institute, Mathematical Department
- Institute for Marine and Atmospheric Research Utrecht (IMAU), Physics Department

Research focusses on climate and ecosystem responses in context of climatic changes using techniques from dynamical systems theory.

EARLIER:

2020-2022: Postdoc @ IMAU on climate response and tipping points

2015-2019: PhD @ Leiden University on vegetation patterns and desertification



# Part 1: Emergence of Spatial Patterns

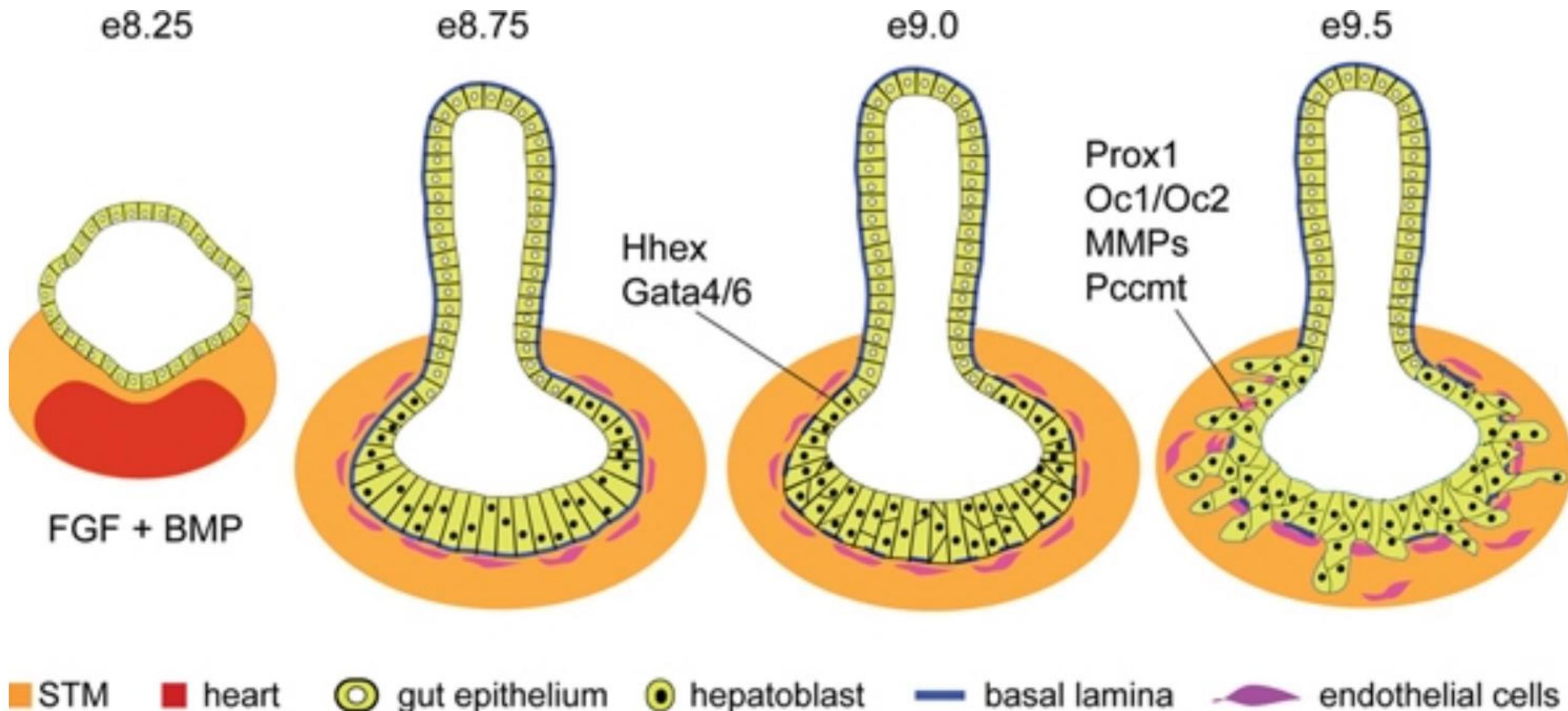
The background image shows a steep hillside with significant environmental damage. The upper portion of the slope is covered in dead, greyish-brown trees, many of which are leaning or fallen. Below this, there is a layer of exposed, eroded earth and debris. The bottom of the slope and the surrounding area are covered in a mix of living green trees and more dead, skeletal remains. The overall scene suggests a major event like a wildfire or a landslip has occurred.

Active Creation

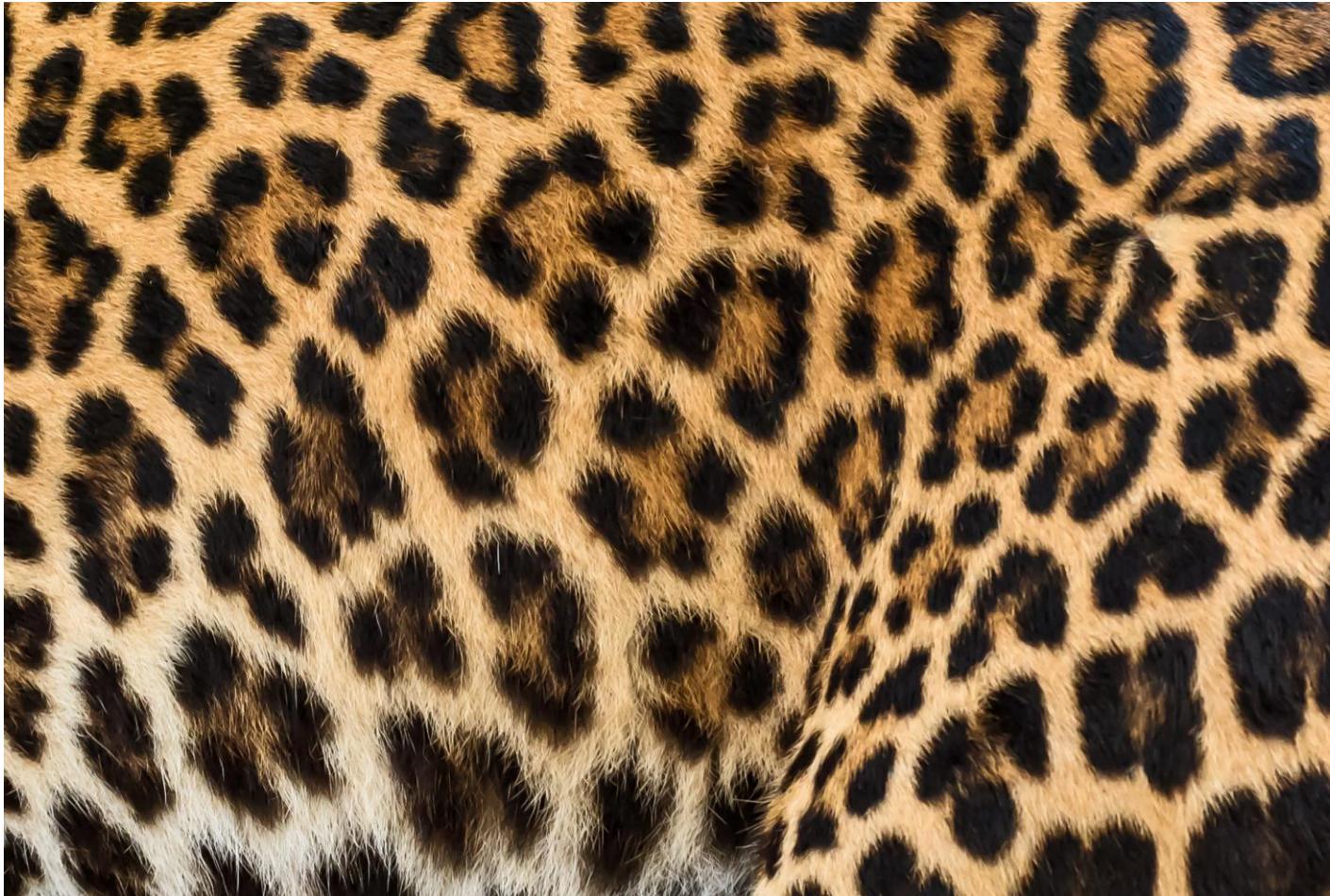


Pre-existing Heterogeneity

# Self-Organisation



# Turing Patterns

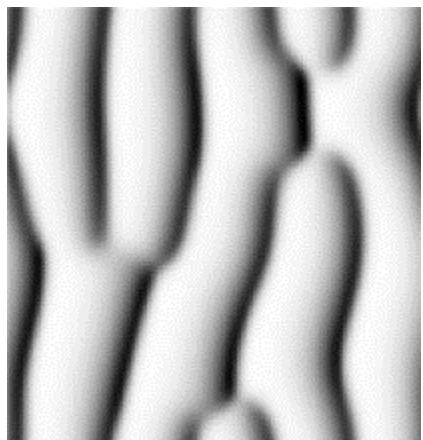


[wikipedia]

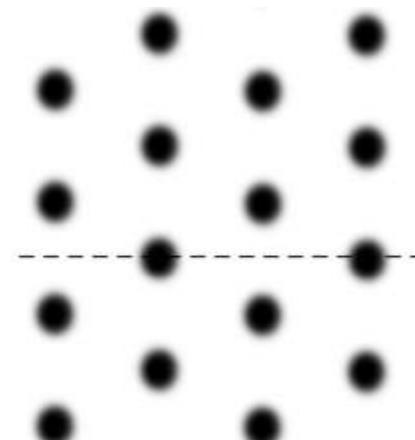
Seminal paper in 1952: “The chemical basis of morphogenesis”

# Reaction-Diffusion Equations

$$\begin{cases} \frac{du}{dt} = f(u, v) + D_u \Delta u \\ \frac{dv}{dt} = g(u, v) + D_v \Delta v \end{cases}$$



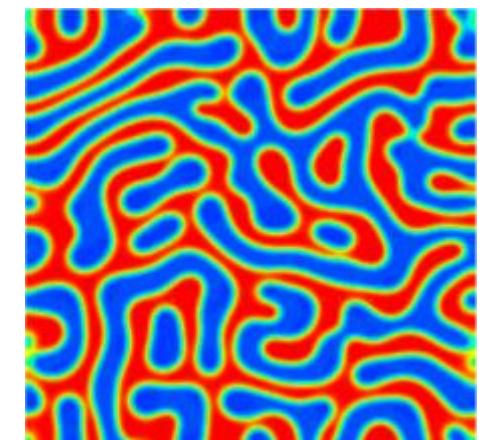
[Klausmeier, 1999]



[Gilad et al, 2004]



[Rietkerk et al, 2002]



[Liu et al, 2013]

# Reaction-Diffusion Equation for Dryland Ecosystems

$$\begin{aligned} w_t &= w_{xx} - w + a - wv^2 \\ v_t &= D^2 v_{xx} - mv + wv^2 \end{aligned}$$

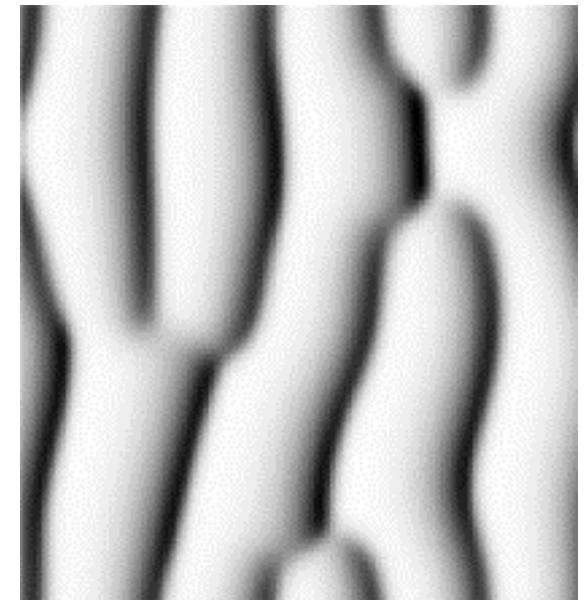
$w$  : water

$v$  : vegetation

$D$  : ratio of diffusion

$a$  : rainfall

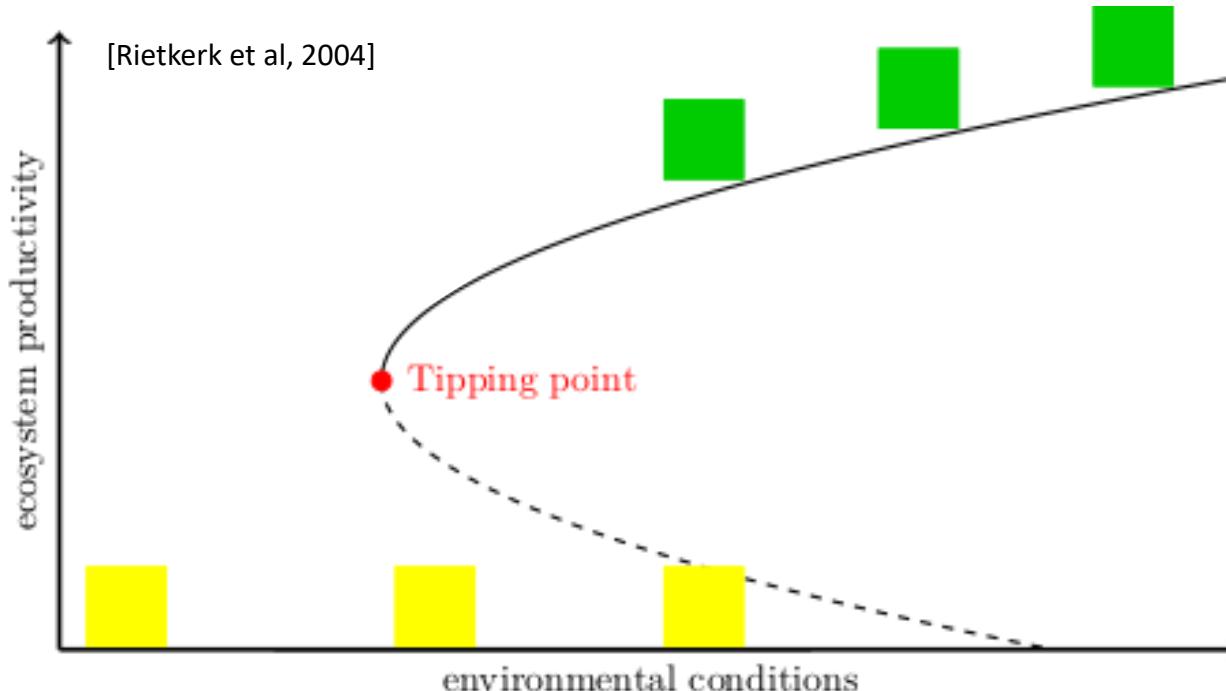
$m$  : mortality



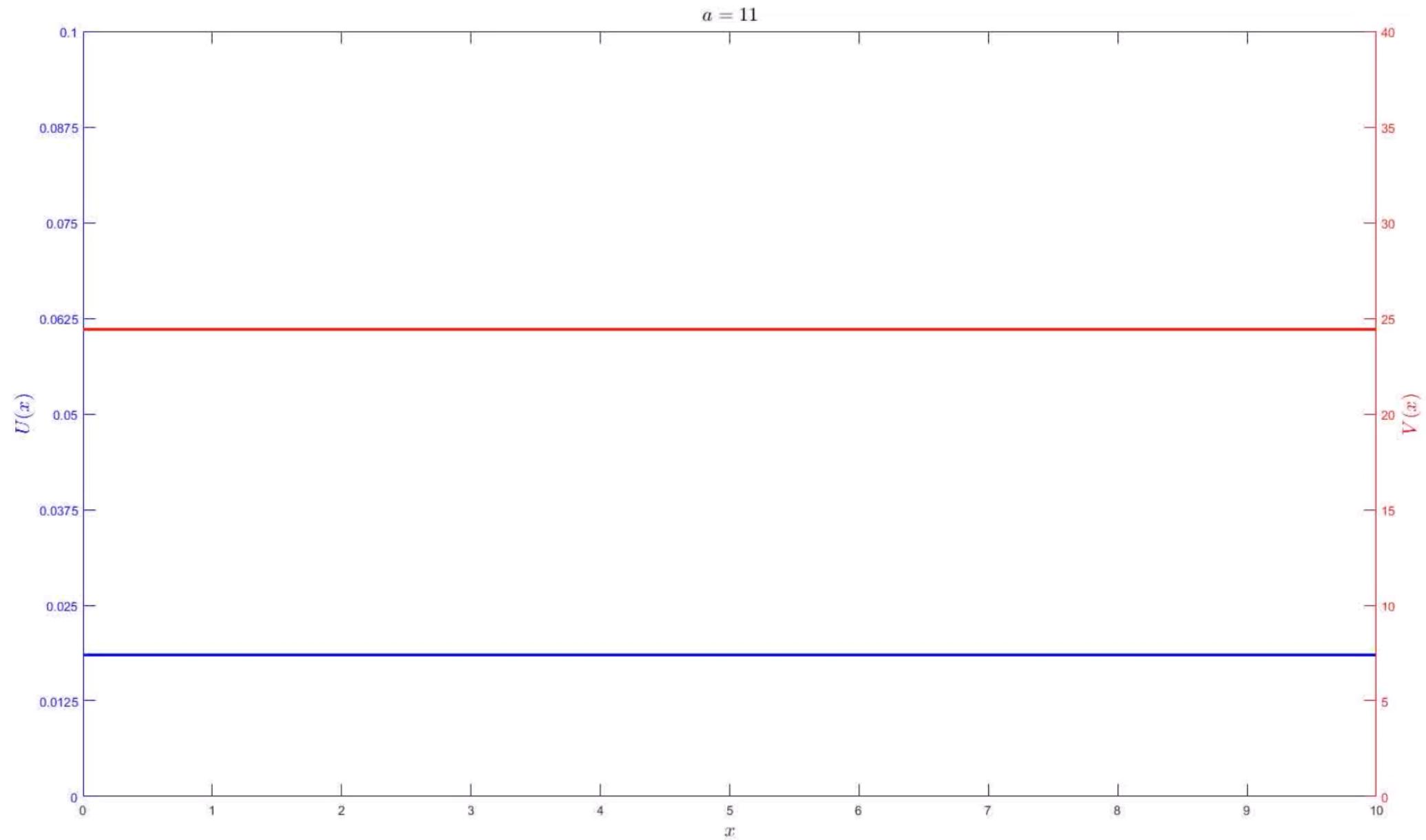
[Klausmeier, 1999]

# Reaction-Diffusion Equation for Dryland Ecosystems

$$w_t = w_{xx} - w + a - wv^2$$
$$v_t = D^2 v_{xx} - mv + wv^2$$



# Spontaneous Pattern Formation



# Turing patterns

$$\begin{cases} \frac{du}{dt} = f(u, v) + D_u \Delta u \\ \frac{dv}{dt} = g(u, v) + D_v \Delta v \end{cases}$$

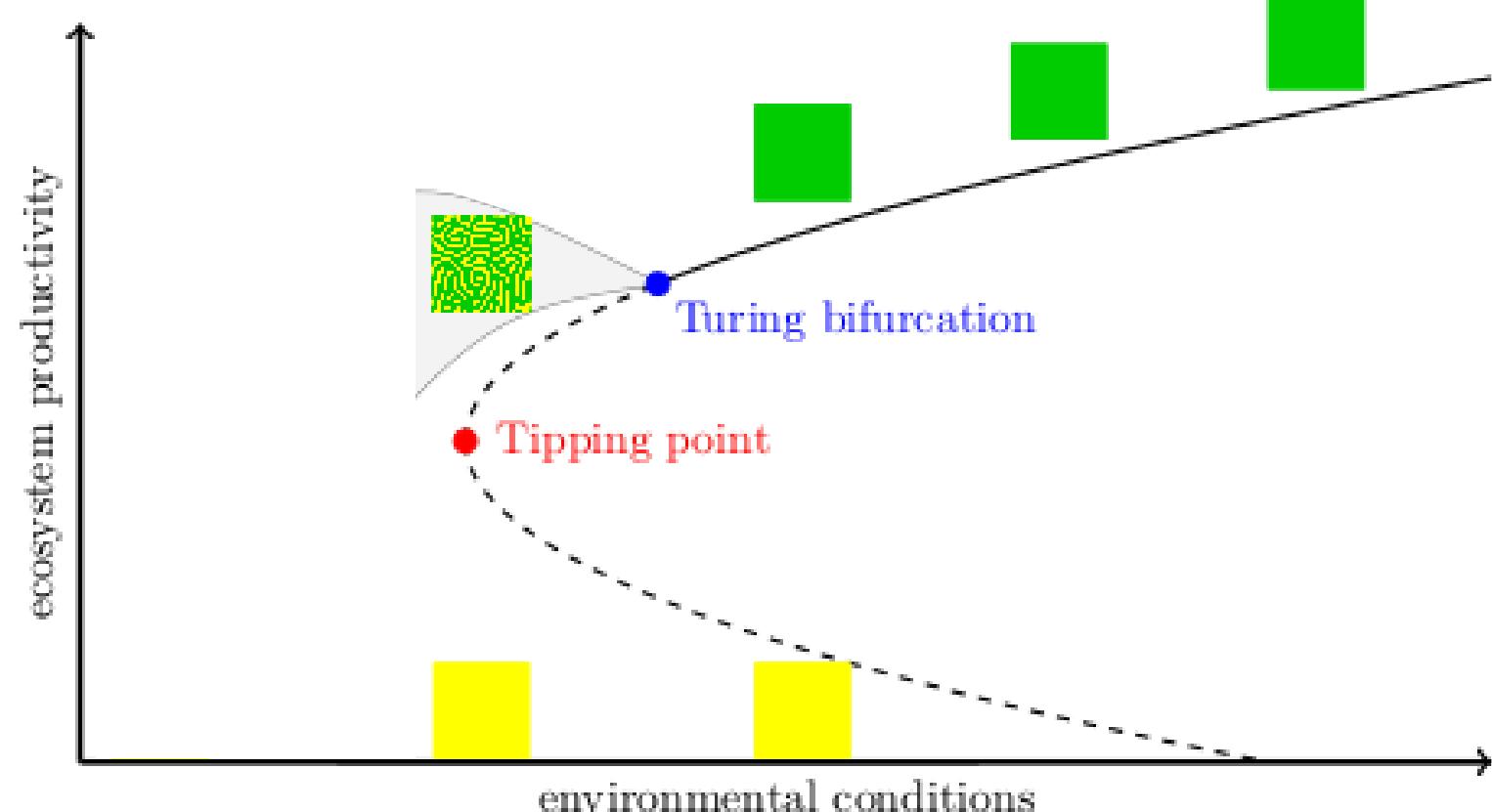
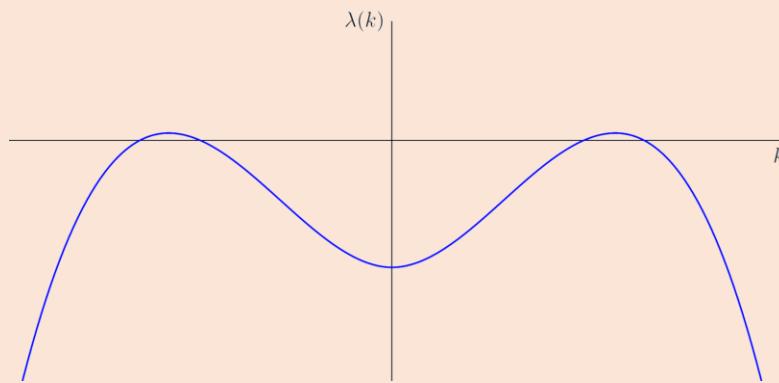
## Turing bifurcation

Instability to non-uniform perturbations

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t} e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

→ Dispersion relation

$$\lambda(k) = \dots$$



**Weakly non-linear analysis**  
Ginzburg-Landau equation / Amplitude Equation  
& Eckhaus/Benjamin-Feir-Newell criterion  
[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

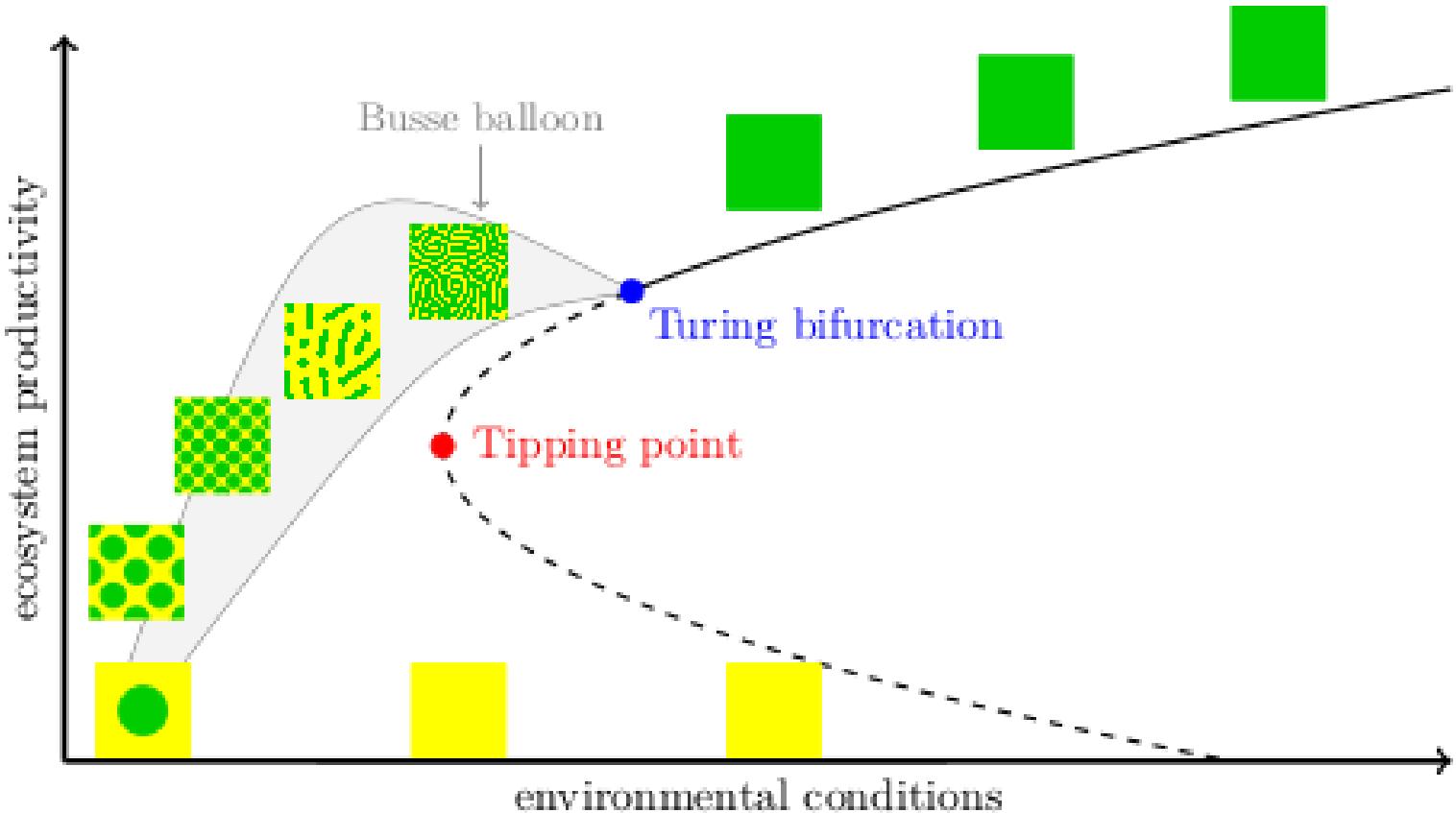
# Busse balloon

## Busse balloon

A model-dependent shape in (*parameter, observable*) space that indicates all stable patterned solutions to the PDE.

**Construction Busse balloon**  
Via numerical continuation  
few general results on the  
shape of Busse balloon

$$\begin{cases} \frac{du}{dt} = f(u, v) + D_u \Delta u \\ \frac{dv}{dt} = g(u, v) + D_v \Delta v \end{cases}$$

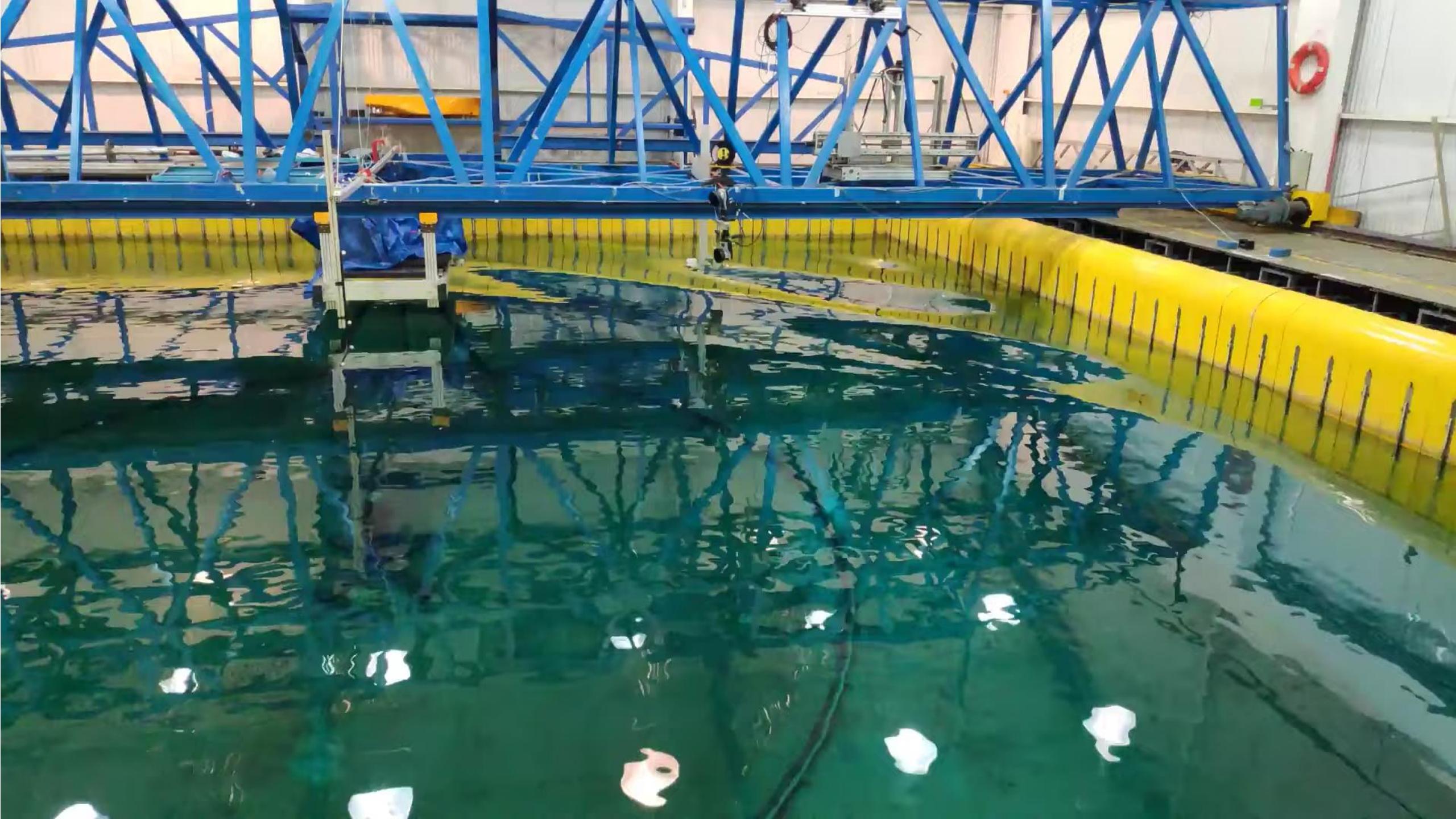


**Busse balloon**  
Idea originates from thermal convection  
[Busse, 1978]

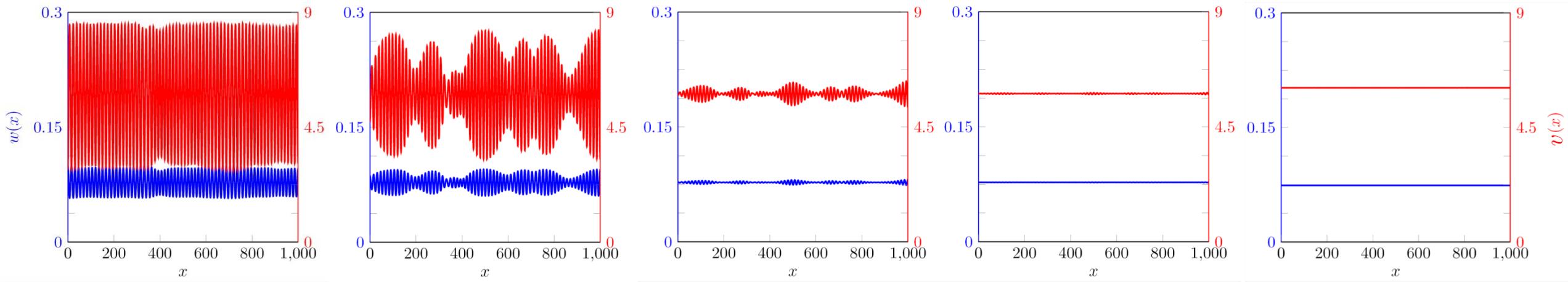
# Rayleigh Bénard thermal convection



Video source: wikiRigaou (wikimedia commons)



# The origin of patterns in dryland model



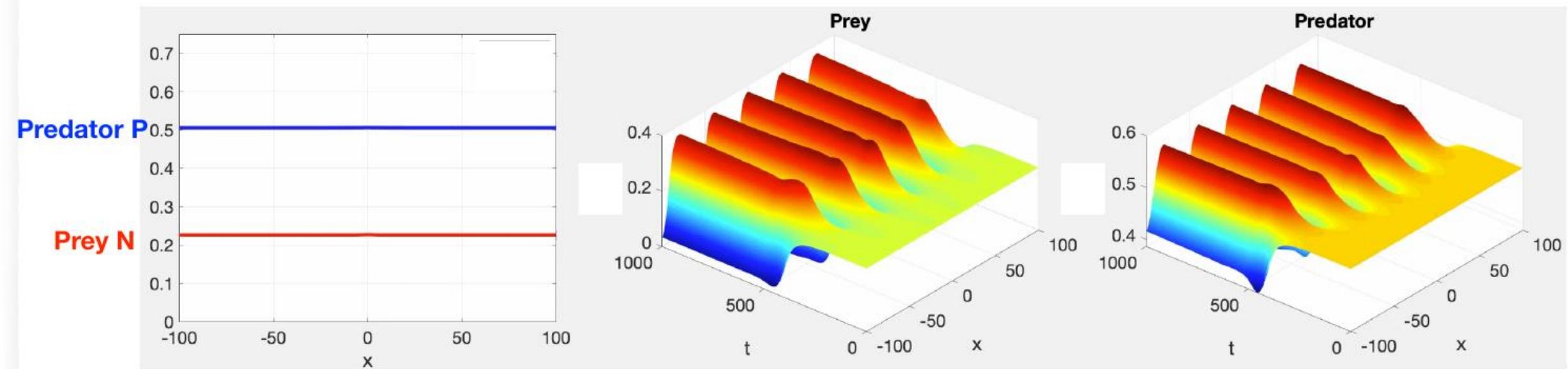
Low rainfall

Critical rainfall  
Onset of patterns

High rainfall

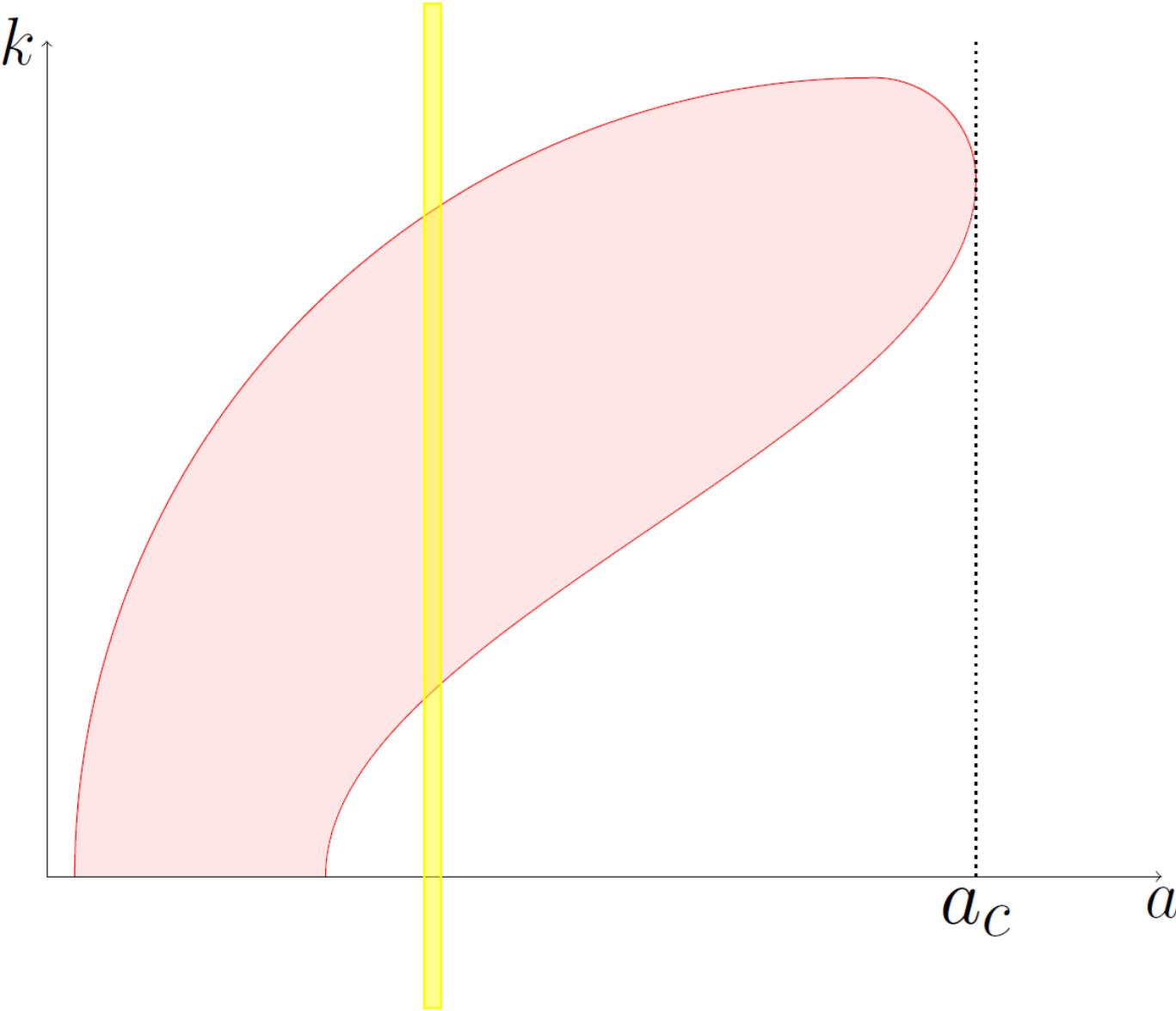
**Example:** Diffusive Holling–Tanner predator-prey model with an alternative food source for the predator

$$N_t = rN \left(1 - \frac{N}{K}\right) - \frac{qNP}{N + a} + D_1 N_{xx},$$
$$P_t = sP \left(1 - \frac{P}{hN + c}\right) + D_2 P_{xx}.$$



[Arancibia-Ibarra et al., 2021]

# Multistability in the Busse balloon



## Observation:

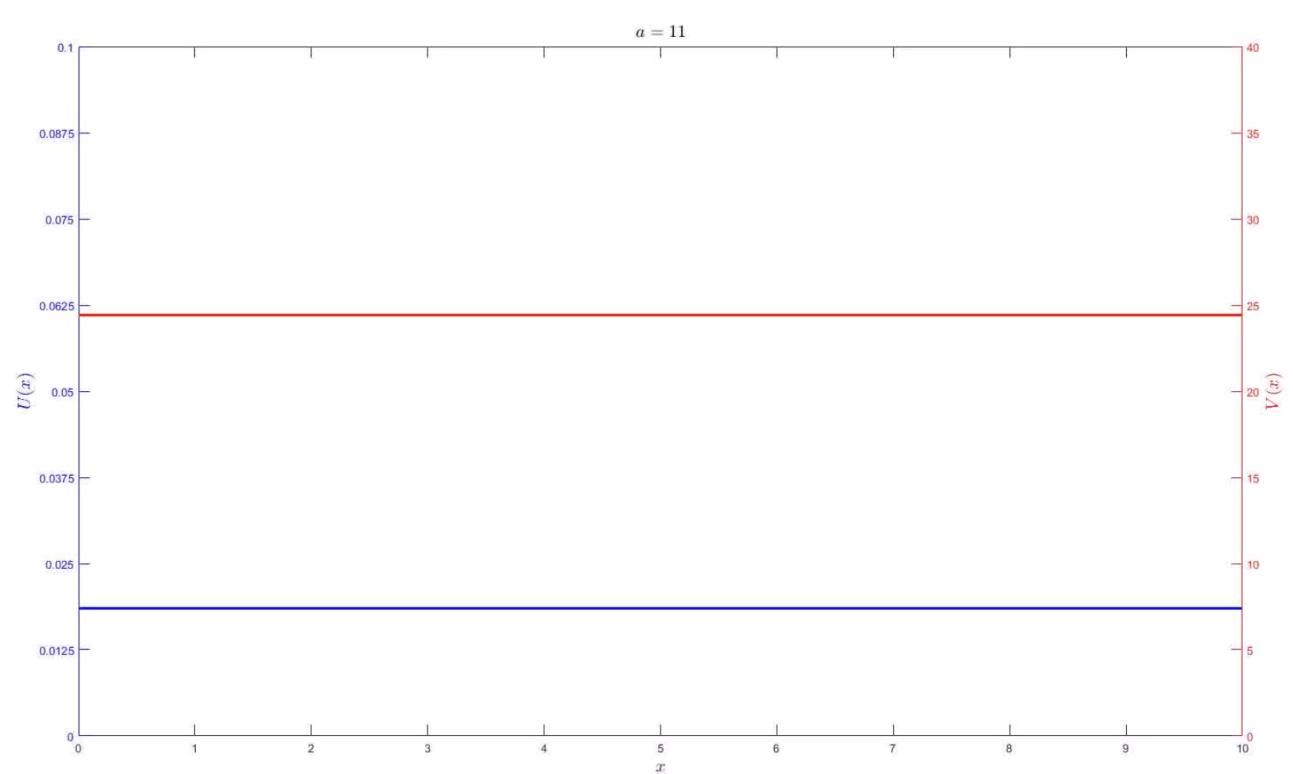
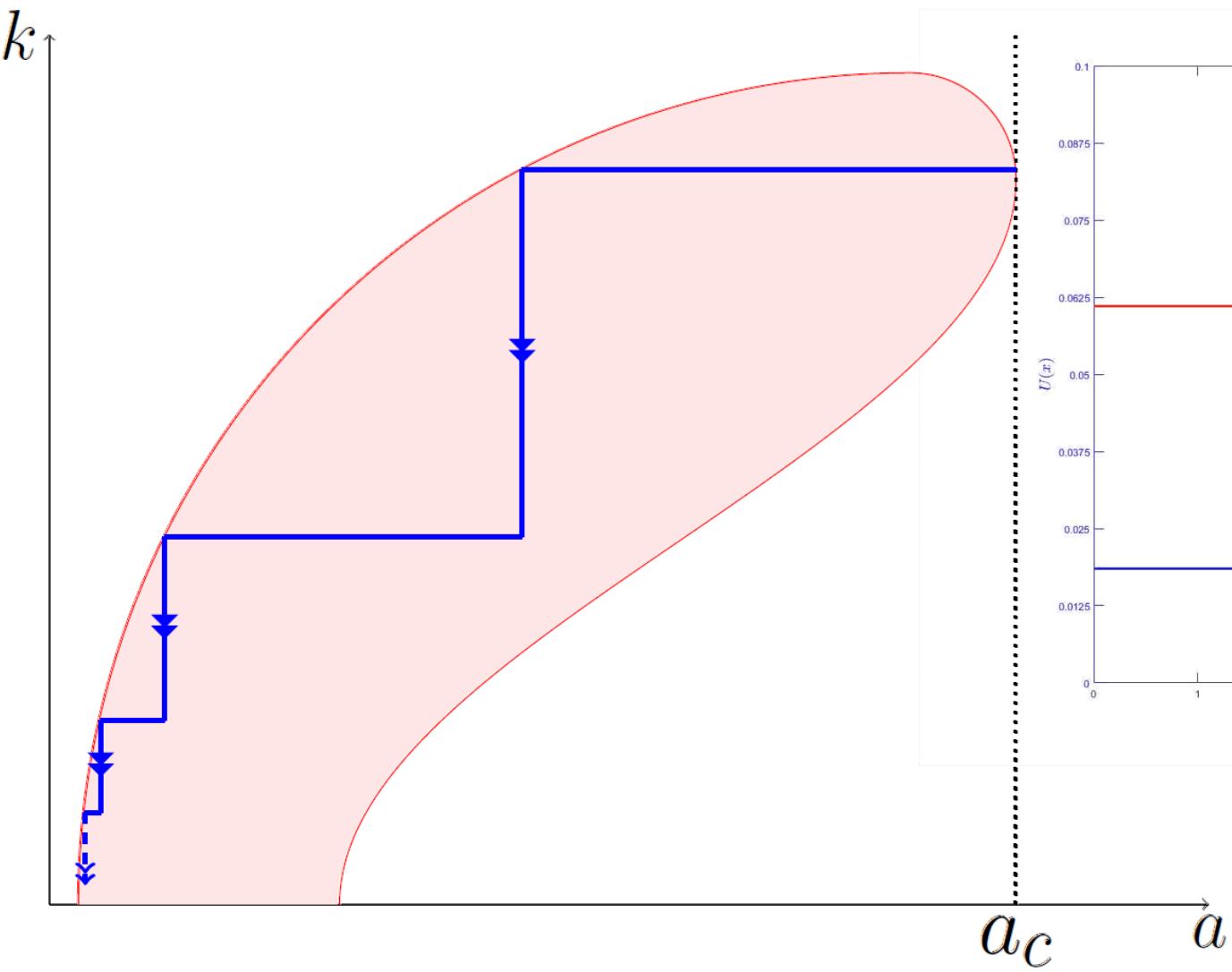
For a fixed parameter value, there is a **continuous** range of wavenumbers possible.

That is, there is a large **multistability** of stable pattern states to the PDE

## Consequence:

Specifying only parameter values is ambiguous, as it does not correspond to only one patterned state.

# A Walk through the Busse balloon



Consequence:  
In patterned shifts, you see smaller transitions from one patterned states to another

A wide-angle photograph of a massive glacier. The ice is a deep, translucent blue, with numerous vertical and horizontal crevasses. The top of the glacier is covered in white snow, and the sky above is filled with soft, white clouds.

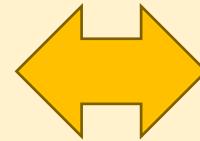
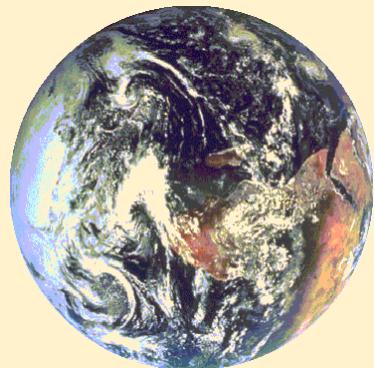
# APPLICATION:

## Tipping in

# Spatially Extended Systems

# Tipping Points

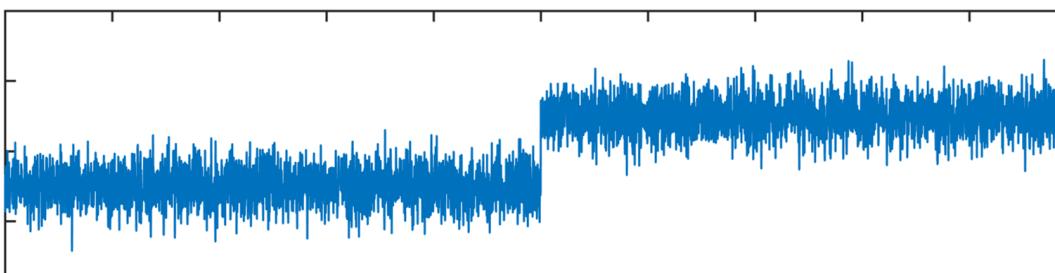
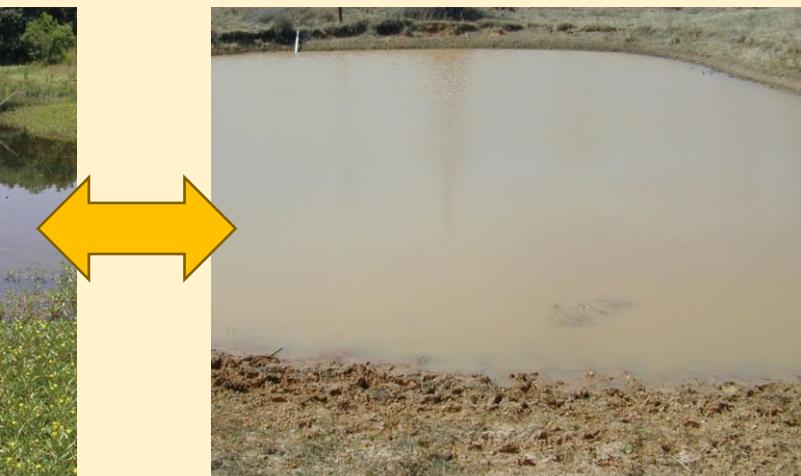
IPCC AR6 (2021) : “a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly”



Planetary transitions

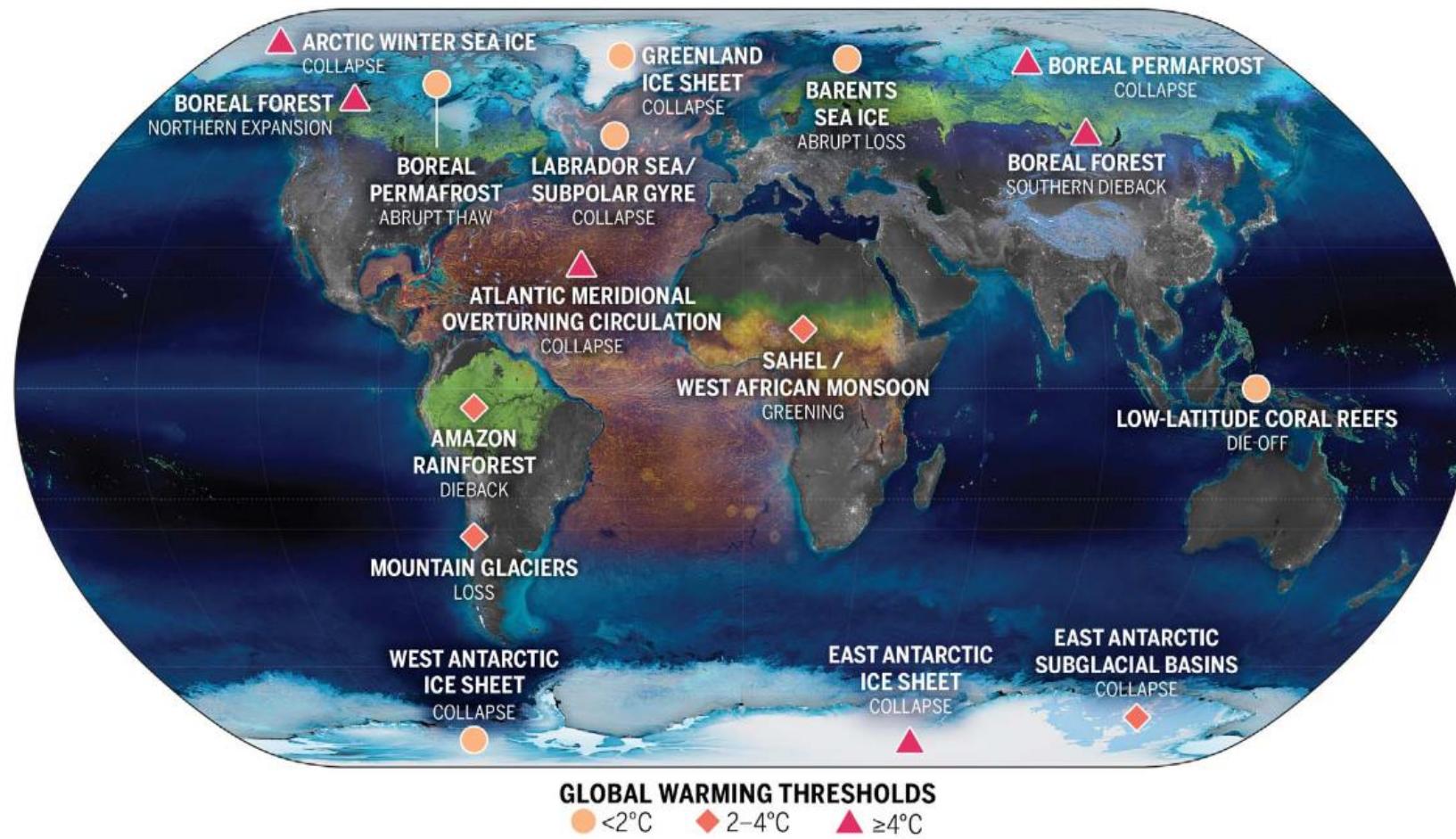


Ecosystem shifts



# Tipping Points

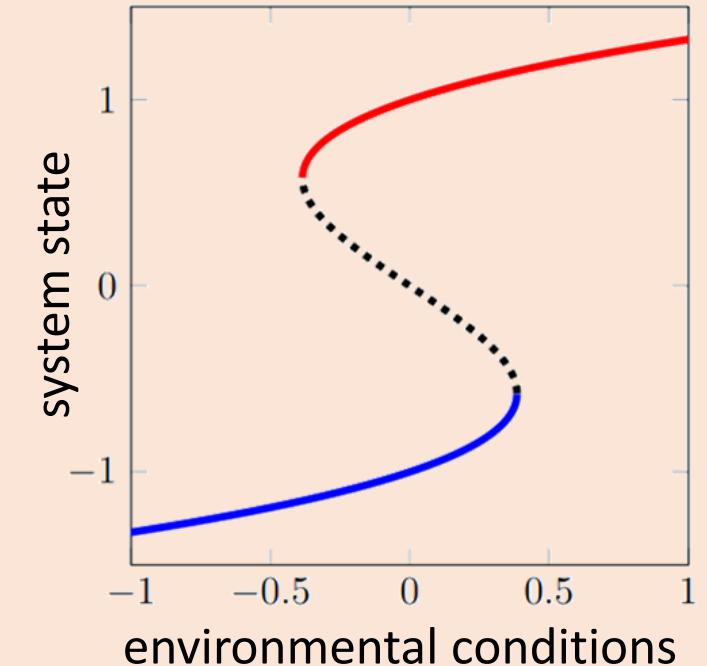
IPCC AR6 (2021) : “a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly”



## Mathematics

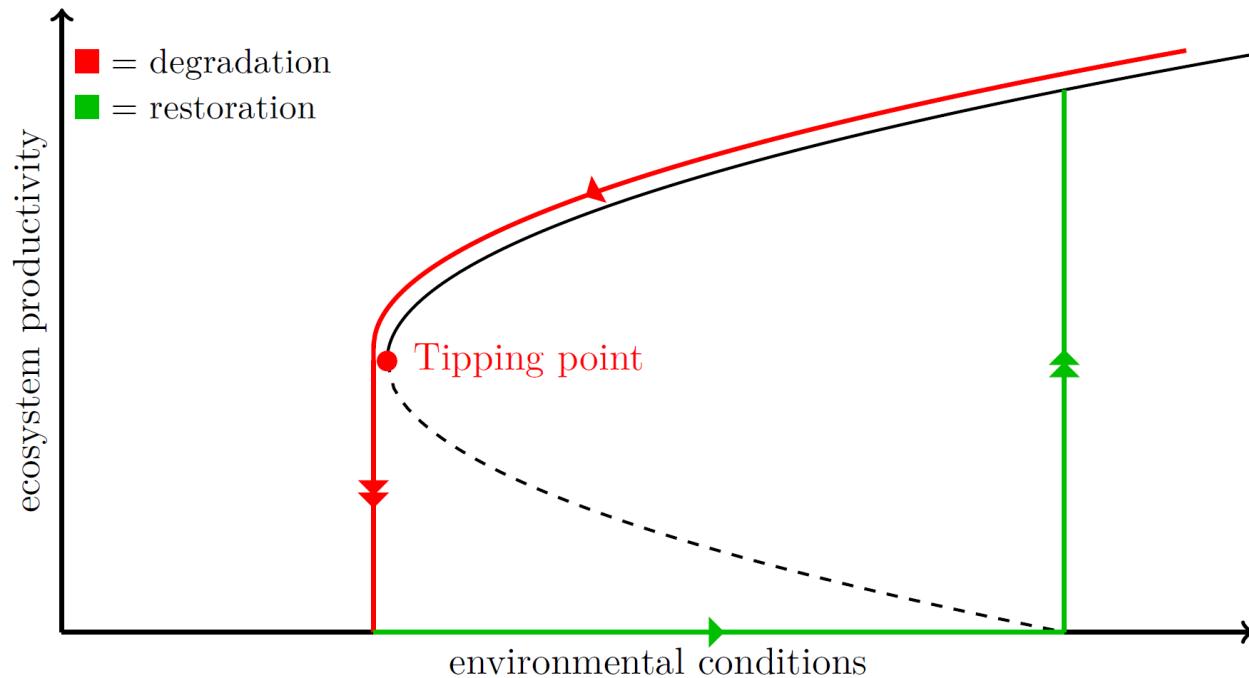
Tipping points  $\leftrightarrow$  Bifurcations

$$\frac{dy}{dt} = f(y, \mu)$$

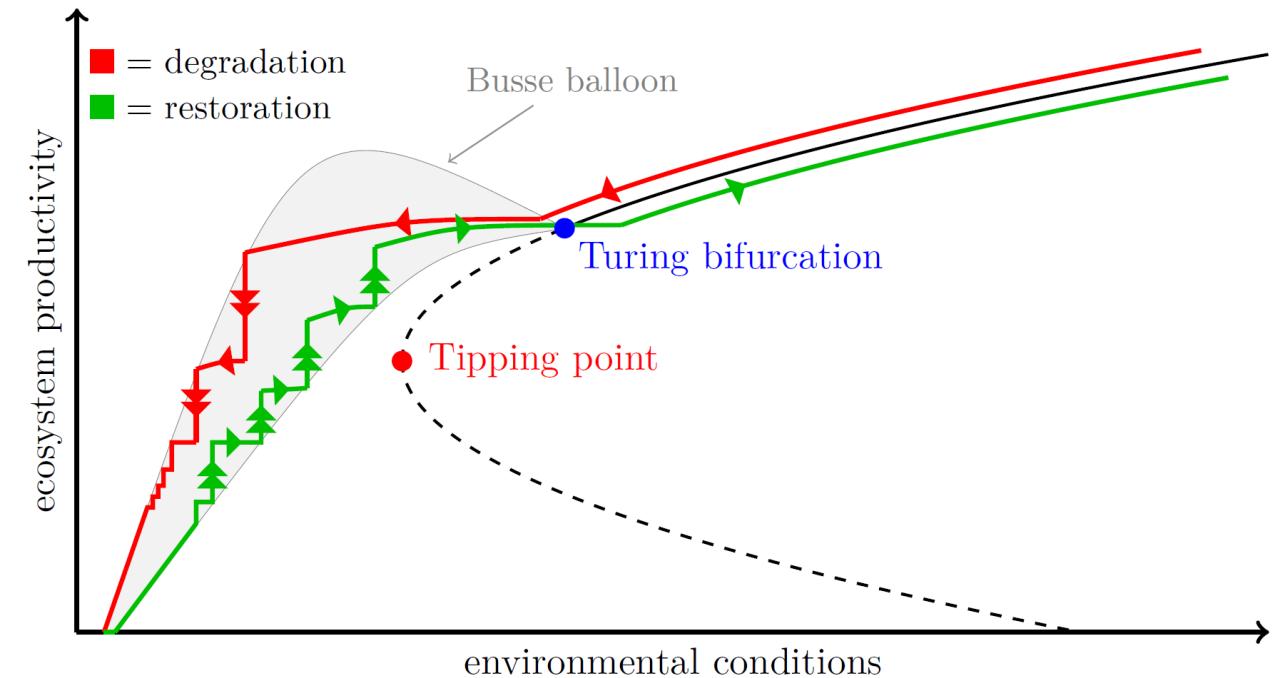




# Tipping of (Turing) patterns



Classic tipping

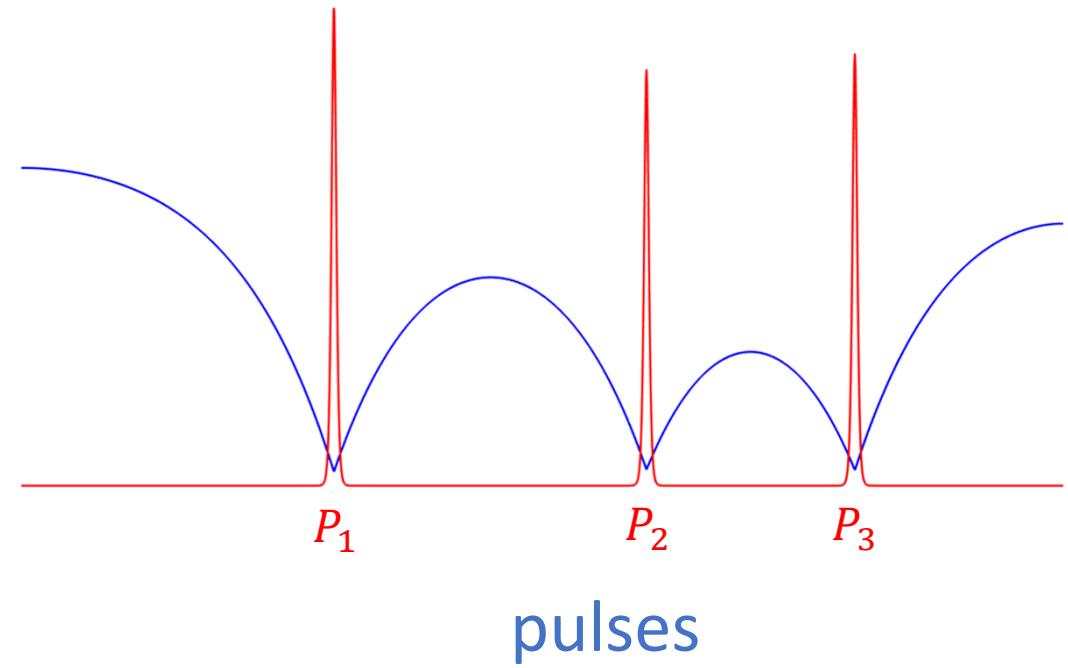
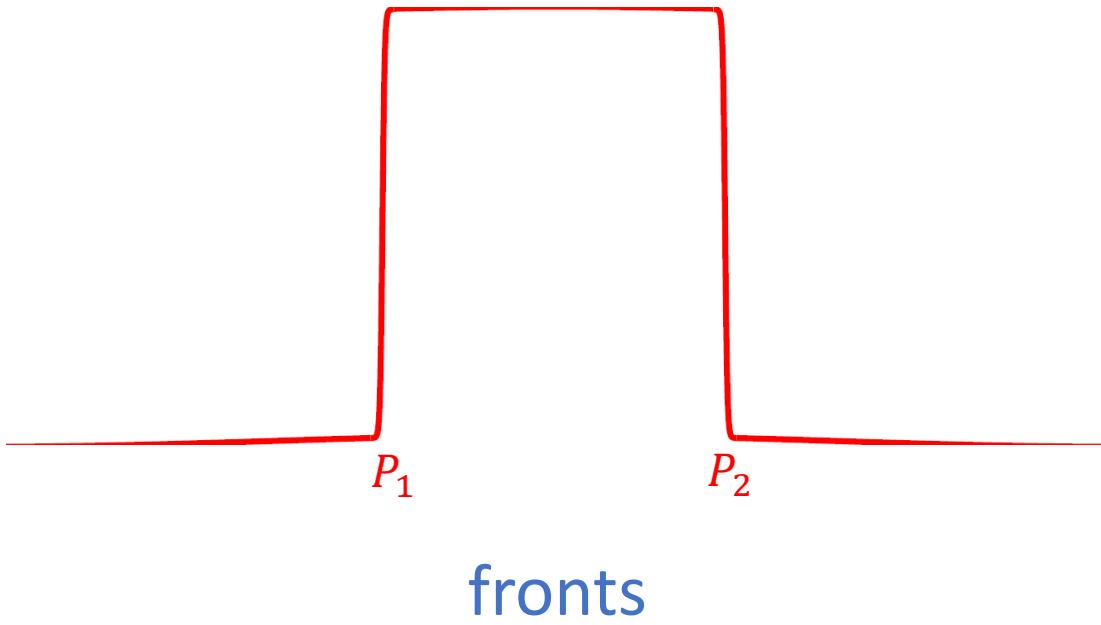


Tipping of patterns

An aerial photograph of a grassland fire on a hillside. The fire is visible as a bright orange and red line of flames moving through the dry, yellowish-brown grass. A large plume of dark smoke is visible on the left side of the fire front. The hillside has a distinct slope, and the surrounding area is covered in green vegetation.

# Part 2: Behaviour of Spatial Patterns

# Dynamics of Patterned States



# Mathematical Study of Localised Structures (1)

Example system:

Allen-Cahn Equation

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + y(1 - y^2) + \mu$$

Introduce travelling wave coordinate(s):

$$\zeta := x - ct$$

Assume state only depends on that:

$$y(x, t) = y(\zeta)$$

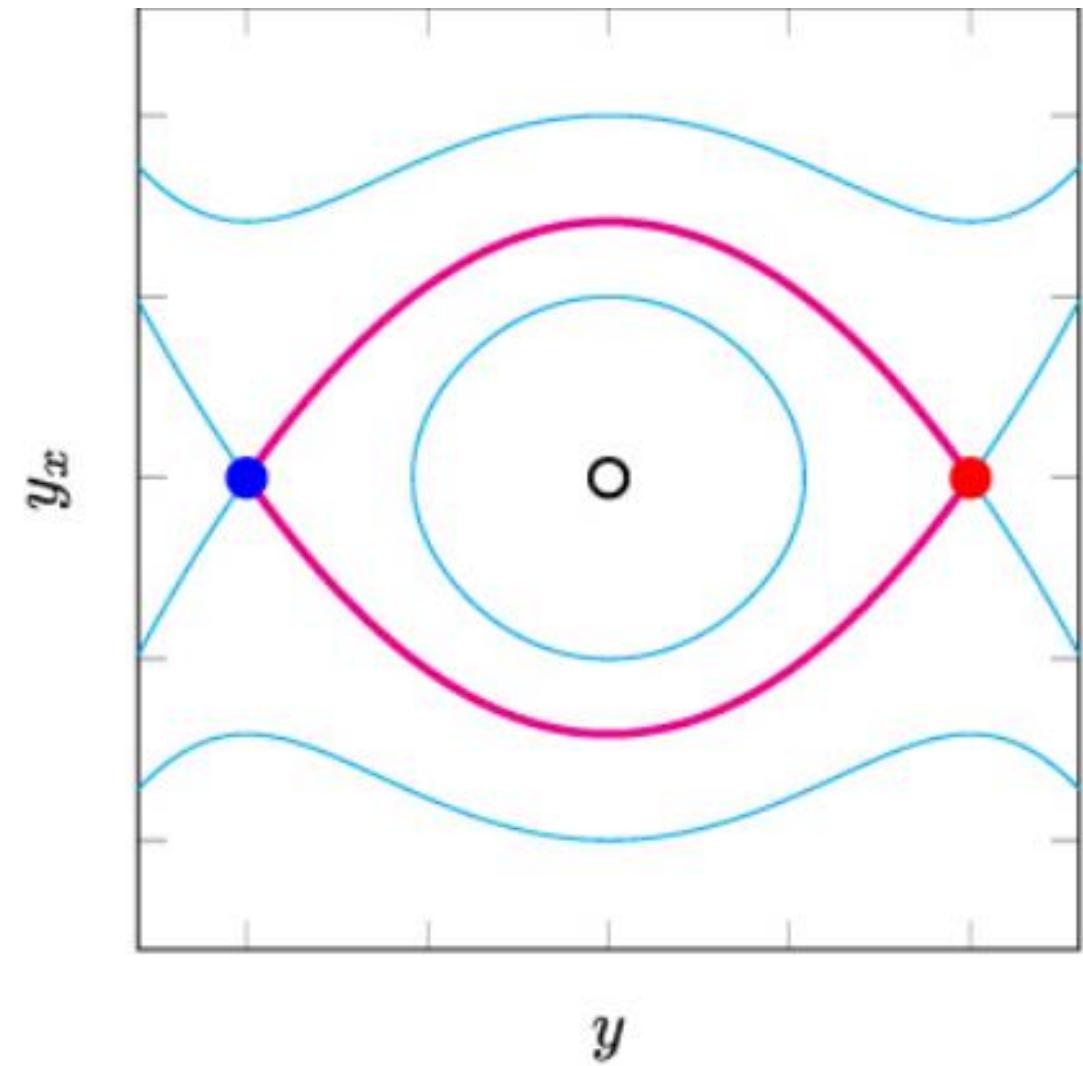
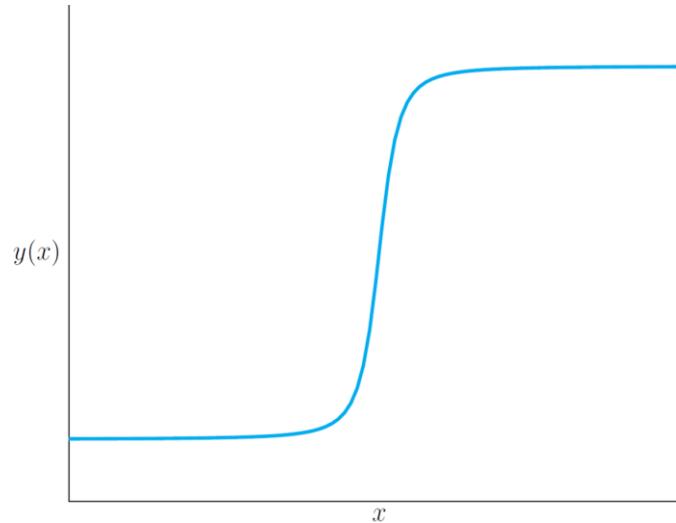
Gives an ordinary differential equation  
often coined the ‘spatial dynamics’:

$$-cy' = y'' + y(1 - y^2) + \mu$$

# Mathematical Study of Localised Structures (2)

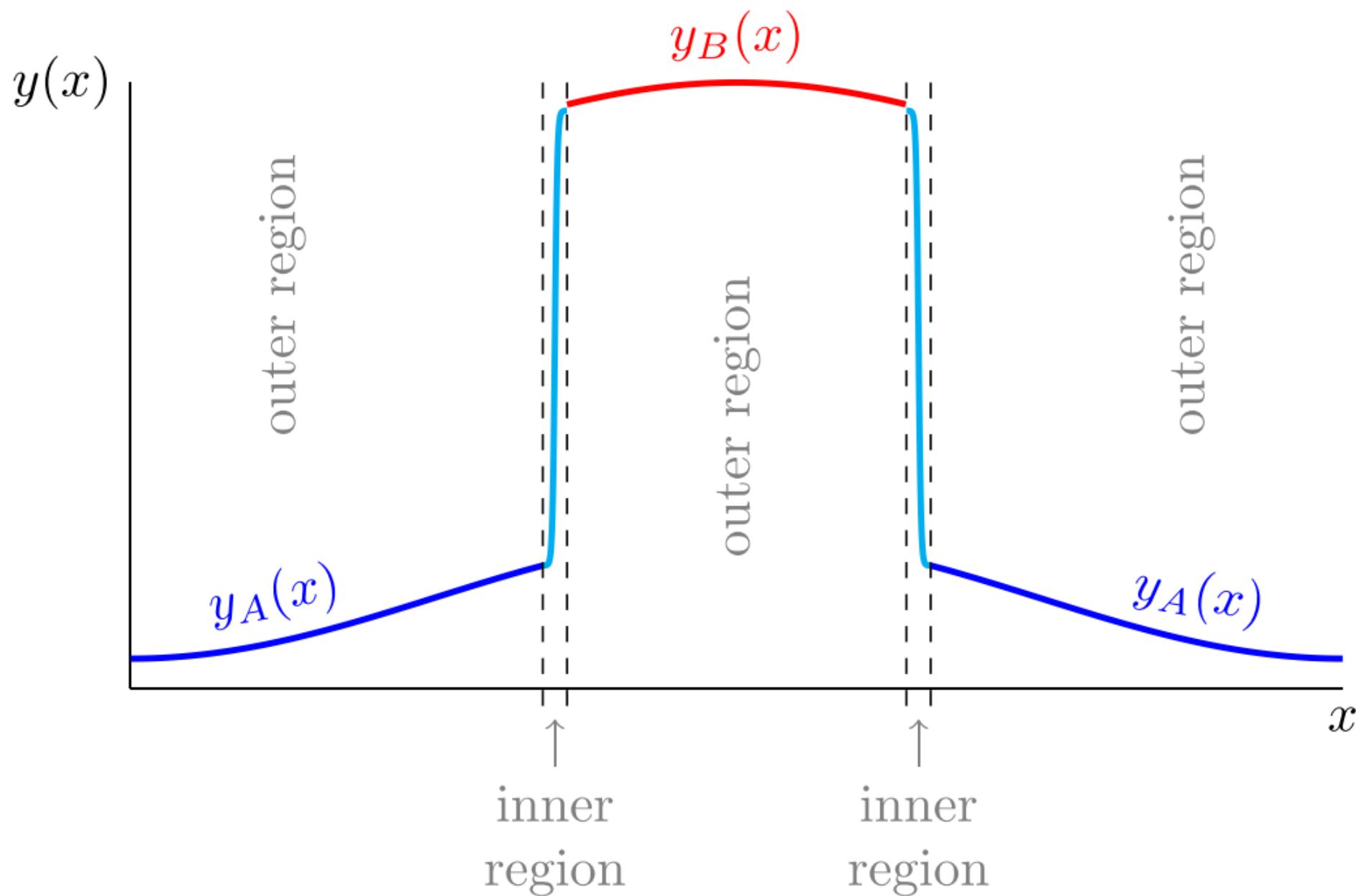
For  $c = 0$  and  $\mu = 0$   
system can be written as

$$\begin{aligned}y' &= p \\p' &= -y(1 - y^2)\end{aligned}$$

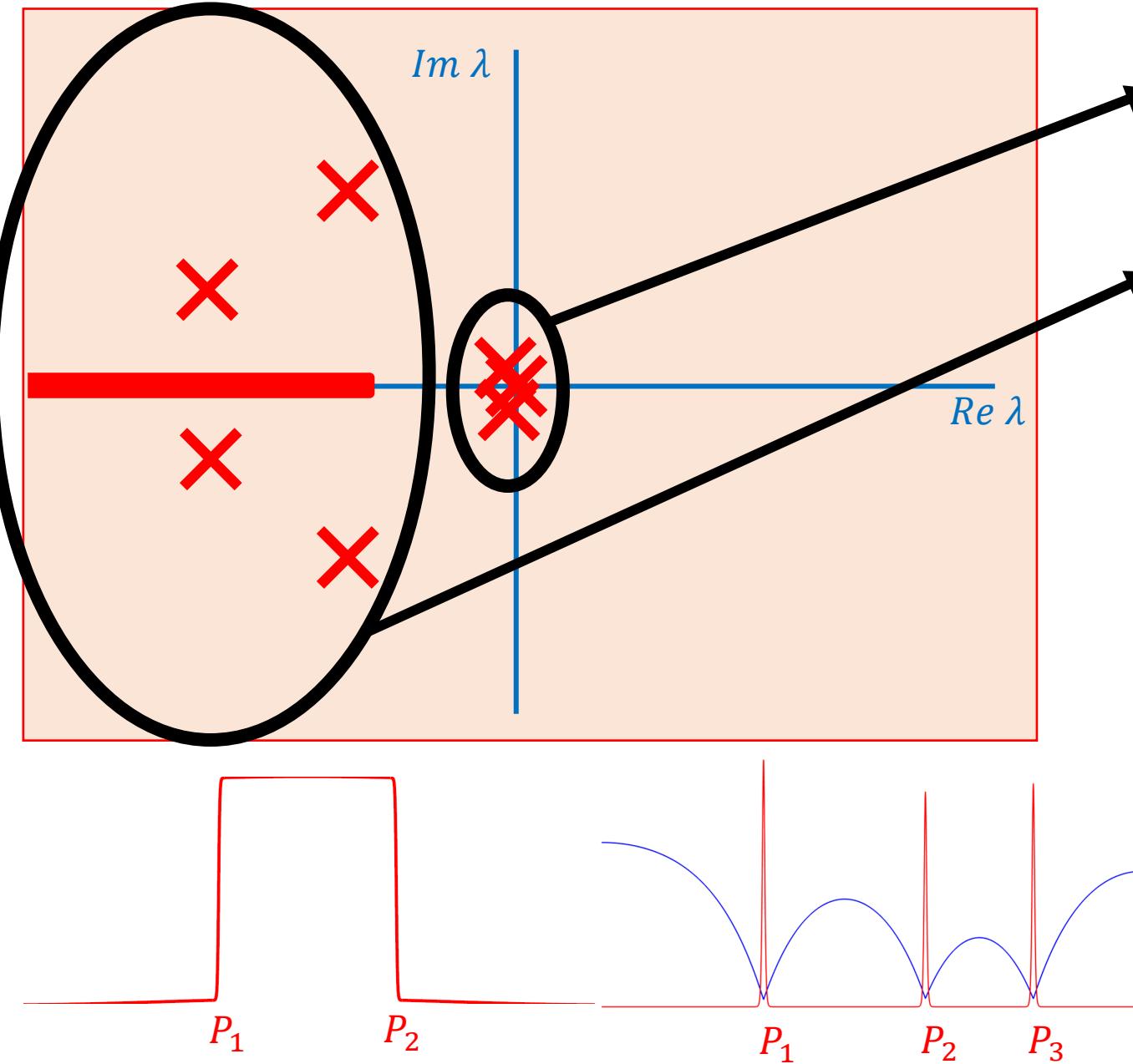


Heteroclinic connections in the spatial dynamics correspond to front solutions to the PDE

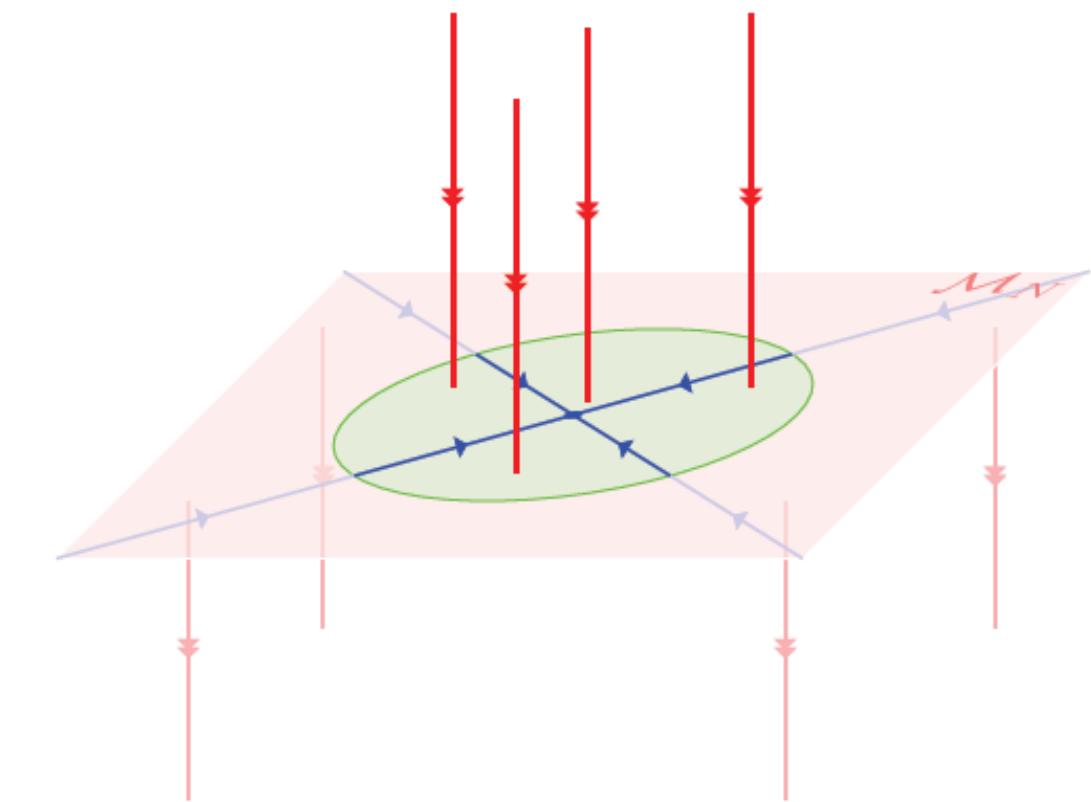
# Mathematical Construction of localised structures



# Dynamics of Patterned States



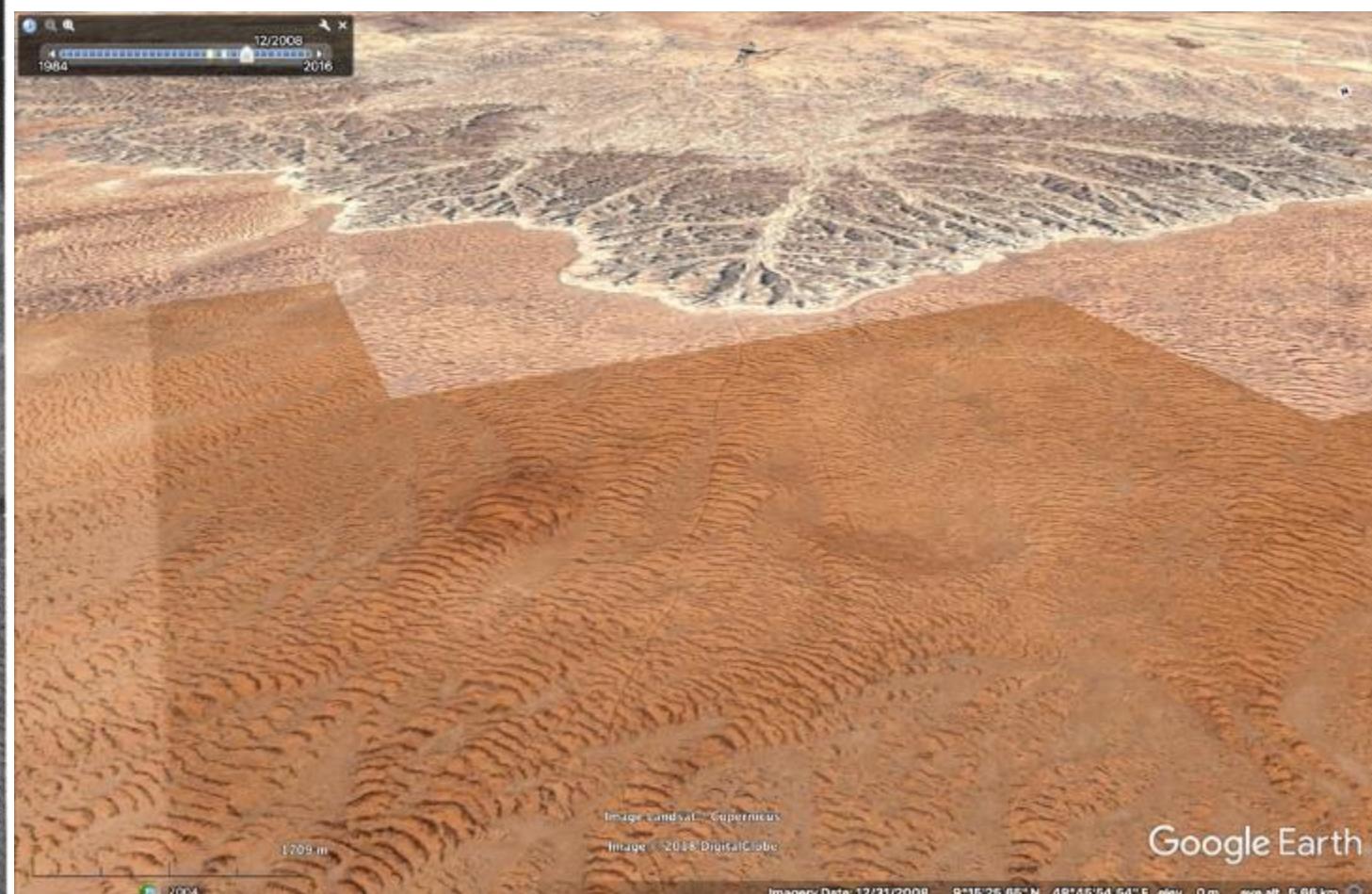
1. SLOW Pattern Adaptation
2. FAST Pattern Degradation



# 1. SLOW pattern adaptation

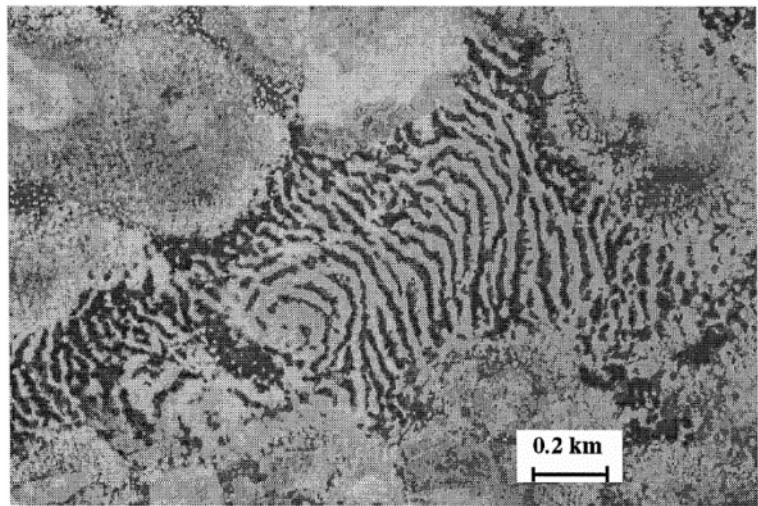


Somaliland, 1948 [Macfadyen, 1950]



Somaliland, 2008

## 2. FAST Pattern Degradation



Niger, 1950 [Valentin, 1999]



Niger, 2008



Niger, 2010



Niger, 2011

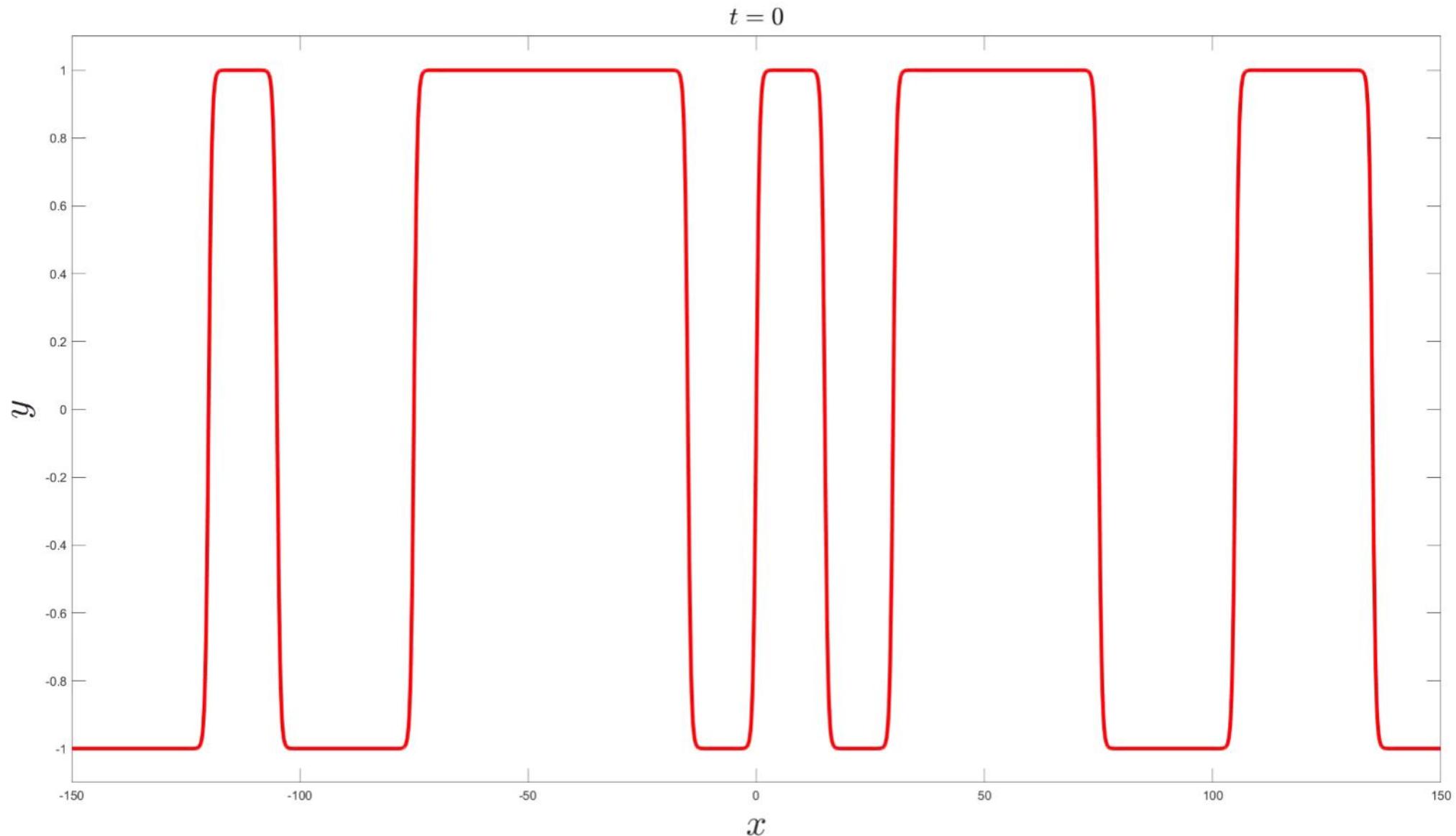


Niger, 2014

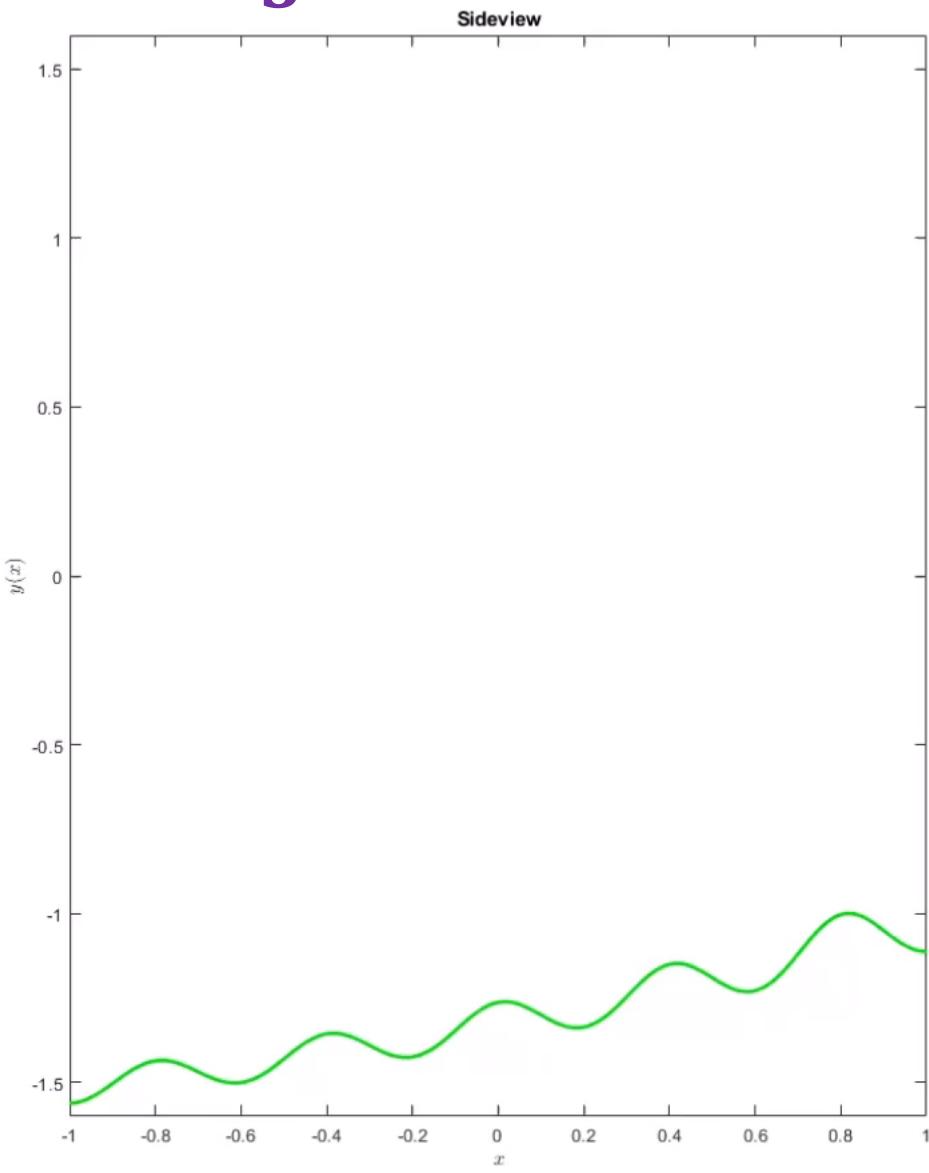
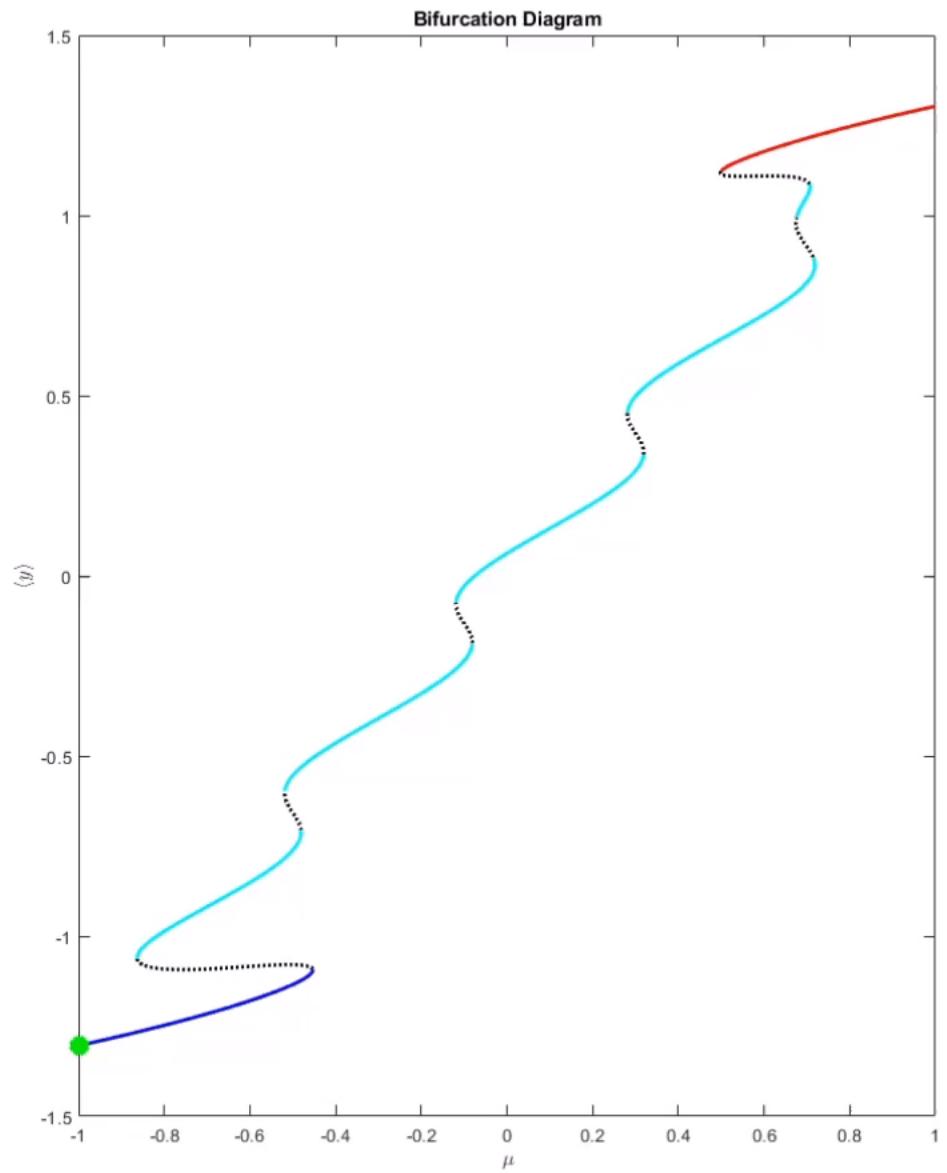


Niger, 2016

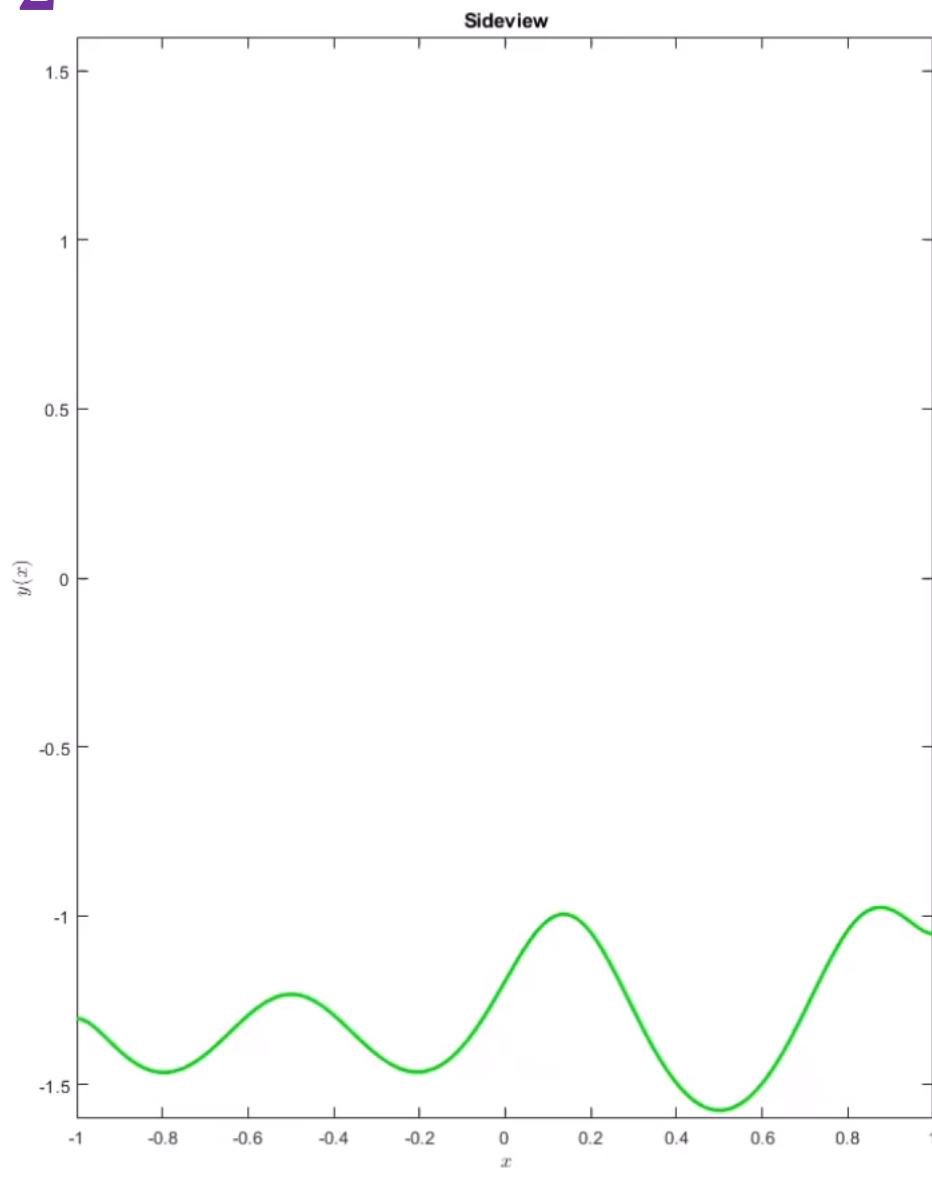
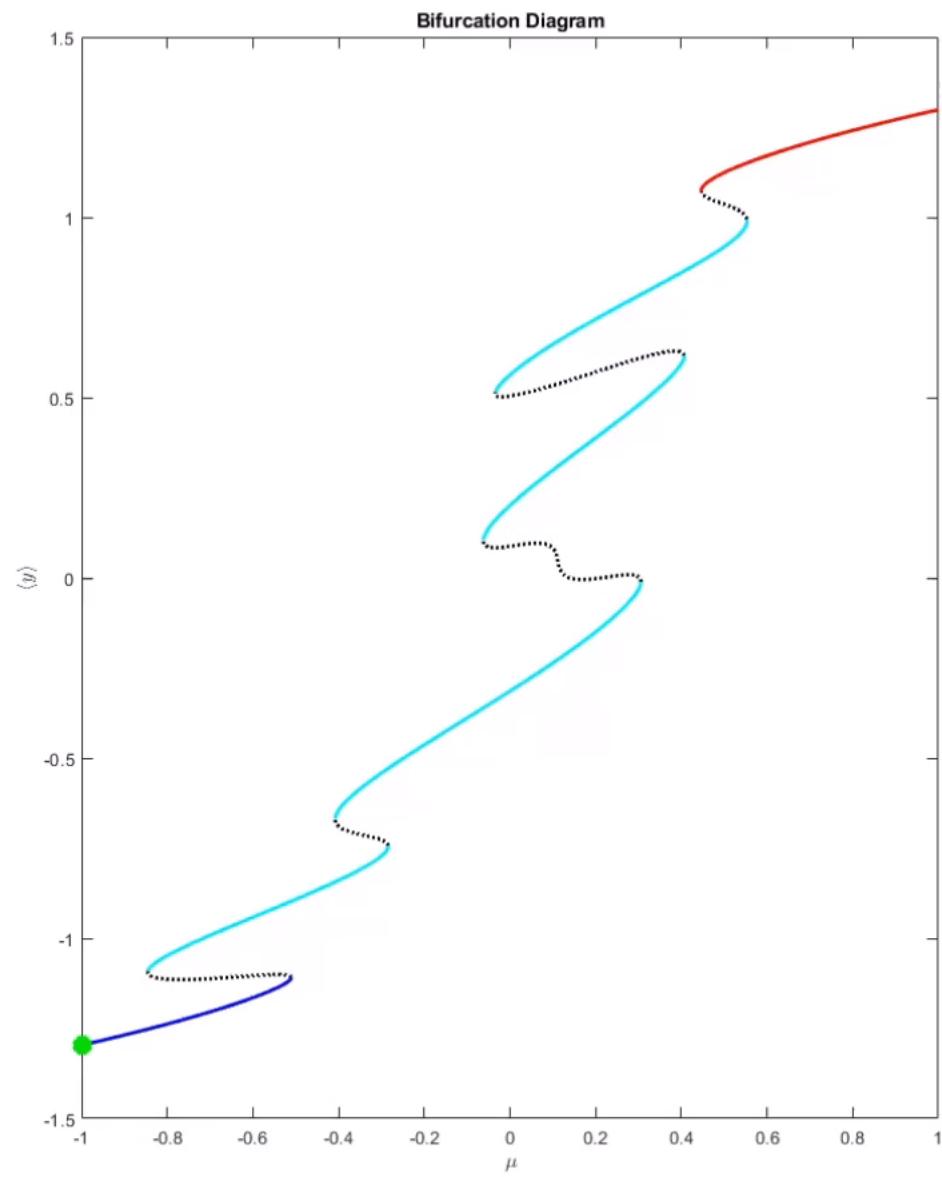
**Dynamics of  $\frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2} + y(1 - y^2) + \mu$**



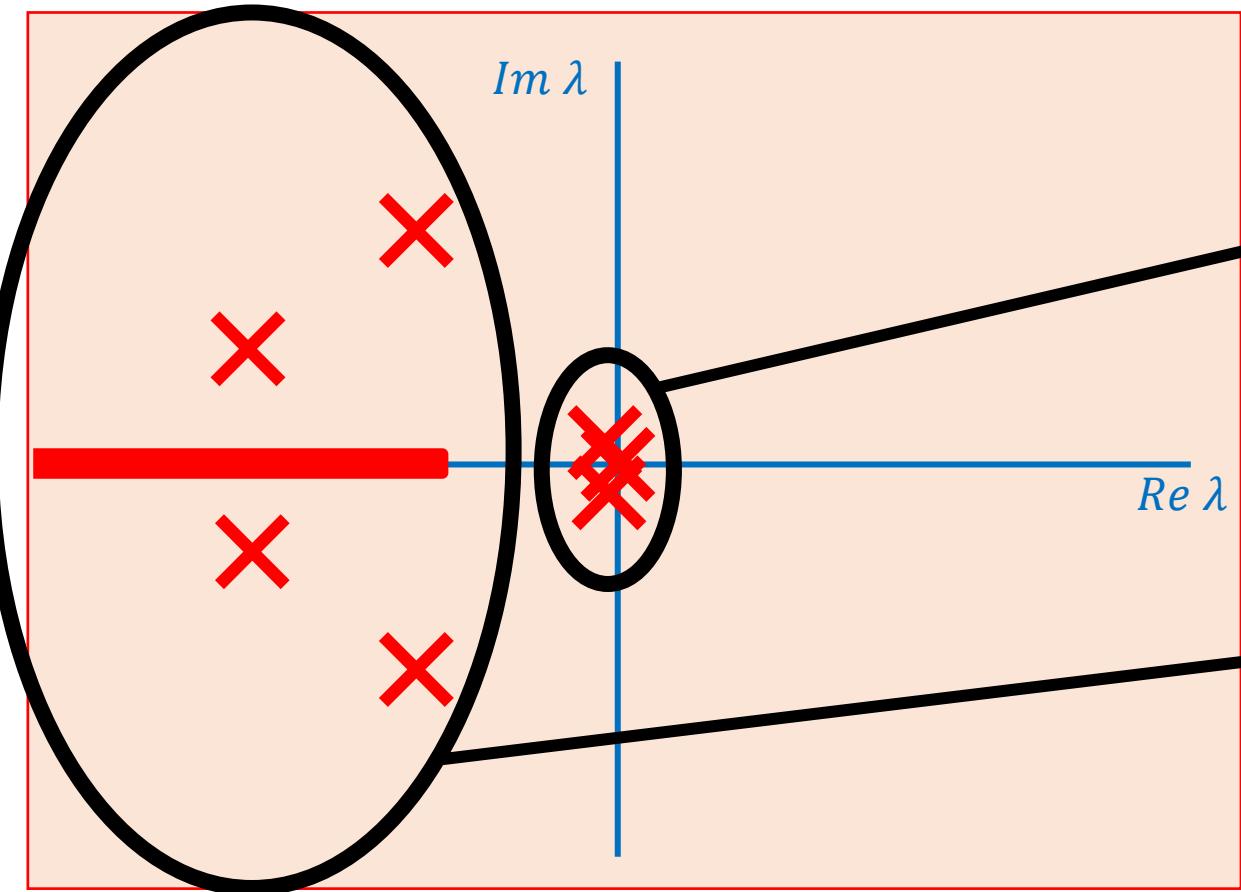
$$y_t = D y_{xx} + y(1 - y^2) + \mu + x + \frac{2}{5} \cos(5\pi x)$$



$$y_t = D y_{xx} + y(1 - y^2) + \mu + \frac{1}{2} \cos(2\pi x) + \sin(3\pi x)$$



# Bifurcations



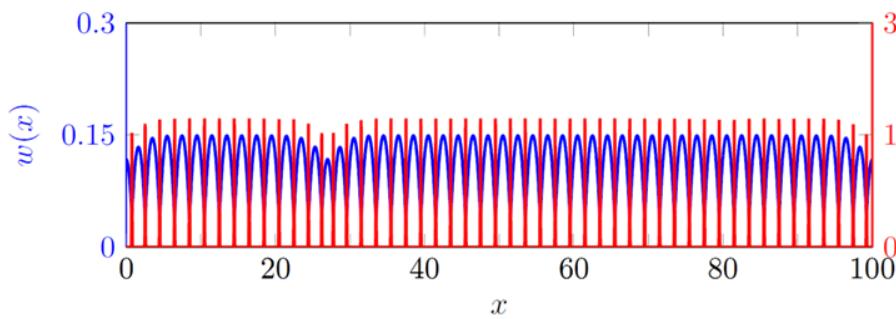
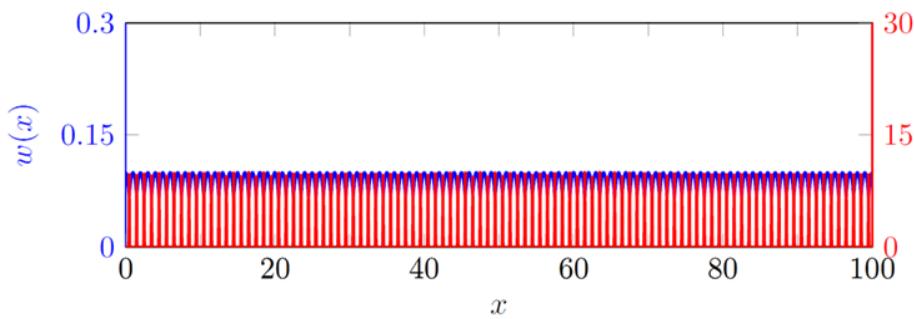
What happens at bifurcation?

## 1. SLOW Pattern Adaptation

At bifurcation:  
→ Location of structure changes

## 2. FAST Pattern Degradation

At bifurcation:  
→ Structures created or destroyed



A close-up photograph of a metal ice cream scoop filled with a scoop of vanilla ice cream. The ice cream is studded with dark chocolate chips. The scoop is held over a large tub of ice cream, which has a similar appearance. The background is a light-colored surface, possibly a kitchen counter.

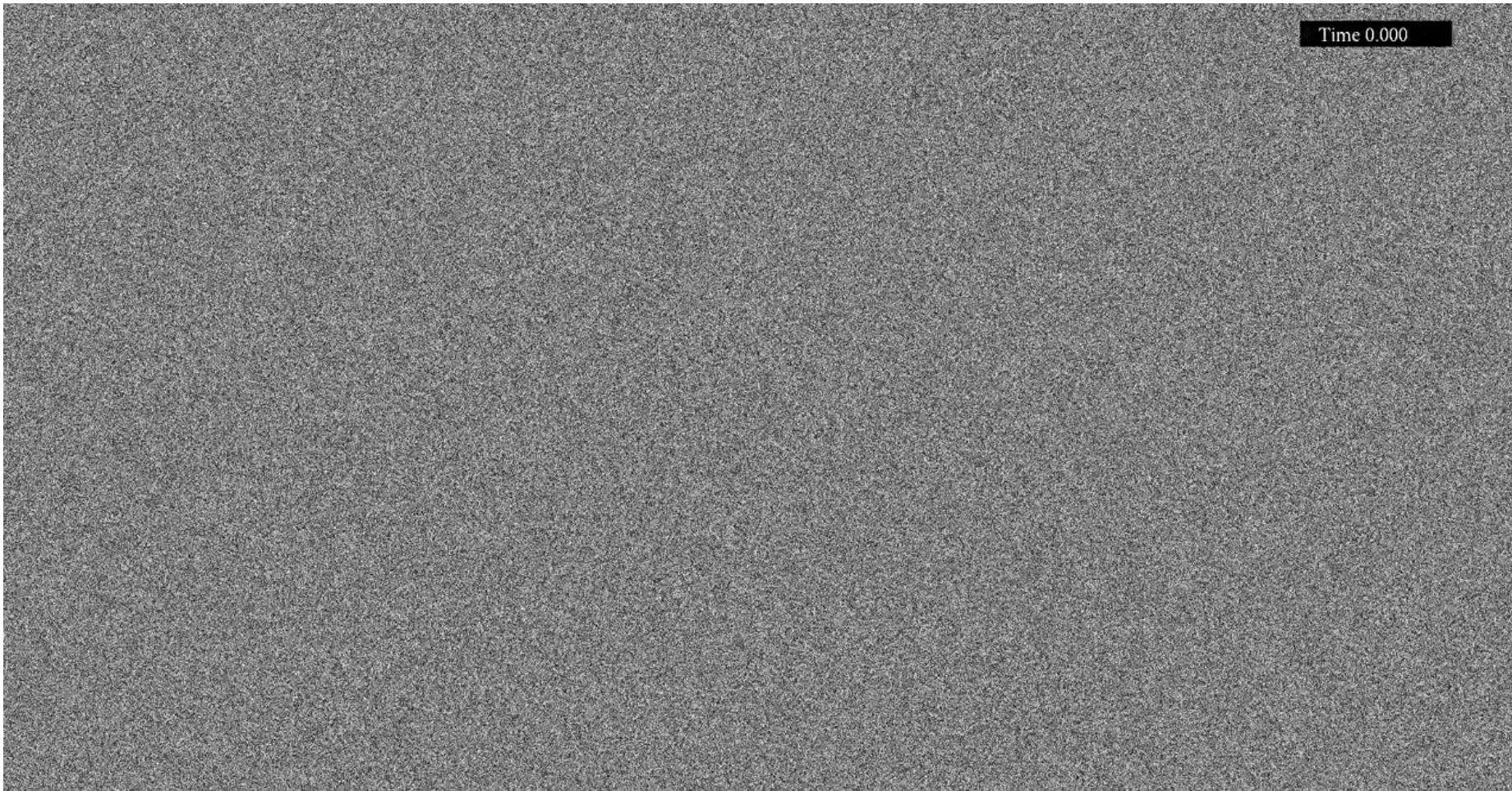
**And what about that ice cream?**

# Phase Separation

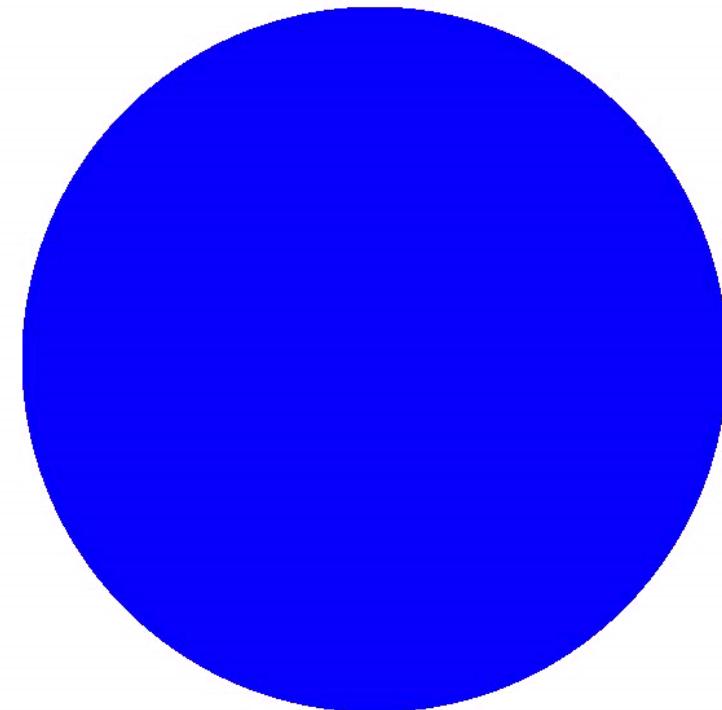
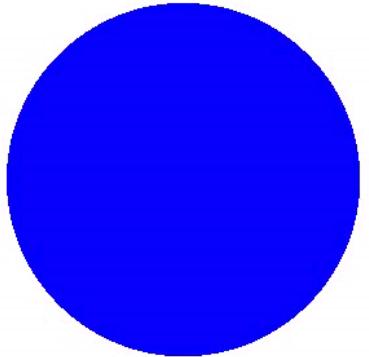


# Cahn-Hilliard Equation

$$\frac{\partial m}{\partial t} = \nabla [f(m) \nabla m - \kappa \nabla (\Delta m)]$$



# Ostwald Ripening



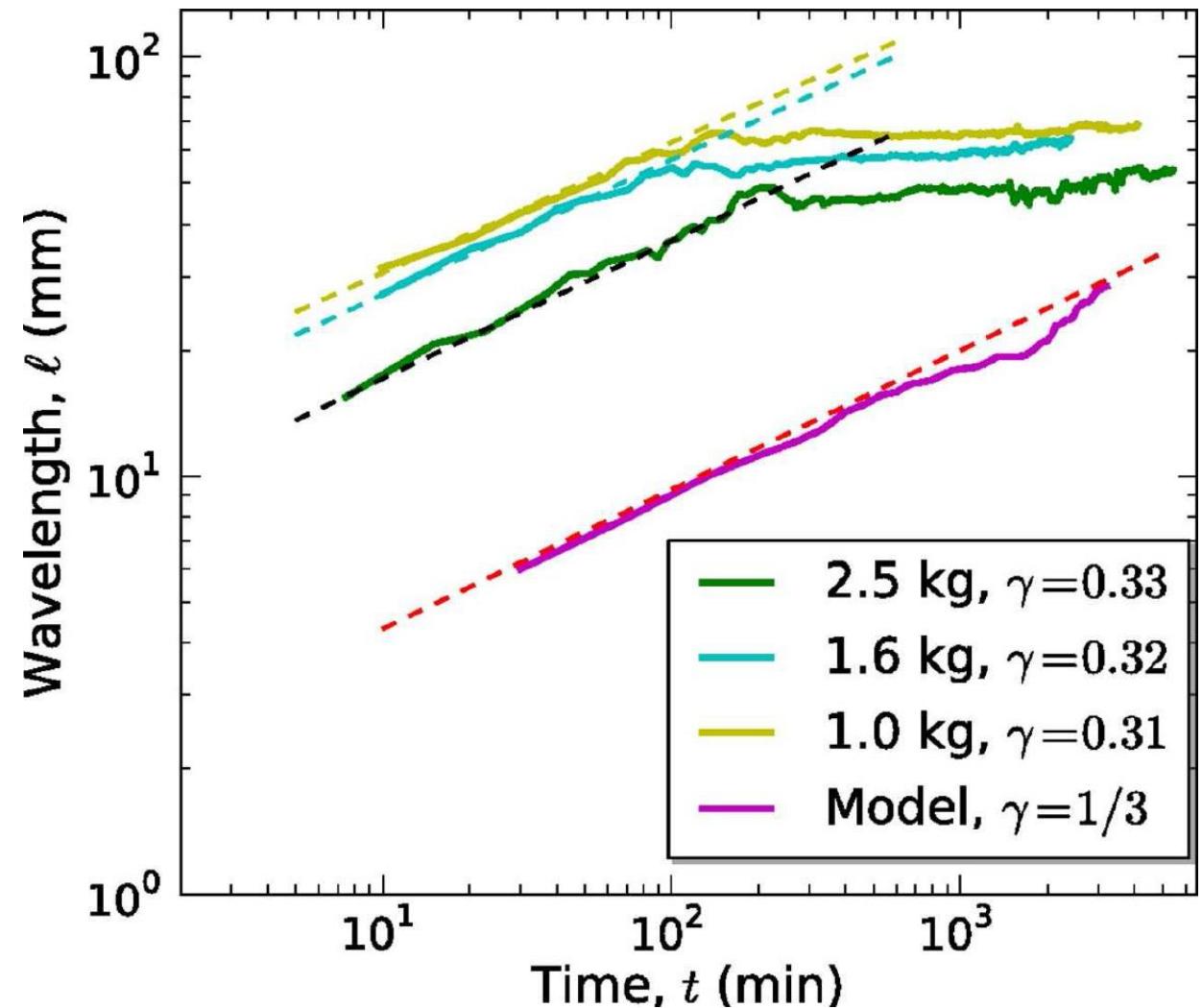
# Ostwald Ripening in mussels

Lifschitz-Slyozov law

Evolution of growth

$$\langle \text{Radius}(t) \rangle \sim t^{1/3}$$

Lifschitz-Slyozov law breaks down  
after few hours.



# Behaviour of Spatial Patterns



# Summary

Patterns in many systems

Emergence of patterns: Turing instability

Dynamics of patterns:

SLOW pattern adaptation

FAST pattern degradation

