7. Simultaneous-Move Games (Mixed Strategies)

Three Rounds of Five Games of Rock-Paper-Scissors

Play 5 games of RPS against a player who decides their strategy based on a roll of the die: 1,2 ~ R, 3,4 ~ P, 5,6 ~ S.
I choose: PSPRS.
Opponent rolls: SRPSR.
1 Win, 3 Losses, 1 Draw

Play another 5 games of RPS but opponent: 1,2,3 ~ R, 4,5 ~ P, 6 ~ S.
I choose: PPPPP.
Opponent rolls: RRSRR.
4 Wins, 1 Loss, 0 Draw

Play another 5 games of RPS but opponent: 1,2,3,4,5 ~ R, 6 ~ P.
I choose: PPPPP.
Opponent rolls: RRPRR.
4 Wins, 0 Losses, 1 Draw

Notice that, when the outcome is even slightly in favor of one strategy over the others (here it is P), I went all in on that strategy, which produced good results (4 Wins in last two rounds).

Soccer Example

Rows are kicker, columns are goalie.

$$\begin{array}{c} \\ L \\ R \end{array} \begin{bmatrix} L & R \\ -1, 1 & 1, -1 \\ 1, -1 & -1, 1 \end{bmatrix}$$

There are no dominant or dominated strategies, nor pure Nash Equilibria.

So, we need to find **Mixed Strategy Nash Equilibria**:
Let the probability that kicker goes left be P, right be 1 - P.
Let the probability that goalies goes left be Q, right be 1 - Q.
Expected payoffs for kicker:
L: (-1)(Q)+(1)(1 - Q) = 1 - 2Q
R: (1)(Q)+(-1)(1 - Q) = 2Q - 1
Thus:
If 1 - 2Q > 2Q - 1 -> L.
If 2Q - 1 > 1 - 2Q -> R.

1 - 2Q = 2Q - 1 -> **Q = .5, 1 - Q = .5**.

Furthermore, **P = .5, 1 - P = .5**.

These are our Mixed Strategy Nash Equilibria.

Football Example

Rows are offense, columns are defense.

$$\begin{bmatrix} & R & P \\ R & -5,5 & 7,-7 \\ P & 18,-18 & 0,0 \end{bmatrix}$$

Let the probability that offense runs be P, pass be 1 - P.

Let the probability that defense defends run be Q, defends pass 1 - Q.

Expected payoffs for offense:

R: (-5)(Q)+(7)(1 - Q) = 7 - 12Q

P: 18(Q)+0(1 - Q) = 18Q

Thus:

7 - 12Q = 18Q -> Q = $\frac{7}{30}$, 1 - Q = $\frac{23}{30}$.

Expected payoffs for defense:

R: (5)(P)+(-18)(1 - P) = 23P - 18

P: (-7)(P)+(0)(1 - P) = -7P

Thus:

23P - 18 = -7P -> P = $\frac{18}{30}$, 1 - P = $\frac{12}{30}$.

Revisiting Battle of Sexes (Non-Zero-Sum)

We know the Pure Nash Equilibria are (Boxing, Boxing), (Ballet, Ballet).

Let the probability that husband does boxing be P, ballet 1 - P.

Let the probability that wife does boxing be Q, ballet 1 - Q.

Expected payoffs for wife:

Boxing: (1)(P)+(0)(1 - P) = P

Ballet: (0)(P)+(2)(1 - P) = 2 - 2P

Thus:

P = 2 - 2P -> P = $\frac{2}{3}$, 1 - P = $\frac{1}{3}$.

Expected payoffs for husband:

Boxing: (2)(Q)+(0)(1 - Q) = 2Q

Ballet: (0)(Q)+(1)(1 - Q) = 1 - Q

Thus:

2Q = 1 - Q -> Q = $\frac{1}{3}$, 1 - Q = $\frac{2}{3}$.

Then, husband's expected payoff is:

(2)($\frac{2}{3}$)($\frac{1}{3}$)+(0)($\frac{2}{3}$)($\frac{2}{3}$)+(0)($\frac{1}{3}$)($\frac{1}{3}$)+(1)($\frac{1}{3}$)($\frac{2}{3}$) = $\frac{6}{9}$.

Similarly, wife's expected payoff is $\frac{2}{3}$.

Revisiting Prisoner's Dilemma

$$\begin{bmatrix} & C & D \\ C & -5,-5 & 0,-10 \\ D & -10,0 & -1,-1 \end{bmatrix}$$

Expected payoffs for A:

C: (-5)(Q)+(0)(1 - Q) = -5Q

D: (-10)(Q)+(-1)(1 - Q) = -9Q - 1

Thus:

-5Q = -9Q - 1 -> Q = $\frac{-1}{4}$, 1 - Q = $\frac{5}{4}$.

These probabilities do not make much sense.

Instead, this game has dominant strategies that lead to the solution (C, C), so we do not calculate mixed strategies like this.

Note Check for dominant/dominated strategies first!

3x3 Example

$$\begin{bmatrix} & L & C & R \\ T & 0,0 & 2,1 & 1,2 \\ M & 1,2 & 0,0 & 2,1 \\ B & 2,1 & 1,2 & 0,0 \end{bmatrix}$$

Let the probability that A chooses T be P_1, M be P_2, B be $1 - P_1 - P_2$.

Let the probability that B chooses L be Q_1, C be Q_2, R be $1 - Q_1 - Q_2$.

Expected payoffs for A:

T: (0)(Q_1)+(2)(Q_2)+(1)($1 - Q_1 - Q_2$) = $Q_2 - Q_1 + 1$

M: (1)(Q_1)+(0)(Q_2)+(2)($1 - Q_1 - Q_2$) = 2 - Q_1 - 2Q_2

B: (2)(Q_1)+(1)(Q_2)+(0)($1 - Q_1 - Q_2$) = 2Q_1 + Q_2

Thus:

$Q_2 - Q_1 + 1 = 2 - Q_1 - 2Q_2$ -> $Q_2 = \frac{1}{3}$.

$2Q_1 + Q_2 = Q_2 - Q_1 + 1$ -> $Q_1 = \frac{1}{3}$.

And, $1 - Q_1 - Q_2 = \frac{1}{3}$.

Similarly, $P_1 = P_2 = 1 - P_1 - P_2 = \frac{1}{3}$.

These are the **fully mixed** NE.

We can also find **partially mixed** NE.

Let the probability that A chooses T be P, M be 1 - P, B be 0.

Expected payoffs for B:

L: (0)(P)+(2)(1 - P) = 2 - 2P

C: (1)(P)+(0)(1 - P) = P

R: (2)(P)+(1)(1 - P) = P + 1

Thus:

B will never choose C since P < P + 1.

This reduces the game:

$$\begin{array}{c} \\ T \\ M \end{array}\begin{bmatrix} & L & R \\ & 0,0 & 1,2 \\ & 1,2 & 2,1 \end{bmatrix}.$$

Then, the NE for this partial mixture is (M, L).

<u>4x4 Example</u>

If there are no dominant or dominated strategies, we consider:

16 Possible Pure NE,

1 Fully Mixed NE,

4 Partially Mixed NE (3 included),

$\binom{4}{2} = 6$ Partially Mixed NE (2 included).