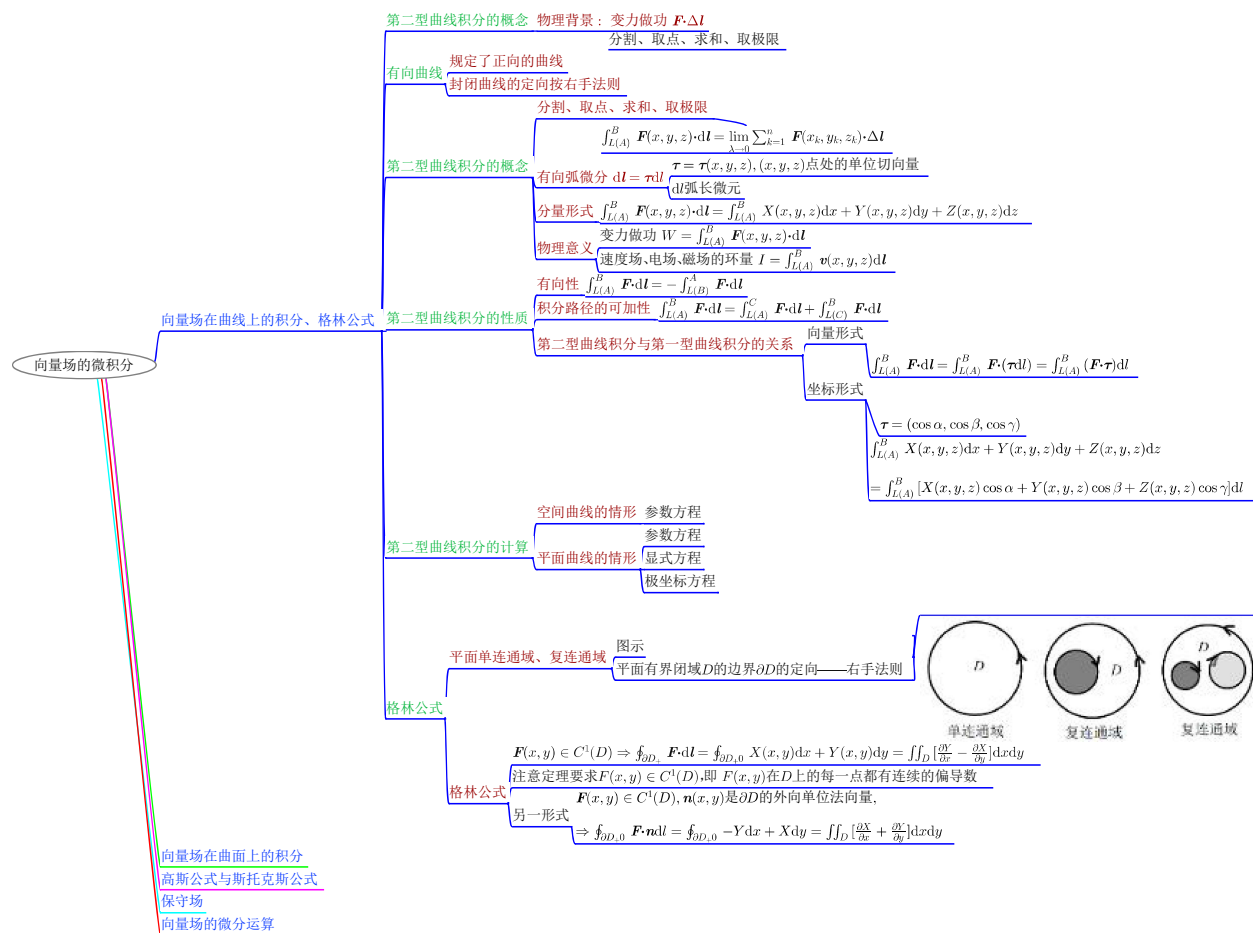


10 第二型曲线积分、格林公式

10.1 复习计划



10.2 知识结构



10.3 重要知识

第二型曲线积分是为了计算力在曲线路径上做功的问题引入的, 是向量场在曲线路径上的积分. 第一型曲线积分是数量场在曲线路径上的积分, 要注意第二型曲线积分与第一型曲线积分的区别.

1. 第二型曲线积分的计算

(a) 空间曲线——已知参数方程:

$$L: \begin{cases} x = x(t), \\ y = y(t), \\ z = z(t), \end{cases} \quad t \in [\alpha, \beta], A: t = \alpha, B: t = \beta, \mathbf{F}(x, y, z) \in C(L),$$

$$\Rightarrow \int_{L(A)}^B \mathbf{F}(x, y, z) \cdot d\mathbf{l} = \int_{L(A)}^B X(x, y, z)dx + Y(x, y, z)dy + Z(x, y, z)dz$$

$$= \int_{\alpha}^{\beta} [X(x(t), y(t), z(t))x'(t) + Y(x(t), y(t), z(t))y'(t) + Z(x(t), y(t), z(t))z'(t)]dt$$

(b) 平面曲线

i. 已知参数方程:

$$L: \begin{cases} x = x(t), \\ y = y(t) \end{cases}, \quad t \in [\alpha, \beta], A: t = \alpha, B: t = \beta, \mathbf{F}(x, y) \in C(L),$$

$$\Rightarrow \int_{L(A)}^B \mathbf{F}(x, y) \cdot d\mathbf{l} = \int_{L(A)}^B X(x, y)dx + Y(x, y)dy$$

$$= \int_{\alpha}^{\beta} [X(x(t), y(t))x'(t) + Y(x(t), y(t))y'(t)]dt$$

ii. 已知显式方程:

$$L: y = f(x) \in C[a, b], A: x = a, B: x = b, \mathbf{F}(x, y) \in C(L),$$

$$\Rightarrow \int_{L(A)}^B \mathbf{F}(x, y) \cdot d\mathbf{l} = \int_{L(A)}^B X(x, y)dx + Y(x, y)dy$$

$$= \int_a^b [X(x, f(x)) + Y(x, f(x))f'(x)]dx$$

iii. 已知极坐标方程:

$$\begin{aligned} L: r = r(\theta), A: t = \alpha, B: t = \beta, \mathbf{F}(x, y) \in C(L), \\ \Rightarrow \int_{L(A)}^B \mathbf{F}(x, y) \cdot d\mathbf{l} = \int_{L(A)}^B X(x, y)dx + Y(x, y)dy \\ = \int_{\alpha}^{\beta} [X(r(\theta) \cos \theta, r(\theta) \sin \theta)(r(\theta) \cos \theta)' + Y(r(\theta) \cos \theta, r(\theta) \sin \theta)(r(\theta) \sin \theta)'] d\theta \end{aligned}$$

2. 格林公式.

(a) 格林公式的第一个形式:

$$\mathbf{F}(x, y) \in C^1(D) \Rightarrow \oint_{\partial D_+} \mathbf{F} \cdot d\mathbf{l} = \oint_{\partial D_+} X(x, y)dx + Y(x, y)dy = \iint_D \left[\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right] dx dy$$

(b) 格林公式的第二个形式:

$$\begin{aligned} \mathbf{F}(x, y) \in C^1(D), \mathbf{n}(x, y) \text{ 是 } \partial D \text{ 的外向单位法向量,} \\ \Rightarrow \oint_{\partial D_+} \mathbf{F} \cdot \mathbf{n} dl = \oint_{\partial D_+} -Y dx + X dy = \iint_D \left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right] dx dy \end{aligned}$$

10.4 习题分类与解题思路

对于第二型空间曲线积分, 目前我们只有参数方程一种类型的曲线可以计算.

对于第二型平面曲线积分, 可以根据曲线的形式(参数方程、显式方程、极坐标方程), 选择合适的公式计算.

对于平面封闭曲线上的第二型曲线积分, 可利用格林公式转化成二重积分, 以简化计算. 应用格林公式时, 须注意平面向量场的两个分量函数应在曲线围成的区域内有一阶连续偏导数, 若不满足该条件, 可选取简单的曲线, 将偏导数不存在的点去掉, 构造可以应用格林公式的区域.

1. 第二型曲线积分.

(a) 空间曲线. 可选择合适的参数将曲线方程写成参数方程的形式, 代入公式计算.

【如习题13.2中的7., 11.】

(b) 平面曲线

i. 显式方程

【如习题13.2中的2., 5.(折线路径, 积分路径可加性), 6.(折线路径, 积分路径可加性), 8.(折线路径, 积分路径可加性)】

ii. 极坐标方程

【如习题13.2中的3., 4.】

iii. 参数方程

【如习题13.2中的1.】

(c) 在计算之前要利用曲线方程将被积函数适当化简.

【如习题13.2中的3., 4., 5., 8.】

(d) 利用对称性可简化计算.

【如习题13.2中的9.(轮换对称性)】

(e) 考查第二型曲线积分的物理意义. (力在曲线路径上做功.)

【如习题13.2中的10.】

2. 格林公式.

(a) 直接代入公式 $\oint_{\partial D} X(x, y)dx + Y(x, y)dy = \iint_D \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dxdy$ 计算. 函数 $X(x, y), Y(x, y)$ 应满足一阶偏导数连续的条件. 初等函数的偏导数在定义域均满足连续的条件, 需要注意初等函数定义域外的点.

【如习题13.3中的1.(1)/(2)/(3)/(4)/(5)/(6)/(7), 2.(1)., 3.(1)/(2)】

(b) 考查函数 $X(x, y), Y(x, y)$ 应满足一阶偏导数连续的条件这一点. 可选取简单曲线将偏导函数的间断点去掉. 这一点会在格林公式、高斯公式、斯托克斯公式的专题中进行总结.

【如习题13.3中的2.(2)】

(c) 考查格林公式的第二个形式.

【如习题13.3中的4.】

10.5 习题13.2解答

1. 计算 $\int_L (x+y)dx - (x-y)dy$, 其中 L 为椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 的上半周, 逆时针方向为正.

$$\begin{aligned} \text{解: } \int_L (x+y)dx - (x-y)dy &= \int_0^\pi (a \cos \theta + b \sin \theta) d(a \cos \theta) - (a \cos \theta - b \sin \theta) d(b \sin \theta) \\ &= \int_0^\pi (-a^2 \sin \theta \cos \theta - ab \sin^2 \theta - ab \cos^2 \theta + b^2 \sin \theta \cos \theta) d\theta \\ &= (b^2 - a^2) \int_0^\pi \sin \theta \cos \theta d\theta - ab \int_0^\pi d\theta = 0 - \pi ab = -\pi ab. \end{aligned}$$

2. 计算 $\int_L y^2 dx - x^2 dy$, 其中 L 为抛物线 $y = x^2$ 自 $x = -1$ 到 $x = 1$ 的一段.

$$\begin{aligned} \text{解: } \int_L y^2 dx - x^2 dy &= \int_{-1}^1 (x^2)^2 dx - x^2 d(x^2) = \int_{-1}^1 x^4 dx - 2x^3 dx = \int_{-1}^1 x^4 - 2x^3 dx \\ &= \left(\frac{1}{5}x^5 - \frac{2}{4}x^4\right)\Big|_{-1}^1 = \frac{2}{5}. \end{aligned}$$

3. 计算 $\oint_L \frac{xdy-ydx}{x^2+y^2}$, 其中 L 为圆周 $x^2 + y^2 = a^2$ (逆时针方向为正).

$$\begin{aligned} \text{解: } \oint_L \frac{xdy-ydx}{x^2+y^2} &= \frac{1}{a^2} \oint_L xdy - ydx = \frac{1}{a^2} \int_0^{2\pi} a \cos \theta d(a \sin \theta) - a \sin \theta d(a \cos \theta) \\ &= \frac{1}{a^2} \int_0^{2\pi} a \cos \theta (a \cos \theta) d\theta - a \sin \theta (-a \sin \theta) d\theta = \frac{1}{a^2} \int_0^{2\pi} (a^2 \cos^2 \theta + a^2 \sin^2 \theta) d\theta = 2\pi. \end{aligned}$$

4. 计算 $\oint_L \frac{ydy - xdx}{x^2 + y^2}$, 其中 L 为圆周 $x^2 + y^2 = a^2$ (逆时针方向为正).

$$\begin{aligned} \text{解: } \oint_L \frac{ydy - xdx}{x^2 + y^2} &= \frac{1}{a^2} \oint_L ydy - xdx = \frac{1}{a^2} \int_0^{2\pi} a \sin \theta d(a \sin \theta) - a \cos \theta d(a \cos \theta) \\ &= \frac{1}{a^2} \int_0^{2\pi} a \sin \theta (a \cos \theta) d\theta - a \cos \theta (-a \sin \theta) d\theta = \frac{1}{a^2} \int_0^{2\pi} (a^2 \sin \theta \cos \theta + a^2 \sin \theta \cos \theta) d\theta \\ &= \int_0^{2\pi} \sin \theta \cos \theta d\theta = \int_0^{2\pi} \sin \theta d \sin \theta = \frac{1}{2} \sin^2 \theta \Big|_0^{2\pi} = 0. \end{aligned}$$

5. 计算 $\int_L \frac{dx+dy}{|x|+|y|}$, 其中 L 为由 $A(0, -1)$ 到 $B(1, 0)$ 再到 $C(0, 1)$ 的折线.

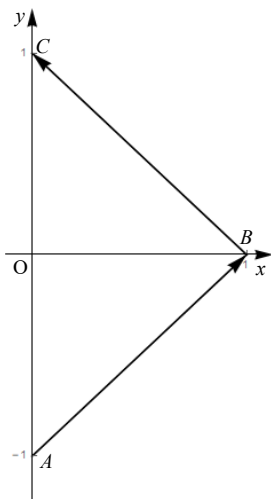


图 1: 习题13.2 5.题图示

解: AB 段的方程为 $\frac{x}{1} + \frac{y}{-1} = 1$, 即 $y = x - 1, x: 0 \rightarrow 1$, BC 段的方程为 $\frac{x}{1} + \frac{y}{1} = 1$, 即 $y = 1 - x, x: 1 \rightarrow 0$,

易知折线 ABC 满足方程 $|x| + |y| = 1$,

$$\begin{aligned} \therefore \int_L \frac{dx+dy}{|x|+|y|} &= \int_L dx + dy = (\int_{AB} + \int_{BC})(dx + dy) = \int_{AB} dx + dy + \int_{BC} dx + dy \\ &= \int_0^1 dx + d(x-1) + \int_1^0 dx + d(1-x) = \int_0^1 dx + dx + \int_1^0 dx - dx = 2 \int_0^1 dx + 0 = 2. \end{aligned}$$

6. 计算 $\int_L (x^2 + y^2)dx + (x^2 - y^2)dy$, 其中 L 为 $y = 1 - |1 - x|, x \in [0, 2]$, 曲线正向为 x 增长的方向.

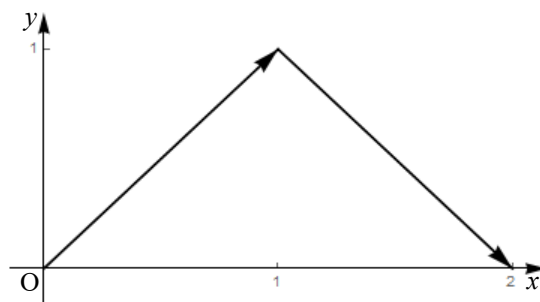


图 2: 习题13.2 6.题图示

$$\text{解: } y = 1 - |1 - x| = \begin{cases} x, & 0 \leq x < 1, \\ 2 - x, & 1 \leq x \leq 2, \end{cases}$$

$$\begin{aligned} & \therefore \int_L (x^2 + y^2)dx + (x^2 - y^2)dy \\ &= \int_0^1 (x^2 + x^2)dx + (x^2 - x^2)dx + \int_1^2 [x^2 + (2-x)^2]dx + [x^2 - (2-x)^2]d(2-x) \\ &= \int_0^1 2x^2 dx + \int_1^2 [x^2 + 4 - 4x + x^2]dx - [x^2 - 4 + 4x - x^2]dx \\ &= \frac{2}{3}x^3 \Big|_0^1 + \int_1^2 [x^2 + 4 - 4x + x^2 - x^2 + 4 - 4x + x^2]dx \\ &= \frac{2}{3} + \int_1^2 [2x^2 - 8x + 8]dx \\ &= \frac{2}{3} + \left(\frac{2}{3}x^3 - 4x^2 + 8x\right) \Big|_1^2 = \frac{2}{3} + \frac{2}{3} \cdot 7 - 4 \cdot 3 + 8 = \frac{4}{3}. \end{aligned}$$

7. 计算 $\oint_L xyz dz$, 其中 L 为 $x^2 + y^2 + z^2 = 1, z = y$, 由 z 轴正向看去为逆时针方向.

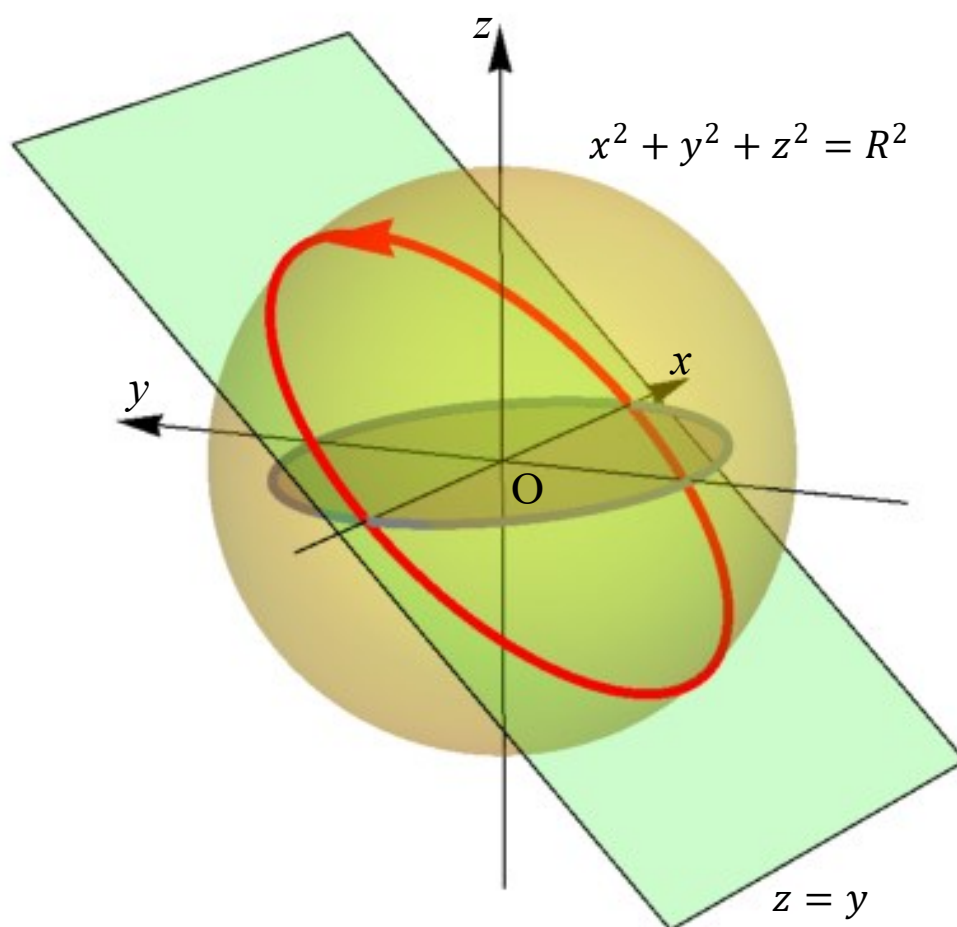


图 3: 习题13.2 7.题图示

解: 由 $\begin{cases} x^2 + y^2 + z^2 = 1, \\ z = y, \end{cases}$ 得 L 所在的投影柱面为 $x^2 + 2y^2 = 1$, 故 L 的方程可表示

为 $\begin{cases} x^2 + 2y^2 = 1, \\ z = y, \end{cases}$ 故可令 $L: \begin{cases} x = \cos \theta, \\ y = \frac{1}{\sqrt{2}} \sin \theta, \\ z = \frac{1}{\sqrt{2}} \sin \theta, \end{cases} \quad \theta: 0 \rightarrow 2\pi,$

$$\therefore \oint_L xyz dz = \int_0^{2\pi} \cos \theta \frac{1}{\sqrt{2}} \sin \theta \cdot \frac{1}{\sqrt{2}} \sin \theta d \frac{1}{\sqrt{2}} \sin \theta = \int_0^{2\pi} \cos \theta \frac{1}{\sqrt{2}} \sin \theta \cdot \frac{1}{\sqrt{2}} \sin \theta \cdot \frac{1}{\sqrt{2}} \cos \theta d\theta$$

$$\begin{aligned}
 &= \frac{1}{2\sqrt{2}} \int_0^{2\pi} \sin^2 \theta \cos^2 \theta d\theta = \frac{1}{2\sqrt{2}} \cdot 4 \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^2 \theta d\theta = \frac{1}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} \sin^2 2\theta d\theta \\
 &= \frac{1}{2\sqrt{2}} \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 2\theta d(2\theta) = \frac{1}{2\sqrt{2}} \cdot \frac{1}{2} \int_0^{\pi} \sin^2 \varphi d\varphi = \frac{1}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} \sin^2 \varphi d\varphi = \frac{1}{2\sqrt{2}} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{8\sqrt{2}}.
 \end{aligned}$$

8. 计算 $\int_L \frac{dx+dy}{|x|+|y|}$, 其中 L 为 $|x|+|y|=1$, 逆时针方向为正.

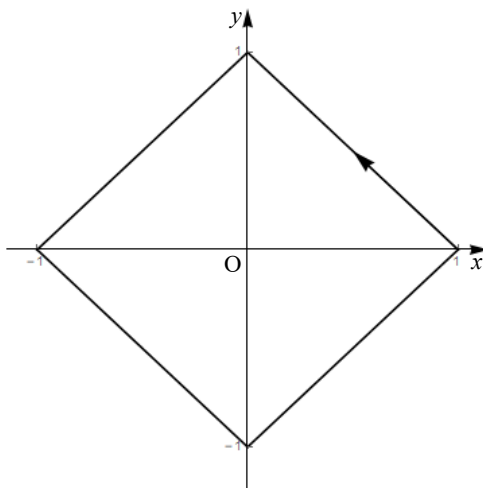


图 4: 习题13.2 8.题图示

$$\text{解: } L \text{ 的方程 } |x|+|y|=1 \text{ 等价于 } y = \begin{cases} 1-x, & 0 \leq x < 1, \\ 1+x, & -1 \leq x < 0, \\ -1-x, & -1 < x \leq 0, \\ x-1, & 0 < x \leq 1, \end{cases}$$

$$\begin{aligned}
 \therefore \int_L \frac{dx+dy}{|x|+|y|} &= \int_L dx + dy \\
 &= \int_0^1 dx + d(1-x) + \int_{-1}^0 dx + d(1+x) + \int_{-1}^0 dx + d(-1-x) + \int_0^1 dx + d(x-1) \\
 &= \int_0^1 dx - dx + \int_{-1}^0 dx + dx + \int_{-1}^0 dx - dx + \int_0^1 dx + dx \\
 &= 0 + 2 \int_0^1 dx + 0 + 2 \int_0^1 dx = -2 + 2 = 0.
 \end{aligned}$$

9. 计算 $\oint_L (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz$, 其中 L 为 $x^2 + y^2 + z^2 = 1$ 在第一卦限与三个坐标面的交线, 方向是 $A(1,0,0) \rightarrow B(0,1,0) \rightarrow C(0,0,1) \rightarrow A(1,0,0)$.

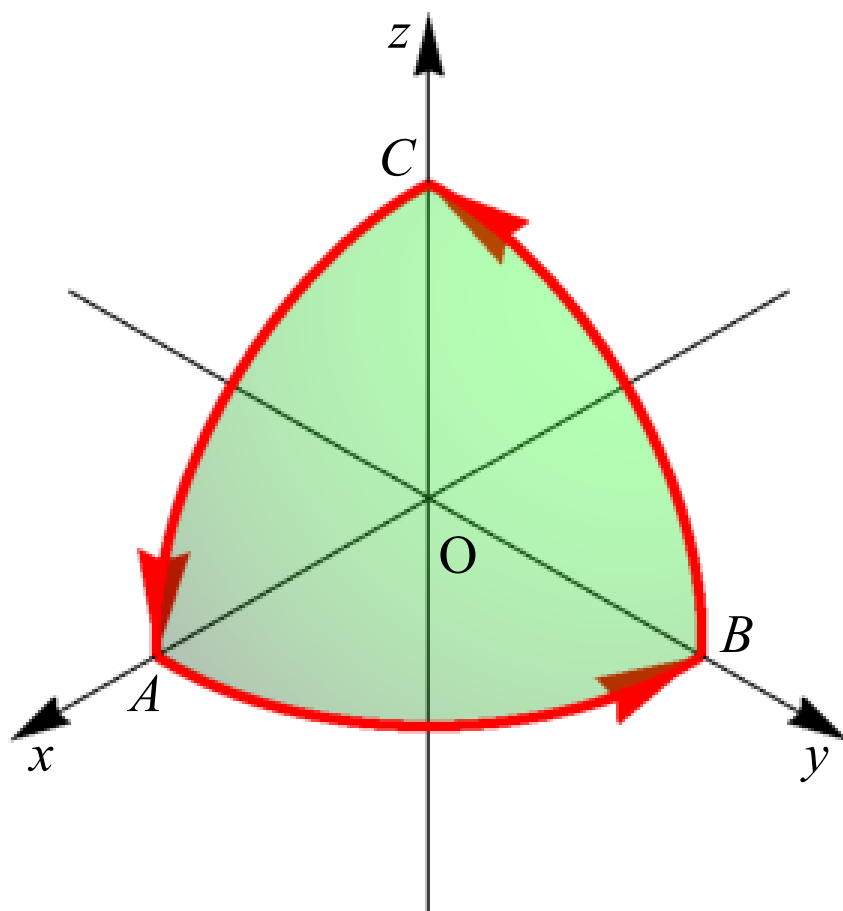


图 5: 习题13.2 9.题图示

$$\begin{aligned}
 &\text{解: } \oint_L (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz \\
 &= (\int_{\widehat{AB}} + \int_{\widehat{BC}} + \int_{\widehat{CA}})[(y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz] \\
 &= \int_{\widehat{AB}} (y^2 - 0^2)dx + (0^2 - x^2)dy + (x^2 - y^2)d0 \\
 &\quad + \int_{\widehat{BC}} (y^2 - z^2)d0 + (z^2 - 0^2)dy + (0^2 + y^2)dz \\
 &\quad + \int_{\widehat{CA}} (0^2 - z^2)dx + (z^2 - x^2)d0 + (x^2 - 0^2)dz \\
 &= \int_{\widehat{AB}} y^2 dx - x^2 dy + \int_{\widehat{BC}} z^2 dy - y^2 dz + \int_{\widehat{CA}} x^2 dz - z^2 dx
 \end{aligned}$$

$$\begin{aligned}
& \left(= \int_{\substack{x^2+y^2=1, z=0 \\ x \geq 0, y \geq 0}} y^2 dx - x^2 dy + \int_{\substack{y^2+z^2=1, x=0 \\ y \geq 0, z \geq 0}} z^2 dy - y^2 dz + \int_{\substack{z^2+x^2=1, y=0 \\ z \geq 0, x \geq 0}} x^2 dz - z^2 dx \right. \\
& \left. = 13 \int_{\substack{x^2+y^2=1, z=0 \\ x \geq 0, y \geq 0}} y^2 dx - x^2 dy \right) \\
& = 3 \int_{\widehat{AB}} y^2 dx - x^2 dy \\
& = 3 \int_0^{\frac{\pi}{2}} \sin^2 \theta d \cos \theta - \cos^2 \theta d \sin \theta = 3 \int_0^{\frac{\pi}{2}} -\sin^2 \theta \sin \theta d\theta - \cos^2 \theta \cos \theta d\theta \\
& = -3 \int_0^{\frac{\pi}{2}} \sin^3 \theta d\theta - 3 \int_0^{\frac{\pi}{2}} \cos^3 \theta d\theta = -3 \cdot \frac{2}{3} - 3 \cdot \frac{2}{3} = -4.
\end{aligned}$$

10. 设平面上的力场 $\mathbf{F}(x, y)$ 指向原点, 每点处力的大小与该点到原点的距离成正比, 比例系数为 k . 设一单位质量的质点沿曲线 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 从点 $A(a, 0)$ 到 $B(0, b)$, 求力场 $\mathbf{F}(x, y)$ 所做的功.

解: 因 $\mathbf{F}(x, y)$ 指向原点, 故可设 $\mathbf{F}(x, y) = -\lambda(x, y)$, $\lambda > 0$,

\therefore 每点处力的大小 $|\mathbf{F}(x, y)|$ 与该点到原点的距离成正比, 比例系数为 k ,

$$\therefore |\mathbf{F}(x, y)| = \lambda \sqrt{x^2 + y^2} = k \sqrt{x^2 + y^2},$$

$$\therefore \lambda = k, \mathbf{F}(x, y) = -k(x, y) = (-kx, -ky),$$

¹这里的依据是轮换对称性, 即按照 $x \rightarrow y, y \rightarrow z, z \rightarrow x$ 可将等号左边第一项变换成第二项, 因表达式的值与数学符号无关, 故第一项和第二项相等. 同理, 按照 $x \rightarrow y, y \rightarrow z, z \rightarrow x$ 可将等号左边第三项变成第一项, 故第三项和第一项相等. 变换顺序可用图 6 表示.

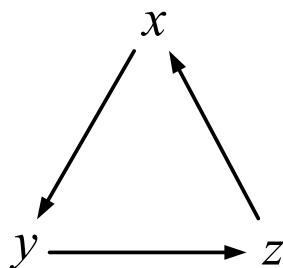


图 6: 轮换对称性的变换顺序

∴力场 $\mathbf{F}(x, y)$ 所做的功

$$\begin{aligned}
 W &= \int_{L(A)}^B \mathbf{F}(x, y) \cdot d\mathbf{l} = \int_{L(A)}^B (-kx, -ky) \cdot (dx, dy) = \int_{L(A)}^B -kx dx - ky dy \\
 &= \int_0^{\frac{\pi}{2}} -ka \cos \theta d(a \cos \theta) - kb \sin \theta d(b \sin \theta) \\
 &= \int_0^{\frac{\pi}{2}} -ka \cos \theta (-a \sin \theta) d\theta - kb \sin \theta (b \cos \theta) d\theta \\
 &= \int_0^{\frac{\pi}{2}} (ka^2 \sin \theta \cos \theta - kb^2 \sin \theta \cos \theta) d\theta \\
 &= k(a^2 - b^2) \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta d\theta = k(a^2 - b^2) \int_0^{\frac{\pi}{2}} \sin \theta d \sin \theta \\
 &= k(a^2 - b^2) \frac{1}{2} \sin^2 \theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{2} k(a^2 - b^2).
 \end{aligned}$$

11. 计算 $\int_L y dx + z dy + x dz$, 其中 L 是螺旋线 $x = a \cos t, y = a \sin t, z = bt (0 \leq t \leq 2\pi)$, 正向为 t 增加的方向.

$$\begin{aligned}
 \text{解: } \int_L y dx + z dy + x dz &= \int_0^{2\pi} a \sin t d(a \cos t) + b t d(a \sin t) + a \cos t d(bt) \\
 &= \int_0^{2\pi} a \sin t (-a \sin t) dt + b t a \cos t dt + ab \cos t dt \\
 &= -a^2 \int_0^{2\pi} \sin^2 t dt + ab \int_0^{2\pi} t \cos t dt + ab \int_0^{2\pi} \cos t dt \\
 &= -4a^2 \int_0^{\frac{\pi}{2}} \sin^2 t dt + ab \int_0^{2\pi} t d \sin t + 0 \\
 &= -4a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + ab(t \sin t \Big|_0^{2\pi} - \int_0^{2\pi} \sin t dt) = -\pi a^2 + (0 - 0) = -\pi a^2.
 \end{aligned}$$

10.6 习题13.3解答

1. 用格林公式计算下列积分:

- (1) $\oint_L xy^2 dy - x^2 y dx$, 其中 L 为圆周 $x^2 + y^2 = a^2$, 逆时针方向;
- (2) $\oint_L (3x + y) dy - (x - y) dx$, 其中 L 为圆周 $(x - 1)^2 + (y - 4)^2 = 9$;
- (3) $\oint_L (x^2 + y^2) dx + (y^2 - x^2) dy$, 其中 L 是区域 D 的边界正向(逆时针), 区域 D 由直线 $y = 0, x = 1, y = x$ 围成;
- (4) $\oint_L (2xy - x^2) dx + (x + y^2) dy$, 其中 L 是区域 D 的边界正向, 区域 D 由曲线 $x = y^2, y = x^2$ 围成;
- (5) $\oint_L (x + y) dx + xy dy$, 其中 L 为椭圆周 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, 正向;
- (6) $\oint_L \sqrt{x^2 + y^2} dx + [x + y \ln(x + \sqrt{x^2 + y^2})] dy$, 其中 L 为圆周 $(x - 2)^2 + y^2 = 1$, 逆时针方向;
- (7) $\oint_L e^x [(1 - \cos y) dx - (y - \sin y) dy]$, 其中 L 为 $D = \{(x, y) \mid 0 \leq x \leq \pi, 0 \leq y \leq \sin x\}$ 的正向边界.

解: (1) 设 D 为以 L 为边界的有界闭域,

$$\therefore \frac{\partial(xy^2)}{\partial x} = y^2, \quad \frac{\partial(-x^2 y)}{\partial y} = -x^2 \text{ 均在 } D \text{ 上连续,}$$

$$\begin{aligned} \therefore \oint_L xy^2 dy - x^2 y dx &= \iint_D \left[\frac{\partial(xy^2)}{\partial x} - \frac{\partial(-x^2 y)}{\partial y} \right] dx dy = \iint_D (y^2 + x^2) dx dy = \int_0^{2\pi} d\theta \int_0^a r^2 \cdot r dr \\ &= 2\pi \cdot \frac{1}{4} r^4 \Big|_0^a = \frac{\pi}{2} a^4. \end{aligned}$$

(2) 设 D 为以 L 为边界的有界闭域,

$$\therefore \frac{\partial(3x+y)}{\partial x} = 3, \quad \frac{\partial[-(x-y)]}{\partial y} = 1 \text{ 均在 } D \text{ 上连续,}$$

$$\begin{aligned} \therefore \oint_L (3x+y) dy - (x-y) dx &= \iint_D \left\{ \frac{\partial(3x+y)}{\partial x} - \frac{\partial[-(x-y)]}{\partial y} \right\} dx dy = \iint_D (3-1) dx dy = 2 \iint_D dx dy \\ &= 2\pi \cdot 9 = 18\pi. \end{aligned}$$

$$(3) \therefore \frac{\partial(y^2-x^2)}{\partial x} = -2x, \quad \frac{\partial(x^2+y^2)}{\partial y} = 2y \text{ 均在 } D \text{ 上连续,}$$

$$\begin{aligned} \therefore \oint_L (x^2+y^2) dx + (y^2-x^2) dy &= \iint_D \left[\frac{\partial(y^2-x^2)}{\partial x} - \frac{\partial(x^2+y^2)}{\partial y} \right] dx dy = \iint_D (-2x-2y) dx dy \\ &= \int_0^1 dx \int_0^x (-2x-2y) dy = \int_0^1 (-2xy - y^2) \Big|_0^x dx = \int_0^1 -2x^2 - x^2 dx = -3x^3 \Big|_0^1 = -3. \end{aligned}$$

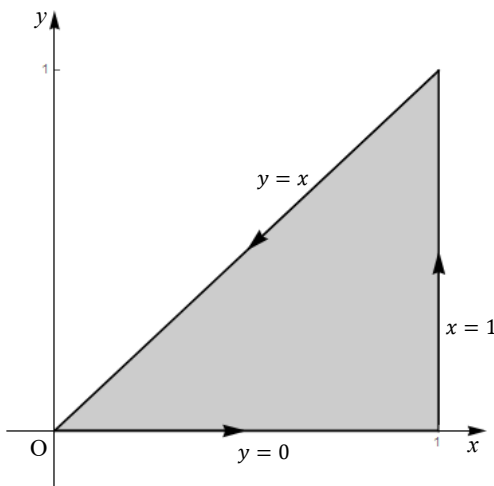


图 7: 习题13.3 1.(3)题图示

$$(4) \therefore \frac{\partial(x+y^2)}{\partial x} = 1, \quad \frac{\partial(2xy-x^2)}{\partial y} = 2x \text{ 均在 } D \text{ 上连续,}$$

$$\begin{aligned} \therefore \oint_L (2xy-x^2) dx + (x+y^2) dy &= \iint_D \left[\frac{\partial(x+y^2)}{\partial x} - \frac{\partial(2xy-x^2)}{\partial y} \right] dx dy = \iint_D (1-2x) dx dy \\ &= \int_0^1 dx \int_{x^2}^{\sqrt{x}} (1-2x) dy = \int_0^1 (y-2xy) \Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 (\sqrt{x} - x^2 - 2x^{\frac{3}{2}} + 2x^3) dx \\ &= \left(\frac{1}{1+\frac{1}{2}} x^{1+\frac{1}{2}} - \frac{1}{3} x^3 - \frac{2}{1+\frac{3}{2}} x^{\frac{3}{2}+1} + \frac{2}{4} x^4 \right) \Big|_0^1 = \frac{2}{3} - \frac{1}{3} - \frac{4}{5} + \frac{1}{2} = \frac{1}{30}. \end{aligned}$$

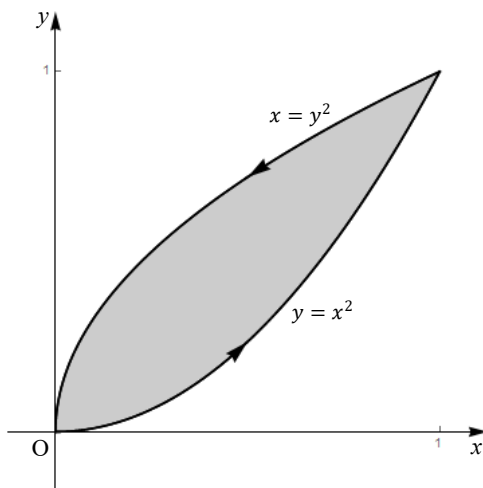


图 8: 习题13.3 1.(4)题图示

(5) 设 D 为以 L 为边界的有界闭域,

$\because \frac{\partial(xy)}{\partial x} = y, \frac{\partial(x+y)}{\partial y} = 1$ 均在 D 上连续,

$$\begin{aligned} \therefore \oint_L (x+y)dx + xydy &= \iint_D \left[\frac{\partial(xy)}{\partial x} - \frac{\partial(x+y)}{\partial y} \right] dxdy = \iint_D (y-1) dxdy \\ &= \int_0^{2\pi} d\theta \int_0^1 (br \sin \theta - 1) \cdot ab r dr = ab^2 \int_0^{2\pi} \sin \theta d\theta \int_0^1 r^2 dr - ab \int_0^{2\pi} d\theta \int_0^1 r dr \\ &= 0 - ab2\pi \cdot \frac{1}{2} = -\pi ab. \end{aligned}$$

(6) 设 D 为以 L 为边界的有界闭域,

$\because \frac{\partial[x+y \ln(x+\sqrt{x^2+y^2})]}{\partial x} = 1 + y \frac{1+\frac{x}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}} = 1 + y \frac{\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}} = 1 + \frac{y}{\sqrt{x^2+y^2}}, \frac{\partial\sqrt{x^2+y^2}}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$ 均在 D 上连续,

$$\begin{aligned} \therefore \oint_L \sqrt{x^2+y^2}dx + [x+y \ln(x+\sqrt{x^2+y^2})]dy &= \iint_D \left\{ \frac{\partial[x+y \ln(x+\sqrt{x^2+y^2})]}{\partial x} - \frac{\partial\sqrt{x^2+y^2}}{\partial y} \right\} dxdy \\ &= \iint_D \left(1 + \frac{y}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}} \right) dxdy = \iint_D dxdy = \pi. \end{aligned}$$

(7) $\because \frac{\partial[-e^x(y-\sin y)]}{\partial x} = -e^x(y-\sin y), \frac{\partial[e^x(1-\cos y)]}{\partial y} = e^x \sin y$ 均在 D 上连续,

$$\begin{aligned} \therefore \oint_L e^x[(1-\cos y)dx - (y-\sin y)dy] &= \iint_D \left\{ \frac{\partial[-e^x(y-\sin y)]}{\partial x} - \frac{\partial[e^x(1-\cos y)]}{\partial y} \right\} dxdy \\ &= \iint_D [-e^x(y-\sin y) - e^x \sin y] dxdy = \iint_D -e^x y dxdy = -\int_0^\pi dx \int_0^{\sin x} e^x y dy = -\int_0^\pi e^x \frac{1}{2} y^2 \Big|_0^{\sin x} dx \\ &= -\frac{1}{2} \int_0^\pi e^x \sin^2 x dx = -\frac{1}{2} \int_0^\pi \sin^2 x de^x = -\frac{1}{2} e^x \sin^2 x \Big|_0^\pi + \frac{1}{2} \int_0^\pi e^x d \sin^2 x = \frac{1}{2} \int_0^\pi e^x 2 \sin x \cos x dx \\ &= \frac{1}{2} \int_0^\pi e^x \sin 2x dx = \frac{1}{2} \int_0^\pi \sin 2x de^x = \frac{1}{2} e^x \sin 2x \Big|_0^\pi - \frac{1}{2} \int_0^\pi e^x d \sin 2x = -\frac{1}{2} \int_0^\pi e^x 2 \cos 2x dx \\ &= -\int_0^\pi \cos 2x de^x = -e^x \cos 2x \Big|_0^\pi + \int_0^\pi e^x d \cos 2x = -e^\pi + 1 - 2 \int_0^\pi e^x \sin 2x dx \\ &= \frac{1}{2} \frac{1}{1+2} (-e^\pi + 1) = \frac{1}{5} (1 - e^\pi). \end{aligned}$$

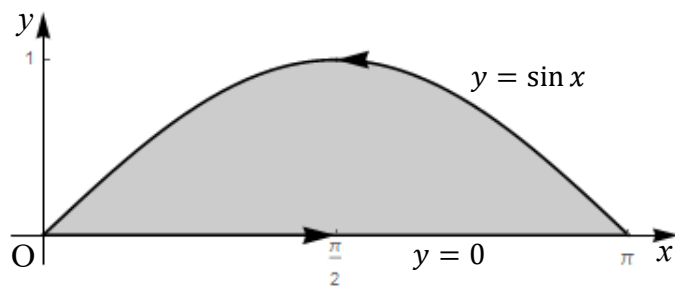


图 9: 习题13.3 1.(7)题图示

2. 计算 $I = \oint_L \frac{(x+y)dx - (x-y)dy}{x^2+y^2}$, 其中 L :

(1) $D = \{(x, y) \mid r^2 \leq x^2 + y^2 \leq R^2\} (0 < r < R)$;

(2) $D = \left\{ (x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \right\}$ 的正向边界.

解: 令 $Y(x, y) = -\frac{x-y}{x^2+y^2}$, $X(x, y) = \frac{x+y}{x^2+y^2}$, 则

$$\begin{aligned} \frac{\partial Y(x, y)}{\partial x} &= \frac{-(x^2 + y^2) + (x - y) \cdot 2x}{(x^2 + y^2)^2} = \frac{x^2 - y^2 - 2xy}{(x^2 + y^2)^2}, \\ \frac{\partial X(x, y)}{\partial y} &= \frac{x^2 + y^2 - (x + y) \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2 - 2xy}{(x^2 + y^2)^2}. \end{aligned}$$

(1) $\because \frac{\partial Y(x, y)}{\partial x}, \frac{\partial X(x, y)}{\partial y}$ 在 $D = \{(x, y) \mid r^2 \leq x^2 + y^2 \leq R^2\} (0 < r < R)$ 上连续,

\therefore

$$I = \iint_D \left[\frac{\partial Y(x, y)}{\partial x} - \frac{\partial X(x, y)}{\partial y} \right] dx dy = 0.$$

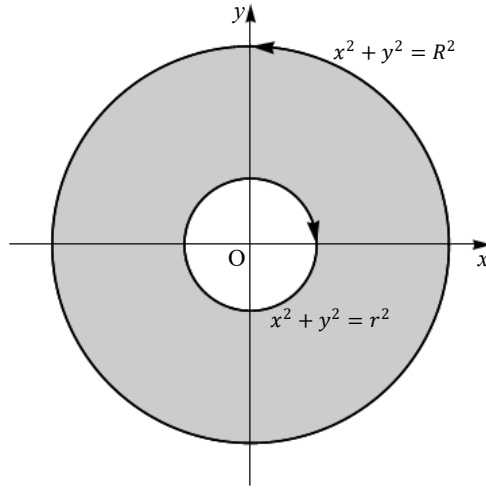


图 10: 习题13.3 2.(1)题图示

(2)取正向圆周 $L_1: x^2 + y^2 = r^2, r < \min\{a, b\}$, 记其围成的区域为 D_1 , 则 $\frac{\partial Y(x,y)}{\partial x}, \frac{\partial X(x,y)}{\partial y}$ 在 L 和 L_1 围成的区域 D^* 内连续,

\therefore

$$\oint_{L+L_1-} \frac{(x+y)dx - (x-y)dy}{x^2 + y^2} = \iint_{D^*} \left[\frac{\partial Y(x,y)}{\partial x} - \frac{\partial X(x,y)}{\partial y} \right] dx dy = 0,$$

\therefore

$$(\oint_L + \oint_{L_1-}) \frac{(x+y)dx - (x-y)dy}{x^2 + y^2} = I + \oint_{L_1-} \frac{(x+y)dx - (x-y)dy}{x^2 + y^2} = 0,$$

\therefore

$$\begin{aligned} I &= -\oint_{L_1-} \frac{(x+y)dx - (x-y)dy}{x^2 + y^2} = \oint_{L_1+} \frac{(x+y)dx - (x-y)dy}{x^2 + y^2} \\ &= \frac{1}{r^2} \oint_{L_1+} (x+y)dx - (x-y)dy \\ &= \frac{1}{r^2} \iint_{D_1} \left[\frac{\partial(-x+y)}{\partial x} - \frac{\partial(x+y)}{\partial y} \right] dx dy \\ &= \frac{1}{r^2} \iint_{D_1} -2 dx dy = \frac{1}{r^2} (-2) \pi r^2 = -2\pi. \end{aligned}$$

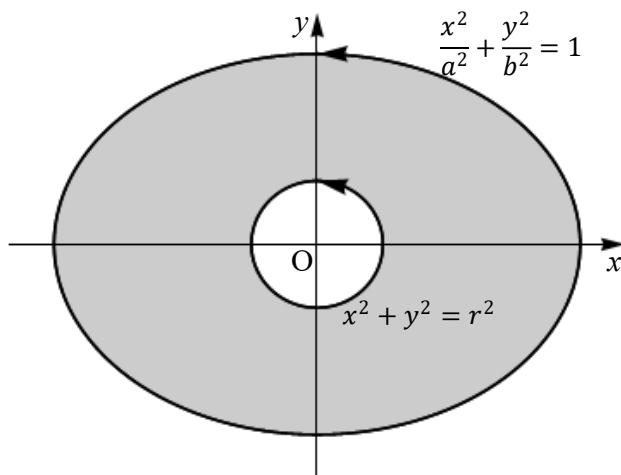


图 11: 习题13.3 2.(2)题图示

3. 设 D 为平面区域, ∂D 为逐段光滑曲线, (\bar{x}, \bar{y}) 是 D 的形心, D 的面积等于 $\sigma(D)$. 试证:

$$(1) \int_{\partial D} x^2 dy = 2\sigma(D)\bar{x};$$

$$(2) \int_{\partial D} xy dy = \sigma(D)\bar{y}.$$

解: (1) $\because x^2, 0 \in C^1(D)$,

$$\therefore \int_{\partial D} x^2 dy = \int_{\partial D} 0 dx + x^2 dy = \iint_D \left[\frac{\partial(x^2)}{\partial x} - \frac{\partial 0}{\partial y} \right] dx dy = \iint_D 2x dx dy = 2 \iint_D x dx dy = 2\sigma(D)\bar{x}.$$

(2) $\because xy, 0 \in C^1(D)$,

$$\therefore \int_{\partial D} xy dy = \int_{\partial D} 0 dx + xy dy = \iint_D \left[\frac{\partial(xy)}{\partial x} - \frac{\partial 0}{\partial y} \right] dx dy = \iint_D y dx dy = \sigma(D)\bar{y}.$$

4. 设 D 为平面区域, ∂D 为逐段光滑曲线, $f \in C^2(\bar{D})$, 求证:

$$\oint_{\partial D} \frac{\partial f}{\partial \mathbf{n}} dl = \iint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx dy.$$

证明: $\because f \in C^2(\bar{D})$,

$\therefore f \in C^1(\bar{D})$,

$\therefore f$ 在 \bar{D} 上可微,

$$\therefore \frac{\partial f}{\partial \mathbf{n}} = \text{grad} f(x, y) \cdot \mathbf{n} = \left(\frac{\partial f(x, y)}{\partial x}, \frac{\partial f(x, y)}{\partial y} \right) \cdot \mathbf{n},$$

\therefore

$$\begin{aligned}\oint_{\partial D} \frac{\partial f}{\partial \mathbf{n}} dl &= \oint_{\partial D} \text{grad} f(x, y) \cdot \mathbf{n} dl = \oint_{\partial D} \left(\frac{\partial f(x, y)}{\partial x}, \frac{\partial f(x, y)}{\partial y} \right) \cdot \mathbf{n} dl \\ &= \iint_D \left[\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \right] dx dy \\ &= \iint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx dy.\end{aligned}$$