

25C 第13章补充题

25C.1 第13章补充题解答

1. 设 D 为 $y = x, y = 4x, xy = 1, xy = 4$ 所围成的区域, F 是一元连续可微函数, $f = F'$.

求证: $\oint_{\partial D} \frac{F(xy)}{y} dy = \ln 2 \int_1^4 f(v) dv$.

证明: $\because D$ 是第一象限内部的区域, x 轴不穿过 D , 且 $F \in C^1$,

$$\therefore \frac{F(xy)}{y} \in C^1(D),$$

$$\therefore \oint_{\partial D} \frac{F(xy)}{y} dy = \iint_D \frac{\partial}{\partial x} \left[\frac{F(xy)}{y} - \frac{\partial 0}{\partial y} \right] dx dy = \iint_D F'(xy) dx dy,$$

$$\text{设} \begin{cases} u = \frac{y}{x}, \\ v = xy, \end{cases} \quad \text{则} D = \{(u, v) \mid 1 \leq u \leq 4, 1 \leq v \leq 4\},$$

$$\frac{D(u, v)}{D(x, y)} = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix} = -\frac{y}{x} - \frac{y}{x} = -2u,$$

$$\begin{aligned} \therefore \iint_D F'(xy) dx dy &= \iint_D f(xy) dx dy = \iint_D f(v) \frac{1}{|\frac{D(u, v)}{D(x, y)}|} du dv = \iint_D f(v) \frac{1}{2u} du dv = \int_1^4 \frac{1}{2u} du \int_1^4 f(v) dv \\ &= \frac{1}{2} \ln u \Big|_1^4 \int_1^4 f(v) dv = \ln 2 \int_1^4 f(v) dv. \end{aligned}$$

2. 设 D 是平面上的有界区域, 函数 $u(x, y)$ 与 $v(x, y)$ 在 \bar{D} 上存在二阶连续偏导数.

$$\text{求证: } \oint_{\partial D} \begin{vmatrix} \frac{\partial u}{\partial n} & \frac{\partial v}{\partial n} \\ u & v \end{vmatrix} dl = \iint_D \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} d\sigma.$$

证明: $\because u(x, y), v(x, y) \in C^2(D)$,

$$\text{方法1: } \therefore \frac{\partial u}{\partial n} = \mathbf{n} \cdot \text{grad} u, \frac{\partial v}{\partial n} = \mathbf{n} \cdot \text{grad} v,$$

$$\begin{aligned} \therefore \oint_{\partial D} \begin{vmatrix} \mathbf{n} \cdot \text{grad} u & \mathbf{n} \cdot \text{grad} v \\ u & v \end{vmatrix} dl &= \oint_{\partial D} [v(\mathbf{n} \cdot \text{grad} u) - u(\mathbf{n} \cdot \text{grad} v)] dl \\ &= \oint_{\partial D} (v \text{grad} u - u \text{grad} v) \cdot \mathbf{n} dl = \oint_{\partial D} (v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}, v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y}) \cdot \mathbf{n} dl \\ &= \iint_D \left[\frac{\partial}{\partial x} (v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}) + \frac{\partial}{\partial y} (v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y}) \right] d\sigma \\ &= \iint_D \left[\frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + v \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} - u \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} + v \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} - u \frac{\partial^2 v}{\partial y^2} \right] d\sigma \\ &= \iint_D (v \frac{\partial^2 u}{\partial x^2} - u \frac{\partial^2 v}{\partial x^2} + v \frac{\partial^2 u}{\partial y^2} - u \frac{\partial^2 v}{\partial y^2}) d\sigma = \iint_D [v(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - u(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})] d\sigma \\ &= \iint_D [v \Delta u - u \Delta v] d\sigma = \iint_D \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} d\sigma. \end{aligned}$$

$$\text{方法2: } \therefore \frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}, \frac{\partial v}{\partial n} = \nabla v \cdot \mathbf{n},$$

$$\begin{aligned} \therefore \oint_{\partial D} \begin{vmatrix} \frac{\partial u}{\partial n} & \frac{\partial v}{\partial n} \\ u & v \end{vmatrix} dl &= \oint_{\partial D} (v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}) dl = \oint_{\partial D} (v \nabla u \cdot \mathbf{n} - u \nabla v \cdot \mathbf{n}) dl \\ &= \oint_{\partial D} (v \nabla u - u \nabla v) \cdot \mathbf{n} dl = \iint_D \nabla \cdot (v \nabla u - u \nabla v) d\sigma \end{aligned}$$

$$\begin{aligned}
&= \iint_D (\nabla v \cdot \nabla u + v \nabla \cdot \nabla u - \nabla u \cdot \nabla v - u \nabla \cdot \nabla v) d\sigma \\
&= \iint_D (v \nabla \cdot \nabla u - u \nabla \cdot \nabla v) d\sigma = \iint_D (v \Delta u - u \Delta v) d\sigma = \iint_D \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} d\sigma.
\end{aligned}$$

3. 计算 $I = \iint_S \frac{\cos \widehat{r\mathbf{n}}}{r^2} dS$, 其中 S 为任意光滑闭曲面, \mathbf{n} 为 S 的外单位法向量, $M_0(x_0, y_0, z_0)$ 是 S 内的一个确定点, \mathbf{r} 是连接 $M_0(x_0, y_0, z_0)$ 和 S 上点 $M(x, y, z)$ 的向量, r 是 \mathbf{r} 的长度.

$$\text{解: } I = \iint_S \frac{\cos \widehat{r\mathbf{n}}}{r^2} dS = \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS,$$

取 $S_1: x^2 + y^2 + z^2 = a^2$, 其中 a 足够小, 使得 S_1 完全包含在 S 内, S_1 的外侧为正, 设 S 和 S_1 围成的区域为 Ω , 设 S_1 围成的区域为 Ω_1 ,

$$\because \frac{\mathbf{r}}{r^3} = \frac{1}{r^3}(x - x_0, y - y_0, z - z_0) \in C^1(\Omega), \quad r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2},$$

$$\text{又} \because \frac{\partial r}{\partial x} = \frac{2(x - x_0)}{2\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} = \frac{x - x_0}{r}, \quad \frac{\partial r}{\partial y} = \frac{y - y_0}{r}, \quad \frac{\partial r}{\partial z} = \frac{z - z_0}{r},$$

$$\begin{aligned}
&\therefore \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS + \iint_{S_1^-} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS \\
&= \iint_{S+S_1^-} \frac{1}{r^3}(x - x_0, y - y_0, z - z_0) \cdot \mathbf{n} dS \\
&= \iiint_{\Omega} \left[\frac{\partial}{\partial x} \left(\frac{x - x_0}{r^3} \right) + \frac{\partial}{\partial y} \left(\frac{y - y_0}{r^3} \right) + \frac{\partial}{\partial z} \left(\frac{z - z_0}{r^3} \right) \right] dx dy dz \\
&= \iiint_{\Omega} \left[\frac{r^3 - (x - x_0)3r^2 \frac{x - x_0}{r}}{r^6} + \frac{r^3 - (y - y_0)3r^2 \frac{y - y_0}{r}}{r^6} + \frac{r^3 - (z - z_0)3r^2 \frac{z - z_0}{r}}{r^6} \right] dx dy dz \\
&= \iiint_{\Omega} \left[\frac{r^2 - 3(x - x_0)^2}{r^5} + \frac{r^2 - 3(y - y_0)^2}{r^5} + \frac{r^2 - 3(z - z_0)^2}{r^5} \right] dx dy dz \\
&= \iiint_{\Omega} \left[\frac{(y - y_0)^2 + (z - z_0)^2 - 2(x - x_0)^2}{r^5} + \frac{(z - z_0)^2 + (x - x_0)^2 - 2(y - y_0)^2}{r^5} + \frac{(x - x_0)^2 + (y - y_0)^2 - 2(z - z_0)^2}{r^5} \right] dx dy dz \\
&= \iiint_{\Omega} 0 dx dy dz = 0,
\end{aligned}$$

$$\begin{aligned}
\therefore I &= \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = - \iint_{S_1^-} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \iint_{S_1} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \iint_{S_1} \frac{\mathbf{r} \cdot \mathbf{n}}{a^3} dS = \frac{1}{a^3} \iint_{S_1} \mathbf{r} \cdot \mathbf{n} dS \\
&= \frac{1}{a^3} \iiint_{\Omega_1} \nabla \cdot \mathbf{r} dx dy dz = \frac{1}{a^3} \iiint_{\Omega_1} \left[\frac{\partial}{\partial x}(x - x_0) + \frac{\partial}{\partial y}(y - y_0) + \frac{\partial}{\partial z}(z - z_0) \right] dx dy dz = \frac{1}{a^3} \iiint_{\Omega_1} 3 dx dy dz \\
&= \frac{3}{a^3} \cdot \frac{4}{3} \pi a^3 = 4\pi.
\end{aligned}$$

4. 设 $\Omega \subset \mathbb{R}^3$ 为有界区域, 其边界 $\partial\Omega$ 为逐片光滑的闭曲面, \mathbf{n} 是 $\partial\Omega$ 的外单位法向量. 函数 u 和 v 在 Ω 中有连续偏导数¹. 求证:

$$\begin{aligned}
(1) & \iint_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS = \iiint_{\Omega} \Delta u dV; \\
(2) & \iint_{\partial\Omega} u \frac{\partial u}{\partial \mathbf{n}} dS = \iiint_{\Omega} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] dV + \iint_{\Omega} u \Delta u dV; \\
(3) & \iint_{\partial\Omega} (u \frac{\partial v}{\partial \mathbf{n}} - v \frac{\partial u}{\partial \mathbf{n}}) dS = \iiint_{\Omega} (u \Delta v - v \Delta u) dV.
\end{aligned}$$

证明: (1) $\because u, v \in C^2(\Omega)$,

¹这里应是二阶连续偏导数

$$\text{方法1: } \therefore \oint_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS = \oint_{\partial\Omega} \text{grad} u \cdot \mathbf{n} dS = \oint_{\partial\Omega} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right) \cdot \mathbf{n} dS = \iiint_{\Omega} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) dV \\ = \iiint_{\Omega} \Delta u dV.$$

$$\text{方法2: } \therefore \oint_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS = \oint_{\partial\Omega} \nabla u \cdot \mathbf{n} dS = \iiint_{\Omega} \nabla \cdot \nabla u dV = \iiint_{\Omega} \Delta u dV.$$

(2) $\therefore u, v \in C^2(\Omega),$

$$\text{方法1: } \therefore \oint_{\partial\Omega} u \frac{\partial u}{\partial \mathbf{n}} dS = \oint_{\partial\Omega} u \text{grad} u \cdot \mathbf{n} dS = \oint_{\partial\Omega} \left(u \frac{\partial u}{\partial x}, u \frac{\partial u}{\partial y}, u \frac{\partial u}{\partial z} \right) \cdot \mathbf{n} dS \\ = \iiint_{\Omega} \left[\frac{\partial}{\partial x} \left(u \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(u \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(u \frac{\partial u}{\partial z} \right) \right] dV = \iiint_{\Omega} \left[\left(\frac{\partial u}{\partial x} \right)^2 + u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial y} \right)^2 + u \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial z} \right)^2 + u \frac{\partial^2 u}{\partial z^2} \right] dV \\ = \iiint_{\Omega} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] dV + \iiint_{\Omega} \left[u \frac{\partial^2 u}{\partial x^2} + u \frac{\partial^2 u}{\partial y^2} + u \frac{\partial^2 u}{\partial z^2} \right] dV \\ = \iiint_{\Omega} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] dV + \iiint_{\Omega} u \Delta u dV.$$

$$\text{方法2: } \therefore \oint_{\partial\Omega} u \frac{\partial u}{\partial \mathbf{n}} dS = \oint_{\partial\Omega} u \nabla u \cdot \mathbf{n} dS = \iiint_{\Omega} \nabla \cdot (u \nabla u) dV \\ = \iiint_{\Omega} (\nabla u \cdot \nabla u + u \nabla \cdot \nabla u) dV = \iiint_{\Omega} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] dV + \iiint_{\Omega} u \Delta u dV.$$

$$(3) \oint_{\partial\Omega} \left(u \frac{\partial v}{\partial \mathbf{n}} - v \frac{\partial u}{\partial \mathbf{n}} \right) dS = \oint_{\partial\Omega} (u \text{grad} v - v \text{grad} u) \cdot \mathbf{n} dS = \oint_{\partial\Omega} \left(u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x}, u \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial y}, u \frac{\partial v}{\partial z} - v \frac{\partial u}{\partial z} \right) \cdot \mathbf{n} dS \\ = \iiint_{\Omega} \left[\frac{\partial}{\partial x} (u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y} (u \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial y}) + \frac{\partial}{\partial z} (u \frac{\partial v}{\partial z} - v \frac{\partial u}{\partial z}) \right] dV \\ = \iiint_{\Omega} \left[\left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} - v \frac{\partial^2 u}{\partial x^2} \right) + \left(\frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + u \frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} - v \frac{\partial^2 u}{\partial y^2} \right) \right. \\ \left. + \left(\frac{\partial u}{\partial z} \frac{\partial v}{\partial z} + u \frac{\partial^2 v}{\partial z^2} - \frac{\partial v}{\partial z} \frac{\partial u}{\partial z} - v \frac{\partial^2 u}{\partial z^2} \right) \right] dV \\ = \iiint_{\Omega} \left[(u \frac{\partial^2 v}{\partial x^2} - v \frac{\partial^2 u}{\partial x^2}) + (u \frac{\partial^2 v}{\partial y^2} - v \frac{\partial^2 u}{\partial y^2}) + (u \frac{\partial^2 v}{\partial z^2} - v \frac{\partial^2 u}{\partial z^2}) \right] dV \\ = \iiint_{\Omega} \left[u \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) - v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \right] dV \\ = \iiint_{\Omega} (u \Delta v - v \Delta u) dV.$$

$$\text{方法2: } \therefore \oint_{\partial\Omega} \left(u \frac{\partial v}{\partial \mathbf{n}} - v \frac{\partial u}{\partial \mathbf{n}} \right) dS = \oint_{\partial\Omega} (u \nabla v \cdot \mathbf{n} - v \nabla u \cdot \mathbf{n}) dS = \oint_{\partial\Omega} (u \nabla v - v \nabla u) \cdot \mathbf{n} dS \\ = \iiint_{\Omega} \nabla \cdot (u \nabla v - v \nabla u) dV = \iiint_{\Omega} (\nabla u \cdot \nabla v + u \nabla \cdot \nabla v - \nabla v \cdot \nabla u - v \nabla \cdot \nabla u) dV \\ = \iiint_{\Omega} (u \nabla \cdot \nabla v - v \nabla \cdot \nabla u) dV = \iiint_{\Omega} (u \Delta v - v \Delta u) dV.$$

5. 设 D 为平面区域, $u(x, y)$ 在 D 上有二阶连续偏导数. 求证下列命题等价:

(1) $u(x, y)$ 在 D 上是调和函数, 即 $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$;

(2) 对于 D 内任意一条圆周 L , 如果 L 包围的区域完全属于 D , 则有 $\oint_L \frac{\partial u}{\partial \mathbf{n}} dl = 0$.

证明: (1) \Rightarrow (2):

$\therefore u(x, y) \in C^2(D),$

\therefore 对于 D 内任意一条圆周 L , 如果 L 包围的区域 D_L 完全属于 D ,

$$\text{则 } \oint_L \frac{\partial u}{\partial \mathbf{n}} dl = \oint_L \nabla u \cdot \mathbf{n} dl = \iint_{D_L} \nabla \cdot \nabla u d\sigma = \iint_{D_L} \Delta u d\sigma = 0.$$

(2) \Rightarrow (1):

方法1: $\forall (x, y) \in D$ 取 D 内包围 (x, y) 的圆周 L , 使得 L 包围的区域 D_L 完全属于 D ,

$$\text{则 } \oint_L \frac{\partial u}{\partial \mathbf{n}} dl = 0,$$

$$\therefore u(x, y) \in C^2(D),$$

$$\therefore \Delta u \in C(D), \text{ 且 } \iint_{D_L} \Delta u d\sigma = \iint_{D_L} \nabla \cdot \nabla u d\sigma = \oint_L \nabla u \cdot \mathbf{n} dl = \oint_L \frac{\partial u}{\partial \mathbf{n}} dl = 0,$$

根据积分中值定理 $\exists (\xi, \eta) \in D_L \subset D, s.t. \iint_{D_L} \Delta u d\sigma = \Delta u(\xi, \eta) A(D_L) = 0$, 其中 $A(D_L)$ 为区域 D_L 的面积,

$$\therefore \lim_{D_L \rightarrow (x, y)} \frac{\iint_{D_L} \Delta u d\sigma}{A(D_L)} = \lim_{(\xi, \eta) \rightarrow (x, y)} \Delta u(\xi, \eta) = \Delta u(x, y) = 0.$$

方法2: 反证. 若 $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ 在 D 上不等于零, 则至少存在一点 M_0 使得

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \Big|_{M_0} \neq 0.$$

不妨设 $\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \Big|_{M_0} > 0$. 由连续性知, 存在位于 D 内的、以 M_0 为中心的一个闭圆域 U , 在 U 上处处有 $\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) > 0$, 从而

$$\iint_U \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx dy > 0.$$

用 ∂U 表示 U 的正向边界(这是 D 内的一个圆周), 于是由格林公式得到

$$\oint_{\partial U} \frac{\partial u}{\partial \mathbf{n}} dl = \iint_U \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx dy > 0.$$

这与假设矛盾.

6. 设 Ω 为光滑曲面 S 围成的有界闭区域, u 是 Ω 上的调和函数 ($\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \equiv 0$). 求证:

$$\iint_S \left[\frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial(\frac{1}{r})}{\partial \mathbf{n}} \right] dS = - \iint_{S_\delta} \left[\frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial(\frac{1}{r})}{\partial \mathbf{n}} \right] dS,$$

其中 $M_0(x_0, y_0, z_0)$ 是 Ω 内一点, S_δ 是以 $M_0(x_0, y_0, z_0)$ 为中心, δ 为半径的球面, 且 $S_\delta \subset \Omega$; $\mathbf{r} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$, r 是 \mathbf{r} 的长度; \mathbf{n} 是 S 的外单位法向量、 S_δ 的内单位法向量.

证明: 设 Ω_1 为 S 内 S_δ 外的区域, 在 Ω_1 上

$$\frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial(\frac{1}{r})}{\partial \mathbf{n}} = \frac{1}{r} \nabla u \cdot \mathbf{n} - u \nabla \left(\frac{1}{r} \right) \cdot \mathbf{n} = \left[\frac{1}{r} \nabla u - u \nabla \left(\frac{1}{r} \right) \right] \cdot \mathbf{n},$$

∴根据高斯公式

$$\begin{aligned}
 & \iint_{S+S_\delta} \left[\frac{1}{r} \nabla u - u \nabla \left(\frac{1}{r} \right) \right] \cdot \mathbf{n} dS \\
 &= \iiint_{\Omega_1} \nabla \cdot \left[\frac{1}{r} \nabla u - u \nabla \left(\frac{1}{r} \right) \right] dV \\
 &= \iiint_{\Omega_1} \left[\nabla \left(\frac{1}{r} \right) \cdot \nabla u + \frac{1}{r} \nabla \cdot \nabla u - \nabla u \cdot \nabla \left(\frac{1}{r} \right) - u \nabla \cdot \nabla \left(\frac{1}{r} \right) \right] dV \\
 &= \iiint_{\Omega_1} \left[\frac{1}{r} \nabla \cdot \nabla u - u \nabla \cdot \nabla \left(\frac{1}{r} \right) \right] dV \\
 &= \iiint_{\Omega_1} \left[\frac{1}{r} \Delta u - u \Delta \left(\frac{1}{r} \right) \right] dV \\
 &= - \iiint_{\Omega_1} u \Delta \left(\frac{1}{r} \right) dV,
 \end{aligned}$$

$$\begin{aligned}
 \because \frac{\partial}{\partial x} \left(\frac{1}{r} \right) &= \frac{-\frac{\partial r}{\partial x}}{r^2} = \frac{1}{r^2} \frac{-2(x-x_0)}{2\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = \frac{-(x-x_0)}{r^3}, \\
 \frac{\partial}{\partial y} \left(\frac{1}{r} \right) &= \frac{-(y-y_0)}{r^3}, \quad \frac{\partial}{\partial z} \left(\frac{1}{r} \right) = \frac{-(z-z_0)}{r^3}, \\
 \frac{\partial^2}{\partial x^2} \left(\frac{1}{r} \right) &= \frac{\partial}{\partial x} \left[\frac{-(x-x_0)}{r^3} \right] = \frac{-r^3+(x-x_0)3r^2 \frac{\partial r}{\partial x}}{r^6} = \frac{\partial}{\partial x} \left[\frac{-(x-x_0)}{r^3} \right] = \frac{-r^3+(x-x_0)3r^2 \frac{2(x-x_0)}{2\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}}{r^6} \\
 &= \frac{\partial}{\partial x} \left[\frac{-(x-x_0)}{r^3} \right] = \frac{-r^3+(x-x_0)3r^2 \frac{(x-x_0)}{r}}{r^6} = \frac{-r^2+3(x-x_0)^2}{r^5} = \frac{2(x-x_0)^2-(y-y_0)^2-(z-z_0)^2}{r^5}, \\
 \frac{\partial^2}{\partial y^2} \left(\frac{1}{r} \right) &= \frac{2(y-y_0)^2-(z-z_0)^2-(x-x_0)^2}{r^5}, \quad \frac{\partial^2}{\partial z^2} \left(\frac{1}{r} \right) = \frac{2(z-z_0)^2-(x-x_0)^2-(y-y_0)^2}{r^5}, \\
 \therefore \Delta \left(\frac{1}{r} \right) &= \frac{\partial^2}{\partial x^2} \left(\frac{1}{r} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{1}{r} \right) + \frac{\partial^2}{\partial z^2} \left(\frac{1}{r} \right) \\
 &= \frac{2(x-x_0)^2-(y-y_0)^2-(z-z_0)^2}{r^5} + \frac{2(y-y_0)^2-(z-z_0)^2-(x-x_0)^2}{r^5} + \frac{2(z-z_0)^2-(x-x_0)^2-(y-y_0)^2}{r^5} = 0,
 \end{aligned}$$

∴

$$\iint_{S+S_\delta} \left[\frac{1}{r} \nabla u - u \nabla \left(\frac{1}{r} \right) \right] \cdot \mathbf{n} dS = - \iiint_{\Omega_1} u \Delta \left(\frac{1}{r} \right) dV = 0,$$

∴

$$\iint_{S+S_\delta} \left[\frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial \left(\frac{1}{r} \right)}{\partial \mathbf{n}} \right] \cdot \mathbf{n} dS = 0,$$

∴

$$\iint_S \left[\frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial \left(\frac{1}{r} \right)}{\partial \mathbf{n}} \right] dS = - \iint_{S_\delta} \left[\frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial \left(\frac{1}{r} \right)}{\partial \mathbf{n}} \right] dS.$$

7. 设 $\Omega \subset \mathbb{R}^3$ 为有界区域, 其边界 $\partial\Omega$ 逐片光滑, \mathbf{n} 是 S 的外单位法向量. f 在 Ω 内调和, 在 $\partial\Omega$ 上有连续的偏导数, 并且在闭区域 $\bar{\Omega}$ 上连续. 求证:

$$(1) \iint_{\partial\Omega} \frac{\partial f}{\partial \mathbf{n}} dS = 0;$$

$$(2) \iint_{\partial\Omega} f \frac{\partial f}{\partial \mathbf{n}} dS = \iiint_{\Omega} \|\nabla f\|^2 dV (\|\nabla f\| \text{ 是向量 } \nabla f \text{ 的长度});$$

(3) 若当 $(x, y, z) \in \partial\Omega$ 时, $f(x, y, z) \equiv 0$, 求证在 Ω 内 $f(x, y, z) \equiv 0$.

证明: (1) $\because f$ 在 Ω 内调和, 即 $\Delta f(x, y, z) = 0, (x, y, z) \in \Omega$,

$$\therefore \oint_{\partial\Omega} \frac{\partial f}{\partial n} dS = \oint_{\partial\Omega} \nabla f \cdot \mathbf{n} dS = \iiint_{\Omega} \nabla \cdot \nabla f dV = \iiint_{\Omega} \Delta f dV = \iiint_{\Omega} 0 dV = 0.$$

$$(2) \oint_{\partial\Omega} f \frac{\partial f}{\partial n} dS = \oint_{\partial\Omega} f \nabla f \cdot \mathbf{n} dS = \iiint_{\Omega} \nabla \cdot (f \nabla f) dV = \iiint_{\Omega} (\nabla f \cdot \nabla f + f \nabla \cdot \nabla f) dV \\ = \iiint_{\Omega} (\|\nabla f\|^2 + f \Delta f) dV = \iiint_{\Omega} \|\nabla f\|^2 dV.$$

(3) \because 当 $(x, y, z) \in \partial\Omega$ 时, $f(x, y, z) \equiv 0$,

$$\therefore \iiint_{\Omega} \|\nabla f\|^2 dV = \oint_{\partial\Omega} f \frac{\partial f}{\partial n} dS = \oint_{\partial\Omega} 0 dS = 0,$$

$$\therefore \|\nabla f\|^2 \geq 0,$$

$$\therefore \|\nabla f\|^2 = 0, \nabla f = \mathbf{0},$$

$$\therefore f(x, y, z) = \text{Const}, (x, y, z) \in \Omega,$$

$\therefore f$ 在闭区域 $\bar{\Omega}$ 上连续,

$$\therefore f(x, y, z) = 0, (x, y, z) \in \Omega.$$

8. 设 $M_0(x_0, y_0, z_0)$ 为空间一确定点, S 是点 M_0 之外的一张逐片光滑曲面. 从点 M_0 出发作射线, 假定每一条这样的射线与曲面最多相交于一点, 则所有与曲面 S 相交的射线构成一个锥体 Λ . 以点 M_0 为中心, 以任意正数 a 为半径做球, 并设该球面含于锥体 Λ 内部的部分面积为 S_a . 定义曲面 S 关于点 M_0 的立体角为 $\Omega_S = \frac{S_a}{a^2}$.

(1) 若 S 是以点 M_0 为中心, 以任意正数 a 为半径的球面, 求 Ω_S ;

(2) 令 $\mathbf{v} = \frac{\mathbf{r}}{r^3}$, 求证: $\Omega_S = \iint_S \mathbf{v} \cdot \mathbf{n} dS$. 其中 $\mathbf{r} = \overrightarrow{M_0 M}, r = \|\mathbf{r}\|$.

解: (1) $\Omega_S = \frac{S_a}{a^2} = \frac{4\pi a^2}{a^2} = 4\pi$.

(2) 证明: 记半径为 a 的球面含于锥体 Λ 内部的部分为 S_a , 不妨设 S_a 包含在锥体 Λ 的由顶点 M_0 到曲面 S 之间的部分, 设 S 和 S_a 与锥体 Λ 侧面围成的区域为 Ω , 锥体 Λ 的侧面在 S 和 S_a 之间的部分记为 S_1 , 易知在 S_1 上 $\mathbf{v} \cdot \mathbf{n} = 0$, 其中 \mathbf{n} 为区域 Ω 边界上的外向单位法向量,

$$\therefore \mathbf{v} = \frac{\mathbf{r}}{r^2} \in C^1(\Omega),$$

$$\therefore \iint_S \mathbf{v} \cdot \mathbf{n} dS + \iint_{S_a} \mathbf{v} \cdot \mathbf{n} dS = \iint_S \mathbf{v} \cdot \mathbf{n} dS + \iint_{S_a} \mathbf{v} \cdot \mathbf{n} dS + \iint_{S_1} \mathbf{v} \cdot \mathbf{n} dS \\ = \oint_{S+S_a+S_1} \mathbf{v} \cdot \mathbf{n} dS = \iiint_{\Omega} \nabla \cdot \mathbf{v} dV = \iiint_{\Omega} \left(\frac{\partial}{\partial x} \left[\frac{x-x_0}{r^3} \right] + \frac{\partial}{\partial y} \left(\frac{y-y_0}{r^3} \right) + \frac{\partial}{\partial z} \left(\frac{z-z_0}{r^3} \right) \right) dV \\ = \iiint_{\Omega} \left(\frac{r^3-(x-x_0)3r^2 \frac{\partial r}{\partial x}}{r^6} + \frac{r^3-(y-y_0)3r^2 \frac{\partial r}{\partial y}}{r^6} + \frac{r^3-(z-z_0)3r^2 \frac{\partial r}{\partial z}}{r^6} \right) dV \\ = \iiint_{\Omega} \left(\frac{r^3-(x-x_0)3r^2 \frac{x-x_0}{r}}{r^6} + \frac{r^3-(y-y_0)3r^2 \frac{y-y_0}{r}}{r^6} + \frac{r^3-(z-z_0)3r^2 \frac{z-z_0}{r}}{r^6} \right) dV \\ = \iiint_{\Omega} \left(\frac{r^2-3(x-x_0)^2}{r^5} + \frac{r^2-3(y-y_0)^2}{r^5} + \frac{r^2-3(z-z_0)^2}{r^5} \right) dV \\ = \iiint_{\Omega} \left(\frac{(y-y_0)^2+(z-z_0)^2-2(x-x_0)^2}{r^5} + \frac{(z-z_0)^2+(x-x_0)^2-2(y-y_0)^2}{r^5} + \frac{(x-x_0)^2+(y-y_0)^2-2(z-z_0)^2}{r^5} \right) dV \\ = \iiint_{\Omega} 0 dV = 0,$$

$$\therefore \iint_S \mathbf{v} \cdot \mathbf{n} dS = - \iint_{S_a} \mathbf{v} \cdot \mathbf{n} dS = - \iint_{S_a} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS = - \iint_{S_a} \frac{\mathbf{r}}{a^3} \cdot \mathbf{n} dS = - \frac{1}{a^3} \iint_{S_a} \mathbf{r} \cdot \mathbf{n} dS,$$

$$\therefore \text{球面 } S_a \text{ 上 } \mathbf{n} = -\left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right) = -\frac{1}{a} \mathbf{r},$$

$$\begin{aligned} \therefore \iint_S \mathbf{v} \cdot \mathbf{n} dS &= -\frac{1}{a^3} \iint_{S_a} \mathbf{r} \cdot \mathbf{n} dS = \frac{1}{a^3} \frac{1}{a} \iint_{S_a} \mathbf{r} \cdot \mathbf{r} dS = \frac{1}{a^3} \frac{1}{a} \iint_{S_a} (x^2 + y^2 + z^2) dS = \frac{1}{a^3} \frac{1}{a} \iint_{S_a} a^2 dS \\ &= \frac{1}{a^2} \iint_{S_a} dS = \frac{S_a}{a^2} = \Omega_S. \end{aligned}$$