2 数列极限

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1. 用数列极限定义证明以下各题:

(1) $\lim\limits_{n\to\infty}\frac{5n^3}{1+n^3}=5$;

(2) $\lim\limits_{n\to\infty}\frac{\sin n^2}{n}=0$.

证明：(1)（**多项式的标准过程.**）

$\because |\frac{5n^3}{1+n^3}-5|=\frac{5}{1+n^3}$

$\therefore \forall\varepsilon>0$，取$N>\sqrt[3]{\frac{5}{\varepsilon}-1}$，则当$n>N$时，$|\frac{5n^3}{1+n^3}-5|<\varepsilon$

故$\lim\limits_{n\to\infty}\frac{5n^3}{1+n^3}=5$.

(2)（**常用$\sin x\leq 1$.**）

$\because |\frac{\sin n^2}{n}-0|\leq\frac{1}{n}$

$\therefore \forall\varepsilon>0$，取$N>\frac{1}{\varepsilon}$，则当$n>N$时，$|\frac{\sin n^2}{n}-0|<\varepsilon$

故$\lim\limits_{n\to\infty}\frac{\sin n^2}{n}=0$.

2. 用极限定义证明以下各题：

(1) 若$\lim\limits_{n\to\infty}a_n=A$，则$\lim\limits_{n\to\infty}|a_n|=|A|$；

(2) 若$\lim\limits_{n\to\infty}a_n=A>0$，则$\lim\limits_{n\to\infty}\sqrt{a_n}=\sqrt{A}$；

(3) 若$\lim\limits_{n\to\infty}a_n=A$，则$\lim\limits_{n\to\infty}a_n^2=A^2$；

(4) 若$\lim\limits_{n\to\infty}a_n=A$，则$\lim\limits_{n\to\infty}\frac{a_n}{n}=0$.

（常用结论的证明.）

证明：(1)$||a_n|-|A||\le|a_n-A|$

$\because\lim\limits_{n\to\infty}a_n=A$

$\therefore\forall\varepsilon>0,\exists N>0,$ 使$n>N$时, $||a_n|-|A||\le|a_n-A|<\varepsilon$

则$\lim\limits_{n\to\infty}|a_n|=|A|$.

(2)$\because\lim\limits_{n\to\infty}a_n=A>0$

$\therefore\exists N_1>0$，使$n>N_1$时，$a_n>0$

$\therefore n>N_1$时，$|\sqrt{a_n}-\sqrt{A}|=\frac{|a_n-A|}{\sqrt{a_n}+\sqrt{A}}<\frac{|a_n-A|}{\sqrt{A}}$

$\because\lim\limits_{n\to\infty}a_n=A$

$\therefore\forall\varepsilon>0$，$\exists N_2>0$，使$n>N_2$时，$|a_n-A|<\varepsilon$

取$N=\max\{N_1,N_2\}$，当$n>N$时，$|\sqrt{a_n}-\sqrt{A}|=\frac{|a_n-A|}{\sqrt{a_n}+\sqrt{A}}<\frac{|a_n-A|}{\sqrt{A}}<\frac{\varepsilon}{\sqrt{A}}$

故$\lim\limits_{n\to\infty}\sqrt{a_n}=\sqrt{A}$.

(3)$|a_n^2-A^2|=|a_n-A||a_n+A|$,

$\because\lim\limits_{n\to\infty}a_n=A$

$\therefore\exists M>0$，使$|a_n|<M(n>0)$

$\therefore|a_n+A|<|a_n|+|A|<M+|A|$

$\therefore\forall\varepsilon>0,\exists N>0$，当$n>N$时，$|a_n-A|<\varepsilon$

$\therefore|a_n^2-A^2|=|a_n-A||a_n+A|<\varepsilon(M+|A|)$

则$\lim\limits_{n\to\infty}a_n^2=A^2$.

(4)$\because\lim\limits_{n\to\infty}a_n=A$

$\therefore\exists M>0$，使$|a_n|<M$

$\forall\varepsilon>0$，取$N>\frac{M}{\varepsilon}$，则当$n>N$时，$|\frac{a_n}{n}-0|<\frac{M}{n}<\varepsilon$.

故$\lim\limits_{n\to\infty}\frac{a_n}{n}=0$.

3. 设$\lim\limits_{n\to\infty} a_n = A, \lim\limits_{n\to\infty} b_n = B$，且$A < B$，则存在正整数$N$，使得当$n > N$时，恒有$a_n < b_n$.

证明：先证明$\lim\limits_{n\to\infty} a_n - b_n = A - B$：

$\because \lim\limits_{n\to\infty} a_n = A, \lim\limits_{n\to\infty} b_n = B$

$\therefore \forall\varepsilon > 0, \exists N_1 > 0$，使得$n > N_1$时$|a_n - A| < \frac{1}{2}\varepsilon$，当$n > N_2$时$|b_n - B| < \varepsilon$，当$n > N_2$时$|b_n - B| < \frac{1}{2}\varepsilon$

取$N =$max$\{N_1, N_2\}$，则当$n > N$时$|a_n - b_n - (A - B)| < |a_n - A| + |b_n - B| < \varepsilon$

$\therefore \lim\limits_{n\to\infty} a_n - b_n = A - B.$

$\because A < B$

$\therefore A - B < 0$

根据数列极限的保号性知存在正整数N，使得当$n > N$时，恒有$a_n < b_n$.

2.3 习题2.2

1. 用夹逼原理求下列极限

(1) $\lim\limits_{n\to\infty} (2 + \frac{1}{n})^{\frac{1}{n}}$；

(2) $\lim\limits_{n\to\infty} n^{\frac{1}{n}}$；

(3) $\lim\limits_{n\to\infty} (\frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n})$；

(4) $\lim\limits_{n\to\infty} (\frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n})$.

证明：(1)$(2+0)^{\frac{1}{n}} < (2 + \frac{1}{n})^{\frac{1}{n}} \leq (2+1)^{\frac{1}{n}}$，即$2^{\frac{1}{n}} < (2 + \frac{1}{n})^{\frac{1}{n}} \leq 3^{\frac{1}{n}}$

$\because \lim\limits_{n\to\infty} 2^{\frac{1}{n}} = 1, \lim\limits_{n\to\infty} 3^{\frac{1}{n}} = 1$

$\therefore \lim\limits_{n\to\infty} (2 + \frac{1}{n})^{\frac{1}{n}} = 1.$

(2)$\because n^{\frac{1}{n}} \geq 1$

$\therefore a_n = n^{\frac{1}{n}} - 1 \geq 0$

$\therefore n = (1 + a_n)^n = 1 + na_n + \frac{n(n+1)}{2!}a_n^2 + \cdots > \frac{n(n+1)}{2!}a_n^2 (n \geq 2)$

$\therefore 0 < a_n < \sqrt{\frac{2}{n+1}} (n \geq 2)$

$\because \lim\limits_{n\to\infty} \sqrt{\frac{2}{n+1}} (n \geq 2) = 0$

$\therefore \lim\limits_{n\to\infty} a_n = 0$

$\therefore \lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1.$

(3)$\frac{1}{n^2+n}+\frac{2}{n^2+n}+\cdots+\frac{n}{n^2+n}<\frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n}<\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}$

即$\frac{\frac{n(n+1)}{2}}{n^2+n}=\frac{1}{2}<\frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n}<\frac{\frac{n(n+1)}{2}}{n^2}=\frac{n^2+n}{2n^2}$

$\because \lim\limits_{n\to\infty}\frac{n^2+n}{2n^2}=\lim\limits_{n\to\infty}\frac{1+\frac{1}{n}}{2}=\frac{1}{2}$

$\therefore \lim\limits_{n\to\infty}(\frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n})=\frac{1}{2}.$

(4)$\frac{n}{\sqrt{n^2+n}}<\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}<\frac{n}{\sqrt{n^2}}=1$

$\because \lim\limits_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=\lim\limits_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}=1$

$\therefore \lim\limits_{n\to\infty}(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}})=1.$

2. 用单调收敛定理求下列极限：

(1) 设$x\neq 0$，令$a_1=\sin x, a_n=\sin a_{n-1}(n=2,3,\cdots)$，求$\lim\limits_{n\to\infty}a_n$.

(2) 设$a>0,k>0,a_1=\frac{1}{2}(a+\frac{k}{a}),a_n=\frac{1}{2}(a_{n-1}+\frac{k}{a_{n-1}})(n=2,3,\cdots)$，求证：$\lim\limits_{n\to\infty}a_n=\sqrt{k}$.

(3) 设$x_1=a>0,y_1=b>0,x_{n+1}=\sqrt{x_ny_n},y_{n+1}=\frac{1}{2}(x_n+y_n)(n=1,2,\ldots)$. 求证：$x_n$和$y_n$收敛于同一个实数.

(1)解：$a_n-a_{n-1}=\sin a_{n-1}-a_{n-1}$，可分以下两种情况讨论：

(i)当$1\geq a_1=\sin x\geq 0$时，$1\geq a_2=\sin a_1\geq 0, 1\geq a_3=\sin a_2\geq 0,\cdots,1\geq a_n=\sin a_{n-1}\geq 0,\cdots$

$\therefore a_n-a_{n-1}=\sin a_{n-1}-a_{n-1}\leq 0(n=2,3,\cdots)$

$\therefore \{a_n\}$单调非增有下界，故收敛，记$\lim\limits_{n\to\infty}a_n=A$.

将$a_n=\sin a_{n-1}$两边取极限得$A=\sin A$，即$\lim\limits_{n\to\infty}a_n=A=0$.

(ii)当$-1\leq a_1=\sin x<0$时，$-1\leq a_2=\sin a_1<0,-1\leq a_3=\sin a_2<0,\cdots,-1\leq a_n=\sin a_{n-1}<0,\cdots$

$\therefore a_n-a_{n-1}=\sin a_{n-1}-a_{n-1}\geq 0(n=2,3,\cdots)$

$\therefore \{a_n\}$单调非减有上界，故收敛，记$\lim\limits_{n\to\infty}a_n=A$.

将$a_n=\sin a_{n-1}$两边取极限得$A=\sin A$，即$\lim\limits_{n\to\infty}a_n=A=0$.

(2)证明：$a_n-a_{n-1}=\frac{1}{2}(a_{n-1}+\frac{k}{a_{n-1}})-a_{n-1}=\frac{1}{2}(\frac{k}{a_{n-1}}-a_{n-1})$

$\because a>0$

$\therefore a_1=\frac{1}{2}(a+\frac{k}{a})>\sqrt{k}$，$a_2=\frac{1}{2}(a_1+\frac{k}{a_1})>\sqrt{k},a_3=\frac{1}{2}(a_2+\frac{k}{a_2})>\sqrt{k},\cdots,a_n=\frac{1}{2}(a_{n-1}+\frac{k}{a_{n-1}})>\sqrt{k},\cdots$

$\therefore a_n-a_{n-1}=\frac{1}{2}(\frac{k}{a_{n-1}}-a_{n-1})<0$

$\therefore \{a_n\}$单调非增有下界，故收敛，记$\lim\limits_{n\to\infty} a_n = A$.

将$a_n = \frac{1}{2}(a_{n-1} + \frac{k}{a_{n-1}})$两边取极限得$A = \frac{1}{2}(A + \frac{k}{A})$，即$\lim\limits_{n\to\infty} a_n = A = \sqrt{k}$.

(3)证明：$y_{n+1} - x_{n+1} = \frac{1}{2}(x_n + y_n) - \sqrt{x_n y_n}$

$\because a > 0, b > 0$

$\therefore x_2 = \sqrt{ab} > 0, y_2 = \frac{1}{2}a + b > 0, x_3 = \sqrt{x_2 y_2}, y_3 = \frac{1}{2}x_2 + y_2, \cdots, x_n = \sqrt{x_{n-1}y_{n-1}} > 0, y_n = \frac{1}{2}(x_{n-1} + y_{n-1}) > 0, \cdots$

$\therefore y_{n+1} - x_{n+1} = \frac{1}{2}(x_n + y_n) - \sqrt{x_n y_n} \geq 0 (n = 1, 2, \cdots)$

$\therefore x_n = \sqrt{x_{n-1}y_{n-1}} \geq \sqrt{x_{n-1}x_{n-1}} = x_{n-1}, y_n = \frac{1}{2}(x_{n-1} + y_{n-1}) \leq y_{n-1} (n = 3, 4, \cdots)$

$\therefore x_2 = \sqrt{ab} \leq x_3 \leq x_4 \leq \cdots \leq x_n \leq \cdots, y_2 = \frac{1}{2}(a+b) \geq y_3 \geq y_4 \geq \cdots \geq y_n \geq \cdots$

且$x_n \leq y_n (n = 2, 3, \cdots)$

$\therefore \{x_n\}, \{y_n\}$均单调有界，故收敛, 记$\lim\limits_{n\to\infty} x_n = A, \lim\limits_{n\to\infty} y_n = B$，则$A, B > 0$.

将$x_{n+1} = \sqrt{x_n y_n}$和$y_{n+1} = \frac{1}{2}(x_n + y_n)$两边取极限得$A = \sqrt{AB}$和$B = \frac{1}{2}(A + B)$，即$A = B$.

3. 设数列$\{a_n\}$具有这样的性质：$\forall p \in \mathbb{Z}^+$，有$\lim\limits_{n\to\infty} |a_{n+p} - a_n| = 0$. 问$\{a_n\}$是不是柯西数列？研究下列数列是否满足上述条件？是否收敛？

   (1) $a_n = \sqrt{n} (n \in \mathbb{Z}^+)$；

   (2) $a_n = \sum_{k=1}^{n} \frac{1}{k}$.

   解：$\{a_n\}$不一定是柯西数列，根据柯西数列的定义可知柯西数列满足该条件，但满足该条件的数列不一定是柯西数列.如：

   (1)$a_n = \sqrt{n} (n \in \mathbb{Z}^+)$，$\forall p \in \mathbb{Z}^+$，有$\lim\limits_{n\to\infty} |a_{n+p} - a_n| = \lim\limits_{n\to\infty} |\sqrt{n+p} - \sqrt{n}| = \lim\limits_{n\to\infty} \frac{p}{\sqrt{n+p}+\sqrt{n}} = \lim\limits_{n\to\infty} \frac{\frac{p}{\sqrt{n}}}{\sqrt{1+\frac{p}{n}}+1} = 0$，但显然$\{a_n\}$不收敛（因为无界），故不是柯西数列.

   (2)$a_n = \sum_{k=1}^{n} \frac{1}{k}$，$\forall p \in \mathbb{Z}^+$，有$\lim\limits_{n\to\infty} |a_{n+p} - a_n| = \lim\limits_{n\to\infty} \sum_{k=n+1}^{n+p} \frac{1}{k} = \lim\limits_{n\to\infty} (\frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+p}) = 0$，但可证明$\{a_n\}$不收敛，故不是柯西数列.

   证明如下：$\because \ln(1 + \frac{1}{k}) < \frac{1}{k}$

   $\therefore a_n = \sum_{k=1}^{n} \frac{1}{k} > \sum_{k=1}^{n} \ln(1 + \frac{1}{k}) = \sum_{k=1}^{n} \ln(1+k) - \ln k = \ln(n+1)$

   $\because \ln(n+1)$无上界

   $\therefore \{a_n\}$无上界

   易知$\{a_n\}$单调增加，故$\{a_n\}$发散.

4. 用柯西收敛准则证明下列级数收敛：

(1) $a_n=\sum_{k=1}^{n}\frac{\sin k}{2^k}(n\in\mathbb{Z}^+)$；

(2) $a_n=\sum_{k=1}^{n}\frac{1}{k(k+1)}$

证明：(1)$|a_{n+p}-a_n|=\sum_{k=n+1}^{n+p}\frac{\sin k}{2^k}<\sum_{k=n+1}^{n+p}\frac{1}{2^k}=\frac{1}{2^{n+1}}\frac{1-\frac{1}{2^p}}{1-\frac{1}{2}}<\frac{1}{2^n}$

$\therefore\forall\varepsilon>0$，取$N>\log_2\frac{1}{\varepsilon}$，当$n>N$时，$\forall p\in\mathbb{Z}^+,|a_{n+p}-a_n|<\frac{1}{2^n}<\varepsilon$

$\therefore\{a_n\}$是柯西数列，故收敛.

(2)$|a_{n+p}-a_n|=\sum_{k=n+1}^{n+p}\frac{1}{k(k+1)}=\frac{1}{n+1}-\frac{1}{n+p}<\frac{1}{n}$

$\therefore\forall\varepsilon>0$，取$N>\frac{1}{\varepsilon}$，当$n>N$时，$\forall p\in\mathbb{Z}^+,|a_{n+p}-a_n|<\frac{1}{n}<\varepsilon$

$\therefore\{a_n\}$是柯西数列，故收敛.

5. 利用四则运算法则求下列极限：

(1) $\lim\limits_{n\to\infty}(\frac{1+2+\cdots+n}{n+2}-\frac{n}{2})$；

(2) $\lim\limits_{n\to\infty}(\sqrt{n^2+n}-n)$；

(3) $\lim\limits_{n\to\infty}(\sqrt[n]{1}+\sqrt[n]{2}+\cdots+\sqrt[n]{100})$.

解：(1)$\lim\limits_{n\to\infty}(\frac{1+2+\cdots+n}{n+2}-\frac{n}{2})=\lim\limits_{n\to\infty}[\frac{\frac{n(n+1)}{2}}{n+2}-\frac{n}{2}]=\lim\limits_{n\to\infty}[\frac{n(n+1)}{2(n+2)}-\frac{n}{2}]=\lim\limits_{n\to\infty}[\frac{-n}{2(n+2)}]=\lim\limits_{n\to\infty}[\frac{-1}{2(1+\frac{2}{n})}]=-\frac{1}{2}$.

(2)$\lim\limits_{n\to\infty}(\sqrt{n^2+n}-n)=\lim\limits_{n\to\infty}\frac{n^2+n-n^2}{\sqrt{n^2+n}+n}=\lim\limits_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}=\frac{1}{2}$.

(3)$\lim\limits_{n\to\infty}(\sqrt[n]{1}+\sqrt[n]{2}+\cdots+\sqrt[n]{100})=\lim\limits_{n\to\infty}\sqrt[n]{1}+\lim\limits_{n\to\infty}\sqrt[n]{2}+\cdots+\lim\limits_{n\to\infty}\sqrt[n]{100}=1+1+\cdots+1=100$.

6. 设$a_n\neq0(n\in\mathbb{Z}^+)$, $\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}|=q<1$，求证：$\lim\limits_{n\to\infty}a_n=0$.

解：$\because\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}|=q$

$\therefore\forall\varepsilon>0$, $\exists N>0$, 使得$n>N$时, $||\frac{a_{n+1}}{a_n}|-q|<\varepsilon$

$\therefore|\frac{a_{n+1}}{a_n}|<q+\varepsilon$

可知当ε足够小时, $|\frac{a_{n+1}}{a_n}|<q+\varepsilon<1(n>N)$, 即在$n>N$时, 数列$\{|a_n|\}$单调减少

$\because a_n\neq0$

$\therefore\lim\limits_{n\to\infty}|a_n|$存在

$\therefore\lim\limits_{n\to\infty}|a_{n+1}|=\lim\limits_{n\to\infty}|a_n|$

$\therefore\lim\limits_{n\to\infty}|a_{n+1}|=q\lim\limits_{n\to\infty}|a_n|=\lim\limits_{n\to\infty}|a_n|$, 即$\lim\limits_{n\to\infty}|a_n|(1-q)=0$

$\because q<1$

$\therefore 1-q\neq 0$

$\therefore \lim\limits_{n\to\infty}|a_n|=0$

$\therefore \forall\varepsilon>0,\ \exists N>0,$ 使得$n>N$时，$|a_n|-0<\varepsilon$，即$|a_n-0|<\varepsilon$

$\therefore \lim\limits_{n\to\infty}a_n=0.$

7. 利用上题结论证明下列结论：

(1) $\lim\limits_{n\to\infty}\frac{a^n}{n!}=0(a>0)$；

(2) $\lim\limits_{n\to\infty}\frac{n^2}{a^n}=0(a>1)$;

(3) $\lim\limits_{n\to\infty}\frac{a^n}{(n!)^2}=0(a>0)$.

证明：(1)$a_n=\frac{a^n}{n!}$, $\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}|=\lim\limits_{n\to\infty}\frac{a}{n+1}=0<1(a>0)$，故$\lim\limits_{n\to\infty}\frac{a^n}{n!}=0$.

(2)$a_n=\frac{n^2}{a^n}$, $\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}|=\lim\limits_{n\to\infty}\frac{1}{a}\frac{(n+1)^2}{n^2}=\frac{1}{a}<1(a>1)$，故$\lim\limits_{n\to\infty}\frac{n^2}{a^n}=0$.

(3)$a_n=\frac{a^n}{(n!)^2}$, $\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}|=\lim\limits_{n\to\infty}\frac{a}{(n+1)^2}=0<1(a>0)$，故$\lim\limits_{n\to\infty}\frac{a^n}{(n!)^2}=0$.

8. 求极限：

(1) $\lim\limits_{n\to\infty}\sin^2(\pi\sqrt{n^2+1})$;

(2) $\lim\limits_{n\to\infty}\sin^2(\pi\sqrt{n^2+n})$.

解：(1)$\lim\limits_{n\to\infty}\sin^2(\pi\sqrt{n^2+1})=\lim\limits_{n\to\infty}\sin^2(\pi\sqrt{n^2+1}-\pi n)=\lim\limits_{n\to\infty}\sin^2\frac{\pi}{\sqrt{n^2+1}+n}=\sin^2 0=0$.

(2)$\lim\limits_{n\to\infty}\sin^2(\pi\sqrt{n^2+n})=\lim\limits_{n\to\infty}\sin^2(\pi\sqrt{n^2+n}-\pi n)=\lim\limits_{n\to\infty}\sin^2\frac{\pi n}{\sqrt{n^2+n}+n}=\lim\limits_{n\to\infty}\sin^2\frac{\pi}{\sqrt{1+\frac{1}{n}}+1}=\sin^2\frac{\pi}{2}=1$.

（这里用到了结论：若$\lim\limits_{n\to\infty}a_n=A$，则$\lim\limits_{n\to\infty}\sin^2 a_n=\sin^2 A$.）