SUPPLEMENTARY MATERIAL FOR "PATTERN INTERFACES CLOSE TO A TURING INSTABILITY IN A TWO-DIMENSIONAL BÉNARD–MARANGONI MODEL"

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1. Coefficients of the reduced equations

We now give the explicit expressions for the coefficients of the formal amplitude equations for a pattern-forming systems close to a (conserved) Turing instability. We consider both the cases of a square Fourier lattice as well as a hexagonal Fourier lattice. Recall that the amplitude equations are obtained by inserting the ansatz

$$\begin{pmatrix} h \\ \theta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \varepsilon \sum_{j=1}^{N} A_j(\varepsilon \mathbf{x}) e^{i\mathbf{k}_j\cdot\mathbf{x}} \boldsymbol{\varphi}_+(\mathbf{k}_j) + c.c. + \varepsilon^2 A_0 \boldsymbol{\varphi}_+(0) + \varepsilon^2 \Psi_0 \boldsymbol{\varphi}_-(0) + \sum_{\gamma\in\Gamma_h} \boldsymbol{\Psi}_\gamma e^{i\gamma\cdot\mathbf{x}}$$

into the thin-film system (1.1). We recall that ε is given by $M - M^* = \varepsilon^2 M_0$ for some $M_0 \in \mathbb{R}$. Additionally, $\boldsymbol{\varphi}_\pm(\mathbf{k})$ for $\mathbf{k} = \mathbf{k}_j, 0$ denotes the eigenvectors to the critical zero eigenvalues. For the calculations, we normalise these such that $\boldsymbol{\varphi}_+(\mathbf{k}_j) = (1,\alpha)^T$ and $\boldsymbol{\varphi}_-(\mathbf{k}_j) = (1,\tilde{\alpha})^T$, where $\alpha, \tilde{\alpha} \in \mathbb{R}$ are given by

$$\begin{aligned}
\alpha &= \frac{12g^2 - 12\beta g + \sqrt{2} g\sqrt{\beta g(-\beta+g+72)} - 12\left((\beta-72)\beta + 6\sqrt{2}\sqrt{\beta g(-\beta+g+72)}\right)}{12(\beta-72)(g-\beta)}, \\
\tilde{\alpha} &= \frac{2\sqrt{2}\sqrt{\beta g(-\beta+g+72)} - 3g(-\beta+g+8)}{8g(g-\beta)}
\end{aligned}$$

For the square lattice, the amplitude equations read as

$$\begin{aligned}
\partial_T A_1 &= -\frac{\lambda_+''(k_m^*)}{2(k_m^*)^2} \partial_X^2 A_1 + M_0\kappa A_1 + K_c A_0 A_1 + K_0 A_1 |A_1|^2 + K_1 A_1 |A_2|^2, \\
\partial_T A_2 &= -\frac{\lambda_+''(k_m^*)}{2(k_m^*)^2} \partial_Y^2 A_2 + M_0\kappa A_2 + K_c A_0 A_2 + K_0 A_2 |A_2|^2 + K_1 A_2 |A_1|^2, \\
\partial_T A_0 &= -\frac{1}{2}\lambda_+''(0)\Delta A_0 + \nabla\cdot\left(p_c(\mathbf{k}_1\mathbf{k}_1^T)\nabla|A_1|^2 + p_c(\mathbf{k}_2\mathbf{k}_2^T)\nabla|A_2|^2\right).
\end{aligned}$$

In the case of a hexagonal lattice, the amplitude equations are given by

$$
\begin{aligned}
\partial_T A_1 &= -\frac{\lambda_+''(k_m^*)}{2(k_m^*)^2}\partial_X^2 A_1 + M_0\kappa A_1 + K_c A_0 A_1 + \frac{N}{\varepsilon}\bar{A}_2\bar{A}_3 + K_0 A_1|A_1|^2 + K_2 A_1(|A_2|^2+|A_3|^2),\\
\partial_T A_2 &= -\frac{\lambda_+''(k_m^*)}{2(k_m^*)^2}\left(-\frac{1}{2}\partial_X + \frac{\sqrt{3}}{2}\partial_Y\right)^2 A_2 + M_0\kappa A_2 + K_c A_0 A_2 + \frac{N}{\varepsilon}\bar{A}_1\bar{A}_3 + K_0 A_2|A_2|^2\\
&\quad + K_2 A_2(|A_1|^2+|A_3|^2),\\
\partial_T A_3 &= -\frac{\lambda_+''(k_m^*)}{2(k_m^*)^2}\left(\frac{1}{2}\partial_X + \frac{\sqrt{3}}{2}\partial_Y\right)^2 A_3 + M_0\kappa A_3 + K_c A_0 A_3 + \frac{N}{\varepsilon}\bar{A}_1\bar{A}_2 + K_0 A_3|A_3|^2\\
&\quad + K_2 A_3(|A_1|^2+|A_2|^2),\\
\partial_T A_0 &= -\frac{1}{2}\lambda_+''(0)\Delta A_0 + \nabla\cdot\left(p_c(\mathbf{k}_1\mathbf{k}_1^T)\nabla|A_1|^2 + p_c(\mathbf{k}_2\mathbf{k}_2^T)\nabla|A_2|^2 + p_c(\mathbf{k}_3\mathbf{k}_3^T)\nabla|A_3|^2\right).
\end{aligned}
$$

Recall that the coefficients of the reduced equations on the center manifold both for the construction of planar patterns as well as for the construction of moving pattern interfaces coincide with the coefficients of the formal amplitude equations.

The calculations leading to the expressions for the coefficients (as well as for α and $\tilde{\alpha}$ above) have been performed using Wolfram Mathematica 14.1 [Inc] and the corresponding notebook can be found at [HJ24].

1.1. **Quadratic coefficients.**

The quadratic coefficient N, which is generated through the resonance on the hexagonal lattice is given by

$$
\begin{aligned}
N &= \mathcal{P}_+(\mathbf{k}_{-3})\left[\mathcal{N}_2(\boldsymbol{\varphi}_+(\mathbf{k}_1)e^{i\mathbf{k}_1\cdot\mathbf{x}}, \boldsymbol{\varphi}_+(\mathbf{k}_2)e^{i\mathbf{k}_2\cdot\mathbf{x}})\right]\\
&= \frac{g\left(-3\beta^2\left(g\left(-4g^2+15\sqrt{2}\sqrt{\beta g(-\beta+g+72)}+62g+15696\right)-24\left(11\sqrt{2}\sqrt{\beta g(-\beta+g+72)}+864\right)\right)-48\sqrt{2}(g-18)(g(g+27)+216)\sqrt{\beta g(-\beta+g+72)}\right)}{(\beta-72)\left(9\beta^2(25g+8)+3\beta(g(421g-5832)-3456)+8(g(g+27)+216)^2\right)}\\
&\quad + \frac{g\left(18\beta^3(37g-24)-2\beta\left(1296\left(23\sqrt{2}\sqrt{\beta g(-\beta+g+72)}+864\right)+g\left(g\left(15\sqrt{2}\sqrt{\beta g(-\beta+g+72)}+4g(g+162)+43416\right)+54\left(49\sqrt{2}\sqrt{\beta g(-\beta+g+72)}+576\right)\right)\right)\right)}{(\beta-72)\left(9\beta^2(25g+8)+3\beta(g(421g-5832)-3456)+8(g(g+27)+216)^2\right)}.
\end{aligned}
$$

The second quadratic coefficient K_c, which is facilitates the coupling of the conservation law into the Ginzburg-Landau equation, reads as

$$\begin{aligned}
K_c &= \mathcal{P}_+(\mathbf{k}_1)\left[\mathcal{N}_2(\boldsymbol{\varphi}_+(0), \boldsymbol{\varphi}_+(k_1)e^{i\mathbf{k}_1\cdot\mathbf{x}})\right] \\
&= -\frac{g\left(8\beta g^4 + 12g^3\left(-\beta^2 + 108\beta + 4\sqrt{2}\sqrt{\beta g(-\beta+g+72)}\right) + 6g^2\left(40\beta^2 + \beta\left(7\sqrt{2}\sqrt{\beta g(-\beta+g+72)} + 13824\right) - 72\sqrt{2}\sqrt{\beta g(-\beta+g+72)}\right)\right)}{(\beta-72)\left(72(\beta-72)^2 + 8g^4 + 432g^3 + 3(421\beta+3096)g^2 + 9\left(25\beta^2 - 1944\beta + 10368\right)g\right)} \\
&\quad - \frac{g\left(-216\sqrt{2}\left(\beta^2 - 96\beta + 1728\right)\sqrt{\beta g(-\beta+g+72)} + 9g\left(-64\beta^3 + 5\beta^2\left(\sqrt{2}\sqrt{\beta g(-\beta+g+72)} + 1152\right) + 48\beta\left(13\sqrt{2}\sqrt{\beta g(-\beta+g+72)} - 1728\right)\right)\right)}{(\beta-72)\left(72(\beta-72)^2 + 8g^4 + 432g^3 + 3(421\beta+3096)g^2 + 9\left(25\beta^2 - 1944\beta + 10368\right)g\right)} \\
&\quad - \frac{9g^2\left(-4032\sqrt{2}\sqrt{\beta g(-\beta+g+72)}\right)}{(\beta-72)\left(72(\beta-72)^2 + 8g^4 + 432g^3 + 3(421\beta+3096)g^2 + 9\left(25\beta^2 - 1944\beta + 10368\right)g\right)}
\end{aligned}$$

1.2. **Coefficients for polynomial in quadratic nonlinearity of conservation law.** Recall that the the leading order quadratic nonlinearity of the conservation law is a potentially non-elliptic, second-order operator of the form $\nabla\cdot(p_c(\mathbf{k}_j\mathbf{k}_j^T)\nabla(|A_j|^2))$. Here $p_c(\mathbf{k}_j\mathbf{k}_j^T) = \kappa_0 + \kappa_1\mathbf{k}_j\mathbf{k}_j^T$ is a polynomial of degree 1 and its coefficients are given by

$$\begin{aligned}
\kappa_0 &= (M-g)(\boldsymbol{\varphi}_+(\mathbf{k}_j))_1^2 + g(\boldsymbol{\varphi}_+(\mathbf{k}_j))_1(\boldsymbol{\varphi}_+(\mathbf{k}_j))_2 - |\mathbf{k}_j|^2(\boldsymbol{\varphi}_+(\mathbf{k}_j))_1^2 \\
&= \frac{g^2}{\beta-g} + \frac{g\sqrt{\beta g(-\beta+g+72)}}{\sqrt{2}(6g-6\beta)} + \frac{g\sqrt{\beta g(-\beta+g+72)}}{6\sqrt{2}(\beta-72)} + \frac{(g+72)g}{\beta-72} - \frac{48\beta(g-72)}{(g+72)^2} + \frac{6\sqrt{2}\sqrt{\beta g(-\beta+g+72)}}{\beta-72} + \frac{576\sqrt{2}\sqrt{\beta g(-\beta+g+72)}}{(g+72)^2} + \frac{(g+120)g}{g+72}, \\
\kappa_1 &= -1
\end{aligned}$$

1.3. **Cubic coefficient: self-interaction.** The self-interaction term K_0 is identical for both square lattice and hexagonal lattice and is given by

$$K_0 = 2\Gamma_0 + 2\Gamma_2 + 3\Lambda_1,$$

where

$$
\begin{aligned}
\Gamma_0 &= \mathcal{P}_+(\mathbf{k}_1)\left[\mathcal{N}_2(\varphi_-(0),\varphi_+(k_m^*)e^{i\mathbf{k}_1\cdot\mathbf{x}};M^*)v_0\right]=0,\\
\Gamma_2 &= \mathcal{P}_+(\mathbf{k}_1)\left[\mathcal{N}_2(\boldsymbol{v}_{2\mathbf{k}_1}e^{2i\mathbf{k}_1\cdot\mathbf{x}},\varphi_+(k_m^*)e^{-i\mathbf{k}_1\cdot\mathbf{x}};M^*)\right]\\
&= \frac{k^6\left(-\left(M^2\left(4g^2+(26g+441)k^2+279g+22k^4+1458\right)\right)-192\left(g+k^2\right)\left((5g+54)k^2+g(g+9)+4k^4\right)-M\left(80(g+9)k^2+g(13g+396)+67k^4\right)\right)}{24\left(3\beta+k^2(g-M+3)+k^4\right)\left(k^2\left(g(M-48)+4k^2(M-48)+72M\right)-12\beta\left(g+4k^2\right)\right)}\\
&\quad+\frac{k^6\left(4M\left(3(3g+116)k^4+3(g+74)(2g+9)k^2+g(g+9)(g+114)+4k^6\right)\right)-72\beta^2k^2\left(g+k^2\right)\left(5g+23k^2\right)-6\beta k^4\left(8\left(g+k^2\right)\left((5g+213)k^2+g(g+42)+4k^4\right)\right)}{24\left(3\beta+k^2(g-M+3)+k^4\right)\left(k^2\left(g(M-48)+4k^2(M-48)+72M\right)-12\beta\left(g+4k^2\right)\right)},\\
\Lambda_1 &= \mathcal{P}_+(\mathbf{k}_1)\left[\mathcal{N}_3(\varphi_+(k_m^*)e^{i\mathbf{k}_1\cdot\mathbf{x}},\varphi_+(k_m^*)e^{i\mathbf{k}_1\cdot\mathbf{x}},\varphi_+(k_m^*)e^{-\mathbf{k}_1\cdot\mathbf{x}};M^*)\right]\\
&= -\frac{k^2\left(g+k^2\right)\left(48\beta+k^2\left(8\left(g+k^2+6\right)-7M\right)\right)}{72\left(3\beta+k^2(g-M+3)+k^4\right)}.
\end{aligned}
$$

1.4. **Cubic coefficient: cross-interaction.** In contrast to the self-interaction coefficients K_0, the cross-interaction terms are different for the square and hexagonal lattice. In the case of the square lattice, the cross-interaction coefficient K_1 is given by

$$
K_1 = 2\Gamma_{0,\mathrm{sq}}+2\Gamma_{3,+,\mathrm{sq}}+2\Gamma_{3,-,\mathrm{sq}}+6\Lambda_{2,\mathrm{sq}},
$$

where

$$
\begin{aligned}
\Gamma_{0,\mathrm{sq}} &= \mathcal{P}_+(\mathbf{k}_1)\left[\mathcal{N}_2(\varphi_-(0),\varphi_+(k_m^*)e^{i\mathbf{k}_1\cdot\mathbf{x}};M^*)v_0\right]=\Gamma_0=0,\\
\Gamma_{3,+,\mathrm{sq}} &= \mathcal{P}_+(\mathbf{k}_1)\left[\mathcal{N}_2(\boldsymbol{v}_{\mathbf{k}_1+\mathbf{k}_2}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}},\varphi_+(k_m^*)e^{-i\mathbf{k}_2\cdot\mathbf{x}};M^*)\right]\\
&= \frac{-27648\beta^2k^2\left(g+k^2\right)\left(g+2k^2\right)-24\beta k^4\left(16\left(g+k^2\right)\left(3(8g+167)k^2+8g(g+30)+16k^4\right)-3M\left((187g+1896)k^2+12g(5g+116)+127k^4\right)\right)}{21\left(3\beta+k^2(g-M+3)+k^4\right)\left(7k^2\left(4g(M-48)+7k^2(M-48)+288M\right)-192\beta\left(4g+7k^2\right)\right)}\\
&\quad+\frac{14k^6\left(-3M^2\left(13(g+24)k^2+4g(g+60)+9k^4+1728\right)-96\left(g+k^2\right)\left((13g+123)k^2+4g(g+15)+9k^4\right)\right)}{21\left(3\beta+k^2(g-M+3)+k^4\right)\left(7k^2\left(4g(M-48)+7k^2(M-48)+288M\right)-192\beta\left(4g+7k^2\right)\right)}\\
&\quad+\frac{14k^6\left(2M\left((26g+993)k^4+(g(19g+1581)+7992)k^2+4g(g(g+147)+1512)+11k^6\right)\right)}{21\left(3\beta+k^2(g-M+3)+k^4\right)\left(7k^2\left(4g(M-48)+7k^2(M-48)+288M\right)-192\beta\left(4g+7k^2\right)\right)},
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{3,-,\mathrm{sq}} &= \mathcal{P}_+(\mathbf{k}_1)\left[\mathcal{N}_2(\boldsymbol{\nu}_{\mathbf{k}_1-\mathbf{k}_2}e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}}, \varphi_+(k_m^*)e^{i\mathbf{k}_2\cdot\mathbf{x}}; M^*)\right] \\
&= \frac{-13824\beta^2k^2\left(g+k^2\right)\left(g+2k^2\right)-12\beta k^4\left(16\left(g+k^2\right)\left(3(8g+167)k^2+8g(g+30)+16k^4\right)-3M\left((187g+1756)k^2+4g(15g+334)+127k^4\right)\right)}{7\left(3\beta+k^2(g-M+3)+k^4\right)\left(7k^2\left(4g(M-48)+7k^2(M-48)+288M\right)-192\beta\left(4g+7k^2\right)\right)} \\
&\quad+\frac{7k^6\left(-3M^2\left((13g+292)k^2+4g(g+58)+9k^4+1440\right)-96\left(g+k^2\right)\left((13g+123)k^2+4g(g+15)+9k^4\right)\right)}{7\left(3\beta+k^2(g-M+3)+k^4\right)\left(7k^2\left(4g(M-48)+7k^2(M-48)+288M\right)-192\beta\left(4g+7k^2\right)\right)} \\
&\quad+\frac{7k^6\left(2M\left((26g+981)k^4+(g(19g+1569)+7380)k^2+4g(g(g+147)+1440)+11k^6\right)\right)}{7\left(3\beta+k^2(g-M+3)+k^4\right)\left(7k^2\left(4g(M-48)+7k^2(M-48)+288M\right)-192\beta\left(4g+7k^2\right)\right)}, \\
\Lambda_{2,\mathrm{sq}} &= \mathcal{P}_+(\mathbf{k}_1)\left[\mathcal{N}_3(\varphi_+(k_m^*)e^{i\mathbf{k}_1\cdot\mathbf{x}}, \varphi_+(k_m^*)e^{i\mathbf{k}_2\cdot\mathbf{x}}, \varphi_+(k_m^*)e^{-\mathbf{k}_2\cdot\mathbf{x}}; M^*)\right] \\
&= -\frac{k^2\left(g+k^2\right)\left(48\beta+k^2\left(8\left(g+k^2+6\right)-7M\right)\right)}{72\left(3\beta+k^2(g-M+3)+k^4\right)} = \Lambda_1.
\end{aligned}
$$

In the case of a hexagonal lattice, the cross-interaction coefficient K_2 is given by

$$
K_2 = 2\Gamma_0 + 2\Gamma_1 + 2\Gamma_3 + 6\Lambda_2,
$$

where

$$
\begin{aligned}
\Gamma_0 &= 0, \\
\Gamma_1 &= \mathcal{P}_+(\mathbf{k}_1)\left[\mathcal{N}_2(\varphi_-(k_m^*)e^{-i\mathbf{k}_3\cdot\mathbf{x}}, \varphi_+(k_m^*)e^{-i\mathbf{k}_2\cdot\mathbf{x}}; M^*)\bar{v}_3\right] \\
&= -\frac{k^4\left(k^2\left(M\left(g+k^2+27\right)-24\left(g+k^2\right)\right)-12\beta\left(g+k^2\right)\right)\left(24\beta\left(g+k^2\right)+k^2\left(4\left(g+k^2\right)\left(g+k^2+9\right)-M\left(5g+5k^2+27\right)\right)\right)}{96\left(3\beta+k^2(g-M+3)+k^4\right)^3} \\
\Gamma_3 &= \mathcal{P}_+(\mathbf{k}_1)\left[\mathcal{N}_2(\boldsymbol{\nu}_{\mathbf{k}_1-\mathbf{k}_2}e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}}, \varphi_+(k_m^*)e^{i\mathbf{k}_2\cdot\mathbf{x}}; M^*)\right] \\
&= \frac{k^6\left(-\left(M^2\left(7g^2+4(8g+159)k^2+456g+25k^4+2700\right)\right)-288\left(g+k^2\right)\left((4g+41)k^2+g(g+11)+3k^4\right)\right)}{24\left(3\beta+k^2(g-M+3)+k^4\right)\left(k^2\left(g(M-48)+3k^2(M-48)+72M\right)-16\beta\left(g+3k^2\right)\right)} \\
&\quad+\frac{k^6\left(6M\left((7g+281)k^4+(g(5g+412)+1896)k^2+g(g(g+131)+1176)+3k^6\right)\right)}{24\left(3\beta+k^2(g-M+3)+k^4\right)\left(k^2\left(g(M-48)+3k^2(M-48)+72M\right)-16\beta\left(g+3k^2\right)\right)} \\
&\quad-\frac{192\beta^2k^2\left(g+k^2\right)\left(4g+13k^2\right)-4\beta k^4\left(24\left(g+k^2\right)\left((4g+127)k^2+g(g+37)+3k^4\right)-M\left(4(41g+387)k^2+g(37g+1008)+127k^4\right)\right)}{24\left(3\beta+k^2(g-M+3)+k^4\right)\left(k^2\left(g(M-48)+3k^2(M-48)+72M\right)-16\beta\left(g+3k^2\right)\right)}
\end{aligned}
$$

2. Eigenvalues of mixed modes

Finally, we give the eigenvalues of the linearisation of the reduced system on the hexagonal lattice on the invariant subspace $\{A_1 \in \mathbb{R}, A_2 = A_3 \in \mathbb{R}\}$ about the fixed point MM corresponding to mixed modes and false hexagons in the full system (1.1). We find that

$$\lambda_{\pm,MM} = \frac{(K_0-K_2)\left(\kappa\,(K_0-K_2)\,(K_0+3K_2)\,M_0 - K_0N_0^2\right)}{2c\,(K_0-K_2)^2\,(K_0+K_2)} \\ \pm \frac{\sqrt{K_0-K_2}\sqrt{\kappa^2\,(K_0-K_2)^3\,(3K_0+K_2)^2M_0^2+2\kappa\,(K_0-K_2)^2\,(11K_0^2+13K_2K_0+4K_2^2)\,M_0N_0^2+K_0\,(17K_0^2+23K_2K_0+8K_2^2)\,N_0^4}}{2c\,(K_0-K_2)^2\,(K_0+K_2)}.$$

A plot of these eigenvalues can be found in Figure 1.

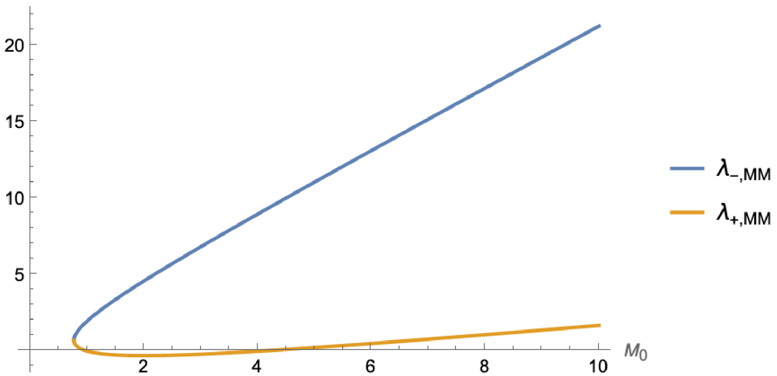

Figure 1. Eigenvalues of the linearisation about MM for $N_0 = 1$, $K_0 = -1$, $K_2 = -2$ and $\kappa = 1$.

References

[HJ24] Bastian Hilder and Jonas Jansen. *Data availability for "Pattern interfaces close to a Turing instability in a two-dimensional Bénard–Marangoni model"*. 2024. URL: `https://github.com/Bastian-Hilder/TuringUnstableThinFilmFronts`.

[Inc] Wolfram Research Inc. *Mathematica, Version 14.1*. Champaign, IL, 2024. URL: `https://www.wolfram.com/mathematica`.

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