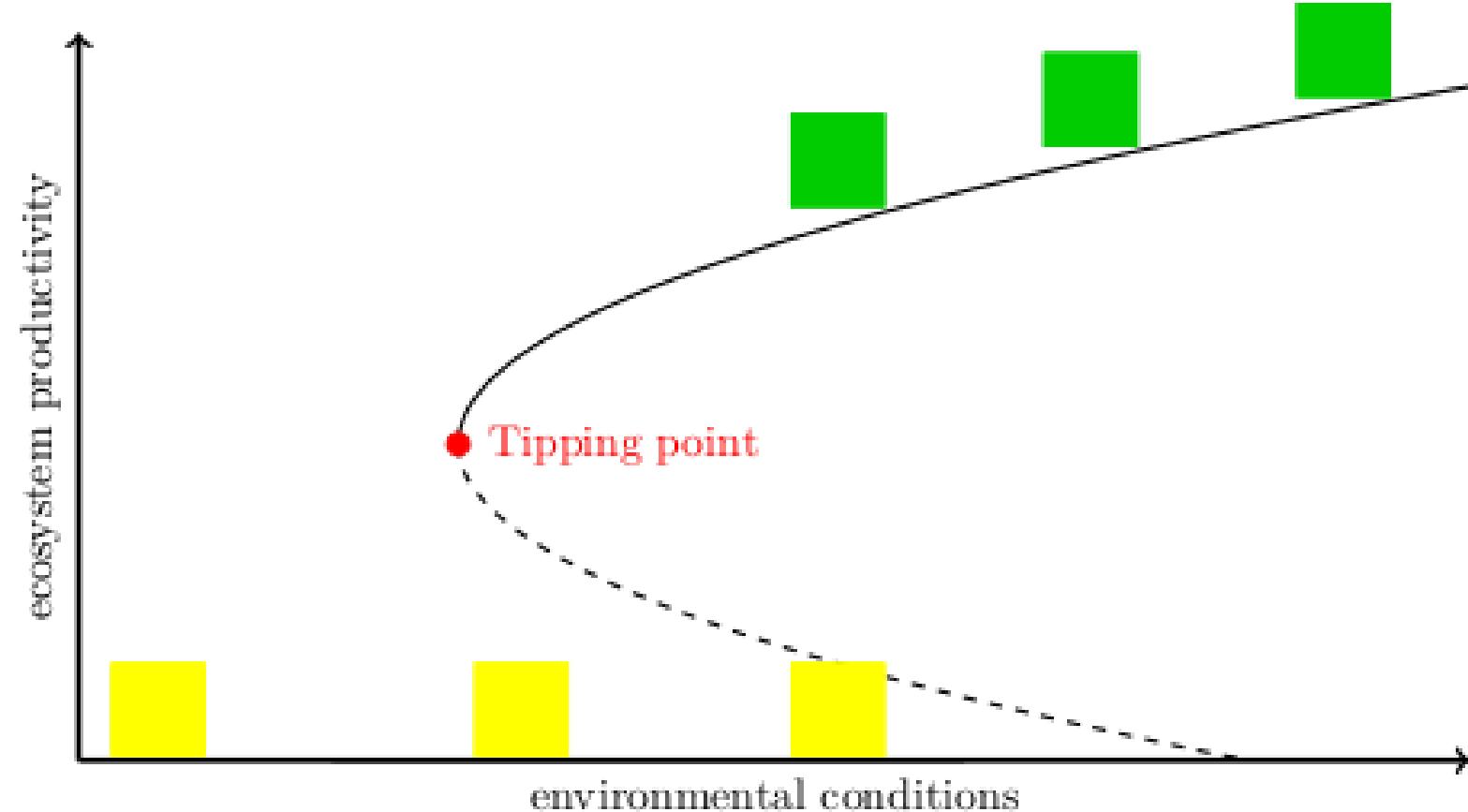
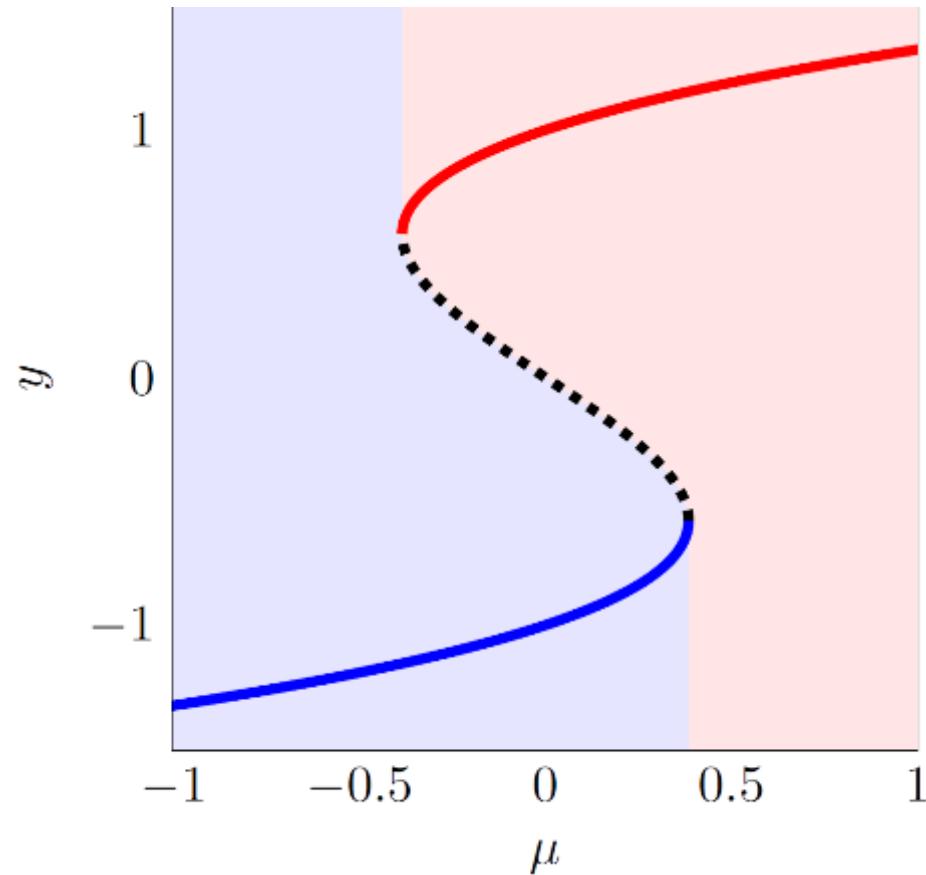
A wide-angle photograph of a massive glacier. The ice is a vibrant turquoise color, with deep blue veins running through it. The surface is covered in white snow and various shades of blue and white from the ice itself. The glacier rises in large, jagged peaks that meet a clear, light blue sky.

Tipping in Spatially Extended Systems

2023-04-19, Colloquium, MI, Potsdam University
Robbin Bastiaansen (r.bastiaansen@uu.nl)

Classic Theory of Tipping



Canonical example:

$$\frac{dy}{dt} = y(1 - y^2) + \mu$$

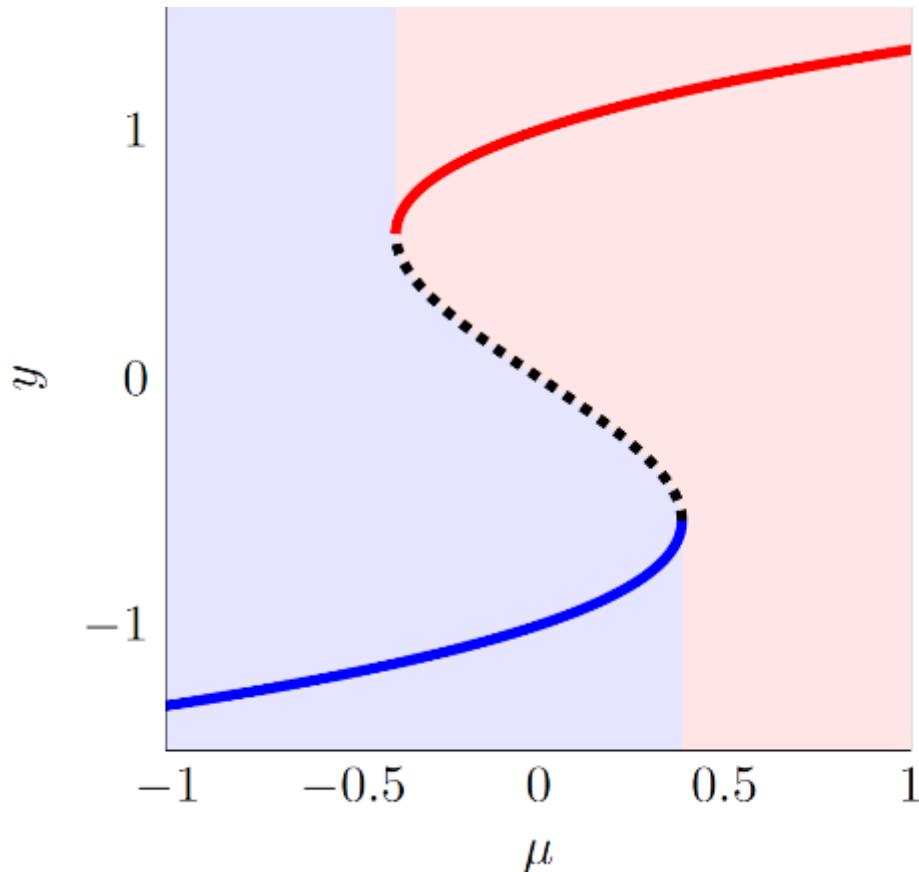
$$\frac{d\vec{y}}{dt} = f(\vec{y}; \mu)$$

$$\begin{cases} \frac{du}{dt} = f(u, v; \mu) \\ \frac{dv}{dt} = g(u, v; \mu) \end{cases}$$

Tipping in ODEs (1)

Canonical example:

$$\frac{dy}{dt} = y(1 - y^2) + \mu$$



Concrete example: Global Energy Balance Model

$$\frac{dT}{dt} = Q(1 - \alpha(T)) - \varepsilon\sigma_0 T^4 + \mu$$

Classic Literature

[Holling, 1973]

[Noy-Meier, 1975]

[May, 1977]

Tipping

[Ashwin et al, 2012]

Bifurcation-Tipping : Basin disappears

Noise-Tipping : Forced outside Basin

Rate-Tipping : *(more complicated)*

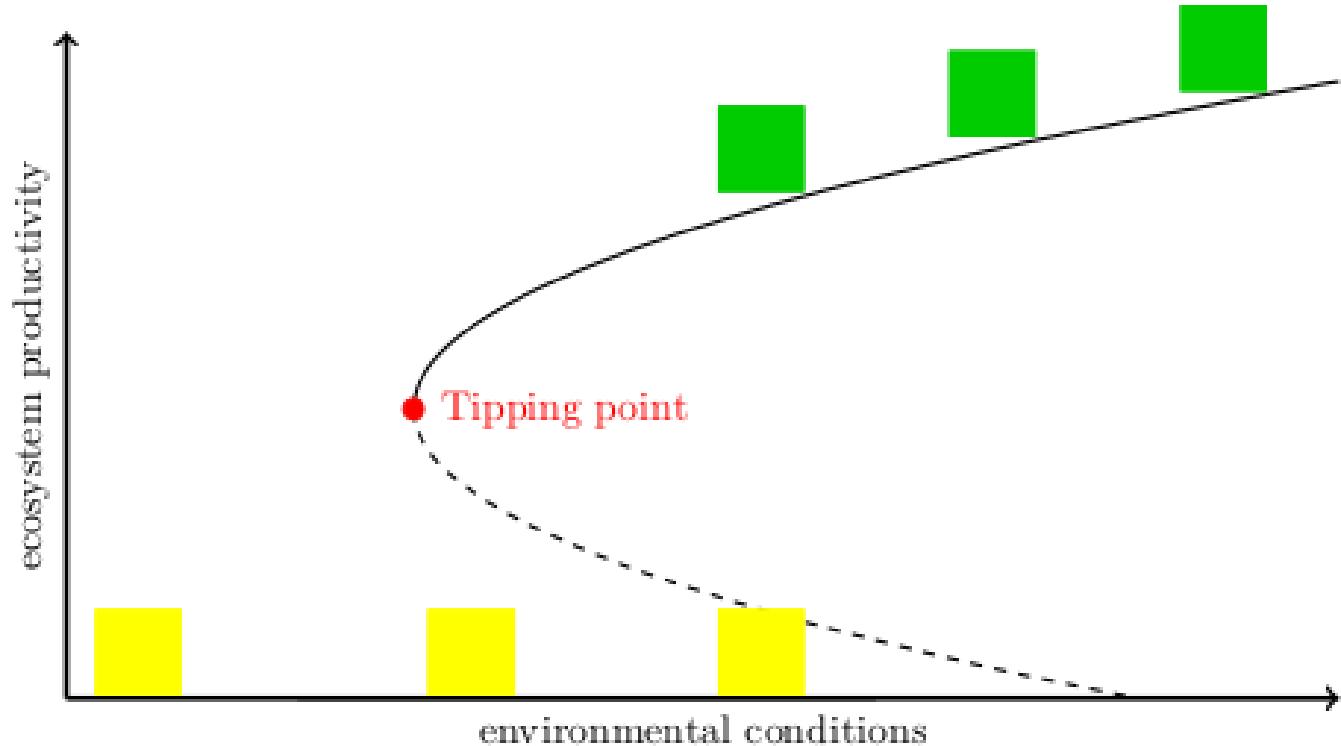
Tipping in ODEs (2)

Two components:

$$\begin{cases} \frac{du}{dt} = f(u, v) \\ \frac{dv}{dt} = g(u, v) \end{cases}$$

includes common models:

- Predator-Prey
- Activator-Inhibitor



Examples of tipping in ODEs include:

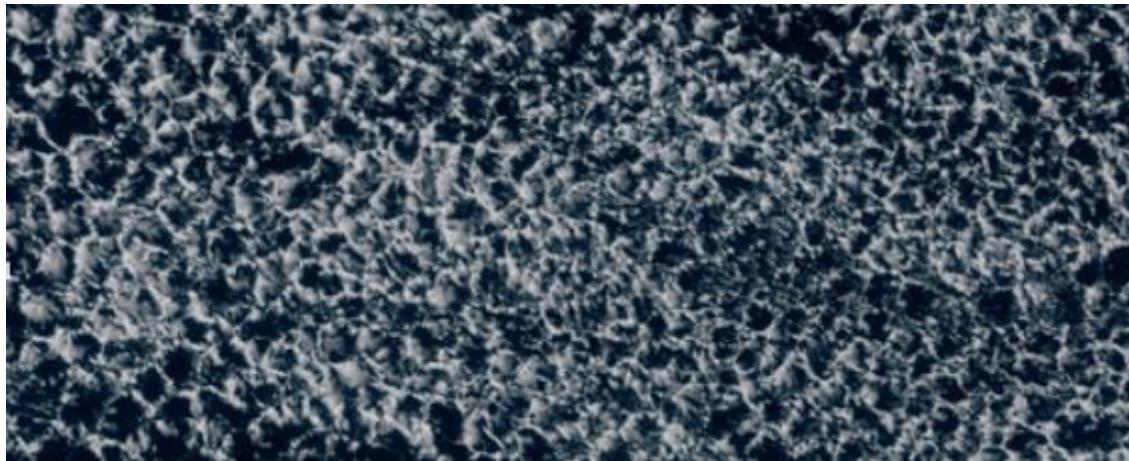
- Forest-Savanna bistability
- Deep ocean exchange
- Cloud formation
- Ice sheet melting
- Turbidity in shallow lakes



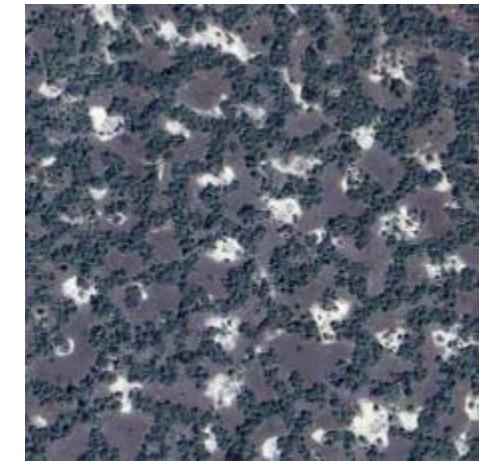
Examples of spatial patterning – regular patterns



mussel beds



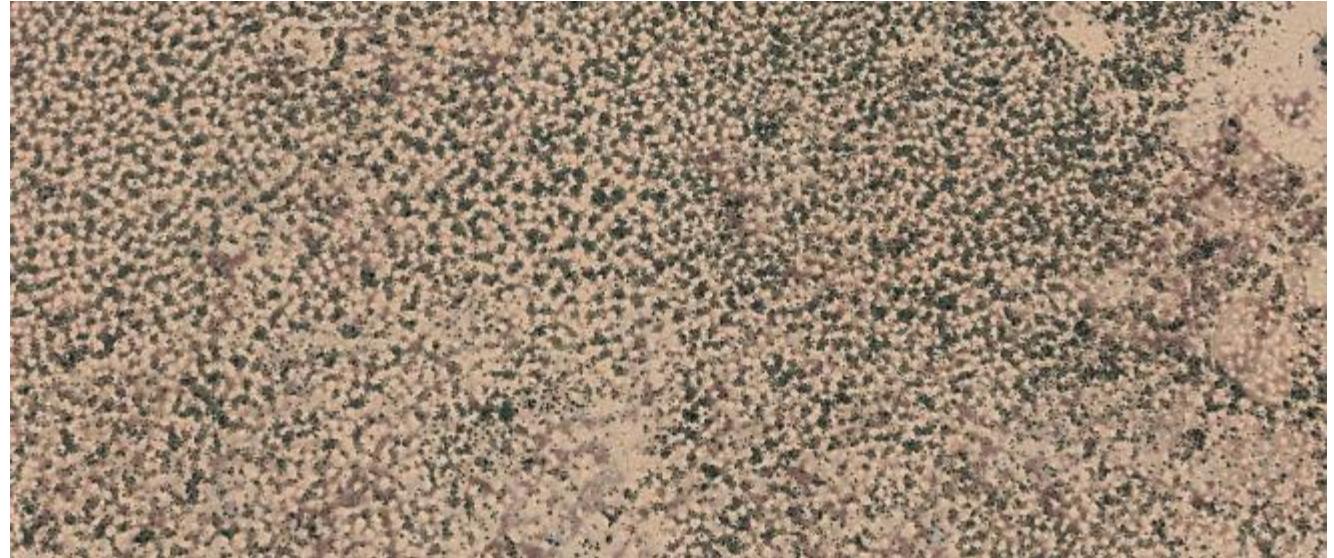
clouds



savannas



melt ponds



drylands

Examples of spatial patterning – spatial interfaces

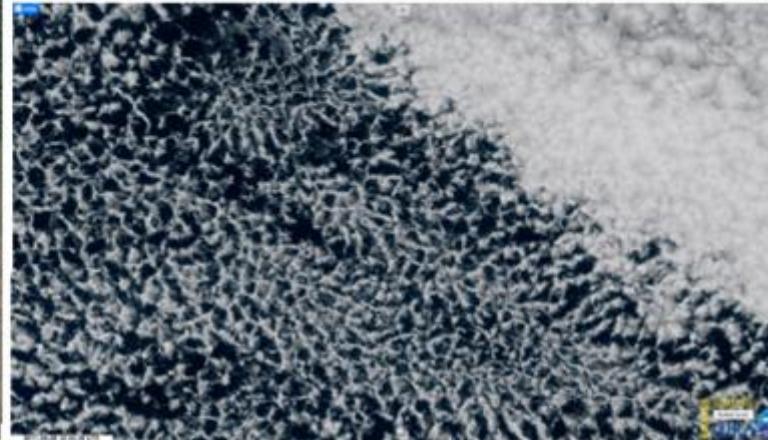
tropical forest
& savanna
ecosystems

[Google Earth]



types of
stratocumulus
clouds

[RAMMB/CIRA SLIDER]



sea-ice & water
at Eltanin Bay

[NASA's Earth observatory]



algae bloom
in Lake St. Clair

[NASA's Earth observatory]





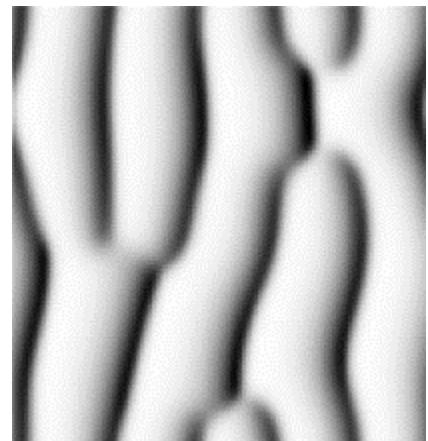
Part 1: Turing Patterns

Patterns in models

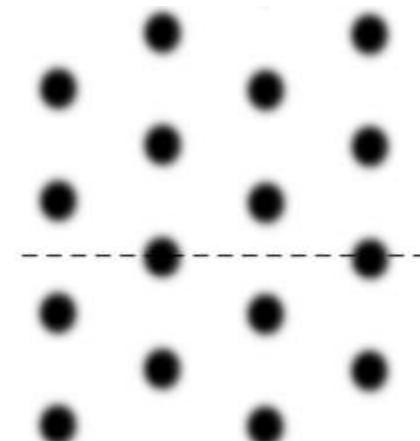
Add spatial transport:

Reaction-Diffusion equations:

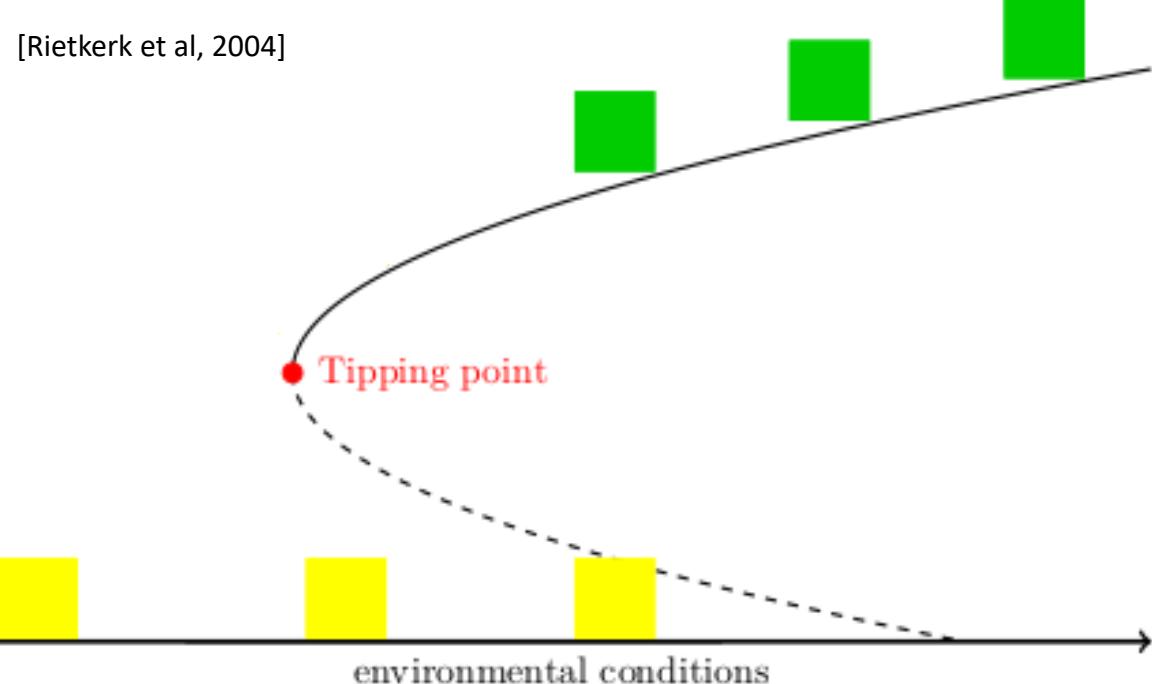
$$\begin{cases} \frac{du}{dt} = f(u, v) + D_u \Delta u \\ \frac{dv}{dt} = g(u, v) + D_v \Delta v \end{cases}$$



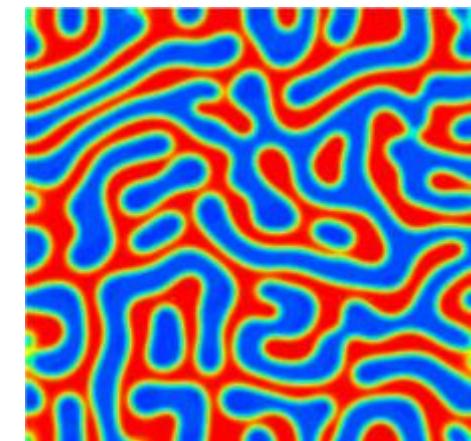
[Klausmeier, 1999]



[Gilad et al, 2004]

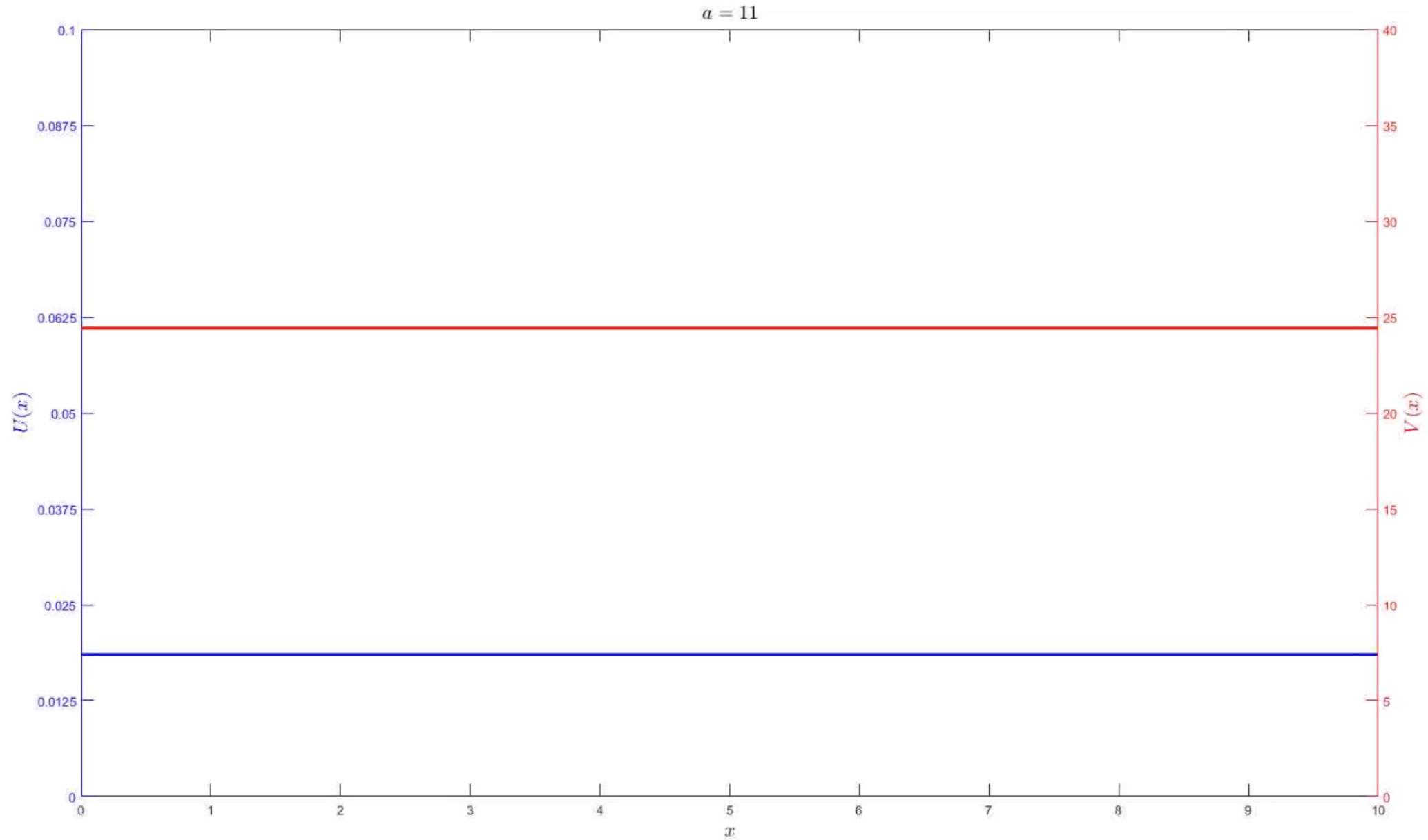


[Rietkerk et al, 2002]



[Liu et al, 2013]

Behaviour of PDEs



Turing patterns

$$\begin{cases} \frac{du}{dt} = f(u, v) + D_u \Delta u \\ \frac{dv}{dt} = g(u, v) + D_v \Delta v \end{cases}$$

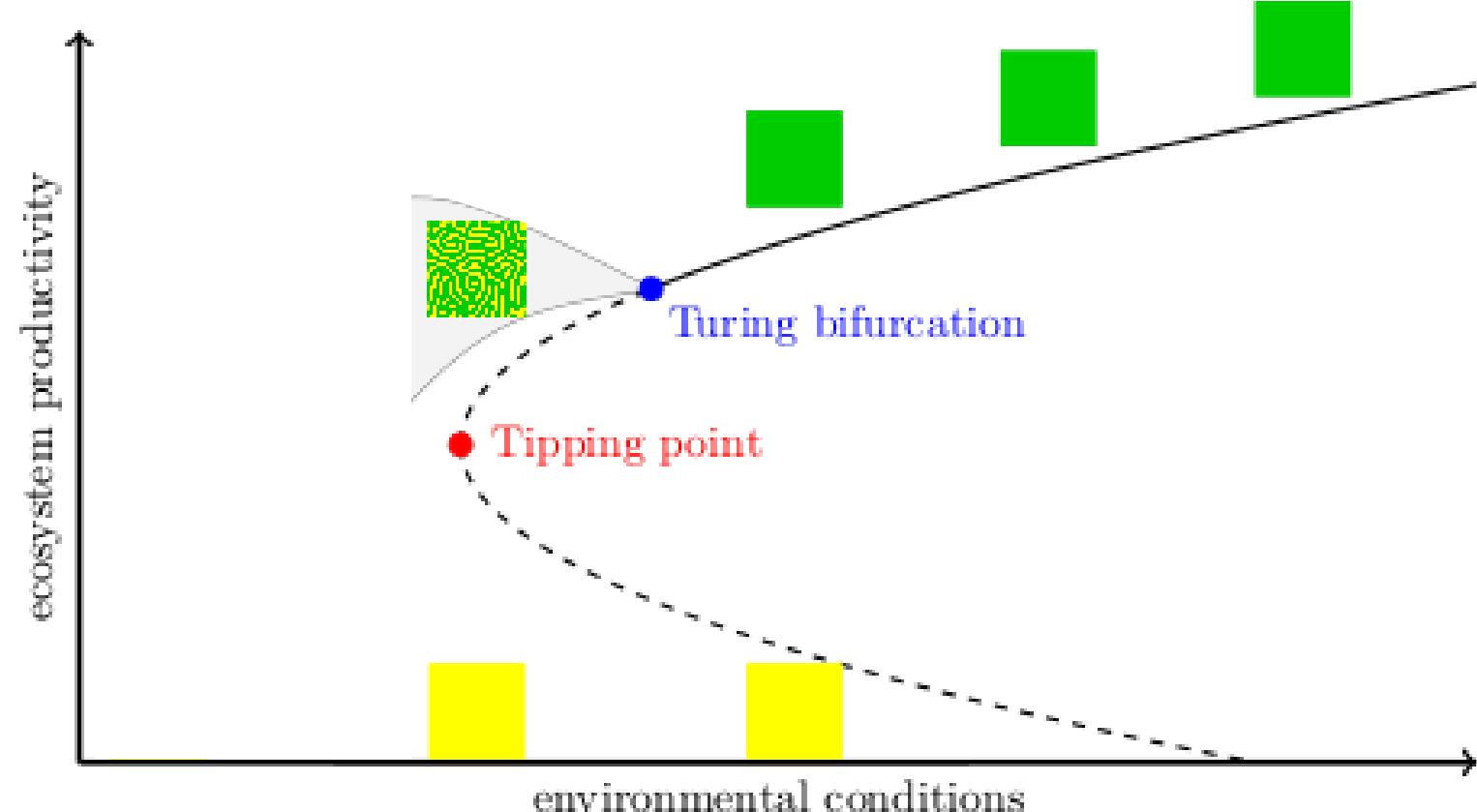
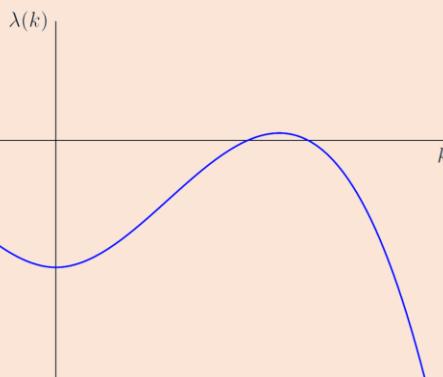
Turing bifurcation

Instability to non-uniform perturbations

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t} e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

→ Dispersion relation

$$\lambda(k) = \dots$$



Weakly non-linear analysis
Ginzburg-Landau equation / Amplitude Equation
& Eckhaus/Benjamin-Feir-Newell criterion
[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Busse balloon

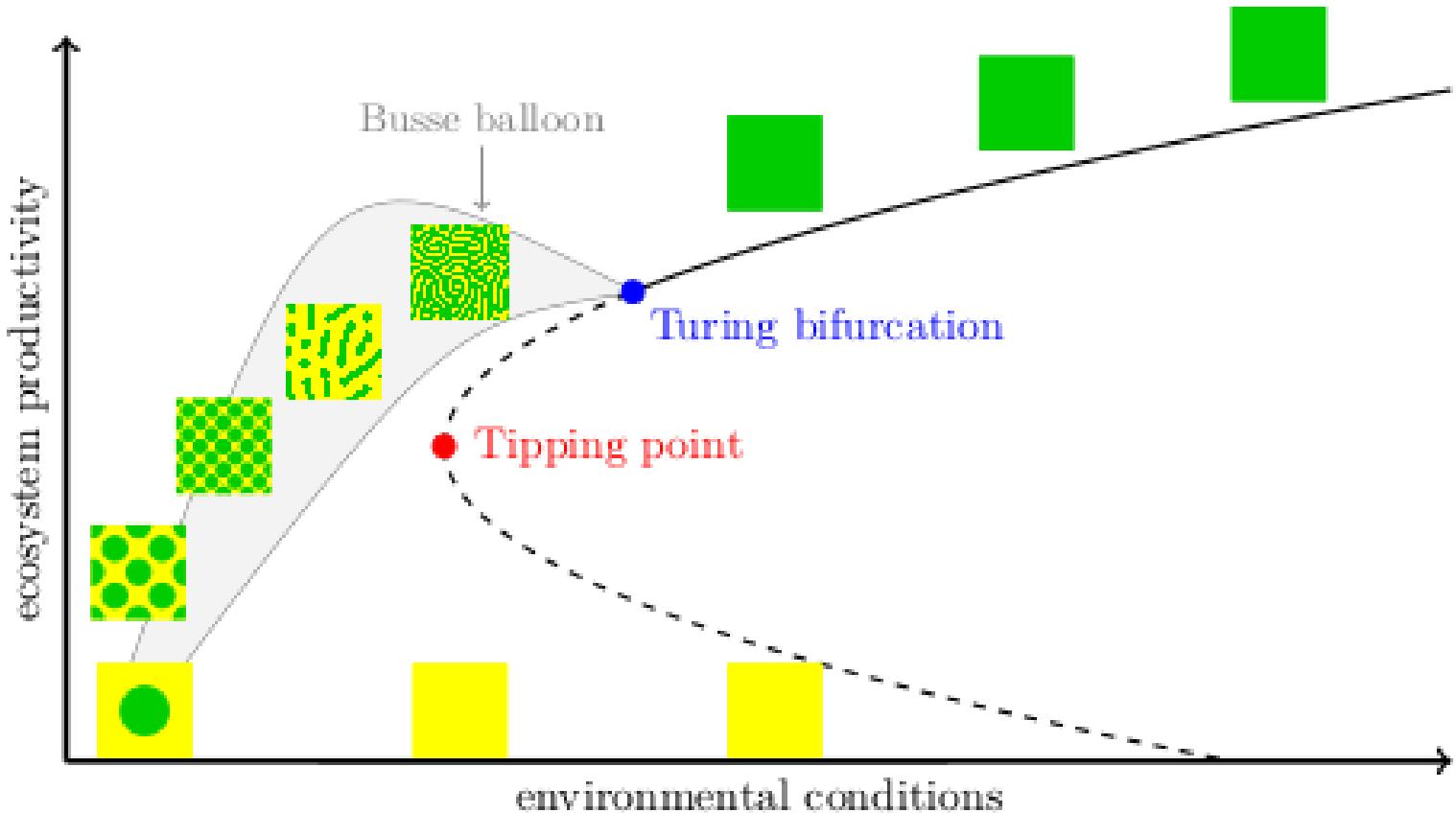
Busse balloon

A model-dependent shape in (*parameter, observable*) space that indicates all stable patterned solutions to the PDE.

Construction Busse balloon

Via numerical continuation
few general results on the
shape of Busse balloon

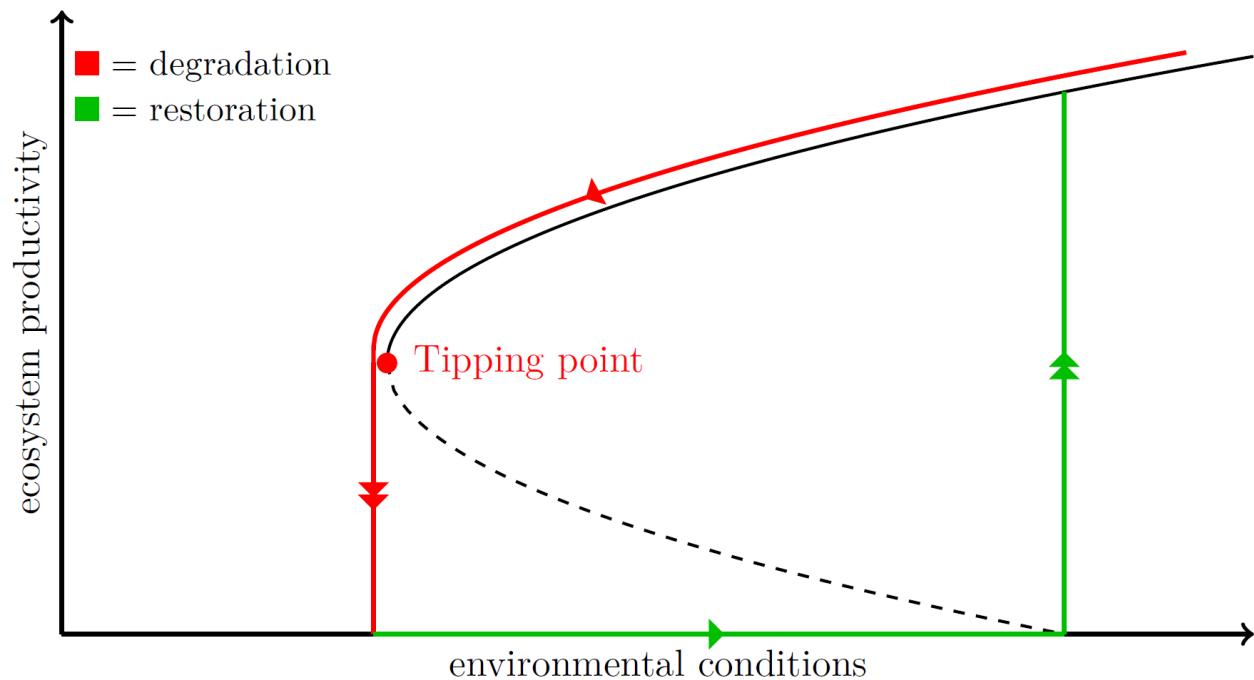
$$\begin{cases} \frac{du}{dt} = f(u, v) + D_u \Delta u \\ \frac{dv}{dt} = g(u, v) + D_v \Delta v \end{cases}$$



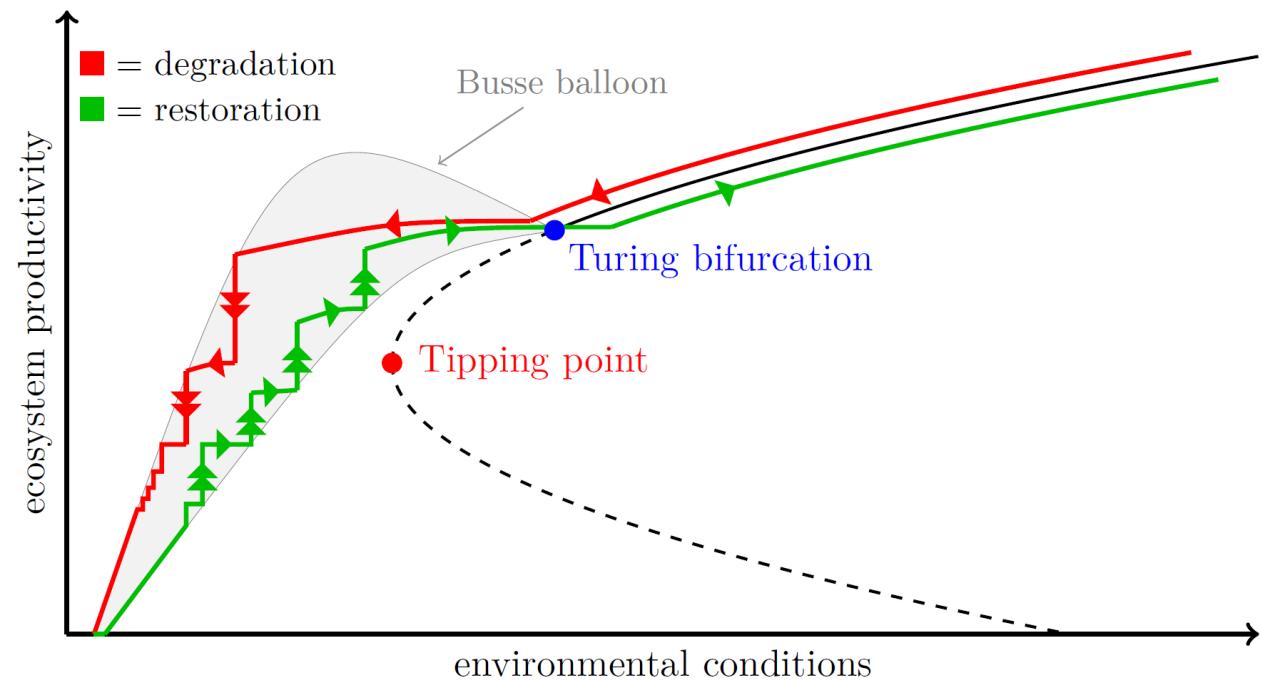
Busse balloon

Idea originates from thermal convection
[Busse, 1978]

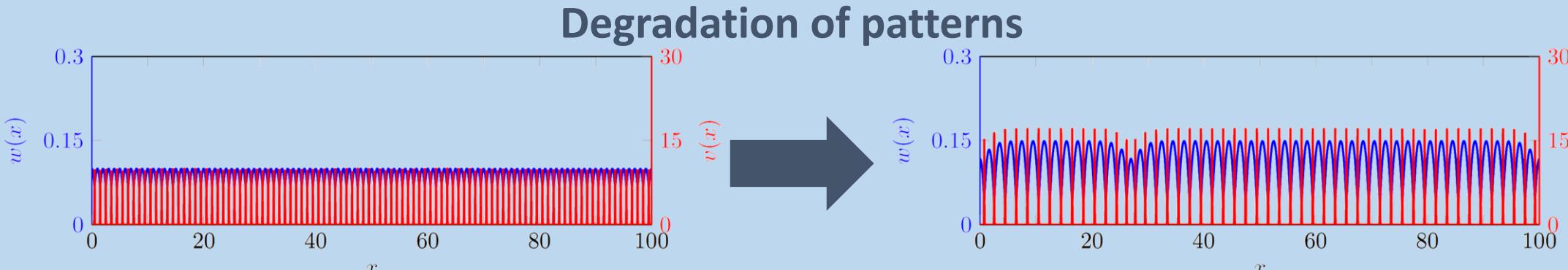
Tipping of (Turing) patterns



Classic tipping



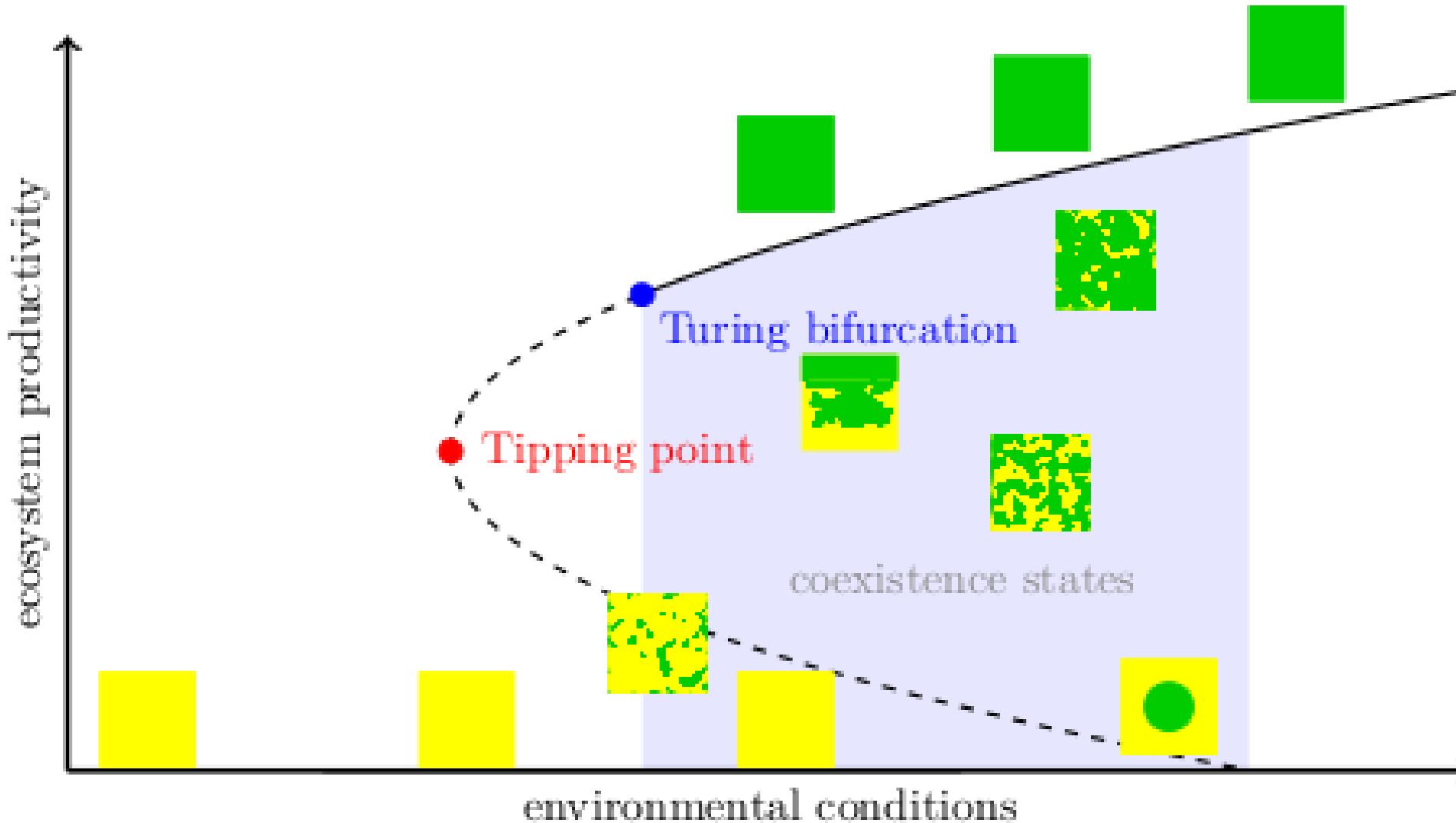
Tipping of patterns





Part 2: Coexistence States and spatial heterogeneities

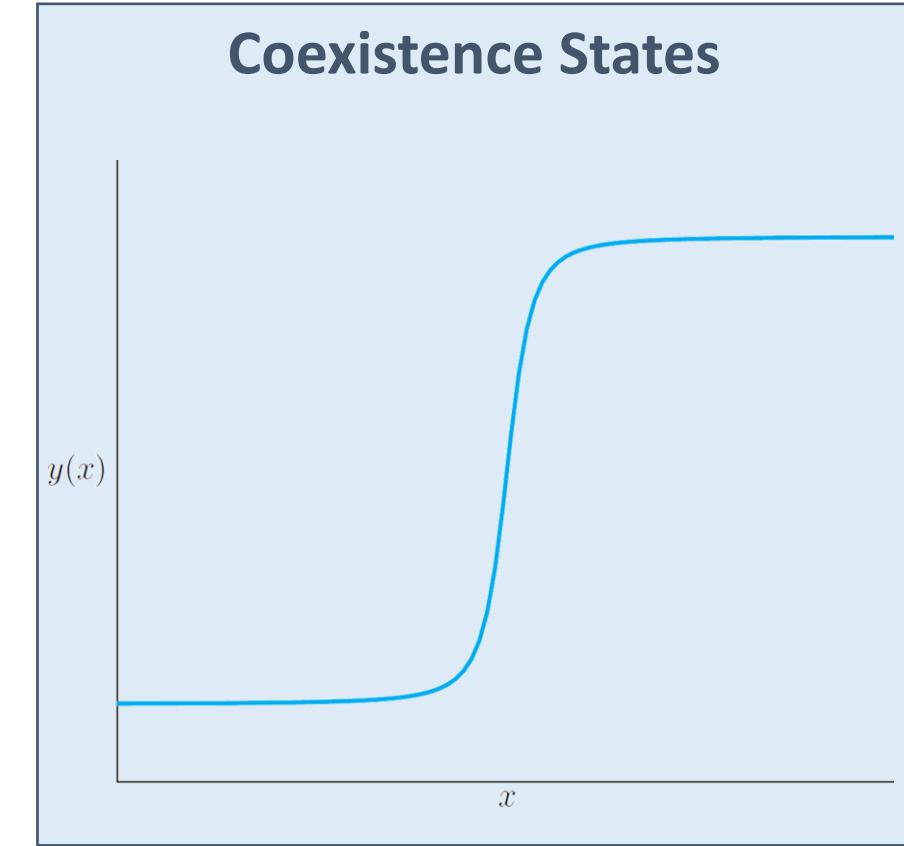
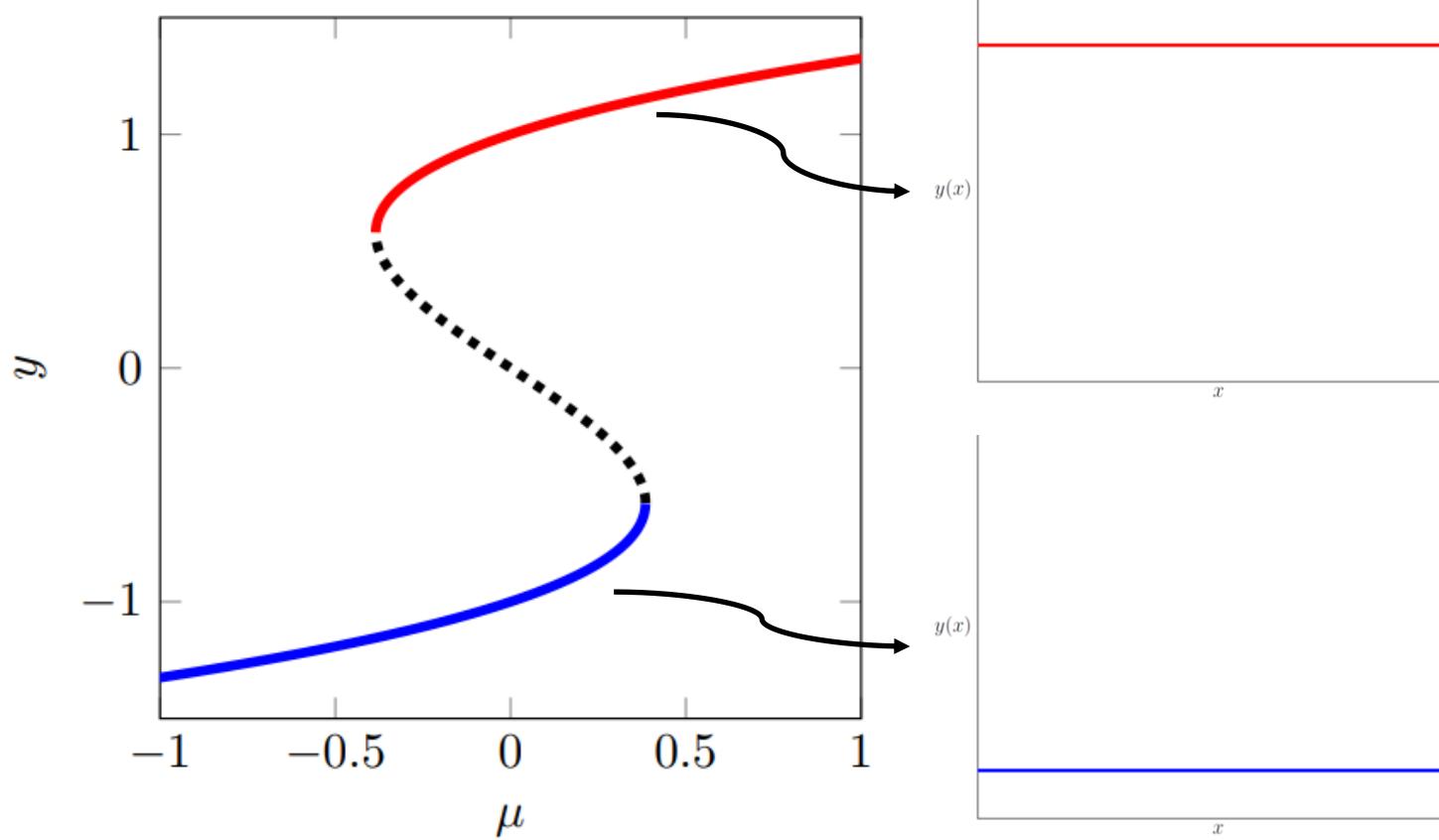
Coexistence states in bifurcation diagram



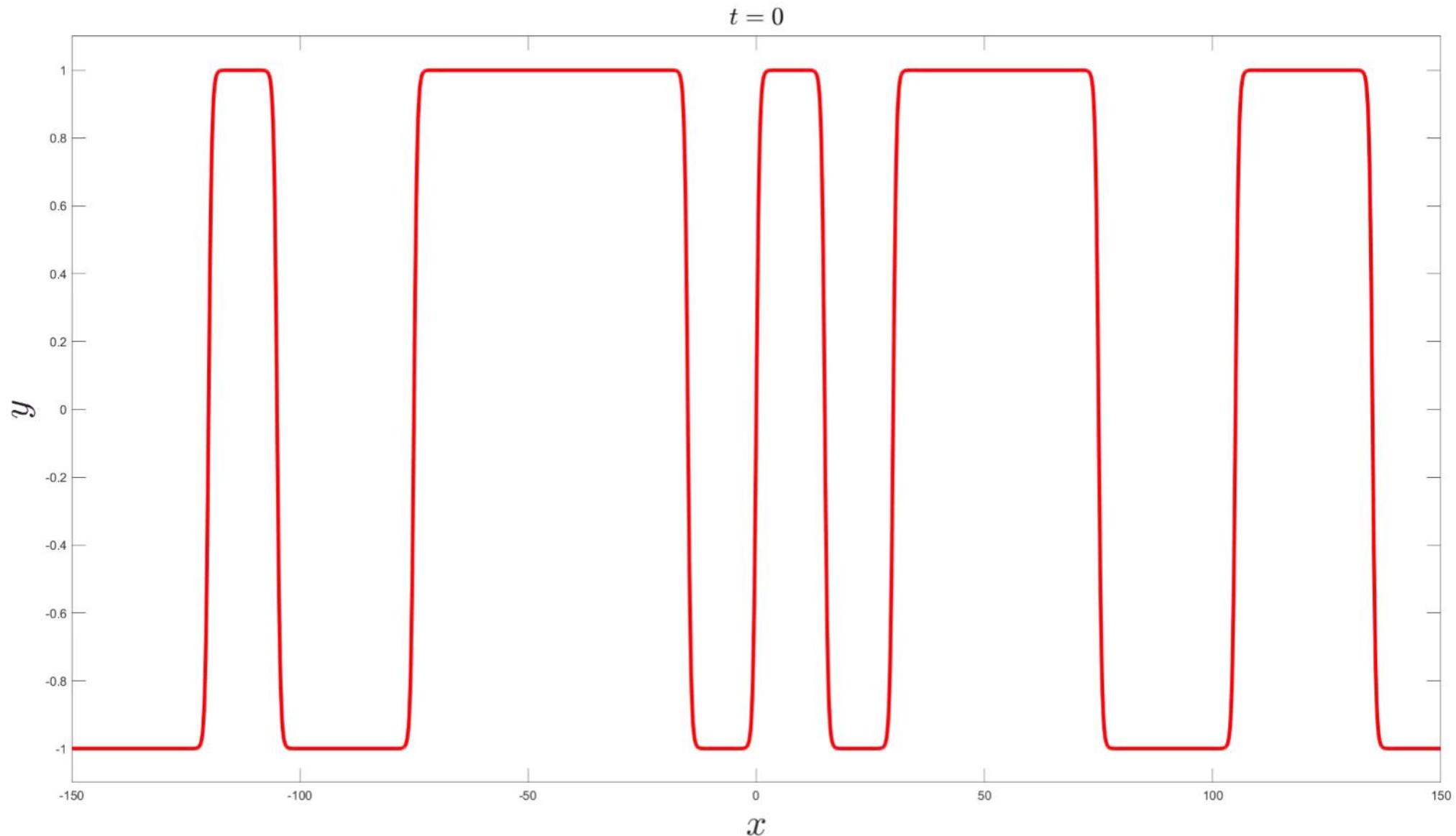
Coexistence states

Bistable (Allen-Cahn/Nagumo) equation:

$$\frac{\partial y}{\partial t} = y(1 - y^2) + \mu + D \frac{\partial^2 y}{\partial x^2}$$



Dynamics of $\frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2} + y(1 - y^2) + \mu$

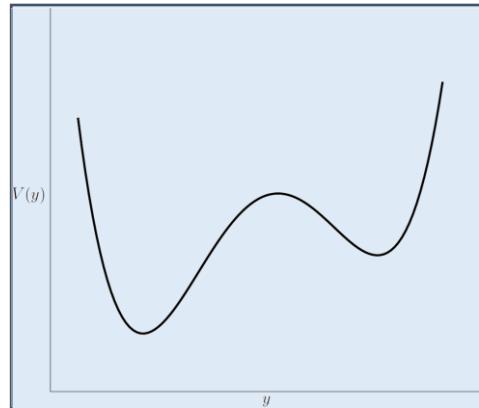


Front Dynamics

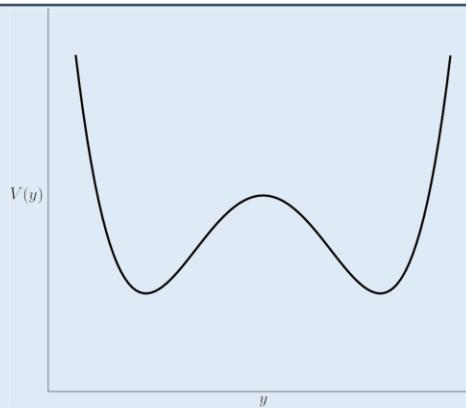
$$\frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2} + f(y; \mu)$$

Potential function $V(y; \mu)$:

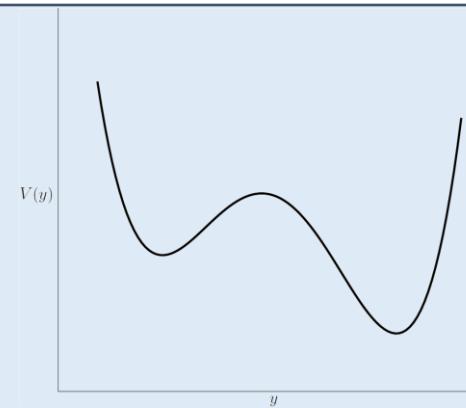
$$\frac{\partial V}{\partial y}(y; \mu) = -f(y; \mu)$$



moves right

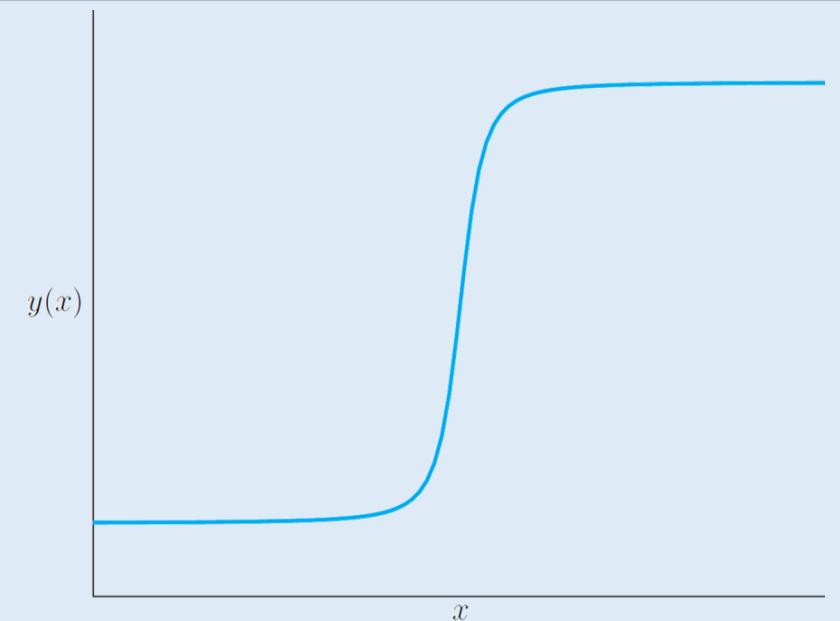
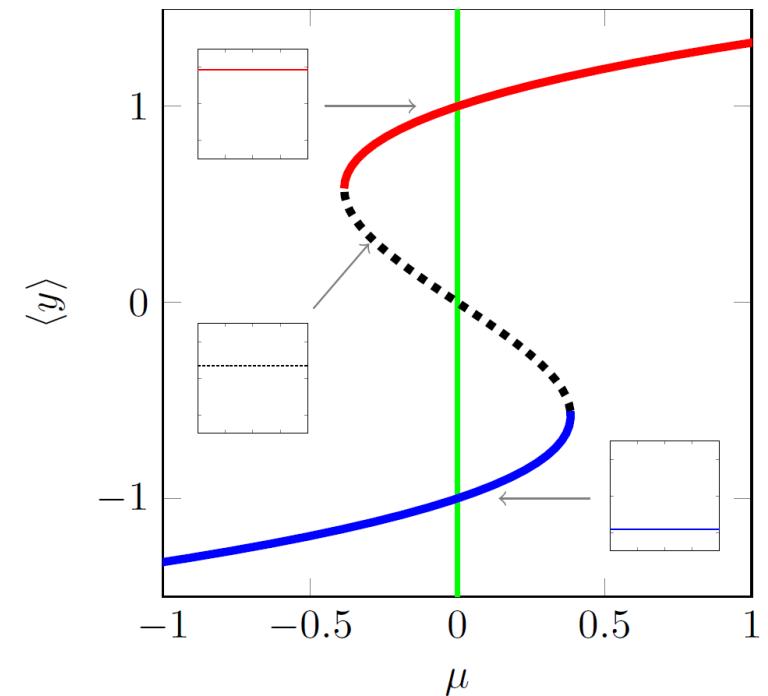


stationary



moves left

Maxwell Point $\mu_{maxwell}$

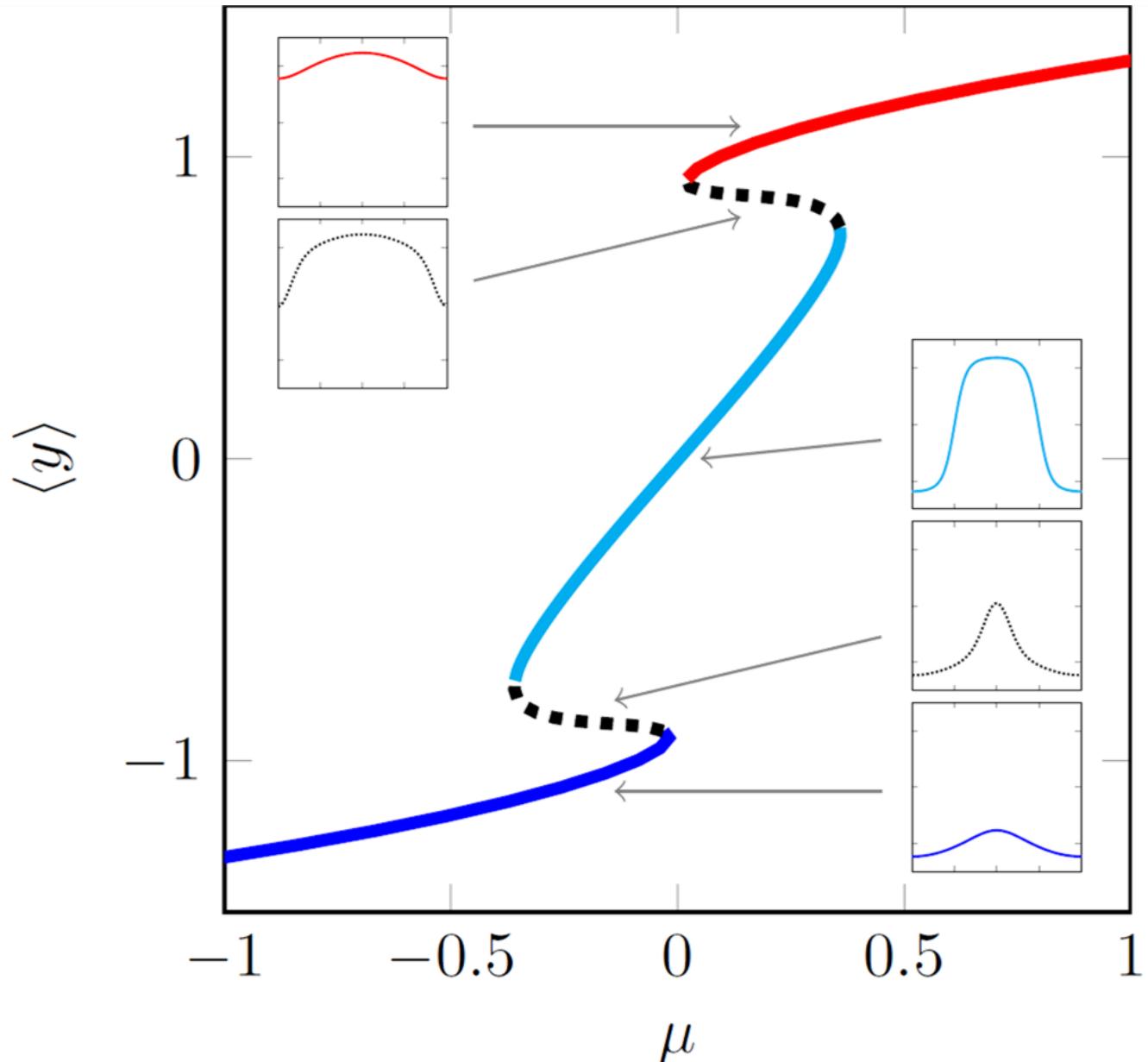


Adding Spatial Heterogeneity

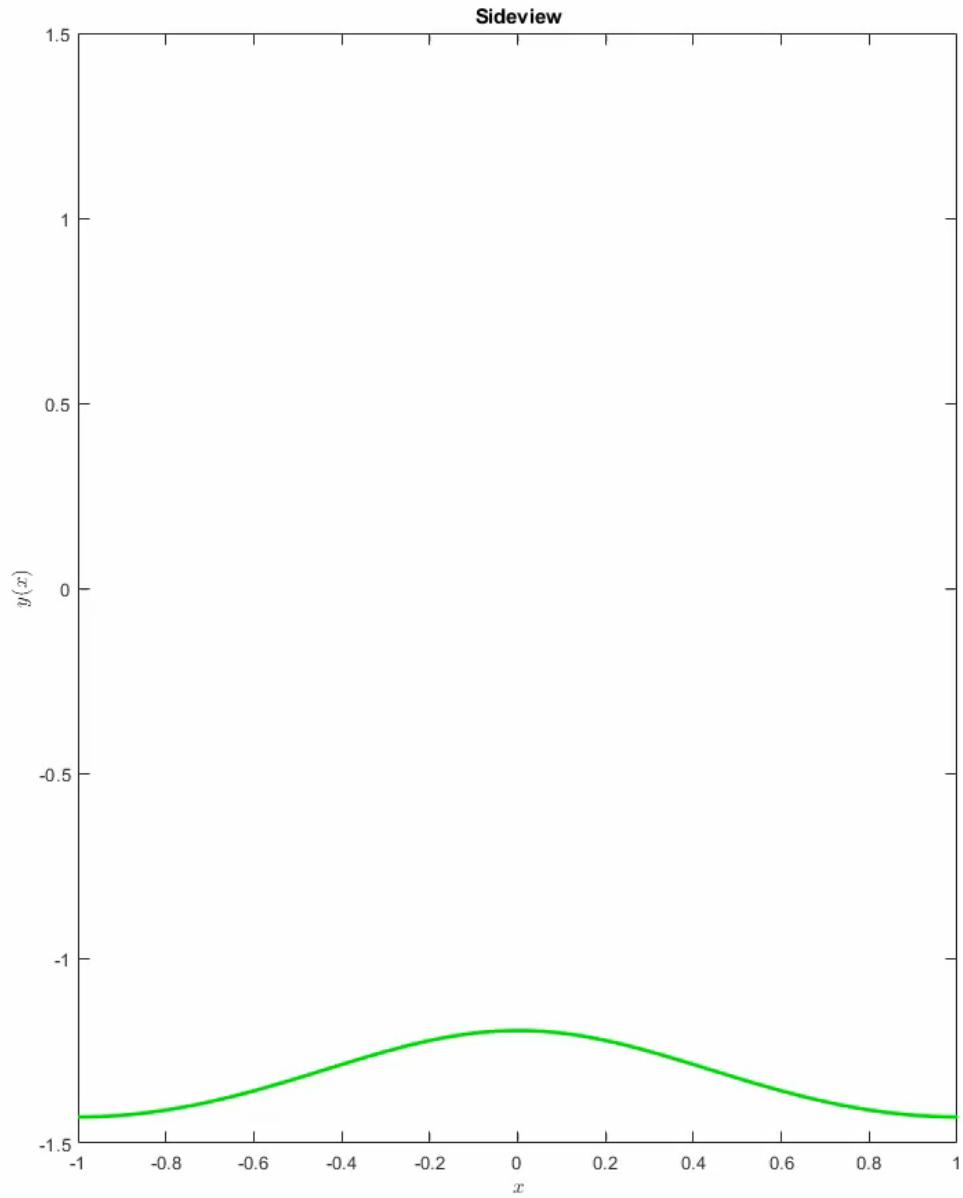
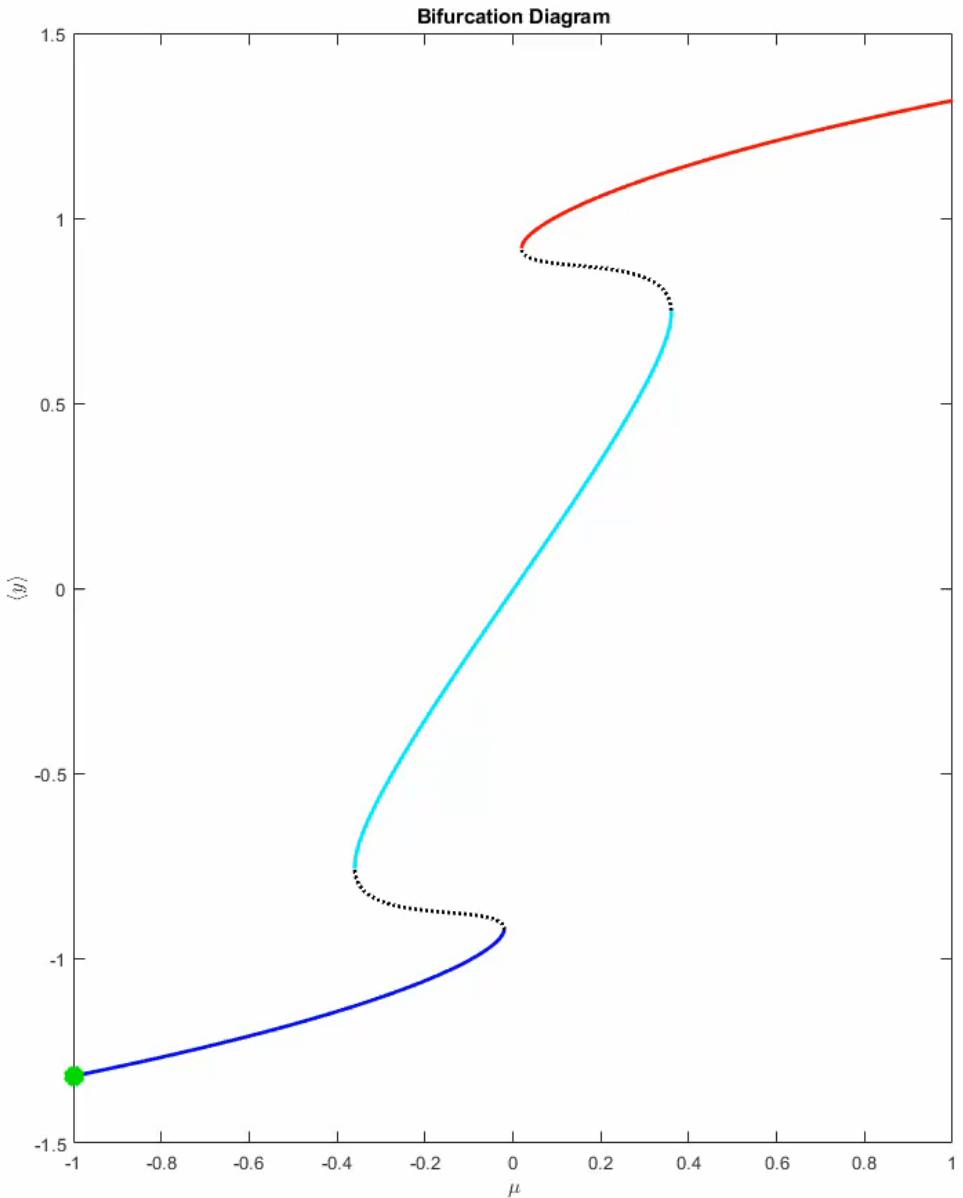
$$\frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2} + f(y, x; \mu)$$

Now, the **local** difference in potentials determines the front movement

- New behaviour:
- Multi-fronts can be stationary
 - Maxwell point is smeared out



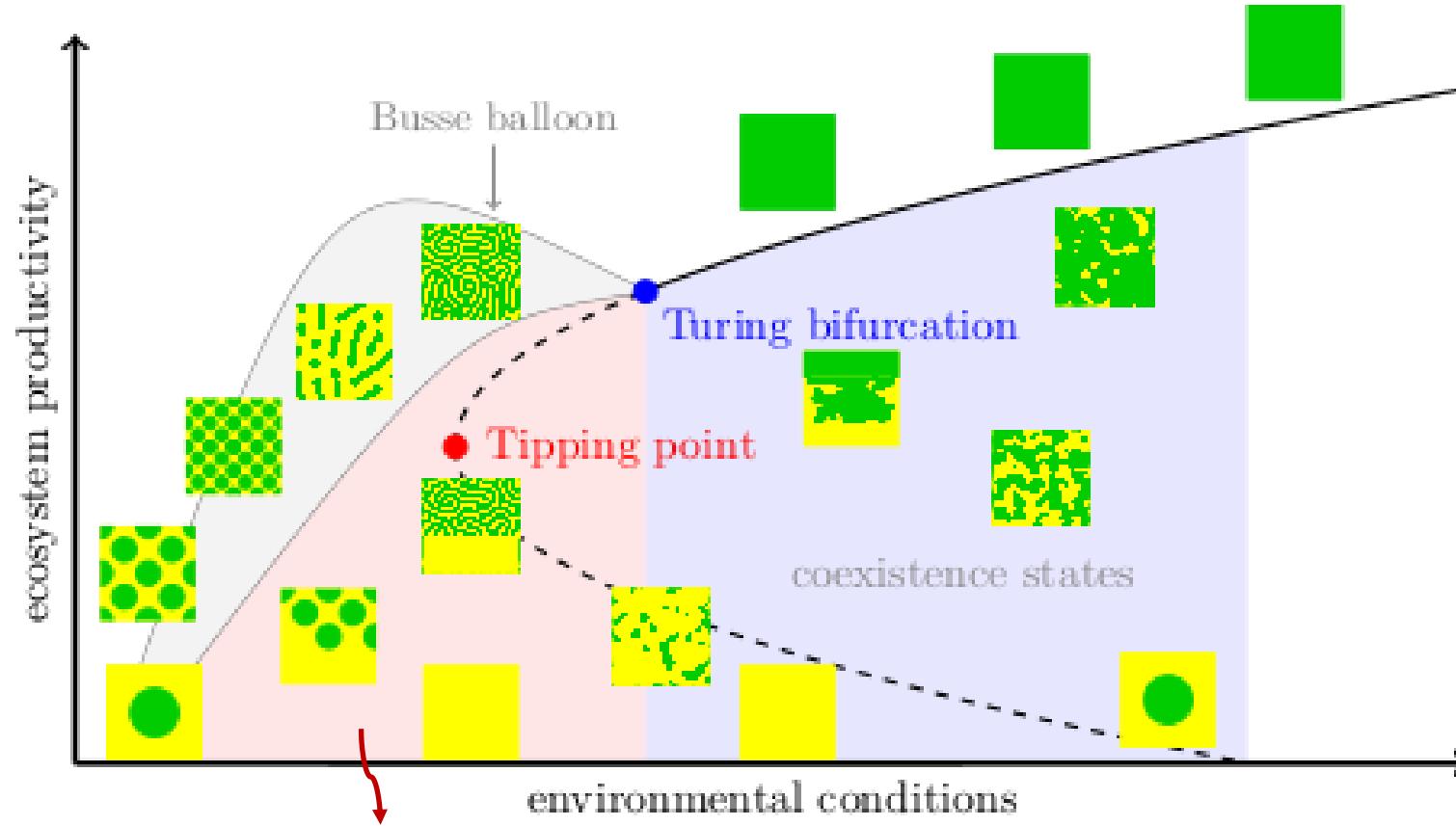
Fragmented Tipping



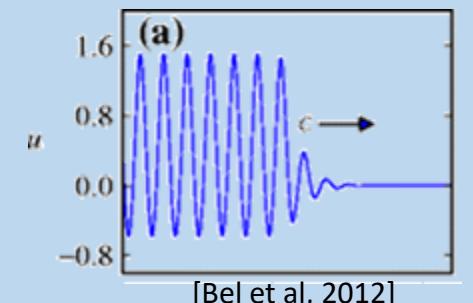
An aerial photograph of a grassland fire on a hillside. The fire is visible as a bright orange and red line of flames moving across the dry, yellowish-brown grass. A large plume of dark smoke is visible in the lower-left foreground, billowing upwards and to the right. The hillside slopes upwards towards the top right of the frame.

Part 3: Tipping in Spatially Extended Systems?

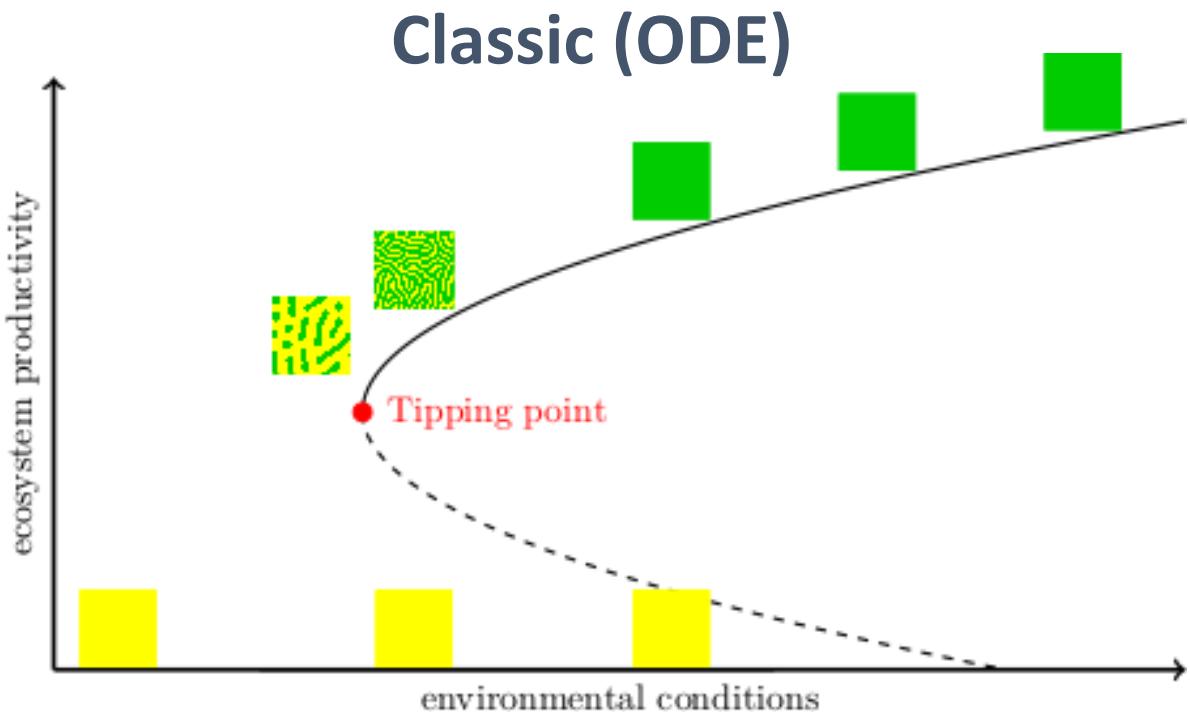
“Bifurcation Diagram” for spatially extended systems



Coexistence states
between patterned and
uniform states also exist

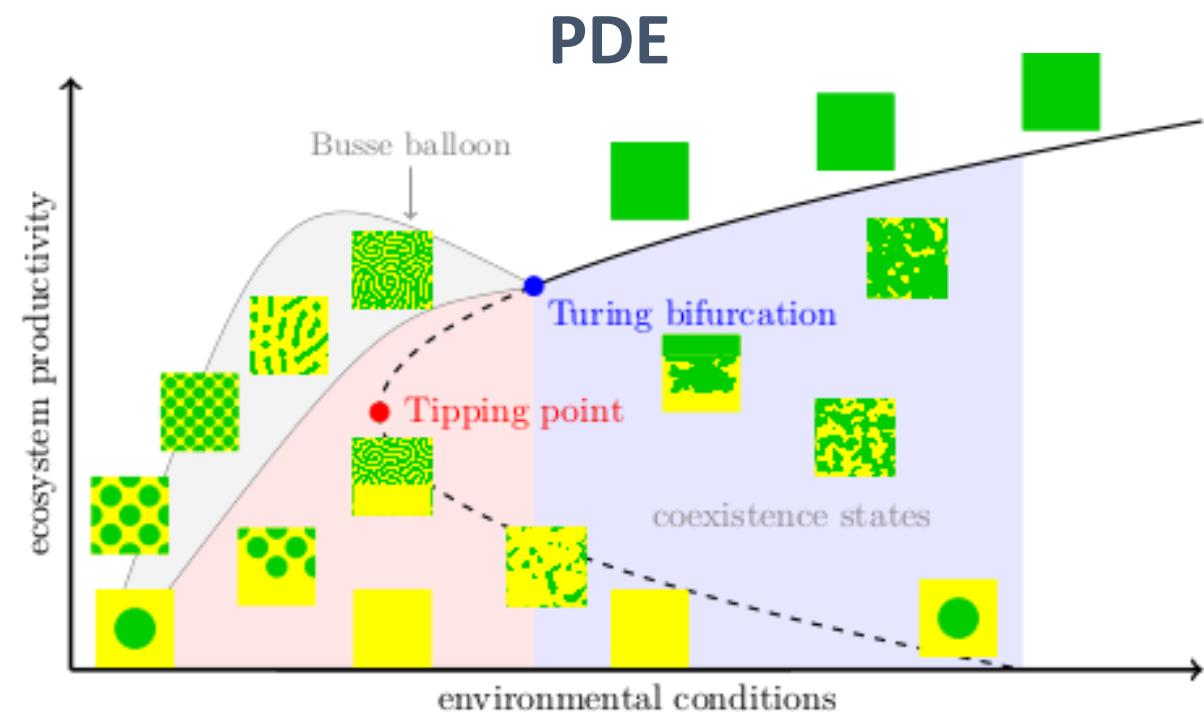


What if the system tips?



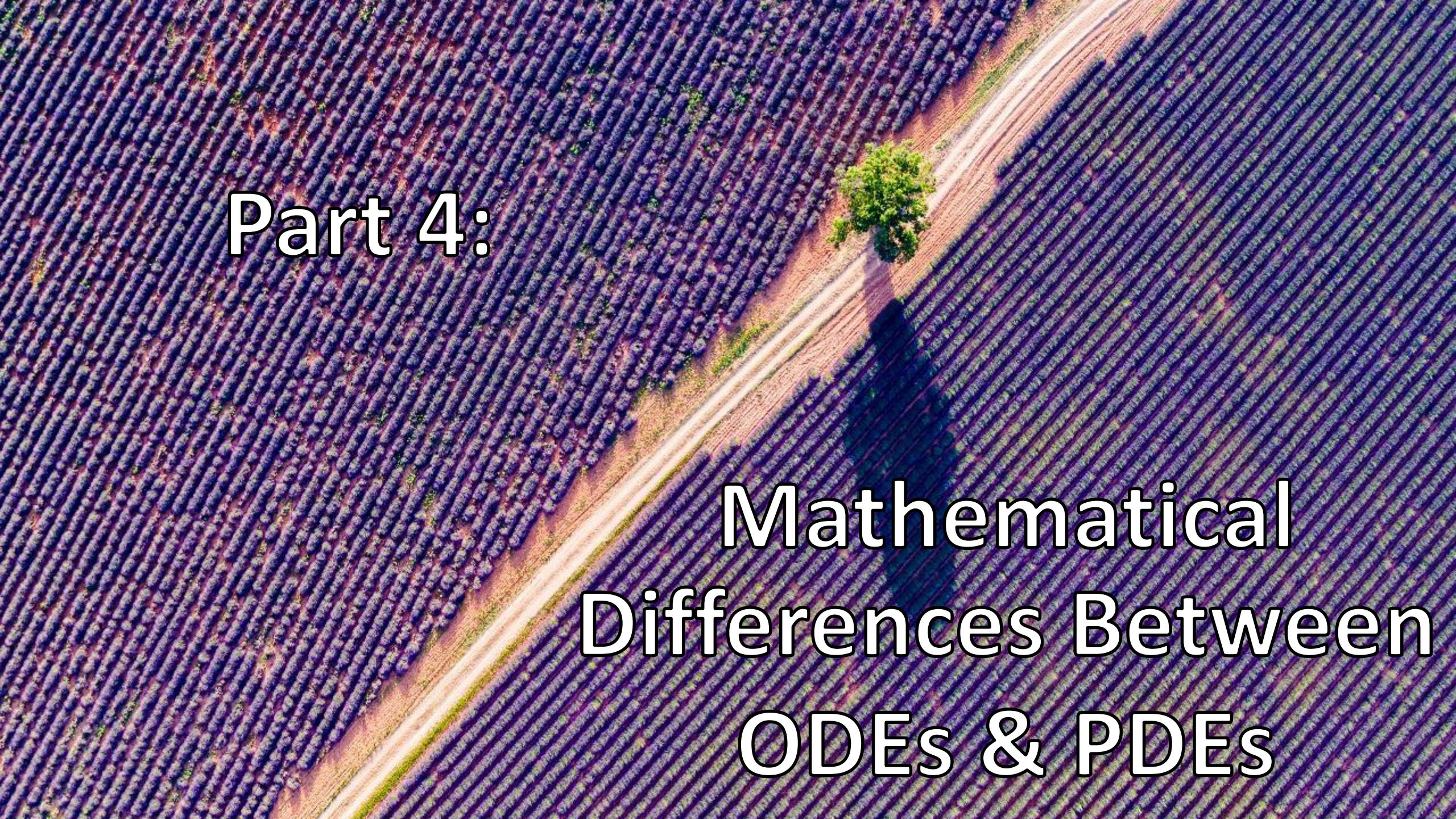
Crossing a Tipping Point:
→ Always full reorganization

Early Warning Signals
signal for WHEN



Crossing a bifurcation:
Now also possible:
→ Spatial reorganization (Turing patterns)
→ Fragmented tipping (coexistence states)

Early Warning Signals
need to signal WHEN & WHAT

An aerial photograph showing a vast, sprawling lavender field. The field is organized into numerous long, narrow, dark purple rows that stretch across the frame. A single, small, bright green tree stands prominently in the center of a dirt road that cuts through the fields. The overall pattern is highly geometric and repetitive.

Part 4:

Mathematical
Differences Between
ODEs & PDEs

Differences between ODEs and PDEs

	<u>ODE</u>	<u>PDE</u>
Stationary States	$0 = f(y^*; \mu)$	$0 = y_{xx}^* + f(y^*; \mu)$
Linear Stability	$\lambda \bar{y} = f_y(y^*; \mu) \bar{y}$	$\lambda \bar{y} = \bar{y}_{xx} + f_y(y^*(x); \mu) \bar{y}$

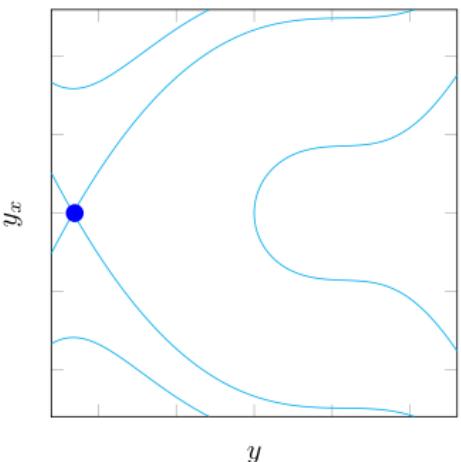
Stationary States

$$y_t = y_{xx} + f(y; \mu)$$

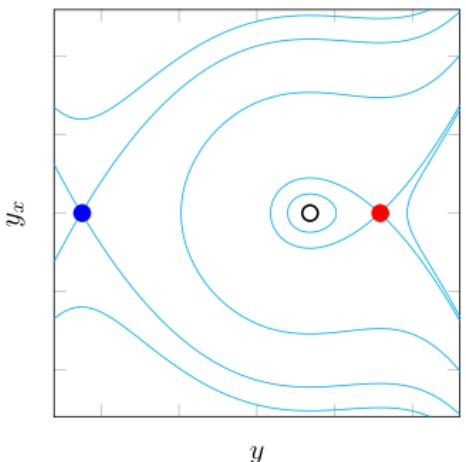
Stationary states

$$0 = y_{xx} + f(y; \mu)$$

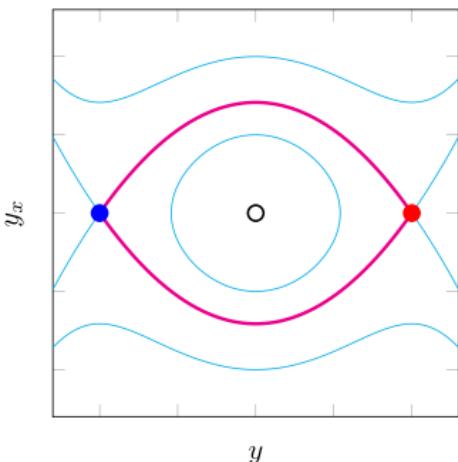
$$\begin{cases} y_x = p \\ p_x = -f(y; \mu) \end{cases}$$



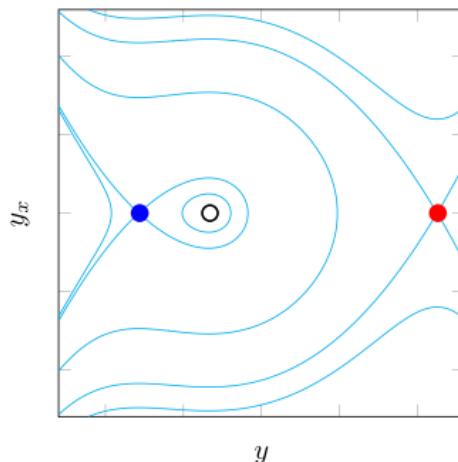
(a) $\mu < \mu_B$



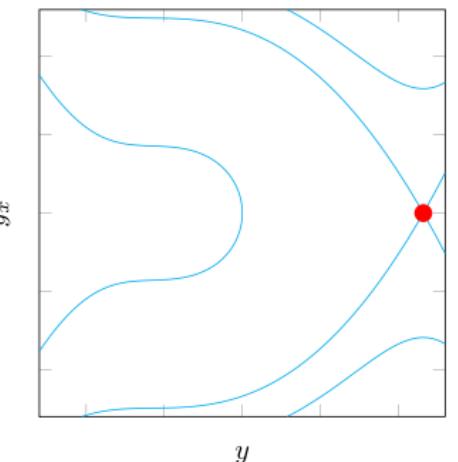
(b) $\mu_B < \mu < \mu_M$



(c) $\mu = \mu_M$



(d) $\mu_M < \mu < \mu_A$

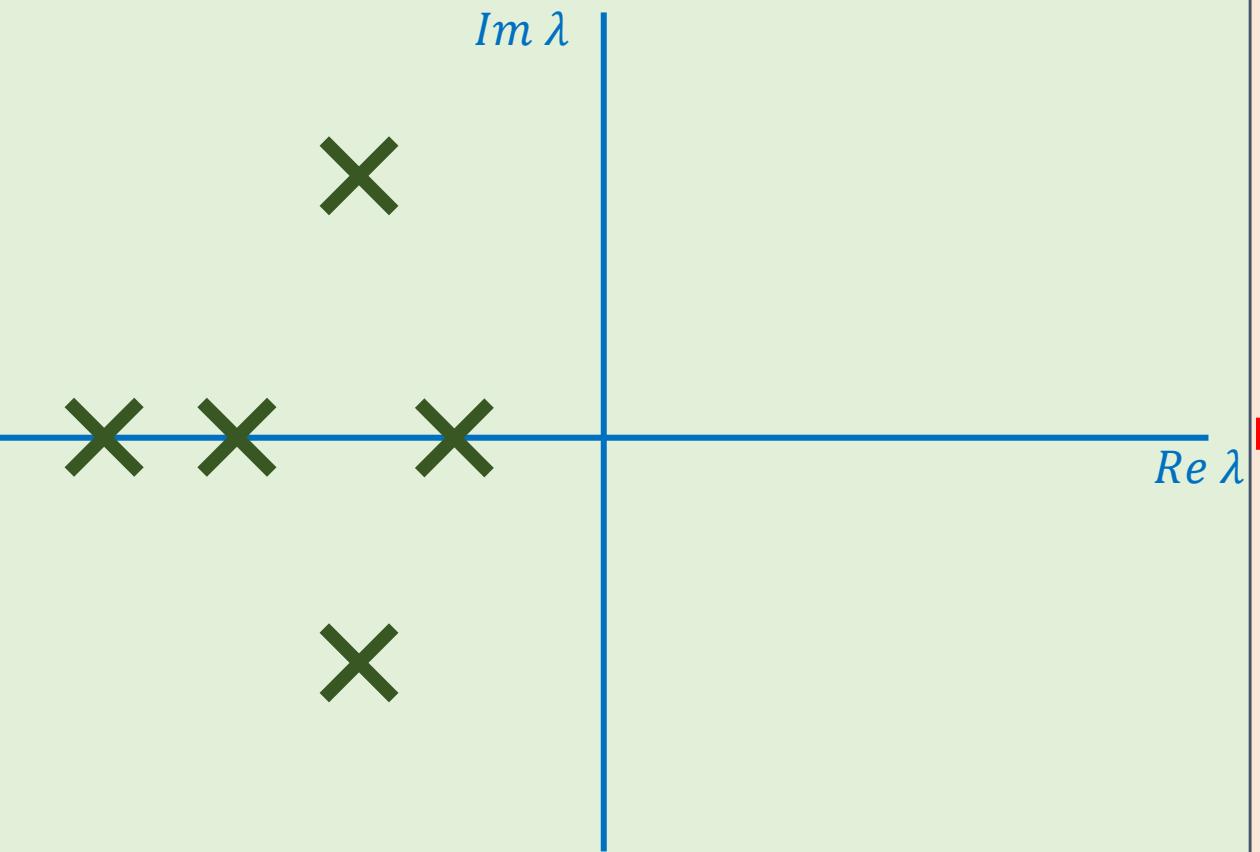


(e) $\mu > \mu_A$

Stability of Stationary States

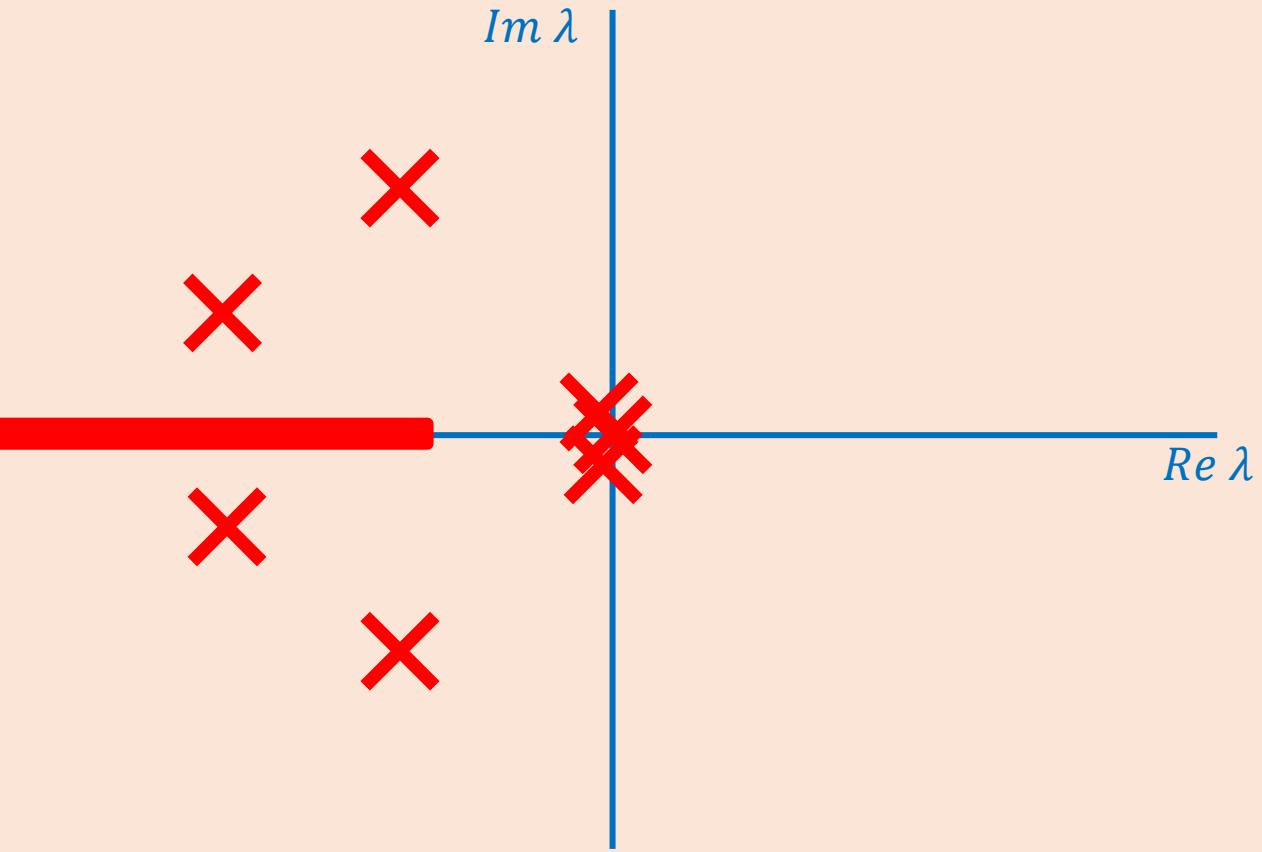
ODE

$$\lambda \bar{y} = f_y(y^*; \mu) \bar{y}$$



PDE

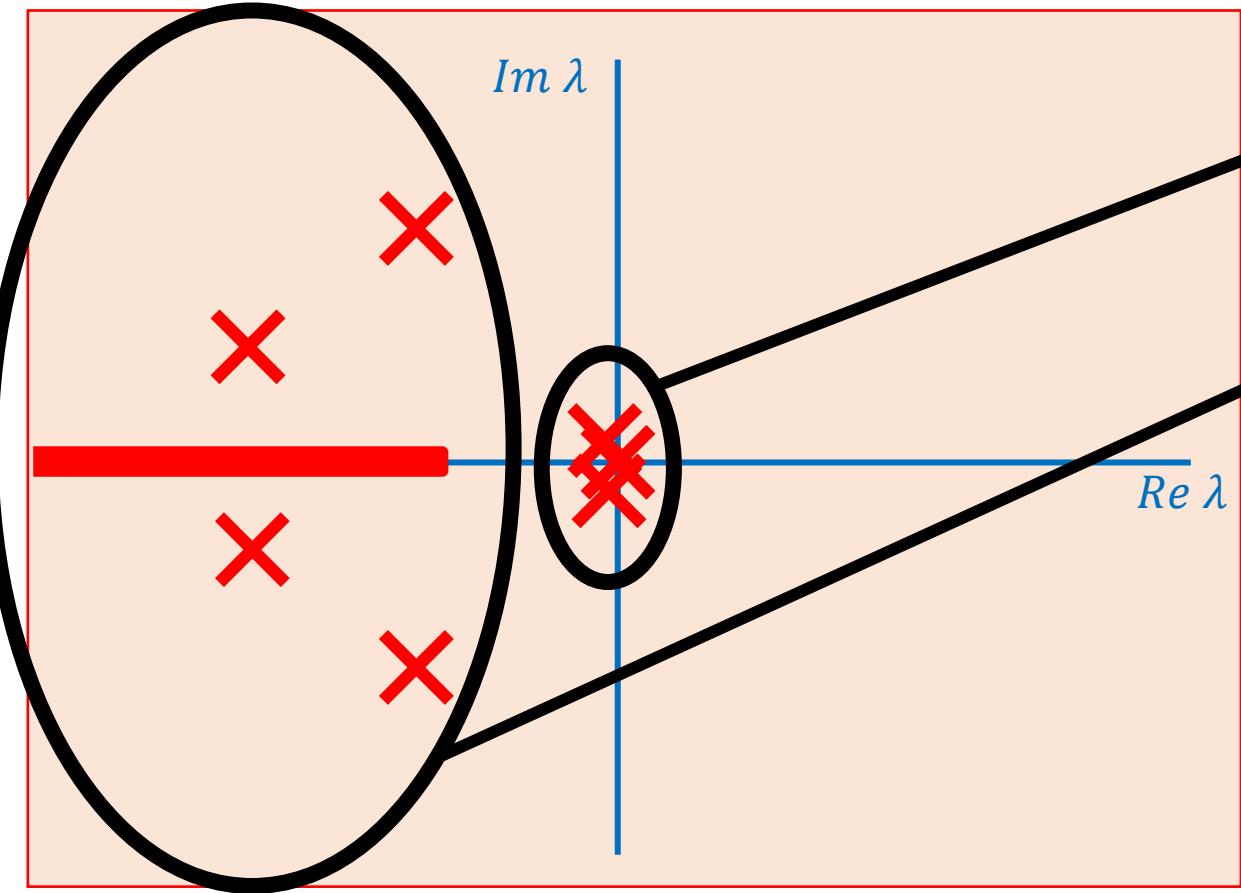
$$\lambda \bar{y} = \bar{y}_{xx} + f_y(y^*(x); \mu) \bar{y}$$



A massive, jagged glacier wall with deep blue ice at its base, set against a dark, rocky mountain background and a calm, light blue lake.

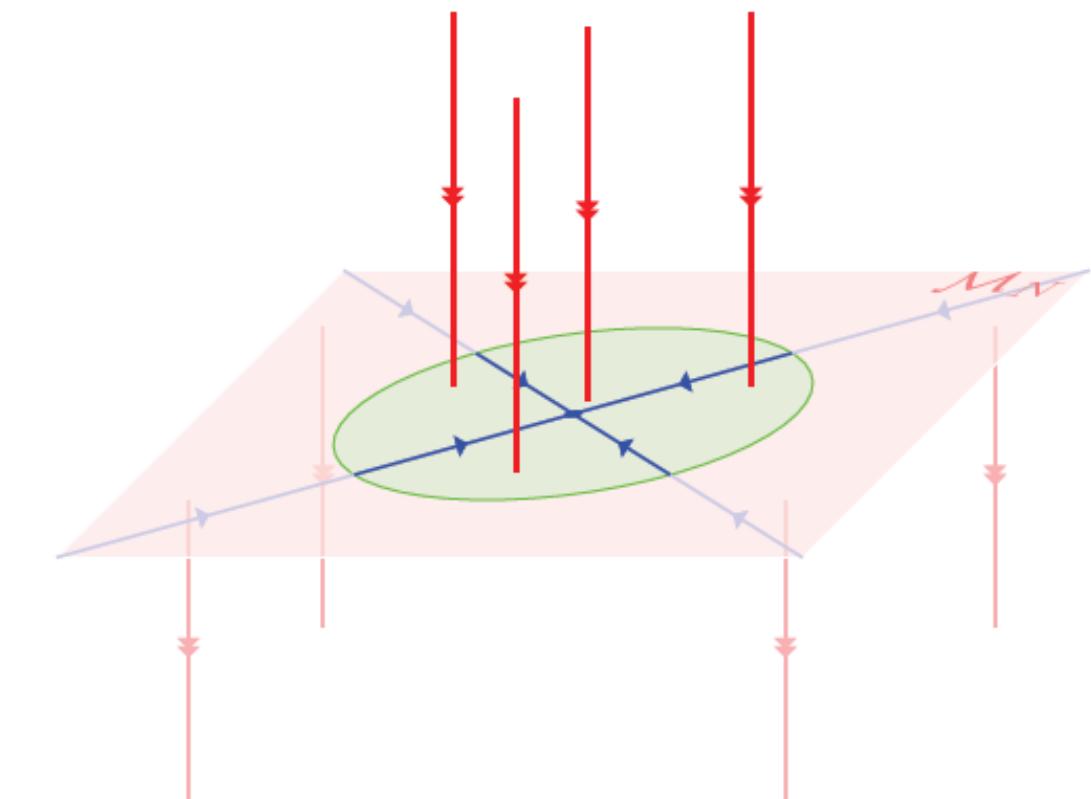
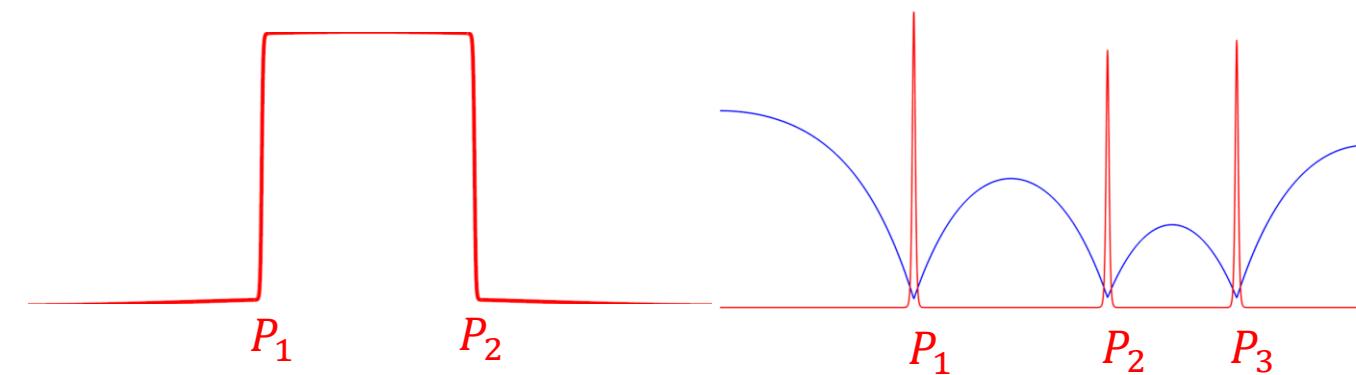
Part 5: Dynamics & Bifurcations of Patterned States

Dynamics of Patterned States



1. SLOW Pattern Adaptation

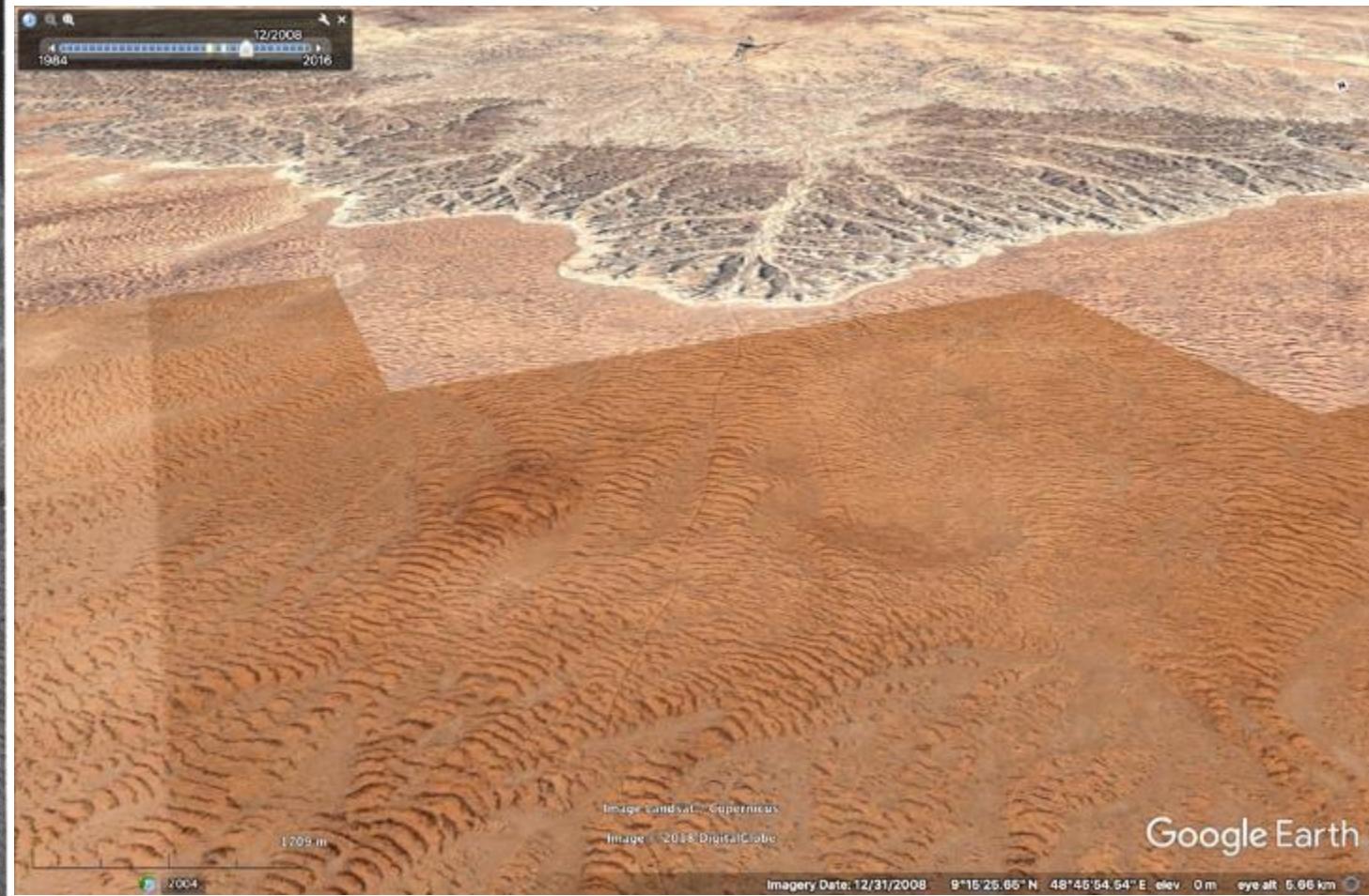
2. FAST Pattern Degradation



1. SLOW pattern adaptation

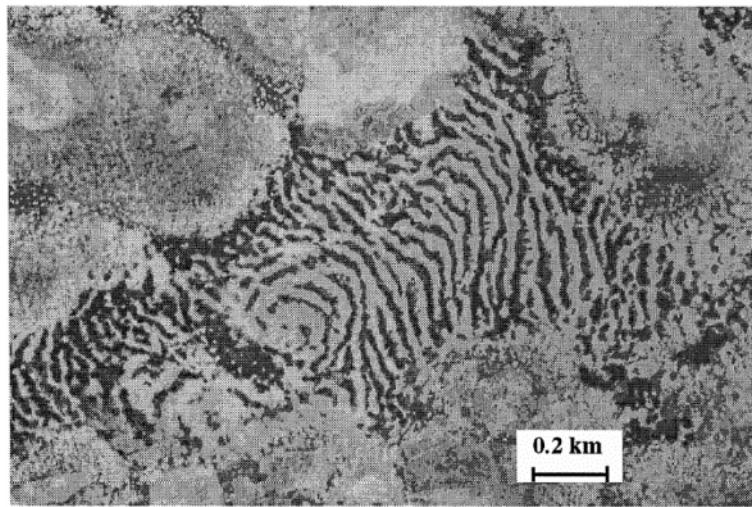


Somaliland, 1948 [Macfadyen, 1950]



Somaliland, 2008

2. FAST Pattern Degradation



Niger, 1950 [Valentin, 1999]



Niger, 2008



Niger, 2010



Niger, 2011

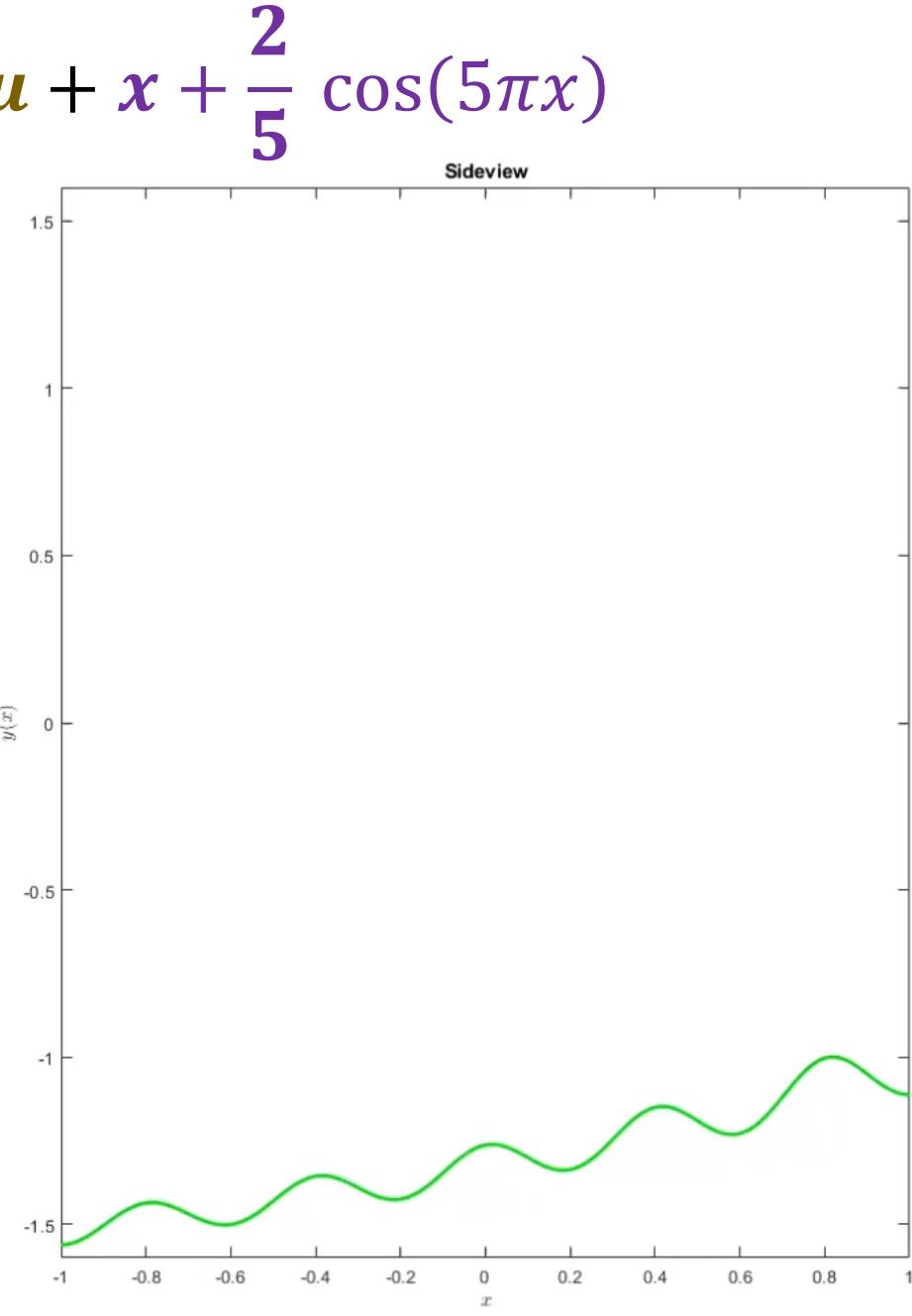
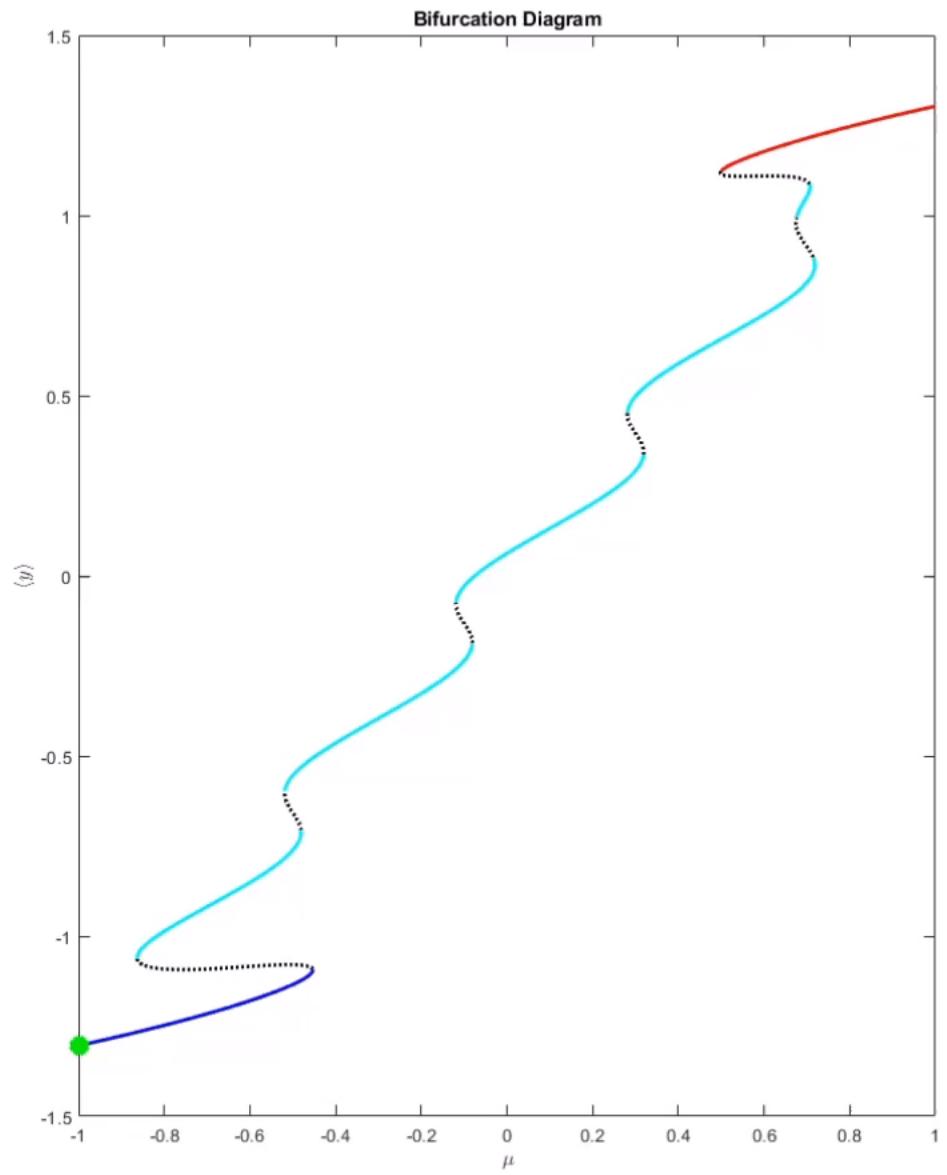


Niger, 2014

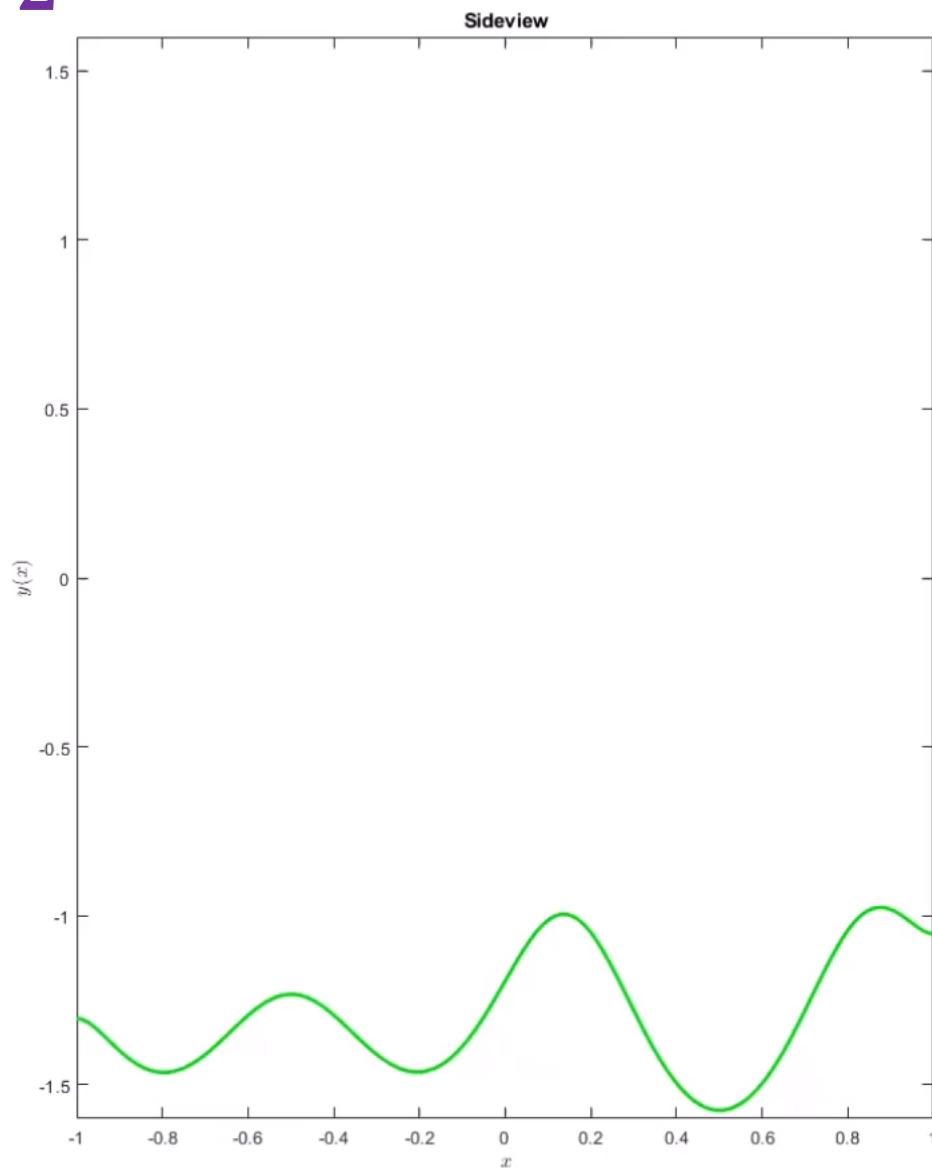
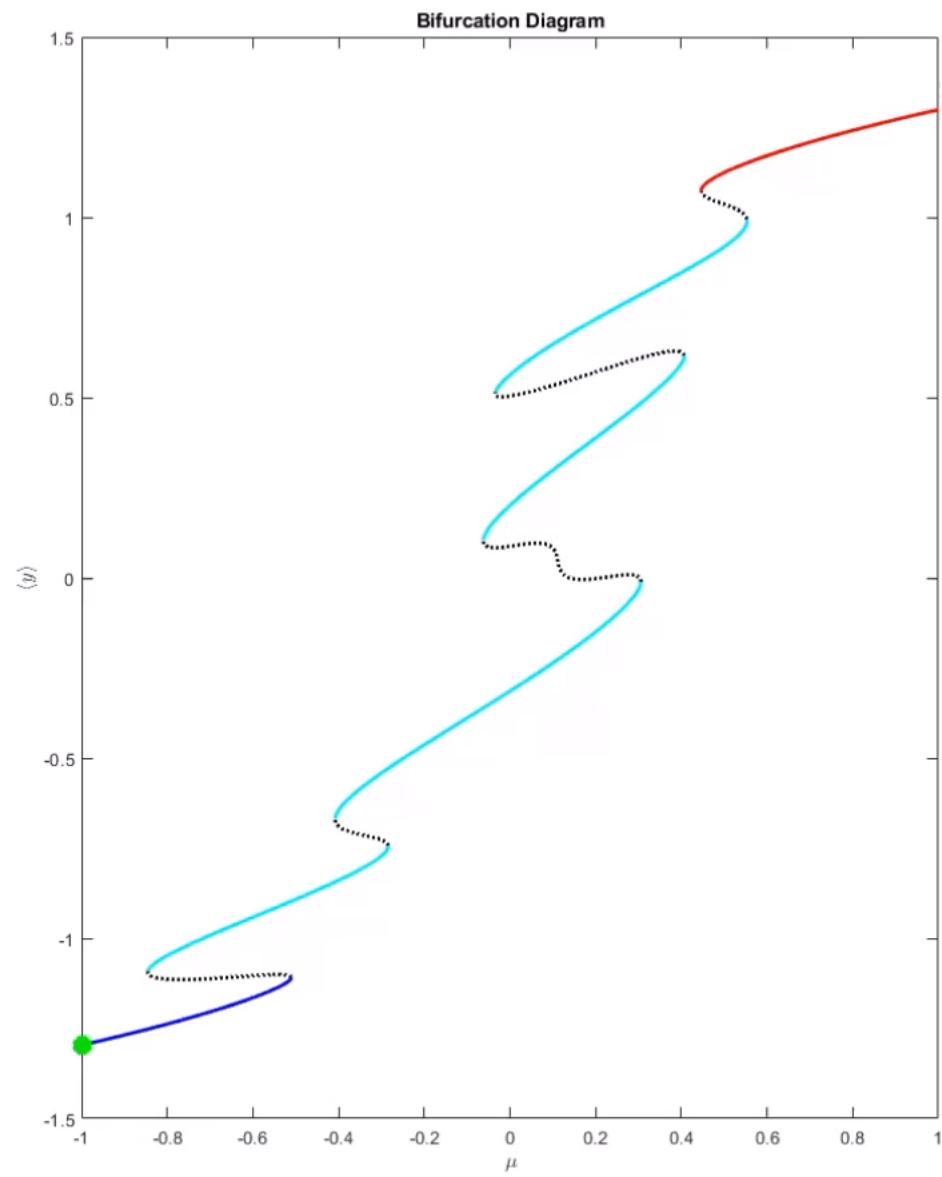


Niger, 2016

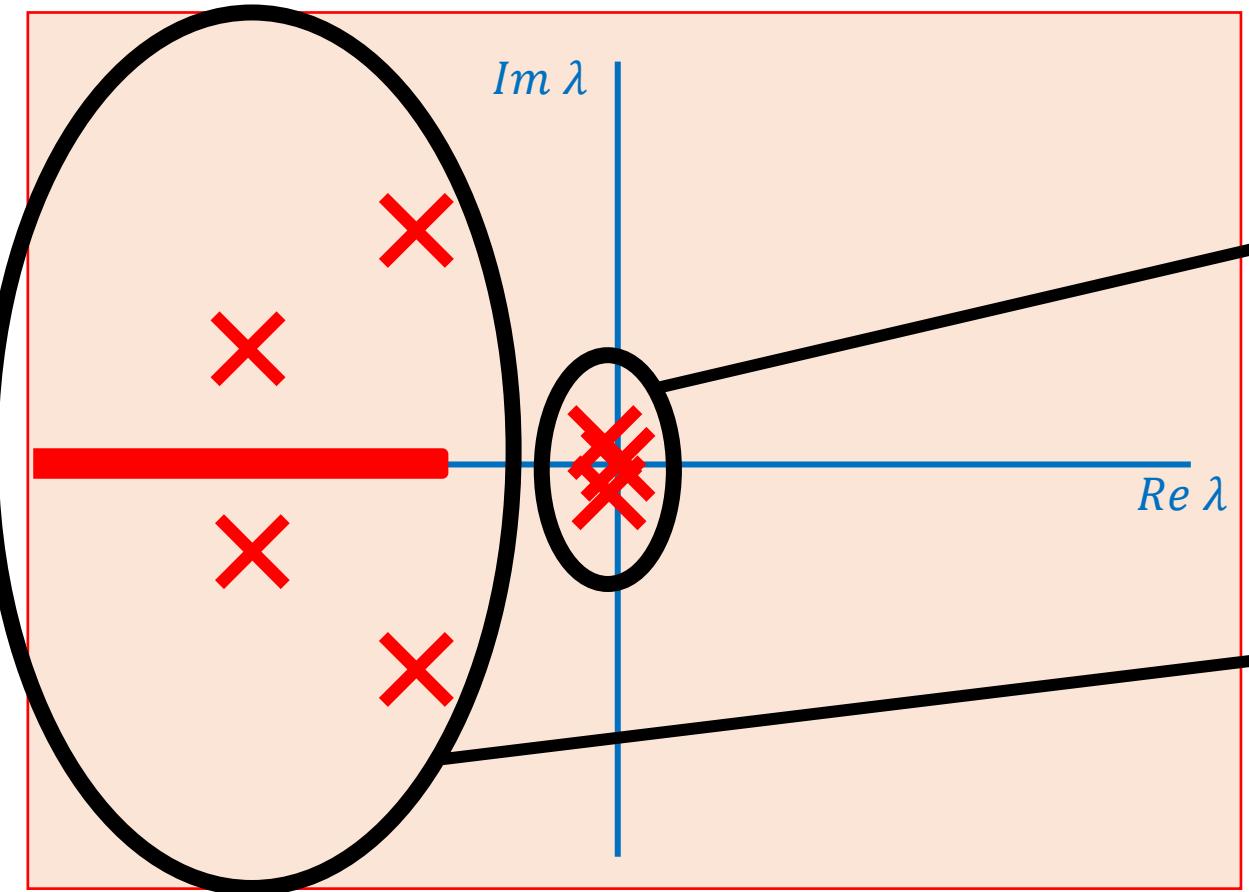
$$y_t = D y_{xx} + y(1 - y^2) + \mu + x + \frac{2}{5} \cos(5\pi x)$$



$$y_t = D y_{xx} + y(1 - y^2) + \mu + \frac{1}{2} \cos(2\pi x) + \sin(3\pi x)$$



Bifurcations



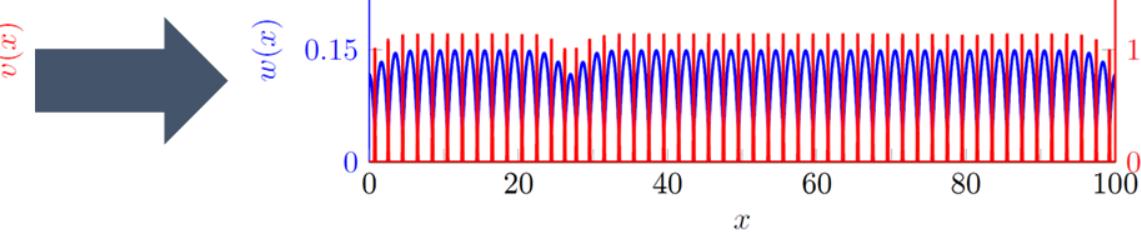
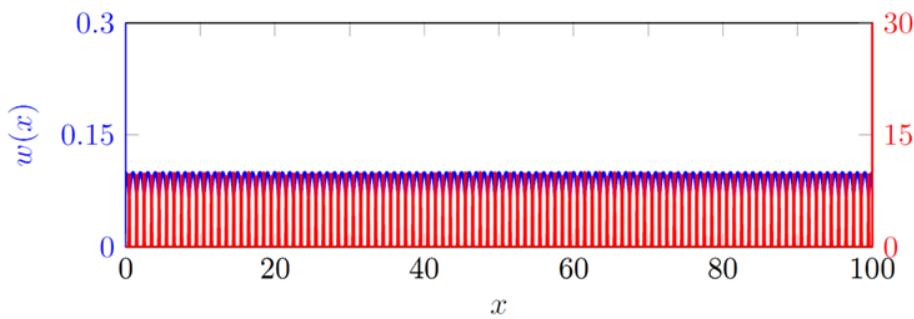
What happens at bifurcation?

1. SLOW Pattern Adaptation

At bifurcation:
→ Location of structure changes

2. FAST Pattern Degradation

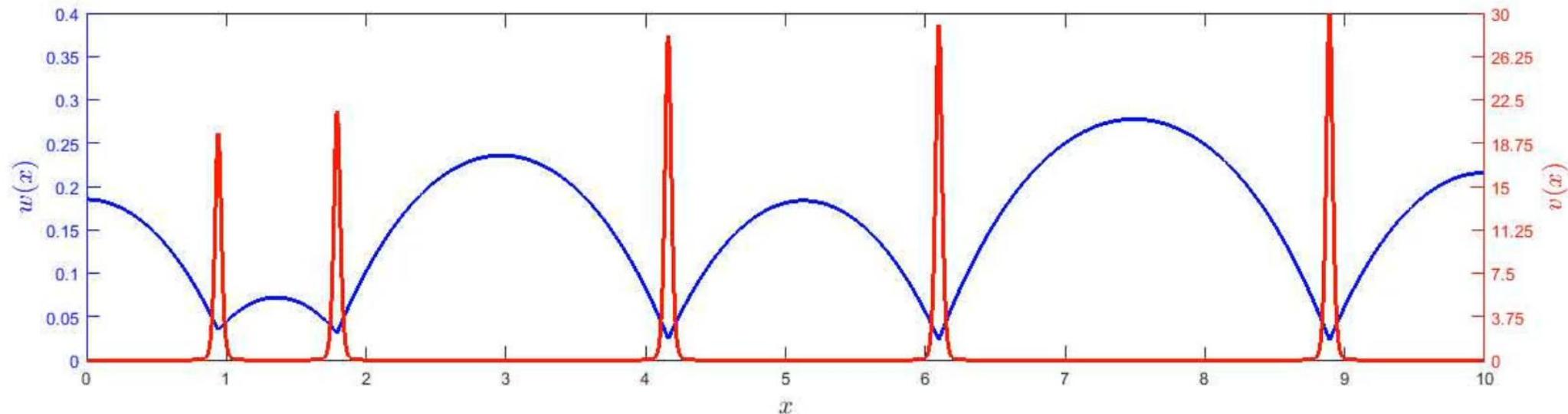
At bifurcation:
→ Structures created or destroyed



Vegetation patches under climate change

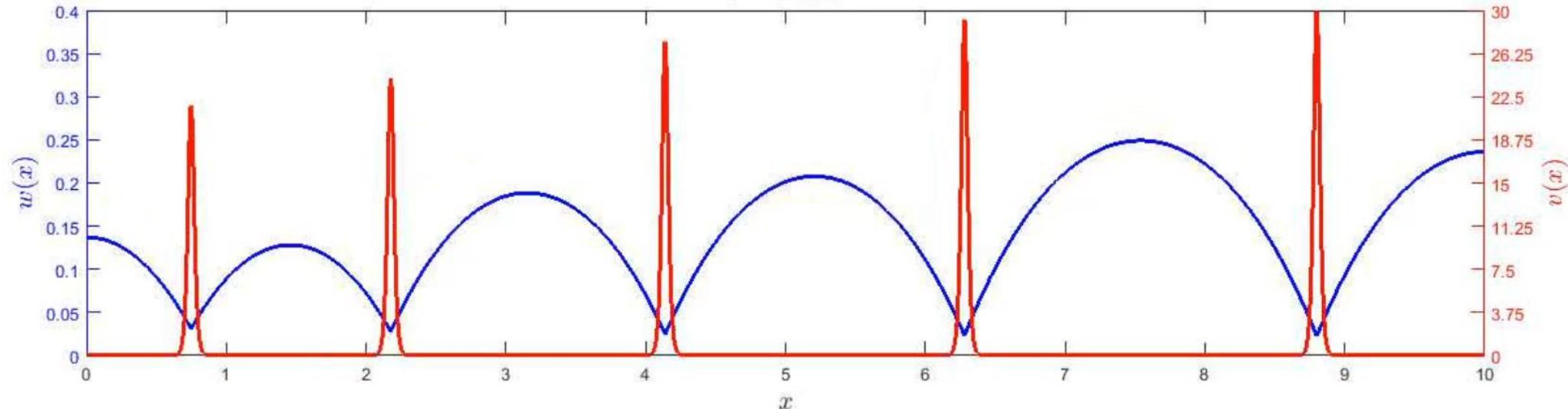
Rate of climate change

FAST



$$a = 0.4995$$

SLOW

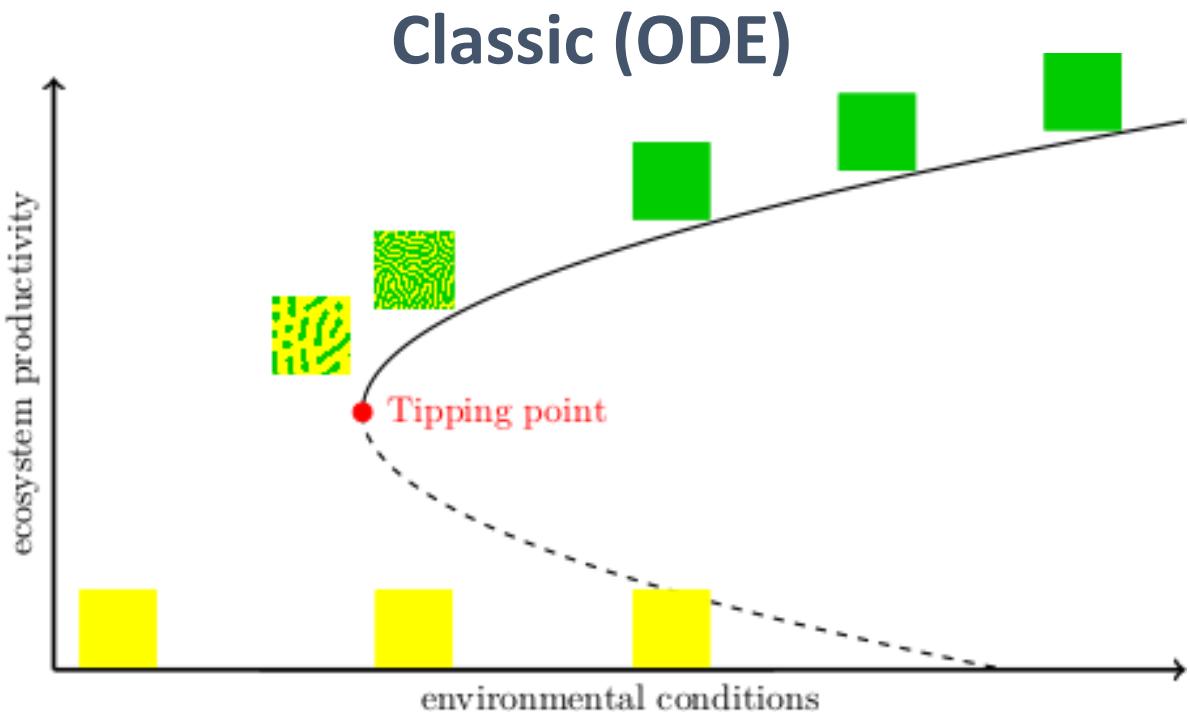


The background image shows a steep hillside with a distinct boundary between two types of vegetation. On the left side, there is a sparse, brownish-green forest where many trees appear dead or severely damaged. On the right side, the hillside is covered in a dense, healthy green forest of coniferous trees. This visual metaphor represents a 'tipping point' or 'phase transition' in a system.

Summary

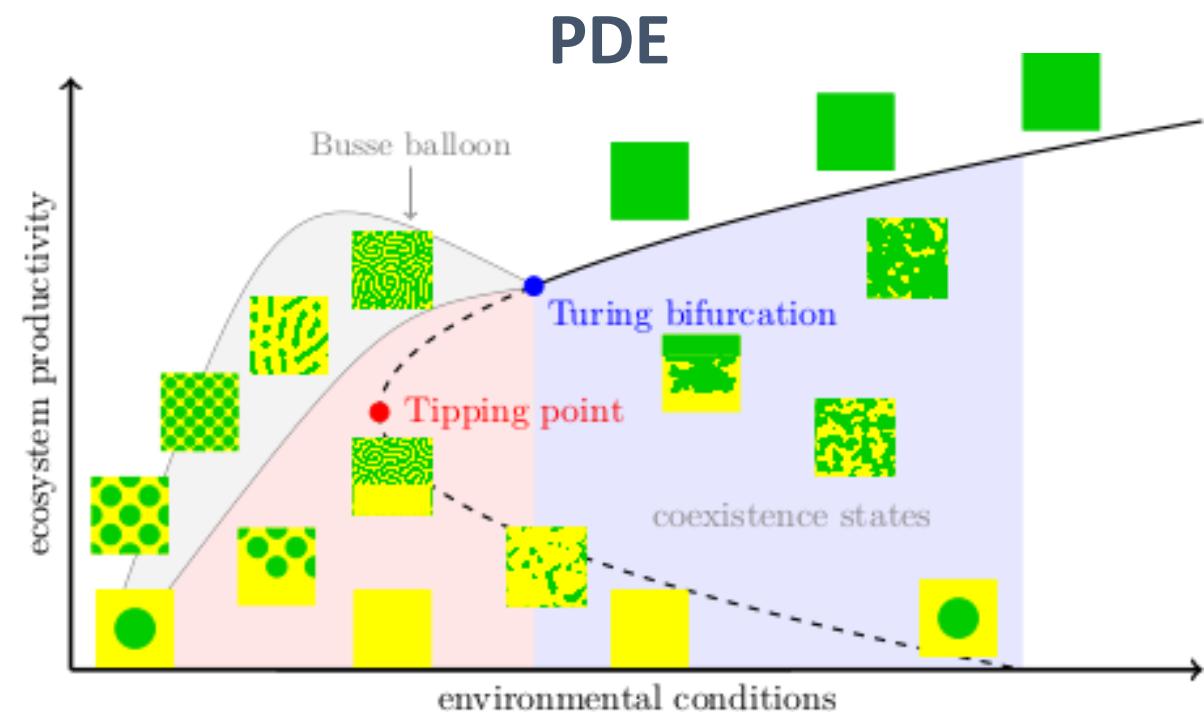
Tipping in Spatially Extended Systems

What if the system tips?



Crossing a Tipping Point:
→ Always full reorganization

Early Warning Signals
signal for WHEN



Crossing a bifurcation:
Now also possible:
→ Spatial reorganization (Turing patterns)
→ Fragmented tipping (coexistence states)

Early Warning Signals
need to signal WHEN & WHAT

Do systems always behave like this?

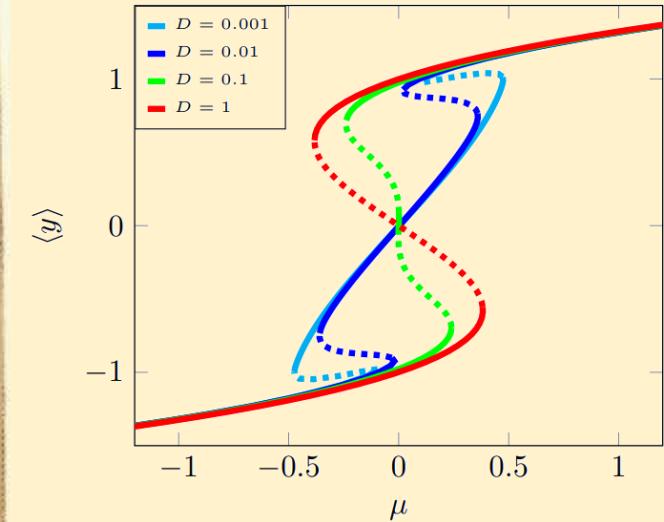
(a.k.a. the small print)

No.

Well-mixed systems



Spatially confined systems



→ Such systems (again) behave like ODEs ←

But even in other systems terms & conditions apply:
System-specific knowledge is required!

Spatial Patterns:

- ❖ Turing Patterns

- ❖ Coexistence States

Tipping can be more subtle:

- ❖ Spatial reorganization

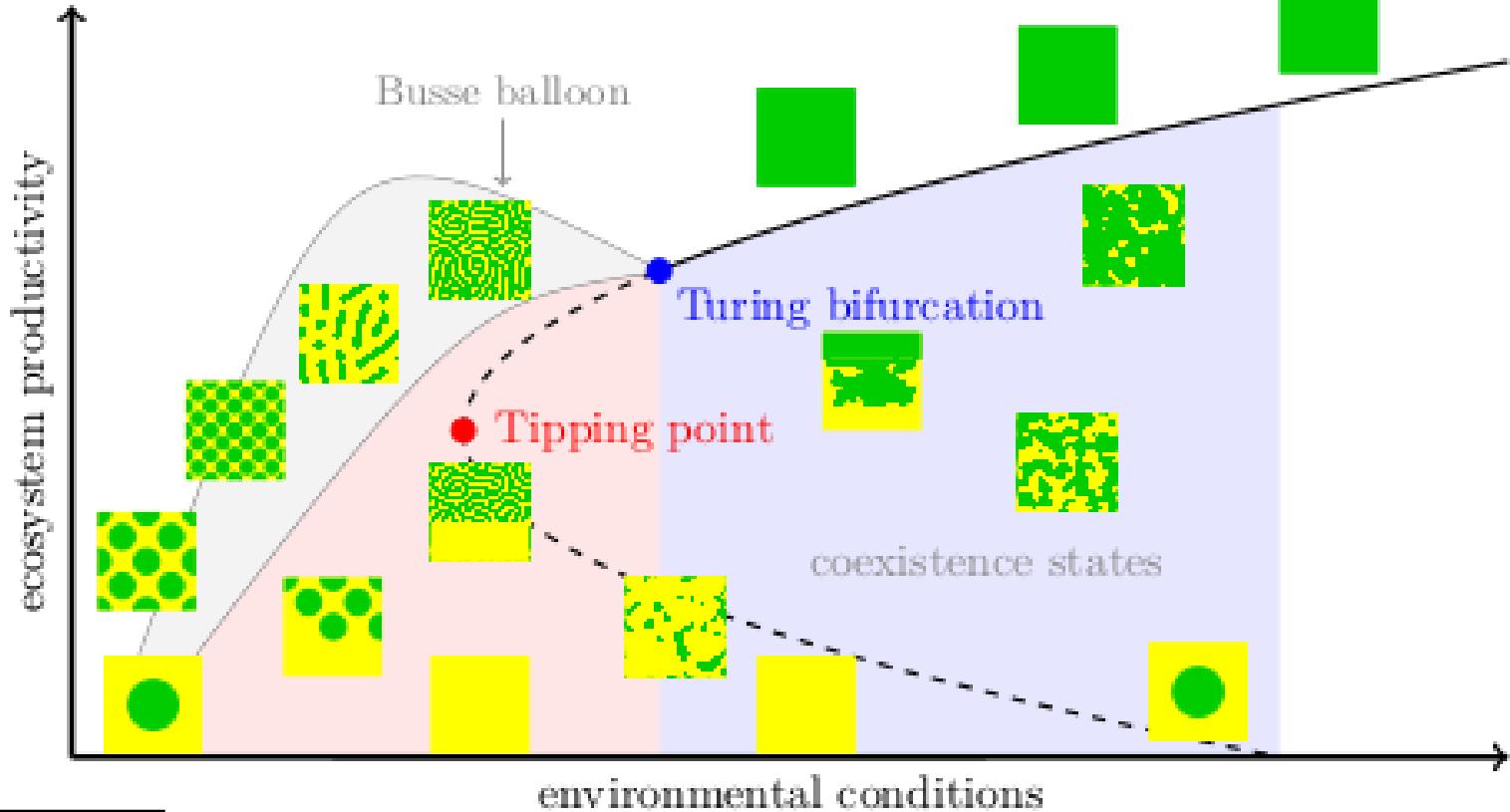
- ❖ Fragmented Tipping

Dynamics of Patterns is:

- ❖ Slow Pattern Adaptation

- ❖ Fast Pattern Degradation

Summary



THANKS TO:

Swarnendu Banerjee

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Max Rietkerk

Mara Baudena

Vincent Deblauwe

Maarten Eppinga

Johan van de Koppel

Eric Siero

Alexandre Bouvet

Arjen Doelman

Anna von der Heydt

Stéphane Mermoz

Koen Siteur

Rietkerk, M., Bastiaansen, R., Banerjee, S., van de Koppel, J., Baudena, M., & Doelman, A. (2021). Evasion of tipping in complex systems through spatial pattern formation. *Science*, 374(6564), eabj0359.



Bastiaansen, R., Dijkstra, H. A., & von der Heydt, A. S. (2021). Fragmented Tipping in a spatially heterogeneous world. *Environmental Research Letters*, 17, 045006



