8C 第5章补充题

8C.1 第5章补充题解答

1. 求证n次拉盖尔多项式
$$\mathrm{L}_n(x)=\mathrm{e}^x\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n\mathrm{e}^{-x})$$
在$(0,+\infty)$上有n个相异实根.

证明：首先证明：若$f\in C[a,+\infty),f(a)=0,\lim\limits_{x\to+\infty}f(x)=0$，且$f(x)$不恒等于0，则$\exists\eta\in[a,+\infty),s.t.f'(\eta)=0$.

若存在一点$x_0\in[a,+\infty),s.t.f(x_0)>0$，由于$\lim\limits_{x\to+\infty}f(x)=0$，所以$\exists X>\max\{a,x_0\}$, $s.t.f(x)<f(x_0)$. 在区间$[a,X]$上，对连续函数$f(x)$应用最大最小值定理可知：$\exists\eta\in[a,X],s.t.f(\eta)=\max\{f(x)|0\le x\le X\}$，则当$x>X$时，$f(x)<f(x_0)\le f(\eta)$，所以$f(\eta)$是$f(x)$在$[a,+\infty)$上的最大值，则$f'(\eta)=0$.

同理，若存在一点$x_0\in[a,+\infty),s.t.f(x_0)<0$，则$\exists\eta',s.t.f(\eta')=\min\{f(x)|a\le x<+\infty\},f'(\eta')=0$.

记$f(x)=x^n\mathrm{e}^{-x}$，则$f(x)$的1到$n-1$阶导数$f'(x),f''(x),\cdots,f^{(n-1)}(x)$都以点$x=0$为零点，且$\lim\limits_{x\to+\infty}f^{(k)}(x)=0,k=0,1,\cdots,n-1$.

根据上面证明的结论，$\exists\xi_1^{[1]},s.t.\frac{\mathrm{d}}{\mathrm{d}x}f(\xi_1^{[1]})=0$，此时$x=0,x=\xi_1^{[1]}$都是$\frac{\mathrm{d}}{\mathrm{d}x}f(x)$的零点，且仍有$\lim\limits_{x\to+\infty}\frac{\mathrm{d}}{\mathrm{d}x}f(x)=0$，根据罗尔定理和上面证明的结论$\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x)$在$(0,+\infty)$上存在两个不同的零点$x=\xi_2^{[1]},x=\xi_2^{[2]}$，此时$x=0,x=\xi_2^{[1]},x=\xi_2^{[2]}$都是$\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x)$的零点，且仍有$\lim\limits_{x\to+\infty}\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x)=0$，故$\frac{\mathrm{d}^3}{\mathrm{d}x^3}f(x)$在$(0,+\infty)$上存在3个不同的零点，以此类推，可知$\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x)$在$(0,+\infty)$上存在$n$个不同的零点.

因为$\mathrm{L}_n(x)$是一个n次多项式，故最多有n个实零点，因此n次拉盖尔多项式在$(0,+\infty)$上有n个相异实零点.

2. 设f在$[a,b]$上可导，且$f'(a)f'(b)<0$，试证存在$\xi\in(a,b)$，使得$f'(\xi)=0$.

证明：$\because f'(a)f'(b)<0$不妨设$f'(a)>0,f'(b)<0$

$\because f'(a)>0$

$\therefore \exists x_1\in(a,b),s.t.f(x_1)>f(a)$

$\because f'(b)<0$

$\therefore \exists x_2\in(x_1,b),s.t.f(x_2)>f(b)$

$\therefore \exists\xi\in(a,b),s.t.f(\xi)=\max\{f(x)|a\le x\le b\}$

$\therefore f'(\xi)=0.$

3. 设f在$[a,b]$上可导，且$f'(a)\neq f'(b)$，试证对于介于$f'(a)$和$f'(b)$之间的每一个实数μ都存在$\xi\in(a,b)$，使$f'(\xi)=\mu$.

证明：令$F(x)=f(x)-\mu x$

$\because F'(a)F'(b)=[f'(a)-\mu][f'(b)-\mu]<0$

$\therefore$根据上题的结论，$\exists\xi\in(a,b),s.t.F'(\xi)=f'(\xi)-\mu=0$，即$f'(\xi)=\mu$.

4. 设f在$(-\infty,+\infty)$上可导，并且满足$\frac{f(x)}{|x|}\to+\infty(x\to\infty)$，试证$\forall a\in\mathbb{R},\exists\xi\in(-\infty,+\infty)$，使得$f'(\xi)=a$.

证明：方法1：$\because\frac{f(x)}{|x|}\to+\infty(x\to\infty)$

$\therefore\frac{f(x)}{x}\to+\infty(x\to+\infty),\frac{f(x)}{x}\to-\infty(x\to-\infty)$

设$x=x_0$是$f(x)$上的任意点

$\because f$在$(-\infty,+\infty)$上可导

$\therefore$根据拉格朗日中值定理，$\frac{f(x)-f(x_0)}{x-x_0}=f'(\eta),\eta$介于$x_0$和$x$之间

$\therefore\lim\limits_{\eta\to+\infty}f'(\eta)=\lim\limits_{x\to+\infty}\frac{f(x)-f(x_0)}{x-x_0}=\lim\limits_{x\to+\infty}\frac{\frac{f(x)}{x}-\frac{f(x_0)}{x}}{1-\frac{x_0}{x}}=+\infty,\lim\limits_{\eta\to-\infty}f'(\eta)=\lim\limits_{x\to-\infty}\frac{f(x)-f(x_0)}{x-x_0}$
$=\lim\limits_{x\to-\infty}\frac{\frac{f(x)}{x}-\frac{f(x_0)}{x}}{1-\frac{x_0}{x}}=-\infty$

$\therefore\forall a\in\mathbb{R},\exists\eta_1>x_0,s.t.f'(\eta_1)>a,\exists\eta_2<x_0,s.t.f'(\eta_2)<f'(a)$

$\therefore\exists\xi\in(\eta_1,\eta_2),s.t.f'(\xi)=a.$

方法2：设$g(x)=f(x)-f(0)$

$\because\frac{f(x)}{|x|}\to+\infty(x\to\infty)$

$\therefore\forall a\in\mathbb{R}$，取$M\geq|a|$，则$\exists N>0,s.t.g(N)-g(0)=f(N)-f(0)=N[\frac{f(N)}{N}-\frac{f(0)}{N}]>NM\geq N|a|,g(-N)-g(0)=f(-N)-f(0)=N[\frac{f(-N)}{N}-\frac{f(0)}{N}]>NM\geq N|a|$

$\because g$在$(-\infty,+\infty)$上可导

$\therefore$根据拉格朗日中值定理$\exists\eta_1\in(0,N),s.t.g(N)-g(0)=g'(\eta_1)N\geq N|a|\geq Na,\exists\eta_2\in(-N,0),s.t.g(-N)-g(0)=-g'(\eta_2)N\geq N|a|\geq-Na$

$\therefore g'(\eta_2)\leq a\leq g'(\eta_1)$

$\therefore\exists\xi\in(\eta_1,\eta_2),s.t.g'(\xi)=f'(\xi)=a.$

方法3：令$g(x)=f(x)-ax$

$\because\frac{f(x)}{|x|}\to+\infty(x\to\infty)$

$\therefore\lim\limits_{x\to-\infty}g(x)=\lim\limits_{x\to-\infty}x(\frac{f(x)}{x}-a)=+\infty,\lim\limits_{x\to+\infty}g(x)=\lim\limits_{x\to+\infty}x(\frac{f(x)}{x}-a)=+\infty$

$\therefore\exists x_1>0,x_2<0,s.t.g(x_1)>0,g(x_2)<0$

$\therefore$对于$g(x_1)>0, g(x_2)>0, \exists N_1>x_2, N_2>-x_2, s.t. g(x)>g(x_1)(x>N_1), g(x)>g(x_2),(x<-N_2)$

$\because f$在$(-\infty,+\infty)$上可导，则$g(x)$在$(-\infty,+\infty)$上可导，故连续

$\therefore \exists\xi\in[N_1,N_2], s.t. g(\xi)=\min\{g(x)|N_1<x<N_2\}$且$g(\xi)<g(x_1), g(\xi)<g(x_2)$

$\therefore g(\xi)\le g(x), x\in(-\infty,+\infty)$，即$g(\xi)=\min\{g(x)|-\infty<x<+\infty\}$

$\therefore g'(\xi)=f'(\xi)-a=0$，即$f'(\xi)=a.$

5. 设f在$[a,b]$上可导，在(a,b)内二阶可导，如果$f'(a)f'(b)>0$，且$f(a)=f(b)$，试证$\exists\xi\in(a,b)$使得$f''(\xi)=0.$

   证明：$\because f'(a)f'(b)>0$，不妨设$f'(a)>0, f'(b)>0$

   $\therefore \exists x_1\in(a,b), s.t. f(x_1)>f(a), \exists x_2\in(x_1,b), s.t. f(x_2)<f(b)=f(a)$

   $\therefore \exists\eta\in(x_1,x_2), s.t. f(\eta)=f(a)=f(b)$

   $\therefore \exists\xi_1\in(a,\eta), \xi_2\in(\eta,b), s.t. f'(\xi_1)=f'(\xi_2)=0$

   $\therefore \exists\xi\in(\xi_1,\xi_2)\subset(a,b), s.t. f''(\xi)=0.$

6. 若f在(a,b)可导，则其导函数$f'(x)$没有第一类间断点.

   证明：方法1：假设$f'(x)$存在第一类间断点$x_0\in(a,b)$，则$\lim\limits_{x\to x_0^+}f'(x)$和$\lim\limits_{x\to x_0^-}f'(x)$都存在

   $\because f$在(a,b)可导

   $\therefore f'_+(x_0)=f'(x_0)=f'_-(x_0)$

   又$\because f'_+(x_0)=\lim\limits_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}=\lim\limits_{\xi\to x_0^+}\frac{f'(\xi)(x-x_0)}{x-x_0}=\lim\limits_{x\to x_0^+}f'(x), f'_-(x_0)=\lim\limits_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}$
   $=\lim\limits_{\xi\to x_0^-}\frac{f'(\xi)(x-x_0)}{x-x_0}=\lim\limits_{x\to x_0^-}f'(x)$

   $\therefore \lim\limits_{x\to x_0^+}f'(x)=f'(x_0)=\lim\limits_{x\to x_0^-}f'(x)$

   $\therefore f'(x)$在x_0处连续，假设不成立

   $\therefore f'(x)$没有第一类间断点.

   方法2：假设$x_0\in(a,b)$是$f'(x)$的第一类间断点，不妨设$\lim\limits_{x\to x_0^+}f'(x)=A, A\ne f'(x_0)$（这里不妨设$A>f'(x_0)$）

   根据极限的保号性，$\exists\delta>0, s.t. f'(x)>\frac{A+f'(x_0)}{2}>f'(x_0), x\in(x_0,x_0+\delta)$

   取$x_1\in(x_0,x_0+\delta), \mu=\frac{A+3f'(x_0)}{4}$，则$f'(x_1)>\frac{A+f'(x_0)}{2}>\mu>f'(x_0)$

   此时不存在$\xi\in(x_0,x_0+\delta), s.t. f'(\xi)=\mu$，这与上述第3题的结论（Darboux定理）矛盾

   故$f'(x)$没有第一类间断点.

7. 试举出一个函数f，它在$(-\infty,+\infty)$上处处可导，其导函数$f'(x)$在$x=0$处有第二类间断点.

解：$f(x)=\begin{cases}x^2\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$，$f'(x)=\begin{cases}2x\sin\frac{1}{x}-\cos\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

8. 设$f(x)$在$[0,a]$二阶可导，$|f''(x)|\leq M, 0\leq x\leq a$. 又设$f(x)$在$(0,a)$取得极大值. 求证$|f'(0)|+|f'(a)|\leq Ma$.

证明：$\because f(x)$在$(0,a)$取得极大值

$\therefore \exists\xi\in(0,a), s.t. f'(\xi)=0$

$\therefore \exists\eta_1\in(0,\xi), s.t. f'(\xi)-f'(0)=f''(\eta_1)\xi, \exists\eta_2\in(\xi,a), s.t. f'(a)-f'(\xi)=f''(\eta_2)(a-\xi)$

$\therefore |f'(0)|+|f'(a)|=|f'(\xi)-f'(0)|+|f'(a)-f'(\xi)|=|f''(\eta_1)|\xi+|f''(\eta_2)|(a-\xi)\leq M\xi+M(a-\xi)=Ma.$

9. 设$f(x)$在$[0,1]$处处可导，$f(0)=0, f(1)=1$且$f(x)\not\equiv x$. 求证$\exists\xi\in(0,1)$使$f'(\xi)>1$.

证明：方法1：假设$\forall x\in(0,1), s.t. f'(x)\leq 1$，令$F(x)=f(x)-x$，则$F'(x)=f'(x)-1\leq 0$

$\therefore F(x)$单调非增

$\because F(0)=0=F(1)$

$\therefore \forall x\in(0,1), 0=F(0)\geq F(x)\geq F(1)=0$

$\therefore F(x)\equiv 0$，矛盾

故$\exists\xi\in(0,1)$使$f'(\xi)>1$.

方法2：$\because f(x)\not\equiv x$

$\therefore \exists x_0\in(0,1), s.t. f(x_0)\neq x_0$

若$f(x_0)>x_0$，则$\exists\eta\in(0,x_0), s.t. f(x_0)-f(0)=f'(\eta)x_0>x_0-f(0)=x_0$

$\therefore [f'(\eta)-1]x_0>0$，即$f'(\eta)>1$.

10. 选择a与b，使得$x-(a+b\cos x)\sin x$为5阶无穷小$(x\to 0)$.

解：

$$\begin{aligned}&x-(a+b\cos x)\sin x\\=&x-\{a+b[\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}x^{2k}+o(x^{2n+1})]\}[\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}x^{2k+1}+o(x^{2n+2})]\\=&x-\{a+b[1-\frac{1}{2}x^2+\frac{1}{4!}x^4+o(x^5)]\}[x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+o(x^6)]\\=&(1-a-b)x+(\frac{b}{2}+\frac{a+b}{3!})x^3+(\frac{b}{4!}-\frac{b}{2\cdot 3!}-\frac{a+b}{5!})x^5+o(x^5)\end{aligned}$$

要使$x-(a+b\cos x)\sin x$为5阶无穷小$(x\to 0)$

则

$$\begin{cases}1-a-b=0\\ \frac{b}{2}+\frac{a+b}{3!}=0\end{cases}$$

则$a=\frac{4}{3},b=-\frac{1}{3}$.

11. 利用泰勒公式求下列极限：

(1)$\lim\limits_{x\to 0}\frac{\sin(\sin x)-\tan(\tan x)}{\sin x-\tan x}$；

(2) $\lim\limits_{x\to 0^+}\frac{\mathrm{e}^x-1-x}{\sqrt{1-x}-\cos\sqrt{x}}$；

(3)$\lim\limits_{x\to 0}\frac{1}{x^4}[\ln(1+\sin^2 x)-6(\sqrt[3]{2-\cos x}-1)]$.

解：(1)

$$\begin{aligned}&\lim_{x\to 0}\frac{\sin(\sin x)-\tan(\tan x)}{\sin x-\tan x}\\ =&\lim_{x\to 0}\frac{[\sin x-\frac{1}{6}\sin^3 x+o(\sin^4 x)]-[\tan x+\frac{1}{3}\tan^3 x+o(\tan^4 x)]}{[x-\frac{1}{6}x^3+o(x^4)]-[x+\frac{1}{3}x^3+o(x^4)]}\\ =&\lim_{x\to 0}\frac{[x-\frac{1}{6}x^3+o(x^4)-\frac{1}{6}x^3+o(x^3)+o(x^4)]-[x+\frac{1}{3}x^3+o(x^4)+\frac{1}{3}x^3+o(x^3)+o(x^4)]}{-\frac{1}{2}x^3+o(x^4)}\\ =&2.\end{aligned}$$

(2)

$$\begin{aligned}&\lim_{x\to 0^+}\frac{\mathrm{e}^x-1-x}{\sqrt{1-x}-\cos\sqrt{x}}\\ =&\lim_{x\to 0^+}\frac{1+x+\frac{x^2}{2!}+o(x^2)-1-x}{1-\frac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2+o(x^2)-[1-\frac{1}{2}x+\frac{1}{4!}x^2+o(x^2)]}\\ =&-3.\end{aligned}$$

(3)

$$
\begin{aligned}
&\lim_{x\to 0}\frac{1}{x^4}[\ln(1+\sin^2 x)-6(\sqrt[3]{2-\cos x}-1)]\\
=&\lim_{x\to 0}\frac{1}{x^4}(\sin^2 x-\frac{1}{2}\sin^4 x+o(\sin^4 x)-6\{1+\frac{1}{3}(1-\cos x)+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}(1-\cos x)^2\\
&+o[(1-\cos x)^2]-1\})\\
=&\lim_{x\to 0}\frac{1}{x^4}([x-\frac{1}{6}x^3+o(x^3)]^2-\frac{1}{2}x^4+o(x^4)+o(x^4)-6\{\frac{1}{3}[\frac{1}{2}x^2-\frac{1}{4!}x^4+o(x^4)]\\
&+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}[\frac{1}{2}x^2+o(x^2)]^2+o[(1-\cos x)^2]\})\\
=&\lim_{x\to 0}\frac{1}{x^4}\{x^2-\frac{1}{3}x^4+o(x^4)-\frac{1}{2}x^4+o(x^4)-6[\frac{1}{6}x^2-\frac{1}{3\cdot 4!}x^4+o(x^4)-\frac{1}{9}\frac{1}{4}x^4\\
&+o(x^4)+o(x^4)]\}\\
&=\lim_{x\to 0}\frac{1}{x^4}[-\frac{1}{3}x^4-\frac{1}{2}x^4-6(-\frac{1}{3\cdot 4!}x^4-\frac{1}{9}\frac{1}{4}x^4)+o(x^4)]\\
=&-\frac{7}{12}.
\end{aligned}
$$

12. 设$f(x)$在$[a,b]$上二阶可导，证明：$\exists x_0\in(a,b)$，使得

$$f(b)-2f(\frac{a+b}{2})+f(a)=\frac{(b-a)^2}{4}f''(x_0).$$

证明：$\because f(x)$在$[a,b]$上二阶可导

$\therefore f(x)=f(\frac{a+b}{2})+f'(\frac{a+b}{2})(x-\frac{a+b}{2})+\frac{f''(\xi)}{2!}(x-\frac{a+b}{2})^2,\xi$介于$x$和$\frac{a+b}{2}$之间

$\therefore f(a)=f(\frac{a+b}{2})+f'(\frac{a+b}{2})(a-\frac{a+b}{2})+\frac{f''(\xi_1)}{2!}(a-\frac{a+b}{2})^2$
$=f(\frac{a+b}{2})+f'(\frac{a+b}{2})\frac{a-b}{2}+\frac{f''(\xi_1)}{2!}(\frac{a-b}{2})^2,\xi_1\in(a,\frac{a+b}{2})$

$f(b)=f(\frac{a+b}{2})+f'(\frac{a+b}{2})(b-\frac{a+b}{2})+\frac{f''(\xi_2)}{2!}(b-\frac{a+b}{2})^2$
$=f(\frac{a+b}{2})+f'(\frac{a+b}{2})\frac{b-a}{2}+\frac{f''(\xi_2)}{2!}(\frac{b-a}{2})^2,\xi_2\in(\frac{a+b}{2},b)$

以上两式相加得$f(a)+f(b)=2f(\frac{a+b}{2})+\frac{1}{2}[f''(\xi_1)+f''(\xi_2)]\frac{(a-b)^2}{4}$

$\because \min\{f''(\xi_1),f''(\xi_2)\}\le\frac{1}{2}[f''(\xi_1)+f''(\xi_2)]\le\max\{f''(\xi_1),f''(\xi_2)\}$

$\therefore \exists x_0\in[\xi_1,\xi_2]\subset(a,b),s.t.f''(x_0)=\frac{1}{2}[f''(\xi_1)+f''(\xi_2)]$

即$f(b)-2f(\frac{a+b}{2})+f(a)=\frac{(b-a)^2}{4}f''(x_0)$.

13. 设$f(x)$在区间$[a,b]$上一阶可导，在(a,b)内二阶可导，且$f'(a)=f'(b)=0$，试证$\exists x_0\in(a,b)$，使得

$$|f''(x_0)|\ge\frac{4}{(b-a)^2}|f(b)-f(a)|.$$

证明：**（该题似乎应加上$f(x)$在a,b两点的一阶导数连续的条件）**

$f(x)=f(a)+f'(a)(x-a)+\frac{1}{2}f''(\xi_1)(x-a)^2=f(a)+\frac{1}{2}f''(\xi_1)(x-a)^2,\xi_1\in(a,x)$

$f(x)=f(b)+f'(b)(x-b)+\frac{1}{2}f''(\xi_2)(x-b)^2=f(b)+\frac{1}{2}f''(\xi_2)(x-b)^2,\xi_2\in(x,b)$

$f(\frac{a+b}{2})=f(a)+\frac{1}{2}f''(\xi_1)(\frac{a+b}{2}-a)^2=f(a)+\frac{1}{2}f''(\xi_1)(\frac{a-b}{2})^2$

$f(\frac{a+b}{2})=f(b)+\frac{1}{2}f''(\xi_2)(\frac{a+b}{2}-b)^2=f(b)+\frac{1}{2}f''(\xi_2)(\frac{a-b}{2})^2$

以上两式相减得$f(b)-f(a)=\frac{1}{2}[f''(\xi_1)-f''(\xi_2)]\frac{(a-b)^2}{4}$

$\therefore|f(b)-f(a)|=\frac{1}{2}|f''(\xi_1)-f''(\xi_2)|\frac{(a-b)^2}{4}\leq\frac{1}{2}[|f''(\xi_1)|+|f''(\xi_2)|]\frac{(a-b)^2}{4}$

记$|f''(\xi)|=\max\{|f''(\xi_1)|,|f''(\xi_2)|\}$

则$|f(b)-f(a)|\leq|f''(\xi)|\frac{(a-b)^2}{4}$

即$|f''(x_0)|\geq\frac{4}{(b-a)^2}|f(b)-f(a)|$.

14. 设$f(x)\in C^2[a,b],f(a)=f(b)=0$，试证：

(1)$\max\limits_{a\leq x\leq b}|f(x)|\leq\frac{1}{8}(b-a)^2\max\limits_{a\leq x\leq b}|f''(x)|$；

(2)$\max\limits_{a\leq x\leq b}|f'(x)|\leq\frac{1}{2}(b-a)\max\limits_{a\leq x\leq b}|f''(x)|$.

证明：(1)设$|f(x_0)=\max\limits_{a\leq x\leq b}|f(x)|$

$\because f(a)=f(b)=0$

$\therefore x_0\in(a,b),|f(x_0)|\geq 0$且$x_0$是$f(x)$的极值点

$\therefore f'(x_0)=0$

$f(x)$在x_0处的一阶泰勒多项式为$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2}f''(\xi)(x-x_0)^2=f(x_0)+\frac{1}{2}f''(\xi)(x-x_0)^2,\xi$介于$x_0$和$x$之间

$\therefore f(a)=f(x_0)+\frac{1}{2}f''(\xi_1)(a-x_0)^2=0(*)$

$f(b)=f(x_0)+\frac{1}{2}f''(\xi_2)(b-x_0)^2=0(**)$

i)当$x_0\in(a,\frac{a+b}{2}]$时，由$(*)$式知$\max\limits_{a\leq x\leq b}|f(x)|=|f(x_0)|=\frac{1}{2}(a-x_0)^2|f''(\xi_1)|\leq\frac{1}{2}(a-\frac{a+b}{2})|f''(\xi_1)|=\frac{1}{8}(a-b)^2|f''(\xi_1)|\leq\frac{1}{8}(b-a)^2\max\limits_{a\leq x\leq b}|f''(x)|$

ii)当$x_0\in(\frac{a+b}{2},b)$时，由$(**)$式知$\max\limits_{a\leq x\leq b}|f(x)|=|f(x_0)|=\frac{1}{2}(b-x_0)^2|f''(\xi_1)|<\frac{1}{2}(b-\frac{a+b}{2})|f''(\xi_1)|=\frac{1}{8}(a-b)^2|f''(\xi_1)|\leq\frac{1}{8}(b-a)^2\max\limits_{a\leq x\leq b}|f''(x)|$

综上所述，$\max\limits_{a\leq x\leq b}|f(x)|\leq\frac{1}{8}(b-a)^2\max\limits_{a\leq x\leq b}|f''(x)|$.

(2)设$|f'(x_0)|=\max\limits_{a\leq x\leq b}|f'(x)|$

$f(x)$在x_0处的一阶泰勒多项式为$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2}f''(\xi)(x-x_0)^2,\xi$介于$x_0$和$x$之间

$\because f(a)=f(b)=0$

$\therefore f(a)=f(x_0)+f'(x_0)(a-x_0)+\frac{1}{2}f''(\xi_1)(a-x_0)^2=0$
$f(b)=f(x_0)+f'(x_0)(b-x_0)+\frac{1}{2}f''(\xi_2)(b-x_0)^2=0$

以上两式相减得$f'(x_0)(a-b)+\frac{1}{2}[f''(\xi_1)(a-x_0)^2-f''(\xi_2)(b-x_0)^2]=0$

所以

$$\begin{aligned}|f'(x_0)|(b-a)&=\frac{1}{2}|f''(\xi_1)(a-x_0)^2-f''(\xi_2)(b-x_0)^2|\\&\le\frac{1}{2}[(a-x_0)^2+(b-x_0)^2]\max_{a\le x\le b}|f''(x)|\\&=\frac{1}{2}(a^2-2ax_0+x_0^2+b^2-2bx_0+x_0^2)\max_{a\le x\le b}|f''(x)|\\&=\frac{1}{2}[2(x_0-\frac{a+b}{2})+\frac{(a-b)^2}{2}]\max_{a\le x\le b}|f''(x)|\\&\le\frac{1}{2}[2(a-\frac{a+b}{2})^2+\frac{(a-b)^2}{2}]\max_{a\le x\le b}|f''(x)|\\&=\frac{1}{2}(b-a)^2\max_{a\le x\le b}|f''(x)|\end{aligned}$$

$\therefore \max\limits_{a\le x\le b}|f'(x)|=|f'(x_0)|\le\frac{1}{2}(b-a)\max\limits_{a\le x\le b}|f''(x)|$

15. 设f在$(-\infty,+\infty)$有定义，并且满足$f(x+y)=f(x)f(y)$，对所有实数x,y都成立，又设$f'(0)=a$. 试求$f'(x)$和$f(x)$的表达式.

解：若存在$x=x_0,s.t.f(x_0)=0$，则$f(x_0+y)=f(x_0)f(y)=0,y\in\mathbb{R}$，即$f(x)\equiv 0$

若$f(x)\neq 0$，则由$f(0+0)=[f(0)]^2$得$f(0)=1$

所以

$$\begin{aligned}f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}&=\lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}=f(x)\lim_{h\to 0}\frac{f(h)-1}{h}\\&=f(x)\lim_{h\to 0}\frac{f(h)-f(0)}{h}=f'(0)f(x)=af(x)\end{aligned}$$

$\therefore f(x)=\mathrm{e}^{ax}$

即$f(x)=0$或$f(x)=\mathrm{e}^{ax}$.

16. 设$f(x)$在区间$[0,+\infty)$有界，处处可导. 求证存在一个单调增加趋向于$+\infty$的点列$\{x_n\}$，使得$\lim\limits_{n\to\infty}f'(x_n)=0$.

证明：$\because f(x)$在区间$[0,+\infty)$有界，处处可导

$\therefore \exists M>0,s.t.|f(x)|\le M,x\in[0,+\infty)$

取$a_n=2^n$，则$|f(a_n)-f(a_{n-1})|\le|f(a_n)|+|f(a_{n-1})|\le 2M$

又$\because f(a_n)-f(a_{n-1})=f'(\xi_n)(a_n-a_{n-1})=f'(\xi_n)2^{n-1},\xi_n\in(a_{n-1},a_n),n>1$

$\therefore |f'(\xi_n)| \leq \frac{2M}{2^{n-1}}, n > 1$

$\because \lim\limits_{n\to\infty} \frac{2M}{2^{n-1}} = 0$

$\therefore \lim\limits_{n\to\infty} f'(\xi_n) = 0$

可取$x_n = \xi_n$，满足$\{x_n\}$单调增加趋向于$+\infty$且使得$\lim\limits_{n\to\infty} f'(x_n) = 0$.