9C 第6章补充题

9C.1 第6章补充题解答

1. 求下列不定积分：

(1)$\int\frac{x\cos x}{\sin^3 x}\mathrm{d}x$；　(2)$\int\frac{x\ln x}{(x^2-1)^{\frac{3}{2}}}\mathrm{d}x$；

(3)$\int\frac{\mathrm{d}x}{(\mathrm{e}^x+1)^2}$；　(4)$\int\frac{x\mathrm{e}^x}{\sqrt{\mathrm{e}^x-1}}\mathrm{d}x$；

(5)$\int\frac{x^2\mathrm{e}^x}{(x+2)^2}\mathrm{d}x$；　(6)$\int\sin^2(\ln x)\mathrm{d}x$；

(7)$\int\mathrm{e}^{2x}(1+\tan x)^2\mathrm{d}x$；　(8)$\int\mathrm{e}^x(\frac{1}{x}+\ln x)\mathrm{d}x$；

(9)$\int\frac{\mathrm{e}^{-\frac{1}{x}}}{x^4}\mathrm{d}x$；　(10)$\int\mathrm{e}^{\sin x}\frac{x\cos^3 x-\sin x}{\cos^2 x}\mathrm{d}x$.

解：(1)$\int\frac{x}{\sin^3 x}\mathrm{d}\sin x=\int x\mathrm{d}(-\frac{1}{2\sin^2 x})=-\frac{x}{2\sin^2 x}+\int\frac{1}{2\sin^2 x}\mathrm{d}x=-\frac{x}{2\sin^2 x}+\frac{1}{2}\int\csc^2 x\mathrm{d}x$
$=-\frac{x}{2\sin^2 x}-\frac{1}{2}\cot x+C.$

(2)方法1：$\int\frac{x\ln x}{(x^2-1)^{\frac{3}{2}}}\mathrm{d}x=\frac{1}{2}\int\frac{\ln x}{(x^2-1)^{\frac{3}{2}}}\mathrm{d}(x^2-1)=\frac{1}{2}\int\ln x\mathrm{d}\frac{-2}{(x^2-1)^{\frac{1}{2}}}=-\int\ln x\mathrm{d}\frac{1}{(x^2-1)^{\frac{1}{2}}}$
$=-\frac{\ln x}{(x^2-1)^{\frac{1}{2}}}+\int\frac{1}{(x^2-1)^{\frac{1}{2}}}\frac{1}{x}\mathrm{d}x\xlongequal[\tan t=\sqrt{x^2-1}]{x=\sec t}-\frac{\ln x}{(x^2-1)^{\frac{1}{2}}}+\int\frac{1}{\tan t}\frac{1}{\sec t}\sec t\tan t\mathrm{d}t$
$=-\frac{\ln x}{(x^2-1)^{\frac{1}{2}}}+t+C=-\frac{\ln x}{(x^2-1)^{\frac{1}{2}}}+\arccos\frac{1}{x}+C=-\frac{\ln x}{(x^2-1)^{\frac{1}{2}}}+\arctan\sqrt{x^2-1}+C.$

方法2：$\int\frac{x\ln x}{(x^2-1)^{\frac{3}{2}}}\mathrm{d}x=\frac{1}{2}\int\frac{\ln x}{(x^2-1)^{\frac{3}{2}}}\mathrm{d}(x^2-1)=\frac{1}{2}\int\ln x\mathrm{d}\frac{-2}{(x^2-1)^{\frac{1}{2}}}=-\int\ln x\mathrm{d}\frac{1}{(x^2-1)^{\frac{1}{2}}}$
$=-\frac{\ln x}{(x^2-1)^{\frac{1}{2}}}+\int\frac{1}{(x^2-1)^{\frac{1}{2}}}\frac{1}{x}\mathrm{d}x\xlongequal[\tan t=\frac{1}{\sqrt{x^2-1}}]{x=\csc t}-\frac{\ln x}{(x^2-1)^{\frac{1}{2}}}-\int\frac{1}{\cot t}\frac{1}{\csc t}\csc t\cot t\mathrm{d}t$
$=-\frac{\ln x}{(x^2-1)^{\frac{1}{2}}}-t+C=-\frac{\ln x}{(x^2-1)^{\frac{1}{2}}}-\arcsin\frac{1}{x}+C=-\frac{\ln x}{(x^2-1)^{\frac{1}{2}}}-\arctan\frac{1}{\sqrt{x^2-1}}+C.$

(3)$\int\frac{\mathrm{d}x}{(\mathrm{e}^x+1)^2}=\int\frac{\mathrm{e}^x+1-\mathrm{e}^x}{(\mathrm{e}^x+1)^2}\mathrm{d}x=\int[\frac{1}{\mathrm{e}^x+1}-\frac{\mathrm{e}^x}{(\mathrm{e}^x+1)^2}]\mathrm{d}x=\int[\frac{\mathrm{e}^x+1-\mathrm{e}^x}{\mathrm{e}^x+1}-\frac{\mathrm{e}^x}{(\mathrm{e}^x+1)^2}]\mathrm{d}x$
$=\int[1-\frac{\mathrm{e}^x}{\mathrm{e}^x+1}-\frac{\mathrm{e}^x}{(\mathrm{e}^x+1)^2}]\mathrm{d}x=x-\ln(\mathrm{e}^x+1)+\frac{1}{\mathrm{e}^x+1}+C.$

(4)$\int\frac{x\mathrm{e}^x}{\sqrt{\mathrm{e}^x-1}}\mathrm{d}x=\int\frac{x}{\sqrt{\mathrm{e}^x-1}}\mathrm{d}\mathrm{e}^x=\int\frac{x}{\sqrt{\mathrm{e}^x-1}}\mathrm{d}\mathrm{e}^x=\int x\mathrm{d}(2\sqrt{\mathrm{e}^x-1})=2x\sqrt{\mathrm{e}^x-1}-2\int\sqrt{\mathrm{e}^x-1}\mathrm{d}x$
$\xlongequal{\sqrt{\mathrm{e}^x-1}=t}2x\sqrt{\mathrm{e}^x-1}-2\int t\mathrm{d}\ln(t^2+1)=2x\sqrt{\mathrm{e}^x-1}-2\int t\frac{2t}{1+t^2}\mathrm{d}t=2x\sqrt{\mathrm{e}^x-1}-4\int\frac{t^2}{1+t^2}\mathrm{d}t$
$=2x\sqrt{\mathrm{e}^x-1}-4\int(1-\frac{1}{1+t^2})\mathrm{d}t=2x\sqrt{\mathrm{e}^x-1}-4(t-\arctan t)+C$
$=2x\sqrt{\mathrm{e}^x-1}-4\sqrt{\mathrm{e}^x-1}+4\arctan\sqrt{\mathrm{e}^x-1}+C.$

(5)$\int\frac{x^2\mathrm{e}^x}{(x+2)^2}\mathrm{d}x=\int x^2\mathrm{e}^x\mathrm{d}\frac{-1}{x+2}=-\frac{x^2\mathrm{e}^x}{x+2}+\int\frac{1}{x+2}\mathrm{d}(x^2\mathrm{e}^x)=-\frac{x^2\mathrm{e}^x}{x+2}+\int\frac{1}{x+2}(2x+x^2)\mathrm{e}^x\mathrm{d}x$
$=-\frac{x^2\mathrm{e}^x}{x+2}+\int x\mathrm{e}^x\mathrm{d}x=-\frac{x^2\mathrm{e}^x}{x+2}+x\mathrm{e}^x-\int\mathrm{e}^x\mathrm{d}x=-\frac{x^2\mathrm{e}^x}{x+2}+x\mathrm{e}^x-\mathrm{e}^x+C$
$=\frac{\mathrm{e}^x(x-2)}{x+2}+C.$

(6)$\int\sin^2(\ln x)\mathrm{d}x=x\sin^2(\ln x)-\int x\mathrm{d}\sin^2(\ln x)=x\sin^2(\ln x)-\int x2\sin(\ln x)\cos(\ln x)\frac{1}{x}\mathrm{d}x$
$=x\sin^2(\ln x)-2\int\sin(\ln x)\cos(\ln x)\mathrm{d}x=x\sin^2(\ln x)-\int\sin(2\ln x)\mathrm{d}x$
$=x\sin^2(\ln x)-[x\sin(2\ln x)-\int x\mathrm{d}\sin(2\ln x)]=x\sin^2(\ln x)-x\sin(2\ln x)+\int x\cos(2\ln x)\frac{2}{x}\mathrm{d}x$
$=x\sin^2(\ln x)-x\sin(2\ln x)+2\int\cos(2\ln x)\mathrm{d}x$
$=x\sin^2(\ln x)-x\sin(2\ln x)+2[x\cos(2\ln x)-\int x\mathrm{d}\cos(2\ln x)]$
$=x\sin^2(\ln x)-x\sin(2\ln x)+2[x\cos(2\ln x)+\int x\sin(2\ln x)\frac{2}{x}\mathrm{d}x]$

$= x\sin^2(\ln x) - x\sin(2\ln x) + 2x\cos(2\ln x) + 4\int\sin(2\ln x)\mathrm{d}x$
$= x\sin^2(\ln x) - \frac{1}{5}x\sin(2\ln x) + \frac{2}{5}x\cos(2\ln x) + C$
$= x\frac{1}{2}[1-\cos(2\ln x)] - \frac{1}{5}x\sin(2\ln x) + \frac{2}{5}x\cos(2\ln x) + C$
$= \frac{1}{2}x - \frac{1}{10}x\cos(2\ln x) - \frac{1}{5}x\sin(2\ln x).$

(7)$\int \mathrm{e}^{2x}(1+\tan x)^2\mathrm{d}x = \int \mathrm{e}^{2x}(1+\tan^2 x + 2\tan x)\mathrm{d}x = \int \mathrm{e}^{2x}(\sec^2 x + 2\tan x)\mathrm{d}x$
$= \int \mathrm{e}^{2x}\mathrm{d}\tan x + \int \tan x\mathrm{d}\mathrm{e}^{2x} = \int(\mathrm{e}^{2x}\mathrm{d}\tan x + \tan x\mathrm{d}\mathrm{e}^{2x}) = \int \mathrm{d}(\mathrm{e}2x\tan x)$
$= \mathrm{e}^{2x}\tan x + C.$

(8)$\int \mathrm{e}^x(\frac{1}{x}+\ln x)\mathrm{d}x = \int \mathrm{e}^x\mathrm{d}\ln x + \int \mathrm{e}^x\ln x\mathrm{d}x = \mathrm{e}^x\ln x - \int \ln x\mathrm{e}^x\mathrm{d}x + \int \mathrm{e}^x\ln x\mathrm{d}x$
$= \mathrm{e}^x\ln x + C.$

(9)$\int \frac{\mathrm{e}^{-\frac{1}{x}}}{x^4}\mathrm{d}x \xlongequal{t=-\frac{1}{x}} \int \mathrm{e}^t t^4\mathrm{d}(-\frac{1}{t}) = \int \mathrm{e}^t t^4\frac{1}{t^2}\mathrm{d}t = \int \mathrm{e}^t t^2\mathrm{d}t = t^2\mathrm{e}^t - \int \mathrm{e}^t 2t\mathrm{d}t$
$= t^2\mathrm{e}^t - 2(t\mathrm{e}^t - \int \mathrm{e}^t\mathrm{d}t) = t^2\mathrm{e}^t - 2t\mathrm{e}^t + 2\mathrm{e}^t + C = \mathrm{e}^{-\frac{1}{x}}(\frac{1}{x^2}+\frac{2}{x}+2) + C.$

(10)$\int \mathrm{e}^{\sin x}\frac{x\cos^3 x - \sin x}{\cos^2 x}\mathrm{d}x = \int \mathrm{e}^{\sin x}x\cos x\mathrm{d}x - \int \mathrm{e}^{\sin x}\frac{\sin x\cos x}{\cos^3 x}\mathrm{d}x$
$= \int \mathrm{e}^{\sin x}x\mathrm{d}\sin x - \int \mathrm{e}^{\sin x}\tan x\sec x\mathrm{d}x = \int x\mathrm{d}\mathrm{e}^{\sin x} - \int \mathrm{e}^{\sin x}\mathrm{d}\sec x$
$= x\mathrm{e}^{\sin x} - \int \mathrm{e}^{\sin x}\mathrm{d}x - (\mathrm{e}^{\sin x}\sec x - \int \sec x\mathrm{d}\mathrm{e}^{\sin x})$
$= x\mathrm{e}^{\sin x} - \int \mathrm{e}^{\sin x}\mathrm{d}x - (\mathrm{e}^{\sin x}\sec x - \int \sec x\mathrm{e}^{\sin x}\cos x\mathrm{d}x)$
$= x\mathrm{e}^{\sin x} - \int \mathrm{e}^{\sin x}\mathrm{d}x - (\mathrm{e}^{\sin x}\sec x - \int \mathrm{e}^{\sin x}\mathrm{d}x)$
$= x\mathrm{e}^{\sin x} - \mathrm{e}^{\sin x}\sec x + C$
$= x\mathrm{e}^{\sin x} - \frac{\mathrm{e}^{\sin x}}{\cos x} + C.$

2. 求不定积分$\int \frac{\mathrm{d}x}{\sin^n x}$的递推公式.

解：$J_n = \int \frac{\mathrm{d}x}{\sin^n x} = \int \csc^n x\mathrm{d}x = -\int \csc^{n-2}x\mathrm{d}\cot x = -\csc^{n-2}x\cot x + \int \cot x\mathrm{d}\csc^{n-2}x$
$= -\csc^{n-2}x\cot x - (n-2)\int \cot x\csc^{n-3}x\csc x\cot x\mathrm{d}x$
$= -\csc^{n-2}x\cot x - (n-2)\int \cot^2 x\csc^{n-2}x\mathrm{d}x$
$= -\csc^{n-2}x\cot x - (n-2)\int(\csc^2 x - 1)\csc^{n-2}x\mathrm{d}x$
$= -\csc^{n-2}x\cot x - (n-2)(\int \csc^n x\mathrm{d}x - \int \csc^{n-2}x\mathrm{d}x)$
$= -\frac{1}{n-1}\csc^{n-2}x\cot x + \frac{n-2}{n-1}J_{n-2}$
$= -\frac{\cos x}{(n-1)\sin^{n-1}x} + \frac{n-2}{n-1}J_{n-2}$

$J_1 = \int \csc x\mathrm{d}x = \ln|\csc x - \cot x| + C$

$J_2 = \int \csc^2 x\mathrm{d}x = -\cot x + C.$