13C 第8章补充题

13C.1 第8章补充题解答

1. 讨论下列级数的收敛性.

(1)$\sum_{n=1}^{\infty}\frac{n!}{n^n}a^n(a>0)$； (2)$\sum_{n=2}^{\infty}\frac{2}{2^{\ln n}}$；

(3)$\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}$； (4)$\sum_{n=1}^{\infty}\frac{\sin^2(\pi\sqrt{n^2+n})}{n}$；

(5)$\sum_{n=2}^{\infty}\ln(1+\frac{(-1)^n}{n^p})(p>0)$；

(6)$\sum_{n=1}^{\infty}(-1)^n(\mathrm{e}^{\frac{1}{\sqrt{n}}}-1-\frac{1}{\sqrt{n}})$；

(7)$\sum_{n=2}^{\infty}\sin(n\pi+\frac{1}{\ln n})$.

解：(1)$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{(n+1)!a^{n+1}}{(n+1)^{n+1}}\frac{n^n}{n!a^n}=\lim\limits_{n\to\infty}a\frac{1}{(1+\frac{1}{n})^n}=\frac{a}{\mathrm{e}}$

$\therefore$当$a<\mathrm{e}$时级数收敛，当$a>\mathrm{e}$时级数发散；

当$a=\mathrm{e}$时，由函数$f(x)=(1+\frac{1}{x})^x$在$x\geq 1$时单调增加(见教材P160例5.1.3)且$\lim\limits_{n\to\infty}(1+\frac{1}{x})^x=\mathrm{e}$可知$(1+\frac{1}{n})^n<\mathrm{e}$

$\therefore\frac{u_{n+1}}{u_n}=\mathrm{e}\frac{1}{(1+\frac{1}{n})^n}>1$

$\therefore u_{n+1}>u_n>\cdots>u_2>u_1=\mathrm{e},\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\frac{n!a^n}{n^n}\neq 0$，级数发散.

(2)$\sum_{n=2}^{\infty}\frac{2}{2^{\ln n}}=\sum_{n=2}^{\infty}\frac{2}{\mathrm{e}^{\ln 2\ln n}}=\sum_{n=2}^{\infty}\frac{2}{n^{\ln 2}}$

$\because 0<\ln 2<1$

$\therefore$级数发散.

(3)$\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}=\sum_{n=2}^{\infty}\frac{1}{\mathrm{e}^{\ln(\ln n)\ln n}}=\sum_{n=2}^{\infty}\frac{1}{n^{\ln(\ln n)}}$

$\because\lim\limits_{n\to\infty}n^2\cdot\frac{1}{n^{\ln(\ln n)}}=\lim\limits_{n\to\infty}\frac{1}{n^{\ln(\ln n)-2}}=0,$

$\therefore$该级数收敛.

(4)$\because\sin(\pi\sqrt{n^2+n})=\sin[(\pi\sqrt{n^2+n}-\pi n)+\pi n]=(-1)^n\sin(\pi\sqrt{n^2+n}-\pi n)$

$=(-1)^n\sin\frac{n\pi}{\sqrt{n^2+n}+n}=(-1)^n\sin\frac{\pi}{\sqrt{1+\frac{1}{n}}+1}$

$\therefore\sin^2(\pi\sqrt{n^2+n})=\sin^2\frac{\pi}{\sqrt{1+\frac{1}{n}}+1}$

$\because 2<\sqrt{1+\frac{1}{n}}+1\leq 1+\sqrt{2}$

$\therefore 1>\sin^2\frac{\pi}{\sqrt{1+\frac{1}{n}}+1}\geq\sin^2\frac{\pi}{1+\sqrt{2}}$

$\therefore\frac{\sin^2(\pi\sqrt{n^2+n})}{n}=\frac{\sin^2\frac{\pi}{\sqrt{1+\frac{1}{n}}+1}}{n}\geq\frac{\sin^2\frac{\pi}{1+\sqrt{2}}}{n}$

$\because\sum_{n=1}^{\infty}\frac{\sin^2\frac{\pi}{1+\sqrt{2}}}{n}$发散

$\therefore\frac{\sin^2(\pi\sqrt{n^2+n})}{n}$发散.

(5)$\because \lim\limits_{n\to\infty} n^p|\ln(1+\frac{(-1)^n}{n^p})| = \lim\limits_{n\to\infty}|n^p\frac{(-1)^n}{n^p}| = 1$

$\therefore$当$p>1$时，该级数绝对收敛

$\ln(1+\frac{(-1)^n}{n^p}) = \frac{(-1)^n}{n^p} - \frac{1}{2n^{2p}} + o(\frac{1}{n^{2p}})$

当$\frac{1}{2}<p\le 1$时，级数$\sum_{n=2}^{\infty}\frac{(-1)^n}{n^p}$条件收敛

$\because \lim\limits_{n\to\infty} n^{2p}|-\frac{1}{2n^{2p}}+o(\frac{1}{n^{2p}})| = \frac{1}{2}$

$\therefore$级数$\sum_{n=2}^{\infty}[-\frac{1}{2n^{2p}}+o(\frac{1}{n^{2p}})]$绝对收敛

$\because |\frac{(-1)^n}{n^p} - \frac{1}{2n^{2p}} + o(\frac{1}{n^{2p}})| \ge \frac{1}{n^p} - |\frac{1}{n^{2p}}+o(\frac{1}{n^{2p}})|$

$\because \sum_{n=2}^{\infty}[\frac{1}{n^p} - |\frac{1}{n^{2p}}+o(\frac{1}{n^{2p}})|] = +\infty$

$\therefore \sum_{n=2}^{\infty}|\ln(1+\frac{(-1)^n}{n^p})| = \sum_{n=2}^{\infty}|\frac{(-1)^n}{n^p} - \frac{1}{2n^{2p}} + o(\frac{1}{n^{2p}})|$发散

$\therefore$级数$\sum_{n=2}^{\infty}\ln(1+\frac{(-1)^n}{n^p})$条件收敛

当$0<p\le\frac{1}{2}$时，级数$\sum_{n=2}^{\infty}\frac{(-1)^n}{n^p}$条件收敛

$\because \lim\limits_{n\to\infty} n^{2p}|-\frac{1}{2n^{2p}}+o(\frac{1}{n^{2p}})| = \frac{1}{2}$

$\therefore$级数$\sum_{n=2}^{\infty}[-\frac{1}{2n^{2p}}+o(\frac{1}{n^{2p}})]$发散

$\therefore$级数$\sum_{n=2}^{\infty}\ln(1+\frac{(-1)^n}{n^p}) = \sum_{n=2}^{\infty}[\frac{(-1)^n}{n^p} - \frac{1}{2n^{2p}} + o(\frac{1}{n^{2p}})]$发散

综上所述，当$p>1$时原级数绝对收敛，当$\frac{1}{2}<p\le 1$时原级数条件收敛，当$0<p\le\frac{1}{2}$时原级数发散.

(6)令$f(x)=\mathrm{e}^x-1-x$, 则当$0<x\le 1$时$f'(x)=\mathrm{e}^x-1>0$

$\therefore |u_{n+1}| = f(\frac{1}{\sqrt{n+1}}) < f(\frac{1}{\sqrt{n}}) = |u_n|$

又$\because \lim\limits_{n\to\infty}(\mathrm{e}^{\frac{1}{\sqrt{n}}}-1-\frac{1}{\sqrt{n}}) = 1-1-0=0$

$\therefore \sum_{n=1}^{\infty}(-1)^n(\mathrm{e}^{\frac{1}{\sqrt{n}}}-1-\frac{1}{\sqrt{n}})$是莱布尼茨交错级数

$\therefore \sum_{n=1}^{\infty}(-1)^n(\mathrm{e}^{\frac{1}{\sqrt{n}}}-1-\frac{1}{\sqrt{n}})$收敛

$\because \lim\limits_{n\to\infty} n\cdot|u_n| = \lim\limits_{n\to\infty} n\cdot|\mathrm{e}^{\frac{1}{\sqrt{n}}}-1-\frac{1}{\sqrt{n}}| = \lim\limits_{n\to\infty} n\cdot|1+\frac{1}{\sqrt{n}}+\frac{1}{2n}+o(\frac{1}{n})-1-\frac{1}{\sqrt{n}}|$
$= \lim\limits_{n\to\infty} n\cdot|1+\frac{1}{\sqrt{n}}+\frac{1}{2n}+o(\frac{1}{n})-1-\frac{1}{\sqrt{n}}| = \frac{1}{2}$

$\therefore \sum_{n=1}^{\infty}|u_n|$发散

$\therefore$级数$\sum_{n=1}^{\infty}(-1)^n(\mathrm{e}^{\frac{1}{\sqrt{n}}}-1-\frac{1}{\sqrt{n}})$条件收敛.

(7)$\sin(n\pi+\frac{1}{\ln n}) = \sin n\pi\sin\frac{1}{\ln n} + \cos n\pi\sin\frac{1}{\ln} = (-1)^n\sin\frac{1}{\ln n}$

$\because \frac{1}{\ln 2} = 1.4427 < \frac{\pi}{2}$

$\therefore$当$n>2$时$\sin\frac{1}{\ln n} > \sin\frac{1}{\ln(n+1)}$

又$\because \lim\limits_{n\to\infty}\sin\frac{1}{\ln n} = 0$

$\therefore$级数$\sum_{n=2}^{\infty}\sin(n\pi+\frac{1}{\ln n})$是莱布尼茨交错级数，故收敛

$\because \lim\limits_{n\to\infty} n\cdot|\sin(n\pi+\frac{1}{\ln n})| = \lim\limits_{n\to\infty} n\cdot|\sin\frac{1}{\ln n}| = \lim\limits_{n\to\infty} n\cdot\frac{1}{\ln n} = +\infty$

$\therefore \sum_{n=2}^{\infty}|u_n|$发散

$\therefore$级数$\sum_{n=2}^{\infty}\sin(n\pi+\frac{1}{\ln n})$条件收敛.

2. 设$a_n\geq 0$且$\sum_{n=1}^{\infty}a_n$发散，讨论下列级数的收敛性：

(1)$\sum_{n=1}^{\infty}\frac{a_n}{1+a_n}$；　(2)$\sum_{n=1}^{\infty}\frac{a_n}{1+n^2a_n}$；　(3)$\sum_{n=1}^{\infty}\frac{a_n}{1+a_n^2}$.

解：(1)假设$\sum_{n=1}^{\infty}\frac{a_n}{1+a_n}$收敛，则$\lim\limits_{n\to\infty}\frac{a_n}{1+a_n} = \lim\limits_{n\to\infty}\frac{1}{\frac{1}{a_n}+1} = 0$

$\therefore \lim\limits_{n\to\infty} a_n = 0$

$\therefore \lim\limits_{n\to\infty}\frac{1}{a_n}\cdot\frac{a_n}{1+a_n} = 1$

$\because$级数$\sum_{n=1}^{\infty}a_n$发散

$\therefore$级数$\sum_{n=1}^{\infty}\frac{a_n}{1+a_n}$发散，与假设矛盾

$\therefore$级数$\sum_{n=1}^{\infty}\frac{a_n}{1+a_n}$发散.

(2)$\because \lim\limits_{n\to\infty} n^2\cdot\frac{a_n}{1+n^2a_n} = \lim\limits_{n\to\infty}\frac{a_n}{\frac{1}{n^2}+a_n} = 1$

$\therefore$级数$\sum_{n=1}^{\infty}\frac{a_n}{1+n^2a_n}$收敛.

(3)当$a_n=n$时，$\sum_{n=1}^{\infty}\frac{a_n}{1+a_n^2} = \sum_{n=1}^{\infty}\frac{n}{1+n^2}$，由$\lim\limits_{n\to\infty} n\cdot\frac{n}{1+n^2} = 1$知该级数发散

当$a_n=n^2$时，$\sum_{n=1}^{\infty}\frac{a_n}{1+a_n^2} = \sum_{n=1}^{\infty}\frac{n^2}{1+n^4}$，由$\lim\limits_{n\to\infty} n^2\cdot\frac{n^2}{1+n^4} = 1$知该级数收敛

$\therefore$该级数可能收敛也可能发散.

3. 设$a>0, b_n = \frac{a^{\frac{n(n+1)}{2}}}{[(1+a)(1+a^2)\cdots(1+a^n)]}$，讨论级数$\sum_{n=1}^{\infty}b_n$的收敛性.

解：$b_n = \frac{a^{1+2+\cdots+n}}{(1+a)(1+a^2)\cdots(1+a^n)} = \frac{a}{1+a}\cdot\frac{a^2}{1+a^2}\cdot\cdots\cdot\frac{a^n}{1+a^n}$

当$a=1$时$b_n=\frac{1}{2^n}$, 级数$\sum_{n=1}^{\infty}b_n$收敛

$\because f(x)=\frac{x}{1+x} = 1-\frac{1}{1+x}$单调增加

$\therefore$当$0<a<1$时$b_n = \frac{a}{1+a}\cdot\frac{a^2}{1+a^2}\cdot\cdots\cdot\frac{a^n}{1+a^n} < \frac{a}{1+a}\cdot\frac{a}{1+a}\cdot\cdots\cdot\frac{a}{1+a} = (\frac{a}{1+a})^n$

$\because$级数$\sum_{n=1}^{\infty}(\frac{a}{1+a})^n$收敛

【或者：$b_n = \frac{a^{1+2+\cdots+n}}{(1+a)(1+a^2)\cdots(1+a^n)} < a^{1+2+\cdots+n} = a^1\cdot a^2\cdot\cdots\cdot a^n < a^n$, 级数$\sum_{n=1}^{\infty}a^n$收敛】

$\therefore$级数$\sum_{n=1}^{\infty}b_n$收敛

当$a>1$时$b_n = \frac{a}{1+a}\cdot\frac{a^2}{1+a^2}\cdot\cdots\cdot\frac{a^n}{1+a^n} = \frac{1}{(1+\frac{1}{a})(1+\frac{1}{a^2})\cdots(1+\frac{1}{a^n})} = \frac{1}{e^{\ln[(1+\frac{1}{a})(1+\frac{1}{a^2})\cdots(1+\frac{1}{a^n})]}}$

$= \frac{1}{e^{\ln(1+\frac{1}{a})+\ln(1+\frac{1}{a^2})+\cdots+\ln(1+\frac{1}{a^n})}} > \frac{1}{e^{\frac{1}{a}+\frac{1}{a^2}+\cdots+\frac{1}{a^n}}} = \frac{1}{e^{\frac{\frac{1}{a}(1-\frac{1}{a^n})}{1-\frac{1}{a}}}} > \frac{1}{e^{\frac{1}{a-1}}}$

$\therefore \lim\limits_{n\to\infty} b_n \neq 0$, 故级数$\sum_{n=1}^{\infty}b_n$发散

综上所述，当$0<a\leq 1$时，级数$\sum_{n=1}^{\infty}b_n$收敛；当$a>1$时，级数$\sum_{n=1}^{\infty}b_n$发散.

4. 设$a_n>0$且$\lim\limits_{n\to\infty}\frac{\ln\frac{1}{a_n}}{\ln n}=l>1$，求证$\sum_{n=1}^{\infty}a_n$收敛.

证明：$\because \lim\limits_{n\to\infty}\frac{\ln\frac{1}{a_n}}{\ln n}=\lim\limits_{n\to\infty}\frac{\ln a_n}{\ln\frac{1}{n}}=l>1$

$\therefore$根据数列极限的保号性知$\exists\varepsilon>0$使得$|\frac{\ln a_n}{\ln\frac{1}{n}}-l|<\varepsilon$即$\frac{\ln a_n}{\ln\frac{1}{n}}>l-\varepsilon$, ε应满足$l-\varepsilon>1$

$\therefore$当$n\geq 2$时$\ln a_n<(l-\varepsilon)\ln\frac{1}{n}=\ln\frac{1}{n^{l-\varepsilon}}$

$\therefore a_n<\frac{1}{n^{l-\varepsilon}}(n\geq 2)$

$\because$当$l-\varepsilon>1$时$\sum_{n=2}^{\infty}\frac{1}{n^{l-\varepsilon}}$收敛

$\therefore\sum_{n=1}^{\infty}a_n$收敛.

5. 已知$\lim\limits_{n\to\infty}(n^{2n\sin\frac{1}{n}}\cdot a_n)=1$，试讨论$\sum_{n=1}^{\infty}a_n$的收敛性.

解：$\because \lim\limits_{n\to\infty}(n^{2n\sin\frac{1}{n}}\cdot a_n)=\lim\limits_{n\to\infty}(n^{2n\sin\frac{1}{n}-1.5}n^{1.5}\cdot a_n)=\lim\limits_{n\to\infty}[\mathrm{e}^{(2n\sin\frac{1}{n}-1.5)\ln n}n^{1.5}\cdot a_n]$

$=\lim\limits_{n\to\infty}\{\mathrm{e}^{[2n(\frac{1}{n}+o(\frac{1}{n^2}))-1.5]\ln n}n^{1.5}\cdot a_n\}=\lim\limits_{n\to\infty}\{\mathrm{e}^{(2+o(\frac{1}{n})-1.5)\ln n}n^{1.5}\cdot a_n\}$

$=\lim\limits_{n\to\infty}\{\mathrm{e}^{(0.5+o(\frac{1}{n}))\ln n}n^{1.5}\cdot a_n\}=1$

$\because \lim\limits_{n\to\infty}\mathrm{e}^{(0.5+o(\frac{1}{n}))\ln n}=+\infty$

$\therefore \lim\limits_{n\to\infty}n^{1.5}\cdot a_n=\lim\limits_{n\to\infty}\frac{1}{\mathrm{e}^{(0.5+o(\frac{1}{n}))\ln n}}=0$

$\therefore$级数$\sum_{n=1}^{\infty}a_n$收敛.

6. 设$p>0,\lim\limits_{n\to\infty}[n^p(\mathrm{e}^{\frac{1}{n}}-1)a_n]=1$，试讨论$\sum_{n=1}^{\infty}a_n$的收敛性.

解：$\because \lim\limits_{n\to\infty}n^p(\mathrm{e}^{\frac{1}{n}}-1)a_n=\lim\limits_{n\to\infty}n^{p-1}\frac{\mathrm{e}^{\frac{1}{n}}-1}{\frac{1}{n}}a_n=\lim\limits_{n\to\infty}n^{p-1}a_n=1$

$\therefore$当$p-1>1$即$p>2$时，级数收敛；

当$0<p-1\leq 1$即$1<p\leq 2$时，级数发散；

当$p=1$时$p-1=0$, $\lim\limits_{n\to\infty}a_n=1$，级数发散；

当$0<p<1$时$p-1<0$, $\lim\limits_{n\to\infty}a_n=+\infty$，级数发散.

7. 设$a_n>0,\sum_{n=1}^{\infty}a_n$发散，令$S_k=a_1+a_2+\cdots+a_k$，试证$\sum_{n=1}^{\infty}\frac{a_n}{S_n}$也发散.

证明：$\because a_n>0,\sum_{n=1}^{\infty}a_n$发散

$\therefore \lim\limits_{n\to\infty}S_n=+\infty$

$\therefore \sum_{k=n+1}^{n+p}\frac{a_k}{S_k}\geq\frac{a_{n+1}+a_{n+2}+\cdots+a_{n+p}}{S_{n+p}}=\frac{S_{n+p}-S_{n+1}}{S_{n+p}}=1-\frac{S_{n+1}}{S_{n+p}}\to 1,p\to\infty$

$\therefore \exists P>0,s.t.$当$p>P$时$\sum_{k=n+1}^{n+p}\frac{a_k}{S_k}\geq\frac{a_{n+1}+a_{n+2}+\cdots+a_{n+p}}{S_{n+p}}>1-\frac{1}{2}=\frac{1}{2}$

$\therefore$对于$\varepsilon_0=\frac{1}{2},\forall N>0$，如取$n=N+1,p=P+1$，则

$$|\sum_{k=n+1}^{n+p}\frac{a_k}{S_k}|\geq\frac{a_{n+1}+a_{n+2}+\cdots+a_{n+p}}{S_{n+p}}>\frac{1}{2}=\varepsilon_0$$

$\therefore$级数$\sum_{n=1}^{\infty}\frac{a_n}{S_n}$发散.

8. 设$\varphi(x)$在$(-\infty,+\infty)$上连续，周期为1，且$\int_0^1\varphi(x)\mathrm{d}x=0, f(x)$在$[0,1]$上连续可导，令$a_n=\int_0^1 f(x)\varphi(nx)\mathrm{d}x$，求证级数$\sum_{n=1}^{\infty}a_n^2$收敛.

证明：令$G(x)=\int_0^x\varphi(nx)\mathrm{d}x$, 则$G(1)=0$

$\therefore$

$$a_n=\int_0^1 f(x)\varphi(nx)\mathrm{d}x=f(x)G(x)\big|_0^1-\int_0^1 G(x)f'(x)\mathrm{d}x=-\int_0^1 G(x)f'(x)\mathrm{d}x$$

$\because f(x)$在$[0,1]$上连续可导，$\varphi(x)$在$(-\infty,+\infty)$上连续

$\therefore \exists M_1=\max|f'(x)|, \exists M_2=\max|\varphi(x)|$

$\therefore$

$$|a_n|\le\int_0^1|G(x)||f(x)|\mathrm{d}x\le M_1\int_0^1|G(x)|\mathrm{d}x$$

$\because\int_0^1\varphi(x)\mathrm{d}x=0$

$\therefore$

$$|G(x)|=|\int_0^x\varphi(nt)\mathrm{d}t|=\frac{1}{n}|\int_0^{nx}\varphi(u)\mathrm{d}u|=\frac{1}{n}|\int_{[nx]}^{nx}\varphi(u)\mathrm{d}u|\le\frac{1}{n}M_2$$

$\therefore a_n\le\frac{1}{n}M_1M_2,\ a_n^2\le\frac{1}{n^2}(M_1M_2)^2$

$\therefore$级数$\sum_{n=1}^{\infty}a_n^2$收敛.

9. 确定下列函数级数的收敛域：

(1)$\sum_{n=1}^{\infty}\frac{x^{n^2}}{2^n}$； (2)$\sum_{n=1}^{\infty}\frac{n}{x^n}$；
(3)$\sum_{n=1}^{\infty}n!(\frac{x}{n})^n$； (4)$\sum_{n=1}^{\infty}(1-\cos\frac{x}{n})$；
(5)$\sum_{n=1}^{\infty}\sin\frac{1}{n^{\frac{x+1}{x}}}$； (6)$\sum_{n=1}^{\infty}\frac{x^n}{1+x^{2n}}$.

解：(1) $\lim\limits_{n\to\infty}\sqrt[n]{|\frac{x^{n^2}}{2^n}|}=\lim\limits_{n\to\infty}\frac{|x|^n}{2}=\begin{cases}0, & |x|<1\\ \frac{1}{2}, & x=1\\ \frac{1}{2}, & x=-1\\ \infty, & |x|>1\end{cases}$

$\therefore$收敛域为$[-1,1]$.

(2) $\lim\limits_{n\to\infty}\sqrt[n]{|\frac{n}{x^n}|}=\lim\limits_{n\to\infty}\frac{\sqrt[n]{n}}{|x|}=\frac{1}{|x|}$

$\therefore$当$|x|>1$时级数绝对收敛；当$|x|<1$时级数发散；当$|x|=1$时，$\lim\limits_{n\to\infty}\frac{n}{x^2}\ne0$, 故发散

$\therefore$级数的收敛域为$(-\infty,-1)\cup(1,+\infty)$.

(3) $\lim\limits_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\frac{(n+1)!(|\frac{x}{n+1}|)^{n+1}}{n!(|\frac{x}{n}|)^n}=\lim\limits_{n\to\infty}(\frac{n}{n+1})^n|x|=\lim\limits_{n\to\infty}\frac{1}{(1+\frac{1}{n})^n}|x|=\frac{|x|}{\mathrm{e}}$

$\therefore$当$|x|<\mathrm{e}$时级数绝对收敛；当$|x|>\mathrm{e}$时级数发散

当$|x|=\mathrm{e}$时，$\frac{|u_{n+1}|}{|u_n|}=\frac{\mathrm{e}}{(1+\frac{1}{n})^n}$

由函数$f(x)=(1+\frac{1}{x})^x$在$x\geq 1$时单调增加(见教材P160例5.1.3)且$\lim\limits_{n\to\infty}(1+\frac{1}{x})^x=\mathrm{e}$

$\therefore \frac{|u_{n+1}|}{|u_n|}>1, |u_{n+1}|>|u_n|>\cdots>|u_1|$

$\therefore \lim\limits_{n\to\infty}|u_n|\neq 0$, 级数发散

综上所述，该级数的收敛域为$(-\mathrm{e},\mathrm{e})$.

(4)$1-\cos\frac{x}{n}=2\sin^2\frac{x}{2n}$

$\because \lim\limits_{n\to\infty}n^2\cdot(1-\cos\frac{x}{n})=\lim\limits_{n\to\infty}n^2\cdot 2\sin^2\frac{x}{2n}=\lim\limits_{n\to\infty}2(\frac{\sin\frac{x}{2n}}{\frac{1}{n}})^2=2(\frac{x}{2})^2=\frac{x^2}{2}$

$\therefore$该级数的收敛域为$(-\infty,+\infty)$.

(5)$\lim\limits_{n\to\infty}n^{\frac{1+1+\frac{1}{x}}{2}}\cdot\sin\frac{1}{n^{\frac{x+1}{x}}}=\lim\limits_{n\to\infty}n^{1+\frac{1}{2x}}\cdot\frac{1}{n^{\frac{x+1}{x}}}=\lim\limits_{n\to\infty}n^{-\frac{1}{2x}}=\begin{cases}0, & x>0\\ +\infty, & x<0\end{cases}$

$\because$当$x>0$时，级数$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1+1+\frac{1}{x}}{2}}}$收敛；当$x<0$时，级数$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1+1+\frac{1}{x}}{2}}}$发散

$\therefore$原级数的收敛域为$(0,+\infty)$.

(6)$\because \lim\limits_{n\to\infty}\sqrt[n]{|\frac{x^n}{1+x^{2n}}|}=\lim\limits_{n\to\infty}\frac{|x|}{\sqrt[n]{1+x^{2n}}}=\begin{cases}|x|, & |x|<1\\ 1, & |x|=1\\ 0, & |x|>1\end{cases}$

$\therefore$当$x\neq 1$时，该级数收敛

当$x=1$时$\sum_{n=1}^{\infty}\frac{x^n}{1+x^{2n}}=\sum_{n=1}^{\infty}\frac{1}{2}$，该级数发散

$\therefore$该级数的收敛域为$(-\infty,1)\cup(1+\infty)$.

10. 讨论下列函数级数在指定区间上的一致收敛性：

(1)$\sum_{n=1}^{\infty}n\mathrm{e}^{-nx},(0,+\infty)$与$[\delta,+\infty),\delta>0$为常数；

(2)$\sum_{n=1}^{\infty}\frac{x^2}{(1+x^2)^n},(0,+\infty)$与$[\delta,+\infty),\delta>0$；

(3)$\sum_{n=1}^{\infty}(-1)^n\frac{1}{x+n},[0,+\infty)$.

解：(1)在区间$(0,+\infty)$上：

$\forall N>0$, 取$n_0=N+1,p_0\geq 1,0<x_0<\frac{\ln[p_0(n_0+1)]}{n_0+p_0}$, 则

$$|\sum_{k=n_0+1}^{n_0+p_0}k\mathrm{e}^{-kx_0}|\geq|\sum_{k=n_0+1}^{n_0+p_0}(n_0+1)\mathrm{e}^{-(n_0+p_0)x_0}|=p_0(n_0+1)\mathrm{e}^{-(n_0+p_0)x_0}=1$$

故级数$\sum_{n=1}^{\infty}n\mathrm{e}^{-nx}$在区间$(0,+\infty)$上不一致收敛.

在区间$[\delta,+\infty),\delta>0$上：

$\because \lim\limits_{n\to\infty} n^2\cdot n\mathrm{e}^{-nx}=0$

$\therefore$级数$\sum_{n=1}^{\infty} n\mathrm{e}^{-nx}$收敛

$\because n\mathrm{e}^{-nx}<n\mathrm{e}^{-n\delta}$

又$\because \lim\limits_{n\to\infty} n^2\cdot n\mathrm{e}^{-n\delta}=0$, 故级数$\sum_{n=1}^{\infty} n\mathrm{e}^{-n\delta}$在区间$[\delta,+\infty)$上收敛

$\therefore$级数$\sum_{n=1}^{\infty} n\mathrm{e}^{-nx}$在区间$[\delta,+\infty)$上一致收敛.

(2)该级数的部分和序列$S_n(x)=\sum_{k=1}^{n}\frac{x^2}{(1+x^2)^k}=\frac{x^2}{1+x^2}\frac{1-\frac{1}{(1+x^2)^n}}{1-\frac{1}{1+x^2}}=1-\frac{1}{(1+x^2)^n}$

和函数$S(x)=\sum_{n=1}^{\infty}\frac{x^2}{(1+x^2)^n}=\frac{x^2}{1+x^2}\frac{1}{1-\frac{1}{1+x^2}}=1$

当$x\in(0,+\infty)$时：

$\forall N>0$, 取$n_0=N+1, 0<x_{n_0}<\sqrt{2^{\frac{1}{n_0}}-1}$, 则

$$|S_{n_0}(x)-S(x)|=\frac{1}{(1+x_{n_0}^2)^{n_0}}>\frac{1}{2}$$

故该级数不一致收敛.

当$x\in[\delta,+\infty)$时：

$\forall\varepsilon>0$, 取$N=\max\{[\frac{\ln\frac{1}{\varepsilon}}{\ln(1+\delta^2)}]+1,1\}$, 当$n>N$时$n>\frac{\ln\frac{1}{\varepsilon}}{\ln(1+\delta^2)}$, 故

$$|S_{n_0}(x)-S(x)|=\frac{1}{(1+x_n^2)^n}<\frac{1}{(1+\delta^2)^n}<\frac{1}{(1+\delta^2)^{\frac{\ln\frac{1}{\varepsilon}}{\ln(1+\delta^2)}}}<\frac{1}{\mathrm{e}^{\ln(1+\delta^2)\frac{\ln\frac{1}{\varepsilon}}{\ln(1+\delta^2)}}}<\varepsilon$$

$\therefore$该级数一致收敛.

(3)$\forall x\in[0,+\infty)$, 级数$\sum_{n=1}^{\infty}(-1)^n\frac{1}{x+n}$是莱布尼茨交错级数，故收敛

$\because \forall\varepsilon>0$, 取$N=\max\{[\frac{1}{\varepsilon}-x-1]+1,1\}$, 则当$n>N$时，$\forall x\in[0,+\infty)$, $\forall p\in\mathbb{Z}^+$,

$$|\sum_{k=n+1}^{n+p}(-1)^n\frac{1}{x+n}|\le\frac{1}{x+n+1}<\varepsilon$$

$\therefore$该级数一致收敛.

11. 设函数$f(x)$在$(-\infty,+\infty)$上有任意阶导数，记$f_n(x)=f^{(n)}(x)(n=1,2,\cdots)$，设函数序列$\{f_n(x)\}$在任意有限区间上一致收敛于某个函数$\varphi(x)$，求证：存在常数$c$，使$\varphi(x)=c\mathrm{e}^x$.

证明：$\because f(x)$在$(-\infty,+\infty)$上有任意阶导数

$\therefore f^{(n)}(x)$在任意区间$[a,b]$上连续可微

$\because \{f^{(n)}\}$在$[a,b]$内一致收敛于$\varphi(x)$

$\therefore$其导函数序列$\{f^{(n)}(x)\}$在$[a,b]$内也一致收敛于$\varphi(x)$

$\therefore$函数$\varphi(x)$在$[a,b]$上可导，且

$$\varphi'(x)=[\lim_{n\to\infty}f^{(n)}(x)]'=\lim_{n\to\infty}f^{(n+1)(x)}=\varphi(x)$$

即

$$\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}=\varphi(x)$$

$\therefore$

$$\frac{\mathrm{d}\varphi(x)}{\varphi(x)}=\mathrm{d}x$$

$\therefore \ln\varphi(x)=x+C$, 即$\varphi(x)=\mathrm{e}^C\mathrm{e}^x=c\mathrm{e}^x$, 其中$c=\mathrm{e}^C$.

12. 已知$\{a_n\}$是一单增有界的正数列，试证级数$\sum_{n=1}^{\infty}(1-\frac{a_n}{a_{n+1}})$收敛.

证明：$\because \{a_n\}$是一单增有界的正数列

$\therefore \lim\limits_{n\to\infty}a_n$存在，不妨设为$A$

$\because \sum_{k=1}^{n}(1-\frac{a_k}{a_{k+1}})=\sum_{k=1}^{n}\frac{a_{k+1}-a_k}{a_{k+1}}\le\sum_{k=1}^{n}\frac{a_{k+1}-a_k}{a_2}=\frac{1}{a_2}(a_{n+1}-a_1)\to\frac{1}{a_2}(A-a_1),n\to\infty$

$\therefore \sum_{k=1}^{n}(1-\frac{a_k}{a_{k+1}})$有界

又$\because 1-\frac{a_n}{a_{n+1}}>0$

$\therefore$根据单调有界收敛定理，级数$\sum_{n=1}^{\infty}(1-\frac{a_n}{a_{n+1}})$收敛.

13. 设a_n是方程$\tan\sqrt{x}=x$的正根$(n=1,2,\cdots)$. 研究$\sum_{n=1}^{\infty}\frac{1}{a_n}$是否收敛.

解：令$y=\sqrt{x}$, 则$\tan\sqrt{x}=x\Leftrightarrow\tan y=y^2$

$\because a_n$是方程$\tan\sqrt{x}=x$的正根$(n=1,2,\cdots)$

$\therefore$由$\tan y$和y^2的图像可知：

$$\pi<\sqrt{a_1}<\frac{3}{2}\pi,\ 2\pi<\sqrt{a_2}<\frac{5}{2}\pi,\cdots,n\pi<\sqrt{a_n}<n\pi+\frac{\pi}{2},\cdots$$

$\therefore a_n>(n\pi)^2,n=1,2,\cdots$

$\therefore \frac{1}{a_n}<\frac{1}{n^2\pi^2},n=1,2,\cdots$

$\because$级数$\sum_{n=1}^{\infty}\frac{1}{n^2}$收敛

$\therefore$级数$\sum_{n=1}^{\infty}\frac{1}{a_n}$收敛.

14. 判定级数$\sum_{n=1}^{\infty}(\ln n+\ln\sin\frac{1}{n})$的收敛性.

解：$\because \sin\frac{1}{n}<\frac{1}{n}(n\ge 1)$

$\therefore \ln n+\ln\sin\frac{1}{n}=\ln\sin\frac{1}{n}-\ln\frac{1}{n}<0$

$\because \ln n+\ln\sin\frac{1}{n}=\ln\sin\frac{1}{n}-\ln\frac{1}{n}=\ln\frac{\sin\frac{1}{n}}{\frac{1}{n}}=\ln\frac{\frac{1}{n}-\frac{1}{3!n^3}+o(\frac{1}{n^3})}{\frac{1}{n}}=\ln[1-\frac{1}{6n^2}+o(\frac{1}{n^2})]$

$\therefore \lim\limits_{n\to\infty}n^2[-(\ln n+\ln\sin\frac{1}{n})]=-\lim\limits_{n\to\infty}n^2\ln[1-\frac{1}{6n^2}+o(\frac{1}{n^2})]=-\lim\limits_{n\to\infty}n^2[-\frac{1}{6n^2}+o(\frac{1}{n^2})]=\frac{1}{6}$

$\therefore$正项级数$\sum_{n=1}^{\infty}-(\ln n+\ln\sin\frac{1}{n})$收敛，故级数$\sum_{n=1}^{\infty}(\ln n+\ln\sin\frac{1}{n})$收敛.

15. 设正项数列$\{a_n\}$单调减少且$\sum_{n=1}^{\infty}(-1)^n a_n$发散，试问级数$\sum_{n=1}^{\infty}(\frac{1}{1+a_n})^n$是否收敛，并说明理由.

解：$\because \{a_n\}$是正项数列且单调减少

$\therefore \{a_n\}$的极限$\lim\limits_{n\to\infty}a_n=A$存在且$A\geq 0$

$\because \sum_{n=1}^{\infty}(-1)^n a_n$发散

$\therefore A\neq 0$, 即$A>0$

$\because \lim\limits_{n\to\infty}\sqrt[n]{(\frac{1}{1+a_n})^n}=\lim\limits_{n\to\infty}\frac{1}{1+a_n}=\frac{1}{1+A}<1$

$\therefore$级数$\sum_{n=1}^{\infty}(\frac{1}{1+a_n})^n$收敛.

16. 试证函数级数$\sum_{n=1}^{\infty}\frac{nx}{1+n^5x^2}$在其收敛域内一致收敛.

证明：$\because |\frac{nx}{1+n^5x^2}|=\frac{n}{\frac{1}{|x|}+n^5|x|}\leq\frac{n}{2\sqrt{n^5}}=\frac{1}{2n^{\frac{3}{2}}}$，$x\neq 0$

又$\because$级数$\sum_{n=1}^{\infty}\frac{1}{2n^{\frac{3}{2}}}$收敛

$\therefore$级数$\sum_{n=1}^{\infty}\frac{nx}{1+n^5x^2}$在其收敛域内一致收敛.

17. 设$u_n>0, v_n>0, \frac{u_{n+1}}{u_n}\leq\frac{v_{n+1}}{v_n}(n=1,2,\cdots)$. 证明由$\sum_{n=1}^{\infty}v_n$收敛可以推出$\sum_{n=1}^{\infty}u_n$收敛.

解：$\because \frac{u_{n+1}}{u_n}\leq\frac{v_{n+1}}{v_n}$

$\therefore \frac{u_{n+1}}{v_{n+1}}\leq\frac{u_n}{v_n}\leq\frac{u_{n-1}}{v_{n-1}}\leq\cdots\leq\frac{u_1}{v_1}$

$\therefore u_n\leq\frac{u_1}{v_1}v_n$

$\because \sum_{n=1}^{\infty}v_n$收敛

$\therefore \sum_{n=1}^{\infty}u_n$收敛.

18. 设$\lim\limits_{n\to\infty}a_n>1$. 求证$\sum_{n=1}^{\infty}\frac{1}{n^{a_n}}$收敛.

证明：$\because \lim\limits_{n\to\infty}a_n=A>1$

$\therefore \exists N>0, s.t. a_n>\frac{1+A}{2}=q>1(n>N)$

$\therefore$当$n>N$时$0<\frac{1}{n^{a_n}}<\frac{1}{n^q}$

$\because \sum_{n=N}^{\infty}\frac{1}{n^q}$收敛，故$\sum_{n=N}^{\infty}\frac{1}{n^{a_n}}$收敛，故$\sum_{n=1}^{\infty}\frac{1}{n^{a_n}}$收敛.

19. 研究下列级数的收敛性：

(1)$\sum_{n=1}^{\infty}\int_0^{n^{-p}}\ln(1+x^2)\mathrm{d}x(p>0)$；　(2)$\sum_{n=1}^{\infty}\int_0^{\frac{1}{n}}(\mathrm{e}^{\sqrt{x}}-1)\mathrm{d}x$；

(3)$\sum_{n=1}^{\infty}\int_0^{\frac{1}{\sqrt{n}}}(\mathrm{e}^{\sqrt{x}}-1)\mathrm{d}x$；　(4)$\sum_{n=1}^{\infty}\int_n^{n+1}\mathrm{e}^{\frac{1}{x}}\mathrm{d}x$.

解：(1)$\int_0^{n^{-p}}\ln(1+x^2)\mathrm{d}x=x\ln(1+x^2)\Big|_0^{n^{-p}}-\int_0^{n^{-p}}x\frac{2x}{1+x^2}\mathrm{d}x$

$=n^{-p}\ln(1+n^{-2p})-2\int_0^{n^{-p}}(1-\frac{1}{1+x^2})\mathrm{d}x=n^{-p}\ln(1+n^{-2p})-2n^{-p}+2\arctan(n^{-p})$

$=n^{-p}[n^{-2p}+o(n^{-2p})]-2n^{-p}+2[n^{-p}-\frac{1}{6}n^{-3p}+o(n^{-3p})]=\frac{2}{3}n^{-3p}+o(n^{-3p})$

$\therefore\lim\limits_{n\to\infty}n^{3p}\int_0^{n^{-p}}\ln(1+x^2)\mathrm{d}x=\lim\limits_{n\to\infty}n^{3p}[\frac{2}{3}n^{-3p}+o(n^{-3p})]=\frac{2}{3}$

$\therefore$当$3p>1$即$p>\frac{1}{3}$时级数收敛，当$0<3p\le 1$即$p\le\frac{1}{3}$时级数发散.

(2)$\because\int_0^{\frac{1}{n}}(\mathrm{e}^{\sqrt{x}}-1)\mathrm{d}x\le\int_0^{\frac{1}{n}}(\mathrm{e}^{\frac{1}{\sqrt{n}}}-1)\mathrm{d}x=\frac{1}{n}(\mathrm{e}^{\frac{1}{\sqrt{n}}}-1)$

$\therefore\lim\limits_{n\to\infty}n^{\frac{3}{2}}\cdot\frac{1}{n}(\mathrm{e}^{\frac{1}{\sqrt{n}}}-1)=\lim\limits_{n\to\infty}\sqrt{n}(\frac{1}{\sqrt{n}})=1$

$\therefore$级数$\sum_{n=1}^{\infty}\frac{1}{n}(\mathrm{e}^{\frac{1}{\sqrt{n}}}-1)$收敛

$\therefore$级数$\sum_{n=1}^{\infty}\int_0^{\frac{1}{n}}(\mathrm{e}^{\sqrt{x}}-1)\mathrm{d}x$收敛.

(3)$\mathrm{e}^{\sqrt{x}}-1=1+\sqrt{x}+\frac{1}{2}(\sqrt{x})^2+\frac{1}{6}(\sqrt{x})^3+\cdots-1=\sqrt{x}+\frac{1}{2}(\sqrt{x})^2+\frac{1}{6}(\sqrt{x})^3+\cdots$

$\therefore\int_0^{\frac{1}{\sqrt{n}}}(\mathrm{e}^{\sqrt{x}}-1)\mathrm{d}x=(\frac{1}{1+\frac{1}{2}}x^{\frac{1}{2}+1}+\frac{1}{2}\frac{1}{1+1}x^{1+1}+\frac{1}{6}\frac{1}{1+\frac{3}{2}}x^{\frac{3}{2}+1}+\cdots)\Big|_0^{\frac{1}{\sqrt{n}}}=\frac{2}{3}\frac{1}{n^{\frac{3}{4}}}+\frac{1}{4}\frac{1}{n}+\frac{1}{15}\frac{1}{n^{\frac{5}{4}}}+\cdots$

$\therefore\lim\limits_{n\to\infty}n^{\frac{3}{4}}\cdot\int_0^{\frac{1}{\sqrt{n}}}(\mathrm{e}^{\sqrt{x}}-1)\mathrm{d}x=\lim\limits_{n\to\infty}n^{\frac{3}{4}}\cdot(\frac{2}{3}\frac{1}{n^{\frac{3}{4}}}+\frac{1}{4}\frac{1}{n}+\frac{1}{15}\frac{1}{n^{\frac{5}{4}}}+\cdots)=\frac{2}{3}$

$\therefore$级数$\sum_{n=1}^{\infty}\int_0^{\frac{1}{\sqrt{n}}}(\mathrm{e}^{\sqrt{x}}-1)\mathrm{d}x$发散.

(4)$\sum_{n=1}^{\infty}\int_n^{n+1}\mathrm{e}^{\frac{1}{x}}\mathrm{d}x=\int_0^{+\infty}\mathrm{e}^{\frac{1}{x}}\mathrm{d}x$

$\because\lim\limits_{n\to\infty}\sqrt{x}\cdot\mathrm{e}^{\frac{1}{x}}=+\infty$

$\therefore$无穷积分$\int_0^{+\infty}\mathrm{e}^{\frac{1}{x}}\mathrm{d}x$发散

$\therefore$级数$\sum_{n=1}^{\infty}\int_n^{n+1}\mathrm{e}^{\frac{1}{x}}\mathrm{d}x$发散.

20. 求函数级数$\sum_{n=1}^{\infty}x^{1+\frac{1}{2}+\cdots+\frac{1}{n}}$的收敛域.

解：【该题可用级数收敛的广义比值判定准则直接得到结果，即：对于正项级数$\sum_{n=1}^{\infty}a_n$，若$\lim\limits_{n\to\infty}(\frac{a_{n+1}}{a_n})^n=q$, 则当$0\le q<\frac{1}{\mathrm{e}}$时，级数$\sum_{n=1}^{\infty}a_n$收敛；当$q>\frac{1}{\mathrm{e}}$时，级数$\sum_{n=1}^{\infty}a_n$发散. 下面的证明过程相当于是广义比值判定准则的证明过程。】

$\because\lim\limits_{n\to\infty}(\frac{a_{n+1}}{a_n})^n=\lim\limits_{n\to\infty}(\frac{x^{1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}}}{x^{1+\frac{1}{2}+\cdots+\frac{1}{n}}})^n=\lim\limits_{n\to\infty}x^{\frac{n}{n+1}}=x$

(1)当$0\le x<\frac{1}{\mathrm{e}}$时$\lim\limits_{n\to\infty}[(\frac{a_{n+1}}{a_n})^n-(\frac{n}{n+1})^n]=x-\frac{1}{\mathrm{e}}<0$

根据数列极限的保号性知$\exists N>0$, 当$n>N$时$(\frac{a_{n+1}}{a_n})^n-(\frac{n}{n+1})^n<0$即$(\frac{a_{n+1}}{a_n})^n<(\frac{n}{n+1})^n$

$\because(\frac{n}{n+1})^n<1$

$\therefore\exists p>1,\ s.t.(\frac{a_{n+1}}{a_n})^n\le[(\frac{n}{n+1})^n]^p<(\frac{n}{n+1})^n$

$\therefore \frac{a_{n+1}}{\frac{1}{(n+1)^p}} \leq \frac{a_n}{\frac{1}{n^p}} \leq \cdots \leq \frac{a_1}{\frac{1}{1^p}}$

$\therefore a_n \leq a_1 \frac{1}{n^p}$

$\therefore \sum_{n=1}^{\infty} a_n$收敛；

(2)当$x \geq \frac{1}{\mathrm{e}}$时$\lim\limits_{n\to\infty}[(\frac{a_{n+1}}{a_n})^n - (\frac{n}{n+1})^n] = x - \frac{1}{\mathrm{e}} \geq 0$

根据数列极限的保号性知$\exists N > 0$, 当$n > N$时$(\frac{a_{n+1}}{a_n})^n - (\frac{n}{n+1})^n \geq 0$即$(\frac{a_{n+1}}{a_n})^n \geq (\frac{n}{n+1})^n$,

即$(\frac{a_{n+1}}{a_n})^n \geq (\frac{n}{n+1})^n$

$\therefore \frac{a_{n+1}}{\frac{1}{n+1}} \geq \frac{a_n}{\frac{1}{n}} \geq \cdots \geq \frac{a_1}{\frac{1}{1}}$

$\therefore a_n \geq a_1 \frac{1}{n}$

$\therefore \sum_{n=1}^{\infty} a_n$发散；

综上所述，级数$\sum_{n=1}^{\infty} a_n$的收敛域为$[0, \frac{1}{\mathrm{e}})$.

21. 设$p > 0$. 求证当且仅当$p > 1$时，曲线$y = x^p \cos\frac{\pi}{x}(0 < x \leq 1)$具有有限的长度.

证明：曲线长度可表示为

$$\int_0^1 \sqrt{1 + [y'(x)]^2}\mathrm{d}x = \int_0^1 \sqrt{1 + [px^{p-1}\cos\frac{\pi}{x} + \pi x^{p-2}\sin\frac{\pi}{x}]^2}\mathrm{d}x,$$

(1)当$p > 2$时该积分的被积函数有界，为一定积分，故原曲线有有限的长度；

(2)当$1 < p \leq 2$时该积分为以0为瑕点的反常积分，$\frac{1}{2} \leq \frac{1}{p} < 1,\ p + \frac{1}{p} > 2$,

$$\begin{aligned}
&\lim_{x\to 0} x^{\frac{1}{p}} \cdot \sqrt{1 + [px^{p-1}\cos\frac{\pi}{x} + \pi x^{p-2}\sin\frac{\pi}{x}]^2} \\
=&\lim_{x\to 0} \sqrt{x^{\frac{2}{p}} + [px^{p+\frac{1}{p}-1}\cos\frac{\pi}{x} + \pi x^{p+\frac{1}{p}-2}\sin\frac{\pi}{x}]^2} \\
=&0,
\end{aligned}$$

故该反常积分收敛，原曲线有有限的长度；

(3)当$0 < p \leq 1$时

$$\begin{aligned}
\int_0^1 \sqrt{1 + [y'(x)]^2}\mathrm{d}x \geq \int_0^1 |y'(x)|\mathrm{d}x = \int_0^1 |[x^p\cos\frac{\pi}{x}]'|\mathrm{d}x &= \sum_{n=1}^{\infty}\int_{\frac{1}{n+1}}^{\frac{1}{n}} |[x^p\cos\frac{\pi}{x}]'|\mathrm{d}x \\
&\geq \sum_{n=1}^{\infty} |\int_{\frac{1}{n+1}}^{\frac{1}{n}} [x^p\cos\frac{\pi}{x}]'\mathrm{d}x| = \sum_{n=1}^{\infty} |x^p\cos\frac{\pi}{x}|_{\frac{1}{n+1}}^{\frac{1}{n}}| \\
&= \sum_{n=1}^{\infty} |\frac{1}{n^p}(-1)^n - \frac{1}{(n+1)^p}(-1)^{n+1}| \\
&= \sum_{n=1}^{\infty} [\frac{1}{n^p} + \frac{1}{(n+1)^p}] \\
&= +\infty,
\end{aligned}$$

故该反常积分发散，原曲线为无限长.

综上所述，当且仅当$p > 1$时，曲线$y = x^p \cos \frac{\pi}{x}(0 < x \leq 1)$具有有限的长度.