

# DIFERENCIAS FINITAS

Buscamos aproximar los operadores diferenciales presentes en las ecuaciones por **operadores en diferencias**, para transformar una ecuación diferencial en un sistema de ecuaciones algebraicas donde las incógnitas son nodos del dominio en los cuales obtendremos la solución.

$$\rho c_p \frac{\partial \phi}{\partial t} = k \frac{\partial^2 \phi}{\partial x^2} - c\phi + G(x) \rightarrow \mathbf{K}\phi = \mathbf{F}$$

Series de Taylor:  $\phi_{i+k} = \phi_i + \sum_{m=1}^{\infty} \frac{(k\Delta x)^m}{m!} \frac{\partial^m \phi}{\partial x^m} \Big|_i$

Ejemplos:

$$\phi_{i+1} = \phi_i + \Delta x \frac{\partial \phi}{\partial x} \Big|_i + \frac{(\Delta x)^2}{2} \frac{\partial^2 \phi}{\partial x^2} \Big|_i + \frac{(\Delta x)^3}{3!} \frac{\partial^3 \phi}{\partial x^3} \Big|_i + \frac{(\Delta x)^4}{4!} \frac{\partial^4 \phi}{\partial x^4} \Big|_i + \dots$$

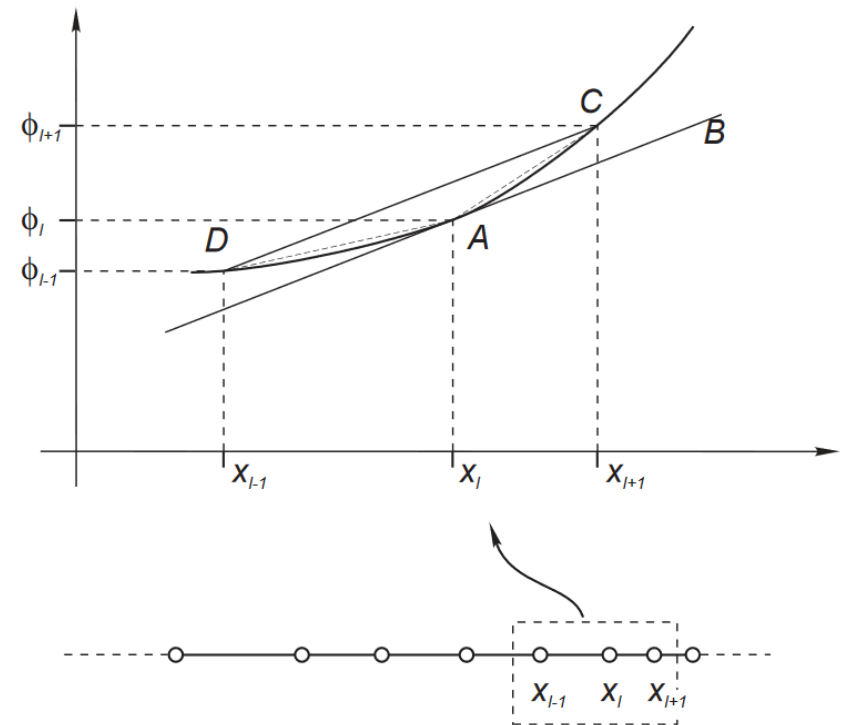
$$\phi_{i-1} = \phi_i - \Delta x \frac{\partial \phi}{\partial x} \Big|_i + \frac{(\Delta x)^2}{2} \frac{\partial^2 \phi}{\partial x^2} \Big|_i - \frac{(\Delta x)^3}{3!} \frac{\partial^3 \phi}{\partial x^3} \Big|_i + \frac{(\Delta x)^4}{4!} \frac{\partial^4 \phi}{\partial x^4} \Big|_i + \dots$$

$$\phi_{i+2} = \phi_i + 2\Delta x \frac{\partial \phi}{\partial x} \Big|_i + \frac{(2\Delta x)^2}{2} \frac{\partial^2 \phi}{\partial x^2} \Big|_i + \frac{(2\Delta x)^3}{3!} \frac{\partial^3 \phi}{\partial x^3} \Big|_i + \frac{(2\Delta x)^4}{4!} \frac{\partial^4 \phi}{\partial x^4} \Big|_i + \dots$$

## DIFERENCIAS FINITAS

Resumen de las aproximaciones más comunes a utilizar:

Operador	Orden del error
$\left. \frac{\partial \phi}{\partial x} \right _i \approx \frac{\phi_i - \phi_{i-1}}{\Delta x}$	$o(\Delta x)$
$\left. \frac{\partial \phi}{\partial x} \right _i \approx \frac{\phi_{i+1} - \phi_i}{\Delta x}$	$o(\Delta x)$
$\left. \frac{\partial \phi}{\partial x} \right _i \approx \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}$	$o(\Delta x^2)$
$\left. \frac{\partial^2 \phi}{\partial x^2} \right _i \approx \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2}$	$o(\Delta x^2)$



Relación entre puntos usados, orden de la derivada, precisión del error:  **$N \geq k + p$**

En aproximaciones centradas y mallas uniformes ganamos un orden de aproximación...

## DIFERENCIAS FINITAS

Retomando la ecuación diferencial:

$$\rho c_p \frac{\partial \phi}{\partial t} = k \frac{\partial^2 \phi}{\partial x^2} - c\phi + G(x)$$

Y sus posibles condiciones de borde: (sumado a una condición inicial)

$$\phi = \bar{\phi}, \quad -k \frac{\partial \phi}{\partial \eta} = q, \quad -k \frac{\partial \phi}{\partial \eta} = h(\phi - \phi_\infty)$$

Discretización completa (mallas uniformes):

$$\rho c_p \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = k \left( \frac{\phi_{i-1}^{n+\theta} - 2\phi_i^{n+\theta} + \phi_{i+1}^{n+\theta}}{\Delta x^2} \right) - c\phi_i^{n+\theta} + G_i$$

$$-k \left( \frac{\phi_{i+1}^{n+\theta} - \phi_{i-1}^{n+\theta}}{2\Delta x} \right) (\pm 1) = q, \quad -k \left( \frac{\phi_{i+1}^{n+\theta} - \phi_{i-1}^{n+\theta}}{2\Delta x} \right) (\pm 1) = h(\phi_i^{n+\theta} - \phi_\infty)$$

## DIFERENCIAS FINITAS

Ejercicio 1a GTP:  $\rho c_p = 0$ ;  $k = 2$ ;  $c = 0$ ;  $G(x) = 100$

$$2 \frac{\partial^2 \phi}{\partial x^2} + 100 = 0; \quad \forall x[0,1]$$

$$\phi(0) = 10; \phi(1) = 50$$

Solución analítica:  $\phi(x) = -25x^2 + 65x + 10$

Discretización propuesta:  $\Delta x = 1/3$



$$2 \left( \frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{\Delta x^2} \right) + 100 = 0$$

Incógnitas del problema?  $\phi_1$  y  $\phi_2$

Siempre buscaremos armar un sistema de N ecuaciones con N incógnitas...

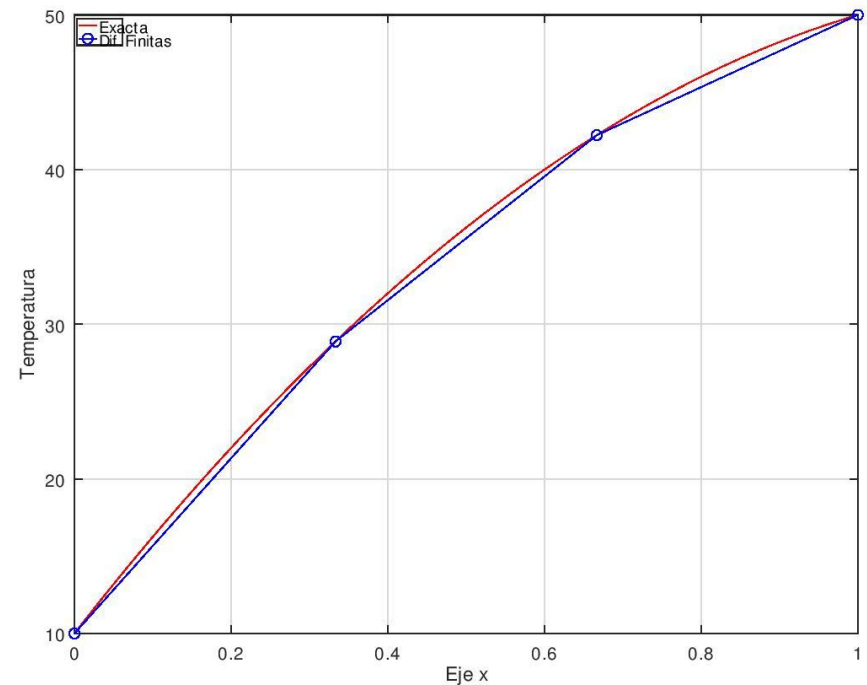
## DIFERENCIAS FINITAS

Sistema de ecuaciones	Sistema matricial	Sistema matricial ampliado
$-2\phi_1 + \phi_2 = -50\Delta x^2 - \phi_0$ $\phi_1 - 2\phi_2 = -50\Delta x^2 - \phi_3$	$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -140/9 \\ -500/9 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 10 \\ -50/9 \\ -50/9 \\ 50 \end{pmatrix}$

$$\phi = K^{-1}F \rightarrow \phi = [10 \quad 28.89 \quad 42.22 \quad 50]$$

Generalizando **K** para n nodos interiores:

1	0	0	0	0	0
1	-2	1	0	0	0
0	1	-2	1	0	0
0	0	1	-2	1	0
0	0	0	1	-2	1
0	0	0	0	0	1



## DIFERENCIAS FINITAS

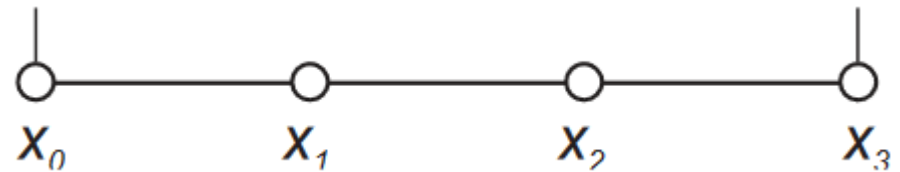
Ejercicio 1c GTP:  $\rho c_p = 0$ ;  $k = 1$ ;  $c = 0$ ;  $G(x) = 100(x - 3)^2$

$$\frac{\partial^2 \phi}{\partial x^2} + 100(x - 3)^2 = 0; \quad \forall x[1,5]$$

$$q(1) = 2; \quad \phi(5) = 0$$

Solución analítica:  $\phi(x) = \frac{-25x^4 + 300x^3 - 1350x^2 + 1906x + 2345}{3}$

Discretización propuesta:  $\Delta x = \frac{L}{N-1} = \frac{4}{3}$



$$(\phi_{i-1} - 2\phi_i + \phi_{i+1}) = -100\Delta x^2(x_i - 3)^2$$

Incógnitas del problema?  $\phi_0$ ,  $\phi_1$  y  $\phi_2$

## DIFERENCIAS FINITAS

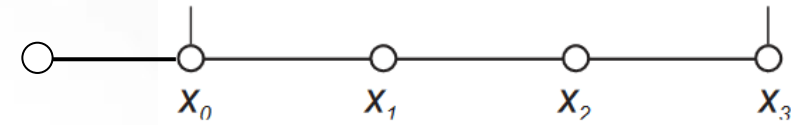
Planteo de condiciones de borde:

- Sin nodo ficticio  $o(\Delta x)$

$$q(1) = 2 \rightarrow -k \frac{\partial \phi}{\partial x}(-1) = 2 \rightarrow k \left( \frac{\phi_1 - \phi_0}{\Delta x} \right) = 2 \quad (\text{pensar el extremo } x=5)$$

- Con nodo ficticio  $o(\Delta x^2)$

$$q(1) = 2 \rightarrow -k \frac{\partial \phi}{\partial x}(-1) = 2 \rightarrow k \left( \frac{\phi_1 - \phi_f}{2\Delta x} \right) = 2 \rightarrow \phi_f = \phi_1 - \frac{4\Delta x}{k}$$

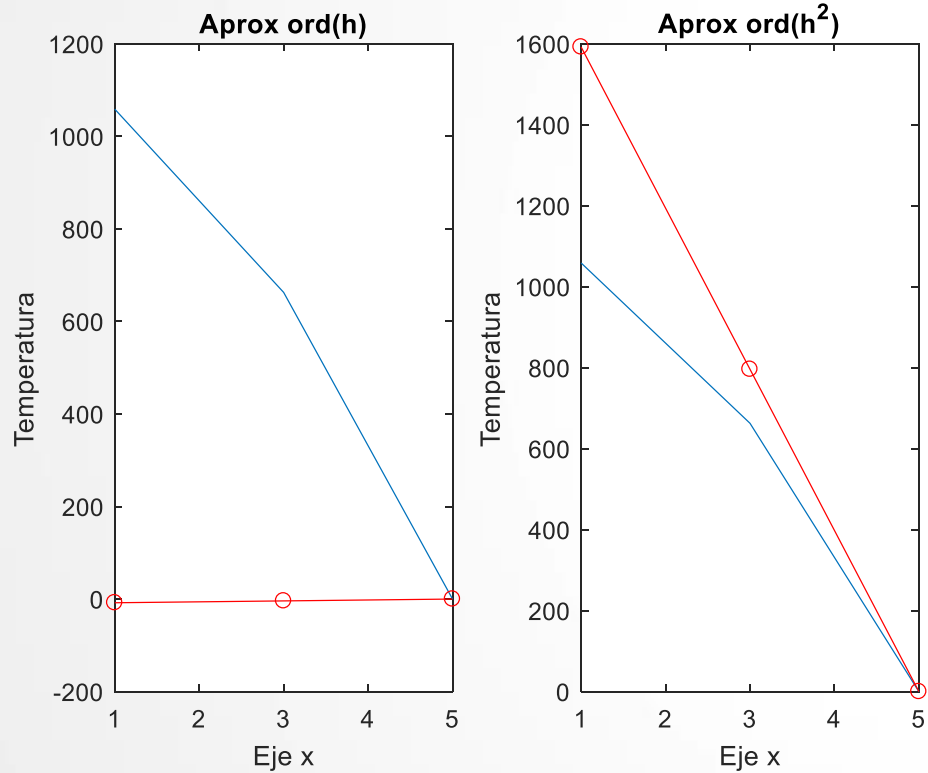


Sistema matricial orden $\Delta x$	Sistema matricial orden $\Delta x^2$
$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 8/3 \\ -79.0123 \\ -79.0123 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} -711.11 + 16/3 \\ -79.0123 \\ -79.0123 \\ 0 \end{pmatrix}$

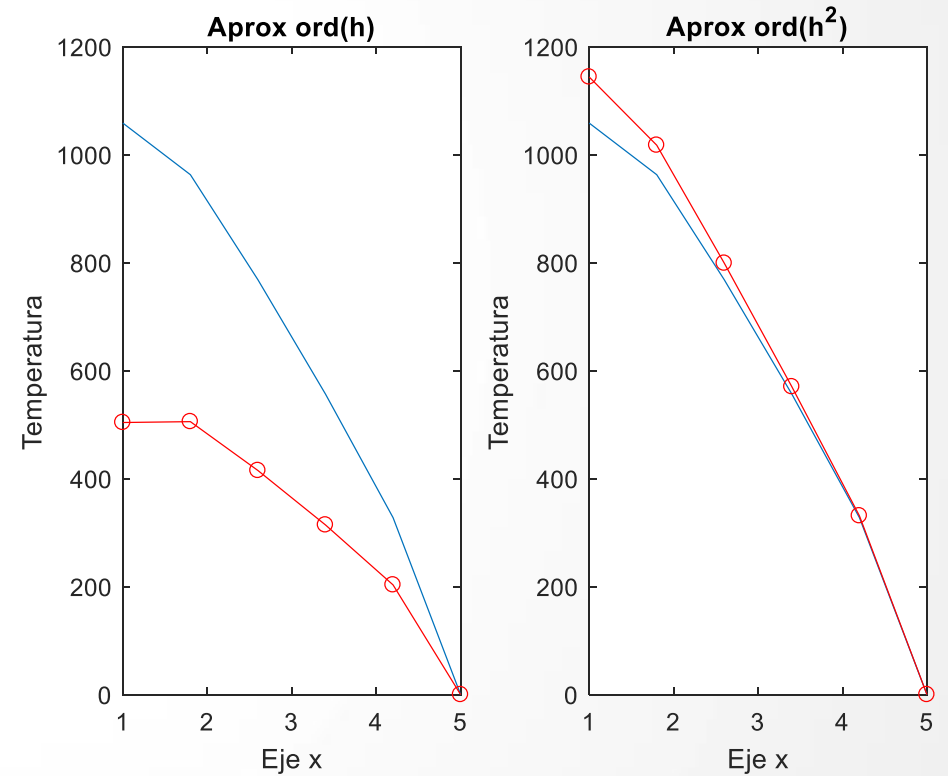
Fila 1: en orden  $\Delta x$  utilizo CB, mientras que en  $\Delta x^2$  escribo la ecuación diferencial y despejo el nodo ficticio de la CB.

## DIFERENCIAS FINITAS

Cant. Puntos: 3



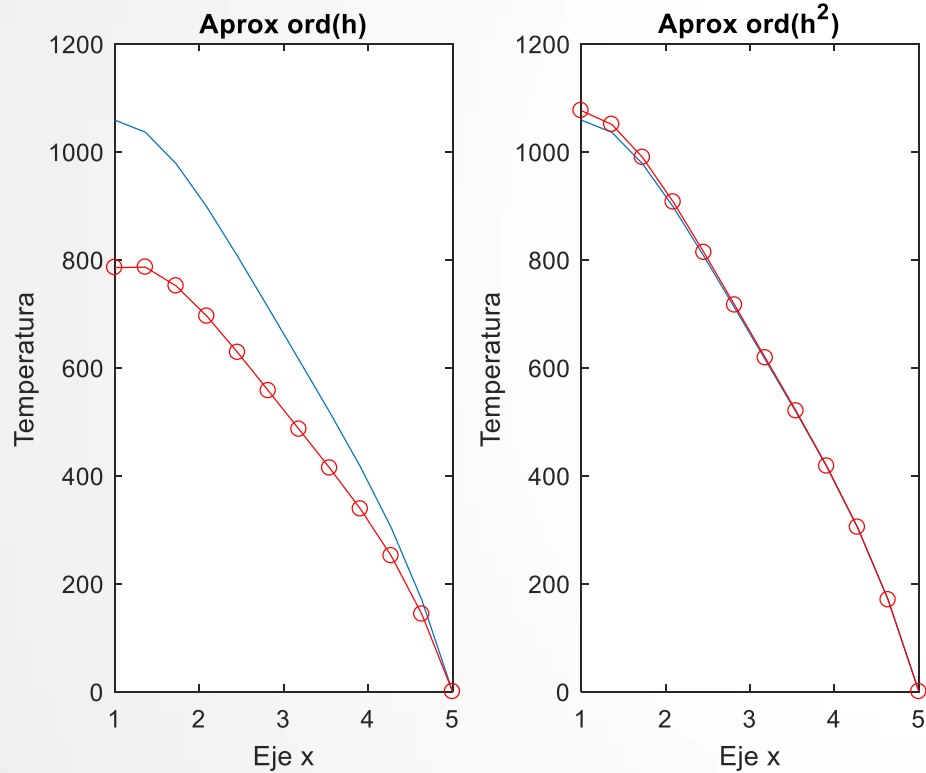
Cant. Puntos: 6



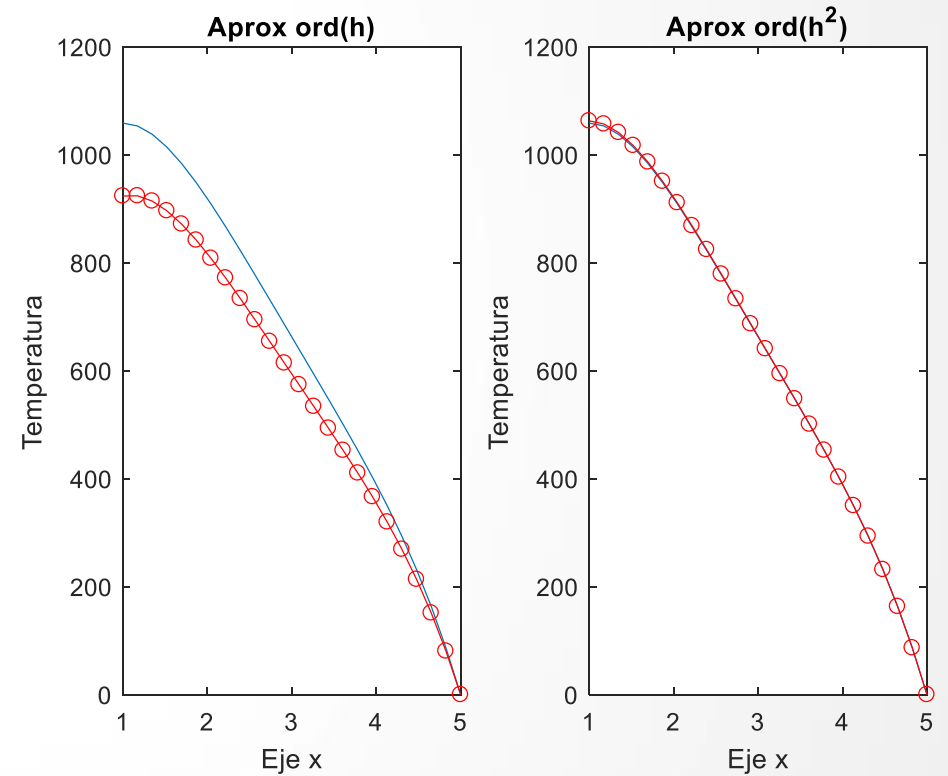


## DIFERENCIAS FINITAS

Cant. Puntos: 12

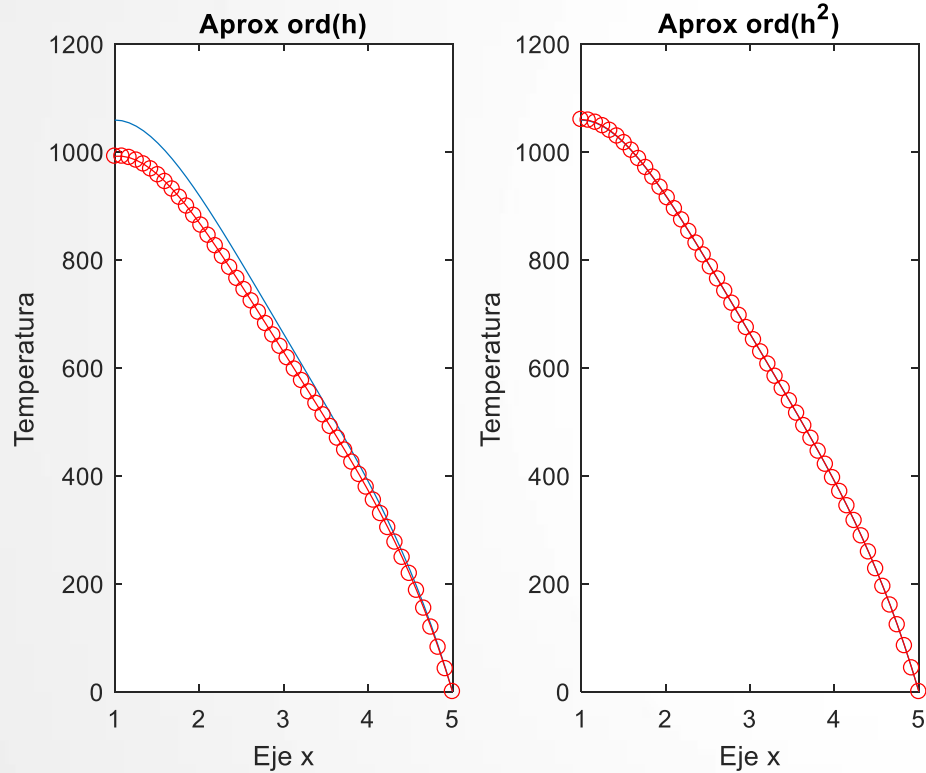


Cant. Puntos: 24

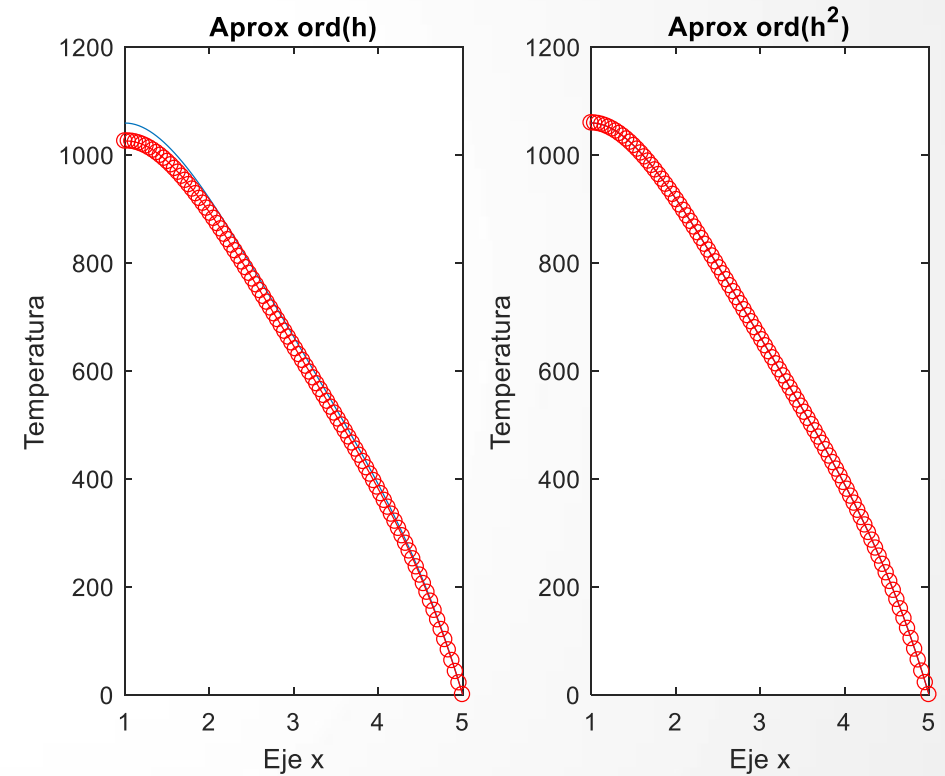


## DIFERENCIAS FINITAS

Cant. Puntos: 48



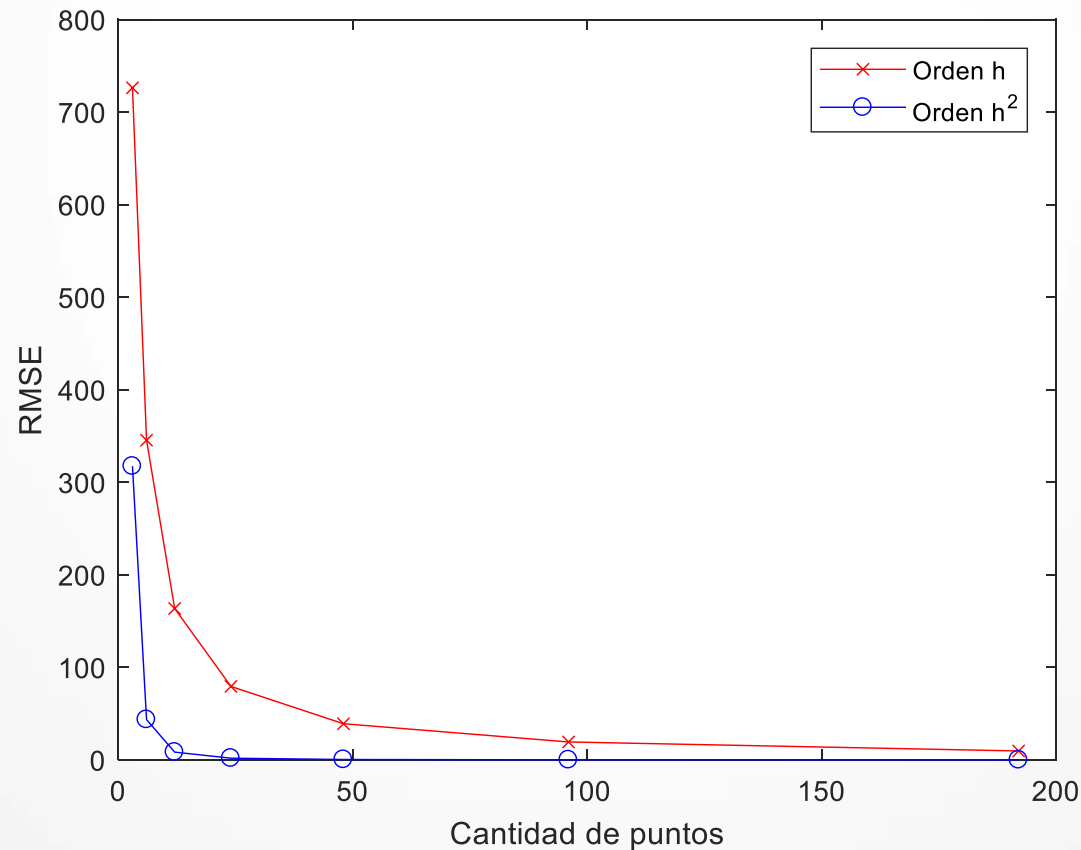
Cant. Puntos: 96



## DIFERENCIAS FINITAS

Medimos el RMSE (raíz del error cuadrático medio) para cada aproximación:

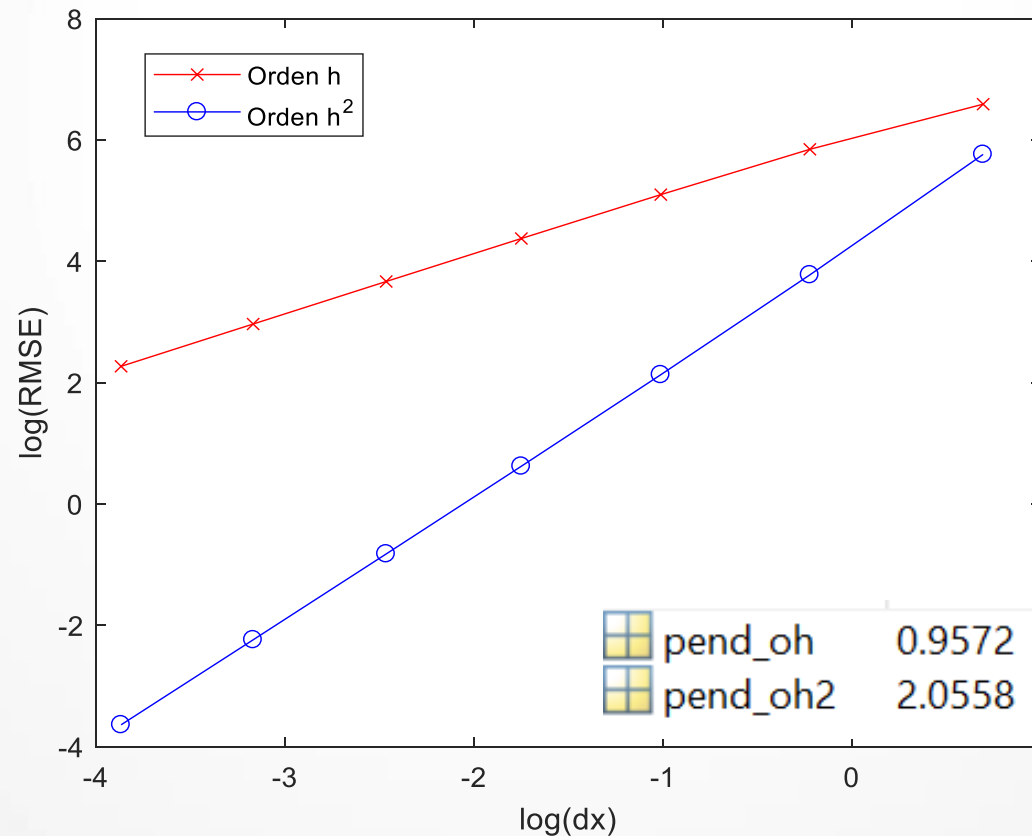
$$Err_{oh} = \sqrt{\frac{\sum_{i=1}^N (\phi ex_i - Qap_{oh}_i)^2}{N}}; Err_{oh2} = \sqrt{\frac{\sum_{i=1}^N (\phi ex_i - Qap_{oh2}_i)^2}{N}}$$



## DIFERENCIAS FINITAS

Sabemos que el error en la aproximación se comporta como:  $E \approx C\Delta x^p$

Corroboramos la convergencia analizando:  $\log E = \log C + p \log \Delta x$ , cuya pendiente  $p$  determina si la solución converge de forma lineal o cuadrática.



## DIFERENCIAS FINITAS

```

1 function [T_ap] = solOrden1(h)
2     k=1;
3     x = 1:h:5;
4     n = length(x);
5     G = 100*((x(1:n-1)-3).^2);
6     K = zeros(n);
7     b = zeros(n,1);
8
9     fila = [1 -2 1];
10    for i=2:(n-1)
11        K(i,i-1:i+1) = fila;
12    end
13    K(1,1:2) = [-1 1];
14    K(n,n) = 1;
15
16    b(1:n-1) = (-G./k)*(h^2);
17    b(1) = 2*h;
18
19    T_ap = K\b;
20 end

```

```

1 function [T_ap] = solOrden2(h)
2     k=1;
3     x = 1:h:5;
4     n = length(x);
5     G = 100*((x(1:n-1)-3).^2);
6     K = zeros(n);
7     b = zeros(n,1);
8
9     fila = [1 -2 1];
10    for i=2:(n-1)
11        K(i,i-1:i+1) = fila;
12    end
13    K(1,1:2) = [-2 2];
14    K(n,n) = 1;
15
16    b(1:n-1) = (-G./k)*(h^2);
17    b(1) = b(1) + 4*h;
18
19    T_ap = K\b;
20 end

```

## DIFERENCIAS FINITAS

Generalizando esquemas numéricos estacionarios (mallas uniformes):

$$k \frac{\partial^2 \phi}{\partial x^2} - c\phi + G(x) = 0 \rightarrow -\phi_{i-1} + \left(2 + \frac{\Delta x^2 c}{k}\right) \phi_i - \phi_{i+1} = \frac{\Delta x^2 G_i}{k}$$

Condición de borde Neumann:  $-k \frac{\partial \phi}{\partial \eta} = q$

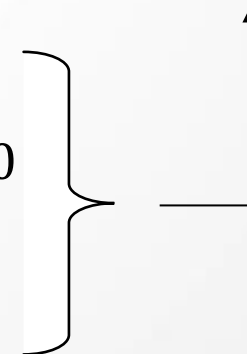
- Extremo izquierdo:  $-k \left( \frac{\phi_{int} - \phi_{fic}}{2\Delta x} \right) (-1) = q$
- Extremo derecho:  $-k \left( \frac{\phi_{fic} - \phi_{int}}{2\Delta x} \right) (1) = q$

$$\left. \begin{array}{l} \text{Extremo izquierdo: } -k \left( \frac{\phi_{int} - \phi_{fic}}{2\Delta x} \right) (-1) = q \\ \text{Extremo derecho: } -k \left( \frac{\phi_{fic} - \phi_{int}}{2\Delta x} \right) (1) = q \end{array} \right\} \rightarrow \boxed{\phi_{fic} = \phi_{int} - \frac{2\Delta x q}{k}}$$

$$\boxed{\phi_{fic} = \phi_{int} - \frac{2\Delta x h}{k} \phi_b + \frac{2\Delta x h}{k} \phi_{inf}}$$

Condición de borde Robin:  $k \frac{\partial \phi}{\partial \eta} + h(\phi_b - \phi_{inf}) = 0$

- Extremo izquierdo:  $k \left( \frac{\phi_{int} - \phi_{fic}}{2\Delta x} \right) (-1) + h(\phi_b - \phi_{inf}) = 0$
- Extremo derecho:  $k \left( \frac{\phi_{fic} - \phi_{int}}{2\Delta x} \right) (1) + h(\phi_b - \phi_{inf}) = 0$



## DIFERENCIAS FINITAS

Entonces para nodos interiores la ecuación a escribir es:

$$-\phi_{i-1} + \left(2 + \frac{\Delta x^2 c}{k}\right) \phi_i - \phi_{i+1} = \frac{\Delta x^2 G_i}{k}$$

Para nodos en el extremo izquierdo:  $-\phi_{fic} + \left(2 + \frac{\Delta x^2 c}{k}\right) \phi_i - \phi_{i+1} = \frac{\Delta x^2 G_i}{k}$

Para nodos en el extremo derecho:  $-\phi_{i-1} + \left(2 + \frac{\Delta x^2 c}{k}\right) \phi_i - \phi_{fic} = \frac{\Delta x^2 G_i}{k}$

Condición de borde Neumann:  $\phi_{fic} = \phi_{int} - \frac{2\Delta x q}{k}$

Condición de borde Robin:  $\phi_{fic} = \phi_{int} - \frac{2\Delta x h}{k} \phi_b + \frac{2\Delta x h}{k} \phi_{inf}$

**Despejar stencil final reemplazando nodo ficticio en la ecuación**

## DIFERENCIAS FINITAS

Generalizando esquemas numéricos no estacionarios (mallas uniformes):

$$\rho c_p \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = k \left( \frac{\phi_{i-1}^{n+\theta} - 2\phi_i^{n+\theta} + \phi_{i+1}^{n+\theta}}{\Delta x^2} \right) - c\phi_i^{n+\theta} + G_i$$

### Esquemas temporales:

- $\theta = 0$ , Forward Euler, esquema explícito de orden  $\Delta t$
- $\theta = 1$ , Backward Euler, esquema implícito de orden  $\Delta t$
- $\theta = 1/2$ , Crank-Nicholson, esquema implícito de orden  $\Delta t^2$



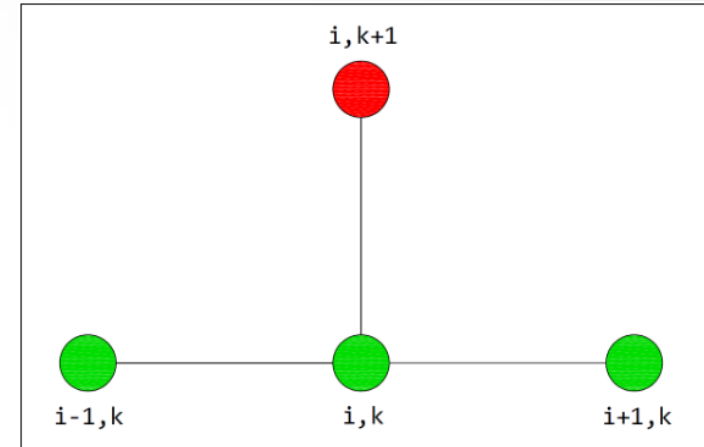
## DIFERENCIAS FINITAS

**Forward Euler:** condicionalmente estable

$$\rho c_p \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = k \left( \frac{\phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n}{\Delta x^2} \right) - c\phi_i^n + G_i$$

$$\phi_i^{n+1} = \frac{\Delta t}{\rho c_p} \left( \frac{k}{\Delta x^2} (\phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n) - c\phi_i^n + G_i \right) + \phi_i^n$$

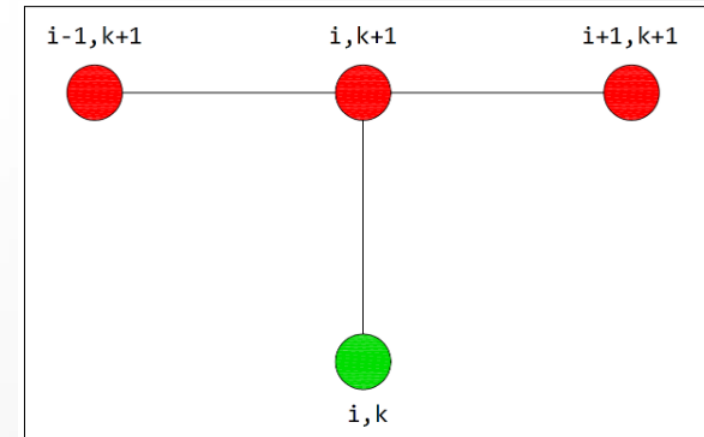
Condición de estabilidad:  $Fo = \frac{\alpha \Delta t}{\Delta x^2} \leq 0.5 \rightarrow \Delta t \leq \frac{0.5 \Delta x^2}{\alpha}$



**Backward Euler:** incondicionalmente estable

$$\rho c_p \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = k \left( \frac{\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1}}{\Delta x^2} \right) - c\phi_i^{n+1} + G_i$$

$$-\frac{k}{\Delta x^2} \phi_{i-1}^{n+1} + \left( \frac{2k}{\Delta x^2} + c + \frac{\rho c_p}{\Delta t} \right) \phi_i^{n+1} - \frac{k}{\Delta x^2} \phi_{i+1}^{n+1} = G_i + \frac{\rho c_p}{\Delta t} \phi_i^n$$

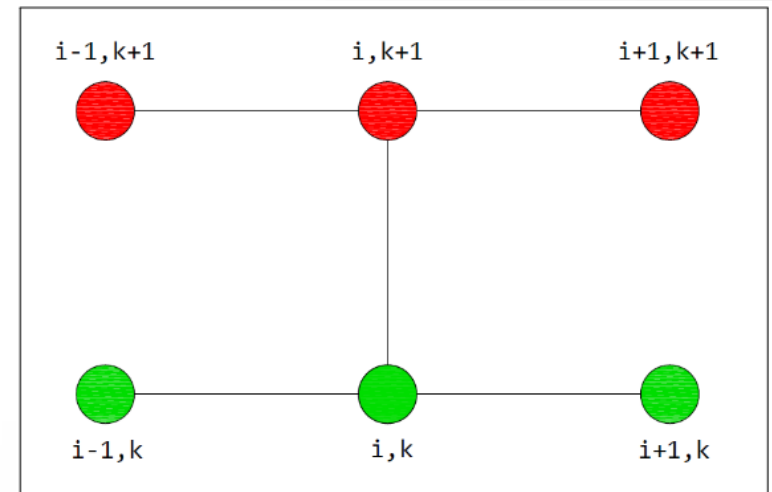


## DIFERENCIAS FINITAS

**Crank-Nicholson:** incondicionalmente estable (oscilaciones espurias)

$$\rho c_p \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \frac{1}{2} \left( k \left( \frac{\phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n}{\Delta x^2} \right) - c\phi_i^n + G_i + k \left( \frac{\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1}}{\Delta x^2} \right) - c\phi_i^{n+1} + G_i \right)$$

Desarrollo...



## DIFERENCIAS FINITAS

**Buenas noticias!**

Discretización nodo a nodo:

$$\rho c_p \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = k \left( \frac{\phi_{i-1}^{n+\theta} - 2\phi_i^{n+\theta} + \phi_{i+1}^{n+\theta}}{\Delta x^2} \right) - c\phi_i^{n+\theta} + G_i$$



Planteo vectorial:

$$\frac{\rho c_p}{\Delta t} (\bar{\phi}^{n+1} - \bar{\phi}^n) = \bar{K} \bar{\phi}^{n+\theta} + \bar{G}$$

Forward Euler

$$\bar{\phi}^{n+1} = \frac{\Delta t}{\rho c_p} (\bar{K} \bar{\phi}^n + \bar{G}) + \bar{\phi}^n$$

Backward Euler

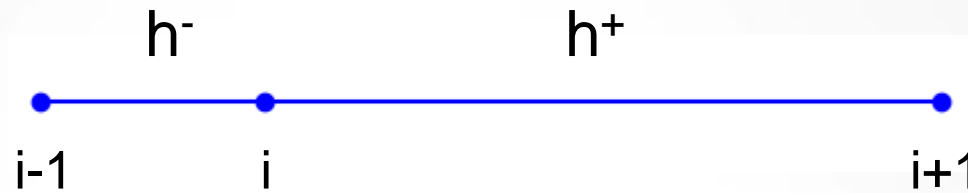
$$\left( \frac{\rho c_p}{\Delta t} \bar{I} - \bar{K} \right) \bar{\phi}^{n+1} = \bar{G} + \frac{\rho c_p}{\Delta t} \bar{\phi}^n$$

Crank-Nicholson

$$\left( \frac{\rho c_p}{\Delta t} \bar{I} - \frac{1}{2} \bar{K} \right) \bar{\phi}^{n+1} = \left( \frac{1}{2} \bar{K} + \frac{\rho c_p}{\Delta t} \bar{I} \right) \bar{\phi}^n + \bar{G}$$

## DIFERENCIAS FINITAS

Mallas no uniformes:



$$\phi_{i+1} = \phi_i + h^+ \frac{\partial \phi}{\partial x} \Big|_i + \frac{(h^+)^2}{2} \frac{\partial^2 \phi}{\partial x^2} \Big|_i + \frac{(h^+)^3}{3!} \frac{\partial^3 \phi}{\partial x^3} \Big|_i + \frac{(h^+)^4}{4!} \frac{\partial^4 \phi}{\partial x^4} \Big|_i + \dots$$

$$\phi_{i-1} = \phi_i - h^- \frac{\partial \phi}{\partial x} \Big|_i + \frac{(h^-)^2}{2} \frac{\partial^2 \phi}{\partial x^2} \Big|_i - \frac{(h^-)^3}{3!} \frac{\partial^3 \phi}{\partial x^3} \Big|_i + \frac{(h^-)^4}{4!} \frac{\partial^4 \phi}{\partial x^4} \Big|_i + \dots$$

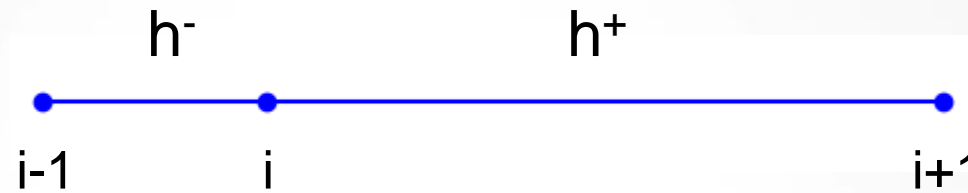
$$\frac{\partial^2 \phi}{\partial x^2} \Big|_i = a\phi_{i+1} + b\phi_i + c\phi_{i-1} = a \left( \phi_i + h^+ \frac{\partial \phi}{\partial x} \Big|_i + \frac{(h^+)^2}{2} \frac{\partial^2 \phi}{\partial x^2} \Big|_i \right) + b\phi_i + c \left( \phi_i - h^- \frac{\partial \phi}{\partial x} \Big|_i + \frac{(h^-)^2}{2} \frac{\partial^2 \phi}{\partial x^2} \Big|_i \right)$$

$$\frac{\partial^2 \phi}{\partial x^2} \Big|_i = (a + b + c)\phi_i + (ah^+ - ch^-) \frac{\partial \phi}{\partial x} \Big|_i + \left( a \frac{(h^+)^2}{2} + c \frac{(h^-)^2}{2} \right) \frac{\partial^2 \phi}{\partial x^2} \Big|_i$$

3 ecuaciones con 3 incógnitas...

## DIFERENCIAS FINITAS

Mallas no uniformes:



$$\begin{pmatrix} 1 & 1 & 1 \\ h^+ & 0 & -h^- \\ (h^+)^2/2 & 0 & (h^+)^2/2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} a &= \frac{2}{h^+(h^++h^-)} \\ b &= \frac{-2}{h^+h^-} \\ c &= \frac{2}{h^-(h^++h^-)} \end{aligned}$$

$$\left. \frac{\partial^2 \phi}{\partial x^2} \right|_i = \frac{2\phi_{i+1}}{h^+(h^++h^-)} - \frac{2\phi_i}{h^+h^-} + \frac{2\phi_{i-1}}{h^-(h^++h^-)}$$

Si  $h^+=h^-$  entonces recuperamos el stencil clásico

# DIFERENCIAS FINITAS

Manipulación de las series de Taylor: podemos aproximar cualquier operador diferencial con cualquier combinación de puntos, equiespaciados o no...

$$\left. \frac{\partial^3 \phi}{\partial x^3} \right|_i = a\phi_{i+3} + b\phi_{i+1} + c\phi_i + d\phi_{i-2} + e\phi_{i-4} \rightarrow n \geq k + p$$

$$\begin{bmatrix} [ & 1, & 1, & 1, & 1, & 1] \\ [ & 3*h, & h, & 0, & -2*h, & -4*h] \\ [ & (9*h^2)/2, & h^2/2, & 0, & 2*h^2, & 8*h^2] \\ [ & (9*h^3)/2, & h^3/6, & 0, & -(4*h^3)/3, & -(32*h^3)/3] \\ [ & (27*h^4)/8, & h^4/24, & 0, & (2*h^4)/3, & (32*h^4)/3] \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left. \frac{\partial^3 \phi}{\partial x^3} \right|_i = \frac{1}{7h^3} \phi_{i+3} - \frac{3}{5h^3} \phi_{i+1} + \frac{1}{2h^3} \phi_i + 0\phi_{i-2} - \frac{3}{70h^3} \phi_{i-4}$$