Systems and control theory
a short crash course
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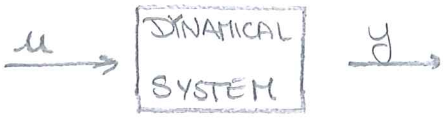
$$\begin{cases} x'(t) = f(x,u,t) \\ y(t) = g(x,u,t) \end{cases}$$



u input, y output, x state variable

Simplest example

time-invariant linear finite dimensional causal systems

future input have no influence in the present

$$\begin{cases} x'(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

A, B, C, D matrices, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$

↳ study A, B, C, D

input to state map $x'(t) = Ax(t) + Bu(t)$ $\quad u(t) \in \mathbb{R}^m$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} B u(s)\, ds \qquad x(t; u, x_0)$$

STABLE if $\forall x(t)$ solution of $x'(t) = Ax$ $\quad x(t) \xrightarrow[t\to\infty]{} 0$

$\iff$ all eigs(A) are in the open left half plane

CONTROLLABLE if $\forall x_0, x_1$ $\exists t_1; u(t)$ for $0 \le t \le t_1 : x(t_1; u, x_0) = x_1$

Thm CONTROLLABLE iff $\operatorname{Im}[B \;\; AB \;\; \dots \;\; A^{n-1}B] = \mathbb{R}^n$

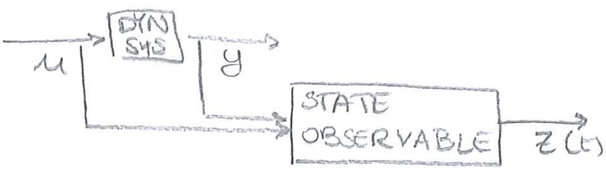
TO STABILIZE a system: choose u: $x(t) \to 0$

STABILIZABLE if $\forall x_0$ $\exists u : x(t, u, x_0) \xrightarrow[t\to\infty]{} 0$

Thm system controllable $\to \exists F$: $A+BF$ has all the eigs in the open left half plain

technically, you can choose where to put the poles

make a state observer



with $z(t) \approx x(t)$ for t large

OBSERVABILITY ~~CONTRO~~ $\iff \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \{0\}$ every initial state can in principle be reconstructed from the outputs

def. $y(x_0; 0, t) = y(x_1; 0, t)\ \forall t \to x_0 = x_1$

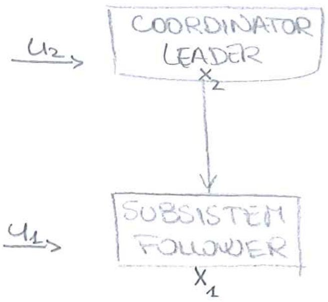
LQ optimal control problem

quadratic cost function $\quad J(x_0, u) = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\; dt$

Q pos. semi-def. R pos. def.

minimize $J(x_0, u)$ over all stabilizing input functions u

assumption: (A, B) ~~stabilizable~~ controllable, (A^T, Q) controllable



information flows downwards
(no information goes up)

leader on follower

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

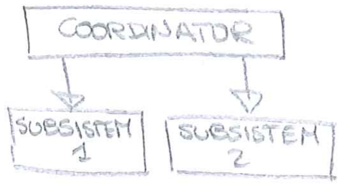
follower on leader

feedback control preserving the structure

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ & F_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

no influence on eigenvalue

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11}+B_{11}F_{11} & \boxed{A_{12}+B_{11}F_{12}+B_{12}F_{22}} \\ & A_{22}+B_{22}F_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$A = \begin{bmatrix} A_{11} & & A_{1c} \\ & A_{22} & A_{2c} \\ & & A_{cc} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & & B_{1c} \\ & B_{22} & B_{2c} \\ & & B_{cc} \end{bmatrix}$$

still an algebra $\rightarrow$ still works

LQ problem for leader - follower

$$J\left(x_0, \begin{bmatrix} u_1(\cdot) \\ u_2(\cdot) \end{bmatrix}\right) = \int_0^\infty [x_1^T \; x_2^T]\begin{bmatrix} Q_{11} & \\ & Q_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [u_1^T \; u_2^T]\begin{bmatrix} R_{11} & \\ & R_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dt$$

$Q_{11} \geq 0, \quad Q_{22} \geq 0, \quad R_{11} > 0, \quad R_{22} > 0$

$$\mathcal{U}^\Delta = \left\{ \begin{bmatrix} F_{11} & F_{12} \\ & F_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} : A+BF \text{ is stable, } F \text{ may depend on } x_0 \right\}$$

problem: $\displaystyle\min_{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathcal{U}^\Delta} J\left(x_0, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)$

suppose F_{22} is fixed $\rightarrow$ u_2 known $\rightarrow$ back to LQ optimal control
find J with F_{22} given
$\rightarrow$ minimise J over all stabilizing F_{22}

Conjecture: $\exists! F_{22}$ that minimises J, the cost
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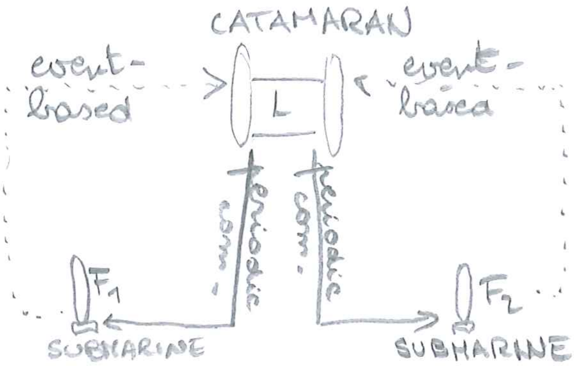
parametrization of the set of stabilization feedback matrices (A_{22}, B_{22}) does not yeld convexity in the straight forward param.

leader-follower problem, but leader has access to informations about the follower when needed $\leftarrow$ event based

catamaran leads the way.



communication through sonic waves, at the moment full communication

goal: limit communication between submarines and catamaran

unrestricted communication: $G=-R^{-1}B^TX$ $\quad G=\begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix}$

$$T(t)=\frac{1}{\|x_2(t)\|^2}x_1(t)\,x_2^T(t) \qquad T(t)\,x_2=x_1$$

allows for the desired form

$$u=Gx$$

idea: set T as piecewise constant

if $$\|\underbrace{G_{21}(x_1(t)}_{\text{where the follower is}} - \underbrace{T(t_{dd})\,x_2(t)}_{\text{where the leader thinks the follower is}}\| \le \underbrace{re^{-\beta t}}_{\text{tollerance}}$$

the communication is reset.

new cost $$J_{r,\beta}-J_0 \le \|R^{1/2}\|^2\,\frac{r^2}{2\beta}$$