12 数项级数

12.1 知识结构

第8章级数

8.1 数项级数的概念与性质

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12.2 习题8.1解答

1. 求下列级数的和：

(1)$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$；

(2)$\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n+1)}$；

解：(1)$\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}=\sum_{k=1}^{n}\frac{1}{2}[\frac{1}{k(k+1)}-\frac{1}{(k+1)(k+2)}]=\frac{1}{2}[\frac{1}{1\times2}-\frac{1}{2\times3}+\frac{1}{2\times3}-\frac{1}{3\times4}+\frac{1}{3\times4}-\frac{1}{4\times5}+\cdots+\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}]=\frac{1}{2}[\frac{1}{1\times2}-\frac{1}{(n+1)(n+2)}]\to\frac{1}{4}(n\to\infty)$.

(2)$\sum_{k=1}^{n}\frac{1}{(3k-2)(3k+1)}=\sum_{k=1}^{n}\frac{1}{3}(\frac{1}{3k-2}-\frac{1}{3k+1})=\frac{1}{3}[1-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\frac{1}{7}-\frac{1}{10}+\cdots+\frac{1}{3n-5}-\frac{1}{3n-2}+\frac{1}{3n-2}-\frac{1}{3n+1}]=\frac{1}{3}[1-\frac{1}{3n+1}]\to\frac{1}{3}(n\to\infty)$.

2. 利用级数的基本性质研究下列级数的收敛性：

(1)$\sum_{n=1}^{\infty}(\frac{3}{2^n}-\frac{4}{3^n})$；　(2)$\sum_{n=1}^{\infty}(\frac{1}{n^2}-\frac{1}{n})$；

(3)$\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n+3})$；　(4)$\sum_{n=1}^{\infty}\frac{n-100}{n}$.

解：(1)$\because\sum_{n=1}^{\infty}\frac{1}{2^n}$和$\sum_{n=1}^{\infty}\frac{1}{3^n}$均收敛

$\therefore \sum_{n=1}^{\infty}(\frac{3}{2^n}-\frac{4}{3^n})=2\sum_{n=1}^{\infty}\frac{1}{2^n}-3\sum_{n=1}^{\infty}\frac{1}{3^n}$收敛.

(2)假设$\sum_{n=1}^{\infty}(\frac{1}{n^2}-\frac{1}{n})=\sum_{n=1}^{\infty}(a_n-b_n)$收敛，则$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}[a_n-(a_n-b_n)]=\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}(a_n-b_n)$收敛，这与$\sum_{n=1}^{\infty}\frac{1}{n}$发散矛盾

故$\sum_{n=1}^{\infty}(\frac{1}{n^2}-\frac{1}{n})$发散.

(3)$\sum_{k=1}^{n}(\frac{1}{k}-\frac{1}{k+3})=1-\frac{1}{4}+\frac{1}{2}-\frac{1}{5}+\frac{1}{3}-\frac{1}{6}+\frac{1}{4}-\frac{1}{7}+\cdots+\frac{1}{n-4}-\frac{1}{n-1}+\frac{1}{n-3}-\frac{1}{n+1}+\frac{1}{n-2}-\frac{1}{n+2}+\frac{1}{n-3}-\frac{1}{n+3}=1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\to 1+\frac{1}{2}+\frac{1}{3}=\frac{11}{6}(n\to\infty)$

$\therefore \sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n+3})$收敛.

(4)$\because \lim\limits_{n\to\infty}\frac{n-100}{n}=1\neq 0$

$\therefore \sum_{n=1}^{\infty}\frac{n-100}{n}$发散.

12.3 习题8.2解答

1. 用比阶判别法判断下列级数的收敛性：

(1)$\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^4+4n-3}}$； (2)$\sum_{n=1}^{\infty}\frac{\sqrt{n+2}-\sqrt{n-1}}{n^\alpha}$；

(3)$\sum_{n=1}^{\infty}(1-\cos\frac{1}{n})$； (4)$\sum_{n=2}^{\infty}n\ln(1-\frac{1}{n^p})$；

(5)$\sum_{n=1}^{\infty}\frac{n^2}{2^n}$； (6)$\sum_{n=2}^{\infty}\frac{1}{(2n-1)^p}$.

解：(1)$\because \lim\limits_{n\to\infty}n^{\frac{4}{3}}\cdot\frac{1}{\sqrt[3]{n^4+4n-3}}=1$

$\therefore \sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^4+4n-3}}$收敛.

(2)$\frac{\sqrt{n+2}-\sqrt{n-1}}{n^\alpha}=\frac{(n+2)-(n-1)}{n^\alpha\cdot(\sqrt{n+2}+\sqrt{n-1})}=\frac{3}{n^\alpha\cdot(\sqrt{n+2}+\sqrt{n-1})}$

$\lim\limits_{n\to\infty}n^{\alpha+\frac{1}{2}}\cdot\frac{\sqrt{n+2}-\sqrt{n-1}}{n^\alpha}=\lim\limits_{n\to\infty}n^{\alpha+\frac{1}{2}}\cdot\frac{3}{n^\alpha\cdot(\sqrt{n+2}+\sqrt{n-1})}=\lim\limits_{n\to\infty}\frac{3}{(\sqrt{1+\frac{2}{n}}+\sqrt{1-\frac{1}{n}})}=\frac{3}{2}$

$\therefore$当$\alpha+\frac{1}{2}>1$即$\alpha>\frac{1}{2}$时，级数收敛；当$\alpha+\frac{1}{2}\leq 1$即$\alpha\leq\frac{1}{2}$时，级数发散.

(3)$\because \lim\limits_{n\to\infty}n^2\cdot(1-\cos\frac{1}{n})=\lim\limits_{n\to\infty}n^2\cdot 2\sin^2\frac{1}{2n}=\lim\limits_{n\to\infty}n^2\cdot 2(\frac{1}{2n})^2=\frac{1}{2}$

$\therefore$级数$\sum_{n=1}^{\infty}(1-\cos\frac{1}{n})$收敛.

(4)由$\ln(1-\frac{1}{n^p}),n=2,3,\cdots$知$p<0$

$\because \lim\limits_{n\to\infty}n^{p-1}\cdot|n\ln(1-\frac{1}{n^p})|=-\lim\limits_{n\to\infty}n^p\cdot\ln(1-\frac{1}{n^p})=1$

$\therefore$当$p-1>1$即$p>2$时，级数收敛；当$p-1\leq 1$即$p\leq 2$时，级数发散.

(5)$\because \lim\limits_{n\to\infty}n^2\cdot\frac{n^2}{2^n}=0$

$\therefore$级数收敛.

(6)$\because \lim\limits_{n\to\infty}n^p\cdot\frac{1}{(2n-1)^p}=\lim\limits_{n\to\infty}\frac{1}{2^p}$

$\therefore$当$p>1$时级数收敛，当$p\leq 1$时级数发散.

2. 利用比值或根值判别法判断下列级数的收敛性：

$(1)\sum_{n=1}^{\infty}\frac{n^p}{a^n}(a>0)$；　$(2)\sum_{n=1}^{\infty}\frac{a^n}{n!}(a>0)$；

$(3)\sum_{n=1}^{\infty}\frac{(2n-1)!!}{3^n\cdot n!}$；　$(4)\sum_{n=1}^{\infty}\frac{2^n+3^n}{n^p}(p>0)$；

$(5)\sum_{n=1}^{\infty}\frac{2n-1}{2^n+2^{-n}}$；　$(6)\sum_{n=1}^{\infty}\frac{a^n}{1+a^{2n}}(a>0)$；

$(7)\sum_{n=2}^{\infty}(\frac{1+\ln n}{1+\sqrt{n}})^n$；　$(8)\sum_{n=1}^{\infty}\frac{1}{3^n}(\frac{n+1}{n})^{n^2}$.

解：(1)$\because \lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{(n+1)^p}{a^{n+1}}\frac{a^n}{n^p}=\lim\limits_{n\to\infty}\frac{1}{a}(1+\frac{1}{n})^p=\frac{1}{a}$

$\therefore$当$\frac{1}{a}<1$即$a>1$时，级数收敛；当$\frac{1}{a}>1$即$a<1$时级数发散；

当$a=1$时$\sum_{n=1}^{\infty}\frac{n^p}{a^n}=\sum_{n=1}^{\infty}n^p$，此时当$p>1$时级数收敛，当$p\le 1$时级数发散.

(2)$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{a^{n+1}}{(n+1)!}\frac{n!}{a^n}=\lim\limits_{n\to\infty}\frac{a}{n+1}=0<1$，级数收敛.

(3)$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{(2n+1)!!}{3^{n+1}\cdot(n+1)!}\frac{3^n\cdot n!}{(2n-1)!!}=\lim\limits_{n\to\infty}\frac{2n+1}{3(n+1)}=\frac{2}{3}<1$，级数收敛.

(4)$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{2^{n+1}+3^{n+1}}{(n+1)^p}\frac{n^p}{2^n+3^n}=\lim\limits_{n\to\infty}\frac{2(\frac{2}{3})^n+3}{(\frac{2}{3})^n+1}(\frac{n}{n+1})^p=3>1$，级数发散.

(5)$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{2n+3}{2^{n+1}+2^{-n-1}}\frac{2^n+2^{-n}}{2n-1}=\lim\limits_{n\to\infty}\frac{2n+3}{2n-1}\frac{1+2^{-2n}}{2+2^{-2n-1}}=\frac{1}{2}<1$，级数收敛.

(6)$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{a^{n+1}}{1+a^{2n+2}}\frac{1+a^{2n}}{a^n}=a\lim\limits_{n\to\infty}\frac{a^{-2n}+1}{a^{-2n}+a^2}=\begin{cases}a<1, & a<1\\ 1, & a=1\\ \frac{1}{a}<1, & a>1\end{cases}$

当$a=1$时$\sum_{n=1}^{\infty}\frac{a^n}{1+a^{2n}}=\sum_{n=1}^{\infty}\frac{1}{2}$发散

故当$a\ne 1$时级数收敛.

(7)$\lim\limits_{n\to\infty}\sqrt[n]{(\frac{1+\ln n}{1+\sqrt{n}})^n}=\lim\limits_{n\to\infty}\frac{1+\ln n}{1+\sqrt{n}}=\lim\limits_{n\to\infty}\frac{\frac{1}{\sqrt{n}}+\frac{\ln n}{\sqrt{n}}}{\frac{1}{\sqrt{n}}+1}=0<1$，级数收敛.

(8)$\lim\limits_{n\to\infty}\sqrt[n]{\frac{1}{3^n}(\frac{n+1}{n})^{n^2}}=\lim\limits_{n\to\infty}\frac{1}{3}(1+\frac{1}{n})^n=\frac{\mathrm{e}}{3}<1$，级数收敛.

3. 设$p>0$，研究级数

$$\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots+\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}+\cdots$$

的收敛性.

解：该级数的前$2n$项和$S_{2n}=\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots+\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}$，$S_{2n}$可视为正项级数$\sum_{n=1}^{\infty}[\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}]$的前$n$项和

$\because \lim\limits_{n\to\infty}n^p[\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}]=\lim\limits_{n\to\infty}[\frac{n^p}{(2n-1)^p}-\frac{n^p}{(2n)^{2p}}]=\frac{1}{2^p}$

$\therefore$i)当$p>1$时级数$\sum_{n=1}^{\infty}[\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}]$收敛，即$\lim\limits_{n\to\infty}S_{2n}$存在

又$\because \lim\limits_{n\to\infty}\frac{1}{(2n+1)^p}=0$

$\therefore \lim\limits_{n\to\infty}S_{2n+1}=\lim\limits_{n\to\infty}[S_{2n}+\frac{1}{(2n+1)^p}]=\lim\limits_{n\to\infty}S_{2n}$

故$\lim\limits_{n\to\infty} S_n=\lim\limits_{n\to\infty} S_{2n}=\lim\limits_{n\to\infty} S_{2n+1}$存在，级数收敛；

ii)当$p\le 1$时级数$\sum_{n=1}^{\infty}[\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}]$发散，即$\lim\limits_{n\to\infty} S_{2n}=+\infty$

又$\because \lim\limits_{n\to\infty}\frac{1}{(2n+1)^p}=0$

$\therefore \lim\limits_{n\to\infty} S_{2n+1}=\lim\limits_{n\to\infty}[S_{2n}+\frac{1}{(2n+1)^p}]=+\infty$

故$\lim\limits_{n\to\infty} S_n=+\infty$，级数发散；

综上所述，当$p>1$时级数收敛，当$p\le 1$时级数发散.

4. 设$a_n>0,\lim\limits_{n\to\infty} a_n>1$，求证$\sum_{n=1}^{\infty}\frac{1}{n^{a_n}}$收敛.

证明：$\because \lim\limits_{n\to\infty} a_n=A>1$

$\therefore \exists N>0, s.t. a_n>\frac{1+A}{2}=q>1(n>N)$

$\therefore$当$n>N$时$0<\frac{1}{n^{a_n}}<\frac{1}{n^q}$

$\because \sum_{n=N}^{\infty}\frac{1}{n^q}$收敛，故$\sum_{n=N}^{\infty}\frac{1}{n^{a_n}}$收敛，故$\sum_{n=1}^{\infty}\frac{1}{n^{a_n}}$收敛.

5. 判定下列级数是否收敛：

(1)$\sum_{n=1}^{\infty}\frac{n!a^n}{n^n}(a>0)$； (2)$\sum_{n=2}^{\infty}\frac{n^{\ln n}}{(\ln n)^n}$；

(3)$\sum_{n=2}^{\infty}\frac{\ln^q n}{n^p}(p>0,q>0)$； (4)$\sum_{n=1}^{\infty}(\frac{1}{n^\alpha}-\sin\frac{1}{n^\alpha})(a>0)$.

解：(1)$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{(n+1)!a^{n+1}}{(n+1)^{n+1}}\frac{n^n}{n!a^n}=\lim\limits_{n\to\infty} a\frac{1}{(1+\frac{1}{n})^n}=\frac{a}{\mathrm{e}}$

$\therefore$当$a<\mathrm{e}$时级数收敛，当$a>\mathrm{e}$时级数发散；

当$a=\mathrm{e}$时，由函数$f(x)=(1+\frac{1}{x})^x$在$x\ge 1$时单调增加(见教材P160例5.1.3)且$\lim\limits_{n\to\infty}(1+\frac{1}{x})^x=\mathrm{e}$可知$(1+\frac{1}{n})^n<\mathrm{e}$

$\therefore \frac{u_{n+1}}{u_n}=\mathrm{e}\frac{1}{(1+\frac{1}{n})^n}>1$

$\therefore u_{n+1}>u_n>\cdots>u_2>u_1=\mathrm{e},\lim\limits_{n\to\infty} u_n=\lim\limits_{n\to\infty}\frac{n!a^n}{n^n}\ne 0$，级数发散.

(2)$\lim\limits_{n\to\infty}\sqrt[n]{\frac{n^{\ln n}}{(\ln n)^n}}=\lim\limits_{n\to\infty}\frac{n^{\frac{\ln n}{n}}}{\ln n}=\lim\limits_{n\to\infty}\frac{\mathrm{e}^{\frac{\ln^2 n}{n}}}{\ln n}=\lim\limits_{n\to\infty}\frac{\mathrm{e}^{(\frac{\ln n}{\sqrt{n}})^2}}{\ln n}=0<1$，级数收敛.

(3)$\lim\limits_{n\to\infty} n^{\frac{p+1}{2}}\cdot\frac{\ln^q n}{n^p}=\lim\limits_{n\to\infty} n^{\frac{1-p}{2}}\ln^q n=\begin{cases}0, & p>1\\ +\infty, & p\le 1\end{cases}$

故当$p>1$时级数收敛，否则发散.

(4)$\lim\limits_{n\to\infty} n^{3\alpha}\cdot(\frac{1}{n^\alpha}-\sin\frac{1}{n^\alpha})=\lim\limits_{n\to\infty} n^\alpha\cdot[\frac{1}{n^\alpha}-\frac{1}{n^\alpha}+\frac{1}{3!}\frac{1}{(n^\alpha)^3}+o(\frac{1}{n^{3\alpha}})]=\lim\limits_{n\to\infty}[\frac{1}{3!}+\frac{o(\frac{1}{n^{3\alpha}})}{\frac{1}{n^{3\alpha}}}]=\frac{1}{3!}$

$\therefore$当$3\alpha>1$即$\alpha>\frac{1}{3}$时，级数收敛；当$3\alpha\le 1$即$\alpha\le\frac{1}{3}$时，级数发散.

12.4 习题8.3解答

1. 判断下列级数的收敛性，对收敛的级数指出绝对收敛，还是条件收敛：

   (1)$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\ln(n+1)}$； (2)$\sum_{n=1}^{\infty}\frac{\sin n\omega}{2^n}$($\omega$为常数)；

   (3)$\sum_{n=1}^{\infty}\frac{(-1)^n\ln(n+1)}{n}$； (4)$\sum_{n=1}^{\infty}\frac{(-1)^n}{n-\ln n}$；

   (5)$1-\ln 2+\frac{1}{2}-\ln\frac{3}{2}+\cdots+\frac{1}{n}-\ln\frac{n+1}{n}+\cdots$；

   (6)$\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}+(-1)^n}$； (7)$\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n+(-1)^n}}$.

   解：(1)$\because \ln(n)>\ln(n+1)$且$\lim\limits_{n\to\infty}\frac{1}{\ln(n+1)}=0$，故$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\ln(n+1)}$是莱布尼茨型交错级数，故收敛

   $\because \lim\limits_{n\to\infty}n\cdot|\frac{(-1)^{n-1}}{\ln(n+1)}|=\lim\limits_{n\to\infty}\frac{n}{\ln(n+1)}=+\infty$，故级数条件收敛.

   (2)$\because |\frac{\sin n\omega}{2^n}|\le\frac{1}{2^n}$且$\sum_{n=1}^{\infty}\frac{1}{2^n}$收敛

   $\therefore \sum_{n=1}^{\infty}\frac{\sin n\omega}{2^n}$绝对收敛.

   (3)令$f(x)=\frac{\ln(x+1)}{x}$，$f'(x)=\frac{\frac{1}{x+1}x-\ln(x+1)}{x^2}=\frac{x[1-\ln(x+1)]-\ln(x+1)}{x^2(x+1)}<0,x\ge 3$

   $\therefore u_n=\frac{\ln(n+1)}{n}>u_{n+1}=\frac{\ln(n+2)}{n+1}(n\ge 3)$且$\lim\limits_{n\to\infty}\frac{\ln(n+1)}{n}=\lim\limits_{n\to\infty}\frac{\ln n(\frac{n+1}{n})}{n}=\lim\limits_{n\to\infty}\frac{\ln n+\ln(\frac{n+1}{n})}{n}$ $=0$

   故$\sum_{n=3}^{\infty}\frac{(-1)^n\ln(n+1)}{n}$是莱布尼茨型交错级数，故收敛，则$\sum_{n=1}^{\infty}\frac{(-1)^n\ln(n+1)}{n}$收敛

   $\because \lim\limits_{n\to\infty}n\cdot|\frac{(-1)^n\ln(n+1)}{n}|=\lim\limits_{n\to\infty}\ln(n+1)=+\infty$，故级数条件收敛.

   (4)令$f(x)=x-\ln x$，$f'(x)=1-\frac{1}{x}=\frac{x-1}{x}>0(x>1)$，则$f(x)$在$[1,+\infty)$上单调增加，且$f(x)\ge f(1)=1>0$

   $\therefore u_n=\frac{1}{n-\ln n}>u_{n+1}=\frac{1}{n+1-\ln(n+1)}$且$\lim\limits_{n\to\infty}\frac{1}{n-\ln n}=\lim\limits_{n\to\infty}\frac{\frac{1}{n}}{1-\frac{\ln n}{n}}=0$，故$\sum_{n=1}^{\infty}\frac{(-1)^n}{n-\ln n}$是莱布尼茨型交错级数，故收敛

   $\because \lim\limits_{n\to\infty}n\cdot|\frac{(-1)^n}{n-\ln n}|=\lim\limits_{n\to\infty}\frac{1}{1-\frac{\ln n}{n}}=1$，故级数条件收敛.

   (5)方法1：原级数的前$2n$项和$S_{2n}=1-\ln 2+\frac{1}{2}-\ln\frac{3}{2}+\cdots+\frac{1}{n}-\ln\frac{n+1}{n}$，$S_{2n}$相当于是正项级数$\sum_{n=1}^{\infty}(\frac{1}{n}-\ln\frac{n+1}{n})$的前$n$项和

   $\because \lim\limits_{n\to\infty}n^2(\frac{1}{n}-\ln\frac{n+1}{n})=\lim\limits_{n\to\infty}n^2[\frac{1}{n}-(\frac{1}{n}-\frac{1}{2n^2}+o(\frac{1}{n^2}))]=\frac{1}{2}$

   $\therefore \sum_{n=1}^{\infty}(\frac{1}{n}-\ln\frac{n+1}{n})$收敛，$\lim\limits_{n\to\infty}S_{2n}=S$存在

   又$\because \lim\limits_{n\to\infty}S_{2n+1}=\lim\limits_{n\to\infty}(S_{2n}+\frac{1}{n+1})=S$

   $\therefore \lim\limits_{n\to\infty}S_n=S$存在

   故原级数收敛.

原级数每项加绝对值得到$\sum_{n=1}^{\infty}|u_n| = 1+\ln 2+\frac{1}{2}+\ln\frac{3}{2}+\cdots+\frac{1}{n}+\ln\frac{n+1}{n}+\cdots$，其前$2n$项和

$$\begin{aligned}\bar{S}_{2n} &= 1+\ln 2+\frac{1}{2}+\ln\frac{3}{2}+\cdots+\frac{1}{n}+\ln\frac{n+1}{n} = \sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{n}\ln\frac{k+1}{k} \\ &= \sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{n}[\ln(k+1)-\ln k] \\ &= \sum_{k=1}^{n}\frac{1}{k}+\ln(k+1)\to+\infty(n\to\infty)\end{aligned}$$

$\therefore \bar{S}_n\to+\infty$

故原级数条件收敛.

方法2：$\because$函数$f(x)=\ln(1+x)-x$在$x>0$时单调减少（$f'(x)=\frac{-x}{1+x}<0(x>0)$），函数$g(x)=\ln(1+x)-\frac{x}{1+x}$在$x>0$时单调增加（$g'(x)=\frac{x}{(1+x)^2}>0(x>0)$）

$\therefore \frac{1}{n}>\ln\frac{n+1}{n}>\frac{1}{n+1}(n\geq 1)$

又$\because \lim\limits_{n\to\infty}\frac{1}{n}=0=\lim\limits_{n\to\infty}\ln\frac{1+n}{n}$

$\therefore$原级数是莱布尼茨型交错级数，故收敛.

$\sum_{n=1}^{\infty}|u_n| = 1+\ln 2+\frac{1}{2}+\ln\frac{3}{2}+\cdots+\frac{1}{n}+\ln\frac{n+1}{n}+\cdots$，其前$2n$项和

$$\bar{S}_{2n} = 1+\ln 2+\frac{1}{2}+\ln\frac{3}{2}+\cdots+\frac{1}{n}+\ln\frac{n+1}{n} = \sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{n}\ln\frac{k+1}{k}\to+\infty(n\to\infty)$$

这里因为$\sum_{n=1}^{\infty}\frac{1}{n}$与$\sum_{n=1}^{\infty}\ln\frac{n+1}{n}$均发散，故$\lim\limits_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}=+\infty, \lim\limits_{n\to\infty}\sum_{k=1}^{n}\ln\frac{k+1}{k}=+\infty$

$\therefore \bar{S}_n\to+\infty$

故原级数条件收敛.

方法3：$\because \int_1^n\frac{1}{x}\mathrm{d}x=\ln n$

$\therefore \ln\frac{1+n}{n}=\ln(1+n)-\ln n=\int_1^{1+n}\frac{1}{x}\mathrm{d}x-\int_1^n\frac{1}{x}\mathrm{d}x=\int_n^{n+1}\frac{1}{x}\mathrm{d}x$

$\because \frac{1}{n}=\frac{1}{n}(n+1-n)>\int_n^{n+1}\frac{1}{x}\mathrm{d}x>\frac{1}{n+1}(n+1-n)=\frac{1}{n+1}$

$\therefore \frac{1}{n}>\ln\frac{n+1}{n}>\frac{1}{n+1}(n\geq 1)$

又$\because \lim\limits_{n\to\infty}\frac{1}{n}=0=\lim\limits_{n\to\infty}\ln\frac{1+n}{n}$

$\therefore$原级数是莱布尼茨型交错级数，故收敛.

$\because \sum_{n=1}^{\infty}\frac{1}{n}$发散

$\therefore$正项级数$1+\ln 2+\frac{1}{2}+\ln\frac{3}{2}+\cdots+\frac{1}{n}+\ln\frac{n+1}{n}+\cdots$发散

故原级数条件收敛.

(6)方法1：原级数的前$2n$项和$S_{2n}=\sum_{k=2}^{2n+1}\frac{(-1)^k}{\sqrt{k}+(-1)^k}=\sum_{m=1}^{n}(\frac{1}{\sqrt{2m}+1}-\frac{1}{\sqrt{2m+1}-1})$，相当于是负项级数$\sum_{n=1}^{\infty}(\frac{1}{\sqrt{2n}+1}-\frac{1}{\sqrt{2n+1}-1})$的前$n$项和

$\because$

$$\frac{1}{\sqrt{2n}+1}-\frac{1}{\sqrt{2n+1}-1}=\frac{\sqrt{2n+1}-\sqrt{2n}-2}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)}=\frac{\sqrt{2n}(\sqrt{\frac{2n+1}{2n}}-1)-2}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)}$$
$$=\frac{\sqrt{2n}(\sqrt{1+\frac{1}{2n}}-1)-2}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)}=\frac{\sqrt{2n}[1+\frac{1}{2}\frac{1}{2n}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\frac{1}{(2n)^2}+o(\frac{1}{n^2})-1]-2}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)}$$
$$=\frac{-2+\frac{1}{2}\frac{1}{\sqrt{2n}}+o(\frac{1}{n^{\frac{1}{2}}})}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)}$$

$\therefore\lim\limits_{n\to\infty}n\cdot[-(\frac{1}{\sqrt{2n}+1}-\frac{1}{\sqrt{2n+1}-1})]=\lim\limits_{n\to\infty}n\cdot\frac{2-\frac{1}{2}\frac{1}{\sqrt{2n}}+o(\frac{1}{n^{\frac{1}{2}}})}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)}=1$

$\therefore$负项级数$\sum_{n=1}^{\infty}(\frac{1}{\sqrt{2n}+1}-\frac{1}{\sqrt{2n+1}-1})$发散

$\therefore\lim\limits_{n\to\infty}S_{2n}=-\infty$

$\therefore$原级数发散.

方法2：假设级数$\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}+(-1)^n}$收敛

$\because$级数$\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}}$收敛

$\therefore$级数$\sum_{n=2}^{\infty}[\frac{(-1)^n}{\sqrt{n}}-\frac{(-1)^n}{\sqrt{n}+(-1)^n}]$收敛

$\because$

$$\sum_{n=2}^{\infty}[\frac{(-1)^n}{\sqrt{n}}-\frac{(-1)^n}{\sqrt{n}+(-1)^n}]=\sum_{n=2}^{\infty}\frac{(-1)^n\sqrt{n}+1-(-1)^n\sqrt{n}}{n+(-1)^n\sqrt{n}}$$
$$=\sum_{n=2}^{\infty}\frac{1}{n+(-1)^n\sqrt{n}}$$

且$\lim\limits_{n\to\infty}n\cdot\frac{1}{n+(-1)^n\sqrt{n}}=1\neq 0$，与$\sum_{n=2}^{\infty}[\frac{(-1)^n}{\sqrt{n}}-\frac{(-1)^n}{\sqrt{n}+(-1)^n}]$收敛矛盾

$\therefore$原级数发散.

(7)原级数的前$2n$项和$S_{2n}=\sum_{k=2}^{2n+1}\frac{(-1)^k}{\sqrt{k+(-1)^k}}=\sum_{m=1}^{n}(\frac{1}{\sqrt{2m+1}}-\frac{1}{\sqrt{2m}})$，相当于是负项级数$\sum_{n=1}^{\infty}(\frac{1}{\sqrt{2n+1}}-\frac{1}{\sqrt{2n}})$的前$n$项和

$\because\frac{1}{\sqrt{2n+1}}-\frac{1}{\sqrt{2n}}=\frac{\sqrt{2n}-\sqrt{2n+1}}{\sqrt{2n}\sqrt{2n+1}(\sqrt{2n}+\sqrt{2n+1})}=\frac{2n-(2n+1)}{\sqrt{2n}\sqrt{2n+1}(\sqrt{2n}+\sqrt{2n+1})}=\frac{-1}{\sqrt{2n}\sqrt{2n+1}(\sqrt{2n}+\sqrt{2n+1})}$

又$\because\lim\limits_{n\to\infty}n^{\frac{3}{2}}\cdot[-(\frac{1}{\sqrt{2n+1}}-\frac{1}{\sqrt{2n}})]=\lim\limits_{n\to\infty}n^{\frac{3}{2}}\cdot\frac{1}{\sqrt{2n}\sqrt{2n+1}(\sqrt{2n}+\sqrt{2n+1})}=\lim\limits_{n\to\infty}\frac{1}{\sqrt{2}\sqrt{2+\frac{1}{n}}(\sqrt{2}+\sqrt{2+\frac{1}{n}})}$
$=\frac{1}{4\sqrt{2}}$

$\therefore\sum_{n=1}^{\infty}(\frac{1}{\sqrt{2n+1}}-\frac{1}{\sqrt{2n}})$收敛，故$\lim\limits_{n\to\infty}S_{2n}=S$存在

$\therefore \lim\limits_{n\to\infty} S_{2n+1} = \lim\limits_{n\to\infty}[S_{2n} + \frac{1}{\sqrt{2n+1}}] = S$

$\therefore \lim\limits_{n\to\infty} S_n = S$存在，级数收敛

$\sum_{n=2}^{\infty}|\frac{(-1)^n}{\sqrt{n+(-1)^n}}|$的前$2n$项和$S_{2n} = \sum_{k=2}^{2n+1}\frac{1}{\sqrt{k+(-1)^k}} = \sum_{m=1}^{n}(\frac{1}{\sqrt{2m+1}}+\frac{1}{\sqrt{2m}}) = \sum_{m=1}^{n}\frac{1}{\sqrt{2m+1}} + \sum_{m=1}^{n}\frac{1}{\sqrt{2m}}$

$\because \lim\limits_{n\to\infty} n^{\frac{1}{2}}\cdot\frac{1}{\sqrt{2n+1}} = \frac{1}{\sqrt{2}}$，故级数$\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n+1}}$发散，$\lim\limits_{n\to\infty}\sum_{m=1}^{n}\frac{1}{\sqrt{2m+1}} = +\infty$，同理$\lim\limits_{n\to\infty}\sum_{m=1}^{n}\frac{1}{\sqrt{2m}} = +\infty$

故$\lim\limits_{n\to\infty} S_{2n} = +\infty$

$\therefore \lim\limits_{n\to\infty} S_n$不存在，级数条件收敛.

2. (1)已知级数$\sum_{n=1}^{\infty}u_n$收敛，能否断定$\sum_{n=1}^{\infty}u_n^2$收敛？
(2)已知级数$\sum_{n=1}^{\infty}u_n$收敛，$\lim\limits_{n\to\infty}\frac{v_n}{u_n} = 1$，能否断定$\sum_{n=1}^{\infty}v_n$收敛？
(3)已知级数$\sum_{n=1}^{\infty}u_n$收敛，$\lim\limits_{n\to\infty}\frac{v_n}{u_n} = 0$，能否断定$\sum_{n=1}^{\infty}v_n$收敛？

解：(1)不能. 如$\sum_{n=1}^{\infty}u_n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$收敛，但$\sum_{n=1}^{\infty}u_n^2 = \sum_{n=1}^{\infty}\frac{1}{n}$发散.

(2)不能. 如$\sum_{n=1}^{\infty}u_n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}, \sum_{n=1}^{\infty}v_n = \sum_{n=1}^{\infty}[\frac{(-1)^{n-1}}{\sqrt{n}} + \frac{1}{n}]$，满足$\sum_{n=1}^{\infty}u_n$收敛且$\lim\limits_{n\to\infty}\frac{v_n}{u_n} = 1$，但$\sum_{n=1}^{\infty}v_n$发散.

(3)不能. 如$\sum_{n=1}^{\infty}u_n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}, \sum_{n=1}^{\infty}v_n = \sum_{n=1}^{\infty}\frac{1}{n}$，满足$\sum_{n=1}^{\infty}u_n$收敛且$\lim\limits_{n\to\infty}\frac{v_n}{u_n} = 0$，但$\sum_{n=1}^{\infty}v_n$发散.

3. 设$a_n = \int_{(n-1)\pi}^{n\pi}\frac{\sin x}{x^p}dx$(其中$p>0$). 研究$\sum_{n=1}^{\infty}a_n$的收敛性.

解：**错误做法**：(1)$a_1 = \int_0^{\pi}\frac{\sin x}{x^p}dx$是一个瑕积分，$x=0$为瑕点

$$\because \lim_{x\to0^+} x^{\frac{1+p}{2}}\frac{\sin x}{x^p} = \lim_{x\to0^+} x^{\frac{1-p}{2}}\sin x = \begin{cases} \mathbf{0}, & \boldsymbol{p<1} \\ \boldsymbol{+\infty}, & \boldsymbol{p>1}\end{cases}$$

【注意：这里当$p<1$时，$x^{\frac{1-p}{2}}\to0(x\to0^+), \sin x\to0(x\to0^+)$，$x^{\frac{1-p}{2}}\sin x\to0$. 但当$p>1$时，虽然$x^{\frac{1-p}{2}}\to+\infty(x\to0^+)$，但是$\sin x\to0(x\to0^+)$，故$x^{\frac{1-p}{2}}\sin x$不一定趋于$+\infty$.
比如当$p=\frac{3}{2}>1$时

$$x^{\frac{1-p}{2}}\sin x = x^{-\frac{1}{4}}[x - \frac{x^3}{3!} + o(x^3)] = x^{\frac{3}{4}} - \frac{x^{\frac{11}{4}}}{3!} + o(x^{\frac{11}{4}}) \to 0(\not\to +\infty)(x\to0^+)$$

所以这里是错误的.】

$\therefore$当$p<1$时，$a_1 = \int_0^{\pi}\frac{\sin x}{x^p}dx$收敛，当$p>1$时，$a_1 = \int_0^{\pi}\frac{\sin x}{x^p}dx$发散

当$p=1$时，$\lim\limits_{x\to0^+} x^{\frac{1}{2}}\frac{\sin x}{x} = \lim\limits_{x\to0^+} x^{\frac{1}{2}}\frac{x-\frac{x^3}{3!}+o(x^3)}{x} = \lim\limits_{x\to0^+} x^{\frac{1}{2}}[1-\frac{x^2}{3!}+o(x^2)] = 0$，$a_1 = \int_0^{\pi}\frac{\sin x}{x^p}dx$收敛

故$p \le 1$.

(2)$\because |a_{n+1}| = |\int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x^p}\mathrm{d}x| = \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x^p}\mathrm{d}x \xlongequal{t=x-\pi} \int_{(n-1)\pi}^{n\pi} \frac{|\sin(t+\pi)|}{(t+\pi)^p}\mathrm{d}t$
$< \int_{(n-1)\pi}^{n\pi} \frac{|\sin t|}{t^p}\mathrm{d}t = |a_n|$

且$\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \int_{(n-1)\pi}^{n\pi} \frac{|\sin x|}{x^p}\mathrm{d}x = \lim\limits_{n\to\infty} \frac{|\sin \xi_n|}{\xi_n^p}[n\pi - (n-1)\pi] = \lim\limits_{n\to\infty} \frac{|\sin \xi_n|}{\xi_n^p}\pi = 0, \xi_n \in ((n-1)\pi, n\pi)$

又$\because a_{n+1}$与a_n异号

$\therefore \sum_{n=1}^{\infty} a_n$是莱布尼茨型交错级数，故收敛.

(3)$\because |a_n| = \int_{(n-1)\pi}^{n\pi} \frac{|\sin x|}{x^p}\mathrm{d}x > \int_{(n-1)\pi}^{n\pi} \frac{|\sin x|}{(n\pi)^p}\mathrm{d}x = \frac{1}{(n\pi)^p} \int_{(n-1)\pi}^{n\pi} |\sin x|\mathrm{d}x = \frac{1}{(n\pi)^p}|\int_{(n-1)\pi}^{n\pi} \sin x\mathrm{d}x|$
$= \frac{1}{(n\pi)^p}|-\cos x\Big|_{(n-1)\pi}^{n\pi}| = \frac{2}{\pi^p}\frac{1}{n^p}$

$\because \lim\limits_{n\to\infty} n^p \cdot \frac{2}{\pi^p}\frac{1}{n^p} = \frac{2}{\pi^p}$且$p \le 1$

$\therefore \sum_{n=1}^{\infty} \frac{2}{\pi^p}\frac{1}{n^p}$发散

$\therefore \sum_{n=1}^{\infty} |a_n|$发散

$\therefore \sum_{n=1}^{\infty} a_n$条件收敛.

正确做法： (1)$a_1 = \int_0^{\pi} \frac{\sin x}{x^p}\mathrm{d}x$是一个瑕积分，$x=0$为瑕点

$\boldsymbol{\because \lim\limits_{x\to 0^+} x^{p-1}\frac{\sin x}{x^p} = \lim\limits_{x\to 0^+} x^{-1}[x - \frac{x^3}{3!} + o(x^3)] = \lim\limits_{x\to 0^+}[1 - \frac{x^2}{3!} + x^2\frac{o(x^2)}{x^2}] = 1}$

【或者：$\boldsymbol{\because \lim\limits_{x\to 0^+} x^{p-1}\frac{\sin x}{x^p} = 1}$】

$\therefore$当$p-1 \ge 1$，即$p \ge 2$时，$a_1 = \int_0^{\pi} \frac{\sin x}{x^p}\mathrm{d}x$发散，当$p-1 < 1$，即$p < 2$时，$a_1 = \int_0^{\pi} \frac{\sin x}{x^p}\mathrm{d}x$收敛

故$\boldsymbol{p < 2}$.

(2)$\because |a_{n+1}| = |\int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x^p}\mathrm{d}x| = \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x^p}\mathrm{d}x \xlongequal{t=x-\pi} \int_{(n-1)\pi}^{n\pi} \frac{|\sin(t+\pi)|}{(t+\pi)^p}\mathrm{d}t$
$< \int_{(n-1)\pi}^{n\pi} \frac{|\sin t|}{t^p}\mathrm{d}t = |a_n|$

且$\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \int_{(n-1)\pi}^{n\pi} \frac{|\sin x|}{x^p}\mathrm{d}x = \lim\limits_{n\to\infty} \frac{|\sin \xi_n|}{\xi_n^p}[n\pi - (n-1)\pi] = \lim\limits_{n\to\infty} \frac{|\sin \xi_n|}{\xi_n^p}\pi = 0, \xi_n \in ((n-1)\pi, n\pi)$

又$\because a_{n+1}$与a_n异号

$\therefore \sum_{n=1}^{\infty} a_n$是莱布尼茨型交错级数，故收敛.

(3)

i)当$0 < p \le 1$时，$|a_n| = \int_{(n-1)\pi}^{n\pi} \frac{|\sin x|}{x^p}\mathrm{d}x > \int_{(n-1)\pi}^{n\pi} \frac{|\sin x|}{(n\pi)^p}\mathrm{d}x = \frac{1}{(n\pi)^p} \int_{(n-1)\pi}^{n\pi} |\sin x|\mathrm{d}x$
$= \frac{1}{(n\pi)^p}|\int_{(n-1)\pi}^{n\pi} \sin x\mathrm{d}x| = \frac{1}{(n\pi)^p}|-\cos x\Big|_{(n-1)\pi}^{n\pi}| = \frac{2}{\pi^p}\frac{1}{n^p}$

$\because \lim\limits_{n\to\infty} n^p \cdot \frac{2}{\pi^p}\frac{1}{n^p} = \frac{2}{\pi^p}$

$\therefore$当$0<p\le 1$时$\sum_{n=1}^{\infty}\frac{2}{\pi^p}\frac{1}{n^p}$发散

$\therefore\sum_{n=1}^{\infty}|a_n|$发散，$\sum_{n=1}^{\infty}a_n$条件收敛

ii)当$1<p<2$时，$|a_n|=\int_{(n-1)\pi}^{n\pi}\frac{|\sin x|}{x^p}\mathrm{d}x<\int_{(n-1)\pi}^{n\pi}\frac{|\sin x|}{[(n-1)\pi]^p}\mathrm{d}x=\frac{1}{[(n-1)\pi]^p}\int_{(n-1)\pi}^{n\pi}|\sin x|\mathrm{d}x$ $=\frac{1}{[(n-1)\pi]^p}|\int_{(n-1)\pi}^{n\pi}\sin x\mathrm{d}x|=\frac{1}{[(n-1)\pi]^p}|-\cos x\Big|_{(n-1)\pi}^{n\pi}|=\frac{2}{\pi^p}\frac{1}{(n-1)^p}$

$\because\lim\limits_{n\to\infty}n^p\cdot\frac{2}{\pi^p}\frac{1}{(n-1)^p}=\frac{2}{\pi^p}$

$\therefore$当$p>1$时$\sum_{n=1}^{\infty}\frac{2}{\pi^p}\frac{1}{(n-1)^p}$收敛

$\therefore\sum_{n=1}^{\infty}|a_n|$收敛，$\sum_{n=1}^{\infty}a_n$绝对收敛

综上所述，当$\mathbf{0<p\le 1}$时$\sum_{\mathbf{n=1}}^{+\infty}\mathbf{a_n}$条件收敛；当$\mathbf{1<p<2}$时$\sum_{\mathbf{n=1}}^{+\infty}\mathbf{a_n}$绝对收敛.

另一种解法：(1)$a_1=\int_0^{\pi}\frac{\sin x}{x^p}\mathrm{d}x$是一个瑕积分$\frac{\sin x}{x^p}\ge 0$，$x=0$为瑕点

$\because\lim\limits_{x\to 0^+}x^{p-1}\frac{\sin x}{x^p}=1$

$\therefore$当$p-1\ge 1$，即$p\ge 2$时，$a_1=\int_0^{\pi}\frac{\sin x}{x^p}\mathrm{d}x$发散，当$p-1<1$，即$p<2$时，$a_1=\int_0^{\pi}\frac{\sin x}{x^p}\mathrm{d}x$收敛，故$p<2$

(2)$\sum_{n=1}^{\infty}a_n=\int_{\pi}^{+\infty}\frac{\sin x}{x^p}\mathrm{d}x$是一个无穷积分

$\because 0\le|\frac{\sin x}{x^p}|\le\frac{1}{x^p}$且当$p>1$时$\int_1^{+\infty}\frac{1}{x^p}\mathrm{d}x$收敛

$\therefore$当$p>1$时$\int_{\pi}^{\infty}\frac{\sin x}{x^p}\mathrm{d}x$绝对收敛

当$0<p\le 1$时

$$\int_{\pi}^{+\infty}\frac{\sin x}{x^p}\mathrm{d}x=-\frac{\cos x}{x^p}\Big|_{\pi}^{+\infty}-p\int_{\pi}^{+\infty}\frac{\cos x}{x^{p+1}}\mathrm{d}x=-\frac{1}{\pi^p}-p\int_{\pi}^{+\infty}\frac{\cos x}{x^{p+1}}\mathrm{d}x$$

且$\int_{\pi}^{+\infty}\frac{\cos x}{x^{p+1}}\mathrm{d}x$绝对收敛(与$\int_{\pi}^{+\infty}\frac{\sin x}{x^{p+1}}\mathrm{d}x,p+1>1$绝对收敛的证法相同)

$\therefore\int_{\pi}^{+\infty}\frac{\sin x}{x^p}\mathrm{d}x$收敛

$\because$

$$\int_{\pi}^{+\infty}\frac{\sin^2 x}{x^p}\mathrm{d}x=\frac{1}{2}\int_{\pi}^{+\infty}\frac{1-\cos 2x}{x^p}\mathrm{d}x$$

且$\int_{\pi}^{+\infty}\frac{1}{x^p}\mathrm{d}x$发散，$\int_{\pi}^{+\infty}\frac{\cos 2x}{x^p}\mathrm{d}x$收敛(与$\int_{\pi}^{+\infty}\frac{\sin x}{x^p}\mathrm{d}x$收敛的证法相同)

$\therefore\int_{\pi}^{+\infty}|\frac{\sin^2 x}{x^p}|\mathrm{d}x$发散

$\therefore$当$0<p\le 1$时$\int_{\pi}^{+\infty}\frac{\sin^2 x}{x^p}\mathrm{d}x$条件收敛

综上所述，当$0<p\le 1$时$\sum_{n=1}^{+\infty}a_n$条件收敛；当$1<p<2$时$\sum_{n=1}^{+\infty}a_n$绝对收敛.

12.5 附录：级数加括号判断收敛性的方法

因为加括号后收敛的级数不一定收敛，所以由加括号后的级数收敛，不能直接得出原级数收敛的结论.

对于一个交错级数而言，如果正负项加括号后得到的新级数收敛，说明级数的偶数项部分和数列$\{S_{2n}\}$收敛，此时若$\lim\limits_{n\to\infty}u_n=0$，则级数的奇数项部分和$S_{2n+1}$也收敛，且$\lim\limits_{n\to\infty}S_{2n+1}=\lim\limits_{n\to\infty}S_{2n}=\lim\limits_{n\to\infty}S_n$，故级数收敛.

若$\lim\limits_{n\to\infty}S_{2n}=\infty$，因为$S_{2n}$是$S_n$的一个子列，则$S_n$发散，可直接得出级数发散的结论.

对于正项级数（或者每一项加了绝对值的级数）而言，如果$\lim\limits_{n\to\infty}S_{2n}=+\infty$，则必有$\lim\limits_{n\to\infty}S_n=+\infty$，级数发散.

1. 设$p>0$，研究级数

$$\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots+\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}+\cdots$$

的收敛性.

【注意：此题不能直接通过$\lim\limits_{n\to\infty}n^p[\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}]=\frac{1}{2^p}$说明当$p>1$时级数收敛，当$p\leq 1$时级数发散. 这相当于是在判断级数$\sum_{n=1}^{\infty}[\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}]$的收敛性. 该级数相当于是将原级数加了括号，由加括号后的级数收敛得不到原来的级数收敛.

比如$\sum_{n=1}^{\infty}(-1)^{n-1}=1-1+1-1+1-1+\cdots$发散，但加括号后$\sum_{n=1}^{\infty}(1-1)=(1-1)+(1-1)+(1-1)+\cdots=0$收敛.

下面是这类题目常用的处理办法.】

解：该级数的前$2n$项和$S_{2n}=\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots+\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}$，$S_{2n}$可视为正项级数$\sum_{n=1}^{\infty}[\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}]$的前$n$项和

$\because\lim\limits_{n\to\infty}n^p[\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}]=\lim\limits_{n\to\infty}[\frac{n^p}{(2n-1)^p}-\frac{n^p}{(2n)^{2p}}]=\frac{1}{2^p}$

$\therefore$i)当$p>1$时级数$\sum_{n=1}^{\infty}[\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}]$收敛，即$\lim\limits_{n\to\infty}S_{2n}$存在

又$\because\lim\limits_{n\to\infty}\frac{1}{(2n+1)^p}=0$

$\therefore\lim\limits_{n\to\infty}S_{2n+1}=\lim\limits_{n\to\infty}[S_{2n}+\frac{1}{(2n+1)^p}]=\lim\limits_{n\to\infty}S_{2n}$

故$\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}S_{2n}=\lim\limits_{n\to\infty}S_{2n+1}$存在，级数收敛；

ii)当$p\leq 1$时级数$\sum_{n=1}^{\infty}[\frac{1}{(2n-1)^p}-\frac{1}{(2n)^{2p}}]$发散，即$\lim\limits_{n\to\infty}S_{2n}=+\infty$

故$\lim\limits_{n\to\infty}S_n$不存在，级数发散；

综上所述，当$p>1$时级数收敛，当$0<p\leq 1$时级数发散.

2. 判断下列级数的收敛性，对收敛的级数指出绝对收敛，还是条件收敛：

(6)$\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}+(-1)^n}$；

解：(6)原级数的前$2n$项和$S_{2n}=\sum_{k=2}^{2n+1}\frac{(-1)^k}{\sqrt{k}+(-1)^k}=\sum_{m=1}^{n}(\frac{1}{\sqrt{2m}+1}-\frac{1}{\sqrt{2m+1}-1})$，相当于是级数$\sum_{n=1}^{\infty}(\frac{1}{\sqrt{2n}+1}-\frac{1}{\sqrt{2n+1}-1})$的前$n$项和

$\because \frac{1}{\sqrt{2n}+1}-\frac{1}{\sqrt{2n+1}-1}=\frac{\sqrt{2n+1}-\sqrt{2n}-2}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)}=\frac{1-2(\sqrt{2n+1}+\sqrt{2n})}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)(\sqrt{2n+1}+\sqrt{2n})}<0$

$\therefore$该级数是负项级数

$\because \frac{1}{\sqrt{2n}+1}-\frac{1}{\sqrt{2n+1}-1}=\frac{\sqrt{2n+1}-\sqrt{2n}-2}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)}=\frac{\sqrt{2n}(\sqrt{\frac{2n+1}{2n}}-1)-2}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)}=\frac{\sqrt{2n}(\sqrt{1+\frac{1}{2n}}-1)-2}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)}$

$=\frac{\sqrt{2n}[1+\frac{1}{2}\frac{1}{2n}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\frac{1}{(2n)^2}+o(\frac{1}{n^2})-1]-2}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)}=\frac{-2+\frac{1}{2}\frac{1}{\sqrt{2n}}+o(\frac{1}{\sqrt{n}})}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)}$

$\therefore \lim\limits_{n\to\infty} n\cdot[-(\frac{1}{\sqrt{2n}+1}-\frac{1}{\sqrt{2n+1}-1})]=\lim\limits_{n\to\infty} n\cdot\frac{2-\frac{1}{2}\frac{1}{\sqrt{2n}}+o(\frac{1}{\sqrt{n}})}{(\sqrt{2n}+1)(\sqrt{2n+1}-1)}=1$

$\therefore$负项级数$\sum_{n=1}^{\infty}(\frac{1}{\sqrt{2n}+1}-\frac{1}{\sqrt{2n+1}-1})$发散

$\therefore \lim\limits_{n\to\infty} S_{2n}=-\infty$

$\therefore \lim\limits_{n\to\infty} S_n$不存在，原级数发散.