In[71]:=
```
ClearAll["`*"]
```
löschee alle

2d patterns in thermocapillary model

Here, we calculate the coefficients on the center manifold, which determines stationary 2d patterns. We distinguish two cases: hexagonal lattices and square lattices. The coefficients on the center manifold are then written as plain text to external files.

Set up the model

In[72]:=
```
(* Vector field s.t. U_t = F(U)*)
F[h_, T_] :=
 FullSimplify[{{-Div[(h^3/3) * (Grad[Laplacian[h, {x1, x2}], {x1, x2}] -
          g * Grad[h, {x1, x2}]) + M * (h^2/2) * Grad[h - T, {x1, x2}], {x1, x2}]},
    {- (-Div[h * Grad[T, {x1, x2}], {x1, x2}] +
        (1/2) * (D[h, x1]^2 + D[h, x2]^2) + β * (T - h) - ((h^3/3) *
            (Grad[Laplacian[h, {x1, x2}], {x1, x2}] - g * Grad[h, {x1, x2}]) +
          M * (h^2/2) * Grad[h - T, {x1, x2}]).Grad[T - h, {x1, x2}] + Div[(h^4/8) *
          (Grad[Laplacian[h, {x1, x2}], {x1, x2}] - g * Grad[h, {x1, x2}]) +
        M * (h^3/6) * Grad[h - T, {x1, x2}], {x1, x2}])}}]
```

In[73]:=
```
(* linearisation about U_0 = (1,1)*)
L[h_, T_] := D[F[1 + ϵ * h, 1 + ϵ * T], ϵ] /. ϵ → 0
               leite ab
Lhat[k1_, k2_] := FullSimplify[
                  vereinfache vollständig
  {{(L[Exp[I * (k1 * x1 + k2 * x2)], 0] * Exp[-I * (k1 * x1 + k2 * x2)])[[1]][[1]],
       Ex⋯  imaginäre Einheit I                Exp⋯  imaginäre Einheit I
    (L[0, Exp[I * (k1 * x1 + k2 * x2)]] * Exp[-I * (k1 * x1 + k2 * x2)])[[1]][[1]]},
          Ex⋯  imaginäre Einheit I             Exp⋯  imaginäre Einheit I
   {(L[Exp[I * (k1 * x1 + k2 * x2)], 0] * Exp[-I * (k1 * x1 + k2 * x2)])[[2]][[1]],
       Ex⋯  imaginäre Einheit I                Exp⋯  imaginäre Einheit I
    (L[0, Exp[I * (k1 * x1 + k2 * x2)]] * Exp[-I * (k1 * x1 + k2 * x2)])[[2]][[1]]}}]
          Ex⋯  imaginäre Einheit I             Exp⋯  imaginäre Einheit I
(* Fourier symbol of linearisation *)
  diskrete Fourier–Transformation
LhatAbs[k_] := Lhat[k, 0]
```

In[76]:=
```
(* quadratic nonlinearity *)
N2[h1_, h2_, T1_, T2_] := FullSimplify[
                          vereinfache vollständig
  (1 / 2) * D[D[F[1 + ϵ * h1 + δ * h2, 1 + ϵ * T1 + δ * T2], ϵ], δ] /. {ϵ → 0, δ → 0}]
            ⋯ leite ab
(* diagonal part *)
N2Diag[h_, T_] := N2[h, h, T, T]
```

In[78]:=
```
(* cubic nonlinearity *)
N3[h1_, h2_, h3_, T1_, T2_, T3_] := FullSimplify[
                                    vereinfache vollständig
  (1 / 6) * D[D[D[F[1 + ϵ * h1 + δ * h2 + γ * h3, 1 + ϵ * T1 + δ * T2 + γ * T3], ϵ], δ], γ] /.
            ⋯ ⋯ leite ab
   {ϵ → 0, δ → 0, γ → 0}]
(* diagonal part *)
N3Diag[h_, T_] := N3[h, h, h, T, T, T]
```

Eigenvalues and eigenvectors

In[80]:=
```
(* Determine coefficient functions a_1 and a_0,
given by the first and zeroth order terms of the determinant,
respectively *)
a1[k_] := FullSimplify[Coefficient[Det[LhatAbs[k] - IdentityMatrix[2] * λ], λ, 1]]
          vereinfache volls⋯ Koeffizient  Determinante       Einheitsmatrix
a0[k_] := FullSimplify[Coefficient[Det[LhatAbs[k] - IdentityMatrix[2] * λ], λ, 0]]
          vereinfache volls⋯ Koeffizient  Determinante       Einheitsmatrix
```

In[82]:=
```
(* Critical Marangoni and wave number -- monotonic instability *)
Mmc[k_] := M /. FullSimplify[Solve[a0[k] == 0, M]][[1]]
                vereinfache volls···  löse
kmc = k /. FullSimplify[Solve[D[Mmc[k], k] == 0, k]][[2]] (* note: β<72 *)
           vereinfache volls···  löse    leite ab
Mcrit = FullSimplify[Mmc[kmc]]
        vereinfache vollständig
```

Out[83]=

$$\sqrt{-\frac{g\,\beta + 6\sqrt{2}\,\sqrt{g\,(72 + g - \beta)\,\beta}}{-72 + \beta}}$$

Out[84]=

$$\frac{48\left(g\,(72 + g - \beta) + 12\left(6\,\beta + \sqrt{2}\,\sqrt{g\,(72 + g - \beta)\,\beta}\right)\right)}{(72 + g)^2}$$

In[85]:=
```
(* Eigenvectors at critical wave numbers and projection operator *)
(* coefficient for eigenvector phi_+(k) *)
α = 1 - 2 * (g + k^2) / (3 * M)
(* coefficient for adjoint eigenvector phi^*_+(k) *)
alphaTilde = FullSimplify[-((1/6) * k^2 * (6 + M) + β) * (2/(k^2 * M))]
aminus =
 amin /. FullSimplify[Solve[-1/2 amin * k^2 * M - 1/6 k^2 (6 + M) - β == a1[k], amin]][[1]][[1]]
PhiPlus = {1, α}
Phim0 = {-1, 1}
PhiP = FullSimplify[NullSpace[LhatAbs[kmc] /. M → Mcrit]][[1]]
PhiPadj = FullSimplify[NullSpace[Adjugate[LhatAbs[kmc]] /. M → Mcrit]][[1]]
PPlus[U_] := FullSimplify[(PhiPadj.U)]
PPlusNormalized[U_] := FullSimplify[(PhiPadj.U) / (PhiP.PhiPadj)]
```
Eigenvektoren | vereinfache vollständig | vereinfache volls··· | löse | vereinfache volls··· | Nullraum | vereinfache volls··· | Nullraum | Adjunkte | vereinfache vollständig | vereinfache vollständig

Out[85]=
$$1-\frac{2\left(g+k^2\right)}{3\,M}$$

Out[86]=
$$-\frac{1}{3}-\frac{2\left(k^2+\beta\right)}{k^2\,M}$$

Out[87]=
$$\frac{-2\left(6+g+k^2\right)+M-\frac{12\,\beta}{k^2}}{3\,M}$$

Out[88]=
$$\left\{1,\ 1-\frac{2\left(g+k^2\right)}{3\,M}\right\}$$

Out[89]=
$$\{-1,\ 1\}$$

Out[90]=
$$\left\{\frac{-72\,g^2+72\,g\,\beta+72\,(-72+\beta)\,\beta-432\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}+6\,\sqrt{2}\,g\,\sqrt{g\,(72+g-\beta)\,\beta}}{g^3+216\,g\,\beta+72\,(-72+\beta)\,\beta},\ 1\right\}$$

Out[91]=
$$\left\{\frac{8\left(3\,g\,(8+g-\beta)+2\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)}{9\,(8+g)^2-(8+9\,g)\,\beta},\ 1\right\}$$

In[94]:=
```
(* Eigenvectors at critical k and M *)
   Eigenvektoren
FullSimplify[(α /. M → Mmc[k]) /. k → kmc]
vereinfache vollständig
FullSimplify[(alphaTilde /. M → Mmc[k]) /. k → kmc]
vereinfache vollständig
```

Out[94]=

$$\frac{12\,g^2 - 12\,g\,\beta + \sqrt{2}\,g\,\sqrt{g\,(72+g-\beta)\,\beta} - 12\left((-72+\beta)\,\beta + 6\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)}{12\,(g-\beta)\,(-72+\beta)}$$

Out[95]=

$$\frac{-3\,g\,(8+g-\beta) + 2\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}}{8\,g\,(g-\beta)}$$

Coefficients on hexagonal lattice

In[96]:=
```
(* Vectors for Fourier lattice *)
   Vektoren           diskrete Fourier-Transformat
k1 = k * {1, 0}
k2 = (k / 2) * {-1, Sqrt[3]}
                     Quadratwurzel
k3 = - (k / 2) * {1, Sqrt[3]}
                     Quadratwurzel
```

Out[96]=

$\{k, 0\}$

Out[97]=

$$\left\{-\frac{k}{2}, \frac{\sqrt{3}\,k}{2}\right\}$$

Out[98]=

$$\left\{-\frac{k}{2}, -\frac{\sqrt{3}\,k}{2}\right\}$$

In[99]:=
```
(* quadratic coefficient - resonance *)
Ncal =
 FullSimplify[((1 / (α + alphaTilde)) * {alphaTilde, 1}.N2[Exp[I * k1.{x1, x2}],
 vereinfache vollständig                                 Ex⋯ imaginäre Einheit I
        Exp[I * k2.{x1, x2}], Exp[I * k1.{x1, x2}] * α, Exp[I * k2.{x1, x2}] * α] *
        Ex⋯ imaginäre Einheit I  Ex⋯ imaginäre Einheit I   Ex⋯ imaginäre Einheit I
      Exp[I * k3.{x1, x2}] /. M → Mmc[k]) /. k → kmc]
      Ex⋯ imaginäre Einheit I
```

Out[99]=

$$\Big\{\Big(g\,\big(18\,(-24+37\,g)\,\beta^3 - 48\,\sqrt{2}\,(-18+g)\,(216+g\,(27+g))\,\sqrt{g\,(72+g-\beta)\,\beta} - 3\,\beta^2\,\big(-24\,\big(864+11\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big) + g\,\big(15\,696+62\,g-4\,g^2+15\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)\big) - 2\,\beta\,\big(1296\,\big(864+23\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big) + g\,\big(g\,\big(43\,416+4\,g\,(162+g)+15\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big) + 54\,\big(576+49\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)\big)\big)\big)\Big) \Big/ \Big((-72+\beta)\,\big(8\,(216+g\,(27+g))^2 + 3\,(-3456+g\,(-5832+421\,g))\,\beta + 9\,(8+25\,g)\,\beta^2\big)\Big)\Big\}$$

In[100]:=

```
(* quadratic coefficient - conservation mode *)
Kc =
 Simplify[Apart[FullSimplify[(Expand[(1 / (α + alphaTilde)) * {alphaTilde, 1}.N2[
 vereinfache  Partial…  vereinfache vollstä…  multipliziere aus
          1, Exp[I * k1.{x1, x2}] * 1, 1, Exp[I * k1.{x1, x2}] * α] *
            Ex…  imaginäre Einheit I        Ex…  imaginäre Einheit I
        Exp[-I * k1.{x1, x2}]] /. M → Mmc[k]) /. k → kmc]]]
        Exp…  imaginäre Einheit I
(* coefficients of polynomal for nonlinearity in conservation law;
recall ϕ+(kj)=(1,α)^T *)
κ0 = FullSimplify[(((M - g) + g * α - k^2) /. M → Mmc[k]) /. k → kmc]
      vereinfache vollständig
κ1 = -1
```

Out[100]=

$$\left\{-\left(\left(g\left(8g^4\beta-216\sqrt{2}\sqrt{g(72+g-\beta)\beta}\left(1728-96\beta+\beta^2\right)+12g^3\left(108\beta-\beta^2+4\sqrt{2}\sqrt{g(72+g-\beta)\beta}\right)+6g^2\left(40\beta^2-72\sqrt{2}\sqrt{g(72+g-\beta)\beta}+\beta\left(13824+7\sqrt{2}\sqrt{g(72+g-\beta)}\right)\right)+9g\left(-64\beta^3-4032\sqrt{2}\sqrt{g(72+g-\beta)\beta}+5\beta^2\left(1152+\sqrt{2}\sqrt{g(72+g-\beta)\beta}\right)+48\beta\left(-1728+13\sqrt{2}\sqrt{g(72+g-\beta)\beta}\right)\right)\right)\right)\Big/\left((-72+\beta)\left(432g^3+8g^4+72(-72+\beta)^2+3g^2(3096+421\beta)+9g\left(10368-1944\beta+25\beta^2\right)\right)\right)\right)\right\}$$

Out[101]=

$$\frac{g(120+g)}{72+g}+\frac{g(72+g)}{-72+\beta}-\frac{48(-72+g)\beta}{(72+g)^2}+\frac{576\sqrt{2}\sqrt{g(72+g-\beta)\beta}}{(72+g)^2}+\frac{g\sqrt{g(72+g-\beta)\beta}}{\sqrt{2}(6g-6\beta)}+\frac{6\sqrt{2}\sqrt{g(72+g-\beta)\beta}}{-72+\beta}+\frac{g\sqrt{g(72+g-\beta)\beta}}{6\sqrt{2}(-72+\beta)}+\frac{g^2}{-g+\beta}$$

Out[102]=

$$-1$$

In[103]:=

```
(* calculate relevant non-central terms *)
ν0 = FullSimplify[(2 / β) * Phim0.N2[Exp[I * k1.{x1, x2}],
     Exp[-I * k1.{x1, x2}], Exp[I * k1.{x1, x2}] * α, Exp[-I * k1.{x1, x2}] * α]]
ν1 = FullSimplify[(1 / (α + alphaTilde)) * (2 / a1[k]) *
    {aminus, 1}.N2[Exp[I * k1.{x1, x2}], Exp[I * k2.{x1, x2}],
     Exp[I * k1.{x1, x2}] * α, Exp[I * k2.{x1, x2}] * α] * Exp[I * k3.{x1, x2}]]
νjj = FullSimplify[-Inverse[Lhat[k1[[1]] + k1[[1]], k1[[2]] + k1[[2]]]].
     N2[Exp[I * k1.{x1, x2}], Exp[I * k1.{x1, x2}], Exp[I * k1.{x1, x2}] * α,
      Exp[I * k1.{x1, x2}] * α] * Exp[-2 * I * k1.{x1, x2}]]
νjl = FullSimplify[-2 * Inverse[Lhat[k1[[1]] - k2[[1]], k1[[2]] - k2[[2]]]].
    N2[Exp[I * k1.{x1, x2}], Exp[-I * k2.{x1, x2}], Exp[I * k1.{x1, x2}] * α,
     Exp[-I * k2.{x1, x2}] * α] * Exp[-I * (k1 - k2).{x1, x2}]]
```

Out[103]=

$$\left\{-\frac{k^2}{\beta}\right\}$$

Out[104]=

$$\left\{\frac{k^4\left(-4\left(g+k^2\right)\left(9+g+k^2\right)+\left(27+5\,g+5\,k^2\right)M\right)-24\,k^2\left(g+k^2\right)\beta}{4\left(k^4+k^2\left(3+g-M\right)+3\,\beta\right)^2}\right\}$$

Out[105]=

$$\left\{\left\{\frac{k^2\left(48\left(g+k^2\right)-\left(27+2\,g+2\,k^2\right)M\right)+6\left(g+k^2\right)\beta}{k^2\left(g\left(-48+M\right)+4\,k^2\left(-48+M\right)+72\,M\right)-12\left(g+4\,k^2\right)\beta}\right\},\right.$$
$$\left\{\left(k^2\left(-16\left(g+k^2\right)\left(g+4\,k^2\right)+\left(g\left(42+g\right)+\left(96+5\,g\right)k^2+4\,k^4\right)M-\left(27+2\,g+2\,k^2\right)M^2\right)+\right.\right.$$
$$\left.\left.6\left(g+k^2\right)M\beta\right)\Big/\left(k^2\,M\left(g\left(-48+M\right)+4\,k^2\left(-48+M\right)+72\,M\right)-12\left(g+4\,k^2\right)M\beta\right)\right\}\Big\}$$

Out[106]=

$$\left\{\left\{\frac{4\,k^2\left(24\left(g+k^2\right)-\left(15+g+k^2\right)M\right)+16\left(g+k^2\right)\beta}{k^2\left(g\left(-48+M\right)+3\,k^2\left(-48+M\right)+72\,M\right)-16\left(g+3\,k^2\right)\beta}\right\},\right.$$
$$\left\{\left(2\,k^2\left(-16\left(g+k^2\right)\left(g+3\,k^2\right)+\left(g\left(44+g\right)+4\left(21+g\right)k^2+3\,k^4\right)M-2\left(15+g+k^2\right)M^2\right)+\right.\right.$$
$$\left.\left.16\left(g+k^2\right)M\beta\right)\Big/\left(k^2\,M\left(g\left(-48+M\right)+3\,k^2\left(-48+M\right)+72\,M\right)-16\left(g+3\,k^2\right)M\beta\right)\right\}\Big\}$$

In[107]:=

```
(* calculate coefficients from quadratic interactions *)
Γ0 = FullSimplify[
     vereinfache vollständig
  Expand[(1 / (α + alphaTilde)) * {alphaTilde, 1}.N2[0, Exp[I * k1.{x1, x2}] * 1,
  multipliziere aus                                    Ex···  imaginäre Einheit I
      1, Exp[I * k1.{x1, x2}] * α] * Exp[-I * k1.{x1, x2}] * ν0]]
         Ex···  imaginäre Einheit I          Exp···  imaginäre Einheit I
Γ1 = FullSimplify[Expand[(1 / (α + alphaTilde)) * {alphaTilde, 1}.
     vereinfache volls···  multipliziere aus
     N2[Exp[-I * k2.{x1, x2}], Exp[-I * k1.{x1, x2}], Exp[-I * k2.{x1, x2}] * α,
        Exp···  imaginäre Einheit I  Exp···  imaginäre Einheit I  Exp···  imaginäre Einheit I
      Exp[-I * k1.{x1, x2}] * α] * Exp[-I * k3.{x1, x2}] * ν1]]
      Exp···  imaginäre Einheit I        Exp···  imaginäre Einheit I
Γ2 = FullSimplify[Expand[(1 / (α + alphaTilde)) *
     vereinfache volls···  multipliziere aus
     {alphaTilde, 1}.N2[Exp[-I * k1.{x1, x2}], Exp[2 * I * k1.{x1, x2}] * η1,
                        Exp···  imaginäre Einheit I  Expone··  imaginäre Einheit I
       Exp[-I * k1.{x1, x2}] * α, Exp[2 * I * k1.{x1, x2}] * η2] *
       Exp···  imaginäre Einheit I    Expone··  imaginäre Einheit I
     Exp[-I * k1.{x1, x2}] /. {η1 → νjj[[1]][[1]], η2 → νjj[[2]][[1]]}]]
     Exp···  imaginäre Einheit I
Γ3 = FullSimplify[Expand[(1 / (α + alphaTilde)) *
     vereinfache volls···  multipliziere aus
     {alphaTilde, 1}.N2[Exp[I * k2.{x1, x2}], Exp[I * (k1 - k2).{x1, x2}] * η1,
                        Ex···  imaginäre Einheit I   Ex···  imaginäre Einheit I
       Exp[I * k2.{x1, x2}] * α, Exp[I * (k1 - k2).{x1, x2}] * η2] *
       Ex···  imaginäre Einheit I    Ex···  imaginäre Einheit I
     Exp[-I * k1.{x1, x2}] /. {η1 → νjl[[1]][[1]], η2 → νjl[[2]][[1]]}]]
     Exp···  imaginäre Einheit I
```

Out[107]=

$$\{0\}$$

Out[108]=

$$\left\{-\left(\left(k^4\left(k^2\left(-24\left(g+k^2\right)+\left(27+g+k^2\right)M\right)-12\left(g+k^2\right)\beta\right)\left(k^2\left(4\left(g+k^2\right)\left(9+g+k^2\right)-\left(27+5g+5k^2\right)M\right)+24\left(g+k^2\right)\beta\right)\right)\Big/\left(96\left(k^4+k^2\left(3+g-M\right)+3\beta\right)^3\right)\right)\right\}$$

Out[109]=

$$\Big\{\Big(k^6\Big(-192\left(g+k^2\right)\left(g\left(9+g\right)+\left(54+5g\right)k^2+4k^4\right)+4\left(g\left(9+g\right)\left(114+g\right)+3\left(74+g\right)\left(9+2g\right)k^2+3\left(116+3g\right)k^4+4k^6\right)M-\left(1458+279g+4g^2+\left(441+26g\right)k^2+22k^4\right)M^2\Big)-6k^4\Big(8\left(g+k^2\right)\left(g\left(42+g\right)+\left(213+5g\right)k^2+4k^4\right)-\left(g\left(396+13g\right)+80\left(9+g\right)k^2+67k^4\right)M\Big)\beta-72k^2\left(g+k^2\right)\left(5g+23k^2\right)\beta^2\Big)\Big/\Big(24\left(k^4+k^2\left(3+g-M\right)+3\beta\right)\left(k^2\left(g\left(-48+M\right)+4k^2\left(-48+M\right)+72M\right)-12\left(g+4k^2\right)\beta\right)\Big)\Big\}$$

Out[110]=

$$\left\{\left(k^6\left(-288\left(g+k^2\right)\left(g(11+g)+(41+4g)k^2+3k^4\right)+6\left(g(1176+g(131+g))+(1896+g(412+5g))k^2+(281+7g)k^4+3k^6\right)M-\left(2700+456g+7g^2+4(159+8g)k^2+25k^4\right)M^2\right)-4k^4\left(24\left(g+k^2\right)\left(g(37+g)+(127+4g)k^2+3k^4\right)-\left(g(1008+37g)+4(387+41g)k^2+127k^4\right)M\right)\beta-192k^2\left(g+k^2\right)\left(4g+13k^2\right)\beta^2\right)\Big/\left(24\left(k^4+k^2(3+g-M)+3\beta\right)\left(k^2\left(g(-48+M)+3k^2(-48+M)+72M\right)-16\left(g+3k^2\right)\beta\right)\right)\right\}$$

In[111]:=

```
(* calculate coefficients from cubic interactions *)
Λ1 = FullSimplify[
  Expand[(1 / (α + alphaTilde)) * {alphaTilde, 1}.N3[Exp[I * k1.{x1, x2}],
      Exp[I * k1.{x1, x2}], Exp[-I * k1.{x1, x2}], Exp[I * k1.{x1, x2}] * α,
      Exp[I * k1.{x1, x2}] * α, Exp[-I * k1.{x1, x2}] * α] * Exp[-I * k1.{x1, x2}]]]
Λ2 = FullSimplify[
  Expand[(1 / (α + alphaTilde)) * {alphaTilde, 1}.N3[Exp[I * k1.{x1, x2}],
      Exp[I * k2.{x1, x2}], Exp[-I * k2.{x1, x2}], Exp[I * k1.{x1, x2}] * α,
      Exp[I * k2.{x1, x2}] * α, Exp[-I * k2.{x1, x2}] * α] * Exp[-I * k1.{x1, x2}]]]
```

vereinfache vollständig · multipliziere aus · Ex··· · imaginäre Einheit I

Out[111]=

$$\left\{-\frac{k^2\left(g+k^2\right)\left(k^2\left(8\left(6+g+k^2\right)-7M\right)+48\beta\right)}{72\left(k^4+k^2(3+g-M)+3\beta\right)}\right\}$$

Out[112]=

$$\left\{-\frac{k^2\left(g+k^2\right)\left(k^2\left(8\left(6+g+k^2\right)-7M\right)+48\beta\right)}{72\left(k^4+k^2(3+g-M)+3\beta\right)}\right\}$$

In[113]:=

```
(* cubic coefficients *)
(* self-interaction term *)
K0 = FullSimplify[Expand[
   FullSimplify[FullSimplify[2 * Γ0 + 2 * Γ2 + 3 * Λ1] /. {M → Mmc[k]}] /. k → kmc]]
(* cross-interaction term *)
K2 = Simplify[Expand[((2 * Γ0 + 2 * Γ1 + 2 * Γ3 + 6 * Λ2) /. M → Mmc[k]) /. k → kmc]]
```

vereinfache volls··· · multipliziere aus · vereinfache vollständig · vereinfache

Out[113]=

$$\begin{aligned}
&\{-((4\,(72+g-\beta)\,(-6912\sqrt{2}\,(-72+\beta)^4\beta^2\sqrt{g\,(72+g-\beta)\,\beta}\,-\\
&\quad 36\,g\,(-72+\beta)^3\beta\,(53\,568\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}+\beta\,(-3888\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}\,+\\
&\qquad 41\sqrt{2}\,\beta\sqrt{g\,(72+g-\beta)\,\beta}+8\,(-72+\beta)\,(216+19\,\beta)))\,+\\
&\quad 32\,g^5\beta\,(4\,852\,224+\beta\,(834\,624+\beta\,(16\,632+43\,\beta)))\,+\\
&\quad 3\,g^2\,(-72+\beta)^2\,(-7\,464\,960\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}\,+\\
&\qquad \beta\,(-20\,736\,(-20\,736+55\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})+\beta\,(864\,(-49\,824+245\sqrt{2}\\
&\qquad\qquad \sqrt{g\,(72+g-\beta)\,\beta})+\beta\,(1\,766\,016+6380\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}\,-\\
&\qquad\qquad 3\,\beta\,(752+70\,\beta-3\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})))))\,+\\
&\quad 16\,g^4\,(4\,478\,976\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}+\beta\,(559\,872\,(816+7\sqrt{2}\\
&\qquad\qquad \sqrt{g\,(72+g-\beta)\,\beta})+\beta\,(864\,(171\,504+245\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,+\\
&\qquad\qquad \beta\,(\beta\,(-48\,456-224\,\beta+\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,+\\
&\qquad\qquad 84\,(29\,808+19\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})))))\,-\\
&\quad 3\,g^3\,(-72+\beta)\,(-2\,985\,984\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}+\beta\,(746\,496\,(-840+13\sqrt{2}\\
&\qquad\qquad \sqrt{g\,(72+g-\beta)\,\beta})+\beta\,(2304\,(87\,696+545\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,+\\
&\qquad\qquad \beta\,(\beta\,(-111\,552-962\,\beta+13\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,+\\
&\qquad\qquad 368\,(28\,080+41\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}))))))))\,/\\
&\quad (3\,(-72+\beta)^3\,(864+12\,g-12\,\beta+\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\\
&\quad (-12\,\beta^4\,+\\
&\quad 1728\sqrt{2}\,(216+g\,(27+g))\sqrt{g\,(72+g-\beta)\,\beta}\,-\\
&\quad \beta^3\,(-2592+4\,g\,(15+g)+15\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,+\\
&\quad \beta^2\,(g\,(8640+4\,(-126+g)\,g-33\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,+\\
&\qquad 72\,(-2592+31\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}))\,+\\
&\quad 72\,\beta\,(72\,(864-17\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})+g\,(-4320+24\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}\,+\\
&\qquad g\,(792+12\,g+\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}))))))\}
\end{aligned}$$

Out[114]=

$$\begin{aligned}
&\{(2\,g\,(72+g-\beta)^2\beta^2\\
&\quad (-60\,466\,176\sqrt{2}\,(-72+\beta)^{10}\beta^3\sqrt{g\,(72+g-\beta)\,\beta}+64\,g^{11}\beta\,(2\,527\,714\,477\,080\,576\,+\\
&\qquad 1\,028\,137\,832\,939\,520\,\beta+120\,437\,013\,086\,208\,\beta^2+6\,502\,290\,702\,336\,\beta^3\,+\\
&\qquad 163\,325\,859\,840\,\beta^4+1\,715\,261\,184\,\beta^5+6\,622\,560\,\beta^6+7308\,\beta^7+\beta^8)\,+\\
&\quad 373\,248\,g\,(-72+\beta)^9\beta^2\,(741\,\beta^3+84\,240\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}-1152\,\beta\\
&\qquad (-621+16\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})+\beta^2\,(-63\,288+281\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}))\,+\\
&\quad 32\,g^{10}\,(-34\,740\,\beta^9-4\,\beta^{10}+2\,166\,612\,408\,926\,208\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}\,+\\
&\qquad 60\,183\,678\,025\,728\,\beta\,(10\,584+47\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,+\\
&\qquad 3\,\beta^8\,(-11\,474\,064+59\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,+\\
&\qquad 20\,961\,607\,680\,\beta^4\,(113\,940+77\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,+\\
&\qquad 320\,246\,784\,\beta^5\,(116\,424+83\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,+\\
&\qquad 404\,352\,\beta^6\,(-788\,616+427\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,+\\
&\qquad 835\,884\,417\,024\,\beta^2\,(309\,384+665\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,+\\
&\qquad 432\,\beta^7\,(-18\,792\,648+865\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,+\\
&\qquad 11\,609\,505\,792\,\beta^3\,(3\,421\,224+3731\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}))\,+\\
&\quad 1296\,g^2\,(-72+\beta)^8\beta\,(128\,016\,\beta^5+456\,855\,552\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta}\,-\\
&\qquad 6912\,\beta^3\,(18\,621+644\sqrt{2}\sqrt{g\,(72+g-\beta)\,\beta})\,-
\end{aligned}$$

$373248\,\beta\,\left(13824+1961\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$5184\,\beta^2\,\left(400032+12257\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$\beta^4\,\left(-7815744+23515\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)\Big)-$
$8\,g^9\,(-72+\beta)\,\Big(-106765\,\beta^9-8\,\beta^{10}+12277470317248512\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}+$
$2\,\beta^8\,\left(-68312520+601\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$10030613004288\,\beta\,\left(43128+1349\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$160123392\,\beta^5\,\left(1196784+1843\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$471744\,\beta^6\,\left(-3916944+3623\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$1048080384\,\beta^4\,\left(11561328+17479\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$17414258688\,\beta^3\,\left(7985248+23933\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$139314069504\,\beta^2\,\left(1274544+25487\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$72\,\beta^7\,\left(-536701536+43487\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)\Big)+$
$27\,g^4\,(-72+\beta)^6\,\Big(-16830\,\beta^8-13838530904064\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}+$
$\beta^7\,\left(-7730256+325\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)-$
$967458816\,\beta^2\,\left(303184+2293\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)-$
$2579890176\,\beta\,\left(-79920+4087\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$13436928\,\beta^3\,\left(436320+18311\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$4\,\beta^6\,\left(-17984304+19495\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$10368\,\beta^4\,\left(218655936+64003\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)-$
$384\,\beta^5\,\left(-51290820+210379\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)\Big)-$
$108\,g^3\,(-72+\beta)^7\,\Big(-309038\,\beta^7-162533081088\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}+$
$644972544\,\beta\,\left(-864+\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)-$
$8957952\,\beta^2\,\left(860976+737\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$9\,\beta^6\,\left(1352976+4409\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$497664\,\beta^3\,\left(2630448+5989\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$4320\,\beta^4\,\left(-13428288+61051\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$24\,\beta^5\,\left(54132624+379153\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)\Big)-$
$9\,g^6\,(-72+\beta)^4\,\Big(-3140646\,\beta^8-416\,\beta^9+1495397222055936\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}-$
$839808\,\beta^4\,\left(-157017472+56353\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$34828517376\,\beta\,\left(3433824+63911\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$\beta^7\,\left(-1237212912+80569\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$644972544\,\beta^2\,\left(38768760+353347\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)-$
$4478976\,\beta^3\,\left(108922752+572069\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$4752\,\beta^5\,\left(1592638560+1231211\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$36\,\beta^6\,\left(-686679216+1563115\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)\Big)-$
$6\,g^7\,(-72+\beta)^3\,\Big(3132\,\beta^9-3751449263603712\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}-$
$3\,\beta^8\,\left(-299823+4\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)-$
$139314069504\,\beta\,\left(6701940+87439\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)-$
$3869835264\,\beta^2\,\left(93960648+295901\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$\beta^7\,\left(-1883494836+395963\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$11197440\,\beta^4\,\left(133719804+786841\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$
$57024\,\beta^5\,\left(219295836+2182057\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$

$$144\,\beta^6\left(-2\,234\,702\,736+3\,364\,705\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$$
$$26\,873\,856\,\beta^3\left(42\,677\,172+4\,951\,153\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)\Big)+$$
$$4\,g^8\,(-72+\beta)^2\Big(-47\,624\,\beta^9+4\,242\,949\,300\,813\,824\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}+$$
$$5\,\beta^8\left(-20\,093\,391+194\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)-$$
$$1\,253\,826\,625\,536\,\beta\left(1\,969\,128+7093\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$$
$$2\,751\,211\,008\,\beta^4\left(4\,795\,048+13\,167\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$$
$$2\,223\,936\,\beta^5\left(90\,353\,448+253\,261\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$$
$$5\,804\,752\,896\,\beta^2\left(-184\,297\,464+266\,129\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$$
$$9\,\beta^7\left(-4\,138\,719\,768+450\,013\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$$
$$2808\,\beta^6\left(-819\,138\,312+1\,009\,543\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$$
$$80\,621\,568\,\beta^3\left(1\,046\,687\,400+8\,456\,773\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)\Big)+$$
$$18\,g^5\,(-72+\beta)^5\Big(-99\,169\,\beta^8+179\,018\,579\,312\,640\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}-$$
$$161\,243\,136\,\beta^3\left(1\,655\,820+587\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$$
$$4\,\beta^7\left(-26\,462\,664+1601\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)-$$
$$644\,972\,544\,\beta^2\left(-3\,829\,068+6883\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$$
$$3\,869\,835\,264\,\beta\left(997\,488+29\,587\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$$
$$18\,\beta^6\left(-240\,229\,488+780\,217\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$$
$$432\,\beta^5\left(2\,004\,690\,960+7\,631\,659\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$$
$$15\,552\,\beta^4\left(350\,915\,328+8\,881\,283\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)\Big)\Big)\Big)\Big/$$
$$\Big((-72+\beta)^3\left(864+12\,g-12\,\beta+\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)$$
$$\left(g\,\beta+6\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)^2$$
$$\Big(12\,\beta^4-1728\,\sqrt{2}\,\left(216+27\,g+g^2\right)\sqrt{g\,(72+g-\beta)\,\beta}+$$
$$\beta^3\left(-2592+60\,g+4\,g^2+15\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+\beta^2\Big(-8640\,g+504\,g^2-$$
$$4\,g^3+33\,\sqrt{2}\,g\,\sqrt{g\,(72+g-\beta)\,\beta}+72\left(2592-31\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)\Big)-$$
$$72\,\beta\Big(12\,g^3+72\left(864-17\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+$$
$$24\,g\left(-180+\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)+g^2\left(792+\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\right)\Big)\Big)^3\Big)\Big\}$$

In[115]:=

```
(* write mathematica for N, K₀ and K₂ to file *)
                  numerischer Wert
Export[FileNameJoin[{NotebookDirectory[], "quadratic-coeff-hex.txt"}],
exportiere setze Dateinamen ⋯ Notebook–Verzeichnis
  {Ncal /. β → b}];
Export[FileNameJoin[
exportiere setze Dateinamen zusammen
   {NotebookDirectory[], "Self-cubic-coeffs-hex.txt"}], {K0 /. β → b}];
    Notebook–Verzeichnis
Export[FileNameJoin[
exportiere setze Dateinamen zusammen
   {NotebookDirectory[], "cross-cubic-coeffs-hex.txt"}], {K2 /. β → b}];
    Notebook–Verzeichnis
```

Plots of hexagonal coefficients on {N=0}

We now analyse the coefficients of the reduced system on the hexagonal lattice. To find the relevant dynamics, we have to restrict the parameter space to a neighborhood of {N=0}. Therefore, we start by deriving a parametric representation of the set {N=0}, that is, we find a function β(g) s.t. N(g,β(g)) = 0. Then, we plot the other parameters on this parameterised curve.

Parameterisation of {N=0}

In[118]:=

```
(* Calculate tildeN,
the quadratic coefficient without inserting the critical wave number *)
tildeN =
 FullSimplify[(1 / (α + alphaTilde)) * {alphaTilde, 1}.N2[Exp[I * k1.{x1, x2}],
 vereinfache vollständig                                  Ex⋯  imaginäre Einheit I
     Exp[I * k2.{x1, x2}], Exp[I * k1.{x1, x2}] * α, Exp[I * k2.{x1, x2}] * α] *
     Ex⋯  imaginäre Einheit I  Ex⋯  imaginäre Einheit I  Ex⋯  imaginäre Einheit I
   Exp[I * k3.{x1, x2}] /. M → Mmc[k]]
   Ex⋯  imaginäre Einheit I
```

Out[118]=

$$\left\{\frac{k^2\left(g+k^2\right)\left(2k^2\left(-18+g+k^2\right)+3\left(12+g+k^2\right)\beta\right)}{2k^2\left(216+27g+g^2+(27+2g)k^2+k^4\right)+432\beta-90\left(g+k^2\right)\beta}\right\}$$

In[119]:=

```
(* Check that this agrees with the expression in Shklyaev et. al. 2012 *)
    prüfe
FullSimplify[
vereinfache vollständig
 2 k^2 (-18 + g + k^2) + 3 (12 + g + k^2) β - ((g + k^2) * (3 * β + 2 * k^2) + 36 * (β - k^2))]
```

Out[119]=

0

In[120]:=

```
(* solve condition for β to obtain a function β(g) *)
(* NOTE: curve is the only solution in a relevant parameter regime,
but is only valid for 0 ≤ g ≤ 18, at g=18,
the curve vanishes and becomes invalid *)
curve = FullSimplify[Solve[2 k^2 (-18 + g + k^2) + 3 (12 + g + k^2) β == 0 /. k → kmc, β]][[3]]
        vereinfache volls…  löse
(* calculate roots of curve *)
Solve[(β /. curve) == 0, g]
löse
(* plot parameter curve on which N=0 for 0 < g < 18 *)
                                        numerischer Wert
Plot[β /. curve, {g, 0, 18}, AxesLabel → {g, β}]
stelle Funktion graphisch dar          Achsenbeschriftungen
```

Solve : There may be values of the parameters for which some or all solutions are not valid.

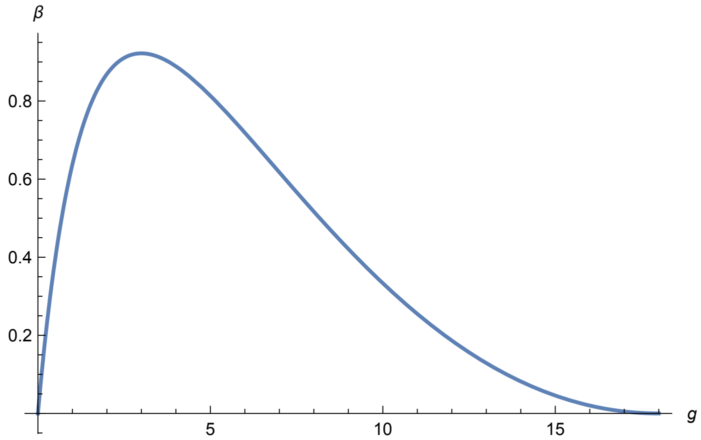
Out[120]=

$$\left\{\beta \to \frac{216 + 87\,g + 2\,g^2 - \sqrt{(9 + 4\,g)\,(72 + 11\,g)^2}}{6 + 3\,g}\right\}$$

Out[121]=

$\{\{g \to 0\}, \{g \to 18\}\}$

Out[122]=

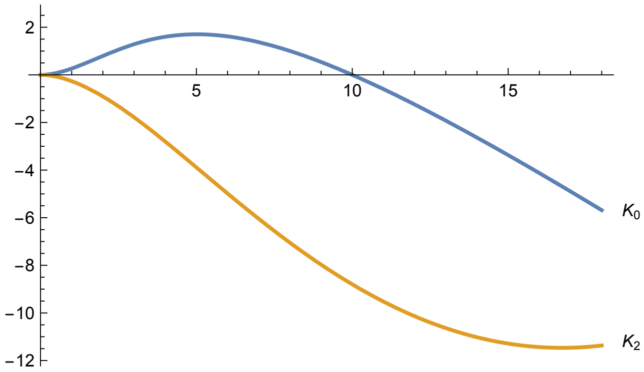


Plots of parameters on {N=0}

In[123]:=

```
(* plot coefficients on the parameter line N(β,g)=0*)
                                           numerischer Wert
Plot[{Labeled[K0 /. curve, "K₀"], Labeled[K2 /. curve, "K₂"]}, {g, 0, 18}]
stelle ··· beschriftet               beschriftet
```

Out[123]=



In[124]:=

```
(* check that miracle is actually true and K_0(β(g),g)=0 at g=10 *)
```

$$\left(\mathrm{K0}\ /.\ \beta \to \frac{216 + 87\,g + 2\,g^2 - \sqrt{46\,656 + 34\,992\,g + 7425\,g^2 + 484\,g^3}}{3\,(2 + g)}\right)\ /.\ g \to 10$$

Out[124]=

```
{0}
```

In[125]:=

```
(* Write coefficients on {N=0} to file *)
   schreibe                     numerischer Wert
Export[FileNameJoin[{NotebookDirectory[], "K0-coeffs-on-Nzero.txt"}],
exportiere setze Dateinamen ··· Notebook-Verzeichnis
  {K0 /. curve}];
Export[
exportiere
  FileNameJoin[{NotebookDirectory[], "K2-coeffs-on-Nzero.txt"}], {K2 /. curve}];
  setze Dateinamen ··· Notebook-Verzeichnis
```

Coefficients on square lattice

Remark: Since the self-interaction term is the same as for the hexagonal lattice, we only calculate the cross-interaction coefficient.

In[127]:=

```
(* wavevectors for square lattice *)
k1Square = k * {1, 0}
k2Square = k * {0, 1}
```

Out[127]=

```
{k, 0}
```

Out[128]=

```
{0, k}
```

In[129]:=

```
(* relevant non-central terms *)
v2k = FullSimplify[
  -Inverse[Lhat[k1Square[[1]] + k1Square[[1]], k1Square[[2]] + k1Square[[2]]]].
    N2[Exp[I*k1Square.{x1, x2}], Exp[I*k1Square.{x1, x2}],
     Exp[I*k1Square.{x1, x2}]*α, Exp[I*k1Square.{x1, x2}]*α]*
   Exp[-2*I*k1Square.{x1, x2}]]
vjmlSquare =
 FullSimplify[-2*Inverse[Lhat[k1Square[[1]] - k2Square[[1]], k1Square[[2]] - k2[[2]]]].
    N2[Exp[I*k1Square.{x1, x2}], Exp[-I*k2.{x1, x2}],
     Exp[I*k1Square.{x1, x2}]*α, Exp[-I*k2.{x1, x2}]*α]*
   Exp[-I*(k1Square - k2).{x1, x2}]]
vjplSquare =
 FullSimplify[-2*Inverse[Lhat[k1Square[[1]] + k2Square[[1]], k1Square[[2]] + k2[[2]]]].
    N2[Exp[I*k1Square.{x1, x2}], Exp[I*k2Square.{x1, x2}],
     Exp[I*k1Square.{x1, x2}]*α, Exp[I*k2Square.{x1, x2}]*α]*
   Exp[-I*(k1Square + k2Square).{x1, x2}]]
```

Out[129]=

$$\left\{\left\{\frac{k^2\left(48\left(g+k^2\right)-\left(27+2g+2k^2\right)M\right)+6\left(g+k^2\right)\beta}{k^2\left(g\left(-48+M\right)+4k^2\left(-48+M\right)+72M\right)-12\left(g+4k^2\right)\beta}\right\},\right.$$
$$\left.\left\{\left(k^2\left(-16\left(g+k^2\right)\left(g+4k^2\right)+\left(g\left(42+g\right)+\left(96+5g\right)k^2+4k^4\right)M-\left(27+2g+2k^2\right)M^2\right)+6\left(g+k^2\right)M\beta\right)\Big/\left(k^2M\left(g\left(-48+M\right)+4k^2\left(-48+M\right)+72M\right)-12\left(g+4k^2\right)M\beta\right)\right\}\right\}$$

Out[130]=

$$\left\{\left\{\frac{192\left(7k^2\left(24\left(g+k^2\right)-\left(15+g+k^2\right)M\right)+48\left(g+k^2\right)\beta\right)}{49k^2\left(4g\left(-48+M\right)+7k^2\left(-48+M\right)+288M\right)-1344\left(4g+7k^2\right)\beta}\right\},\right.$$
$$\left.\left\{\left(168k^2\left(-16\left(g+k^2\right)\left(4g+7k^2\right)+\left(4g\left(44+g\right)+\left(236+11g\right)k^2+7k^4\right)M-8\left(15+g+k^2\right)M^2\right)+9216\left(g+k^2\right)M\beta\right)\Big/\left(7M\left(7k^2\left(4g\left(-48+M\right)+7k^2\left(-48+M\right)+288M\right)-192\left(4g+7k^2\right)\beta\right)\right)\right\}\right\}$$

Out[131]=

$$\left\{\left\{\frac{896k^2\left(24\left(g+k^2\right)-\left(18+g+k^2\right)M\right)+6144\left(g+k^2\right)\beta}{49k^2\left(4g\left(-48+M\right)+7k^2\left(-48+M\right)+288M\right)-1344\left(4g+7k^2\right)\beta}\right\},\right.$$
$$\left.\left\{\left(112k^2\left(-16\left(g+k^2\right)\left(4g+7k^2\right)+\left(4g\left(48+g\right)+11\left(24+g\right)k^2+7k^4\right)M-8\left(18+g+k^2\right)M^2\right)+6144\left(g+k^2\right)M\beta\right)\Big/\left(7M\left(7k^2\left(4g\left(-48+M\right)+7k^2\left(-48+M\right)+288M\right)-192\left(4g+7k^2\right)\beta\right)\right)\right\}\right\}$$

In[132]:=

```
(* calculate cross coefficients from quadratic interactions *)
Γ0Square = Γ0
Γ3pSquare = FullSimplify[
   Expand[(1 / (α + alphaTilde)) * {alphaTilde, 1}.N2[Exp[-I * k2Square.{x1, x2}],
        Exp[I * (k1Square + k2Square).{x1, x2}] * η1, Exp[-I * k2Square.{x1, x2}] * α,
        Exp[I * (k1Square + k2Square).{x1, x2}] * η2] * Exp[-I * k1Square.{x1, x2}] /.
     {η1 → vjplSquare[[1]][[1]], η2 → vjplSquare[[2]][[1]]}]]
Γ3mSquare = FullSimplify[
   Expand[(1 / (α + alphaTilde)) * {alphaTilde, 1}.N2[Exp[I * k2Square.{x1, x2}],
        Exp[I * (k1Square - k2Square).{x1, x2}] * η1, Exp[I * k2Square.{x1, x2}] * α,
        Exp[I * (k1Square - k2Square).{x1, x2}] * η2] * Exp[-I * k1Square.{x1, x2}] /.
     {η1 → vjmlSquare[[1]][[1]], η2 → vjmlSquare[[2]][[1]]}]]
```

Out[132]=

$$\{0\}$$

Out[133]=

$$\Big\{\Big(14\,k^6\,\big(-96\,(g+k^2)\,\big(4\,g\,(15+g)+(123+13\,g)\,k^2+9\,k^4\big)+2\,\big(4\,g\,(1512+g\,(147+g))+(7992+g\,(1581+19\,g))\,k^2+(993+26\,g)\,k^4+11\,k^6\big)\,M-3\,\big(1728+4\,g\,(60+g)+13\,(24+g)\,k^2+9\,k^4\big)\,M^2\big)-24\,k^4\,\big(16\,(g+k^2)\,\big(8\,g\,(30+g)+3\,(167+8\,g)\,k^2+16\,k^4\big)-3\,\big(12\,g\,(116+5\,g)+(1896+187\,g)\,k^2+127\,k^4\big)\,M\big)\,\beta-27648\,k^2\,(g+k^2)\,(g+2\,k^2)\,\beta^2\Big)\Big/\Big(21\,\big(k^4+k^2\,(3+g-M)+3\,\beta\big)\,\big(7\,k^2\,(4\,g\,(-48+M)+7\,k^2\,(-48+M)+288\,M)-192\,(4\,g+7\,k^2)\,\beta\big)\Big)\Big\}$$

Out[134]=

$$\Big\{\Big(7\,k^6\,\big(-96\,(g+k^2)\,\big(4\,g\,(15+g)+(123+13\,g)\,k^2+9\,k^4\big)+2\,\big(4\,g\,(1440+g\,(147+g))+(7380+g\,(1569+19\,g))\,k^2+(981+26\,g)\,k^4+11\,k^6\big)\,M-3\,\big(1440+4\,g\,(58+g)+(292+13\,g)\,k^2+9\,k^4\big)\,M^2\big)-12\,k^4\,\big(16\,(g+k^2)\,\big(8\,g\,(30+g)+3\,(167+8\,g)\,k^2+16\,k^4\big)-3\,\big(4\,g\,(334+15\,g)+(1756+187\,g)\,k^2+127\,k^4\big)\,M\big)\,\beta-13824\,k^2\,(g+k^2)\,(g+2\,k^2)\,\beta^2\Big)\Big/\Big(7\,\big(k^4+k^2\,(3+g-M)+3\,\beta\big)\,\big(7\,k^2\,(4\,g\,(-48+M)+7\,k^2\,(-48+M)+288\,M)-192\,(4\,g+7\,k^2)\,\beta\big)\Big)\Big\}$$

In[135]:=

```
(* calculate cross coefficients from cubic interactions *)
Λ2Square = FullSimplify[
  Expand[(1 / (α + alphaTilde)) * {alphaTilde, 1}.N3[Exp[I * k1Square.{x1, x2}],
      Exp[I * k2Square.{x1, x2}], Exp[-I * k2Square.{x1, x2}],
      Exp[I * k1Square.{x1, x2}] * α, Exp[I * k2Square.{x1, x2}] * α,
      Exp[-I * k2Square.{x1, x2}] * α] * Exp[-I * k1Square.{x1, x2}]]]
```

vereinfache vollständig; multipliziere aus; Ex⋯ imaginäre Einheit I; Exp⋯ imaginäre Einheit I

Out[135]=

$$\left\{-\frac{k^2\left(g+k^2\right)\left(k^2\left(8\left(6+g+k^2\right)-7\,M\right)+48\,\beta\right)}{72\left(k^4+k^2\,(3+g-M)+3\,\beta\right)}\right\}$$

In[136]:=

```
(* cross interaction coefficient *)
K1Square = Simplify[
  Expand[((2 * Γ0Square + 2 * Γ3pSquare + 2 * Γ3mSquare + 6 * Λ2Square) /. M → Mmc[k]) /.
    k → kmc]]
```

vereinfache; multipliziere aus

Out[136]=

$$\Big\{\Big(16\,g\,(72+g-\beta)\,\beta\,\Big(8\,584\,704\,\sqrt{2}\,(-72+\beta)^6\,\beta^2\,\sqrt{g\,(72+g-\beta)\,\beta}\,+$$
$$280\,g^7\,\beta\,\big(1\,934\,917\,632+1\,209\,323\,520\,\beta+78\,382\,080\,\beta^2+1\,088\,640\,\beta^3+3240\,\beta^4+\beta^5\big)-$$
$$2\,g^6\,\Big(1\,170\,348\,\beta^6+407\,\beta^7-290\,304\,\beta^4\,\big(15\,039+25\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)-$$
$$240\,\beta^5\,\big(-1\,126\,098+35\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)-$$
$$22\,394\,880\,\beta^3\,\big(58\,956+49\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)-$$
$$1\,289\,945\,088\,\beta\,\big(-20\,682+175\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)-$$
$$107\,495\,424\,\beta^2\,\big(146\,889+350\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)\Big)+31\,104\,g\,(-72+\beta)^5\,\beta$$
$$\Big(120\,\beta^3+190\,944\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}-36\,\beta\,\big(-6768+451\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$\beta^2\,\big(-12\,024+485\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)\Big)-$$
$$6\,g^5\,(-72+\beta)\,\Big(-295\,697\,\beta^6-89\,\beta^7-180\,592\,312\,320\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}+$$
$$26\,873\,856\,\beta\,\big(-903\,672+167\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$13\,436\,928\,\beta^2\,\big(-53\,994+1087\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$134\,784\,\beta^3\,\big(2\,114\,856+4501\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$864\,\beta^4\,\big(2\,939\,076+5599\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$\beta^5\,\big(-65\,656\,152+6383\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)\Big)+$$
$$216\,g^2\,(-72+\beta)^4\,\beta\,\Big(43\,648\,\beta^4-25\,920\,\beta\,\big(-115\,344+173\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$373\,248\,\big(-50\,256+227\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)-$$
$$3\,\beta^3\,\big(686\,256+1669\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)-$$
$$24\,\beta^2\,\big(4\,831\,056+27\,745\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)\Big)+$$
$$108\,g^3\,(-72+\beta)^3\,\Big(-1012\,\beta^6-5\,518\,098\,432\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}+$$
$$\beta^5\,\big(-277\,972+19\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$2\,\beta^4\,\big(2\,808\,288+1849\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$746\,496\,\beta\,\big(-117\,360+2479\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$48\,\beta^3\,\big(23\,620\,032+15\,155\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$3456\,\beta^2\,\big(6\,197\,472+43\,909\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)\Big)+$$
$$3\,g^4\,(-72+\beta)^2\,\Big(-76\,086\,\beta^6-162\,533\,081\,088\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}+$$
$$1\,990\,656\,\beta^2\,\big(-2\,568\,861+304\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$\beta^5\,\big(-23\,199\,408+5501\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)-$$
$$8\,957\,952\,\beta\,\big(1\,482\,192+21\,533\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$24\,192\,\beta^3\,\big(2\,900\,880+21\,977\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$96\,\beta^4\,\big(22\,001\,112+52\,033\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)\Big)\Big)\Big)\Big/$$
$$\Big(63\,(-72+\beta)^4\,\big(864+12\,g-12\,\beta+\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)$$
$$\big(g\,\beta+6\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)^2$$
$$\Big(12\,\beta^4-1728\,\sqrt{2}\,\big(216+27\,g+g^2\big)\,\sqrt{g\,(72+g-\beta)\,\beta}+$$
$$\beta^3\,\big(-2592+60\,g+4\,g^2+15\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$\beta^2\,\Big(-8640\,g+504\,g^2-4\,g^3+33\,\sqrt{2}\,g\,\sqrt{g\,(72+g-\beta)\,\beta}+$$
$$72\,\big(2592-31\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)\Big)-$$
$$72\,\beta\,\Big(12\,g^3+72\,\big(864-17\,\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+$$
$$24\,g\,\big(-180+\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)+g^2\,\big(792+\sqrt{2}\,\sqrt{g\,(72+g-\beta)\,\beta}\big)\Big)\Big)\Big)\Big\}$$

In[137]:=

```
(* Write expression for K₁ to file *)
 ⌊schreibe
Export[FileNameJoin[{NotebookDirectory[], "cross-cubic-coeffs-square.txt"}],
⌊exportiere ⌊setze Dateinamen ⋯ ⌊Notebook-Verzeichnis
  {K1Square /. β → b}];
```

Plot coefficients on the square lattice

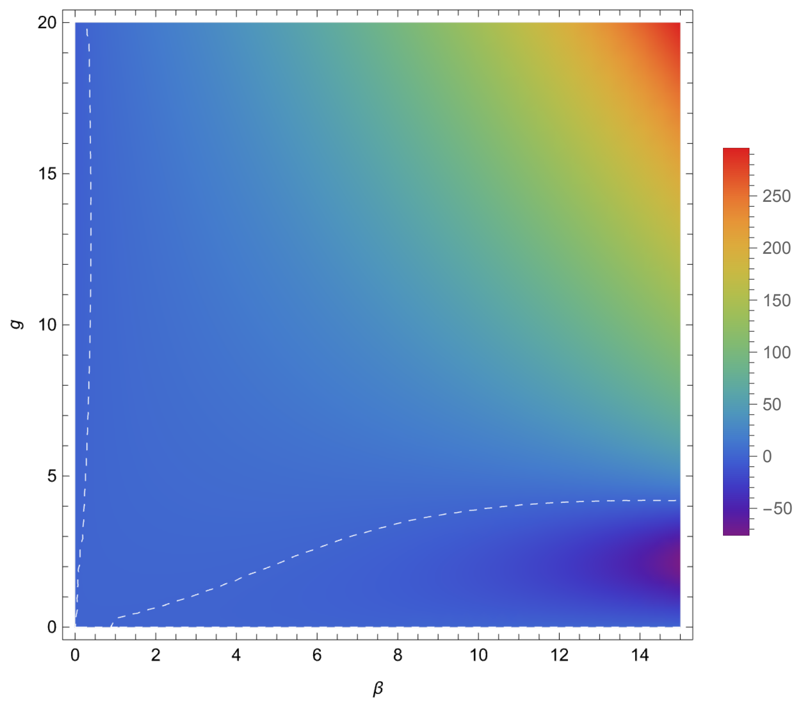
Here, we plot the coefficients K_0 and K_1 in the (β,g)-parameter plane. We then plot the regions of parameter space, where certain existence and selection criteria for roll waves and square patterns are satisfied.

Density plots of coefficients K_0 and K_1

In[138]:=

```
(* Density plot of self-interaction coefficient K₀ *)
DensityPlot[K0, {β, 0.001, 15}, {g, 0, 20}, ColorFunction → "Rainbow",
⌊graphische Darstellung der Dichte                 ⌊Farbfunktion
 PlotPoints → 35, PlotLegends → Automatic, MeshFunctions → {#3 &},
 ⌊Anzahl der Punkte in ⋯ ⌊Legenden der Gr⋯ ⌊automatisch ⌊Funktion zur Netzunterteilung
 Mesh → {{0}}, MeshStyle → {White, Dashed}, FrameLabel → {β, g}]
 ⌊Gitternetz    ⌊Netzstil    ⌊weiß   ⌊gestrichelt ⌊Rahmenbeschriftung
(* Density plot of self-interaction coefficient K₁ *)
DensityPlot[K1Square, {β, 0.001, 15}, {g, 0, 20}, ColorFunction → "Rainbow",
⌊graphische Darstellung der Dichte                       ⌊Farbfunktion
 PlotPoints → 35, PlotLegends → Automatic, MeshFunctions → {#3 &},
 ⌊Anzahl der Punkte in ⋯ ⌊Legenden der Gr⋯ ⌊automatisch ⌊Funktion zur Netzunterteilung
 Mesh → {{0}}, MeshStyle → {White, Dashed}, FrameLabel → {β, g}]
 ⌊Gitternetz    ⌊Netzstil    ⌊weiß   ⌊gestrichelt ⌊Rahmenbeschriftung
```

Out[138]=



Out[139]=