9 原函数与不定积分

9.1 知识结构

第6章用原函数与不定积分

6.1 概念和性质

6.1.1 原函数

6.1.2 不定积分

6.1.3 基本积分公式

6.1.4 原函数存在的条件

6.1.5 不定积分的线性性质

6.2 换元积分法

6.2.1 第一换元法

6.2.2 第二换元法

6.3 分部积分法

6.4 有理函数的积分

6.4.1 分式函数的积分

6.4.2 三角函数有理式的积分

6.5 简单无理式的积分、不定积分小结

6.5.1 简单无理式的积分

6.5.2 不定积分小结

9.2 习题6.1解答

1. 证明$f(x)=\frac{1}{2}x^2\mathrm{sgn}x$是$|x|$在$(-\infty,+\infty)$的一个原函数.

   证明：$f(x)=\frac{1}{2}x^2\mathrm{sgn}x=\begin{cases}\frac{1}{2}x^2, & x>0\\ 0, & x=0\\ -\frac{1}{2}x^2, & x<0\end{cases}$

$$f'(x)=\begin{cases}x, & x>0\\ \lim\limits_{x\to0}\frac{f(x)-0}{x-0}=\lim\limits_{x\to0}\frac{\frac{1}{2}x^2\mathrm{sgn}x-0}{x-0}=\lim\limits_{x\to0}\frac{1}{2}x\mathrm{sgn}x=0, & x=0\\ -x, & x<0\end{cases}=|x|$$

故$f(x)=\frac{1}{2}x^2\mathrm{sgn}x$是$|x|$在$(-\infty,+\infty)$的一个原函数.

2. 求下列不定积分：

(1)$\int\cos^2x\mathrm{d}x$；

(2)$\int\tan^2x\mathrm{d}x$；

(3)$\int\frac{\mathrm{d}x}{\sin^2x\cos^2x}$；

(4)$\int\frac{x^2-2}{x+1}\mathrm{d}x$；

(5)$\int\frac{x-2}{\sqrt{1+x}}\mathrm{d}x$.

解：(1)$\int\cos^2x\mathrm{d}x=\int\frac{1}{2}(1+\cos2x)\mathrm{d}x=\frac{1}{2}(\int1\mathrm{d}x+\int\cos2x\mathrm{d}x)=\frac{1}{2}(x+\frac{1}{2}\sin2x)+C=\frac{1}{2}x+\frac{1}{4}\sin2x+C$.

(2)$\int\tan^2x\mathrm{d}x=\int\frac{\sin^2x}{\cos^2x}\mathrm{d}x=\int\frac{1-\cos^2x}{\cos^2x}\mathrm{d}x=\int(\sec^2x-1)\mathrm{d}x=\tan x-x+C$.

(3)$\int\frac{\mathrm{d}x}{\sin^2x\cos^2x}=\int\frac{\sin^2x+\cos^2x}{\sin^2x\cos^2x}\mathrm{d}x=\int(\sec^2x+\csc^2x)\mathrm{d}x=\tan x-\cot x+C$.

(4)$\int\frac{x^2-2}{x+1}\mathrm{d}x=\int\frac{x^2-1-1}{x+1}\mathrm{d}x=\int(\frac{x^2-1}{x+1}-\frac{1}{x+1})\mathrm{d}x=\int(x-1-\frac{1}{x+1})\mathrm{d}x=\frac{1}{2}x^2-x-\ln|x+1|+C$

(5)$\int\frac{x-2}{\sqrt{1+x}}\mathrm{d}x=\int\frac{x+1-3}{\sqrt{1+x}}\mathrm{d}x=\int(\sqrt{1+x}-\frac{3}{\sqrt{1+x}})\mathrm{d}(1+x)=\frac{3}{2}\sqrt{(1+x)^3}-3\cdot2\sqrt{1+x}+C=\frac{3}{2}\sqrt{(1+x)^3}-6\sqrt{1+x}+C$.

3. 设$f(x)=\begin{cases}e^x, & x\geq0\\ x+1, & x<0\end{cases}$.求$\int f(x)\mathrm{d}x$.

解：$f(x)=e^x,x\geq0$的原函数是$F(x)=e^x+C_1$，$f(x)=x+1,x<0$的原函数是$F(x)=\frac{1}{2}x^2+C_2$，

由$f(0)=1$知原函数$F(x)$在$x=0$点可导，故连续，故$1+C_1=C_2$，可取$C_1=0,C_2=1$，得$f(x)$的一个原函数$F(x)=\begin{cases}e^x, & x\geq0\\ \frac{1}{2}x^2+1, & x<0\end{cases}$.

故$\int f(x)\mathrm{d}x=\begin{cases}e^x+C, & x\geq0\\ \frac{1}{2}x^2+1+C, & x<0\end{cases}$.

4. 求$\int\max\{x,x^2\}\mathrm{d}x$.

解：$\max\{x,x^2\}=\begin{cases}x^2, & x\leq0\\ x, & 0<x\leq1\\ x^2, & x>1\end{cases}$

$$\int \max\{x,x^2\}\mathrm{d}x=\begin{cases}\frac{1}{3}x^3+C_1, & x\le 0\\ \frac{1}{2}x^2+C_2, & 0<x\le 1\\ \frac{1}{3}x^3+C_3, & x>1\end{cases}$$

$\because \max\{x,x^2\}(0)=0,\max\{x,x^2\}(1)=1$

故$\max\{x,x^2\}$的原函数在$x=0$和$x=1$处可导，故连续，故$C_1=C_2,\frac{1}{2}+C_2=\frac{1}{3}+C_3$，

取$C_1=\frac{1}{2}+C,C_2=\frac{1}{2}+C,C_3=\frac{2}{3}+C$得$\int \max\{x,x^2\}\mathrm{d}x=\begin{cases}\frac{1}{3}x^3+\frac{1}{2}+C, & x\le 0\\ \frac{1}{2}x^2+\frac{1}{2}+C, & 0<x\le 1\\ \frac{1}{3}x^3+\frac{2}{3}+C, & x>1\end{cases}$.

9.3 习题6.2解答

1. 求下列不定积分：

(1)$\int(2x+3)^4\mathrm{d}x$；

(2)$\int x3^{x^2+1}\mathrm{d}x$；

(3)$\int\frac{\ln x}{x}\mathrm{d}x$；

(4)$\int\frac{1}{x(2+x)}\mathrm{d}x$；

(5)$\int\cos x\cos 3x\mathrm{d}x$；

(6)$\int(\frac{1}{\sqrt{4-x^2}}+\frac{1}{1+2x^2})$；

(7)$\int\frac{1}{1-\sin x}\mathrm{d}x$；

(8)$\int\frac{3x}{1+x^2}\mathrm{d}x$；

(9)$\int\frac{e^x}{1+e^x}\mathrm{d}x$；

(10)$\int\frac{1}{\sqrt{1-x^2}\arccos x}\mathrm{d}x$；

(11)$\int\frac{\sin\sqrt{x}}{\sqrt{x}}\mathrm{d}x$；

(12)$\int\frac{1}{(1+x^2)\arctan x}\mathrm{d}x$；

(13)$\int\cos^5 x\mathrm{d}x$；

(14)$\int\frac{1}{\cos^2 x-\sin^2 x}\mathrm{d}x$；

(15)$\int\frac{1}{3-2x^2}\mathrm{d}x$；

(16)$\int\frac{1}{x^2-4x-12}$；

(17)$\int\frac{2}{\mathrm{e}^x+\mathrm{e}^{-x}}\mathrm{d}x$；

(18)$\int\frac{1}{\sin^2 x+4\cos^2 x}\mathrm{d}x$；

(19)$\int\frac{1}{1+\cos x}\mathrm{d}x$；

(20)$\int\frac{\mathrm{d}x}{1+\cos x}$；

(21)$\int\frac{\sin 2x}{1+\cos^4 x}\mathrm{d}x$；

(22)$\int\frac{\sqrt{x}}{1-\sqrt[3]{x}}\mathrm{d}x$；

(23)$\int\frac{x}{\sqrt{1-x}}\mathrm{d}x$；

(24)$\int\frac{1}{(4-x^2)^{\frac{3}{2}}}\mathrm{d}x$；

(25)$\int\frac{\sqrt{3-x^2}}{x}\mathrm{d}x$；

(26)$\int\frac{1}{1+\sqrt{3x}}\mathrm{d}x$；

(27)$\int\frac{1}{\sqrt{1+\sqrt{x}}}\mathrm{d}x$；

(28)$\int\frac{\mathrm{e}^{2x}}{\sqrt[3]{1+\mathrm{e}^x}}\mathrm{d}x$；

(29)$\int\frac{1}{\sqrt{1+\mathrm{e}^x}}\mathrm{d}x$；

(30)$\int\frac{x^5}{\sqrt{1+x^2}}\mathrm{d}x$.

解：(1)$\int(2x+3)^4\mathrm{d}x=\frac{1}{2}\int(2x+3)^4\mathrm{d}(2x+3)=\frac{1}{2}\frac{1}{5}(2x+3)^5+C=\frac{1}{10}(2x+3)^5+C$.

(2)$\int x3^{x^2+1}\mathrm{d}x=\frac{1}{2}\int 3^{x^2+1}\mathrm{d}(x^2+1)=\frac{1}{2\ln 3}3^{x^2+1}+C$.

(3)$\int\frac{\ln x}{x}\mathrm{d}x=\int\ln x\mathrm{d}\ln x=\frac{1}{2}(\ln x)^2+C$.

(4)$\int\frac{1}{x(2+x)}\mathrm{d}x=\frac{1}{2}\int(\frac{1}{x}-\frac{1}{x+2})\mathrm{d}x=\frac{1}{2}(\ln|x|-\ln|x+2|)+C=\frac{1}{2}\ln|\frac{x}{x+2}|+C$.

(5)$\int\cos x\cos 3x\mathrm{d}x=\int\frac{1}{2}[\cos(x+3x)+\cos(x-3x)]\mathrm{d}x=\int\frac{1}{2}[\cos 4x+\cos 2x]\mathrm{d}x=\frac{1}{2}(\frac{1}{4}\sin 4x+\frac{1}{2}\sin 2x)+C=\frac{1}{8}\sin 4x+\frac{1}{4}\sin 2x+C$.

(6)$\int(\frac{1}{\sqrt{4-x^2}}+\frac{1}{1+2x^2})=\int[\frac{1}{2\sqrt{1-(\frac{x}{2})^2}}+\frac{1}{1+(\sqrt{2}x)^2}]\mathrm{d}x=\int\frac{\mathrm{d}\frac{x}{2}}{\sqrt{1-(\frac{x}{2})^2}}+\frac{1}{\sqrt{2}}\int\frac{\mathrm{d}\sqrt{2}x}{1+(\sqrt{2}x)^2}=\arcsin\frac{x}{2}+\frac{1}{\sqrt{2}}\arctan\sqrt{2}x+C.$

(7)$\int\frac{1}{1-\sin x}\mathrm{d}x=\int\frac{1+\sin x}{\cos^2 x}\mathrm{d}x=\int(\sec^2 x+\sec x\tan x)\mathrm{d}x=\tan x+\sec x+C.$

(8)$\int\frac{3x}{1+x^2}\mathrm{d}x=\frac{3}{2}\int\frac{1}{1+x^2}\mathrm{d}(x^2+1)=\frac{3}{2}\ln(x^2+1)+C.$

(9)$\int\frac{\mathrm{e}^x}{1+\mathrm{e}^x}\mathrm{d}x=\int\frac{\mathrm{d}(\mathrm{e}^x+1)}{\mathrm{e}^x+1}=\ln(\mathrm{e}^x+1)+C.$

(10)$\int\frac{1}{\sqrt{1-x^2}\arccos x}\mathrm{d}x=-\int\frac{\mathrm{d}\arccos x}{\arccos x}=-\ln|\arccos x|+C.$

(11)$\int\frac{\sin\sqrt{x}}{\sqrt{x}}\mathrm{d}x=2\int\sin\sqrt{x}\mathrm{d}\sqrt{x}=-2\cos\sqrt{x}+C.$

(12)$\int\frac{1}{(1+x^2)\arctan x}\mathrm{d}x=\int\frac{\mathrm{d}\arctan x}{\arctan x}=\ln|\arctan x|+C.$

(13)$\int\cos^5 x\mathrm{d}x=\int\cos^4 x\mathrm{d}\sin x=\int(1-\sin^2 x)^2\mathrm{d}\sin x\overset{u=\sin x}{=\!=\!=}\int(1-u^2)^2\mathrm{d}u=\int(1-2u^2+u^4)\mathrm{d}u=u-\frac{2}{3}u^3+\frac{1}{5}u^5+C=\sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+C.$

(14)$\int\frac{1}{\cos^2 x-\sin^2 x}\mathrm{d}x=\int\frac{1}{\cos 2x}\mathrm{d}x=\frac{1}{2}\int\sec 2x\mathrm{d}2x=\frac{1}{2}\ln|\sec 2x+\tan 2x|+C=\frac{1}{2}\ln|\frac{1+\sin 2x}{\cos 2x}|+C=\frac{1}{2}\ln|\frac{(\sin x+\cos x)^2}{\cos^2 x-\sin^2 x}|+C=\frac{1}{2}\ln|\frac{\sin x+\cos x}{\sin x-\cos x}|+C.$

(15)$\int\frac{1}{3-2x^2}\mathrm{d}x=\frac{1}{2}\int\frac{1}{\frac{3}{2}-x^2}\mathrm{d}x=\frac{1}{2}\frac{1}{2\sqrt{\frac{3}{2}}}\int[\frac{1}{\sqrt{\frac{3}{2}}-x}+\frac{1}{\sqrt{\frac{3}{2}}+x}]\mathrm{d}x=\frac{1}{4\sqrt{\frac{3}{2}}}\ln|\frac{x+\sqrt{\frac{3}{2}}}{x-\sqrt{\frac{3}{2}}}|+C=\frac{1}{2\sqrt{6}}\ln|\frac{x+\sqrt{\frac{3}{2}}}{x-\sqrt{\frac{3}{2}}}|+C.$

(16)$\int\frac{1}{x^2-4x-12}=\int\frac{1}{(x-6)(x+2)}\mathrm{d}x=\frac{1}{8}\int(\frac{1}{x-6}-\frac{1}{x+2})\mathrm{d}x=\frac{1}{8}\ln|\frac{x-6}{x+2}|+C.$

(17)$\int\frac{2}{\mathrm{e}^x+\mathrm{e}^{-x}}\mathrm{d}x=\int\frac{2e^x}{e^{2x}+1}\mathrm{d}x=2\int\frac{\mathrm{d}e^x}{e^{2x}+1}=2\arctan e^x+C.$

(18)方法1：$\int\frac{1}{\sin^2 x+4\cos^2 x}\mathrm{d}x=\int\frac{\sec^2 x}{\tan^2 x+4}\mathrm{d}x=\frac{1}{2}\int\frac{\mathrm{d}\frac{\tan x}{2}}{(\frac{\tan x}{2})^2+1}=\frac{1}{2}\arctan\frac{\tan x}{2}+C.$

方法2：$\int\frac{1}{\sin^2 x+4\cos^2 x}\mathrm{d}x=\int\frac{\csc^2 x}{1+(2\cot x)^2}\mathrm{d}x=-\frac{1}{2}\int\frac{\mathrm{d}2\cot x}{1+(2\cot x)^2}=-\frac{1}{2}\arctan(2\cot x)+C.$

(19)$\int\frac{1}{1+\cos x}\mathrm{d}x=\int\frac{1-\cos x}{\sin^2 x}\mathrm{d}x=\int(\csc^2 x-\cot x\csc x)\mathrm{d}x=-\cot x+\csc x+C=\frac{1-\cos x}{\sin x}+C=\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}+C=\tan\frac{x}{2}+C.$

(20)$\int\frac{\mathrm{d}x}{1+\cos x}=\int\frac{1-\cos x}{\sin^2 x}\mathrm{d}x=\int(\csc^2 x-\cot x\csc x)\mathrm{d}x=-\cot x+\csc x+C.$

(21)$\int\frac{\sin 2x}{1+\cos^4 x}\mathrm{d}x=-\frac{1}{2}\int\frac{\mathrm{d}\cos 2x}{1+(\frac{1+\cos 2x}{2})^2}=-\int\frac{\mathrm{d}\frac{\cos 2x+1}{2}}{1+(\frac{1+\cos 2x}{2})^2}=-\arctan\frac{1+\cos 2x}{2}+C=-\arctan(\cos^2 x)+C.$

(22)$\int\frac{\sqrt{x}}{1-\sqrt[3]{x}}\mathrm{d}x\overset{x=t^6}{=\!=\!=}\int\frac{t^3}{1-t^2}6t^5\mathrm{d}t=6\int\frac{t^8}{1-t^2}\mathrm{d}t=-6\int\frac{t^8-1+1}{t^2-1}\mathrm{d}t=-6\int\frac{(t^2-1)(t^2+1)(t^4+1)+1}{t^2-1}\mathrm{d}t=-6\int[(t^2+1)(t^4+1)+\frac{1}{t^2-1}]\mathrm{d}t=-6\int[t^6+t^4+t^2+1+\frac{1}{2}(\frac{1}{t-1}-\frac{1}{t+1})]\mathrm{d}t=-6(\frac{1}{7}t^7+\frac{1}{5}t^5+\frac{1}{3}t^3+t+\frac{1}{2}\ln|\frac{t-1}{t+1}|)+C=-\frac{6}{7}t^7-\frac{6}{5}t^5-2t^3-6t-3\ln|\frac{t-1}{t+1}|+C=-\frac{6}{7}x^{\frac{7}{6}}-\frac{6}{5}x^{\frac{5}{6}}-2x^{\frac{1}{2}}-6x^{\frac{1}{6}}-3\ln|\frac{x^{\frac{1}{6}}-1}{x^{\frac{1}{6}}+1}|+C.$

(23)$\int\frac{x}{\sqrt{1-x}}\mathrm{d}x=\int\frac{x-1+1}{\sqrt{1-x}}\mathrm{d}x=\int(-\sqrt{1-x}+\frac{1}{\sqrt{1-x}})\mathrm{d}x=\frac{2}{3}(1-x)^{\frac{3}{2}}-2\sqrt{1-x}+C.$

(24)$\int\frac{1}{(4-x^2)^{\frac{3}{2}}}\mathrm{d}x\overset{x=2\sin t}{=\!=\!=\!=}\int\frac{2\cos t\mathrm{d}t}{(4-4\sin^2 t)^{\frac{3}{2}}}=\int\frac{2\cos t\mathrm{d}t}{8\cos^3 t}=\int\frac{\mathrm{d}t}{4\cos^2 t}=\frac{1}{4}\tan t+C=\frac{1}{4}\frac{2\sin x}{\sqrt{4-4\sin^2 x}}+C=\frac{1}{4}\frac{x}{\sqrt{4-x^2}}+C.$

(25)$\int\frac{\sqrt{3-x^2}}{x}\mathrm{d}x\xlongequal{x=\sqrt{3}\sin t}\int\frac{\sqrt{3}\cos t}{\sqrt{3}\sin t}\sqrt{3}\cos t\mathrm{d}t=\sqrt{3}\int\frac{\cos^2 t}{\sin t}\mathrm{d}t=\sqrt{3}\int\frac{1-\sin^2 t}{\sin t}\mathrm{d}t=\sqrt{3}\int(\csc t-\sin t)\mathrm{d}t=\sqrt{3}(\ln|\csc t-\cot t|+\cos t)+C=\sqrt{3}(\ln|\frac{\sqrt{3}}{x}-\frac{\sqrt{1-\frac{x^2}{3}}}{\frac{x}{\sqrt{3}}}|+\sqrt{1-\frac{x^2}{3}})+C=\sqrt{3}\ln|\frac{\sqrt{3}-\sqrt{3-x^2}}{x}|+\sqrt{3-x^2}+C.$

(26)$\int\frac{1}{1+\sqrt{3x}}\mathrm{d}x\xlongequal{\sqrt{3x}=t}\int\frac{1}{1+t}\frac{2t}{3}\mathrm{d}t=\frac{2}{3}\int\frac{t+1-1}{1+t}\mathrm{d}t=\frac{2}{3}\int(1-\frac{1}{1+t})\mathrm{d}t=\frac{2}{3}[t-\ln(1+t)]+C=\frac{2}{3}[\sqrt{3x}-\ln(1+\sqrt{3x})]+C.$

(27)$\int\frac{1}{\sqrt{1+\sqrt{x}}}\mathrm{d}x\xlongequal{t=\sqrt{x}}\int\frac{1}{\sqrt{1+t}}2t\mathrm{d}t=2\int\frac{1+t-1}{\sqrt{1+t}}\mathrm{d}t=2\int(\sqrt{1+t}-\frac{1}{\sqrt{1+t}})\mathrm{d}(1+t)=2[\frac{2}{3}(1+t)^{\frac{3}{2}}-2\sqrt{1+t}]+C=2[\frac{2}{3}(1+\sqrt{x})^{\frac{3}{2}}-2\sqrt{1+\sqrt{x}}]+C=4\sqrt{1+\sqrt{x}}(\frac{\sqrt{x}-2}{3})+C.$

(28)$\int\frac{\mathrm{e}^{2x}}{\sqrt[3]{1+\mathrm{e}^x}}\mathrm{d}x\xlongequal{t=\sqrt[3]{1+\mathrm{e}^x}}\int\frac{(t^3-1)^2}{t}\mathrm{d}\ln(t^3-1)=\int\frac{(t^3-1)^2}{t}\frac{3t^2}{t^3-1}\mathrm{d}t=\int 3t(t^3-1)\mathrm{d}t=\int(3t^4-3t)\mathrm{d}t=\frac{3}{5}t^5-\frac{3}{2}t^2+C=\frac{3}{5}(1+\mathrm{e}^x)^{\frac{5}{3}}-\frac{3}{2}(1+\mathrm{e}^x)^{\frac{2}{3}}+C=(1+e^x)^{\frac{2}{3}}(\frac{3e^x}{5}-\frac{9}{10})+C.$

(29)$\int\frac{1}{\sqrt{1+\mathrm{e}^x}}\mathrm{d}x\xlongequal{\sqrt{1+\mathrm{e}^x}=t}\int\frac{1}{t}\mathrm{d}\ln(t^2-1)=\int\frac{1}{t}\frac{2t}{t^2-1}\mathrm{d}t=\int\frac{2}{t^2-1}\mathrm{d}t=\int(\frac{1}{t-1}-\frac{1}{t+1})\mathrm{d}t=\ln|\frac{t-1}{t+1}|+C=\ln\frac{\sqrt{1+\mathrm{e}^x}-1}{\sqrt{1+\mathrm{e}^x}+1}+C.$

(30)$\int\frac{x^5}{\sqrt{1+x^2}}\mathrm{d}x\xlongequal{x=\tan t}\int\frac{\tan^5 t}{\sec t}\sec^2 t\mathrm{d}t=\int\tan^5 t\sec t\mathrm{d}t=\int\frac{\sin^5 t}{\cos^6 t}\mathrm{d}t=-\int\frac{\sin^4 t}{\cos^6 t}\mathrm{d}\cos t=-\int\frac{(1-\cos^2 t)^2}{\cos^6 t}\mathrm{d}\cos t=-\int\frac{1-2\cos^2 t+\cos^4 t}{\cos^6 t}\mathrm{d}\cos t=-\int(\frac{1}{\cos^6 t}-\frac{2}{\cos^4 t}+\frac{1}{\cos^2 t})\mathrm{d}\cos t=-(\frac{1}{-5\cos^5 t}-\frac{2}{-3\cos^3 t}+\frac{1}{-\cos t})+C=\frac{1}{5\cos^5 t}-\frac{2}{3\cos^3 t}+\frac{1}{\cos t}+C=\frac{1}{5}\sec^5 t-\frac{2}{3}\sec^3 t+\sec t+C=\frac{1}{5}(1+x^2)^2\sqrt{1+x^2}-\frac{2}{3}(1+x^2)\sqrt{1+x^2}+\sqrt{1+x^2}+C.$

2. 证明：

$$\int\frac{a_1\sin x+b_1\cos x}{a\sin x+b\cos x}\mathrm{d}x=Ax+B\ln|a\sin x+b\cos x|+C,$$

其中A,B为常数，$a^2+b^2>0$.

证明：$(Ax+B\ln|a\sin x+b\cos x|+C)'=A+B\frac{a\cos x-b\sin x}{a\sin x+b\cos x}=\frac{(Aa-Bb)\sin x+(Ab+Ba)\cos x}{a\sin x+b\cos x}=\frac{a_1\sin x+b_1\cos x}{a\sin x+b\cos x}$

其中$a_1=Aa-Bb,b_1=Ab+Ba$，证毕.

9.4 习题6.3解答

求下列不定积分

1.$\int x\cos x\mathrm{d}x$； 2.$\int\frac{\ln x}{x^2}\mathrm{d}x$；
3.$\int(\ln x)^2\mathrm{d}x$； 4.$\int(\ln(\ln x)+\frac{1}{\ln x})\mathrm{d}x$；
5.$\int\arctan\sqrt{x}\mathrm{d}x$； 6.$\int\frac{x\tan x}{\cos^4 x}\mathrm{d}x$；
7.$\int x^2\mathrm{e}^{-2x}\mathrm{d}x$； 8.$\int\mathrm{e}^{2x}\sin x\mathrm{d}x$；
9.$\int\sin\sqrt{x}\mathrm{d}x$； 10.$\int(\arccos x)^2\mathrm{d}x$；
11.$\int x\sin x\cos 2x\mathrm{d}x$； 12.$\int\frac{x}{\cos^2 x}\mathrm{d}x$；
13.$\int\mathrm{e}^{\sqrt[3]{x}}\mathrm{d}x$； 14.$\int\frac{\cos^2 x}{e^x}\mathrm{d}x$；
15.$\int\sin(\ln x)\mathrm{d}x$； 16.$\int\frac{\ln(1+x^2)}{x^3}\mathrm{d}x$；
17.$\int(\frac{\ln x}{x})^2$； 18.$\int x\ln(1+x^2)\mathrm{d}x$；
19.$\int\mathrm{e}^{2x}(1+\tan x)^2\mathrm{d}x$； 20.$\int x\tan^2 2x\mathrm{d}x$；
21.$\int\ln(x+\sqrt{1+x^2})\mathrm{d}x$； 22.$\int\frac{\arctan x}{x^2\sqrt{1-x^2}}\mathrm{d}x$.

解：1.$\int x\cos x\mathrm{d}x=\int x\mathrm{d}\sin x=x\sin x-\int\sin x\mathrm{d}x=x\sin x+\cos x+C.$

2.$\int\frac{\ln x}{x^2}\mathrm{d}x=\int\ln x\mathrm{d}(-\frac{1}{x})=-\frac{1}{x}\ln x+\int\frac{1}{x}\mathrm{d}\ln x=-\frac{\ln x}{x}+\int\frac{1}{x^2}\mathrm{d}x=-\frac{\ln x}{x}-\frac{1}{x}+C.$

3.$\int(\ln x)^2\mathrm{d}x=x(\ln x)^2-\int x\mathrm{d}(\ln x)^2=x(\ln x)^2-\int x(\frac{2\ln x}{x})\mathrm{d}x=x(\ln x)^2-2\int\ln x\mathrm{d}x$
$=x(\ln x)^2-2(x\ln x-\int x\mathrm{d}\ln x)=x(\ln x)^2-2x\ln x+2\int\mathrm{d}x=x(\ln x)^2-2x\ln x+2x+C.$

4.$\int(\ln(\ln x)+\frac{1}{\ln x})\mathrm{d}x=\int\ln(\ln x)\mathrm{d}x+\int\frac{1}{\ln x}\mathrm{d}x=x\ln(\ln x)-\int x\frac{1}{x\ln x}+\int\frac{1}{\ln x}\mathrm{d}x$
$=x\ln(\ln x)+C.$

5.$\int\arctan\sqrt{x}\mathrm{d}x=x\arctan\sqrt{x}-\int x\frac{1}{1+x}\frac{1}{2\sqrt{x}}\mathrm{d}x=x\arctan\sqrt{x}-\frac{1}{2}\int\frac{\sqrt{x}}{1+x}\mathrm{d}x$
$=x\arctan\sqrt{x}-\frac{1}{2}\int\frac{\sqrt{x}}{1+x}\mathrm{d}(\sqrt{x})^2=x\arctan\sqrt{x}-\frac{1}{2}\int\frac{\sqrt{x}}{1+x}2\sqrt{x}\mathrm{d}\sqrt{x}$
$=x\arctan\sqrt{x}-\int\frac{(\sqrt{x})^2}{1+(\sqrt{x})^2}\mathrm{d}\sqrt{x}=x\arctan\sqrt{x}-\int 1-\frac{1}{1+(\sqrt{x})^2}\mathrm{d}\sqrt{x}$
$=x\arctan\sqrt{x}-\sqrt{x}+\arctan\sqrt{x}+C.$

6.$\int\frac{x\tan x}{\cos^4 x}\mathrm{d}x=\int\frac{x\sin x}{\cos^5 x}\mathrm{d}x=-\int\frac{x\mathrm{d}\cos x}{\cos^5 x}=\int x\mathrm{d}(\frac{1}{4\cos^4 x})=\frac{x}{4\cos^4 x}-\int\frac{1}{4\cos^4 x}\mathrm{d}x$
$=\frac{x}{4\cos^4 x}-\frac{1}{4}\int\sec^4 x\mathrm{d}x=\frac{x}{4\cos^4 x}-\frac{1}{4}\int\sec^2 x\mathrm{d}\tan x=\frac{x}{4\cos^4 x}-\frac{1}{4}(\sec^2 x\tan x-\int\tan x\mathrm{d}\sec^2 x)$
$=\frac{x}{4\cos^4 x}-\frac{1}{4}(\sec^2 x\tan x-\int\tan x\mathrm{d}\sec^2 x)=\frac{x}{4\cos^4 x}-\frac{1}{4}(\sec^2 x\tan x-\int\tan x2\sec x\sec x\tan x\mathrm{d}x)$
$=\frac{x}{4\cos^4 x}-\frac{1}{4}(\sec^2 x\tan x-2\int\tan^2 x\sec^2 x\mathrm{d}x)=\frac{x}{4\cos^4 x}-\frac{1}{4}(\sec^2 x\tan x-2\int\tan^2 x\mathrm{d}\tan x)$
$=\frac{x}{4\cos^4 x}-\frac{1}{4}(\sec^2 x\tan x-\frac{2}{3}\tan^3 x)+C=\frac{x}{4\cos^4 x}-\frac{1}{4}\sec^2 x\tan x+\frac{1}{6}\tan^3 x+C$
$=\frac{x}{4\cos^4 x}-\frac{1}{4}\tan x-\frac{1}{12}\tan^3 x+C.$

7.$\int x^2\mathrm{e}^{-2x}\mathrm{d}x=\int x^2\mathrm{d}(\frac{-1}{2}\mathrm{e}^{-2x})=-\frac{1}{2}(x^2\mathrm{e}^{-2x}-\int\mathrm{e}^{-2x}2x\mathrm{d}x)=-\frac{1}{2}x^2e^{-2x}+\int\mathrm{e}^{-2x}x\mathrm{d}x$
$=-\frac{1}{2}x^2\mathrm{e}^{-2x}-\frac{1}{2}\int x\mathrm{d}\mathrm{e}^{-2x}=-\frac{1}{2}x^2\mathrm{e}^{-2x}-\frac{1}{2}(x\mathrm{e}^{-2x}-\int\mathrm{e}^{-2x}\mathrm{d}x)$
$=-\frac{1}{2}x^2\mathrm{e}^{-2x}-\frac{1}{2}x\mathrm{e}^{-2x}+\frac{1}{2}\int\mathrm{e}^{-2x}\mathrm{d}x=-\frac{1}{2}x^2\mathrm{e}^{-2x}-\frac{1}{2}x\mathrm{e}^{-2x}-\frac{1}{4}\mathrm{e}^{-2x}+C.$

8.$\int\mathrm{e}^{2x}\sin x\mathrm{d}x=-\int\mathrm{e}^{2x}\mathrm{d}\cos x=-\mathrm{e}^{2x}\cos x+\int\cos x2\mathrm{e}^{2x}\mathrm{d}x=-\mathrm{e}^{2x}\cos x+2\int\mathrm{e}^{2x}\mathrm{d}\sin x$
$=-\mathrm{e}^{2x}\cos x+2(\mathrm{e}^{2x}\sin x-\int\sin x2\mathrm{e}^{2x}\mathrm{d}x)=-\mathrm{e}^{2x}\cos x+2\mathrm{e}^{2x}\sin x-4\int\mathrm{e}^{2x}\sin x\mathrm{d}x$
$=\frac{1}{5}\mathrm{e}^{2x}(2\sin x-\cos x)+C.$

9.$\int \sin\sqrt{x}\mathrm{d}x \xlongequal{t=\sqrt{x}} 2\int t\sin t\mathrm{d}t = -2\int t\mathrm{d}\cos t = -2(t\cos t - \int \cos t\mathrm{d}t)$
$= -2t\cos t + 2\sin t + C = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C.$

10.方法1：$\int(\arccos x)^2\mathrm{d}x \xlongequal{t=\arccos x} \int t^2\mathrm{d}\cos t = t^2\cos t - 2\int t\cos t\mathrm{d}t$
$= t^2\cos t - 2\int t\mathrm{d}\sin t = t^2\cos t - 2(t\sin t - \int \sin t\mathrm{d}t) = t^2\cos t - 2t\sin t + 2\int \sin t\mathrm{d}t$
$= t^2\cos t - 2t\sin t - 2\cos t + C = (\arccos x)^2 x - 2(\arccos x)\sqrt{1-x^2} - 2x + C.$

方法2：$\int(\arccos x)^2\mathrm{d}x = x(\arccos x)^2 - \int x\mathrm{d}(\arccos x)^2 = x(\arccos x)^2 - \int x(2\arccos x)\frac{-1}{\sqrt{1-x^2}}\mathrm{d}x$
$= x(\arccos x)^2 + \int \arccos x\frac{\mathrm{d}x^2}{\sqrt{1-x^2}} = x(\arccos x)^2 - \int \arccos x\mathrm{d}(2\sqrt{1-x^2})$
$= x(\arccos x)^2 - (2\sqrt{1-x^2}\arccos x - 2\int \sqrt{1-x^2}\frac{-1}{\sqrt{1-x^2}}\mathrm{d}x)$
$= x(\arccos x)^2 - 2\sqrt{1-x^2}\arccos x - 2x + C.$

11.$\int x\sin x\cos 2x\mathrm{d}x = \int x\sin x(2\cos^2 x - 1)\mathrm{d}x = 2\int x\sin x\cos^2 x\mathrm{d}x - \int x\sin x\mathrm{d}x$
$= -2\int x\cos^2 x\mathrm{d}\cos x + \int x\mathrm{d}\cos x = -\frac{2}{3}\int x\mathrm{d}\cos^3 x + (x\cos x - \int \cos x\mathrm{d}x)$
$= -\frac{2}{3}(x\cos^3 x - \int \cos^3 x\mathrm{d}x) + x\cos x - \sin x$
$= -\frac{2}{3}[x\cos^3 x - \int(1-\sin^2 x)\mathrm{d}\sin x] + x\cos x - \sin x$
$= -\frac{2}{3}x\cos^3 x + \frac{2}{3}\int(1-\sin^2 x)\mathrm{d}\sin x + x\cos x - \sin x$
$= -\frac{2}{3}x\cos^3 x + \frac{2}{3}\sin x - \frac{2}{9}\sin^3 x + x\cos x - \sin x + C$
$= -\frac{2}{3}x\cos^3 x - \frac{2}{9}\sin^3 x + x\cos x - \frac{1}{3}\sin x + C.$

12.$\int \frac{x}{\cos^2 x}\mathrm{d}x = \int x\mathrm{d}\tan x = x\tan x - \int \tan x\mathrm{d}x = x\tan x + \ln|\cos x| + C.$

13.$\int \mathrm{e}^{\sqrt[3]{x}}\mathrm{d}x \xlongequal{\sqrt[3]{x}=t} \int \mathrm{e}^t 3t^2\mathrm{d}t = 3\int t^2\mathrm{d}\mathrm{e}^t = 3(t^2 e^x - \int \mathrm{e}^t 2t\mathrm{d}t) = 3t^2 e^t - 6\int t\mathrm{d}e^t$
$= 3t^2\mathrm{e}^t - 6(t\mathrm{e}^t - \int \mathrm{e}^t\mathrm{d}t) = 3t^2\mathrm{e}^t - 6t\mathrm{e}^t + 6\mathrm{e}^t + C = 3e^{\sqrt[3]{x}}(\sqrt[3]{x^2} - 2\sqrt[3]{x} + 2) + C.$

14.$\int \frac{\cos^2 x}{\mathrm{e}^x}\mathrm{d}x = \int \mathrm{e}^{-x}\frac{1+\cos 2x}{2}\mathrm{d}x = \frac{1}{2}\int \mathrm{e}^{-x}\mathrm{d}x + \frac{1}{2}\int \mathrm{e}^{-x}\cos 2x\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-x} - \frac{1}{2}\int \mathrm{e}^{-x}\cos 2x\mathrm{d}x$

$\because \frac{1}{2}\int \mathrm{e}^{-x}\cos 2x\mathrm{d}x = -\frac{1}{2}\int \cos 2x\mathrm{d}\mathrm{e}^{-x}$
$= -\frac{1}{2}(\mathrm{e}^{-x}\cos 2x + \int \mathrm{e}^{-x}2\sin 2x\mathrm{d}x) = -\frac{1}{2}\mathrm{e}^{-x}\cos 2x - \int \mathrm{e}^{-x}\sin 2x\mathrm{d}x$
$= -\frac{1}{2}\mathrm{e}^{-x}\cos 2x + \int \sin 2x\mathrm{d}\mathrm{e}^{-x} = -\frac{1}{2}\mathrm{e}^{-x}\cos 2x + \mathrm{e}^{-x}\sin 2x - 2\int \mathrm{e}^{-x}\cos 2x\mathrm{d}x$

$\therefore \int \mathrm{e}^{-x}\cos 2x\mathrm{d}x = \frac{2}{5}(-\frac{1}{2}\mathrm{e}^{-x}\cos 2x + \mathrm{e}^{-x}\sin 2x) + C$

$\therefore \int \frac{\cos^2 x}{\mathrm{e}^x}\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-x} - \frac{1}{2}\int \mathrm{e}^{-x}\cos 2x\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-x} - \frac{1}{5}(-\frac{1}{2}\mathrm{e}^{-x}\cos 2x + \mathrm{e}^{-x}\sin 2x) + C$
$= \mathrm{e}^{-x}(\frac{1}{10}\cos 2x - \frac{1}{5}\sin 2x - \frac{1}{2}) + C.$

15.$\int \sin(\ln x)\mathrm{d}x = x\sin(\ln x) - \int x\mathrm{d}\sin(\ln x) = x\sin(\ln x) - \int x\cos(\ln x)\frac{1}{x}\mathrm{d}x$
$= x\sin(\ln x) - \int \cos(\ln x)\mathrm{d}x = x\sin(\ln x) - [x\cos(\ln x) + \int x\sin(\ln x)\frac{1}{x}\mathrm{d}x]$
$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\mathrm{d}x = \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C.$

16.$\int \frac{\ln(1+x^2)}{x^3}\mathrm{d}x = \int \ln(1+x^2)\mathrm{d}(-\frac{1}{2x^2}) = -\frac{1}{2x^2}\ln(1+x^2) + \int \frac{1}{2x^2}\frac{2x}{1+x^2}\mathrm{d}x$
$= -\frac{\ln(1+x^2)}{2x^2} + \int \frac{1}{x(1+x^2)}\mathrm{d}x = -\frac{\ln(1+x^2)}{2x^2} + \int(\frac{1}{x} - \frac{x}{1+x^2})\mathrm{d}x = -\frac{\ln(1+x^2)}{2x^2} + \int \frac{1}{x}\mathrm{d}x - \frac{1}{2}\int \frac{\mathrm{d}(1+x^2)}{1+x^2}\mathrm{d}x$
$= -\frac{\ln(1+x^2)}{2x^2} + \ln|x| - \frac{1}{2}\ln(1+x^2) + C.$

17.$\int(\frac{\ln x}{x})^2\mathrm{d}x=\int(\ln x)^2\mathrm{d}(-\frac{1}{x})=-\frac{1}{x}(\ln x)^2+\int\frac{1}{x}(2\ln x)\frac{1}{x}\mathrm{d}x=-\frac{1}{x}(\ln x)^2+2\int\frac{\ln x}{x^2}\mathrm{d}x$
$=-\frac{1}{x}(\ln x)^2+2\int\ln x\mathrm{d}(-\frac{1}{x})=-\frac{1}{x}(\ln x)^2+2(-\frac{1}{x}\ln x+\int\frac{1}{x}\frac{1}{x}\mathrm{d}x)$
$=-\frac{1}{x}(\ln x)^2-\frac{2}{x}\ln x-\frac{2}{x}+C.$

18.$\int x\ln(1+x^2)\mathrm{d}x=\int\ln(1+x^2)\mathrm{d}(\frac{1}{2}x^2)=\frac{1}{2}x^2\ln(1+x^2)-\int\frac{1}{2}x^2\frac{2x}{1+x^2}\mathrm{d}x$
$=\frac{1}{2}x^2\ln(1+x^2)-\int\frac{x^3}{1+x^2}\mathrm{d}x=\frac{1}{2}x^2\ln(1+x^2)-\int(x-\frac{x}{1+x^2})\mathrm{d}x$
$=\frac{1}{2}x^2\ln(1+x^2)-\int x\mathrm{d}x+\int\frac{x}{1+x^2}\mathrm{d}x=\frac{1}{2}x^2\ln(1+x^2)-\frac{1}{2}x^2+\frac{1}{2}\int\frac{\mathrm{d}(1+x^2)}{1+x^2}$
$=\frac{1}{2}x^2\ln(1+x^2)-\frac{1}{2}x^2+\frac{1}{2}\ln(1+x^2)+C=\frac{1}{2}(1+x^2)\ln(1+x^2)-\frac{1}{2}x^2+C.$

19.$\int\mathrm{e}^{2x}(1+\tan x)^2\mathrm{d}x=\int\mathrm{e}^{2x}(\sec^2x+2\tan x)\mathrm{d}x=\int\mathrm{e}^{2x}\mathrm{d}\tan x+\int\tan x\mathrm{d}\mathrm{e}^{2x}=\int\mathrm{d}(\mathrm{e}^{2x}\tan x)$
$=\mathrm{e}^{2x}\tan x+C.$

20.$\int x\tan^2 2x\mathrm{d}x=\int x(\sec^2 2x-1)\mathrm{d}x=\frac{1}{2}\int x\mathrm{d}\tan 2x-\frac{1}{2}x^2=\frac{1}{2}(x\tan 2x-\int\tan 2x\mathrm{d}x)-\frac{1}{2}x^2=\frac{1}{2}x\tan 2x+\frac{1}{4}\ln|\cos 2x|-\frac{1}{2}x^2+C.$

21.$\int\ln(x+\sqrt{1+x^2})\mathrm{d}x=x\ln(x+\sqrt{1+x^2})-\int x\frac{1+\frac{2x}{2\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}\mathrm{d}x$
$=x\ln(x+\sqrt{1+x^2})-\int\frac{x}{\sqrt{1+x^2}}\mathrm{d}x=x\ln(x+\sqrt{1+x^2})-\frac{1}{2}\int\frac{\mathrm{d}(1+x^2)}{\sqrt{1+x^2}}$
$=x\ln(x+\sqrt{1+x^2})-\frac{1}{2}2\sqrt{1+x^2}+C=x\ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}+C.$

22.$\int\frac{\arcsin x}{x^2\sqrt{1-x^2}}\mathrm{d}x\xlongequal{t=\arcsin x}\int\frac{t}{\sin^2 t\cos t}\cos t\mathrm{d}t=\int t\csc^2 t\mathrm{d}t=-\int t\mathrm{d}\cot t=-t\cot t+\int\cot t\mathrm{d}t=-t\cot t+\ln|\sin t|+C=-\frac{\sqrt{1-x^2}}{x}\arcsin x+\ln|x|+C.$

9.5 习题6.4解答

求下列不定积分：

1.$\int\frac{x^2}{(1+x^2)^2}\mathrm{d}x$；	2.$\int\frac{x-1}{3+x^2}\mathrm{d}x$；
3.$\int\frac{x^3+1}{x(x-1)^3}\mathrm{d}x$；	4.$\int\frac{x^5}{x^6-x^3-2}\mathrm{d}x$；
5.$\int\frac{x^4\mathrm{d}x}{x^2+1}$；	6.$\int\frac{x^3}{(x-1)^{100}}$；
7.$\int\frac{x^9}{(x^{10}+2x^5+2)^2}\mathrm{d}x$；	8.$\int\frac{x}{x^2+2x-8}\mathrm{d}x$；
9.$\int\frac{x-1}{1+2x^2}\mathrm{d}x$；	10.$\int\frac{x}{x^2+4x+13}\mathrm{d}x$；
11.$\int\frac{\sin 2x}{1+\cos x}\mathrm{d}x$；	12.$\int\frac{2+\cos x}{1+\cos x}\mathrm{d}x$；
13.$\int\frac{\mathrm{d}x}{\sin x\cos^3 x}\mathrm{d}x$；	14.$\int\frac{\tan x\mathrm{d}x}{3\sin^2 x+2\cos^2 x}\mathrm{d}x$；
15.$\int\cot^3 x\mathrm{d}x$；	16.$\int\frac{\mathrm{d}x}{1+\sin x+\cos x}$；
17.$\int\frac{\mathrm{d}x}{2+\cos x}$；	18.$\int\cos^4 x\mathrm{d}x$；
19.$\int\frac{\mathrm{d}x}{\cos^2 x-\sin^2 x}$；	20.$\int\frac{\mathrm{d}x}{\sin 2x+1}$.

解：1.方法1：$\int\frac{x^2}{(1+x^2)^2}\mathrm{d}x=\int[\frac{1}{1+x^2}-\frac{1}{(1+x^2)^2}]\mathrm{d}x=\arctan x-\int\frac{1}{(1+x^2)^2}\mathrm{d}x=\arctan x-[\frac{1}{2}\arctan x+\frac{x}{2(1+x^2)}]+C=\frac{1}{2}\arctan x-\frac{x}{2(1+x^2)}+C.$

方法2：$\int\frac{x^2}{(1+x^2)^2}\mathrm{d}x\xlongequal{x=2\tan t}\int\frac{\tan^2 t}{\sec^4 x}\sec^2 t\mathrm{d}t=\int\sin^2 t\mathrm{d}t=\int\frac{1}{2}(1-\cos 2t)\mathrm{d}t=\frac{1}{2}t-\frac{1}{4}\sin 2t+C=\frac{1}{2}\arctan x-\frac{x}{2(1+x^2)}+C.$

2.$\int \frac{x-1}{3+x^2}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(3+x^2)}{3+x^2} - \frac{1}{3}\int \frac{1}{1+(\frac{x}{\sqrt{3}})^2}\mathrm{d}x = \frac{1}{2}\ln(3+x^2) - \frac{1}{\sqrt{3}}\arctan(\frac{x}{\sqrt{3}}) + C.$

3.记$\frac{x^3+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$

则$x^3+1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx$

可得$A=-1, B=2, C=1, D=2$

$\therefore \int[\frac{-1}{x} + \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{2}{(x-1)^3}]\mathrm{d}x = -\ln|x| + 2\ln|x-1| - \frac{1}{x-1} - \frac{1}{(x-1)^2} + C.$

4.$\int \frac{x^5}{x^6-x^3-2}\mathrm{d}x = \frac{1}{6}\int \frac{\mathrm{d}(x^6-x^3-2)}{x^6-x^3-2} + \frac{1}{2}\int \frac{x^2}{x^6-x^3-2}\mathrm{d}x = \frac{1}{6}\ln|x^6-x^3-2| + \frac{1}{6}\int \frac{\mathrm{d}x^3}{(x^3+1)(x^3-2)} = \frac{1}{6}\ln|x^6-x^3-2| - \frac{1}{18}\int(\frac{1}{x^3+1} - \frac{1}{x^3-2})\mathrm{d}x^3 = \frac{1}{6}\ln|x^6-x^3-2| - \frac{1}{18}(\ln|x^3+1| - \ln|x^3-2|) + C$
$= \frac{1}{9}\ln|x^3+1| + \frac{2}{9}\ln|x^3-2| + C.$

5.$\int \frac{x^4\mathrm{d}x}{x^2+1} = \int(x^2 - 1 + \frac{1}{x^2+1})\mathrm{d}x = \frac{1}{3}x^3 - x + \arctan x + C.$

6.方法1：记$\frac{x^3}{(x-1)^{100}} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \cdots + \frac{A_{96}}{(x-1)^{96}} + \frac{A_{97}}{(x-1)^{97}} + \frac{A_{98}}{(x-1)^{98}} + \frac{A_{99}}{(x-1)^{99}} + \frac{A_{100}}{(x-1)^{100}}$
$= \frac{A_1(x-1)^{99} + A_2(x-1)^{98} + \cdots + A_{96}(x-1)^4 + A_{97}(x-1)^3 + A_{98}(x-1)^2 + A_{99}(x-1) + A_{100}}{(x-1)^{100}}$

则$A_1 = A_2 = \cdots = A_{96} = 0$

$A_{97} = 1, -3A_{97} + A_{98} = 0, 3A_{97} - 2A_{98} + A_{99} = 0, -A_{97} + A_{98} - A_{99} + A_{100} = 0$

$\therefore A_{97} = 1, A_{98} = 3, A_{99} = 3, A_{100} = 1$

$\therefore \int \frac{x^3}{(x-1)^{100}}\mathrm{d}x = \int[\frac{1}{(x-1)^{97}} + \frac{3}{(x-1)^{98}} + \frac{3}{(x-1)^{99}} + \frac{1}{(x-1)^{100}}]\mathrm{d}x$
$= -\frac{1}{96(x-1)^{96}} - \frac{3}{97(x-1)^{97}} - \frac{3}{98(x-1)^{98}} - \frac{1}{99(x-1)^{99}} + C$
$= -\frac{1}{(x-1)^{96}}[\frac{1}{96} + \frac{3}{97(x-1)} + \frac{3}{98(x-1)^2} + \frac{1}{99(x-1)^3}] + C.$

方法2：$\int \frac{x^3}{(x-1)^{100}}\mathrm{d}x \xlongequal{t=x-1} \int \frac{(t+1)^3}{t^{100}}\mathrm{d}t = \int \frac{t^3+3t^2+3t+1}{t^{100}}\mathrm{d}t = \int(\frac{1}{t^{97}} + \frac{3}{t^{98}} + \frac{3}{t^{99}} + \frac{1}{t^{100}})\mathrm{d}t$
$= -\frac{1}{96t^{96}} - \frac{3}{97t^{97}} - \frac{3}{98t^{98}} - \frac{1}{99t^{99}} + C = -\frac{1}{(x-1)^{96}}[\frac{1}{96} + \frac{3}{97(x-1)} + \frac{3}{98(x-1)^2} + \frac{1}{99(x-1)^3}] + C$

7.$\int \frac{x^9}{(x^{10}+2x^5+2)^2}\mathrm{d}x = \frac{1}{10}\int \frac{\mathrm{d}(x^{10}+2x^5+2)}{(x^{10}+2x^5+2)^2} - \int \frac{x^4\mathrm{d}x}{(x^{10}+2x^5+2)^2} = -\frac{1}{10}\frac{1}{x^{10}+2x^5+2} - \frac{1}{5}\int \frac{\mathrm{d}(x^5+1)}{[(x^5+1)^2+1]^2}$
$= -\frac{1}{10}\frac{1}{x^{10}+2x^5+2} - \frac{1}{5}[\frac{1}{2}\arctan(x^5+1) + \frac{x^5+1}{2[(x^5+1)^2+1]}] + C$
$= -\frac{x^5+2}{10[(x^5+1)^2+1]} - \frac{1}{10}\arctan(x^5+1) + C.$

8.$\int \frac{x}{x^2+2x-8}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(x^2+2x-8)}{x^2+2x-8} - \int \frac{1}{x^2+2x-8}\mathrm{d}x = \frac{1}{2}\ln|x^2+2x-8| - \int \frac{1}{(x+4)(x-2)}\mathrm{d}x$
$= \frac{1}{2}\ln|x^2+2x-8| + \frac{1}{6}\int(\frac{1}{x+4} - \frac{1}{x-2})\mathrm{d}x = \frac{1}{2}\ln|x^2+2x-8| + \frac{1}{6}(\ln|x+4| - \ln|x-2|) + C$
$= \frac{2}{3}\ln|x+4| + \frac{1}{3}\ln|x-2| + C.$

9.$\int \frac{x-1}{1+2x^2}\mathrm{d}x = \frac{1}{4}\int \frac{\mathrm{d}(1+2x^2)}{1+2x^2} - \frac{1}{\sqrt{2}}\int \frac{1}{1+(\sqrt{2}x)^2}\mathrm{d}(\sqrt{2}x) = \frac{1}{4}\ln(1+2x^2) - \frac{1}{\sqrt{2}}\arctan(\sqrt{2}x) + C.$

10.$\int \frac{x}{x^2+4x+13}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(x^2+4x+13)}{x^2+4x+13} - 2\int \frac{1}{x^2+4x+13}\mathrm{d}x = \frac{1}{2}\ln(x^2+4x+13) - \frac{2}{3}\int \frac{1}{(\frac{x+2}{3})^2+1}\mathrm{d}\frac{x+2}{3}$
$= \frac{1}{2}\ln(x^2+4x+13) - \frac{2}{3}\arctan\frac{x+2}{3} + C.$

11.$\int \frac{\sin 2x}{1+\cos x}\mathrm{d}x = -\int \frac{2\cos x}{1+\cos x}\mathrm{d}\cos x = -\int(2 - \frac{2}{1+\cos x})\mathrm{d}\cos x = -2\cos x + 2\ln(1+\cos x) + C.$

12.$\int \frac{2+\cos x}{1+\cos x}\mathrm{d}x = \int(1 + \frac{1}{1+\cos x})\mathrm{d}x = x + \int \frac{1}{2\cos^2\frac{x}{2}}\mathrm{d}x = x + \int \sec^2\frac{x}{2}\mathrm{d}\frac{x}{2} = x + \tan\frac{x}{2} + C$
$= x - \cot x + \csc x + C.$

13.$\int\frac{\mathrm{d}x}{\sin x\cos^3 x}=\int\frac{\sec^2 x\mathrm{d}x}{\sin x\cos x}=\int\frac{\mathrm{d}\tan x}{\frac{\tan x}{1+\tan^2 x}}=\int(\frac{1}{\tan x}+\tan x)\mathrm{d}\tan x=\ln|\tan x|+\frac{1}{2}\tan^2 x+C.$

14.$\int\frac{\tan x\mathrm{d}x}{3\sin^2 x+2\cos^2 x}=\int\frac{\tan x\sec^2 x}{3\tan^2 x+2}\mathrm{d}x=\frac{1}{2}\int\frac{\tan x}{1+(\frac{\sqrt{3}}{\sqrt{2}}\tan x)^2}\mathrm{d}\tan x=\frac{1}{3}\int\frac{\frac{\sqrt{3}}{\sqrt{2}}\tan x}{1+(\frac{\sqrt{3}}{\sqrt{2}}\tan x)^2}\mathrm{d}\frac{\sqrt{3}}{\sqrt{2}}\tan x$
$=\frac{1}{6}\int\frac{\mathrm{d}[1+(\frac{\sqrt{3}}{\sqrt{2}}\tan x)^2]}{1+(\frac{\sqrt{3}}{\sqrt{2}}\tan x)^2}=\frac{1}{6}\ln(1+\frac{3}{2}\tan^2 x)+C.$

15.$\int\cot^3 x\mathrm{d}x=\int(\csc^2 x-1)\cot x\mathrm{d}x=\int\csc^2 x\cot x\mathrm{d}x-\int\cot x\mathrm{d}x=-\int\cot\mathrm{d}\cot x-\ln|\sin x|=-\frac{1}{2}\cot^2 x-\ln|\sin x|+C.$

16.$\int\frac{\mathrm{d}x}{1+\sin x+\cos x}=\int\frac{\mathrm{d}x}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}=\int\frac{\sec^2\frac{x}{2}\mathrm{d}\frac{x}{2}}{1+\tan\frac{x}{2}}=\int\frac{\mathrm{d}\tan\frac{x}{2}}{1+\tan\frac{x}{2}}=\ln|1+\tan\frac{x}{2}|+C.$

17.$\int\frac{\mathrm{d}x}{2+\cos x}=\int\frac{\mathrm{d}x}{\sin^2\frac{x}{2}+3\cos^2\frac{x}{2}}=2\int\frac{\sec^2\frac{x}{2}\mathrm{d}\frac{x}{2}}{\tan^2\frac{x}{2}+3}=\frac{2}{\sqrt{3}}\int\frac{\mathrm{d}\frac{1}{\sqrt{3}}\tan\frac{x}{2}}{(\frac{1}{\sqrt{3}}\tan\frac{x}{2})^2+1}=\frac{2}{\sqrt{3}}\arctan\frac{\tan\frac{x}{2}}{\sqrt{3}}+C.$

18.$\int\cos^4 x\mathrm{d}x=\int\frac{\cos^4 x}{(\sin^2 x+\cos^2 x)^3}\mathrm{d}x=\int\frac{\sec^2 x\mathrm{d}x}{(\tan^2 x+1)^3}=\int\frac{\mathrm{d}\tan x}{(\tan^2 x+1)^3}=\frac{3}{4}\int\frac{\mathrm{d}\tan x}{(\tan^2 x+1)^2}+\frac{\tan x}{4(\tan^2 x+1)^2}$
$=\frac{3}{4}[\frac{1}{2}x+\frac{\tan x}{2(\tan^2 x+1)}]+\frac{\tan x}{4(\tan^2 x+1)^2}+C=\frac{3}{8}x+\frac{3\tan x}{8(\tan^2 x+1)}+\frac{\tan x}{4(\tan^2 x+1)^2}+C$
$=\frac{3}{8}x+\frac{3}{8}\sin x\cos x+\frac{1}{4}\sin x\cos^3 x+C=\frac{3}{8}x+\frac{3}{16}\sin 2x+\frac{1}{16}\sin 2x(1+\cos 2x)+C$
$=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C.$

19.$\int\frac{\mathrm{d}x}{\cos^2 x-\sin^2 x}=\int\frac{\mathrm{d}x}{\cos 2x}=\frac{1}{2}\int\sec 2x\mathrm{d}2x=\frac{1}{2}\ln|\sec 2x+\tan 2x|+C$
$=\frac{1}{2}\ln|\frac{\cos x+\sin x}{\cos x-\sin x}|+C.$

20.$\int\frac{\mathrm{d}x}{\sin 2x+1}=\int\frac{\mathrm{d}x}{(\sin x+\cos x)^2}=\int\frac{\sec^2 x\mathrm{d}x}{(\tan x+1)^2}=\int\frac{\mathrm{d}\tan x}{(\tan x+1)^2}=-\frac{1}{1+\tan x}+C=-\frac{\sin x}{\sin x+\cos x}+C.$

9.6 习题6.5解答

求下列不定积分：

(1)$\int\frac{\mathrm{d}x}{1+\sqrt{x}}$； (2)$\int\frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}}$；

(3)$\int\frac{\sqrt{1-x}}{x}\mathrm{d}x$； (4)$\int\frac{\mathrm{d}x}{x\sqrt{2x+1}}$；

(5)$\int\frac{\mathrm{d}x}{\sqrt[3]{1-3x}}$； (6)$\int\frac{\sqrt{x}}{\sqrt{1-x}}\mathrm{d}x$；

(7)$\int\frac{\mathrm{d}x}{\sqrt{x^2-x}}$； (8)$\int\frac{1}{x^2+2x+3}\mathrm{d}x$；

(9)$\int\frac{\mathrm{d}x}{\sqrt{x^2-4x}}$； (10)$\int\frac{x^3}{\sqrt{x^8}}\mathrm{d}x$；

(11)$\int\frac{\mathrm{d}x}{\sqrt{1+\mathrm{e}^x}}$； (12)$\int\frac{\sqrt{1+x^2}}{x}\mathrm{d}x$；

(13)$\int 2\mathrm{e}^x\sqrt{1-\mathrm{e}^{2x}}\mathrm{d}x$； (14)$\int\frac{\mathrm{d}x}{x^2\sqrt{x^2+9}}$；

(15)$\int\mathrm{e}^{\sqrt{2x-1}}\mathrm{d}x$.

解：(1)$\int\frac{\mathrm{d}x}{1+\sqrt{x}}\xlongequal{t=\sqrt{x}}\int\frac{2t\mathrm{d}t}{1+t}=2\int(1-\frac{1}{1+t})\mathrm{d}t=2(t-\ln|1+t|)+C=2t-2\ln|1+t|+C$
$=2\sqrt{x}-2\ln(1+\sqrt{x})|+C.$

(2)$\int\frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}}\xlongequal{x=t^6}\int\frac{6t^5\mathrm{d}t}{t^3+t^2}=6\int\frac{t^3}{t+1}\mathrm{d}t=6\int(t^2-t+1-\frac{1}{t+1})\mathrm{d}t$
$=6(\frac{1}{3}t^3-\frac{1}{2}t^2+t-\ln|1+t|)+C=2t^3-3t^2+6t-6\ln|1+t|+C$
$=2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln(1+\sqrt[6]{x})+C.$

(3)$\int \frac{\sqrt{1-x}}{x}\mathrm{d}x \xlongequal{\sqrt{1-x}=t} \int \frac{-2t^2\mathrm{d}t}{1-t^2} = 2\int \frac{t^2}{t^2-1}\mathrm{d}t = 2\int(1+\frac{1}{t^2-1})\mathrm{d}t = 2t+\int(\frac{1}{t-1}-\frac{1}{t+1})\mathrm{d}t$
$= 2t+\ln|\frac{t-1}{t+1}|+C = 2\sqrt{1-x}+\ln|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}|+C.$

(4)$\int \frac{\mathrm{d}x}{x\sqrt{2x+1}} \xlongequal{\sqrt{2x+1}=t} \int \frac{t\mathrm{d}t}{\frac{1}{2}(t^2-1)t} = \int \frac{2\mathrm{d}t}{t^2-1} = \int(\frac{1}{t-1}-\frac{1}{t+1})\mathrm{d}t = \ln|\frac{t-1}{t+1}|+C$
$= \ln|\frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1}|+C.$

(5)$\int \frac{x\mathrm{d}x}{\sqrt[3]{1-3x}} \xlongequal{t=\sqrt[3]{1-3x}} \int \frac{\frac{1}{3}(1-t^3)(-t^2)\mathrm{d}t}{t} = \frac{1}{3}\int(t^4-t)\mathrm{d}t = \frac{1}{3}(\frac{1}{5}t^5-\frac{1}{2}t^2)+C$
$= \frac{1}{3}\sqrt[3]{(1-3x)^2}[\frac{1}{5}(1-3x)-\frac{1}{2}]+C = -\frac{1}{10}\sqrt[3]{1-3x}(1+2x)+C.$

(6)$\int \frac{\sqrt{x}}{1-x}\mathrm{d}x \xlongequal{t=\sqrt{x}} \int \frac{2t^2}{1-t^2}\mathrm{d}t = 2\int(-1-\frac{1}{t^2-1})\mathrm{d}t = -2t-\int(\frac{1}{t-1}-\frac{1}{t+1})\mathrm{d}t$
$= -2t-\ln|\frac{t-1}{t+1}|+C = -2\sqrt{x}-\ln|\frac{\sqrt{x}-1}{\sqrt{x}+1}|+C.$

(7)方法1：$\int \frac{\mathrm{d}x}{\sqrt{x^2-x}} = \int \frac{\mathrm{d}x}{\sqrt{(x-\frac{1}{2})^2-\frac{1}{4}}} \xlongequal{x-\frac{1}{2}=\frac{1}{2}\sec t} \int \frac{\frac{1}{2}\sec t\tan t\mathrm{d}t}{\frac{1}{2}\tan t} = \int \sec t\mathrm{d}t$
$= \ln|\sec t+\tan t|+C = \ln|2(x-\frac{1}{2})+\sqrt{4(x-\frac{1}{2})^2-1}|+C = \ln|x-\frac{1}{2}+\sqrt{x^2-x}|+C.$

方法2：$\int \frac{\mathrm{d}x}{\sqrt{x^2-x}} = \int \sqrt{\frac{x-1}{x}}\frac{1}{x-1}\mathrm{d}x \xlongequal[x=\frac{1}{1-t^2}]{t=\sqrt{\frac{x-1}{x}}} \int t\frac{1}{\frac{1-1+t^2}{1-t^2}}\frac{2t}{(1-t^2)^2}\mathrm{d}t = \int \frac{2}{1-t^2}\mathrm{d}t$
$= \int(\frac{1}{1+t}+\frac{1}{1-t})\mathrm{d}t = \ln|\frac{1+t}{1-t}|+C = \ln|\frac{1+\sqrt{\frac{x-1}{x}}}{1-\sqrt{\frac{x-1}{x}}}|+C = \ln|\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}-\sqrt{x-1}}|+C$
$= 2\ln(\sqrt{x}+\sqrt{x-1})+C.$ [1]

(8)$\int \frac{1}{\sqrt{x^2+2x+3}}\mathrm{d}x = \int \frac{1}{\sqrt{(x+1)^2+2}}\mathrm{d}x \xlongequal{x+1=\sqrt{2}\tan t} \int \frac{\sqrt{2}\sec^t\mathrm{d}t}{\sqrt{2}\sec t} = \int \sec t\mathrm{d}t$
$= \ln|\sec t+\tan t|+C = \ln|\frac{x+1}{\sqrt{2}}+\sqrt{\frac{(x+1)^2}{2}+1}|+C = \ln(x+1+\sqrt{x^2+2x+3})+C$
$= \mathrm{arcsinh}\frac{x+1}{\sqrt{2}}+C.$

(9)$\int \frac{\mathrm{d}x}{\sqrt{x^2-4x}} = \int \frac{\mathrm{d}x}{\sqrt{(x-2)^2-4}} \xlongequal{x-2=2\sec t} \int \frac{2\sec t\tan t\mathrm{d}t}{2\tan t} = \int \sec t\mathrm{d}t = \ln|\sec t+\tan t|+C$
$= \ln|\frac{1}{2}(x-2)+\sqrt{\frac{1}{4}(x-2)^2-1}|+C = \ln|x-2+\sqrt{x^2-4x}|+C.$

---

[1]对方法2而言这里可以只考虑$x>1$的情况。如果需要考虑符号，则可这样做：

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-x}} = \mathrm{sgn}(x-1)\int \sqrt{\frac{x-1}{x}}\frac{1}{x-1}\mathrm{d}x \xlongequal[x=\frac{1}{1-t^2}]{t=\sqrt{\frac{x-1}{x}}} \mathrm{sgn}(1-t)\int t\frac{1}{\frac{1-1+t^2}{1-t^2}}\frac{2t}{(1-t^2)^2}\mathrm{d}t = \mathrm{sgn}(1-t)\int \frac{2}{1-t^2}\mathrm{d}t$$

$$= \mathrm{sgn}(1-t)\int(\frac{1}{1+t}+\frac{1}{1-t})\mathrm{d}t = \mathrm{sgn}(1-t)\ln|\frac{1+t}{1-t}|+C = \mathrm{sgn}(x-1)\ln|\frac{1+\sqrt{\frac{x-1}{x}}}{1-\sqrt{\frac{x-1}{x}}}|+C$$

$$= \begin{cases}\ln|\frac{1+\sqrt{\frac{x-1}{x}}}{1-\sqrt{\frac{x-1}{x}}}|+C, & x>1\\ -\ln|\frac{1+\sqrt{\frac{x-1}{x}}}{1-\sqrt{\frac{x-1}{x}}}|+C, & x<0\end{cases} = \begin{cases}\ln|\frac{(1+\sqrt{\frac{x-1}{x}})^2}{\frac{1}{x}}|+C, & x>1\\ -\ln|\frac{(1+\sqrt{\frac{x-1}{x}})^2}{\frac{1}{x}}|+C, & x<0\end{cases} = \begin{cases}\ln|x(1+\sqrt{\frac{x-1}{x}})^2|+C, & x>1\\ -\ln|x(1+\sqrt{\frac{x-1}{x}})^2|+C, & x<0\end{cases}$$

$$= \begin{cases}\ln|x+x-1+2x\sqrt{\frac{x-1}{x}}|+C, & x>1\\ -\ln|x+x-1+2x\sqrt{\frac{x-1}{x}}|+C, & x<0\end{cases} = \begin{cases}\ln|2x-1+2\sqrt{x^2-x}|+C, & x>1\\ -\ln|2x-1-2\sqrt{x^2-x}|+C, & x<0\end{cases}$$

$$= \begin{cases}\ln|2x-1+2\sqrt{x^2-x}|+C, & x>1\\ \ln|2x-1+2\sqrt{x^2-x}|+C, & x<0\end{cases} = \ln|2x-1+2\sqrt{x^2-x}|+C.$$

其中由于$t=\sqrt{\frac{x-1}{x}} = \begin{cases}>1, & x<0\\ <1, & x>1\end{cases}$，故$\mathrm{sgn}(x-1)=\mathrm{sgn}(1-t)$.

(10)$\int \frac{x^3}{\sqrt{x^8+1}}\mathrm{d}x = \frac{1}{4}\int \frac{\mathrm{d}(x^4)}{\sqrt{(x^4)^2+1}} \xlongequal{x^4=\tan t} \frac{1}{4}\int \frac{\sec^2 t\mathrm{d}t}{\sec t} = \frac{1}{4}\int \sec t\mathrm{d}t = \frac{1}{4}\ln|\sec t+\tan t|+C$
$= \frac{1}{4}\ln|t^4+\sqrt{t^8+1}|+C = \frac{1}{4}\mathrm{arcsinh}(x^4)+C.$

(11)$\int \frac{\mathrm{d}x}{\sqrt{1+\mathrm{e}^x}} \xlongequal{t=\sqrt{1+\mathrm{e}^x}} \int \frac{1}{t}\frac{2t}{t^2-1}\mathrm{d}t = \int(\frac{1}{t-1}-\frac{1}{t+1})\mathrm{d}t = \ln|\frac{t-1}{t+1}|+C = \ln(\frac{\sqrt{1+\mathrm{e}^x}-1}{\sqrt{1+\mathrm{e}^x}+1})+C.$

(12)$\int \frac{\sqrt{1+x^2}}{x}\mathrm{d}x \xlongequal{x=\tan t} \int \frac{\sec t\sec^2 t\mathrm{d}t}{\tan t} = \int \frac{\mathrm{d}t}{\sin t\cos^2 t} = \int \frac{\sin t\mathrm{d}t}{\sin^2 t\cos^2 t} = -\int \frac{\mathrm{d}\cos t}{(1-\cos^2 t)\cos^2 t}$
$= \int \frac{\mathrm{d}\cos t}{(\cos^2 t-1)\cos^2 t} \xlongequal{u=\cos t} \int \frac{\mathrm{d}u}{(u^2-1)u^2} = \int(-\frac{1}{u^2}+\frac{1}{2(u-1)}-\frac{1}{2(u+1)})\mathrm{d}u = \frac{1}{u}+\frac{1}{2}\ln|\frac{u-1}{u+1}|+C$
$= \sec t+\frac{1}{2}\ln|\frac{1-\sec t}{1+\sec t}|+C = \sqrt{x^2+1}+\frac{1}{2}\ln(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1})+C.$

(13)$\int 2\mathrm{e}^x\sqrt{1-\mathrm{e}^{2x}}\mathrm{d}x = \int 2\sqrt{1-\mathrm{e}^{2x}}\mathrm{d}\mathrm{e}^x \xlongequal{\mathrm{e}^x=\sin t} \int 2\cos t\mathrm{d}\sin t = \int 2\cos^2 t\mathrm{d}t$
$= \int(1+\cos 2t)\mathrm{d}t = t+\frac{1}{2}\sin 2t+C = t+\sin t\cos t+C = \arcsin \mathrm{e}^x+\mathrm{e}^x\sqrt{1-\mathrm{e}^{2x}}+C.$

(14)$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+9}} \xlongequal{x=3\tan t} \int \frac{3\sec^2 t\mathrm{d}t}{9\tan^2 t 3\sec t} = \frac{1}{9}\int \frac{\cos t}{\sin^2 t}\mathrm{d}t = \frac{1}{9}\int \mathrm{d}\sin t\sin^2 t = -\frac{1}{9\sin t}+C$
$= -\frac{1}{9}\sqrt{\frac{\sin^2 t+\cos^2 t}{\sin^2 t}}\mathrm{sgn}(\sin t)+C = -\frac{1}{9}\sqrt{\frac{\tan^2 t+1}{\tan^2 t}}\mathrm{sgn}(\sin t)+C = -\frac{1}{9}\sqrt{\frac{x^2+9}{x^2}}\mathrm{sgn}(x)+C$
$= -\frac{1}{9}\frac{\sqrt{x^2+9}}{x}+C$（这里用到了$\mathrm{sgn}(x)=\mathrm{sgn}(\tan t)=\mathrm{sgn}(\sin t), t\in(-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2})$）.

(15)$\int \mathrm{e}^{\sqrt{2x-1}}\mathrm{d}x \xlongequal{t=\sqrt{2x-1}} \int \mathrm{e}^t t\mathrm{d}t = \int t\mathrm{d}\mathrm{e}^t = t\mathrm{e}^t-\int \mathrm{e}^t\mathrm{d}t = t\mathrm{e}^t-\mathrm{e}^t+C$
$= e^{\sqrt{2x-1}}(\sqrt{2x-1}-1)+C.$