13 函数级数

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13.2 习题8.4解答

1. 求下列函数项级数的收敛域，并指出使级数绝对收敛、条件收敛的x的范围：

(1)$\sum_{n=1}^{\infty} n\mathrm{e}^{-nx}$；　(2)$\sum_{n=1}^{\infty}(\frac{\lg x}{2})^n$；

(3)$\sum_{n=1}^{\infty} x^n\ln(1+\frac{1}{2^n})$；　(4)$\sum_{n=1}^{\infty}\frac{1}{n+x^n}$；

(5)$\sum_{n=6}^{\infty}\frac{(-1)^n}{(n^2-4n-5)^x}$；　(6)$\sum_{n=1}^{\infty}\frac{x^n}{1+x^{2n}}$；

(7)$\sum_{n=1}^{\infty}\frac{(-1)^n}{n+x^2}$；　(8)$\sum_{n=1}^{\infty}\frac{n5^{2n}}{6^n}x^n(1-x)^n$.

解：(1)$\because \lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{(n+1)\mathrm{e}^{-(n+1)x}}{n\mathrm{e}^{-nx}}=\frac{1}{\mathrm{e}^x}$

$\therefore$当$\frac{1}{\mathrm{e}^x}<1$即$x>0$时，该正项级数绝对收敛；当$\frac{1}{\mathrm{e}^x}>1$即$x<0$时，级数发散；当$x=0$时，级数为$\sum_{n=1}^{\infty}n$发散，故级数的收敛域为$(0,+\infty)$.

(2)$\because \lim\limits_{n\to\infty}\sqrt[n]{|(\frac{\lg x}{2})^n|}=|\frac{\lg x}{2}|$

$\therefore$当$|\frac{\lg x}{2}|<1$即$\frac{1}{100}<x<100$时，级数绝对收敛；

当$|\frac{\lg x}{2}|>1$即$0<x<\frac{1}{100}$或$x>100$时，$\lim\limits_{n\to\infty}(\frac{\lg x}{2})^n=+\infty$级数发散；

当$|\frac{\lg x}{2}|=1$即$x=\frac{1}{100}$或$x=100$时，$\lim\limits_{n\to\infty}(\frac{\lg x}{2})^n=\lim\limits_{n\to\infty}(\frac{\pm 2}{2})^n\neq 0$级数发散.

故级数的收敛域为$(\frac{1}{100},100)$.

(3)$\because \lim\limits_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\frac{|x|^{n+1}\ln(1+\frac{1}{2^{n+1}})}{|x|^n\ln(1+\frac{1}{2^n})}=|x|\lim\limits_{n\to\infty}\frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}}=\frac{|x|}{2}$

$\therefore$当$|x|<2$时级数绝对收敛

当$|x|>2$时$\lim\limits_{n\to\infty}x^n\ln(1+\frac{1}{2^n})=\lim\limits_{n\to\infty}x^n\frac{1}{2^n}=\lim\limits_{n\to\infty}(\frac{x}{2})^n=\infty$，级数发散；

当$|x|=2$时$\lim\limits_{n\to\infty}x^n\ln(1+\frac{1}{2^n})=\lim\limits_{n\to\infty}(\pm 2)^n\frac{1}{2^n}\neq 0$，级数发散.

故级数的收敛域为$(-2,2)$.

(4)当$|x|>1$时$\lim\limits_{n\to\infty}n^2\cdot|\frac{1}{n+x^n}|=\lim\limits_{n\to\infty}\frac{\frac{n^2}{|x|^n}}{\frac{n}{|x|^n}+(-1)^n}=0$，级数绝对收敛；

当$|x|<1$时$\lim\limits_{n\to\infty}n\cdot\frac{1}{n+x^n}=1$，级数发散；

当$x=1$时级数$\sum_{n=1}^{\infty}\frac{1}{n+x^n}=\sum_{n=1}^{\infty}\frac{1}{n+1}$发散；

当$x=-1$时$\frac{1}{n+x^n}=\frac{1}{n+(-1)^n}\geq\frac{1}{n+1}$，级数发散.

故级数的收敛域为$(-\infty,-1)\cup(1,+\infty)$.

(5)$\because \lim\limits_{n\to\infty}n^{2x}\cdot|\frac{(-1)^n}{(n^2-4n-5)^x}|=\lim\limits_{n\to\infty}(\frac{n^2}{n^2-4n-5})^x=1$

$\therefore$当$2x>1$即$x>\frac{1}{2}$时，级数绝对收敛；

当$0<2x\leq 1$即$0<x\leq\frac{1}{2}$时，由$\frac{1}{(n^2-4n-5)^x}$单调减少且$\lim\limits_{n\to\infty}\frac{1}{(n^2-4n-5)^x}=0$知，级数条件收敛；

当$x=0$时，$\sum_{n=6}^{\infty}\frac{(-1)^n}{(n^2-4n-5)^x}=\sum_{n=6}^{\infty}(-1)^n$发散；

当$x<0$时，$\lim\limits_{n\to\infty}\frac{(-1)^n}{(n^2-4n-5)^x}=\infty$，级数发散.

故级数的收敛域为$(0,+\infty)$.

(6)$\because \lim\limits_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\frac{|x|^{n+1}}{1+x^{2n+2}}\frac{1+x^{2n}}{|x|^n}=\lim\limits_{n\to\infty}|x|\frac{1+x^{2n}}{1+x^{2n+2}}=\lim\limits_{n\to\infty}|x|\frac{\frac{1}{x^{2n}}+1}{\frac{1}{x^{2n}}+x^2}=\begin{cases}|x|<1, & |x|<1\\ 1, & |x|=1\\ \frac{1}{|x|}<1, & |x|>1\end{cases}$

$\therefore$当$|x|\neq 1$时级数绝对收敛

当$|x|=1$时$\lim\limits_{n\to\infty}\frac{x^n}{1+x^{2n}}=\lim\limits_{n\to\infty}\frac{(\pm 1)^n}{1+1}\neq 0$，级数发散.

故级数的收敛域为$(-\infty,-1)\cup(-1,1)\cup(1,+\infty)$.

(7)由$|u_n|=\frac{1}{n+x^2}>\frac{1}{n+1+x^2}=|u_{n+1}|$且$\lim\limits_{n\to\infty}\frac{1}{n+x^2}=0$知级数为莱布尼茨型交错级数，故收敛，

又由$\lim\limits_{n\to\infty}n\cdot|\frac{(-1)^n}{n+x^2}|=\lim\limits_{n\to\infty}\frac{n}{n+x^2}=1$可知级数条件收敛.

级数的收敛域为$(-\infty,+\infty)$.

(8)$\lim\limits_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\frac{(n+1)|\frac{25}{6}x(1-x)|^{n+1}}{n|\frac{25}{6}x(1-x)|^n}=\frac{25}{6}|x(1-x)|$

当$\frac{25}{6}|x(1-x)|<1$即$-\frac{1}{5}x<\frac{2}{5}$或$\frac{3}{5}<x<\frac{6}{5}$时级数绝对收敛；

当$\frac{25}{6}|x(1-x)|>1$即$x<-\frac{1}{5}$或$\frac{2}{5}<x<\frac{3}{5}$或$x>\frac{6}{5}$时，$\lim\limits_{n\to\infty}\frac{n5^{2n}}{6^n}x^n(1-x)^n=\infty$，级数发散；

当$\frac{25}{6}|x(1-x)|=1$即$x<-\frac{1}{5}$或$x=\frac{2}{5}$或$x=\frac{3}{5}$或$x=\frac{6}{5}$时，$\lim\limits_{n\to\infty}\frac{n5^{2n}}{6^n}x^n(1-x)^n=\lim\limits_{n\to\infty}n(\pm1)^n\neq0$，级数发散.

收敛域为$(-\frac{1}{5},\frac{2}{5})\cup(\frac{3}{5},\frac{6}{5})$.

2. 用魏尔斯特拉斯判别法证明下列函数项级数在收敛域内一致收敛：

(1)$\sum_{n=1}^{\infty}\frac{nx}{1+n^5x^2}$；　(2)$\sum_{n=1}^{\infty}\frac{\cos nx+\sin n^2x}{n^{1.001}}$；

(3)$\sum_{n=1}^{\infty}\ln(1+\frac{2|x|}{x^2+n^3})$；　(4)$\sum_{n=1}^{\infty}x^2\mathrm{e}^{-nx}$.

证明：(1)$\lim\limits_{n\to\infty}n^2\cdot|\frac{nx}{1+n^5x^2}|=\lim\limits_{n\to\infty}\frac{n^3|x|}{1+n^5x^2}=0$，故级数在收敛域内绝对收敛，收敛域为$(-\infty,+\infty)$

$\because|\frac{nx}{1+n^5x^2}|=\frac{n}{\frac{1}{|x|}+n^5|x|}\leq\frac{n}{2\sqrt{n^5}}=\frac{1}{2n^{\frac{3}{2}}}$

又$\because$正项级数$\sum_{n=1}^{\infty}\frac{1}{2n^{\frac{3}{2}}}$收敛，故原级数一致收敛.

(2)$\lim\limits_{n\to\infty}n^{1.0005}\cdot|\frac{\cos nx+\sin n^2x}{n^{1.001}}|=\lim\limits_{n\to\infty}\frac{|\cos nx+\sin n^2x|}{n^{0.0005}}=0$，故级数在收敛域内绝对收敛，收敛域为$(-\infty,+\infty)$

$\because|\frac{\cos nx+\sin n^2x}{n^{1.001}}|\leq\frac{1}{n^{1.001}}$，且正项级数$\sum_{n=1}^{\infty}\frac{1}{n^{1.001}}$收敛，故原级数一致收敛.

(3)$\lim\limits_{n\to\infty}n^2\cdot|\ln(1+\frac{2|x|}{x^2+n^3})|=\lim\limits_{n\to\infty}n^2\cdot\frac{2|x|}{x^2+n^3}=0$，故级数在其收敛域内绝对收敛，收敛域为$(-\infty,+\infty)$

$\because 0<\ln(1+\frac{2|x|}{x^2+n^3})<\frac{2|x|}{x^2+n^3}=\frac{2}{|x|+\frac{n^3}{|x|}}\leq\frac{2}{2\sqrt{n^3}}=\frac{1}{n^{\frac{3}{2}}}$，且正项级数$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$收敛，故原级数一致收敛.

(4)当$x=0$时级数收敛，当$x\neq0$时$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{x^2\mathrm{e}^{-(n+1)x}}{x^2\mathrm{e}^{-nx}}=\mathrm{e}^{-x}$，故当$\mathrm{e}^{-x}<1$即$x>0$时级数绝对收敛，当$\mathrm{e}^{-x}>1$即$x<0$时，级数发散. 故级数的收敛域为$(0,+\infty)$

$\because u_n'(x)=(2x-nx^2)\mathrm{e}^{-nx}$，当$x=0$或$x=\frac{2}{n}$时$u_n'(x)=0$，故$x=\frac{2}{n}$是函数$u_n(x)$在区间$(0,+\infty)$上的唯一驻点，且是极大值点，故是最大值点

故$0<u_n(x)=x^2\mathrm{e}^{-nx}\leq u_n(\frac{2}{n})=\frac{4}{\mathrm{e}n^2}$

$\because$正项级数$\sum_{n=1}^{\infty}\frac{4}{\mathrm{e}n^2}$收敛，故原级数在收敛域内一致收敛.

3. (1)已知级数$\sum_{n=1}^{\infty}u_n(x)$在某区间一致收敛，能否断定$\sum_{n=1}^{\infty}u_n(x)$在此区间内绝对收敛？试研究级数$\sum_{n=1}^{\infty}(-1)^n\frac{1}{x^2+n}$在$(-\infty,+\infty)$上的一致收敛性与绝对收敛性.

   (2)设级数$\sum_{n=1}^{\infty}u_n(x)$在某区间绝对收敛，能否断定该级数在该区间内一致收敛？

   解：(1)不能.

   $\because |u_n(x)|=\frac{1}{n+x^2}>\frac{1}{n+1+x^2}=|u_{n+1}|$且$\lim\limits_{n\to\infty}\frac{1}{n+x^2}=0$故级数$\sum_{n=1}^{\infty}(-1)^n\frac{1}{x^2+n}$为莱布尼茨型交错级数，故收敛

   $\because \forall\varepsilon>0$,取$N=\max\{[\frac{1}{\varepsilon}-1]+1,1\}>0,s.t.|S(x)-S_n(x)|\le|u_{n+1}|=\frac{1}{n+1+x^2}<\frac{1}{n+1}<\varepsilon(\forall n>N,\forall x\in\mathbb{R})$

   $\therefore$级数$\sum_{n=(}^{\infty}-1)^n\frac{1}{x^2+n}$在$(-\infty,+\infty)$内一致收敛

   $\because \lim\limits_{n\to\infty}n\cdot|\frac{(-1)^n}{n+x^2}|=1$，故级数条件收敛.

   (2)不能.

   考虑级数$\sum_{n=1}^{\infty}n\mathrm{e}^{-nx}$. 因为$\lim\limits_{n\to\infty}n^2\cdot n\mathrm{e}^{-nx}=\lim\limits_{n\to\infty}n^3\mathrm{e}^{-nx}=0(x>0)$，故正项级数$\sum_{n=1}^{\infty}n\mathrm{e}^{-nx}$在$(0,+\infty)$上绝对收敛.

   但$|S(x)-S_n(x)|=\sum_{k=n+1}^{\infty}k\mathrm{e}^{-kx}>(n+1)\mathrm{e}^{-(n+1)x},\ n=1,2,\cdots$

   对于$\varepsilon_0=\frac{1}{2},\forall N\in\mathbb{Z}^+$，若取$x_n=\frac{1}{n+1},n>N$，则有$|S(x_n)-S_n(x_n)|\ge\frac{n+1}{\mathrm{e}}>\varepsilon_0$，级数在$(0,+\infty)$上不一致收敛.

4. 直接证明例8.4.7及例8.4.8的结论.

   证明：(1)例8.4.7：级数$\sum_{n=0}^{\infty}x(1-x)^n$的部分和$S_n(x)=\frac{x[1-(1-x)^n]}{1-(1-x)}=1-(1-x)^n$，和函数$S(x)=\begin{cases}0, & x=0\\ 1, & 0<x\le 1\end{cases}$

   对于$\varepsilon_0=\frac{1}{4}>0,\forall N\in\mathbb{Z}^+$，取$x_n=1-\frac{1}{2^{\frac{1}{n}}},n>N$，则

   $$|S_n(x_n)-S(x_n)|=|[1-(\frac{1}{2^{\frac{1}{n}}})^n]-0|=\frac{1}{2}>\varepsilon_0,$$

   故级数$\sum_{n=0}^{\infty}x(1-x)^n$不一致收敛.

   (2)例8.4.8：$f_n(x)=nx(1-x^2)^n,f(x)=\lim\limits_{n\to\infty}f_n(x)=\lim\limits_{n\to\infty}nx\mathrm{e}^{n\ln(1-x^2)}=\lim\limits_{n\to\infty}\frac{nx}{\mathrm{e}^{n\ln\frac{1}{1-x^2}}}$
   $=0,x\in[0,1]$

   对于$\varepsilon_0=\frac{1}{4},\forall N\in\mathbb{Z}^+$，取$x_n=\frac{1}{\sqrt{n}},n>N$，则

   $$|f_n(x)-f(x)|=\sqrt{n}(1-\frac{1}{n})^n\ge\sqrt{2}(1-\frac{1}{2})^2=\frac{\sqrt{2}}{4}>\varepsilon_0,n\ge 2,$$

   故函数列$f_n(x)$不一致收敛.

13.3 习题8.5解答

1. 求下列幂级数的收敛半径和收敛域：

(1)$\sum_{n=1}^{\infty}\frac{2^{n+1}}{n^2}x^n$； (2)$\sum_{n=1}^{\infty}\frac{x^{2n+1}}{9^n}$；

(3)$\sum_{n=1}^{\infty}n!(x-1)^n$； (4)$\sum_{n=1}^{\infty}\frac{\ln(n+1)}{n+1}x^n$；

(5)$\sum_{n=1}^{\infty}(\frac{1}{2^n}+(-2)^n)x^n$； (6)$\sum_{n=1}^{\infty}(\frac{n+1}{n})^{n^2}x^n$；

(7)$\sum_{n=1}^{\infty}x^{n^2}$； (8)$\sum_{n=1}^{\infty}(1+\frac{1}{2}+\cdots+\frac{1}{n})x^n$.

解：(1) $\lim\limits_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\frac{2^{n+2}|x|^{n+1}}{(n+1)^2}\frac{n^2}{2^{n+1}|x|^n}=2|x|$

当$2|x|<1$即$-\frac{1}{2}<x<\frac{1}{2}$时，级数绝对收敛；

当$2|x|=1$即$x=\pm\frac{1}{2}$时，$\sum_{n=1}^{\infty}|\frac{2^{n+1}}{n^2}x^n|=\sum_{n=1}^{\infty}\frac{2}{n^2}$，级数绝对收敛；

当$2|x|>1$即$x<-\frac{1}{2}$或$x>\frac{1}{2}$时，$\lim\limits_{n\to\infty}\frac{2^{n+1}}{n^2}x^n=\lim\limits_{n\to\infty}\frac{2(2x)^n}{n^2}=\infty$，级数发散.

故收敛半径为$R=\frac{1}{2}$，收敛域为$[-\frac{1}{2},\frac{1}{2}]$.

(2) $\lim\limits_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\frac{|x|^{2n+3}}{9^{n+1}}\frac{9^n}{|x|^{2n+1}}=\frac{x^2}{9}$

当$\frac{x^2}{9}<1$即$-3<x<3$时，级数绝对收敛；

当$\frac{x^2}{9}=1$即$x=\pm3$时，$\lim\limits_{n\to\infty}\frac{x^{2n+1}}{9^n}=\lim\limits_{n\to\infty}\frac{(\pm3)^{2n+1}}{9^n}=\pm3\neq0$，级数发散；

当$\frac{x^2}{9}>1$即$x>3$或$x<-3$时，$\lim\limits_{n\to\infty}\frac{x^{2n+1}}{9^n}=\lim\limits_{n\to\infty}(\frac{x^2}{9})^nx=\infty$，级数发散.

故收敛半径为$R=3$，收敛域为$(-3,3)$.

(3) $\lim\limits_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\frac{(n+1)!|x-1|^{n+1}}{n!|x-1|^n}=\lim\limits_{n\to\infty}(n+1)|x-1|=+\infty$

当$x-1=0$即$x=1$时，级数绝对收敛；

当$x-1\neq0$即$x\neq1$时，$\lim\limits_{n\to\infty}n!(x-1)^n=\infty$，级数发散.

故收敛半径为$R=0$，收敛域为$\{1\}$.

(4) $\lim\limits_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\frac{\ln(n+2)|x|^{n+1}}{n+2}\frac{n+1}{\ln(n+1)|x|^n}=\lim\limits_{n\to\infty}|x|\frac{n+1}{n+2}\frac{\ln(n+2)}{\ln(n+1)}=\lim\limits_{n\to\infty}|x|\frac{n+1}{n+2}\frac{\frac{1}{n+2}}{\frac{1}{n+1}}=|x|$

当$|x|<1$即$-1<x<1$时，级数绝对收敛；

当$x=1$时，$\sum_{n=1}^{\infty}\frac{\ln(n+1)}{n+1}x^n=\sum_{n=1}^{\infty}\frac{\ln(n+1)}{n+1}$，$\lim\limits_{n\to\infty}n\cdot\frac{\ln(n+1)}{n+1}=+\infty$，级数发散；

当$x=-1$时，$\sum_{n=1}^{\infty}\frac{\ln(n+1)}{n+1}x^n=\sum_{n=1}^{\infty}\frac{\ln(n+1)}{n+1}(-1)^n$，由$f(x)=\frac{\ln(x+1)}{x+1},f'(x)=\frac{1-\ln(x+1)}{(x+1)^2}<0(x\geq2)$知$|u_n|>|u_{n+1}|$，又因为$\lim\limits_{n\to\infty}\frac{\ln(n+1)}{n+1}=0$，故$\sum_{n=1}^{\infty}\frac{\ln(n+1)}{n+1}(-1)^n$为莱布尼茨型交错级数，级数条件收敛；

当$|x|>1$即$x>1$或$x<-1$时，$\lim\limits_{n\to\infty}\frac{\ln(n+1)}{n+1}x^n=\infty$，级数发散.

故收敛半径为$R=1$，收敛域为$[-1,1)$.

(5) $\lim\limits_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\frac{|\frac{1}{2^{n+1}}+(-2)^{n+1}||x|^{n+1}}{|\frac{1}{2^n}+(-2)^n||x|^n}=\lim\limits_{n\to\infty}|x|\frac{|\frac{1}{2^{2n+1}}+2(-1)^{n+1}|}{|\frac{1}{2^{2n}}+(-1)^n|}=2|x|$

当$2|x|<1$即$-\frac{1}{2}<x<\frac{1}{2}$时，级数绝对收敛；

当$2|x|=1$即$x=\pm\frac{1}{2}$时，$\lim\limits_{n\to\infty}[\frac{1}{2^n}+(-2)^n]x^n=\lim\limits_{n\to\infty}[\frac{1}{2^n}+(-2)^n](\pm\frac{1}{2})^n=\lim\limits_{n\to\infty}[\frac{1}{2^{2n}}+(-1)^n])(\pm1)^n\neq0$级数发散；

当$2|x|>1$即$x>\frac{1}{2}$或$x<-\frac{1}{2}$时，$\lim\limits_{n\to\infty}[\frac{1}{2^n}+(-2)^n]x^n=\lim\limits_{n\to\infty}[(\frac{x}{2})^n+(-2x)^n]\neq0$，级数发散.

故收敛半径为$R=\frac{1}{2}$，收敛域为$(-\frac{1}{2},\frac{1}{2})$.

(6) $\lim\limits_{n\to\infty}\sqrt[n]{|(\frac{n+1}{n})^{n^2}||x|^n}=\lim\limits_{n\to\infty}(1+\frac{1}{n})^n|x|=\mathrm{e}|x|$

当$\mathrm{e}|x|<1$即$-\frac{1}{\mathrm{e}}<x<\frac{1}{\mathrm{e}}$时，级数绝对收敛；

当$\mathrm{e}|x|=1$即$x=\pm\frac{1}{\mathrm{e}}$时，$\lim\limits_{n\to\infty}(\frac{n+1}{n})^{n^2}x^n=\lim\limits_{n\to\infty}(\frac{n+1}{n})^{n^2}(\pm\frac{1}{\mathrm{e}})^n=\lim\limits_{n\to\infty}[\pm\frac{1}{\mathrm{e}}(\frac{n+1}{n})^n]^n$

$=\lim\limits_{n\to\infty}(\pm1)^n\mathrm{e}^{n[\ln\frac{1}{\mathrm{e}}+n\ln(1+\frac{1}{n})]}=\lim\limits_{n\to\infty}(\pm1)^n\mathrm{e}^{n[-1+n(\frac{1}{n}-\frac{1}{2n^2}+o(\frac{1}{n^2}))]}=\lim\limits_{n\to\infty}(\pm1)^n\mathrm{e}^{[-\frac{1}{2}+\frac{o(\frac{1}{n})}{\frac{1}{n}}]}$

$=\lim\limits_{n\to\infty}(\pm1)^n\mathrm{e}^{-\frac{1}{2}}\neq0$，级数发散.

故该幂级数的收敛半径为$R=\frac{1}{\mathrm{e}}$，收敛域为$(-\frac{1}{\mathrm{e}},\frac{1}{\mathrm{e}})$.

(7) $\lim\limits_{n\to\infty}\sqrt[n]{|x^{n^2}|}=\lim\limits_{n\to\infty}|x|^n=\begin{cases}0, & |x|<1\\ 1, & |x|=1\\ \infty, & |x|>1\end{cases}$

当$|x|<1$时，级数绝对收敛；当$|x>1|$时级数发散；当$|x|=1$时，$\lim\limits_{n\to\infty}(\pm1)^{n^2}\neq0$，级数发散.

收敛半径为$R=1$，收敛域为$(-1,1)$.

(8) $\lim\limits_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\frac{(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1})|x|^{n+1}}{(1+\frac{1}{2}+\cdots+\frac{1}{n})|x|^n}=\lim\limits_{n\to\infty}|x|(1+\frac{\frac{1}{n+1}}{1+\frac{1}{2}+\cdots+\frac{1}{n}})=\lim\limits_{n\to\infty}|x|[1+\frac{1}{(n+1)(1+\frac{1}{2}+\cdots+\frac{1}{n})}]$
$=|x|$

当$|x|<1$时，级数绝对收敛；

当$|x|=1$时，$\lim\limits_{n\to\infty}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})x^n=\lim\limits_{n\to\infty}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})(\pm1)^n=\infty$，级数发散.

故该幂级数的收敛半径为$R=1$，收敛域为$(-1,1)$.

2. 求下列幂级数的收敛区间及和函数：

(1)$\sum_{n=0}^{\infty}\frac{x^n}{2^n}$；　(2)$\sum_{n=1}^{\infty}(2n+1)x^n$；

(3)$\sum_{n=1}^{\infty}\frac{2n-1}{2^n}x^{2n-2}$；　(4)$\sum_{n=1}^{\infty}\frac{n(n+1)}{2}x^{n-1}$.

解：(1)$\sum_{n=0}^{\infty}\frac{x^n}{2^n}$是以$\frac{x}{2}$为公比的几何级数，故收敛域为$(-2,2)$，和函数为$S(x)=\frac{1}{1-\frac{x}{2}}=\frac{2}{2-x}$.

(2) $\lim\limits_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\frac{(2n+3)|x|^{n+1}}{(2n+1)|x^n|}=|x|$，当$|x|<1$时级数绝对收敛，当$|x|=1$时，$\lim\limits_{n\to\infty}(2n+$

1)$(\pm 1)^n \neq 0$，级数发散，故该幂级数的收敛域为$(-1,1)$

$$\begin{aligned}
&\sum_{n=1}^{\infty}(2n+1)x^n=\sum_{n=1}^{\infty}(n+n+1)x^n=x\sum_{n=1}^{\infty}nx^{n-1}+\sum_{n=1}^{\infty}(n+1)x^n\\
=&x\sum_{n=1}^{\infty}(x^n)'+\sum_{n=1}^{\infty}(x^{n+1})'=x(\sum_{n=1}^{\infty}x^n)'+(\sum_{n=1}^{\infty}x^{n+1})'\\
=&x(\frac{1}{1-x}-1)'+(\frac{1}{1-x}-1-x)'=\frac{x}{(1-x)^2}+\frac{1}{(1-x)^2}-1\\
=&\frac{3x-x^2}{(1-x)^2}.
\end{aligned}$$

(3) $\lim\limits_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\frac{(2n+1)|x|^{2n}}{2^{n+1}}\frac{2^n}{(2n-1)|x|^{2n-2}}=\frac{x^2}{2}$，当$\frac{x^2}{2}<1$时级数绝对收敛，当$\frac{x^2}{2}=1$时，级数$\sum_{n=1}^{\infty}\frac{2n-1}{2^n}x^{2n-2}=\sum_{n=1}^{\infty}\frac{2n-1}{2}(\frac{x^2}{2})^{n-1}=\sum_{n=1}^{\infty}\frac{2n-1}{2}$发散，故该幂级数的收敛域为$(-\sqrt{2},\sqrt{2})$

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{2n-1}{2^n}x^{2n-2}=\sum_{n=1}^{\infty}(\frac{x^{2n-1}}{2^n})'=(\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2^n})'=(\frac{1}{x}\sum_{n=1}^{\infty}\frac{x^{2n}}{2^n})'=[\frac{1}{x}\sum_{n=1}^{\infty}(\frac{x^2}{2})^n]'\\
=&[\frac{1}{x}(\frac{1}{1-\frac{x^2}{2}}-1)]'=\frac{2+x^2}{(2-x^2)^2}.
\end{aligned}$$

(4) $\lim\limits_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\frac{(n+1)(n+2)|x|^{n+1}}{2}\frac{2}{n(n+1)|x|^n}=|x|$，当$|x|<1$时，级数绝对收敛，当$|x=1|$时，级数$\sum_{n=1}^{\infty}\frac{n(n+1)}{2}x^{n-1}=\sum_{n=1}^{\infty}\frac{n(n+1)}{2}(-1)^{n-1}$发散，故该幂级数的收敛域为$(-1,1)$

$$\sum_{n=1}^{\infty}\frac{n(n+1)}{2}x^{n-1}=\sum_{n=1}^{\infty}(\frac{1}{2}x^{n+1})''=\frac{1}{2}(\sum_{n=1}^{\infty}x^n)''=\frac{1}{2}(\frac{1}{1-x}-1)''=\frac{1}{2}[\frac{1}{(1-x)^2}]'=\frac{1}{(1-x)^3}.$$

3. 利用直接展开法求下列函数在指定点处的泰勒级数：

(1)$f(x)=a^x(a>0),x_0=0$；

(2)$g(x)=\frac{1}{2}(\mathrm{e}^x-\mathrm{e}^{-x}),x_0=0$；

(3)$\varphi(x)=\cos x,x_0=\frac{\pi}{2}$；

(4)$\psi(x)=\sin x,x_0=a$.

解：(1)$f'(x)=a^x\ln a,f''(x)=a^x(\ln a)^2,\cdots,f^{(n)}(x)=a^x(\ln a)^n,\cdots$

$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{(\ln a)^n}{n!}x^n,x\in(-\infty,+\infty)$.

(2)$g'(x)=\frac{1}{2}(\mathrm{e}^x+\mathrm{e}{-}x),g''(x)=\frac{1}{2}(\mathrm{e}^x-\mathrm{e}^{-x})=g(x),g'''(x)=g'(x),\cdots,$

$= g^{(2m)}(x) = g(x) = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}), g^{(2m+1)}(x) = g'(x) = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}), \cdots$

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = \sum_{m=0}^{\infty} [\frac{g^{(2m)}(0)}{(2m)!} x^{2m} + \frac{g^{(2m+1)}(0)}{(2m+1)!} x^{2m+1}] \\ &= \sum_{m=0}^{\infty} [\frac{g^{(2m)}(0)}{(2m)!} x^{2m} + \frac{g^{(2m+1)}(0)}{(2m+1)!} x^{2m+1}] = \sum_{m=0}^{\infty} [\frac{0}{(2m)!} x^{2m} + \frac{1}{(2m+1)!} x^{2m+1}] \\ &= \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} x^{2m+1}, x \in (-\infty, +\infty). \end{aligned}$$

(3)方法1：$\varphi'(x) = -\sin x, \varphi''(x) = -\cos x, \varphi'''(x) = \sin x, \varphi^{(4)}(x) = \cos x, \cdots$,
$\varphi^{(4k)}(x) = \cos x, \varphi^{(4k+1)}(x) = -\sin x, \varphi^{(4k+2)}(x) = -\cos x, \varphi^{(4k+3)}(x) = \sin x, \cdots$

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(\frac{\pi}{2})}{n!} (x - \frac{\pi}{2})^n$$

$$= \sum_{k=0}^{\infty} [\frac{\varphi^{(4k)}(\frac{\pi}{2})}{(4k)!} (x - \frac{\pi}{2})^{4k} + \frac{\varphi^{(4k+1)}(\frac{\pi}{2})}{(4k+1)!} (x - \frac{\pi}{2})^{4k+1} + \frac{\varphi^{(4k+2)}(\frac{\pi}{2})}{(4k+2)!} (x - \frac{\pi}{2})^{4k+2}$$
$$+ \frac{\varphi^{(4k+3)}(\frac{\pi}{2})}{(4k+3)!} (x - \frac{\pi}{2})^{4k+3}]$$
$$= \sum_{k=0}^{\infty} [\frac{\cos\frac{\pi}{2}}{(4k)!} (x - \frac{\pi}{2})^{4k} + \frac{-\sin\frac{\pi}{2}}{(4k+1)!} (x - \frac{\pi}{2})^{4k+1} + \frac{-\cos\frac{\pi}{2}}{(4k+2)!} (x - \frac{\pi}{2})^{4k+2} + \frac{\sin\frac{\pi}{2}}{(4k+3)!} (x - \frac{\pi}{2})^{4k+3}]$$
$$= \sum_{k=0}^{\infty} [\frac{-1}{(4k+1)!} (x - \frac{\pi}{2})^{4k+1} + \frac{1}{(4k+3)!} (x - \frac{\pi}{2})^{4k+3}]$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(2m+1)!} (x - \frac{\pi}{2})^{2m+1}, x \in (-\infty, +\infty).$$

方法2：$\varphi^{(n)}(x) = \cos(x + \frac{n\pi}{2})$,

$$\varphi^{(n)}(\frac{\pi}{2}) = \cos[\frac{(n+1)\pi}{2}] = \begin{cases} 0, & x = 4k \\ -1, & x = 4k+1 \\ 0, & x = 4k+2 \\ 1, & x = 4k+3 \end{cases}, k = 0, 1, \cdots$$

$$= \begin{cases} 0, & x = 2m \\ (-1)^{m-1}, & x = 2m+1 \end{cases}, m = 0, 1, \cdots$$

$$\begin{aligned}\varphi(x) &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(\frac{\pi}{2})}{n!}(x - \frac{\pi}{2})^n \\ &= \sum_{m=0}^{\infty} [\frac{\varphi^{(2m)}(\frac{\pi}{2})}{(2m)!}(x - \frac{\pi}{2})^{2m} + \frac{\varphi^{(2m+1)}(\frac{\pi}{2})}{(2m+1)!}(x - \frac{\pi}{2})^{2m+1}] \\ &= \sum_{m=0}^{\infty} \frac{(-1)^{m-1}}{(2m+1)!}(x - \frac{\pi}{2})^{2m+1}, x \in (-\infty, +\infty).\end{aligned}$$

(4)$\psi^{(n)} = \sin(x + \frac{n\pi}{2})$

$$\psi(x) = \sum_{n=0}^{\infty} \frac{\psi^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{\sin(x + \frac{n\pi}{2})}{n!}(x-a)^n, x \in (-\infty, +\infty).$$

4. 利用间接展开法求下列函数在指定点的泰勒级数，并指出收敛区间：

(1)$f_1(x) = \frac{1}{1-x^2}, x_0 = 0$；

(2)$f_2(x) = \ln x, x_0 = 1$；

(3)$f_3(x) = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}), x_0 = 0$；

(4)$f_4(x) = \frac{1}{2x^2+x-3}, x_0 = 3$；

(5)$f_5(x) = (x-2)\mathrm{e}^{-x}, x_0 = 1$；

(6)$f_6(x) = \frac{1}{(1+x)^2}, x_0 = 0$；

(7)$f_7(x) = x(1-x^2)^{-\frac{1}{2}}, x_0 = 0$；

(8)$f_8(x) = \ln(x + \sqrt{1+x^2}), x_0 = 0$.

解：(1)$f_1(x) = \sum_{n=0}^{\infty}(x^2)^n = \sum_{n=0}^{\infty} x^{2n}$，收敛半径为$R = 1$，当$|x| = 1$，时级数$\sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} 1$发散，故收敛域为$(-1, 1)$.

(2)$f_2(x) = \ln[(1 + (x-1)] = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-1)^n$，收敛半径为$R = 1$，当$x = 0$时级数$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}(-1)^n = \sum_{n=1}^{\infty} \frac{1}{n}$，级数发散；当$x = 2$时，级数$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$收敛. 故收敛域为$(0, 2]$.

(3)$f_3(x) = \frac{1}{2}[\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}] = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{(2k)!}x^{2k}$，收敛半径为$R = +\infty$，收敛域为$(-\infty, +\infty)$.

(4)

$$\begin{aligned}f_4(x) &= \frac{1}{(2x+3)(x-1)} = \frac{1}{5}\frac{1}{x-1} - \frac{2}{5}\frac{1}{2x+3} = \frac{1}{5}\frac{1}{(x-3)+2} - \frac{2}{5}\frac{1}{2(x-3)+9} \\ &= \frac{1}{5 \cdot 2}\frac{1}{1 + \frac{x-3}{2}} - \frac{2}{5 \cdot 9}\frac{1}{1 + \frac{2}{9}(x-3)} = \frac{1}{5 \cdot 2}\sum_{n=0}^{\infty}(-\frac{x-3}{2})^n - \frac{2}{5 \cdot 9}\sum_{n=0}^{\infty}[-\frac{2}{9}(x-3)]^n \\ &= \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n(\frac{1}{2^{n+1}} - \frac{2}{9^{n+1}})(x-3)^n\end{aligned}$$

收敛区间为$(3-2,3+2)\cap(3-\frac{9}{2},3+\frac{9}{2})$，即$(1,5)$.

当$x=1$时，级数$\frac{1}{5}\sum_{n=0}^{\infty}(-1)^n(\frac{1}{2^{n+1}}-\frac{2}{9^{n+1}})(x-3)^n=\frac{1}{5}\sum_{n=0}^{\infty}(-1)^n(\frac{1}{2^{n+1}}-\frac{2}{9^{n+1}})(-2)^n$ $=\frac{1}{5}\sum_{n=0}^{\infty}[\frac{1}{2}-(\frac{2}{9})^{n+1}]$发散；

当$x=5$时，级数$\frac{1}{5}\sum_{n=0}^{\infty}(-1)^n(\frac{1}{2^{n+1}}-\frac{2}{9^{n+1}})(x-3)^n=\frac{1}{5}\sum_{n=0}^{\infty}[\frac{1}{2}-(\frac{2}{9})^{n+1}](-1)^n$发散.

故收敛域为$(2,5)$.

(5)

$$\begin{aligned}
f_5(x)&=[(x-1)-1]\mathrm{e}^{-(x-1)-1}=\frac{1}{\mathrm{e}}[(x-1)-1]\mathrm{e}^{-(x-1)}\\
&=\frac{1}{\mathrm{e}}[(x-1)-1]\sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^n}{n!}\\
&=\frac{1}{\mathrm{e}}[(x-1)\sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^n}{n!}-\sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^n}{n!}]\\
&=\frac{1}{\mathrm{e}}[\sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^{n+1}}{n!}-\sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^n}{n!}]\\
&=\frac{1}{\mathrm{e}}[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{(n-1)!}-\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^n}{n!}-1]\\
&=-\frac{1}{\mathrm{e}}+\frac{1}{\mathrm{e}}\sum_{n=1}^{\infty}[\frac{1}{(n-1)!}+\frac{1}{n!}](-1)^{n-1}(x-1)^n,
\end{aligned}$$

收敛域为$(-\infty,+\infty)$.

(6)$f_6(x)=-(\frac{1}{x+1})'=-[\sum_{n=0}^{\infty}(-x)^n]'=\sum_{n=1}^{\infty}n(-x)^{n-1}=\sum_{n=1}^{\infty}(-1)^{n-1}nx^{n-1}$,收敛半径为$R=1$，当$x=\pm1$时，级数$\sum_{n=1}^{\infty}n(-x)^{n-1}=\sum_{n=1}^{\infty}n(\mp1)^{n-1}$发散，故收敛域为$(-1,1)$.

(7)

$$\begin{aligned}
f_7(x)&=x(1-x^2)^{-\frac{1}{2}}\\
&=x[1+\sum_{n=1}^{\infty}\frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-n+1)}{n!}(-x^2)^n]\\
&=x[1+\sum_{n=1}^{\infty}(-1)^n\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\cdots\cdot\frac{2n-1}{2}}{n!}(-x^2)^n]\\
&=x[1+\sum_{n=1}^{\infty}(-1)^{2n}\frac{(2n-1)!!}{2^n n!}x^{2n}]\\
&=x+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!}x^{2n+1},
\end{aligned}$$

收敛半径为$R=1$，当$|x|=1$时，级数$x+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!}x^{2n+1}=(\pm1)[1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}]$

$\because \frac{(2n-1)!!}{(2n)!!} = \frac{(2n-1)\cdot(2n-3)\cdot\cdots\cdot3\cdot1}{(2n)\cdot(2n-2)\cdot\cdots\cdot4\cdot2} = \frac{1}{2n}\frac{2n-1}{2n-2}\cdots\frac{5}{4}\frac{3}{2} > \frac{1}{2n}$，级数$\sum_{n=1}^{\infty}\frac{1}{2n}$发散，故级数$x+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!}x^{2n+1} = (\pm1)[1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}]$发散.

故收敛域为$(-1,1)$.

(8)

$$\begin{aligned}
f_8(x) &= \int_0^x \frac{1}{\sqrt{1+x^2}}\mathrm{d}x \\
&= \int_0^x [1+\sum_{n=1}^{\infty}\frac{(-\frac{1}{2})\cdot(-\frac{1}{2}-1)\cdot(-\frac{1}{2}-2)\cdot\cdots\cdot(-\frac{1}{2}-n+1)}{n!}x^{2n}]\mathrm{d}x \\
&= x+\sum_{n=1}^{\infty}\frac{(-\frac{1}{2})\cdot(-\frac{3}{2})\cdot(-\frac{5}{2})\cdot\cdots\cdot(-\frac{2n-1}{2})}{n!}\int_0^x x^{2n}\mathrm{d}x \\
&= x+\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}\frac{1}{2n+1}x^{2n+1} \\
&= x+\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!!}{(2n+1)2^n n!}x^{2n+1},
\end{aligned}$$

幂级数的收敛半径为$R=1$

当$|x|=1$时，级数$x+\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!!}{(2n+1)2^n n!}x^{2n+1} = (\pm1)[1+\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!!}{(2n+1)(2n)!!}]$

记$u_n = \frac{(2n-1)!!}{(2n+1)(2n)!!}$，则$u_{n+1} = \frac{(2n+1)!!}{(2n+3)(2n+2)!!} = \frac{2n+1}{2n+3}\frac{2n+1}{2n+2}\frac{(2n-1)!!}{(2n+1)(2n)!!} = \frac{2n+1}{2n+3}\frac{2n+1}{2n+2}u_n < u_n$且$\lim\limits_{n\to\infty}u_n = \lim\limits_{n\to\infty}\frac{1}{2n+1}\cdot\frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\cdots\cdot\frac{1}{2} = 0$，故$\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!!}{(2n+1)(2n)!!}$为莱布尼茨型交错级数，则$x+\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!!}{(2n+1)2^n n!}x^{2n+1} = (\pm1)[1+\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!!}{(2n+1)(2n)!!}]$收敛.

故收敛域为$[-1,1]$.

5. 求下列函数的麦克劳林级数中指定的项：

(1)$f(x)=\mathrm{e}^x\sin x, 0$至4项；

(2)$g(x)=\tan x, 0$至3项.

解：(1)$f(x)=\mathrm{e}^x\sin x = (1+x+\frac{1}{2}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\cdots)(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots)$
$=x+x^2+(\frac{1}{2}-\frac{1}{6})x^3+(\frac{1}{6}-\frac{1}{6})x^4 = x+x^2+\frac{1}{3}x^3$.

(2)$g'(x)=\sec^2 x, g''(x)=2\sec x\sec x\tan x = 2\sec^2 x\tan x = 2(\tan^2 x+1)\tan x = 2\tan^3 x+2\tan x, g'''(x)=6\tan^2 x\sec^2 x+2\sec^2 x$

$g(0)=0, g'(0)=1, g''(0)=0, g'''(0)=2$

$\therefore g(x)=g(0)+g'(0)x+\frac{g''(0)}{2!}x^2+\frac{g'''(0)}{3!}x^3 = x+\frac{1}{3}x^3$.

6. 求下列数的近似值，精确到指定的误差范围：

(1)e，误差不超过10^{-4}；

(2)$\sqrt[3]{500}$，误差不超过10^{-3}.

解：(1)$\mathrm{e}=\mathrm{e}^1=1+1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots+\frac{1}{n!}+\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+\cdots$

取前$n+1$项，误差$E=\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+\cdots=\frac{1}{(n+1)!}[1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+\cdots]$
$<\frac{1}{(n+1)!}[1+\frac{1}{n+2}+\frac{1}{(n+2)^2}+\cdots]=\frac{1}{(n+1)!}\frac{1}{1-\frac{1}{n+2}}=\frac{1}{(n+1)!}\frac{n+2}{n+1}$

当$n=7$时，$\frac{1}{(n+1)!}\frac{n+2}{n+1}=2.8\times10^{-5}<10^{-4}$

则$e\approx1+1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}\approx2.7183.$

(2)

$$\begin{aligned}\sqrt[3]{500}=&\sqrt[3]{8^3-12}=8\sqrt[3]{1-\frac{12}{8^3}}\\
=&8[1+\frac{1}{3}(-\frac{12}{8^3})+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}(-\frac{12}{8^3})^2+\frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}(-\frac{12}{8^3})^3+\cdots\\
&+\frac{\frac{1}{3}(\frac{1}{3}-1)\cdots(\frac{1}{3}-n+1)}{n!}(-\frac{12}{8^3})^n+\frac{\frac{1}{3}(\frac{1}{3}-1)\cdots(\frac{1}{3}-n+1)(\frac{1}{3}-n)}{(n+1)!}(-\frac{12}{8^3})^{n+1}+\cdots]\\
=&8[1-\frac{1}{3}(\frac{12}{8^3})-\frac{2}{3\cdot2!}(\frac{12}{8^3})^2-\frac{2\cdot5}{3^2\cdot3!}(\frac{12}{8^3})^3-\cdots\\
&-\frac{2\cdot5\cdot\cdots\cdot(3n-4)}{3^nn!}(\frac{12}{8^3})^n-\frac{2\cdot5\cdot\cdots\cdot(3n-4)(3n-1)}{3^{n+1}(n+1)!}(\frac{12}{8^3})^{n+1}-\cdots]\end{aligned}$$

取前$n+1$项，误差

$$\begin{aligned}E=&8|-\frac{2\cdot5\cdot\cdots\cdot(3n-4)(3n-1)}{3^{n+1}(n+1)!}(\frac{12}{8^3})^{n+1}-\frac{2\cdot5\cdot\cdots\cdot(3n-4)(3n-1)(3n+2)}{3^{n+2}(n+2)!}(\frac{12}{8^3})^{n+2}-\cdots|\\
=&8[\frac{2\cdot5\cdot\cdots\cdot(3n-4)(3n-1)}{3^{n+1}(n+1)!}(\frac{12}{8^3})^{n+1}+\frac{2\cdot5\cdot\cdots\cdot(3n-4)(3n-1)(3n+2)}{3^{n+2}(n+2)!}(\frac{12}{8^3})^{n+2}+\cdots]\\
=&8[(\frac{1}{3}\cdot\frac{2}{3\cdot2}\cdot\frac{5}{3\cdot3}\cdot\cdots\cdot\frac{3n-4}{3n}\cdot\frac{3n-1}{3n+3})(\frac{12}{8^3})^{n+1}\\
&+(\frac{1}{3}\cdot\frac{2}{3\cdot2}\cdot\frac{5}{3\cdot3}\cdot\cdots\cdot\frac{3n-4}{3n}\cdot\frac{3n-1}{3n+3}\cdot\frac{3n+2}{3n+7})(\frac{12}{8^3})^{n+2}+\cdots]\\
&<8[(\frac{12}{8^3})^{n+1}+(\frac{12}{8^3})^{n+2}+\cdots]\\
&=8(\frac{12}{8^3})^{n+1}[1+\frac{12}{8^3}+(\frac{12}{8^3})^2+\cdots]\\
&=8(\frac{12}{8^3})^{n+1}\frac{1}{1-\frac{12}{8^3}}=(\frac{12}{8^3})^n\frac{96}{8^3-12}\end{aligned}$$

当$n=2$时$E<(\frac{12}{8^3})^n\frac{96}{8^3-12}\approx0.0001<10^{-3}$

则$\sqrt[3]{500}\approx8[1+\frac{1}{3}(-\frac{12}{8^3})+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}(-\frac{12}{8^3})^2]\approx7.937.$

7. 求下列积分的级数表达式，取前三项求其近似值并估计误差：
(1)$\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1+x^4}}\mathrm{d}x$；
(2)$\int_0^{\frac{1}{4}}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x.$

解：(1)

$$\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1+x^4}}\mathrm{d}x=\int_0^{\frac{1}{2}}[1+\sum_{n=1}^{\infty}\frac{(-\frac{1}{2})\cdot(-\frac{1}{2}-1)\cdot(-\frac{1}{2}-2)\cdot\cdots\cdot(-\frac{1}{2}-n+1)}{n!}x^{4n}]\mathrm{d}x$$

$$=\int_0^{\frac{1}{2}}[1+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n n!}x^{4n}]\mathrm{d}x$$

$$\approx\int_0^{\frac{1}{2}}[1-\frac{1}{2}x^4+\frac{3}{8}x^8]\mathrm{d}x=(x-\frac{1}{10}x^5+\frac{3}{72}x^9)\Big|_0^{\frac{1}{2}}=(\frac{1}{2}-\frac{1}{10}\frac{1}{32}+\frac{1}{24}\frac{1}{512})$$

≈ 0.496953

误差$E\le\int_0^{\frac{1}{2}}[\frac{(2\cdot3-1)!!}{2^3 3!}x^{4\cdot3}]\mathrm{d}x=\int_0^{\frac{1}{2}}\frac{3}{8}x^8\mathrm{d}x=\frac{15}{48\times13}x^{13}\Big|_0^{\frac{1}{2}}=\frac{5}{16\times13}\frac{1}{8192}\approx2.93438\times10^{-6}$.

(2)

$$\int_0^{\frac{1}{4}}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x=\int_0^{\frac{1}{4}}\sum_{n=0}^{\infty}\frac{(-\frac{x^2}{2})^n}{n!}\mathrm{d}x=\int_0^{\frac{1}{4}}\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!}x^{2n}\mathrm{d}x$$

$$\approx\int_0^{\frac{1}{4}}[1-\frac{1}{2}x^2+\frac{1}{8}x^4]\mathrm{d}x\approx(x-\frac{1}{6}x^3+\frac{1}{40}x^5)\Big|_0^{\frac{1}{4}}=(\frac{1}{4}-\frac{1}{6\times64}+\frac{1}{40\times1024})$$

≈ 0.2474

误差$E\le\int_0^{\frac{1}{4}}\frac{1}{2^3 3!}x^6\mathrm{d}x=\frac{1}{48\times7}x^7\Big|_0^{\frac{1}{4}}\approx1.81652\times10^{-7}$.

8. 设$R>0$，证明：若$\sum_{n=0}^{\infty}a_nx^n$与$\sum_{n=0}^{\infty}b_nx^n$在$(-R,R)$有相同的和函数，则$a_n=b_n(n=0,1,2,\cdots)$.

证明：$\because\sum_{n=0}^{\infty}a_nx^n$与$\sum_{n=0}^{\infty}b_nx^n$在$(-R,R)$有相同的和函数

$\therefore\forall x\in(-R,R),\sum_{n=0}^{\infty}a_nx^n,\sum_{n=0}^{\infty}b_nx^n$均收敛，且$\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}b_nx^n$

$\therefore\sum_{n=0}^{\infty}a_nx^n-\sum_{n=0}^{\infty}b_nx^n=\sum_{n=0}^{\infty}(a_n-b_n)x^n=0,\forall x\in(-R,R)$

$\therefore a_n=b_n,n=0,1,2,\cdots$.

9. 设$f(x)=\sum_{n=0}^{\infty}a_nx^n,x\in(-R,R),R>0$，证明：
(1)若$f(x)$为偶函数，则$a_{2k+1}=0(k=0,1,2,\cdots)$；
(2)若$f(x)$为奇函数，则$a_{2k}=0(k=0,1,2,\cdots)$.

证明：(1)$\because f(x)=\sum_{n=0}^{\infty}a_nx^n,x\in(-R,R),R>0$是偶函数

$\therefore f(x)=f(-x),\forall x\in(-R,R)$

$\therefore\forall x\in(-R,R),\sum_{n=0}^{\infty}a_nx^n,\sum_{n=0}^{\infty}a_n(-x)^n$均收敛，且$\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_n(-x)^n$

$\therefore\forall x\in(-R,R),\sum_{n=0}^{\infty}a_nx^n-\sum_{n=0}^{\infty}a_n(-x)^n=0=\sum_{n=0}^{\infty}[a_n-(-1)^na_n]x^n=\sum_{k=0}^{\infty}2a_{2k+1}x^{2k+1}$

$\therefore a_{2k+1}=0,k=0,1,2,\cdots$.

(2)$\because f(x)=\sum_{n=0}^{\infty}a_nx^n, x\in(-R,R), R>0$是奇函数

$\therefore f(x)=-f(-x), \forall x\in(-R,R)$

$\therefore \forall x\in(-R,R), \sum_{n=0}^{\infty}a_nx^n, \sum_{n=0}^{\infty}a_n(-x)^n$均收敛，且$\sum_{n=0}^{\infty}a_nx^n=-\sum_{n=0}^{\infty}a_n(-x)^n$

$\therefore \forall x\in(-R,R), \sum_{n=0}^{\infty}a_nx^n+\sum_{n=0}^{\infty}a_n(-x)^n=0=\sum_{n=0}^{\infty}[a_n+(-1)^na_n]x^n=\sum_{k=0}^{\infty}2a_{2k}x^{2k}$

$\therefore a_{2k}=0, k=0,1,2,\cdots.$

10. 设幂级数$\sum_{n=0}^{\infty}a_nx^n$在点$x=3$处条件收敛，试求幂级数$\sum_{n=1}^{\infty}na_n(x-1)^{n+1}$的收敛半径.

解：$\because \sum_{n=0}^{\infty}a_nx^n$在点$x=3$处条件收敛

$\therefore \sum_{n=0}^{\infty}a_nx^n$的收敛半径$R=3$

$\because \sum_{n=1}^{\infty}na_n(x-1)^{n+1}=(x-1)^2\sum_{n=1}^{\infty}na_n(x-1)^{n-1}=(x-1)^2\sum_{n=1}^{\infty}[a_n(x-1)^n]'=(x-1)^2[\sum_{n=1}^{\infty}a_n(x-1)^n]'=(x-1)^2[\sum_{n=0}^{\infty}a_n(x-1)^n-a_0]'$

$\therefore$幂级数$\sum_{n=1}^{\infty}na_n(x-1)^{n+1}$的收敛半径$R=3.$

13.4 习题8.6解答

1. 将下列函数在所给长度为$2l$的区间上，展成以$2l$为周期的傅里叶级数：

(1)$f(x)=\frac{\pi}{4}-\frac{x}{2}, x\in(-\pi,\pi)$；
(2)$f(x)=x^2, x\in(0,2\pi)$；
(3)$f(x)=|x|, x\in(-l,l)$；
(4)$f(x)=\begin{cases}-\pi, & -\pi\le x<0,\\ 3x^2+1, & 0\le x<\pi.\end{cases}$

解：(1)$l=\pi, a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(\frac{\pi}{4}-\frac{x}{2})\mathrm{d}x=\frac{1}{\pi}(\frac{\pi}{4}x-\frac{1}{4}x^2)\Big|_{-\pi}^{\pi}=\frac{\pi}{2}$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(\frac{\pi}{4}-\frac{x}{2})\cos(nx)\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\pi}{4}\cos(nx)\mathrm{d}x=0$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(\frac{\pi}{4}-\frac{x}{2})\sin(nx)\mathrm{d}x=-\frac{1}{2\pi}\int_{-\pi}^{\pi}x\sin(nx)\mathrm{d}x=-\frac{1}{\pi}\int_0^{\pi}x\sin(nx)\mathrm{d}x=\frac{1}{n\pi}\int_0^{\pi}x\mathrm{d}\cos(nx)$
$=\frac{1}{n\pi}(x\cos nx\Big|_0^{\pi}-\int_0^{\pi}\cos nx\mathrm{d}x)=\frac{1}{n\pi}(\pi\cos n\pi-0)=\frac{(-1)^n}{n}$

$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$

(2)$l=\pi, a_0=\frac{1}{\pi}\int_0^{2\pi}x^2\mathrm{d}x=\frac{1}{\pi}\frac{1}{3}x^3\Big|_0^{2\pi}=\frac{8\pi^2}{3}$

$a_n=\frac{1}{\pi}\int_0^{2\pi}x^2\cos nx\mathrm{d}x=\frac{1}{n\pi}\int_0^{2\pi}x^2\mathrm{d}\sin nx=\frac{1}{n\pi}(x^2\sin nx\Big|_0^{2\pi}-\int_0^{2\pi}2x\sin nx\mathrm{d}x)$
$=-\frac{2}{n\pi}\int_0^{2\pi}x\sin nx\mathrm{d}x=\frac{2}{n^2\pi}\int_0^{2\pi}x\mathrm{d}\cos nx=\frac{2}{n^2\pi}(x\cos nx\Big|_0^{2\pi}-\int_0^{2\pi}\cos nx\mathrm{d}x)=\frac{4}{n^2}$

$b_n=\frac{1}{\pi}\int_0^{2\pi}x^2\sin nx\mathrm{d}x=-\frac{1}{n\pi}\int_0^{2\pi}x^2\mathrm{d}\cos nx=-\frac{1}{n\pi}(x^2\cos nx\Big|_0^{2\pi}-\int_0^{2\pi}2x\cos nx\mathrm{d}x)$
$=-\frac{1}{n\pi}(4\pi^2-\frac{2}{n}\int_0^{2\pi}x\mathrm{d}\sin nx)=-\frac{4\pi}{n}+\frac{2}{n^2\pi}(x\sin nx\Big|_0^{2\pi}-\int_0^{2\pi}\sin nx\mathrm{d}x)=-\frac{4\pi}{n}-\frac{2}{n^2\pi}\int_0^{2\pi}\sin nx\mathrm{d}x$
$=-\frac{4\pi}{n}$

$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}(\frac{4}{n^2}\cos nx-\frac{4\pi}{n}\sin nx).$

$(3)a_0=\frac{1}{l}\int_{-l}^{l}f(x)\mathrm{d}x=\frac{1}{l}\int_{-l}^{l}|x|\mathrm{d}x=\frac{2}{l}\int_0^{l}x\mathrm{d}x=\frac{1}{l}x^2\Big|_0^{l}=l$

$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi}{l}x\mathrm{d}x=\frac{1}{l}\int_{-l}^{l}|x|\cos\frac{n\pi}{l}x\mathrm{d}x=\frac{2}{l}\int_0^{l}x\cos\frac{n\pi}{l}x\mathrm{d}x=\frac{2}{n\pi}\int_0^{l}x\mathrm{d}\sin\frac{n\pi}{l}x$
$=\frac{2}{n\pi}(x\sin\frac{n\pi}{l}x\Big|_0^{l}-\int_0^{l}\sin\frac{n\pi}{l}x\mathrm{d}x)=-\frac{2}{n\pi}\int_0^{l}\sin\frac{n\pi}{l}x\mathrm{d}x=\frac{2l}{n^2\pi^2}\cos\frac{n\pi}{l}x\Big|_0^{l}=\frac{2l}{n^2\pi^2}[(-1)^n-1]$

$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi}{l}x\mathrm{d}x=\frac{1}{l}\int_{-l}^{l}|x|\sin\frac{n\pi}{l}x\mathrm{d}x=0$

$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi}{l}x+b_n\sin\frac{n\pi}{l}x)=\frac{l}{2}+\sum_{n=1}^{\infty}[(-1)^n-1]\frac{2l}{n^2\pi^2}\cos\frac{n\pi}{l}x$
$=\frac{l}{2}+\sum_{n=1}^{\infty}(-2)\frac{2l}{(2n-1)^2\pi^2}\cos\frac{(2n-1)\pi}{l}x=\frac{l}{2}+\sum_{n=1}^{\infty}\frac{-4l}{(2n-1)^2\pi^2}\cos\frac{(2n-1)\pi}{l}x.$

$(4)l=\pi,a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\mathrm{d}x=\frac{1}{\pi}[\int_{-\pi}^{0}(-\pi)\mathrm{d}x+\int_0^{\pi}(3x^2+1)\mathrm{d}x]=\frac{1}{\pi}[-\pi^2+(x^3+x)\Big|_0^{\pi}]$
$=\pi^2-\pi+1$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\mathrm{d}x=\frac{1}{\pi}[\int_{-\pi}^{0}(-\pi)\cos nx\mathrm{d}x+\int_0^{\pi}(3x^2+1)\cos nx\mathrm{d}x]$
$=-\int_{-\pi}^{0}\cos nx\mathrm{d}x+\frac{1}{\pi}\int_0^{\pi}(3x^2+1)\cos nx\mathrm{d}x=-\frac{1}{n}\sin nx\Big|_{-\pi}^{0}+\frac{1}{n\pi}\int_0^{\pi}(3x^2+1)\mathrm{d}\sin nx$
$=\frac{1}{n\pi}(3x^2+1)\sin nx\Big|_0^{\pi}-\frac{1}{n\pi}\int_0^{\pi}6x\sin nx\mathrm{d}x=\frac{6}{n^2\pi}\int_0^{\pi}x\mathrm{d}\cos nx$
$=\frac{6}{n^2\pi}(x\cos nx\Big|_0^{\pi}-\int_0^{\pi}\cos nx\mathrm{d}x)=\frac{6}{n^2\pi}[\pi(-1)^n-\int_0^{\pi}\cos nx\mathrm{d}x]=\frac{6}{n^2\pi}[\pi(-1)^n-\frac{1}{n}\sin nx\Big|_0^{\pi}]$
$=\frac{6\pi(-1)^n}{n^2\pi}$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\mathrm{d}x=\frac{1}{\pi}[\int_{-\pi}^{0}(-\pi)\sin nx\mathrm{d}x+\int_0^{\pi}(3x^2+1)\sin nx\mathrm{d}x]$
$=-\int_{-\pi}^{0}\sin nx\mathrm{d}x+\frac{1}{\pi}\int_0^{\pi}(3x^2+1)\sin nx\mathrm{d}x=\frac{1}{n}\cos nx\Big|_{-\pi}^{0}-\frac{1}{n\pi}\int_0^{\pi}(3x^2+1)\mathrm{d}\cos nx$
$=\frac{1-(-1)^n}{n}-\frac{1}{n\pi}[(3x^2+1)\cos nx\Big|_0^{\pi}-\int_0^{\pi}6x\cos nx\mathrm{d}x]$
$=\frac{1-(-1)^n}{n}-\frac{1}{n\pi}[(3\pi^2+1)(-1)^n-1-\frac{6}{n}\int_0^{\pi}x\mathrm{d}\sin nx]$
$=\frac{1-(-1)^n}{n}-\frac{(3\pi^2+1)(-1)^n-1}{n\pi}+\frac{6}{n^2\pi}(x\sin nx\Big|_0^{\pi}-\int_0^{\pi}\sin nx\mathrm{d}x)$
$=\frac{1-(-1)^n}{n}-\frac{(3\pi^2+1)(-1)^n-1}{n\pi}+\frac{6}{n^3\pi}\cos nx\Big|_0^{\pi}$
$=\frac{1-(-1)^n}{n}-\frac{(3\pi^2+1)(-1)^n-1}{n\pi}+\frac{6[(-1)^n-1]}{n^3\pi}$
$=\frac{3\pi(-1)^{n-1}}{n}+(\frac{1}{n}+\frac{1}{n\pi}-\frac{6}{n^3\pi})[1-(-1)^n]$

$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$
$=\frac{\pi^2-\pi+1}{2}+\sum_{n=1}^{\infty}\left(\frac{6\pi(-1)^n}{n^2\pi}\cos nx+\{\frac{3\pi(-1)^{n-1}}{n}+(\frac{1}{n}+\frac{1}{n\pi}-\frac{6}{n^3\pi})[1-(-1)^n]\}\sin nx\right)$

2. 将下列函数在所给长度为l的区间上，展成以$2l$为周期的余弦级数：

$(1)f(x)=\mathrm{e}^x,x\in(0,\pi)$；

$(2) f(x)=\begin{cases}1, & 0 \leq x \leq h, \\ 0, & h<x<\pi ;\end{cases}$

$(3) f(x)=\begin{cases}x, & 0 \leq x \leq \pi, \\ 0, & \pi<x<2 \pi .\end{cases}$

解：$(1) l=\pi, a_0=\frac{2}{\pi} \int_0^\pi \mathrm{e}^x \mathrm{d} x=\left.\frac{2}{\pi} \mathrm{e}^x\right|_0^\pi=\frac{2}{\pi}(\mathrm{e}^\pi-1)$

$a_n=\frac{2}{\pi} \int_0^\pi \mathrm{e}^x \cos n x \mathrm{d} x=\frac{2}{\pi} \int_0^\pi \cos n x \mathrm{d e}^x=\frac{2}{\pi}(\left.\mathrm{e}^x \cos n x\right|_0^\pi+n \int_0^\pi \mathrm{e}^x \sin n x \mathrm{d} x)$
$=\frac{2}{\pi}[\mathrm{e}^\pi(-1)^n-1]+\frac{2 n}{\pi} \int_0^\pi \sin n x \mathrm{de}^x=\frac{2}{\pi}[\mathrm{e}^\pi(-1)^n-1]+\frac{2 n}{\pi}(\left.\mathrm{e}^x \sin n x\right|_0^\pi-n \int_0^\pi \mathrm{e}^x \cos n x \mathrm{d} x)$
$=\frac{2}{\pi}[\mathrm{e}^\pi(-1)^n-1]-\frac{2 n^2}{\pi} \int_0^\pi \mathrm{e}^x \cos n x \mathrm{d} x=\frac{2}{\pi} \frac{\frac{2}{\pi}[\mathrm{e}^\pi(-1)^n-1]}{\frac{2}{\pi}+\frac{2 n^2}{\pi}}=\frac{2[\mathrm{e}^\pi(-1)^n-1]}{\pi(1+n^2)}$

$f(x) \sim \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n x=\frac{1}{\pi}(\mathrm{e}^\pi-1)+\sum_{n=1}^{\infty} \frac{2[\mathrm{e}^\pi(-1)^n-1]}{\pi(1+n^2)} \cos n x .$

$(2) l=\pi, a_0=\frac{2}{\pi} \int_0^\pi f(x) \mathrm{d} x=\frac{2}{\pi}(\int_0^h \mathrm{d} x+\int_h^\pi 0 \mathrm{d} x)=\frac{2 h}{\pi} .$

$a_n=\frac{2}{\pi} \int_0^\pi f(x) \cos n x \mathrm{d} x=\frac{2}{\pi} \int_0^h \cos n x \mathrm{d} x=\left.\frac{2}{n \pi} \sin n x\right|_0^h=\frac{2}{n \pi} \sin n h$

$f(x) \sim \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n x=\frac{h}{\pi}+\sum_{n=1}^{\infty} \frac{2}{n \pi} \sin n h \cos n x .$

$(3) l=2 \pi, a_0=\frac{2}{2 \pi} \int_0^{2 \pi} f(x) \mathrm{d} x=\frac{1}{\pi}(\int_0^\pi x \mathrm{d} x+\int_\pi^{2 \pi} 0 \mathrm{d} x)=\left.\frac{1}{\pi} \frac{1}{2} x^2\right|_0^\pi=\frac{\pi}{2}$

$a_n=\frac{2}{2 \pi} \int_0^{2 \pi} f(x) \cos \frac{n \pi}{2 \pi} x \mathrm{d} x=\frac{1}{\pi} \int_0^\pi x \cos \frac{n}{2} x \mathrm{d} x=\frac{2}{n \pi} \int_0^\pi x \mathrm{d} \sin \frac{n}{2} x=\frac{2}{n \pi}(\left.x \sin \frac{n}{2} x\right|_0^\pi-\int_0^\pi \sin \frac{n}{2} x \mathrm{d} x)=\frac{2}{n} \sin \frac{n}{2} \pi-\frac{2}{n \pi} \int_0^\pi \sin \frac{n}{2} x \mathrm{d} x=\frac{2}{n} \sin \frac{n}{2} \pi-\left.\frac{4}{n^2 \pi} \cos \frac{n}{2} x\right|_0^\pi$
$=\frac{2}{n} \sin \frac{n}{2} \pi-\frac{4}{n^2 \pi}(\cos \frac{n}{2} \pi-1)$

$f(x) \sim \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos \frac{n}{2} x=\frac{\pi}{4}+\sum_{n=0}^{\infty}[\frac{2}{n} \sin \frac{n}{2} \pi-\frac{4}{n^2 \pi}(\cos \frac{n}{2} \pi-1)] \cos \frac{n}{2} x .$

3. 将下列函数在所给长度为l的区间上，展成以$2l$为周期的正弦级数：

$(1) f(x)=\frac{\pi-x}{2}, x \in(0, \pi)$；

$(2) f(x)=x(\pi-x), x \in(0, \pi)$.

解：$(1) l=\pi$

$b_n=\frac{2}{\pi} \int_0^\pi f(x) \sin n x \mathrm{d} x=\frac{2}{\pi} \int_0^\pi \frac{\pi-x}{2} \sin n x \mathrm{d} x=\frac{1}{n \pi} \int_0^\pi(x-\pi) \mathrm{d} \cos n x$
$=\frac{1}{n \pi}[\left.(x-\pi) \cos n x\right|_0^\pi-\int_0^\pi \cos n x \mathrm{d} x]=\frac{1}{n \pi}[\pi-\left.\frac{1}{n} \sin n x\right|_0^\pi]=\frac{1}{n}$

$f(x) \sim \sum_{n=1}^{\infty} b_n \sin n x=\sum_{n=1}^{\infty} \frac{1}{n} \sin n x .$

$(2) l=\pi$

$b_n=\frac{2}{\pi} \int_0^\pi f(x) \sin n x \mathrm{d} x=\frac{2}{\pi} \int_0^\pi x(\pi-x) \sin n x \mathrm{d} x=\frac{2}{\pi} \int_0^\pi(\pi x-x^2) \sin n x \mathrm{d} x$
$=\frac{2}{n \pi} \int_0^\pi(x^2-\pi x) \mathrm{d} \cos n x=\frac{2}{n \pi}[\left.(x^2-\pi x) \cos n x\right|_0^\pi-\int_0^\pi(2 x-\pi) \cos n x \mathrm{d} x]$
$=-\frac{2}{n \pi} \int_0^\pi(2 x-\pi) \cos n x \mathrm{d} x=-\frac{2}{n^2 \pi} \int_0^\pi(2 x-\pi) \mathrm{d} \sin n x$

$= -\frac{2}{n^2\pi}[(2x-\pi)\sin nx\Big|_0^\pi - \int_0^\pi 2\sin nx\mathrm{d}x] = \frac{4}{n^2\pi}\int_0^\pi \sin nx\mathrm{d}x = -\frac{4}{n^3\pi}\cos n\pi\Big|_0^\pi$
$= \frac{4[1-(-1)^n]}{n^3\pi}$

$f(x) \sim \sum_{n=1}^\infty b_n \sin nx = \sum_{n=1}^\infty \frac{4[1-(-1)^n]}{n^3\pi}\sin nx = \sum_{n=1}^\infty \frac{8}{(2n-1)^3\pi}\sin(2n-1)x.$

4. 把函数$f(x) = -x+1$按下列要求展开，并比较各和函数的图形.
(1)在$(0,2\pi)$展成以2π为周期的傅里叶级数；
(2)在$(0,\pi)$展成以2π为周期的正弦级数；
(3)在$(0,\pi)$展成以π为周期的傅里叶级数；
(4)在$(-1,1)$展成以2为周期的傅里叶级数.

解：(1)$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}(-x+1)\mathrm{d}x = \frac{1}{\pi}(-\frac{1}{2}x^2+x)\Big|_0^{2\pi} = \frac{1}{\pi}(-\frac{1}{2}4\pi^2+2\pi)$
$= -2\pi+2$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}(-x+1)\cos nx\mathrm{d}x = \frac{1}{n\pi}\int_0^{2\pi}(-x+1)\mathrm{d}\sin nx$
$= \frac{1}{n\pi}[(-x+1)\sin nx\Big|_0^{2\pi} + \int_0^{2\pi}\sin nx\mathrm{d}x] = \frac{1}{n\pi}\int_0^{2\pi}\sin nx\mathrm{d}x = -\frac{1}{n^2\pi}\cos nx\Big|_0^{2\pi} = 0$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}(-x+1)\sin nx\mathrm{d}x = -\frac{1}{\pi}\int_0^{2\pi} x\sin nx\mathrm{d}x = \frac{1}{n\pi}\int_0^{2\pi} x\mathrm{d}\cos nx$
$= \frac{1}{n\pi}(x\cos nx\Big|_0^{2\pi} - \int_0^{2\pi}\cos nx\mathrm{d}x) = \frac{1}{n\pi}(2\pi - \frac{1}{n}\sin nx\Big|_0^{2\pi}) = \frac{2}{n}$

$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^\infty(a_n\cos nx + b_n\sin nx) = 1-\pi+\sum_{n=1}^\infty \frac{2}{n}\sin nx.$

和函数如图 1所示.

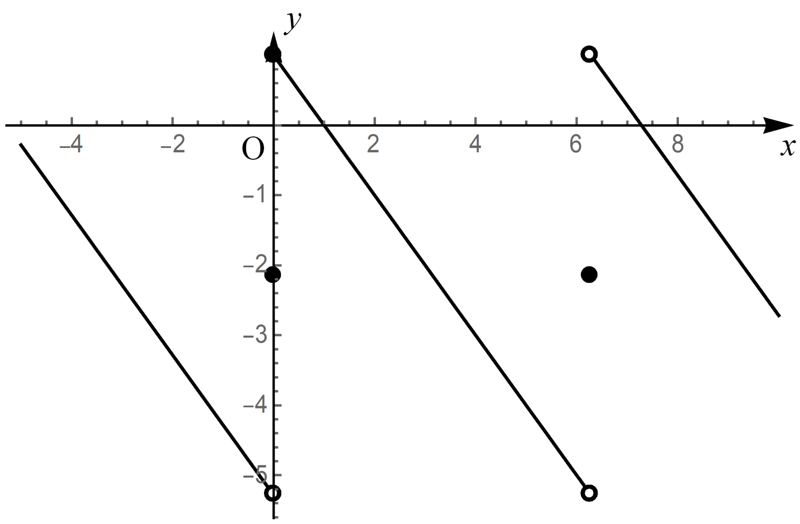

图 1: 习题8.6 4.(1)图示

(2)$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\mathrm{d}x = \frac{2}{\pi}\int_0^\pi(-x+1)\sin nx\mathrm{d}x = \frac{2}{n\pi}\int_0^\pi(x-1)\mathrm{d}\cos nx$
$= \frac{2}{n\pi}[(x-1)\cos nx\Big|_0^\pi - \int_0^\pi\cos nx\mathrm{d}x] = \frac{2}{n\pi}[(\pi-1)(-1)^n+1-\frac{1}{n}\sin nx\Big|_0^\pi]$
$= \frac{2}{n\pi}[(\pi-1)(-1)^n+1]$

$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{2}{n\pi}[(\pi-1)(-1)^n+1]\sin nx.$

和函数如图 2所示.

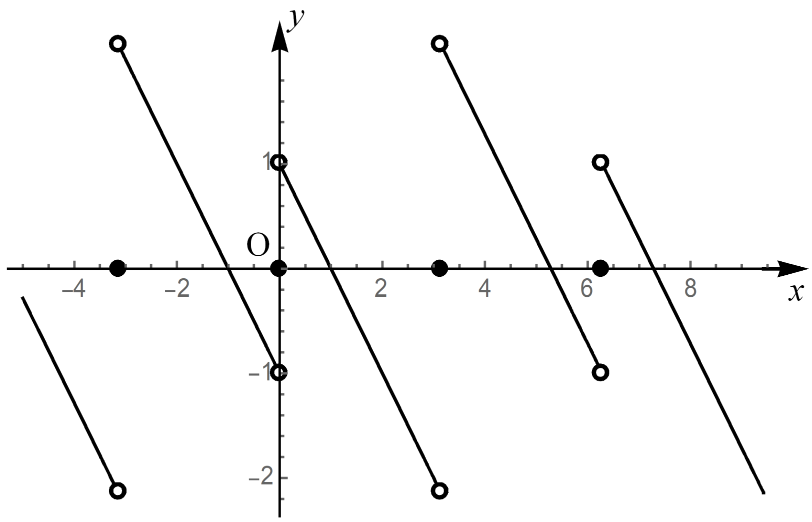

图 2: 习题8.6 4.(2)图示

(3)$l=\frac{\pi}{2}, a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}(-x+1)\mathrm{d}x=\frac{2}{\pi}(-\frac{1}{2}x^2+x)\Big|_0^{\pi}=-\pi+2$

$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos\frac{n\pi}{\frac{\pi}{2}}x\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}(-x+1)\cos 2nx\mathrm{d}x=\frac{2}{2n\pi}\int_0^{\pi}(-x+1)\mathrm{d}\sin 2nx$

$=\frac{2}{2n\pi}[(-x+1)\sin 2nx\Big|_0^{\pi}+\int_0^{\pi}\sin 2nx\mathrm{d}x]=\frac{1}{n\pi}\int_0^{\pi}\sin 2nx\mathrm{d}x=-\frac{1}{2n^2\pi}\cos 2nx\Big|_0^{\pi}=0$

$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin 2nx\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}(-x+1)\sin 2nx\mathrm{d}x=\frac{2}{2n\pi}\int_0^{\pi}(x-1)\mathrm{d}\cos 2nx$

$=\frac{1}{n\pi}[(x-1)\cos 2nx\Big|_0^{\pi}-\int_0^{\pi}\cos 2nx\mathrm{d}x]=\frac{1}{n}$

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos 2nx+b_n\sin 2nx)=1-\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{1}{n}\sin 2nx.$

和函数如图 3所示.

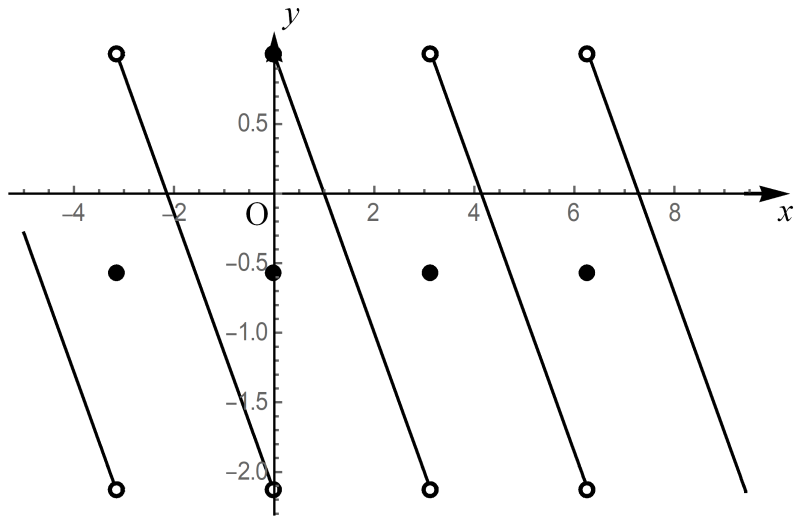

图 3: 习题8.6 4.(3)图示

(4) $l=1, a_0=\frac{1}{1}\int_{-1}^{1}f(x)\mathrm{d}x=\int_{-1}^{1}(-x+1)\mathrm{d}x=2$

$a_n=\frac{1}{1}\int_{-1}^{1}f(x)\cos\frac{n\pi}{1}x\mathrm{d}x=\int_{-1}^{1}(-x+1)\cos n\pi x\mathrm{d}x=\int_{-1}^{1}\cos n\pi x\mathrm{d}x=\frac{1}{n\pi}\sin n\pi x\Big|_{-1}^{1}$

$=\frac{1}{n\pi}(\sin n\pi+\sin n\pi)=0$

$b_n=\frac{1}{1}\int_{-1}^{1}f(x)\sin\frac{n\pi}{1}x\mathrm{d}x=\int_{-1}^{1}(-x+1)\sin n\pi x\mathrm{d}x=-\int_{-1}^{1}x\sin n\pi x\mathrm{d}x=\frac{1}{n\pi}\int_{-1}^{1}x\mathrm{d}\cos n\pi x$

$=\frac{1}{n\pi}(x\cos n\pi x\Big|_{-1}^{1}-\int_{-1}^{1}\cos n\pi x\mathrm{d}x)=\frac{1}{n\pi}(\cos n\pi+\cos n\pi-\frac{1}{n\pi}\sin n\pi x\Big|_{-1}^{1})=\frac{2(-1)^n}{n\pi}$

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\pi x+b_n\sin n\pi x)=1+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\sin n\pi x.$

和函数如图 4所示.

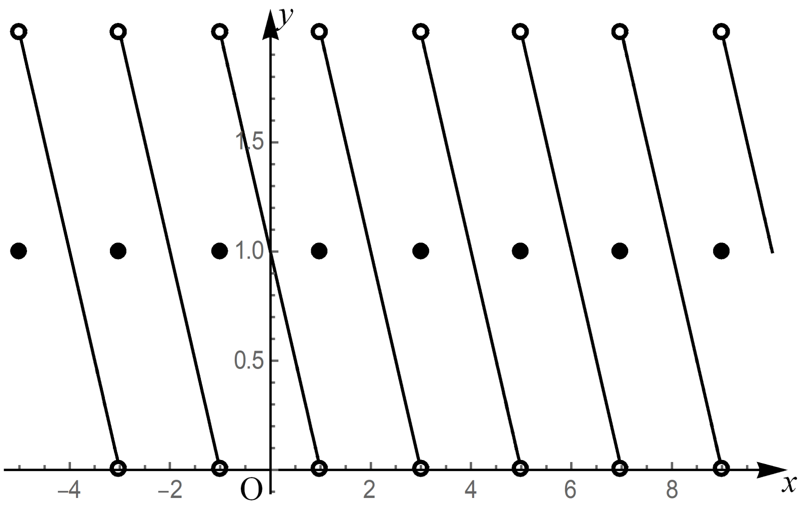

图 4: 习题8.6 4.(4)图示