21 含参变量的积分、向量场的微分运算

21.1 知识结构

第12章重积分

12.7 含参变量的积分

12.7.1 引言

12.7.2 含参变量的定积分

12.7.3 含参变量的广义积分

第13章向量场的微积分

13.1 向量场的微分运算

(a) 数量场的梯度算子

(b) 向量场的散度算子

(c) 向量场的旋度算子

21.2 习题12.7解答

1. 求下列含参变量积分的导数：

$(1)f(x)=\int_0^{\pi}\sin(xy)\mathrm{d}y$; $\quad(2)f(x)=\int_0^{x}\sin(xy)\mathrm{d}y$;

$(3)f(x)=\int_0^1\frac{x\mathrm{d}y}{\sqrt{1-x^2y^2}}$; $\quad(4)f(x)=\int_0^x f(y+x,y-x)\mathrm{d}y$.

解：$(1)f'(x)=\int_0^{\pi}\frac{\partial}{\partial x}\sin(xy)\mathrm{d}y=\int_0^{\pi}y\cos(xy)\mathrm{d}y=\frac{1}{x}\int_0^1 y\cos(xy)\mathrm{d}(xy)=\frac{1}{x}\int_0^{\pi}y\mathrm{d}\sin(xy)$

$=\frac{y}{x}\sin(xy)\Big|_0^{\pi}-\frac{1}{x}\int_0^{\pi}\sin(xy)\mathrm{d}y=\frac{\pi\sin(\pi x)}{x}+\frac{1}{x^2}\cos(xy)\Big|_0^{\pi}=\frac{\pi\sin(\pi x)}{x}+\frac{\cos(\pi x)}{x^2}-\frac{1}{x^2}$.

$(2)f'(x)=\int_0^{x}\frac{\partial}{\partial x}\sin(xy)\mathrm{d}y+\sin(x\cdot x)\cdot\frac{\mathrm{d}x}{\mathrm{d}x}-\sin(x\cdot 0)\cdot\frac{\mathrm{d}0}{\mathrm{d}x}=\int_0^x y\cos(xy)\mathrm{d}y-\sin(x^2)$

$=\frac{1}{x}\int_0^x y\mathrm{d}\sin(xy)+\sin(x^2)=\frac{y}{x}\sin(xy)\Big|_0^x-\frac{1}{x}\int_0^x\sin(xy)\mathrm{d}y+\sin(x^2)$

$=\sin(x^2)-\frac{1}{x^2}\int_0^x\sin(xy)\mathrm{d}(xy)+\sin(x^2)=2\sin(x^2)+\frac{1}{x^2}\cos(xy)\Big|_0^x$

$=2\sin(x^2)+\frac{\cos(x^2)}{x^2}-\frac{1}{x^2}$.

(3)方法1：$f'(x)=\int_0^1\frac{\partial}{\partial x}[\frac{x}{\sqrt{1-x^2y^2}}]\mathrm{d}y=\int_0^1\frac{\sqrt{1-x^2y^2}-x\frac{-2xy^2}{2\sqrt{1-x^2y^2}}}{1-x^2y^2}\mathrm{d}y=\int_0^1\frac{1}{(1-x^2y^2)^{\frac{3}{2}}}\mathrm{d}y$

$=\frac{1}{x}\int_0^1\frac{1}{(1-x^2y^2)^{\frac{3}{2}}}\mathrm{d}(xy)\xlongequal{xy=\sin\theta}\frac{1}{x}\int_0^{\arcsin x}\frac{1}{\cos^3\theta}\mathrm{d}\sin\theta=\frac{1}{x}\int_0^{\arcsin x}\frac{\cos\theta}{\cos^3\theta}\mathrm{d}\theta=\frac{1}{x}\int_0^{\arcsin x}\sec^2\theta\mathrm{d}\theta$

$=\frac{1}{x}\tan\theta\Big|_0^{\arcsin x}=\frac{1}{x}\frac{x}{\sqrt{1-x^2}}=\frac{1}{\sqrt{1-x^2}}$.

方法2：$\because f(x)=\int_0^1\frac{x\mathrm{d}y}{\sqrt{1-x^2y^2}}=\int_0^1\frac{\mathrm{d}(xy)}{\sqrt{1-x^2y^2}}=\arcsin(xy)\Big|_0^1=\arcsin x$,

$\therefore f'(x)=\frac{1}{\sqrt{1-x^2}}$.

(4)$f'(x)=\int_0^x \frac{\partial}{\partial x}f(y+x,y-x)\mathrm{d}y+f(x+x,x-x)\frac{\mathrm{d}x}{\mathrm{d}x}-f(0+x,0-x)\frac{\mathrm{d}0}{\mathrm{d}x}$
$=\int_0^x[f_1'(y+x,y-x)-f_2'(y+x,y-x)]\mathrm{d}y+f(2x,0).$

2. 设$f(x)$是$[0,1]$上的连续函数，对$x\in[0,1]$，令$F(x)=\int_0^x f(t)(x-t)^{n-1}\mathrm{d}t$，求$F^{(n)}(x)$.

解：$F'(x)=\int_0^x \frac{\partial}{\partial x}[f(t)(x-t)^{n-1}]\mathrm{d}t+f(x)(x-x)^{n-1}\frac{\mathrm{d}x}{\mathrm{d}x}-f(0)(x-0)\frac{\mathrm{d}0}{\mathrm{d}x}$
$=(n-1)\int_0^x f(t)(x-t)^{n-2}\mathrm{d}t,$

$F''(x)=(n-1)\int_0^x \frac{\partial}{\partial x}[f(t)(x-t)^{n-2}]\mathrm{d}t+(n-1)f(x)(x-x)^{n-2}\frac{\mathrm{d}x}{\mathrm{d}x}-(n-1)f(0)(x-0)^{n-2}\frac{\mathrm{d}0}{\mathrm{d}x}$
$=(n-1)(n-2)\int_0^x f(t)(x-t)^{n-3}\mathrm{d}t,$

$\cdots$

$F^{(n-1)}(x)=(n-1)(n-2)\cdots[n-(n-1)]\int_0^x f(t)(x-t)^{n-(n-1)-1}\mathrm{d}t=(n-1)!\int_0^x f(t)\mathrm{d}t,$

$F^{(n)}(x)=(n-1)!f(x).$

3. 设$f(y)=\int_0^1(x-1)x^y\ln^{-1}x\mathrm{d}x$，求$f'(y)$和$\lim\limits_{y\to+\infty}f(y)$，并证明$f(y)=\ln\frac{2+y}{1+y}(y>-1)$.

证明：$f'(y)=\int_0^1\frac{\partial}{\partial y}[(x-1)x^y\ln^{-1}x]\mathrm{d}x=\int_0^1(x-1)x^y\ln x\ln^{-1}x\mathrm{d}x=\int_0^1(x-1)x^y\mathrm{d}x$
$=\int_0^1(x^{y+1}-x^y)\mathrm{d}x=(\frac{1}{y+2}x^{y+2}-\frac{1}{y+1}x^{y+1})\big|_0^1=\frac{1}{y+2}-\frac{1}{y+1},$

$\lim\limits_{y\to+\infty}f(y)=\int_0^1\lim\limits_{y\to+\infty}(x-1)x^y\ln^{-1}x\mathrm{d}x=\int_0^1 0\mathrm{d}x=0,$

$\therefore f(y)=\int_{+\infty}^y f'(t)\mathrm{d}t=\int_{+\infty}^y(\frac{1}{t+2}-\frac{1}{t+1})\mathrm{d}t=\ln\frac{t+2}{t+1}\big|_{+\infty}^y=\ln\frac{y+2}{y+1}-\lim\limits_{y\to+\infty}\ln(1+\frac{1}{t+1})$
$=\ln\frac{y+2}{y+1}.$

21.3 第12章补充题解答

1. 设$f(u)$是连续函数，求证：

$$\int_a^b\mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{n-1}}f(x_n)\mathrm{d}x_n=\frac{1}{(n-1)!}\int_a^b(b-x)^{n-1}f(x)\mathrm{d}x.$$

证明：当$n=1$时$\int_a^b f(x_1)\mathrm{d}x_1=\frac{1}{(1-1)!}\int_a^b(b-x)^{1-1}f(x)\mathrm{d}x$，命题成立，

当$n=2$时$\int_a^b\mathrm{d}x_1\int_a^{x_1}f(x_2)\mathrm{d}x_2=\int_a^b\mathrm{d}x_2\int_{x_2}^b f(x_2)\mathrm{d}x_1=\int_a^b(b-x_2)f(x_2)\mathrm{d}x_2$
$=\int_a^b(b-x)f(x)\mathrm{d}x$，命题成立，

假设当$n=k$时命题成立，即

$$\int_a^b\mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{k-1}}f(x_k)\mathrm{d}x_k=\frac{1}{(k-1)!}\int_a^b(b-x)^{k-1}f(x)\mathrm{d}x.$$

则当$n=k+1$时，令$F(x)=\int_a^x f(x_{k+1})\mathrm{d}x_{k+1}$

$$
\begin{aligned}
&\int_a^b \mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{k-1}}\mathrm{d}x_k\int_a^{x_k}f(x_{k+1})\mathrm{d}x_{k+1}\\
=&\int_a^b \mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{k-1}}F(x_k)\mathrm{d}x_k\\
=&\frac{1}{(k-1)!}\int_a^b(b-x)^{k-1}F(x)\mathrm{d}x\\
=&\frac{1}{(k-1)!}\int_a^b(b-x)^{k-1}[\int_a^x f(x_{k+1})\mathrm{d}x_{k+1}]\mathrm{d}x\\
=&\frac{1}{(k-1)!}\int_a^b f(x_{k+1})[\int_{x_{k+1}}^b(b-x)^{k-1}\mathrm{d}x]\mathrm{d}x_{k+1}\\
=&\frac{1}{(k-1)!}\int_a^b f(x_{k+1})[-\frac{1}{k-1+1}(b-x)^{k-1+1}\Big|_{x_{k+1}}^b]\mathrm{d}x_{k+1}\\
=&\frac{1}{(k-1)!}\int_a^b f(x_{k+1})\frac{1}{k}(b-x_{k+1})^k\mathrm{d}x_{k+1}\\
=&\frac{1}{k!}\int_a^b f(x)(b-x)^k\mathrm{d}x,
\end{aligned}
$$

故当$n=k+1$时命题也成立. 证毕.

2. 求椭圆$\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1$(质量均匀)绕直线$y=kx$的转动惯量，并说明$k$为何值时转动惯量最大.

解：设$D=\{(x,y)\mid\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1\}$，$\forall(x,y)\in D$到直线$y=kx$的距离为$d=\frac{|kx-y|}{\sqrt{1+k^2}}$，设椭圆的密度为$\mu$,

则已知椭圆绕直线$y=kx$的转动惯量

$J=\iint\limits_D d^2\mu\mathrm{d}x\mathrm{d}y=\iint\limits_D\frac{(kx-y)^2}{1+k^2}\mu\mathrm{d}x\mathrm{d}y$
$=\frac{\mu}{1+k^2}\iint\limits_D(kar\cos\theta-br\sin\theta)^2\cdot abr\mathrm{d}r\mathrm{d}\theta=\frac{ab\mu}{1+k^2}\int_0^1 r^3\mathrm{d}r\int_0^{2\pi}(ka\cos\theta-b\sin\theta)^2\mathrm{d}\theta$
$=\frac{ab\mu}{4(1+k^2)}\int_0^{2\pi}(k^2a^2\cos^2\theta+b^2\sin^2\theta-2kab\sin\theta\cos\theta)\mathrm{d}\theta$
$=\frac{ab\mu}{4(1+k^2)}(k^2a^2\int_0^{2\pi}\cos^2\theta\mathrm{d}\theta+b^2\int_0^{2\pi}\sin^2\theta\mathrm{d}\theta-2ab\int_0^{2\pi}\sin\theta\cos\theta\mathrm{d}\theta)$
$=\frac{ab\mu}{4(1+k^2)}(k^2a^24\cdot\frac{\pi}{4}+b^24\cdot\frac{\pi}{4}-0)=\frac{\pi ab\mu}{4(1+k^2)}(k^2a^2+b^2)=\frac{\pi}{4}ab\mu\frac{k^2a^2+b^2}{1+k^2}=\frac{\pi}{4}ab\mu(a^2+\frac{b^2-a^2}{1+k^2})$,

i)若$a>b>0$，则$k=\infty$时转动惯量J取得最大值；

ii)若$b>a>0$，则$k=0$时转动惯量J取得最大值；

iii)若$a=b>0$，则转动惯量J恒为常数.

3. 设有半径为R，高为H的正圆锥体(质量均匀)，试求：
(1)该圆锥体对位于其顶点处质量为m的质点的引力；
(2)该圆锥体关于它的中心轴的转动惯量.

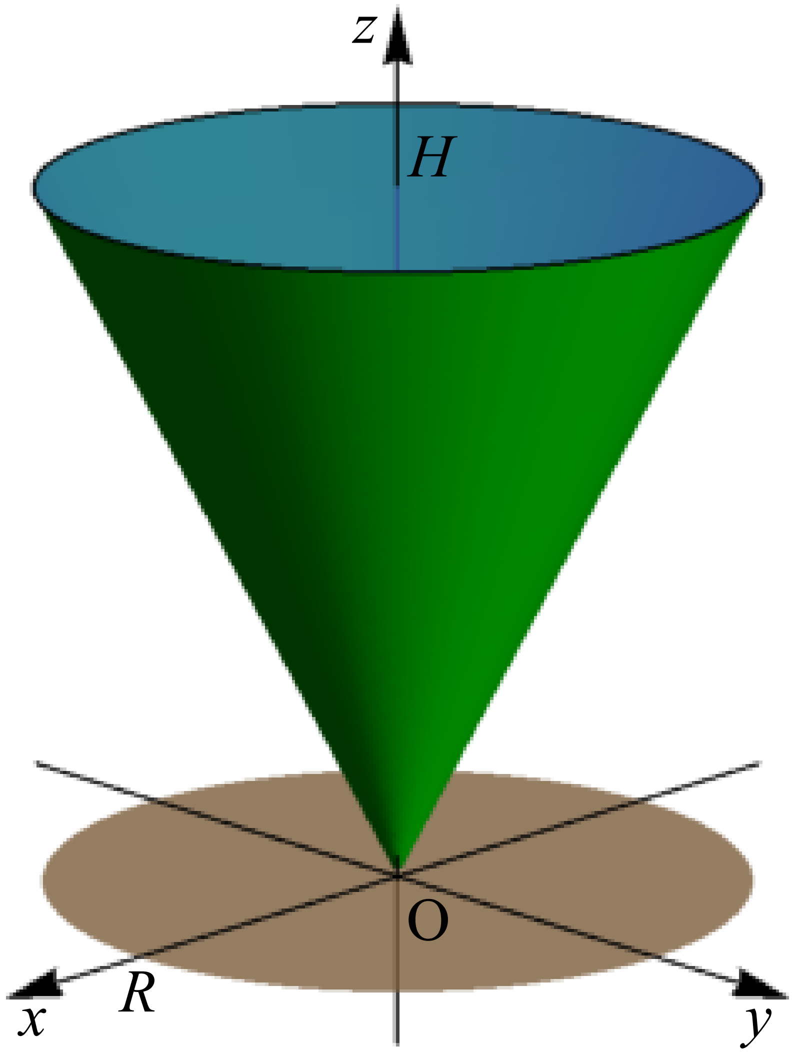

图 1: 第12章补充题 3.题图示

解：以该圆锥体的顶点为坐标原点，该圆锥体的旋转轴为z轴，建立空间直角坐标系，使该圆锥体在xOy平面上方，该圆锥体占据的空间区域可表示为$\Omega=\{(x,y,z)\mid \frac{H}{R}\sqrt{x^2+y^2}\leqslant z\leqslant H, x^2+y^2\leqslant R^2\}=\{(\theta,r,z)\mid 0\leqslant\theta\leqslant 2\pi, 0\leqslant r\leqslant R, \frac{H}{R}r\leqslant z\leqslant H\}$根据对称性可知引力的$x$和$y$分量$F_x=F_y=0$，设该圆锥体的密度为$\rho$，则引力的$z$分量

$F_z=\iiint\limits_{\Omega}\frac{G\rho m}{x^2+y^2+z^2}\frac{z}{\sqrt{x^2+y^2+z^2}}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_0^{2\pi}\mathrm{d}\theta\int_0^R\mathrm{d}r\int_{\frac{H}{R}r}^{H}\frac{G\rho m}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}}r\mathrm{d}z$
$=2\pi G\rho m\int_0^R\mathrm{d}r\int_{\frac{H}{R}r}^{H}\frac{rz}{(r^2+z^2)^{\frac{3}{2}}}\mathrm{d}z=2\pi G\rho m\int_0^R\frac{1}{2}r\frac{1}{1-\frac{3}{2}}(r^2+z^2)^{-\frac{3}{2}+1}\Big|_{\frac{H}{R}r}^{H}\mathrm{d}r$

$= 2\pi G\rho m\int_0^R r(\frac{1}{\sqrt{r^2+\frac{H^2}{R^2}r^2}} - \frac{1}{\sqrt{r^2+H^2}})\mathrm{d}r = 2\pi G\rho m\int_0^R (\frac{1}{\sqrt{1+\frac{H^2}{R^2}}} - \frac{r}{\sqrt{r^2+H^2}})\mathrm{d}r$
$= 2\pi G\rho m(\frac{1}{\sqrt{1+\frac{H^2}{R^2}}}r - \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}}\sqrt{r^2+H^2})\big|_0^R = 2\pi G\rho m(\frac{R}{\sqrt{1+\frac{H^2}{R^2}}} - \sqrt{R^2+H^2}+H)$
$= 2\pi G\rho m(H + \frac{R^2-R^2-H^2}{\sqrt{R^2+H^2}}) = 2\pi G\rho m(H - \frac{H^2}{\sqrt{R^2+H^2}})$,

故该圆锥体对位于其顶点处质量为m的质点的引力为$(0,0,2\pi G\rho m(H-\frac{H^2}{\sqrt{R^2+H^2}}))$.

(2)该圆锥体关于它的中心轴的转动惯量

$J_z = \iiint\limits_\Omega (x^2+y^2)\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta\int_0^R\mathrm{d}r\int_{\frac{H}{R}r}^H r^2\rho\cdot r\mathrm{d}z = 2\pi\rho\int_0^R r^3(H-\frac{H}{R}r)\mathrm{d}r$
$= 2\pi\rho(\frac{1}{4}Hr^4 - \frac{1}{5}\frac{H}{R}r^5)\big|_0^R = 2\pi\rho(\frac{1}{4}HR^4 - \frac{1}{5}HR^4) = \frac{1}{10}\pi\rho HR^4.$

4. 计算积分$\iint\limits_D (x^2+y^2)^{\frac{1}{2}}\mathrm{d}x\mathrm{d}y$，其中$D=\{(x,y)\mid 0\leqslant x,y\leqslant a\}$.

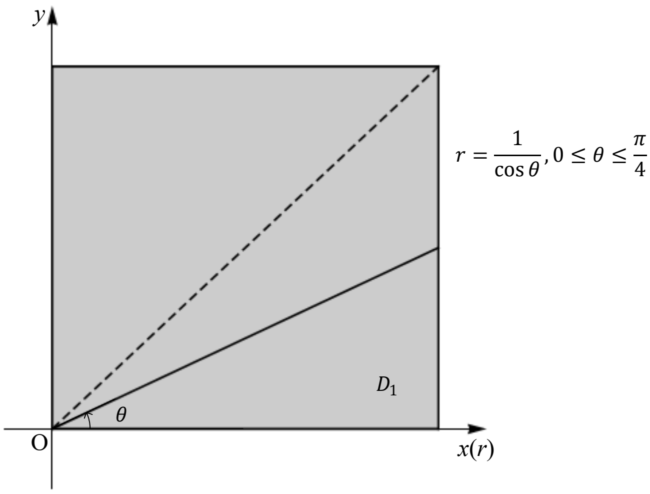

图 2: 第12章补充题 4.题图示

解：设$D_1=\{(x,y)\mid 0\leqslant x\leqslant 1, 0\leqslant y\leqslant x\}=\{(r,\theta)\mid 0\leqslant\theta\leqslant\frac{\pi}{4}, 0\leqslant r\leqslant\frac{a}{\cos\theta}\}$，根据对称性可知

$\iint\limits_D (x^2+y^2)^{\frac{1}{2}}\mathrm{d}x\mathrm{d}y = 2\iint\limits_{D_1}(x^2+y^2)^{\frac{1}{2}}\mathrm{d}x\mathrm{d}y = 2\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^{\frac{a}{\cos\theta}} r\cdot r\mathrm{d}r = 2\int_0^{\frac{\pi}{4}}\frac{1}{3}r^3\big|_0^{\frac{a}{\cos\theta}}\mathrm{d}\theta$
$= 2\int_0^{2\pi}\frac{1}{3}\frac{a^3}{\cos^3\theta}\mathrm{d}\theta = \frac{2}{3}a^3\int_0^{\frac{\pi}{4}}\sec^3\theta\mathrm{d}\theta = \frac{2}{3}a^3\int_0^{\frac{\pi}{4}}\sec\theta\mathrm{d}\tan\theta = \frac{2}{3}a^3(\sec\theta\tan\theta\big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\tan\theta\mathrm{d}\sec\theta)$
$= \frac{2}{3}a^3(\sqrt{2} - \int_0^{\frac{\pi}{4}}\tan^2\theta\sec\theta\mathrm{d}\theta) = \frac{2}{3}a^3[\sqrt{2} - \int_0^{\frac{\pi}{4}}(\sec^2\theta-1)\sec\theta\mathrm{d}\theta]$
$= \frac{2}{3}a^3[\sqrt{2} - \int_0^{\frac{\pi}{4}}(\sec^3\theta - \sec\theta)\mathrm{d}\theta] = \frac{2}{3}a^3(\sqrt{2} - \int_0^{\frac{\pi}{4}}\sec^3\theta\mathrm{d}\theta + \int_0^{\frac{\pi}{4}}\sec\theta\mathrm{d}\theta)$
$= \frac{\frac{2}{3}a^3}{\frac{2}{3}a^3+\frac{2}{3}a^3}(\frac{2}{3}\sqrt{2}a^3 + \frac{2}{3}a^3\int_0^{\frac{\pi}{4}}\sec\theta\mathrm{d}\theta) = \frac{1}{2}(\frac{2}{3}\sqrt{2}a^3 + \frac{2}{3}a^3\ln|\tan\theta+\sec\theta|\big|_0^{\frac{\pi}{4}})$
$= \frac{1}{3}a^3[\sqrt{2}+\ln(1+\sqrt{2})].$

5. 设$t>0, f(x)$在$[0,1]$上连续，求证$\int_0^t\mathrm{d}x\int_0^x\mathrm{d}y\int_0^y f(x)f(y)f(z)\mathrm{d}z = \frac{1}{6}(\int_0^t f(s)\mathrm{d}s)^3$.

证明：方法1：设$F(x)=\int_0^x f(s)\mathrm{d}s$，则

$$\begin{aligned}
&\int_0^t \mathrm{d}x\int_0^x \mathrm{d}y\int_0^y f(x)f(y)f(z)\mathrm{d}z=\int_0^t f(x)\mathrm{d}x\int_0^x f(y)\mathrm{d}y\int_0^y f(z)\mathrm{d}z\\
=&\int_0^t f(x)\mathrm{d}x\int_0^x f(y)\mathrm{d}y\int_0^y \mathrm{d}F(z)=\int_0^t f(x)\mathrm{d}x\int_0^x f(y)[F(y)-F(0)]\mathrm{d}y\\
=&\int_0^t f(x)\mathrm{d}x\int_0^x f(y)F(y)\mathrm{d}y=\int_0^t f(x)\mathrm{d}x\int_0^x F(y)\mathrm{d}F(y)\\
=&\int_0^t f(x)\{\frac{1}{2}[F(x)]^2-[F(0)]^2\}\mathrm{d}x=\frac{1}{2}\int_0^t [F(x)]^2\mathrm{d}F(x)=\frac{1}{6}[F(x)]^3\Big|_0^t=\frac{1}{6}[F(t)]^3\\
=&\frac{1}{6}(\int_0^t f(s)\mathrm{d}s)^3.
\end{aligned}$$

方法2：记$\varphi(y)=(\int_0^y f(z)\mathrm{d}z)^2$，则$\varphi'(y)=2f(y)\int_0^y f(z)\mathrm{d}z$. 于是

$$\begin{aligned}
\int_0^x \mathrm{d}y\int_0^y f(y)f(z)\mathrm{d}z=\int_0^x \mathrm{d}y[f(y)\int_0^y f(z)\mathrm{d}z]=\frac{1}{2}\int_0^x \varphi' y\mathrm{d}y=\frac{1}{2}\int_0^x \mathrm{d}\varphi(y)=\frac{1}{2}\varphi(x)\\
=\frac{1}{2}(\int_0^x f(s)\mathrm{d}s)^2,
\end{aligned}$$

$$\begin{aligned}
\int_0^t \mathrm{d}x\int_0^x \mathrm{d}y\int_0^y f(x)f(y)f(z)\mathrm{d}z=\frac{1}{2}\int_0^t \mathrm{d}x[f(x)(\int_0^x f(s)\mathrm{d}s)^2]=\frac{1}{6}\int_0^t \mathrm{d}(\int_0^x f(s)\mathrm{d}s)^3\\
=\frac{1}{6}(\int_0^t f(s)\mathrm{d}s)^3.
\end{aligned}$$

6. 计算$\int_L(x^{\frac{4}{3}}+y^{\frac{4}{3}})\mathrm{d}l$，其中$L$为星形线$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}(a>0)$.

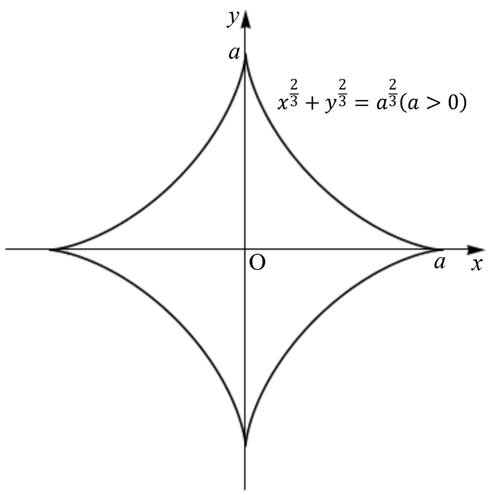

图 3: 第12章补充题 6.题图示

解：L的参数方程可表示为$\begin{cases}x=a\cos^3\theta,\\ y=a\sin^3\theta,\end{cases}\quad 0\leqslant\theta\leqslant 2\pi,$

$\therefore \int_L (x^{\frac{4}{3}}+y^{\frac{4}{3}})\mathrm{d}l=\int_0^{2\pi}(a^{\frac{4}{3}}\cos^4\theta+a^{\frac{4}{3}}\sin^4\theta)\sqrt{[(a\cos^3\theta)']^2+[(a\sin^3\theta)']^2}\mathrm{d}\theta$
$=\int_0^{2\pi}(a^{\frac{4}{3}}\cos^4\theta+a^{\frac{4}{3}}\sin^4\theta)\sqrt{[-3a\cos^2\theta\sin\theta]^2+[3a\sin^2\theta\cos\theta]^2}\mathrm{d}\theta$
$=\int_0^{2\pi}(a^{\frac{4}{3}}\cos^4\theta+a^{\frac{4}{3}}\sin^4\theta)\sqrt{9a^2\cos^2\theta\sin^2\theta}\mathrm{d}\theta=4\int_0^{\frac{\pi}{2}}(a^{\frac{4}{3}}\cos^4\theta+a^{\frac{4}{3}}\sin^4\theta)\sqrt{9a^2\cos^2\theta\sin^2\theta}\mathrm{d}\theta$
$=4\int_0^{\frac{\pi}{2}}(a^{\frac{4}{3}}\cos^4\theta+a^{\frac{4}{3}}\sin^4\theta)3a\cos\theta\sin\theta\mathrm{d}\theta=12a^{\frac{7}{3}}\int_0^{\frac{\pi}{2}}(\cos^5\theta\sin\theta+\sin^5\theta\cos\theta)\mathrm{d}\theta$
$=12a^{\frac{7}{3}}(\int_0^{\frac{\pi}{2}}\cos^5\theta\sin\theta\mathrm{d}\theta+\int_0^{\frac{\pi}{2}}\sin^5\theta\cos\theta\mathrm{d}\theta)=12a^{\frac{7}{3}}(-\frac{1}{6}\cos^6\theta\Big|_0^{\frac{\pi}{2}}+\frac{1}{6}\sin^6\theta\Big|_0^{\frac{\pi}{2}})$
$=4a^{\frac{7}{3}}.$

7. 设$f(u)$是连续函数，$D=\{(x,y)\mid x^2+y^2\leqslant a^2\}$. 试求

$$\iint\limits_D \frac{af(x)+bf(y)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y.$$

解：$\because$积分域D关于$y=x$对称，且在关于$y=x$的对称点(x,y)和$(x',y')=(y,x)$处
$\frac{f(x')}{f(x')+f(y')}=\frac{f(y)}{f(y)+f(x)}$,
$\therefore \iint\limits_D \frac{f(x)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y=\iint\limits_D \frac{f(y)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y=\frac{1}{2}\iint\limits_D \frac{f(x)+f(y)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y=\frac{1}{2}\iint\limits_D \mathrm{d}x\mathrm{d}y=\frac{1}{2}\pi a^2$,
$\therefore \iint\limits_D \frac{af(x)+bf(y)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y=a\iint\limits_D \frac{f(x)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y+b\iint\limits_D \frac{f(y)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y=\frac{a+b}{2}\pi a^2.$

8. 设$D=\{(x,y)\mid |x|+|y|\leqslant 1\}$，将$\iint\limits_D f(x+y)\mathrm{d}x\mathrm{d}y$化为定积分.

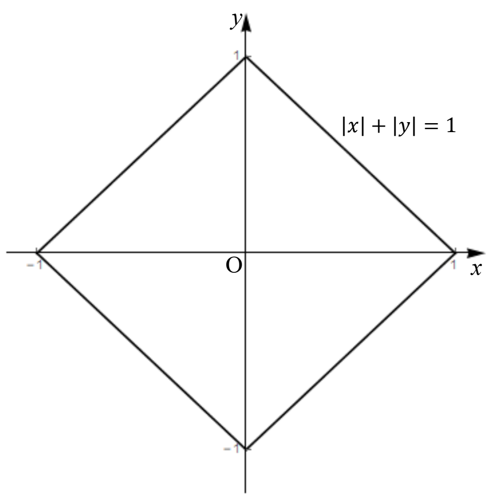

图 4: 第12章补充题 8.题图示

解：方法1：令$\begin{cases} u=x+y, \\ v=x-y, \end{cases}$ 则$D=\{(u,v)\mid -1\leqslant u\leqslant 1,-1\leqslant v\leqslant 1\}$,

$\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}=\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}=-2,$

$\therefore \iint\limits_{D} f(x+y)\mathrm{d}x\mathrm{d}y = \iint\limits_{D} f(u)\frac{1}{|\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}|}\mathrm{d}u\mathrm{d}v = \iint\limits_{D} \frac{1}{2}f(u)\mathrm{d}u\mathrm{d}v = \frac{1}{2}\int_{-1}^{1}\mathrm{d}v\int_{-1}^{1}f(u)\mathrm{d}u$
$= \int_{-1}^{1}f(u)\mathrm{d}u.$

方法2：将重积分化为累次积分，有

$$\iint\limits_{|x|+|y|\leqslant 1} f(x+y)\mathrm{d}x\mathrm{d}y = \int_{-1}^{0}\mathrm{d}x\int_{-1-x}^{1+x}f(x+y)\mathrm{d}y + \int_{0}^{1}\mathrm{d}x\int_{-1+x}^{1-x}f(x+y)\mathrm{d}y,$$

令$x+y=u$,

$$\begin{aligned}\text{上式} &= \int_{-1}^{0}\mathrm{d}x\int_{-1}^{1+2x}f(u)\mathrm{d}u + \int_{0}^{1}x\mathrm{d}\int_{2x-1}^{1}f(u)\mathrm{d}u \\ &= \int_{-1}^{1}f(u)\mathrm{d}u\int_{\frac{u-1}{2}}^{\frac{u+1}{2}}\mathrm{d}x = \int_{-1}^{1}f(u)\mathrm{d}u.\end{aligned}$$

9. 设一元函数f连续，$\Omega_t = \{(x,y,z) \mid x^2+y^2 \leqslant t^2, 0 \leqslant z \leqslant h\}$. 令$F(t) = \iiint\limits_{\Omega_t}[z^2+f(x^2+y^2)]\mathrm{d}V$，试求$\frac{\mathrm{d}F}{\mathrm{d}t}$和$\lim\limits_{t\to 0}\frac{F(t)}{t^2}$.

解：$F(t) = \iiint\limits_{\Omega_t}[z^2+f(x^2+y^2)]\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta\int_0^t\mathrm{d}r\int_0^h[z^2+f(r^2)]r\mathrm{d}z$
$= 2\pi\int_0^t[\frac{1}{3}rz^3+rf(r^2)z]\big|_0^h\mathrm{d}r = 2\pi\int_0^t[\frac{1}{3}h^3r+hrf(r^2)]\mathrm{d}r,$
$\therefore \frac{\mathrm{d}F}{\mathrm{d}t} = \frac{2\pi}{3}h^3t+2\pi htf(t^2),$
$\therefore \lim\limits_{t\to 0}\frac{1}{t^2}F(t) = \lim\limits_{t\to 0}\frac{\frac{\mathrm{d}F}{\mathrm{d}t}}{2t} = \lim\limits_{t\to 0}\frac{2\pi[\frac{1}{3}h^3t+htf(t^2)]}{2t} = \lim\limits_{t\to 0}\pi[\frac{1}{3}h^3+hf(t^2)] = \frac{\pi}{3}h^3+\pi hf(0).$

10. 计算下列积分：
(1)$\iiint\limits_{\Omega}(ax+by+cz)^2\mathrm{d}V$，其中$\Omega = \{(x,y,z) \mid x^2+y^2+z^2 \leqslant R^2\}$；
(2)$\iiint\limits_{\Omega}(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2})\mathrm{d}V$，其中$\Omega = \{(x,y,z) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1\}$.

解：(1)方法1：由对称性可知
$\iiint\limits_{\Omega}(ax+by+cz)^2\mathrm{d}V = \iiint\limits_{\Omega}(a^2x^2+b^2y^2+c^2z^2+2abxy+2acxz+2bcyz)\mathrm{d}V$
$= \iiint\limits_{\Omega}(a^2x^2+b^2y^2+c^2z^2)\mathrm{d}V = (a^2+b^2+c^2)\iiint\limits_{\Omega}z^2\mathrm{d}V$
$= (a^2+b^2+c^2)\iiint\limits_{\Omega}r^2\cos^2\varphi\cdot r^2\sin\varphi\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r$
$= (a^2+b^2+c^2)\int_0^{2\pi}\mathrm{d}\theta\int_0^R r^4\mathrm{d}r\int_0^{\pi}\cos^2\varphi\sin\varphi\mathrm{d}\varphi$
$= \frac{2}{5}\pi R^5(a^2+b^2+c^2)(-\frac{1}{3}\cos^3\varphi)\big|_0^{\pi} = \frac{4}{15}\pi R^5(a^2+b^2+c^2).$

方法2：$\iiint\limits_{\Omega}(ax+by+cz)^2\mathrm{d}V = \iiint\limits_{\Omega}(a^2x^2+b^2y^2+c^2z^2+2abxy+2acxz+2bcyz)\mathrm{d}V$
$= \iiint\limits_{\Omega}(a^2x^2+b^2y^2+c^2z^2)\mathrm{d}V = (a^2+b^2+c^2)\iiint\limits_{\Omega}z^2\mathrm{d}V = \frac{1}{3}(a^2+b^2+c^2)\iiint\limits_{\Omega}(x^2+y^2+z^2)\mathrm{d}V$
$= \frac{1}{3}(a^2+b^2+c^2)\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^R r^2\cdot r^2\sin\varphi\mathrm{d}r = \frac{2}{3}\pi(a^2+b^2+c^2)\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_0^R r^4\mathrm{d}r$
$= \frac{2}{3}\pi(a^2+b^2+c^2)(-\cos\varphi)\big|_0^{\pi}\frac{1}{5}r^5\big|_0^R = \frac{4}{15}\pi R^5(a^2+b^2+c^2).$

(2)令$\begin{cases} x = au, \\ y = bv, \\ z = cw, \end{cases}$ 则$\Omega = \{(u,v,w) \mid u^2+v^2+w^2 \leqslant 1\}$,

$\frac{\mathrm{D}(x,y,z)}{\mathrm{D}(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$,

$\therefore \iiint\limits_{\Omega}(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2})\mathrm{d}V = \iiint\limits_{\Omega}(u^2+v^2+w^2)|\frac{\mathrm{D}(u,v,w)}{\mathrm{D}(x,y,z)}|\mathrm{d}u\mathrm{d}v\mathrm{d}w = abc\iiint\limits_{\Omega}(u^2+v^2+w^2)\mathrm{d}u\mathrm{d}v\mathrm{d}w$
$= abc\frac{4}{15}\pi 1^4(1^2+1^2+1^2) = \frac{4}{5}\pi abc.$

21.4 习题13.1解答

1. 验证梯度算子$\boldsymbol{\nabla}$的下列性质，其中α,β为任意常数，f,g为任意可微函数：
   (1)$\boldsymbol{\nabla}(\alpha f+\beta g) = \alpha\boldsymbol{\nabla}+\beta\boldsymbol{\nabla}g$；
   (2)$\boldsymbol{\nabla}(fg) = g\boldsymbol{\nabla}f+f\boldsymbol{\nabla}g$；
   (3)$\boldsymbol{\nabla}(\frac{f}{g}) = \frac{g\boldsymbol{\nabla}f-f\boldsymbol{\nabla}g}{g^2}$(在$g$不等于零处成立).

   证明：(1)

$$\begin{aligned}
\boldsymbol{\nabla}(\alpha f+\beta g) &= (\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})(\alpha f+\beta g) \\
&= \frac{\partial(\alpha f+\beta g)}{\partial x}\boldsymbol{i}+\frac{\partial(\alpha f+\beta g)}{\partial y}\boldsymbol{j}+\frac{\partial(\alpha f+\beta g)}{\partial z}\boldsymbol{k} \\
&= (\alpha\frac{\partial f}{\partial x}+\beta\frac{\partial g}{\partial x})\boldsymbol{i}+(\alpha\frac{\partial f}{\partial y}+\beta\frac{\partial g}{\partial y})\boldsymbol{j}+(\alpha\frac{\partial f}{\partial z}+\beta\frac{\partial g}{\partial z})\boldsymbol{k} \\
&= \alpha(\frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}+\frac{\partial f}{\partial z}\boldsymbol{k})+\beta(\frac{\partial g}{\partial x}\boldsymbol{i}+\frac{\partial g}{\partial y}\boldsymbol{j}+\frac{\partial g}{\partial z}\boldsymbol{k}) \\
&= \alpha\boldsymbol{\nabla}f+\beta\boldsymbol{\nabla}g.
\end{aligned}$$

   (2)

$$\begin{aligned}
\boldsymbol{\nabla}(fg) &= (\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})(fg) \\
&= \frac{\partial(fg)}{\partial x}\boldsymbol{i}+\frac{\partial(fg)}{\partial y}\boldsymbol{j}+\frac{\partial(fg)}{\partial z}\boldsymbol{k} \\
&= (g\frac{\partial f}{\partial x}+f\frac{\partial g}{\partial x})\boldsymbol{i}+(g\frac{\partial f}{\partial y}+f\frac{\partial g}{\partial y})\boldsymbol{j}+(g\frac{\partial f}{\partial z}+f\frac{\partial g}{\partial z})\boldsymbol{k} \\
&= g(\frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}+\frac{\partial f}{\partial z}\boldsymbol{k})+f(\frac{\partial g}{\partial x}\boldsymbol{i}+\frac{\partial g}{\partial y}\boldsymbol{j}+\frac{\partial g}{\partial z}\boldsymbol{k}) \\
&= g\boldsymbol{\nabla}f+f\boldsymbol{\nabla}g.
\end{aligned}$$

(3)

$$\begin{aligned}
\boldsymbol{\nabla}(\frac{f}{g}) &= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k})(\frac{f}{g}) \\
&= \frac{\partial}{\partial x}(\frac{f}{g})\boldsymbol{i} + \frac{\partial}{\partial y}(\frac{f}{g})\boldsymbol{j} + \frac{\partial}{\partial z}(\frac{f}{g})\boldsymbol{k} \\
&= \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\boldsymbol{i} + \frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\boldsymbol{j} + \frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\boldsymbol{k} \\
&= \frac{g(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}) - f(\frac{\partial g}{\partial x}\boldsymbol{i} + \frac{\partial g}{\partial y}\boldsymbol{j} + \frac{\partial g}{\partial z}\boldsymbol{k})}{g^2} \\
&= \frac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2}.
\end{aligned}$$

2. 验证散度算子的下列性质(其中f为函数，$\boldsymbol{u}, \boldsymbol{v}$是向量场)：

$$\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = -\boldsymbol{u} \cdot \boldsymbol{\nabla} \times \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \times \boldsymbol{u}.$$

证明：

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{u}\times\boldsymbol{v}) &= (\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})\cdot\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ u_1(x,y,z) & u_2(x,y,z) & u_3(x,y,z)\\ v_1(x,y,z) & v_2(x,y,z) & v_3(x,y,z)\end{vmatrix}\\
&= (\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})\cdot[(u_2v_3-u_3v_2)\boldsymbol{i}+(u_3v_1-u_1v_3)\boldsymbol{j}+(u_1v_2-u_2v_2)\boldsymbol{k}]\\
&= \frac{\partial(u_2v_3-u_3v_2)}{\partial x}+\frac{\partial(u_3v_1-u_1v_3)}{\partial y}+\frac{\partial(u_1v_2-u_2v_1)}{\partial z}\\
&= \frac{\partial u_2}{\partial x}v_3+u_2\frac{\partial v_3}{\partial x}-\frac{\partial u_3}{\partial x}v_2-u_3\frac{\partial v_2}{\partial x}\\
&+\frac{\partial u_3}{\partial y}v_1+u_3\frac{\partial v_1}{\partial y}-\frac{\partial u_1}{\partial y}v_3-u_1\frac{\partial v_3}{\partial y}\\
&+\frac{\partial u_1}{\partial z}v_2+u_1\frac{\partial v_2}{\partial z}-\frac{\partial u_2}{\partial z}v_1-u_2\frac{\partial v_1}{\partial z}\\
&= u_1(\frac{\partial v_2}{\partial z}-\frac{\partial v_3}{\partial y})+u_2(\frac{\partial v_3}{\partial x}-\frac{\partial v_1}{\partial z})+u_3(\frac{\partial v_1}{\partial y}-\frac{\partial v_2}{\partial x})\\
&+v_1(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z})+v_2(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x})+v_3(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y})\\
&= -u_1(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z})-u_2(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x})-u_3(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y})\\
&\quad+v_1(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z})+v_2(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x})+v_3(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y})\\
&= -(u_1\boldsymbol{i}+u_2\boldsymbol{j}+u_3\boldsymbol{k})\cdot[(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z})\boldsymbol{i}+(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x})\boldsymbol{j}+(\frac{\partial v_3}{\partial x}-\frac{\partial v_1}{\partial y})\boldsymbol{k}]\\
&\quad+(v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k})\cdot[(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z})\boldsymbol{i}+(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x})\boldsymbol{j}+(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y})\boldsymbol{k}]\\
&= -(u_1\boldsymbol{i}+u_2\boldsymbol{j}+u_3\boldsymbol{k})\cdot\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}+(v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k})\cdot\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u_1 & u_2 & u_3\end{vmatrix}\\
&= -\boldsymbol{u}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})+\boldsymbol{v}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u})
\end{aligned}
$$

3. 设$\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}, r=\sqrt{x^2+y^2+z^2}$：

(1)设$f(u)$为可微函数，求$\boldsymbol{\nabla}f(r)$；

(2)设$\boldsymbol{F}=f(r)\boldsymbol{r}$，求证$\boldsymbol{\nabla}\times\boldsymbol{F}\equiv\boldsymbol{0}$. 又问当$f$满足什么条件时，$\boldsymbol{\nabla}\cdot\boldsymbol{F}=0$?

解：(1)

$$\begin{aligned}\boldsymbol{\nabla} f(r) &= \frac{\partial f(r)}{\partial x}\boldsymbol{i} + \frac{\partial f(r)}{\partial y}\boldsymbol{j} + \frac{\partial f(r)}{\partial z}\boldsymbol{k} \\ &= f'(r)\frac{\partial r}{\partial x}\boldsymbol{i} + f'(r)\frac{\partial r}{\partial y}\boldsymbol{j} + f'(r)\frac{\partial r}{\partial z}\boldsymbol{k} = f'(r)(\frac{\partial r}{\partial x}\boldsymbol{i} + \frac{\partial r}{\partial y}\boldsymbol{j} + \frac{\partial r}{\partial z}\boldsymbol{k}) \\ &= f'(r)(\frac{x}{\sqrt{x^2+y^2+z^2}}\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\boldsymbol{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\boldsymbol{k}) \\ &= \frac{f'(r)}{\sqrt{x^2+y^2+z^2}}(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) = \frac{f'(r)}{r}\boldsymbol{r}.\end{aligned}$$

(2)

i)证明：

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{F} = \boldsymbol{\nabla}\times[f(r)\boldsymbol{r}] &= \boldsymbol{\nabla} f(r)\times\boldsymbol{r} + f(r)\boldsymbol{\nabla}\times\boldsymbol{r} \\ &= \frac{f'(r)}{r}\boldsymbol{r}\times\boldsymbol{r} + f(r)\boldsymbol{\nabla}\times\boldsymbol{r} \\ &= 0 + f(r)\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\ &= f(r)[(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z})\boldsymbol{i} + (\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x})\boldsymbol{j} + (\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y})\boldsymbol{k}] \\ &= f(r)(0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}) = 0.\end{aligned}$$

ii)

$\because$

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{F} = \boldsymbol{\nabla}\cdot[f(r)\boldsymbol{r}] &= \boldsymbol{\nabla} f(r)\cdot\boldsymbol{r} + f(r)\boldsymbol{\nabla}\cdot\boldsymbol{r} = \frac{f'(r)}{r}\boldsymbol{r}\cdot\boldsymbol{r} + f(r)(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}) \\ &= \frac{\mathrm{d}f(r)}{\mathrm{d}r}r + 3f(r) = 0,\end{aligned}$$

$\therefore$当$f(r)\neq 0$时，$r\neq 0$,

$$\frac{\mathrm{d}f(r)}{f(r)} = -3\frac{\mathrm{d}r}{r},$$

$\therefore$

$$\ln|f(r)| = -3\ln|r| + C_1,$$

$\therefore$

$$f(r) = \frac{C}{r^3}, C\text{为任意常数}.$$

4. 验证旋度算子的下列基本公式：

(1)$\boldsymbol{\nabla}\times(\alpha\boldsymbol{u} + \beta\boldsymbol{v}) = \alpha\boldsymbol{\nabla}\times\boldsymbol{u} + \beta\boldsymbol{\nabla}\times\boldsymbol{v}$;

(2)$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \mathbf{0}$;

(3)$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) = 0$.

证明：(1)

$$\begin{aligned}
\boldsymbol{\nabla} \times (\alpha \boldsymbol{u} + \beta \boldsymbol{v}) &= \boldsymbol{\nabla} \times (\alpha u_1 \boldsymbol{i} + \alpha u_2 \boldsymbol{j} + \alpha u_3 \boldsymbol{k} + \beta v_1 \boldsymbol{i} + \beta v_2 \boldsymbol{j} + \beta v_3 \boldsymbol{k}) \\
&= \boldsymbol{\nabla} \times [(\alpha u_1 + \beta v_1)\boldsymbol{i} + (\alpha u_2 + \beta v_2)\boldsymbol{j} + (\alpha u_3 + \beta v_3)\boldsymbol{k}] \\
&= \frac{\partial(\alpha u_1 + \beta v_1)}{\partial x} + \frac{\partial(\alpha u_2 + \beta v_2)}{\partial y} + \frac{\partial(\alpha u_3 + \beta v_3)}{\partial z} \\
&= \alpha\frac{\partial u_1}{\partial x} + \beta\frac{\partial v_1}{\partial x} + \alpha\frac{\partial u_2}{\partial y} + \beta\frac{\partial v_2}{\partial y} + \alpha\frac{\partial u_3}{\partial z} + \beta\frac{\partial v_3}{\partial z} \\
&= \alpha(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}) + \beta(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}) \\
&= \alpha \boldsymbol{\nabla} \cdot \boldsymbol{u} + \beta \boldsymbol{\nabla} \cdot \boldsymbol{v}.
\end{aligned}$$

(2)当$f \in C^2$时

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \boldsymbol{\nabla} \times (\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= (\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x})\boldsymbol{i} + (\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z})\boldsymbol{j} + (\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x})\boldsymbol{k} \\
&= 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k} = \mathbf{0}.
\end{aligned}$$

(3)当$\boldsymbol{v} \in C^2$时

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) &= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}) \cdot \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1(x,y,z) & v_2(x,y,z) & v_3(x,y,z) \end{vmatrix} \\
&= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}) \cdot [(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z})\boldsymbol{i} + (\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x})\boldsymbol{j} + (\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y})\boldsymbol{k}] \\
&= (\frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z}) + (\frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_3}{\partial y \partial x}) + (\frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}) \\
&= (\frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_3}{\partial y \partial x}) + (\frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_1}{\partial z \partial y}) + (\frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_2}{\partial x \partial z}) \\
&= 0.
\end{aligned}$$

5. 求下列向量场的散度：

(1)$\boldsymbol{v} = xyz(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$;

(2)$\boldsymbol{v} = (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \times \boldsymbol{c}$;

(3)$\boldsymbol{v} = [(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{c}](x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$(其中$\boldsymbol{c}$为常值向量).

解：(1)

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{v}&=\boldsymbol{\nabla}(xyz)\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})+xyz\boldsymbol{\nabla}\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\\&=(yz\boldsymbol{i}+xz\boldsymbol{j}+xy\boldsymbol{k})\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})+xyz(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z})\\&=xyz+xyz+xyz+3xyz=6xyz.\end{aligned}$$

(2)

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{v}&=\boldsymbol{c}\cdot[\boldsymbol{\nabla}\times(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})]-(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot(\boldsymbol{\nabla}\times\boldsymbol{c})\\&=\boldsymbol{c}\cdot\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\x&y&z\end{vmatrix}-(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{0}\\&=\boldsymbol{c}\cdot[(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z})\boldsymbol{i}+(\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x})\boldsymbol{j}+(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y})\boldsymbol{k}]-0\\&=\boldsymbol{c}\cdot\boldsymbol{0}-0=0.\end{aligned}$$

(3)记$\boldsymbol{c}=(c_1,c_2,c_3)$

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{v}&=[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}]\boldsymbol{\nabla}\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})+(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{\nabla}[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}]\\&=[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}](\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z})+(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{\nabla}(c_1x+c_2y+c_3z)\\&=3[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}]+(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot(c_1\boldsymbol{i}+c_2\boldsymbol{j}+c_3\boldsymbol{k})\\&=3[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}]+(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}\\&=4[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}.\end{aligned}$$

6. 求下列向量场的旋度：

(1)$\boldsymbol{v}=y^2z\boldsymbol{i}+z^2x\boldsymbol{j}+x^2y\boldsymbol{k}$;

(2)$\boldsymbol{v}=f(\sqrt{x^2+y^2+z^2})\boldsymbol{c}$(其中$\boldsymbol{c}$)为常值向量.

解：(1)

$$\boldsymbol{\nabla}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\y^2z&z^2x&x^2y\end{vmatrix}=(x^2-2xz)\boldsymbol{i}+(y^2-2xy)\boldsymbol{j}+(z^2-2yz)\boldsymbol{k}.$$

(2)

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{v}&=\boldsymbol{\nabla}\times[f(\sqrt{x^2+y^2+z^2})\boldsymbol{c}]\\&=\boldsymbol{\nabla}f(\sqrt{x^2+y^2+z^2})\times\boldsymbol{c}+f(\sqrt{x^2+y^2+z^2})\boldsymbol{\nabla}\times\boldsymbol{c}\\&=f'(\sqrt{x^2+y^2+z^2})(\frac{x}{\sqrt{x^2+y^2+z^2}}\boldsymbol{i}+\frac{y}{\sqrt{x^2+y^2+z^2}}\boldsymbol{j}+\frac{z}{\sqrt{x^2+y^2+z^2}}\boldsymbol{k})\times\boldsymbol{c}+\boldsymbol{0}\\&=\frac{f'(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\times\boldsymbol{c}.\end{aligned}$$