11C 第7章补充题

11C.1 第7章补充题解答

1. 设$f \in R[a,b], g \in R[a,b]$，求证：

$$\Big(\int_a^b f(x)g(x)\mathrm{d}x\Big)^2 \le \int_a^b f^2(x)\mathrm{d}x \int_a^b g^2(x)\mathrm{d}x$$

证明：令$F(x) = \Big(\int_a^x f(t)g(t)\mathrm{d}t\Big)^2 - \int_a^x f^2(t)\mathrm{d}t \int_a^x g^2(t)\mathrm{d}t$

$F'(x) = 2\int_a^x f(t)g(t)\mathrm{d}t f(x)g(x) - f^2(x)\int_a^x g^2(t)\mathrm{d}t - g^2(x)\int_a^x f^2(t)\mathrm{d}t$

$= \int_a^x [2f(t)g(t)f(x)g(x) - f^2(x)g^2(t) - g^2(x)f^2(t)]\mathrm{d}t$

$= -\int_a^x [f(x)g(t) - f(t)g(x)]^2\mathrm{d}t \le 0$

$\therefore F(x) \le F(0) = 0$，即$\Big(\int_a^b f(x)g(x)\mathrm{d}x\Big)^2 \le \int_a^b f^2(x)\mathrm{d}x \int_a^b g^2(x)\mathrm{d}x.$

2. 设$f(x)$在区间$[0,1]$上连续且单调减少，又设$f(x) > 0$，求证对于任意满足$0 < \alpha < \beta < 1$的α和β，有

$$\beta\int_0^\alpha f(x)\mathrm{d}x > \alpha\int_0^\beta f(x)\mathrm{d}x$$

证明：方法1：令$F(x) = x\int_0^\alpha f(t)\mathrm{d}t - \alpha\int_0^x f(t)\mathrm{d}t$

$F'(x) = \int_0^\alpha f(t)\mathrm{d}t - \alpha f(x), F''(x) = -\alpha f'(x)$

$\because f(x)$在区间$[0,1]$上连续且单调减少

$\therefore F''(x) = -\alpha f'(x) > 0$

$\therefore$当$x > \alpha$时$F'(x) = \int_0^\alpha f(t)\mathrm{d}t - \alpha f(x) > F(\alpha) = \int_0^\alpha f(t)\mathrm{d}t - \alpha f(\alpha)$

$= \int_0^\alpha [f(t) - f(\alpha)]\mathrm{d}t > 0$

$\therefore$当$x > \alpha$时$F(x) > F(\alpha) = 0$

$\therefore$对于任意满足$0 < \alpha < \beta < 1$的α和β，有$\beta\int_0^\alpha f(x)\mathrm{d}x > \alpha\int_0^\beta f(x)\mathrm{d}x.$

方法2：$\beta\int_0^\alpha f(x)\mathrm{d}x - \alpha\int_0^\beta f(x)\mathrm{d}x = \beta\int_0^\alpha f(x)\mathrm{d}x - \alpha\int_0^\alpha f(x)\mathrm{d}x - \alpha\int_\alpha^\beta f(x)\mathrm{d}x$

$= (\beta-\alpha)\int_0^\alpha f(x)\mathrm{d}x - \alpha\int_\alpha^\beta f(x)\mathrm{d}x = (\beta-\alpha)\alpha f(\xi_1) - \alpha(\beta-\alpha)f(\xi_2)$

$= \alpha(\beta-\alpha)[f(\xi_1) - f(\xi_2)] > 0, \xi_1 \in (0,\alpha), \xi_2 \in (\alpha,\beta)$

$\therefore$对于任意满足$0 < \alpha < \beta < 1$的α和β，有$\beta\int_0^\alpha f(x)\mathrm{d}x > \alpha\int_0^\beta f(x)\mathrm{d}x.$

3. 设$f(x), g(x)$在区间$[0,+\infty)$上连续，其中$f(x) > 0 (0 \le x < +\infty)$，$g(x)$在区间$[0,+\infty)$上单调增加，令

$$\varphi(x) = \frac{\int_0^x f(t)g(t)\mathrm{d}t}{\int_0^x f(t)\mathrm{d}t}.$$

求证$\varphi(x)$在区间$[0,+\infty)$上单调增加.

证明：$\because f(x)>0(0\le x<+\infty)$，$g(x)$在区间$[0,+\infty)$上单调增加

$\therefore \varphi'(x)=\frac{f(x)g(x)\int_0^x f(t)\mathrm{d}t-f(x)\int_0^x f(t)g(t)\mathrm{d}t}{[\int_0^x f(t)\mathrm{d}t]^2}=f(x)\frac{\int_0^x f(t)[g(x)-g(t)]\mathrm{d}t}{[\int_0^x f(t)\mathrm{d}t]^2}>0, x\in[0,+\infty)$

【或者$\varphi'(x)=\frac{f(x)g(x)\int_0^x f(t)\mathrm{d}t-f(x)\int_0^x f(t)g(t)\mathrm{d}t}{[\int_0^x f(t)\mathrm{d}t]^2}=f(x)\frac{\int_0^x f(t)[g(x)-g(t)]\mathrm{d}t}{[\int_0^x f(t)\mathrm{d}t]^2}$
$=f(x)\frac{xf(\xi)[g(x)-g(\xi)]}{[\int_0^x f(t)\mathrm{d}t]^2}>0, \xi\in(0,x), x\in[0,+\infty)$】

$\therefore \varphi(x)$在区间$[0,+\infty)$上单调增加.

4. 设$f'(x)$在区间$[a,b]$上连续，且$f(a)=f(b)=0$，求证：

$$\Big|\int_a^b f(x)\mathrm{d}x\Big|\le\frac{(b-a)^2}{4}\max_{a\le x\le b}|f'(x)|$$

证明：$\because f'(x)$在区间$[a,b]$上连续，且$f(a)=f(b)=0$

$\therefore f(x)=f'(\xi_1)(x-a), f(x)=f'(\xi_2)(x-b), \xi_1\in(a,x), \xi_2\in(x,b)$

$\therefore \big|\int_a^{\frac{a+b}{2}} f(x)\mathrm{d}x\big|=\big|\int_a^{\frac{a+b}{2}} f'(\xi_1)(x-a)\mathrm{d}x\big|\le\int_a^{\frac{a+b}{2}}\big|f'(\xi_1)\big|(x-a)\mathrm{d}x$
$\le\int_a^{\frac{a+b}{2}}\max\limits_{a\le x\le b}\big|f'(x)\big|(x-a)\mathrm{d}x=\max\limits_{a\le x\le b}\big|f'(x)\big|\int_a^{\frac{a+b}{2}}(x-a)\mathrm{d}x=\frac{1}{8}(b-a)^2\max\limits_{a\le x\le b}\big|f'(x)\big|$

$\big|\int_{\frac{a+b}{2}}^b f(x)\mathrm{d}x\big|=\big|\int_{\frac{a+b}{2}}^b f'(\xi_2)(x-b)\mathrm{d}x\big|\le\int_{\frac{a+b}{2}}^b\big|f'(x)\big|(b-x)\mathrm{d}x$
$\le\int_{\frac{a+b}{2}}^b\max\limits_{a\le x\le b}\big|f'(x)\big|(b-x)\mathrm{d}x=\max\limits_{a\le x\le b}\big|f'(x)\big|\int_{\frac{a+b}{2}}^b(b-x)\mathrm{d}x=\frac{1}{8}(b-a)^2\max\limits_{a\le x\le b}\big|f'(x)\big|$

$\therefore \big|\int_a^b f(x)\mathrm{d}x\big|=\big|\int_a^{\frac{a+b}{2}} f(x)\mathrm{d}x+\int_{\frac{a+b}{2}}^b f(x)\mathrm{d}x\big|\le\big|\int_a^{\frac{a+b}{2}} f(x)\mathrm{d}x\big|+\big|\int_{\frac{a+b}{2}}^b f(x)\mathrm{d}x\big|$
$\le\frac{1}{8}(b-a)^2\max\limits_{a\le x\le b}\big|f'(x)\big|+\frac{1}{8}(b-a)^2\max\limits_{a\le x\le b}\big|f'(x)\big|=\frac{1}{4}(b-a)^2\max\limits_{a\le x\le b}\big|f'(x)\big|$.

5. 设$f(x)$在区间$[a,b]$上连续且单调增加，证明：

$$\int_a^b xf(x)\mathrm{d}x\ge\frac{a+b}{2}\int_a^b f(x)\mathrm{d}x$$

证明：$\int_a^b(x-\frac{a+b}{2})f(x)\mathrm{d}x=\int_a^{\frac{a+b}{2}}(x-\frac{a+b}{2})f(x)\mathrm{d}x+\int_{\frac{a+b}{2}}^b(x-\frac{a+b}{2})f(x)\mathrm{d}x$

$\because f(x)$在$[a,b]$上连续，$g(x)=x-\frac{a+b}{2}$在$[a,\frac{a+b}{2}]$上连续且非正，$g(x)=x-\frac{a+b}{2}$在$[\frac{a+b}{2},b]$上连续且非负

$\therefore \int_a^b(x-\frac{a+b}{2})f(x)\mathrm{d}x=f(\xi_1)\int_a^{\frac{a+b}{2}}(x-\frac{a+b}{2})\mathrm{d}x+f(\xi_2)\int_{\frac{a+b}{2}}^b(x-\frac{a+b}{2})\mathrm{d}x$
$=f(\xi_1)(\frac{1}{2}x^2-\frac{a+b}{2}x)\Big|_a^{\frac{a+b}{2}}+f(\xi_2)(\frac{1}{2}x^2-\frac{a+b}{2}x)\Big|_{\frac{a+b}{2}}^b$
$=\frac{1}{8}(b-a)^2[f(\xi_2)-f(\xi_1)], \xi_2\in(\frac{a+b}{2},b), \xi_1\in(a,\frac{a+b}{2})$

$\because f(x)$在$[a,b]$上单调增加

$\therefore \int_a^b(x-\frac{a+b}{2})f(x)\mathrm{d}x=\frac{1}{8}(b-a)^2[f(\xi_2)-f(\xi_1)]>0$，即$\int_a^b xf(x)\mathrm{d}x\ge\frac{a+b}{2}\int_a^b f(x)\mathrm{d}x$.

6. 设$f(x)$在区间$[0,1]$上连续，下凸且非负，$f(0)=0$，求证：

$$\int_0^{\frac{1}{2}} f(x)\mathrm{d}x \le \frac{1}{4}\int_0^1 f(x)\mathrm{d}x.$$

证明：$\because f(x)$下凸

$\therefore \forall x_1,x_2\in[0,1]$有$f(\frac{x_1+x_2}{2})\le\frac{1}{2}[f(x_1)+f(x_2)]$

$\therefore \int_0^{\frac{1}{2}} f(x)\mathrm{d}x=\frac{1}{2}\int_0^1 f(\frac{u}{2})\mathrm{d}u=\frac{1}{2}\int_0^1 f(\frac{u+0}{2})\mathrm{d}u\le\frac{1}{2}\int_0^1\frac{1}{2}[f(u)+f(0)]\mathrm{d}u=\frac{1}{4}\int_0^1 f(u)\mathrm{d}u$
$=\frac{1}{4}\int_0^1 f(x)\mathrm{d}x.$

7. 设$f(x)$在区间$[a,b]$上连续且$f(x)\ge 0$，又令$M=\max\limits_{a\le x\le b}\{f(x)\}$，求证：

$$\lim_{n\to\infty}\Big(\int_a^b f^n(x)\mathrm{d}x\Big)^{\frac{1}{n}}=M.$$

证明：记$f(x_0)=M=\max\limits_{a\le x\le b}\{f(x)\}$

$\because f(x)$在区间$[a,b]$上连续

$\therefore \forall\varepsilon>0$且$\varepsilon<M$存在$x_0$的邻域$I\subset[a,b], s.t. f(x)>M-\varepsilon>0, x\in I$

$\therefore M=(M^n)^{\frac{1}{n}}=\Big(\int_a^b M^n\mathrm{d}x\Big)^{\frac{1}{n}}\ge\Big(\int_a^b f^n(x)\mathrm{d}x\Big)^{\frac{1}{n}}\ge\Big(\int_I f^n(x)\mathrm{d}x\Big)^{\frac{1}{n}}$

$>\Big(\int_I (M-\varepsilon)^n\mathrm{d}x\Big)^{\frac{1}{n}}=(M-\varepsilon)\Big(\int_I\mathrm{d}x\Big)^{\frac{1}{n}}=M-\varepsilon$

$\therefore -\varepsilon<\Big(\int_a^b f^n(x)\mathrm{d}x\Big)^{\frac{1}{n}}-M<0<\varepsilon$

$\therefore \lim\limits_{n\to\infty}\Big(\int_a^b f^n(x)\mathrm{d}x\Big)^{\frac{1}{n}}=M.$

8. 设$f\in C[a,b]$，如果对于任意一个满足$g(a)=g(b)=0$的$g\in C[a,b]$，都有$\int_a^b f(x)g(x)\mathrm{d}x=0$. 求证：$f(x)\equiv 0$.

证明：假设$\exists x_0\in[a,b], s.t. f(x_0)\ne 0$. 不妨设$x_0\in(a,b), f(x_0)>0$. 这时存在包含$x_0$的区间$[x_1,x_2]\subset[a,b]$，使得$\forall x\in[x_1,x_2]$，有$f(x)>0$. 构造函数

$$g(x)=\begin{cases}(x-x_1)^2(x-x_2)^2, & x_1<x<x_2,\\ 0, & \text{otherwise.}\end{cases}$$

则有$\int_a^b f(x)g(x)\mathrm{d}x=\int_{x_1}^{x_2} f(x)g(x)\mathrm{d}x=f(\xi)\int_{x_1}^{x_2} g(x)\mathrm{d}x>0$. 这与对于任意一个满足$g(a)=g(b)=0$的$g\in C[a,b]$，都有$\int_a^b f(x)g(x)\mathrm{d}x=0$矛盾.

故$f(x)\equiv 0$.

9. 设$f\in C(0,+\infty)$，并且对任意的$a>0$和$b>1$，积分值$\int_a^{ab}f(x)\mathrm{d}x$与$a$无关. 求证：存在常数$c$使得$f(x)=\frac{c}{x}$.

证明：令$F(a)=\int_a^{ab}f(x)\mathrm{d}x, a>0, b>10$

$\because$对任意的$a>0$和$b>1$，积分值$\int_a^{ab}f(x)\mathrm{d}x$与$a$无关

$\therefore \frac{\mathrm{d}F}{\mathrm{d}a}=bf(ab)-f(a)\equiv 0$

取$a=1$，则$bf(b)=f(1), f(b)=\frac{f(1)}{b}, b>1$

$\because f\in C(0,+\infty)$

$\therefore f(b)=\frac{f(1)}{b}$在$b=1$时也成立

即$f(x)=\frac{c}{x}, x\geq 1, c=f(1)$

当$0<b<1$时，$\int_1^b f(x)\mathrm{d}x \xlongequal{u=\frac{1}{x}} -\int_{\frac{1}{b}}^1 f(\frac{1}{u})\frac{1}{u^2}\mathrm{d}u=\int_1^{\frac{1}{b}} f(\frac{1}{u})\frac{1}{u^2}\mathrm{d}u=\int_1^{\frac{1}{b}} \frac{c}{\frac{1}{u}}\frac{1}{u^2}\mathrm{d}u$

$=c\int_1^{\frac{1}{b}}\frac{1}{u}\mathrm{d}u=-c\ln b$

两端对b求导，得到$f(b)=\frac{c}{b}, 0<b<1$，即$\forall x\in(0,1), f(x)=\frac{c}{x}$

综上所述：$f(x)=\frac{c}{x}, x\in(0,+\infty)$.

10. 设$f\in R[a,b]$，其中$b-a=1$，求证：

(1)$\mathrm{e}^{\int_a^b f(x)\mathrm{d}x}\leq\int_a^b \mathrm{e}^{f(x)}\mathrm{d}x$；

(2)若$f(x)\geq c>0$，则$\int_0^1\ln f(x)\mathrm{d}x\leq\ln\int_0^1 f(x)\mathrm{d}x$.

证明：(1)记$T: a=x_0<x_1<x_2<\cdots<x_n=b$为区间$[a,b]$的一个分割，则$\int_a^b f(x)\mathrm{d}x=\lim\limits_{\lambda\to 0} f(\xi_i)\Delta x_i, \Delta x_i=x_i-x_{i-1}, \xi\in(x_{i-1},x_i)$

$\because g(x)=\mathrm{e}^x$下凸

故

$$\begin{aligned}\mathrm{e}^{\frac{\Delta x_1}{b-a}f(\xi_1)+\frac{\Delta x_2}{b-a}f(\xi_2)+\cdots+\frac{\Delta x_n}{b-a}f(\xi_n)} &= \mathrm{e}^{\Delta x_1 f(\xi_1)+\Delta x_2 f(\xi_2)+\cdots+\Delta x_n f(\xi_n)}\\ &\leq \Delta x_1\mathrm{e}^{f(\xi_1)}+\Delta x_2\mathrm{e}^{f(\xi_2)}+\cdots+\Delta x_n\mathrm{e}^{f(\xi_n)}\end{aligned}$$

即$\mathrm{e}^{\sum_{i=1}^n\Delta x_i f(\xi_i)}\leq\sum_{i=1}^n\mathrm{e}^{f(\xi_i)}\Delta x_i$，两边取极限得

$$\mathrm{e}^{\int_a^b f(x)\mathrm{d}x}=\lim_{\lambda\to 0}\mathrm{e}^{\sum_{i=1}^n\Delta x_i f(\xi_i)}\leq\lim_{\lambda\to 0}\sum_{i=1}^n\mathrm{e}^{f(\xi_i)}\Delta x_i=\int_a^b\mathrm{e}^{f(x)}\mathrm{d}x.$$

(2)$\because h(x)=\ln x$上凸

$\therefore$对于区间$[0,1]$的分割$T^*: 0=x_0<x_1<x_2<\cdots<x_n=1, \Delta x_i=x_i-x_{i-1}, \eta_i \in (x_{i-1},x_i)$

$$\begin{aligned}&\ln[\frac{\Delta x_1}{1-0}f(\eta_1)+\frac{\Delta x_2}{1-0}f(\eta_2)+\cdots+\frac{\Delta x_n}{1-0}f(\eta_n)]\\=&\ln[\Delta x_1 f(\eta_1)+\Delta x_2 f(\eta_2)+\cdots+\Delta x_n f(\eta_n)]\\\geq&\Delta x_1\ln f(\eta_1)+\Delta x_2\ln f(\eta_2)+\cdots+\Delta x_n f(\eta_n)\end{aligned}$$

即$\ln[\sum_{i=1}^n \Delta x_i f(\eta_i)] \geq \sum_{i=1}^n \Delta x_i \ln f(\eta_i)$，两边取极限得

$$\ln\int_0^1 f(x)\mathrm{d}x=\lim_{\lambda\to 0}\ln[\sum_{i=1}^n \Delta x_i f(\eta_i)]\geq \lim_{\lambda\to 0}\sum_{i=1}^n \Delta x_i\ln f(\eta_i)=\int_0^1 \ln f(x)\mathrm{d}x$$

即$\int_0^1 \ln f(x)\mathrm{d}x \leq \ln\int_0^1 f(x)\mathrm{d}x$.

11. 设$f\in C[0,\pi]$，求证：

$$\lim_{n\to\infty}\int_0^\pi f(x)|\sin nx|\mathrm{d}x=\frac{2}{\pi}\int_0^\pi f(x)\mathrm{d}x.$$

【注意：原题给的条件为$f\in R[0,\pi]$，这里改成了$f\in C[0,\pi]$，以便应用推广的积分中值定理。】

证明：$\int_0^\pi f(x)|\sin nx|\mathrm{d}x=\sum_{k=1}^n\int_{\frac{k-1}{n}\pi}^{\frac{k}{n}\pi} f(x)|\sin nx|\mathrm{d}x$

$\because f(x)\in C[0,\pi], g(x)=|\sin nx|$在$[\frac{k-1}{n}\pi,\frac{k}{n}\pi]$上连续且非负

$\therefore$根据推广的积分中值定理

$$\int_{\frac{k-1}{n}\pi}^{\frac{k}{n}\pi} f(x)|\sin nx|\mathrm{d}x=f(\xi_k)\int_{\frac{k-1}{n}\pi}^{\frac{k}{n}\pi}|\sin nx|\mathrm{d}x=\frac{2}{n}f(\xi_k),\xi_k\in(\frac{k}{n}\pi,\frac{k-1}{n}\pi)$$

$\therefore \int_0^\pi f(x)|\sin nx|\mathrm{d}x=\sum_{k=1}^n\frac{2}{n}f(\xi_k)=\frac{2}{\pi}\sum_{k=1}^n f(\xi_k)\frac{\pi}{n}$

$\therefore \lim\limits_{n\to\infty}\int_0^\pi f(x)|\sin nx|\mathrm{d}x=\lim\limits_{n\to\infty}\frac{2}{\pi}\sum_{k=1}^n f(\xi_k)\frac{\pi}{n}=\frac{2}{\pi}\int_0^\pi f(x)\mathrm{d}x.$

12. 若f为连续函数，求证：

$$\int_0^x f(u)(x-u)\mathrm{d}u=\int_0^x\big(\int_0^u f(t)\mathrm{d}t\big)\mathrm{d}u.$$

证明：$\int_0^x f(u)(u-x)\mathrm{d}u=\int_0^x(u-x)\mathrm{d}[\int_0^u f(t)\mathrm{d}t]=(u-x)\int_0^u f(t)\mathrm{d}t\Big|_0^x-\int_0^x\big(\int_0^u f(t)\mathrm{d}t\big)\mathrm{d}u$
$=-\int_0^x\big(\int_0^u f(t)\mathrm{d}t\big)\mathrm{d}u$

$\therefore \int_0^x f(u)(x-u)\mathrm{d}u=\int_0^x\big(\int_0^u f(t)\mathrm{d}t\big)\mathrm{d}u.$

13. 设$f(x)$在区间$[0,+\infty)$上一致连续且非负，如果无穷积分$\int_0^{+\infty} f(x)\mathrm{d}x$收敛，
求证：$\lim\limits_{x\to+\infty} f(x)=0.$

证明：假设$\lim\limits_{x\to+\infty} f(x)\neq 0$，则$\exists\varepsilon_0=2a>0,\forall X>0$，当$x>X$时，
$|f(x)-0|=f(x)>\varepsilon_0=2a$

可取一单调增加且趋于$+\infty$的点列$\{x_n\}, s.t. f(x_n)\geq 2a, x_{n+1}-x_n>1, x_1>1$

$\because f(x)$在区间$[0,+\infty)$上一致连续

$\therefore$对于$\varepsilon_1=a>0,\exists b>0$（不妨设$b<\frac{1}{2}$），当$|x-x_n|<b$时，$|f(x)-f(x_n)|<\varepsilon_1=a$

此时$f(x)-f(x_n)>-a, f(x)>f(x_n)-a\geq a$

则$\int_0^{x_{n+1}} f(x)\mathrm{d}x\geq\sum_{k=1}^n\int_{x_k-b}^{x_k+b} f(x)\mathrm{d}x\geq\sum_{k=1}^n\int_{x_k-b}^{x_k+b} a\mathrm{d}x=n\cdot a\cdot 2b\to+\infty, n\to\infty$

$\therefore\lim\limits_{A\to+\infty}\int_0^A f(x)\mathrm{d}x=+\infty,\int_0^{+\infty} f(x)\mathrm{d}x$发散，假设不成立

$\therefore\lim\limits_{x\to+\infty} f(x)=0.$

14. 计算两椭圆$\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1$和$\frac{x^2}{b^2}+\frac{y^2}{a^2}\leq 1(a>0,b>0)$公共部分的面积.

解：由$\begin{cases}\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\\ \frac{x^2}{b^2}+\frac{y^2}{a^2}=1\end{cases}$ 得两椭圆在第一象限的交点为$P(\frac{ab}{\sqrt{a^2+b^2}},\frac{ab}{\sqrt{a^2+b^2}})$

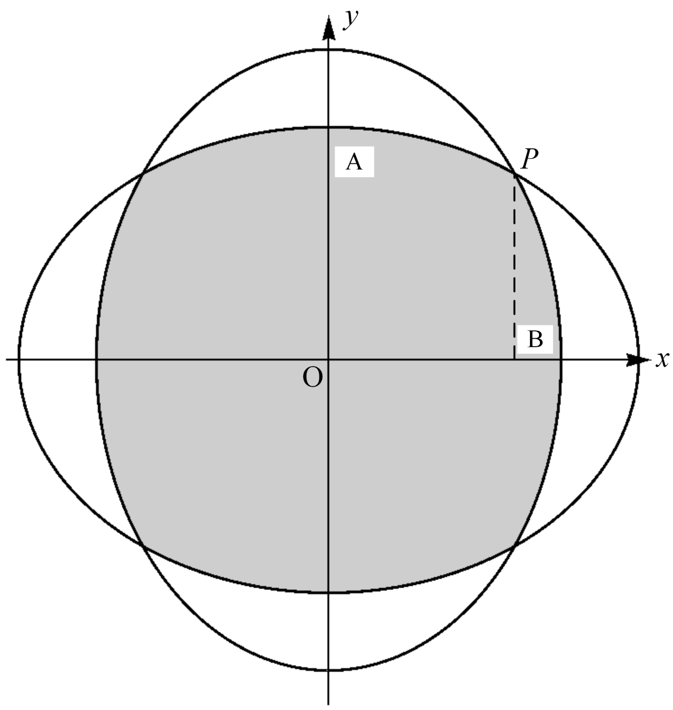

图 1: 第7章补充题 14题图示

如图1所示，图形在第一象限内可以分成A,B两部分.

A部分的面积$S_A=\int_0^{\frac{ab}{\sqrt{a^2+b^2}}}y\mathrm{d}x=\int_0^{\frac{ab}{\sqrt{a^2+b^2}}}b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$

$\xlongequal{x=a\sin t}\int_0^{\arcsin\frac{b}{\sqrt{a^2+b^2}}}b\cos t\cdot a\cos t\mathrm{d}t=ab\int_0^{\arcsin\frac{b}{\sqrt{a^2+b^2}}}\cos^2 t\mathrm{d}t$

$=ab\int_0^{\arcsin\frac{b}{\sqrt{a^2+b^2}}}\frac{1}{2}(1+\cos 2t)\mathrm{d}t=\frac{ab}{2}(t+\frac{1}{2}\sin 2t)\Big|_0^{\arcsin\frac{b}{\sqrt{a^2+b^2}}}$

$=\frac{ab}{2}(t+\sin t\sqrt{1-\sin^2 t})\Big|_0^{\arcsin\frac{b}{\sqrt{a^2+b^2}}}=\frac{ab}{2}(\arcsin\frac{b}{\sqrt{a^2+b^2}}+\frac{b}{\sqrt{a^2+b^2}}\sqrt{1-\frac{b^2}{a^2+b^2}})$

$=\frac{ab}{2}\arcsin\frac{b}{\sqrt{a^2+b^2}}+\frac{a^2b^2}{2(a^2+b^2)}$

方法1：B部分的面积$S_B=\int_{\frac{ab}{\sqrt{a^2+b^2}}}^{b}y\mathrm{d}x=\int_{\frac{ab}{\sqrt{a^2+b^2}}}^{b}a\sqrt{1-\frac{x^2}{b^2}}\mathrm{d}x$

$\xlongequal{x=b\sin t}\int_{\arcsin\frac{a}{\sqrt{a^2+b^2}}}^{\frac{\pi}{2}}a\cos t\cdot b\cos t\mathrm{d}t=ab\int_{\arcsin\frac{a}{\sqrt{a^2+b^2}}}^{\frac{\pi}{2}}\cos^2 t\mathrm{d}t$

$=ab\int_{\arcsin\frac{a}{\sqrt{a^2+b^2}}}^{\frac{\pi}{2}}\frac{1}{2}(1+\cos 2t)\mathrm{d}t=\frac{ab}{2}(t+\frac{1}{2}\sin 2t)\Big|_{\arcsin\frac{a}{\sqrt{a^2+b^2}}}^{\frac{\pi}{2}}$

$=\frac{ab}{2}(t+\sin t\sqrt{1-\sin^2 t})\Big|_{\arcsin\frac{a}{\sqrt{a^2+b^2}}}^{\frac{\pi}{2}}=\frac{ab}{2}(\frac{\pi}{2}-\arcsin\frac{a}{\sqrt{a^2+b^2}}+0-\frac{a}{\sqrt{a^2+b^2}}\sqrt{1-\frac{a^2}{a^2+b^2}})$

$=\frac{ab}{2}\arcsin\frac{b}{\sqrt{a^2+b^2}}+0-\frac{a^2b^2}{2(a^2+b^2)}$

则两椭圆围成的公共部分的面积

$S=4(S_A+S_B)=4ab(\arcsin\frac{b}{\sqrt{a^2+b^2}}.$

方法2：则两椭圆围成的公共部分的面积

$S=4(2S_A-Square)=4[2(\frac{ab}{2}\arcsin\frac{b}{\sqrt{a^2+b^2}}+\frac{a^2b^2}{2(a^2+b^2)})-\frac{a^2b^2}{a^2+b^2}]=4ab\arcsin\frac{b}{\sqrt{a^2+b^2}}$

其中$Square=\frac{a^2b^2}{a^2+b^2}$为两椭圆交点$P$与坐标轴围成的正方形的面积.

15. 求曲线$L:x^3+y^3-3axy=0(a>0)$

(1)自闭部分围成的面积；

(2)与其渐近线围成的面积.

解：(1)将$x=r\cos\theta,y=r\sin\theta$代入$x^3+y^3-3axy=0(a>0)$得$r^3\cos^3\theta+r^3\sin^3\theta-3ar^2\cos\theta\sin\theta=0$，即曲线的极坐标方程为

$$r(\theta)=\frac{3a\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}$$

下面做出函数的图形：

(i)定义域、奇偶性、周期性. 令$\cos^3\theta+\sin^3\theta=0$得$\tan^3\theta=-1$即$\theta=-\frac{\pi}{4}$或$\theta=\frac{3}{4}\pi$，故定义域为$(-\frac{\pi}{4},\frac{3}{4}\pi)\cup(\frac{3}{4}\pi,\frac{7}{4}\pi)$，

因$r(\theta+\pi)=\frac{3a\cos(\theta+\pi)\sin(\theta+\pi)}{\cos^3(\theta+\pi)+\sin^3(\theta+\pi)}=-r(\theta)$，所以$(r(\theta+\pi)\cos(\theta+\pi),r(\theta+\pi)\sin(\theta+\pi))=(r(\theta)\cos\theta,r(\theta)\sin\theta)$故只需做出$(-\frac{\pi}{4},\frac{3}{4}\pi)$内的图形即可，

因$r(\frac{\pi}{4}-\theta)=\frac{3a\cos(\frac{\pi}{4}-\theta)\sin(\frac{\pi}{4}-\theta)}{\cos^3(\frac{\pi}{4}-\theta)+\sin^3(\frac{\pi}{4}-\theta)}=\frac{3a\cos[\frac{\pi}{2}-(\frac{\pi}{4}-\theta)]\sin[\frac{\pi}{2}-(\frac{\pi}{4}-\theta)]}{\cos^3[\frac{\pi}{2}-(\frac{\pi}{4}-\theta)]+\sin^3[\frac{\pi}{2}-(\frac{\pi}{4}-\theta)]}=\frac{3a\cos(\frac{\pi}{4}+\theta)\sin(\frac{\pi}{4}+\theta)}{\cos^3(\frac{\pi}{4}+\theta)+\sin^3(\frac{\pi}{4}+\theta)}=r(\frac{\pi}{4}+\theta)$，故曲线关于$\theta=\frac{\pi}{4}$对称，

当$-\frac{\pi}{4}<\theta<0$时，$r(\theta)<0$，当$0<\theta<\frac{\pi}{2}$时，$r(\theta)>0$，当$\frac{\pi}{2}<\theta<\frac{3}{4}\pi$时，$r(\theta)<0$，

$r(0)=r(\frac{\pi}{2})=0$；

(ii)渐近线. $\because \lim\limits_{\theta\to-\frac{\pi}{4}^+}r(\theta)=-\infty, \lim\limits_{\theta\to\frac{3}{4}\pi^-}r(\theta)=+\infty,$

$\lim\limits_{\theta\to-\frac{\pi}{4}^+}x=\lim\limits_{\theta\to-\frac{\pi}{4}^+}r(\theta)\cos\theta=\lim\limits_{\theta\to-\frac{\pi}{4}^+}\frac{3a\cos^2\theta\sin\theta}{\cos^3\theta+\sin^3\theta}=-\infty,$

$\lim\limits_{\theta\to\frac{3}{4}\pi^-}x=\lim\limits_{\theta\to\frac{3}{4}\pi^-}r(\theta)\cos\theta=\lim\limits_{\theta\to\frac{3}{4}\pi^-}\frac{3a\cos^2\theta\sin\theta}{\cos^3\theta+\sin^3\theta}=+\infty,$

$\therefore \lim\limits_{x\to+\infty}\frac{y}{x}=\lim\limits_{\theta\to\frac{3}{4}\pi^-}\tan\theta=-1, \lim\limits_{x\to-\infty}\frac{y}{x}=\lim\limits_{\theta\to-\frac{\pi}{4}^+}\tan\theta=-1$，即$\lim\limits_{x\to\infty}\frac{y}{x}=-1,$

$\therefore \lim\limits_{x\to+\infty}(x+y)=\lim\limits_{\theta\to\frac{3}{4}\pi^-}[r(\theta)\cos\theta+r(\theta)\sin\theta]=\lim\limits_{\theta\to\frac{3}{4}\pi^-}r(\theta(\cos\theta+\sin\theta)$

$=\lim\limits_{\theta\to\frac{3}{4}\pi^-}\frac{3a\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}(\cos\theta+\sin\theta)=\lim\limits_{\theta\to\frac{3}{4}\pi^-}\frac{3a\cos\theta\sin\theta}{\cos^2\theta-\sin\theta\cos\theta+\sin^2\theta}=-a,$

$\lim\limits_{x\to-\infty}(x+y)=\lim\limits_{\theta\to-\frac{\pi}{4}^+}[r(\theta)\cos\theta+r(\theta)\sin\theta]=\lim\limits_{\theta\to-\frac{\pi}{4}^+}r(\theta)(\cos\theta+\sin\theta)$

$\lim\limits_{\theta\to-\frac{\pi}{4}^+}\frac{3a\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}(\cos\theta+\sin\theta)=\lim\limits_{\theta\to-\frac{\pi}{4}^+}\frac{3a\cos\theta\sin\theta}{\cos^2\theta-\sin\theta\cos\theta+\sin^2\theta}=-a,$

$\therefore \lim\limits_{x\to\infty}(x+y)=-a,$

故曲线有斜渐近线$x+y+a=0$；

(iii)极值点与单调区间. $r'(\theta)=\frac{3a\cos2\theta(\cos^3\theta+\sin^3\theta)-3a\cos\theta\sin\theta(-3\cos^2\theta\sin\theta+3\sin^2\theta\cos\theta)}{(\cos^3\theta+\sin^3\theta)^2}$

$=\frac{3a(\cos^2\theta-\sin^2\theta)(\cos^3\theta+\sin^3\theta)-3a\cos\theta\sin\theta(-3\cos^2\theta\sin\theta+3\sin^2\theta\cos\theta)}{(\cos^3\theta+\sin^3\theta)^2}$

$=\frac{3a(\cos^2\theta-\sin^2\theta)(\cos^3\theta+\sin^3\theta)+9a\cos^2\theta\sin^2\theta(\cos\theta-\sin\theta)}{(\cos^3\theta+\sin^3\theta)^2}$

$=3a(\cos\theta-\sin\theta)\frac{(\cos\theta+\sin\theta)(\cos^3\theta+\sin^3\theta)+3\cos^2\theta\sin^2\theta}{(\cos^3\theta+\sin^3\theta)^2}$

$=3a(\cos\theta-\sin\theta)\frac{(\cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta)(\cos^2\theta-\cos\theta\sin\theta+\sin^2\theta)+3\cos^2\theta\sin^2\theta}{(\cos^3\theta+\sin^3\theta)^2}$

$=3a(\cos\theta-\sin\theta)\frac{(1+2\cos\theta\sin\theta)(1-\cos\theta\sin\theta)+3\cos^2\theta\sin^2\theta}{(\cos^3\theta+\sin^3\theta)^2}$

$=3a(\cos\theta-\sin\theta)\frac{1-2\cos^2\theta\sin^2\theta+\cos\theta\sin\theta+3\cos^2\theta\sin^2\theta}{(\cos^3\theta+\sin^3\theta)^2}$

$=3a(\cos\theta-\sin\theta)\frac{1+\cos^2\theta\sin^2\theta+\cos\theta\sin\theta}{(\cos^3\theta+\sin^3\theta)^2}$

$=3a(\cos\theta-\sin\theta)\frac{1+\frac{1}{4}\sin^2 2\theta+\frac{1}{2}\sin2\theta}{(\cos^3\theta+\sin^3\theta)^2}$

$=3a(\cos\theta-\sin\theta)\frac{4+\sin^2 2\theta+2\sin2\theta}{4(\cos^3\theta+\sin^3\theta)^2}$

$=3a(\cos\theta-\sin\theta)\frac{(\sin2\theta+1)^2+3}{4(\cos^3\theta+\sin^3\theta)^2},$

令$r'(\theta)=0$得$\theta=\frac{\pi}{4}$，当$-\frac{\pi}{4}<\theta<\frac{\pi}{4}$时$r'(\theta)>0$，当$\frac{\pi}{4}<\theta<\frac{3}{4}\pi$时$r'(\theta)<0$，故$\theta=\frac{\pi}{4}$是$r(\theta)$的最大值点.

据此可作出下表：

θ	$\frac{\pi}{4}$	$(-\frac{\pi}{4},0)$	0	$(0,\frac{\pi}{4})$	$\frac{\pi}{4}$	$(\frac{\pi}{4},\frac{\pi}{2})$	$\frac{\pi}{2}$	$(\frac{\pi}{2},\frac{3}{4}\pi)$	$\frac{3}{4}\pi$
$r'(\theta)$		$+$		$+$	0	$-$		$-$	
$r(\theta)$	$-\infty$	$(-\infty,0)$	0	$(0,\frac{3\sqrt{2}}{2}a)$	最大值$\frac{3\sqrt{2}}{2}a$	$(\frac{3\sqrt{2}}{2}a,0)$	0	$(0,-\infty)$	$-\infty$
$x=r(\theta)\cos\theta$	$-\infty$	$(-\infty,0)$	0	$(0,\frac{3}{2}a)$	$\frac{3}{2}a$	$(\frac{3}{2}a,0)$	0	$(0,+\infty)$	$+\infty$
$y=r(\theta)\sin\theta$	$-x-a$	$(+\infty,0)$	0	$(0,\frac{3}{2}a)$	$\frac{3}{2}a$	$(\frac{3}{2}a,0)$	0	$(0,-\infty)$	$-x-a$

可据此画出如下曲线.

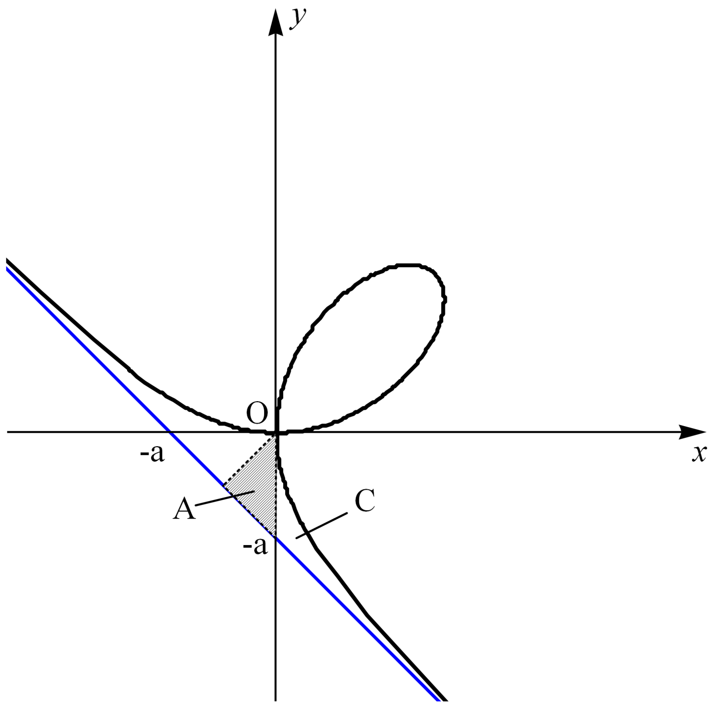

图 2: 第7章补充题 15题图示

由上图可知$r=r(\theta),0\le\theta\le\frac{\pi}{2}$是曲线的自闭部分.

自闭部分的面积为$S_1=\frac{1}{2}\int_0^{\frac{\pi}{2}}r^2(\theta)\mathrm{d}\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{9a^2\sin^2\theta\cos^2\theta}{(\cos^3\theta+\sin^3\theta)^2}\mathrm{d}\theta\ \frac{t=\tan\theta}{\overline{\overline{\mathrm{d}t=\sec^2\theta\mathrm{d}\theta}}}\ \frac{9a^2}{2}\int_0^{+\infty}\frac{t^2}{(1+t^3)^2}\mathrm{d}t$

$=\frac{3a^2}{2}\int_0^{+\infty}\frac{\mathrm{d}(1+t^3)}{(1+t^3)^2}=-\frac{3a^2}{2}\frac{1}{1+t^3}\Big|_0^{+\infty}=\frac{3}{2}a^2.$

(2)利用对称性，曲线与渐近线围成的面积为图2中A和C两部分面积之和的二倍，

渐近线的参数方程为$r_1(\theta)=-\frac{a}{\sin\theta+\cos\theta}$，

则C部分的面积$S_C=\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3}{4}\pi}[r_1^2(\theta)-r^2(\theta)]\mathrm{d}\theta=\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3}{4}\pi}[\frac{a^2}{(\sin\theta+\cos\theta)^2}-\frac{9a^2\sin^2\theta\cos^2\theta}{(\sin^3\theta+\cos^3\theta)^2}]\mathrm{d}\theta$

$\frac{t=\tan\theta}{\overline{\overline{\mathrm{d}t=\frac{1}{2}\sec^2\theta\mathrm{d}\theta}}}\ \frac{1}{2}\int_{-\infty}^{-1}[\frac{a^2}{(t+1)^2}-\frac{9a^2t^2}{(t^3+1)^2}]\mathrm{d}t=\frac{1}{2}[-\frac{a^2}{t+1}+\frac{3a^2}{t^3+1}]\Big|_{-\infty}^{-1}=\frac{1}{2}\frac{a^2(2-t)(1+t)}{(1+t)(t^2-t+1)}\Big|_{-\infty}^{-1}=\frac{1}{2}\frac{a^2(2-t)}{(t^2-t+1)}\Big|_{-\infty}^{-1}$

$=\frac{a^2}{2}$,

A部分的面积$S_A=\frac{1}{4}a^2$,

曲线与渐近线围成的面积$S=2(S_A+S_C)=\frac{3}{2}a^2$.

【注意：教材答案中$r''(\theta)<0$的结论有误，$r''(\theta)$在$(0,\frac{\pi}{2})$内有正有负.】

16. 设$f(x)\in C[0,1]$，且$f(x)<1$，证明：方程$2x-\int_0^x f(t)\mathrm{d}t=1$在区间$(0,1)$上有且只有一个根.

证明：令$F(x)=2x-\int_0^x f(t)\mathrm{d}t-1$

$\because f(x)\in C[0,1]$

$\therefore f(x)\in R[0,1],\int_0^x f(t)\mathrm{d}t\in C[0,1],F(x)\in C[0,1]$

$\because f(x)<1$

$\therefore F(1)=1-\int_0^1 f(t)\mathrm{d}t>0$

又$\because F(0)=-1<0$

$\therefore \exists\xi\in(0,1),s.t.F(\xi)=0$

$\because F'(x)=2-f(x)>1>0$

$\therefore F(x)$在$[0,1]$上单调增加

$\therefore \xi$唯一，即方程$2x-\int_0^x f(t)\mathrm{d}t=1$在区间$(0,1)$上有且只有一个根.

17. 计算积分$\int_0^{\frac{\pi}{2}}\ln(\sin x)\mathrm{d}x$和$\int_0^{\pi}\ln(1+\cos x)\mathrm{d}x$.

解：(1)$I=\int_0^{\frac{\pi}{2}}\ln(\sin x)\mathrm{d}x\xlongequal{x=\frac{\pi}{2}-t}\int_{\frac{\pi}{2}}^0\ln[\sin(\frac{\pi}{2}-t)]\mathrm{d}(\frac{\pi}{2}-t)=\int_0^{\frac{\pi}{2}}\ln(\cos x)\mathrm{d}x$,

$I=\int_0^{\frac{\pi}{2}}\ln(\sin x)\mathrm{d}x\xlongequal{u=\pi-x}\int_{\pi}^{\frac{\pi}{2}}\ln[\sin(\pi-u)]\mathrm{d}(\pi-u)=\int_{\frac{\pi}{2}}^{\pi}\ln(\sin x)\mathrm{d}x$

$=\frac{1}{2}[\int_0^{\frac{\pi}{2}}\ln(\sin x)\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}\ln(\sin x)\mathrm{d}x]=\frac{1}{2}\int_0^{\pi}\ln(\sin x)\mathrm{d}x$,

$2I=\int_0^{\frac{\pi}{2}}\ln(\sin x)\mathrm{d}x+\int_0^{\frac{\pi}{2}}\ln(\cos x)\mathrm{d}x=\int_0^{\frac{\pi}{2}}\ln(\sin x\cos x)\mathrm{d}x$,

$2I+\frac{\pi}{2}\ln 2=\int_0^{\frac{\pi}{2}}\ln(\sin x)\mathrm{d}x+\int_0^{\frac{\pi}{2}}\ln(\cos x)\mathrm{d}x+\int_0^{\frac{\pi}{2}}\ln 2\mathrm{d}x=\int_0^{\frac{\pi}{2}}\ln(\sin 2x)\mathrm{d}x$

$=\frac{1}{2}\int_0^{\pi}\ln(\sin x)\mathrm{d}x=I$,

$\therefore \int_0^{\frac{\pi}{2}}\ln(\sin x)\mathrm{d}x=I=-\frac{\pi}{2}\ln 2$.

(2)$\int_0^{\pi}\ln(1+\cos x)\mathrm{d}x\xlongequal{x=2u}\int_0^{\frac{\pi}{2}}\ln(1+\cos 2u)\mathrm{d}2u=2\int_0^{\frac{\pi}{2}}\ln(1+\cos 2x)\mathrm{d}x$

$=2\int_0^{\frac{\pi}{2}}\ln(2\cos^2 x)\mathrm{d}x=2\int_0^{\frac{\pi}{2}}\ln 2\mathrm{d}x+2\int_0^{\frac{\pi}{2}}\cos x\mathrm{d}x=\pi\ln 2+2I=\pi\ln 2-\pi\ln 2=0$.

18. 设$f(x)$连续，$\varphi(x)=\int_0^1 f(xt)\mathrm{d}t$，且$\lim\limits_{x\to 0}\frac{f(x)}{x}=A$($A$为常数). 求$\varphi'(x)$且讨论$\varphi'(x)$在$x=0$的连续性.

解：$\varphi(x)=\int_0^1 f(xt)\mathrm{d}t\xlongequal{xt=u}\int_0^x f(u)\mathrm{d}\frac{u}{x}=\frac{1}{x}\int_0^x f(u)\mathrm{d}u(x\neq 0)$

$\because f(x)$连续且$\lim\limits_{x\to 0}\frac{f(x)}{x}=A$($A$为常数)

$\therefore f(0)=0$，当$x=0$时，$\varphi(0)=\int_0^1 f(0)\mathrm{d}t=0$

$\therefore \varphi(x)=\begin{cases}\frac{1}{x}\int_0^x f(u)\mathrm{d}u, & x\neq 0\\ 0, & x=0\end{cases}$

$\therefore \varphi'(x)=\begin{cases}-\frac{1}{x^2}\int_0^x f(u)\mathrm{d}u+\frac{1}{x}f(x), & x\neq 0\\ \lim\limits_{x\to 0}\frac{\varphi(x)-\varphi(0)}{x-0}=\lim\limits_{x\to 0}\frac{\frac{1}{x}\int_0^x f(u)\mathrm{d}u-0}{x-0}=\lim\limits_{x\to 0}\frac{\int_0^x f(u)\mathrm{d}u}{x^2}=\lim\limits_{x\to 0}\frac{f(x)}{2x}=\frac{A}{2}, & x=0\end{cases}$

$\because \lim\limits_{x\to 0}-\frac{1}{x^2}\int_0^x f(u)\mathrm{d}u=\lim\limits_{x\to 0}-\frac{f(x)}{2x}=-\frac{A}{2}$

$\therefore \lim\limits_{x\to 0}\varphi'(x)=\lim\limits_{x\to 0}[-\frac{1}{x^2}\int_0^x f(u)\mathrm{d}u+\frac{1}{x}f(x)]=-\frac{A}{2}+A=\frac{A}{2}=\varphi'(0)$

故$\varphi'(x)$在$x=0$连续.

19. 设$f(x)\in C^2[a,b]$，试证：$\exists\xi\in[a,b]$，使得

$$\int_a^b f(x)\mathrm{d}x=(b-a)f(\frac{a+b}{2})+\frac{1}{24}(b-a)^3f''(\xi).$$

证明：$\because f(x)\in C^2[a,b]$

$\therefore f(x)=f(\frac{a+b}{2})+f'(\frac{a+b}{2})(x-\frac{a+b}{2})+\frac{f''(\eta)}{2!}(x-\frac{a+b}{2})^2, \eta$介于$x$和$\frac{a+b}{2}$之间

$\because f''(\eta)$作为关于x的函数在区间$[a,b]$上连续，$g(x)=(x-\frac{a+b}{2})^2$在$[a,b]$上可积且非负

$\therefore \exists\xi\in(a,b), s.t. \int_a^b\frac{f''(\eta)}{2!}(x-\frac{a+b}{2})^2\mathrm{d}x=\frac{f''(\xi)}{2!}\int_a^b(x-\frac{a+b}{2})^2\mathrm{d}x=\frac{f''(\xi)}{2!}\frac{1}{3}(x-\frac{a+b}{2})^3\Big|_a^b$
$=\frac{1}{24}(b-a)^3f''(\xi)$

$\because \int_a^b f'(\frac{a+b}{2})(x-\frac{a+b}{2})\mathrm{d}x=f'(\frac{a+b}{2})\int_a^b(x-\frac{a+b}{2})\mathrm{d}x=f'(\frac{a+b}{2})\frac{1}{2}(x-\frac{a+b}{2})^2\Big|_a^b=0$

$\therefore \int_a^b f(x)\mathrm{d}x=\int_a^b[f(\frac{a+b}{2})+f'(\frac{a+b}{2})(x-\frac{a+b}{2})+\frac{f''(\eta)}{2!}(x-\frac{a+b}{2})^2]\mathrm{d}x$
$=(b-a)f(\frac{a+b}{2})+\frac{1}{24}(b-a)^3f''(\xi).$