10 定积分(1)

10.1 知识结构

第7章定积分

7.1 积分概念和积分存在的条件

- 7.1.1 曲边梯形的面积
- 7.1.2 黎曼积分的定义
- 7.1.3 积分存在的条件

7.2 定积分的性质

- (a) 方向性
- (b) 线性性质
- (c) 区间可加性
- (d) 特殊函数定积分的性质（奇偶性、周期性）
- (e) 比较定理（保号性）
  - 估值定理
- (f) 绝对值函数的可积性
- (g) 积分中值定理
  - 第一积分中值定理

7.3 变上限积分与牛顿-莱布尼茨公式

- 7.3.1 变上限积分
- 7.3.2 牛顿-莱布尼茨公式

7.4 定积分的换元积分法和分部积分法

- 7.4.1 定积分的换元积分法
- 7.4.2 定积分的分部积分法

10.2 习题7.1解答

1. 用积分的几何意义计算下列定积分：
   (1)$\int_1^3(1+2x)\mathrm{d}x$；
   (2)$\int_{-3}^0\sqrt{9-x^2}\mathrm{d}x$.

   解：(1)积分$\int_1^3(1+2x)\mathrm{d}x$表示直线$y=1+2x, x=1, x=3$和$x$轴围成的梯形面积，故$\int_1^3(1+2x)\mathrm{d}x=(3+7)\times 2/2=10$.

   (2)积分$\int_{-3}^0\sqrt{9-x^2}\mathrm{d}x$表示圆$x^2+y^2=9$在第二象限部分的面积，则$\int_{-3}^0\sqrt{9-x^2}\mathrm{d}x=\frac{1}{4}\pi\times 9=\frac{9}{4}\pi$.

2. 利用定理7.1.1证明狄利克雷函数在区间$[0,1]$上不可积.

   证明：对于狄利克雷函数$D(x)=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x\notin\mathbb{Q}\end{cases}$在区间$[0,1]$上的任意一个分割$T: a=x_0<x_1<\cdots<x_n=b$，其中每一个区间$[x_{i-1},x_i]$上的振幅$\omega_i$均为1，则$\sum_{i=1}^n\omega_i\Delta x_i=\sum_{i=1}^n\Delta x_i=1$，故$\lim\limits_{\lambda\to 0}\sum_{i=1}^n\omega_i\Delta x_i=1\neq 0$，故狄利克雷函数在区间$[0,1]$上不可积.

3. 利用定理7.1.1证明：若$f\in R[a,b]$，则$|f|\in R[a,b], f^2\in R[a,b]$.

   证明：$\because f\in R[a,b]$

   $\therefore$对于区间$[a,b]$上的任意一个分割$T: a=x_0<x_1<\cdots<x_n=b$，有$\lim\limits_{\lambda\to 0}\sum_{i=1}^n\omega_i\Delta x_i=0$

   对于同一个分割T，记$|f|$的振幅为ω_i^*，易知$\omega_i^*\leq\omega_i$

   $\therefore 0\leq\sum_{i=1}^n\omega_i^*\Delta x_i\leq\sum_{i=1}^n\omega_i\Delta x_i$

   $\therefore \lim\limits_{\lambda\to 0}\sum_{i=1}^n\omega_i^*\Delta x_i=0$

   $\therefore |f|\in R[a,b]$

   对于上述分割T，记f^2的振幅为ω_i^{**}，对于区间$[x_{i-1},x_i]$，设$|f|$在其中的最大值和最小值分别为M_i, m_i，则$\omega_i^{**}=M_i^2-m_i^2=(M_i-m_i)(M_i+m_i)=(M_i+m_i)\omega_i^{**}$

   由$f\in R[a,b]$知f有界，从而$|f|$有界，故$\exists A>0, s.t. M_i+m_i<A$

   $\therefore \omega_i^{**}\leq A\omega_i^{**}$

   $\therefore 0\leq\sum_{i=1}^n\omega_i^{**}\Delta x_i\leq\sum_{i=1}^n A\omega_i^*\Delta x_i=A\sum_{i=1}^n\omega_i^*\Delta x_i$

   $\therefore \lim\limits_{\lambda\to 0}\sum_{i=1}^n\omega_i^{**}\Delta x_i=0$

   $\therefore f^2\in R[a,b]$.

4. 举例说明：由$|f|\in R[a,b]$一般不能推出$f\in R[a,b]$.

解：比如函数$f(x)=\begin{cases}1, & x\in\mathbb{Q}\\ -1, & x\notin\mathbb{Q}\end{cases}$，$|f(x)|=1, x\in\mathbb{R}$，$|f|\in C[0,1]$故$|f|\in R[0,1]$，但对区间$[0,1]$上的任意一个分割$T: a=x_0<x_1<\cdots<x_n=b$，$f$在其中每一个区间$[x_{i-1},x_i]$上的振幅$\omega_i$均为2，则$\sum_{i=1}^n\omega_i\Delta x_i=\sum_{i=1}^n\Delta x_i=2$，故$\lim\limits_{\lambda\to 0}\sum_{i=1}^n\omega_i\Delta x_i=2\neq 0$，故$f\notin R[a,b]$.

10.3 习题7.2解答

1. 比较下列每组中两个积分的大小：

   (1)$\int_0^1 \mathrm{e}^x\mathrm{d}x, \int_0^1 \mathrm{e}^{x^2}\mathrm{d}x$.

   (2)$\int_0^{\frac{\pi}{2}}\sin x\mathrm{d}x, \int_0^1\sin(\sin x)\mathrm{d}x$.

   解：(1)当$x\in(0,1)$时，$x-x^2=x(1-x)>0$，故$\mathrm{e}^x-\mathrm{e}^{x^2}=\mathrm{e}^{x^2}(\mathrm{e}^{x-x^2}-1)>0$

   $\therefore \int_0^1\mathrm{e}^x\mathrm{d}x>\int_0^1\mathrm{e}^{x^2}\mathrm{d}x$

   (2)当$x\in(0,\frac{\pi}{2})$时，$0<\sin x<x<\frac{\pi}{2}$，故$\sin x>\sin(\sin x)$

   $\therefore \int_0^{\frac{\pi}{2}}\sin x\mathrm{d}x>\int_0^{\frac{\pi}{2}}\sin(\sin x)\mathrm{d}x>\int_0^1\sin(\sin x)\mathrm{d}x$.

2. 证明下列不等式

   (1)$\frac{2}{\sqrt[4]{\mathrm{e}}}<\int_0^2\mathrm{e}^{x^2-x}\mathrm{d}x<2\mathrm{e}^2$；

   (2)$\int_0^{2\pi}|a\sin x+b\cos x|\mathrm{d}x\leq 2\pi\sqrt{a^2+b^2}$.

   证明：(1)当$x\in[0,2]$时，$-\frac{1}{4}\leq x^2-x\leq 2$，故$-\frac{1}{\sqrt[4]{\mathrm{e}}}\leq\mathrm{e}^{x^2-x}\leq\mathrm{e}^2$

   $\therefore \int_0^2\frac{1}{\sqrt[4]{\mathrm{e}}}\mathrm{d}x<\int_0^2\mathrm{e}^{x^2-x}\mathrm{d}x<\int_0^2\mathrm{e}^2\mathrm{d}x$

   即$\frac{2}{\sqrt[4]{\mathrm{e}}}<\int_0^2\mathrm{e}^{x^2-x}\mathrm{d}x<2\mathrm{e}^2$.

   (2)$|a\sin x+b\cos x|=\sqrt{a^2+b^2}|\frac{a}{\sqrt{a^2+b^2}}\sin x+\frac{b}{\sqrt{a^2+b^2}}\cos x|=\sqrt{a^2+b^2}\sin(x+\phi)\leq\sqrt{a^2+b^2}$

   $\therefore \int_0^{2\pi}|a\sin x+b\cos x|\mathrm{d}x\leq\int_0^{2\pi}\pi\sqrt{a^2+b^2}\mathrm{d}x=2\pi\sqrt{a^2+b^2}$.

3. 证明下列等式

   (1)$\lim\limits_{A\to+\infty}\int_A^{A+1}\frac{\cos x}{x}\mathrm{d}x=0$.

   (2)$\lim\limits_{p\to+\infty}\int_0^{\frac{\pi}{2}}\sin^p x\mathrm{d}x=0$.

   证明：(1)$0\leq|\int_A^{A+1}\frac{\cos x}{x}|\mathrm{d}x\leq\int_A^{A+1}|\frac{\cos x}{x}|\mathrm{d}x\leq\int_A^{A+1}\frac{1}{x}\mathrm{d}x=\frac{1}{\xi}(A+1-A)=\frac{1}{\xi}, A<\xi<A+1$

   $\because \lim\limits_{A\to+\infty}\frac{1}{\xi}=\lim\limits_{\xi\to+\infty}\frac{1}{\xi}=0$

   $\therefore \lim\limits_{A\to+\infty}|\int_A^{A+1}\frac{\cos x}{x}|=0$

$\therefore \lim\limits_{A\to+\infty}\int_A^{A+1}\frac{\cos x}{x}=0.$

(2)$\forall\varepsilon>0$(不妨设$\varepsilon<\frac{\pi}{2}$), $\int_0^{\frac{\pi}{2}}\sin^p x\mathrm{d}x=\int_0^{\frac{\pi}{2}-\varepsilon}\sin^p x\mathrm{d}x+\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}}\sin^p x\mathrm{d}x$

$\because \int_0^{\frac{\pi}{2}-\varepsilon}\sin^p x\mathrm{d}x\le\int_0^{\frac{\pi}{2}-\varepsilon}\sin^p(\frac{\pi}{2}-\varepsilon)\mathrm{d}x=(\frac{\pi}{2}-\varepsilon)\sin^p(\frac{\pi}{2}-\varepsilon)$

$\because 0<\sin x<1$

$\therefore$对于该$\varepsilon,\exists P>0,s.t.\int_0^{\frac{\pi}{2}-\varepsilon}\sin^p x\mathrm{d}x\le(\frac{\pi}{2}-\varepsilon)\sin^p(\frac{\pi}{2}-\varepsilon)<\varepsilon(p>P)$

$\because \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}}\sin^p x\mathrm{d}x\le\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}}1\mathrm{d}x=\varepsilon$

$\therefore |\int_0^{\frac{\pi}{2}}\sin^p x\mathrm{d}x-0|=\int_0^{\frac{\pi}{2}}\sin^p x\mathrm{d}x=\int_0^{\frac{\pi}{2}-\varepsilon}\sin^p x\mathrm{d}x+\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}}\sin^p x\mathrm{d}x<2\varepsilon$

$\therefore \lim\limits_{p\to+\infty}\int_0^{\frac{\pi}{2}}\sin^p x\mathrm{d}x=0.$

4. 证明下列不等式：

(1)$\int_1^n\ln x\mathrm{d}x<\ln n!$；

(2)$f\in C[0,1],f(0)=0,f(1)=1,f''(x)>0$则$\int_0^1 f(x)\mathrm{d}x<\frac{1}{2}$.

证明：(1)$\int_1^n\ln x\mathrm{d}x=\int_1^2\ln x\mathrm{d}x+\int_2^3\ln x\mathrm{d}x+\cdots+\int_{n-1}^n\ln x\mathrm{d}x<\int_1^2\ln 1\mathrm{d}x+\int_2^3\ln 2\mathrm{d}x+\cdots+\int_{n-1}^n\ln n\mathrm{d}x=\ln 2+\ln 3+\cdots+\ln n=\ln n!$.

(2)$\because f''(x)>0,x\in[0,1]$

$\therefore f(x)$在区间$[0,1]$上严格下凸

又$\because f(0)=0,f(1)=1$

$\therefore f(x)$在直线$y=x$的下方，即$f(x)<x,x\in(0,1)$

$\therefore \int_0^1 f(x)\mathrm{d}x<\int_0^1 x\mathrm{d}x=\frac{1}{2}$.

10.4 习题7.3解答

1. 求下列变限积分的导数：

(1)$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\sqrt{1+t}\mathrm{d}t$；

(2)$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x^2}\frac{\mathrm{d}t}{\sqrt{1+t}}$；

(3)$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\sin x\cos t^2\mathrm{d}t$；

(4)$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2}\sqrt{1+t}\mathrm{d}t$.

解：(1)$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\sqrt{1+t}\mathrm{d}t=\sqrt{1+x}$.

(2)$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x^2}\frac{\mathrm{d}t}{\sqrt{1+t}}=\frac{\mathrm{d}}{\mathrm{d}x}(\int_x^0\frac{\mathrm{d}t}{\sqrt{1+t}}+\int_0^{x^2}\frac{\mathrm{d}t}{\sqrt{1+t}})=-\frac{1}{\sqrt{1+x}}+\frac{2x}{\sqrt{1+x^2}}$.

(3)$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\sin x\cos t^2\mathrm{d}t=\cos x\int_0^x\cos t^2\mathrm{d}t+\sin x\cos x^2$.

(4)$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2}\sqrt{1+t}\mathrm{d}t=\sqrt{1+x^2}\cdot 2x=2x\sqrt{1+x^2}$.

2. 求下列极限：

(1)$\lim\limits_{x\to 0}\frac{\int_0^x \cos t^2\mathrm{d}t}{\ln(1+x)}$；

(2)$\lim\limits_{x\to 0}\frac{(\int_0^x \sin t\mathrm{d}t)^2}{\int_0^x \sin t^2\mathrm{d}t}$.

解：(1)$\lim\limits_{x\to 0}\frac{\int_0^x \cos t^2\mathrm{d}t}{\ln(1+x)}=\lim\limits_{x\to 0}\frac{\int_0^x \cos t^2\mathrm{d}t}{x}=\lim\limits_{x\to 0}\frac{\cos x^2}{1}=1$.

(2)$\lim\limits_{x\to 0}\frac{(\int_0^x \sin t\mathrm{d}t)^2}{\int_0^x \sin t^2\mathrm{d}t}=\lim\limits_{x\to 0}\frac{2(\int_0^x \sin t\mathrm{d}t)\sin x}{\sin x^2}=\lim\limits_{x\to 0}\frac{2(\int_0^x \sin t\mathrm{d}t)x}{x^2}=\lim\limits_{x\to 0}\frac{2\int_0^x \sin t\mathrm{d}t}{x}=\lim\limits_{x\to 0}\frac{2\sin t}{1}=0$.

3. 用牛顿-莱布尼茨公式计算下列积分(m,k是整数)：

(1)$\int_0^1 x(1-2x^2)^8\mathrm{d}x$；　(2)$\int_0^\pi(a\cos x+b\sin x)\mathrm{d}x$；

(3)$\int_{\mathrm{e}}^{\mathrm{e}^2}\frac{\mathrm{d}x}{x\ln x}$；　(4)$\int_{-1}^0(x+1)\sqrt{1-x-\frac{1}{2}x^2}\mathrm{d}x$；

(5)$\int_{-\pi}^{\pi}\sin mx\sin kx\mathrm{d}x$；　(6)$\int_{-\pi}^{\pi}\cos mx\cos kx\mathrm{d}x$；

(7)$\int_{-\pi}^{\pi}\sin mx\cos kx\mathrm{d}x$；　(8)$\int_{-\pi}^{\pi}\sqrt{1-\cos^2 x}\mathrm{d}x$.

解：(1)$\int x(1-2x^2)^8\mathrm{d}x=-\frac{1}{4}\int(1-2x^2)^8\mathrm{d}(1-2x^2)=-\frac{1}{4}\frac{1}{9}(1-2x^2)^9+C=-\frac{1}{36}(1-2x^2)^9$

$\int_0^1 x(1-2x^2)^8\mathrm{d}x=[-\frac{1}{36}(1-2x^2)^9]|_{x=1}-[-\frac{1}{36}(1-2x^2)^9]|_{x=0}=\frac{1}{18}$.

(2)$\int_0^\pi(a\cos x+b\sin x)\mathrm{d}x=(a\sin x-b\cos x)|_{x=\pi}-(a\sin x-b\cos x)|_{x=0}=2b$.

(3)$\int_{\mathrm{e}}^{\mathrm{e}^2}\frac{\mathrm{d}x}{x\ln x}=\ln\ln x|_{x=\mathrm{e}^2}-\ln\ln x|_{x=\mathrm{e}}=\ln 2$.

(4)$\int(x+1)\sqrt{1-x-\frac{1}{2}x^2}\mathrm{d}x=-\int\sqrt{\frac{3}{2}-\frac{1}{2}(x+1)^2}\mathrm{d}[\frac{3}{2}-\frac{1}{2}(1+x)^2]$

$=-\frac{2}{3}[\frac{3}{2}-\frac{1}{2}(1+x)^2]^{\frac{3}{2}}+C$

$\int_{-1}^0(x+1)\sqrt{1-x-\frac{1}{2}x^2}\mathrm{d}x=-\frac{2}{3}[\frac{3}{2}-\frac{1}{2}(1+x)^2]^{\frac{3}{2}}|_{x=0}+\frac{2}{3}[\frac{3}{2}-\frac{1}{2}(1+x)^2]^{\frac{3}{2}}|_{x=-1}=-\frac{2}{3}[\frac{3}{2}-\frac{1}{2}(1+0)^2]^{\frac{3}{2}}+\frac{2}{3}[\frac{3}{2}-\frac{1}{2}(1-1)^2]^{\frac{3}{2}}=-\frac{2}{3}+\frac{2}{3}(\frac{3}{2})^{\frac{3}{2}}=\sqrt{\frac{3}{2}}-\frac{2}{3}$.

(5)$\int\sin mx\sin kx\mathrm{d}x=\frac{1}{2}\int[\cos(m-k)x-\cos(m+k)x]\mathrm{d}x$

$$=\begin{cases}\frac{1}{2}(x-\frac{1}{2m}\sin 2mx)+C, & m=k,\\ \frac{1}{2(m-k)}\sin(m-k)x-\frac{1}{2(m+k)}\sin(m+k)x+C, & m\neq k,\end{cases}\quad m,n\in\mathbb{Z}^+$$

$$\int_{-\pi}^{\pi}\sin mx\sin kx\mathrm{d}x=\begin{cases}\pi, & m=k\neq 0,\\ 0, & m\neq k,\end{cases}\quad m,n\in\mathbb{Z}^+.$$

(6)$\int\cos mx\cos kx\mathrm{d}x=\frac{1}{2}\int[\cos(m-k)x+\cos(m+k)x]\mathrm{d}x$

$$=\begin{cases}\frac{1}{2}(x+\frac{1}{2m}\sin 2mx)+C, & m=k,\\ \frac{1}{2}[\frac{1}{m-k}\sin(m-k)x+\frac{1}{m+k}\sin(m+k)x]+C, & m\neq k,\end{cases}\quad m,n\in\mathbb{Z}^+$$

$$\int_{-\pi}^{\pi}\cos mx\cos kx\mathrm{d}x=\begin{cases}\pi, & m=k\neq 0,\\ 0, & m\neq k,\end{cases}\quad m,n\in\mathbb{Z}^+.$$

(7)$\int\sin mx\cos kx\mathrm{d}x=\frac{1}{2}\int[\sin(m+k)x+\sin(m-k)x]\mathrm{d}x$

$=\begin{cases}-\frac{1}{2}\frac{1}{2m}\cos 2mx+C, & m=k,\\ -\frac{1}{2}[\frac{1}{m+k}\cos(m+k)x+\frac{1}{m-k}\cos(m-k)x]+C, & m\neq k,\end{cases} m,n\in\mathbb{Z}^+$

$\int_{-\pi}^{\pi}\sin mx\cos kx\mathrm{d}x=0.$

(8)$\int_{-\pi}^{\pi}\sqrt{1-\cos^2x}\mathrm{d}x=\int_{-\pi}^{\pi}|\sin x|\mathrm{d}x=-\int_{-\pi}^{0}\sin x\mathrm{d}x+\int_0^{\pi}\sin x\mathrm{d}x$

$=\cos x|_{x=0}-\cos x|_{x=-\pi}+(-\cos x)|_{x=\pi}-(-\cos x)|_{x=0}=4.$

4. 计算$\int_{-1}^2\max\{x,x^2\}\mathrm{d}x$.

解：$\int_{-1}^2\max\{x,x^2\}\mathrm{d}x=\int_{-1}^0x^2\mathrm{d}x+\int_0^1x\mathrm{d}x+\int_1^2x^2\mathrm{d}x$

$=\frac{1}{3}x^3|_{x=0}-\frac{1}{3}x^3|_{x=-1}+\frac{1}{2}x^2|_{x=1}-\frac{1}{2}x^2|_{x=0}+\frac{1}{3}x^3|_{x=2}-\frac{1}{3}x^3|_{x=1}=\frac{1}{3}+\frac{1}{2}+\frac{8}{3}-\frac{1}{3}=\frac{19}{6}.$

5. 用定积分求下列极限：

(1) $\lim\limits_{n\to\infty}\frac{1^p+2^p+\cdots+n^p}{n^{p+1}},p>0$；

(2) $\lim\limits_{n\to\infty}(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n})$；

(3) $\lim\limits_{n\to\infty}\frac{1}{n}(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n})$；

(4) $\lim\limits_{n\to\infty}\frac{\sqrt[n]{(2n)!}}{n\sqrt[n]{n!}}$.

解：(1) $\lim\limits_{n\to\infty}\frac{1^p+2^p+\cdots+n^p}{n^{p+1}}=\lim\limits_{n\to\infty}\frac{1}{n}\sum_{k=1}^n(\frac{k}{n})^p=\lim\limits_{n\to\infty}\sum_{k=1}^n(\frac{k}{n})^p\cdot\frac{1}{n}=\int_0^1x^p\mathrm{d}x=\frac{1}{p+1}.$

(2) $\lim\limits_{n\to\infty}(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n})=\lim\limits_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\frac{1}{1+\frac{k}{n}}=\lim\limits_{n\to\infty}\sum_{k=1}^n\frac{1}{1+\frac{k}{n}}\cdot\frac{1}{n}=\int_1^2\frac{1}{x}\mathrm{d}x=\ln 2.$

(3) $\lim\limits_{n\to\infty}\frac{1}{n}(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n})=\lim\limits_{n\to\infty}\frac{1}{n}(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n}+\sin\frac{n\pi}{n})$

$=\lim\limits_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\sin\frac{k\pi}{n}=\lim\limits_{n\to\infty}\sum_{k=1}^n\sin\frac{k\pi}{n}\cdot\frac{1}{n}=\int_0^{\pi}\sin\pi x\mathrm{d}x=\frac{2}{\pi}.$

(4) $\lim\limits_{n\to\infty}\frac{\sqrt[n]{(2n)!}}{n\sqrt[n]{n!}}=\lim\limits_{n\to\infty}\sqrt[n]{\frac{2n\cdot(2n-1)\cdot(2n-2)\cdot\cdots\cdot(n+1)}{n^n}}$

$=\lim\limits_{n\to\infty}\sqrt[n]{(1+\frac{n}{n})\cdot(1+\frac{n-1}{n})\cdot(1+\frac{n-2}{n})\cdot\cdots\cdot(1+\frac{1}{n})}=\lim\limits_{n\to\infty}\mathrm{e}^{\frac{1}{n}\sum_{k=1}^n\ln(1+\frac{k}{n})}$

$=\lim\limits_{n\to\infty}\mathrm{e}^{\sum_{k=1}^n\ln(1+\frac{k}{n})\cdot\frac{1}{n}}=\lim\limits_{n\to\infty}\mathrm{e}^{\int_1^2\ln x\mathrm{d}x}=\mathrm{e}^{(x\ln x-x)|_{x=2}-(x\ln x-x)|_{x=1}}=\mathrm{e}^{2\ln 2-1}=\frac{4}{\mathrm{e}}.$

6. 假设$f(x)$连续、单调增加. 求证：$\int_{-\pi}^{\pi}f(x)\sin x\mathrm{d}x>0.$

证明：$\int_{-\pi}^{\pi}f(x)\sin x\mathrm{d}x=\int_{-\pi}^0f(x)\sin x\mathrm{d}x+\int_0^{\pi}f(x)\sin x\mathrm{d}x$

$=\int_0^{\pi}f(t-\pi)\sin(t-\pi)\mathrm{d}t+\int_0^{\pi}f(x)\sin x\mathrm{d}x$

$=-\int_0^{\pi}f(t-\pi)\sin t\mathrm{d}t+\int_0^{\pi}f(x)\sin x\mathrm{d}x$

$=\int_0^{\pi}[f(x)-f(x-\pi)]\sin x\mathrm{d}x$

当$0<x<\pi$时，$x-\pi<x,\sin x>0$，故$f(x-\pi)-f(x)<0$，$[f(x)-f(x-\pi)]\sin x>0$

$\therefore\int_{-\pi}^{\pi}f(x)\sin x\mathrm{d}x=\int_0^{\pi}[f(x)-f(x-\pi)]\sin x\mathrm{d}x>0.$

10.5 习题7.4解答

1. 求下列定积分：

(1)$\int_0^3 \frac{x}{1+\sqrt{1+x}}\mathrm{d}x$； (2)$\int_1^{\mathrm{e}} \frac{1+\ln x}{x}\mathrm{d}x$；

(3)$\int_{\frac{1}{\pi}}^{\frac{2}{\pi}} \frac{\sin\frac{1}{x}}{x^2}\mathrm{d}x$； (4)$\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{\arctan\sqrt{x}}{\sqrt{x(1-x)}}\mathrm{d}x$；

(5)$\int_1^2 \frac{\sqrt{4-x^2}}{x^2}\mathrm{d}x$； (6)$\int_0^2 \sqrt{(4-x^2)^3}\mathrm{d}x$；

(7)$\int_0^4 \frac{\sqrt{x}}{1+x\sqrt{x}}\mathrm{d}x$； (8)$\int_0^{\frac{\pi}{4}} \frac{\mathrm{d}x}{1+\cos^2 x}$；

(9)$\int_0^{\ln 2} \sqrt{\mathrm{e}^x-1}\mathrm{d}x$； (10)$\int_{\sqrt{2}}^2 \frac{\mathrm{d}x}{x\sqrt{x^2-1}}$；

(11)$\int_0^1 \ln(1+x^2)\mathrm{d}x$； (12)$\int_0^{\mathrm{e}} x(\ln x)^2\mathrm{d}x$；

(13)$\int_0^4 \cos(\sqrt{x}-1)\mathrm{d}x$； (14)$\int_0^1 x\arctan x\mathrm{d}x$；

(15)$\int_0^1 \arcsin x\mathrm{d}x$； (16)$\int_0^1 \mathrm{e}^{\sqrt{x}}\mathrm{d}x$；

(17)$\int_0^{\sqrt{\ln 2}} x^3\mathrm{e}^{-x^2}\mathrm{d}x$； (18)$\int_0^1 \frac{x\mathrm{e}^x}{(1+x)^2}\mathrm{d}x$；

(19)$\int_1^{\mathrm{e}} \cos(\ln x)\mathrm{d}x$； (20)$\int_1^2 \frac{x\mathrm{e}^x}{(\mathrm{e}^x-1)^2}\mathrm{d}x$.

解：(1)$\int_0^3 \frac{x}{1+\sqrt{1+x}}\mathrm{d}x \xlongequal{t=\sqrt{1+x}} \int_1^2 \frac{t^2-1}{1+t}2t\mathrm{d}t = 2\int_1^2(t^2-t)\mathrm{d}t = 2(\frac{1}{3}t^3-\frac{1}{2}t^2)\Big|_1^2 = \frac{5}{3}$.

(2)$\int_1^{\mathrm{e}} \frac{1+\ln x}{x}\mathrm{d}x = \int_1^{\mathrm{e}}(1+\ln x)\mathrm{d}\ln x = [\ln x+\frac{1}{2}(\ln x)^2]_1^{\mathrm{e}} = \frac{3}{2}$.

(3)$\int_{\frac{1}{\pi}}^{\frac{2}{\pi}} \frac{\sin\frac{1}{x}}{x^2}\mathrm{d}x = -\int_{\frac{1}{\pi}}^{\frac{2}{\pi}} \sin\frac{1}{x}\mathrm{d}(\frac{1}{x}) = \cos\frac{1}{x}\Big|_{\frac{1}{\pi}}^{\frac{2}{\pi}} = 1$.

(4)$\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}\mathrm{d}x \xlongequal{t=\arcsin\sqrt{x}} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{t}{\sqrt{\sin^2 t(1-\sin^2 t)}}2\sin t\cos t\mathrm{d}t = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2t\mathrm{d}t = t^2\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{7\pi^2}{144}$.

(5)$\int_1^2 \frac{\sqrt{4-x^2}}{x^2}\mathrm{d}x \xlongequal{x=2\sin t} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{2\cos t}{4\sin^2 t}2\cos t\mathrm{d}t = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(\csc^2 t-1)\mathrm{d}t = -\cot t-t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \sqrt{3}-\frac{\pi}{3} = 3\pi$.

(6)$\int_0^2 \sqrt{(4-x^2)^3}\mathrm{d}x \xlongequal{x=2\sin t} \int_0^{\frac{\pi}{2}} \sqrt{(4-4\sin^2 t)^3}2\cos t\mathrm{d}t = 16\int_0^{\frac{\pi}{2}} \cos^4 t\mathrm{d}t = 16(\frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2})$
$= 3\pi$.

(7)$\int_0^4 \frac{\sqrt{x}}{1+x\sqrt{x}}\mathrm{d}x \xlongequal{t=\sqrt{x}} \int_0^2 \frac{t}{1+t^3}2t\mathrm{d}t = 2\int_0^2 \frac{t^2}{1+t^3}\mathrm{d}t = \frac{2}{3}\int_0^2 \frac{\mathrm{d}t^3}{1+t^3} = \frac{2}{3}\ln|1+t^3|\Big|_0^2 = \frac{4}{3}\ln 3$.

(8)$\int_0^{\frac{\pi}{4}} \frac{\mathrm{d}x}{1+\cos^2 x} = \int_0^{\frac{\pi}{4}} \frac{\mathrm{d}x}{\sin^2 x+2\cos^2 x} = \int_0^{\frac{\pi}{4}} \frac{\sec^2 x\mathrm{d}x}{\tan^2 x+2} = \int_0^{\frac{\pi}{4}} \frac{\mathrm{d}\tan x}{\tan^2 x+2} = \frac{1}{\sqrt{2}}\arctan(\frac{1}{\sqrt{2}}\tan x)\Big|_0^{\frac{\pi}{4}}$
$= \frac{\sqrt{2}}{2}\arctan\frac{\sqrt{2}}{2}$.

(9)$\int_0^{\ln 2} \sqrt{\mathrm{e}^x-1}\mathrm{d}x \xlongequal{t=\sqrt{\mathrm{e}^x-1}} \int_0^1 t\frac{2t}{t^2+1}\mathrm{d}t = 2\int_0^1(1-\frac{1}{t^2+1})\mathrm{d}t = 2(t-\arctan t)\Big|_0^1 = 2-\frac{\pi}{2}$.

(10)$\int_{\sqrt{2}}^2 \frac{\mathrm{d}x}{x\sqrt{x^2-1}} \xlongequal{x=\sec t} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan t\sec t\mathrm{d}t}{\sec t\tan t} = t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\pi}{12}$.

(11)$\int_0^1 \ln(1+x^2)\mathrm{d}x = x\ln(1+x^2)\Big|_0^1 - \int_0^1 x\mathrm{d}\ln(1+x^2) = \ln 2-\int_0^1 \frac{2x^2}{1+x^2}\mathrm{d}x$
$= \ln 2-2\int_0^1(1-\frac{1}{1+x^2})\mathrm{d}x = \ln 2-2(x-\arctan x)\Big|_0^1 = \ln 2-2(1-\frac{\pi}{4}) = \ln 2-2+\frac{\pi}{2}$.

(12)$\int_1^{\mathrm{e}} x(\ln x)^2\mathrm{d}x = \frac{1}{2}\int_1^{\mathrm{e}}(\ln x)^2\mathrm{d}(x^2) = \frac{1}{2}[x^2(\ln x)^2\Big|_1^{\mathrm{e}} - \int_1^{\mathrm{e}} x^2\mathrm{d}(\ln x)^2] = \frac{1}{2}[\mathrm{e}^2-\int_1^{\mathrm{e}} x2\ln x\mathrm{d}x]$
$= \frac{1}{2}\mathrm{e}^2-\frac{1}{2}\int_1^{\mathrm{e}}\ln x\mathrm{d}x^2 = \frac{1}{2}\mathrm{e}^2-\frac{1}{2}(x^2\ln x\Big|_1^{\mathrm{e}} - \int_1^{\mathrm{e}} x^2\frac{1}{x}\mathrm{d}x) = \frac{1}{2}\int_1^{\mathrm{e}} x\mathrm{d}x = \frac{1}{4}x^2\Big|_1^{\mathrm{e}} = \frac{1}{4}(\mathrm{e}^2-1)$.

(13)$\int_0^4\cos(\sqrt{x}-1)\mathrm{d}x \xlongequal{t=\sqrt{x}-1} \int_{-1}^1(\cos t)2(t+1)\mathrm{d}t = 2\int_{-1}^1(t+1)\mathrm{d}\sin t$
$=2[(t+1)\sin t\Big|_{-1}^1-\int_{-1}^1\sin t\mathrm{d}t]=4\sin 1+\cos t\Big|_{-1}^1=4\sin 1.$

(14)$\int_0^1 x\arctan x\mathrm{d}x=\frac{1}{2}\int_0^1\arctan x\mathrm{d}x^2=\frac{1}{2}x^2\arctan x\Big|_0^1-\frac{1}{2}\int_0^1 x^2\frac{1}{1+x^2}\mathrm{d}x$
$=\frac{\pi}{8}-\frac{1}{2}\int_0^1(1-\frac{1}{1+x^2})\mathrm{d}x=\frac{\pi}{8}-\frac{1}{2}(x-\arctan x)\Big|_0^1=\frac{\pi}{4}-\frac{1}{2}.$

(15)$\int_0^1\arcsin x\mathrm{d}x=x\arcsin x\Big|_0^1-\int_0^1 x\mathrm{d}\arcsin x=\frac{\pi}{2}-\int_0^1 x\frac{1}{\sqrt{1-x^2}}\mathrm{d}x=\frac{\pi}{2}+\frac{1}{2}\int_0^1\frac{\mathrm{d}(1-x^2)}{\sqrt{1-x^2}}$
$=\frac{\pi}{2}+\frac{1}{2}2\sqrt{1-x^2}\Big|_0^1=\frac{\pi}{2}-1.$

(16)$\int_0^1\mathrm{e}^{\sqrt{x}}\mathrm{d}x \xlongequal{t=\sqrt{x}} \int_0^1\mathrm{e}^t 2t\mathrm{d}t=2\int_0^1 t\mathrm{d}\mathrm{e}^t=2(t\mathrm{e}^t\Big|_0^1-\int_0^1\mathrm{e}^t\mathrm{d}t)=2\mathrm{e}-2\mathrm{e}^t\Big|_0^1=2.$

(17)$\int_0^{\sqrt{\ln 2}}x^3\mathrm{e}^{-x^2}\mathrm{d}x=\frac{1}{2}\int_0^{\sqrt{\ln 2}}x^2\mathrm{e}^{-x^2}\mathrm{d}x^2 \xlongequal{t=x^2} \frac{1}{2}\int_0^{\ln 2}t\mathrm{e}^{-t}\mathrm{d}t=\frac{1}{2}(-t\mathrm{e}^{-t}\Big|_0^{\ln 2}+\int_0^{\ln 2}\mathrm{e}^{-t}\mathrm{d}t)$
$=-\frac{1}{4}\ln 2+\frac{1}{2}(-\mathrm{e}^{-t})\Big|_0^{\ln 2}=-\frac{1}{4}\ln 2+\frac{1}{4}.$

(18)$\int_0^1\frac{x\mathrm{e}^x}{(1+x)^2}\mathrm{d}x=\int_0^1 x\mathrm{e}^x\mathrm{d}(-\frac{1}{1+x})=-\frac{x\mathrm{e}^x}{1+x}\Big|_0^1+\int_0^1\frac{1}{1+x}(\mathrm{e}^x x+\mathrm{e}^x)\mathrm{d}x=-\frac{\mathrm{e}}{2}+\mathrm{e}^x\Big|_0^1=\frac{\mathrm{e}}{2}-1.$

(19)$\int_1^{\mathrm{e}}\cos(\ln x)\mathrm{d}x=x\cos(\ln x)\Big|_1^{\mathrm{e}}+\int_1^{\mathrm{e}}\sin(\ln x)\mathrm{d}x=\mathrm{e}\cos 1-1+[x\sin(\ln x)\Big|_1^{\mathrm{e}}-\int_1^{\mathrm{e}}\cos(\ln x)\mathrm{d}x]$
$=\mathrm{e}\cos 1-1+\mathrm{e}\sin 1-\int_1^{\mathrm{e}}\cos(\ln x)\mathrm{d}x=\frac{1}{2}(\mathrm{e}\cos 1+\mathrm{e}\sin 1-1).$

(20)$\int_1^2\frac{x\mathrm{e}^x}{(\mathrm{e}^x-1)^2}\mathrm{d}x=\int_1^2\frac{x}{(\mathrm{e}^x-1)^2}\mathrm{d}(\mathrm{e}^x-1)=\int_1^2 x\mathrm{d}(-\frac{1}{\mathrm{e}^x-1})=-\frac{x}{\mathrm{e}^x-1}\Big|_1^2+\int_1^2\frac{1}{\mathrm{e}^x-1}\mathrm{d}x$
$=-\frac{2}{\mathrm{e}^2-1}+\frac{1}{\mathrm{e}-1}+\int_1^2(-1+\frac{\mathrm{e}^x}{\mathrm{e}^x-1})\mathrm{d}x=-\frac{2}{\mathrm{e}^2-1}+\frac{1}{\mathrm{e}-1}+[-x+\ln(\mathrm{e}^x-1)]\Big|_1^2$
$=\frac{1}{\mathrm{e}+1}-1+\ln(\mathrm{e}^2-1)-\ln(\mathrm{e}-1)=\frac{1}{\mathrm{e}+1}-1+\ln(\mathrm{e}+1).$

2. 设$f(x)$在$[0,1]$上连续，证明：
(1)$\int_0^{\frac{\pi}{2}}f(\sin x)\mathrm{d}x=\int_0^{\frac{\pi}{2}}f(\cos x)\mathrm{d}x$；
(2)$\int_0^{\pi}xf(\sin x)\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi}f(\sin x)\mathrm{d}x.$

证明：(1)$\int_0^{\frac{\pi}{2}}f(\sin x)\mathrm{d}x \xlongequal{x=\frac{\pi}{2}-t} \int_{\frac{\pi}{2}}^0 f(\sin(\frac{\pi}{2}-t))\mathrm{d}(\frac{\pi}{2}-t)=-\int_{\frac{\pi}{2}}^0 f(\cos t)\mathrm{d}t=\int_0^{\frac{\pi}{2}}f(\cos x)\mathrm{d}x.$

(2)$\int_0^{\pi}xf(\sin x)\mathrm{d}x \xlongequal{x=\pi-t} \int_{\pi}^0(\pi-t)f(\sin(\pi-t))\mathrm{d}(\pi-t)=-\int_{\pi}^0\pi f(\sin t)\mathrm{d}t+\int_{\pi}^0 tf(\sin t)\mathrm{d}t$
$=\pi\int_0^{\pi}f(\sin x)\mathrm{d}x-\int_0^{\pi}xf(\sin x)\mathrm{d}x$

$\therefore\int_0^{\pi}xf(\sin x)\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi}f(\sin t)\mathrm{d}t.$

3. 证明：
(1)连续奇函数的一切原函数都是偶函数；
(2)连续偶函数的原函数中有一个是奇函数.

证明：(1)设$f(x)$为一个连续奇函数，则$f(x)$的任意原函数可表示为$F(x)=\int_0^x f(t)\mathrm{d}t+C$

$F(-x)=\int_0^{-x}f(t)\mathrm{d}t+C \xlongequal{u=-t} -\int_0^x f(-u)\mathrm{d}u+C=\int_0^x f(u)\mathrm{d}u+C=F(x)$

故$f(x)$的任意原函数$F(x)$均为偶函数.

(2)设$f(x)$为一个连续偶函数，则$f(x)$的任意原函数可表示为$F(x)=\int_0^x f(t)\mathrm{d}t+C$

$F(-x)=\int_0^{-x} f(t)\mathrm{d}t+C \xlongequal{u=-t} -\int_0^x f(-u)\mathrm{d}u+C=-\int_0^x f(u)\mathrm{d}u+C=-F(x)+2C$

当且仅当$C=0$时$F(x)$是奇函数.

4. 设$f(x)$连续，证明：

$$\int_0^R x^3 f(x^2)\mathrm{d}x=\frac{1}{2}\int_0^{R^2} xf(x)\mathrm{d}x.$$

证明：$\int_0^R x^3 f(x^2)\mathrm{d}x=\int_0^R x^2 f(x^2)\mathrm{d}(\frac{1}{2}x^2) \xlongequal{t=x^2} \frac{1}{2}\int_0^{R^2} tf(t)\mathrm{d}t=\frac{1}{2}\int_0^{R^2} xf(x)\mathrm{d}x.$