28 线性常微分方程组

28.1 知识结构

第14章常微分方程

14.5 线性微分方程组

14.5.1 一般理论

14.5.2 线性微分方程组解的结构

14.5.3 常系数微分方程组的解

28.2 线性微分方程组

- 线性常系数微分方程组
  - 一般理论
    - 矩阵函数 $\boldsymbol{A}(t)$
      - $\boldsymbol{A}(t)=(a_{ij}(t))_{i=1,\cdots,m;j=1,\cdots,n}$ a_{ij}连续（可微）$\Rightarrow$ $\boldsymbol{A}(t)$连续（可微）
      - $\frac{\mathrm{d}\boldsymbol{A}(t)}{\mathrm{d}t}=(a_{ij}{}'(t))_{i=1,\cdots,m;j=1,\cdots,n}$
      - $\int_{t_0}^{t}\boldsymbol{A}(s)\mathrm{d}s=(\int_{t_0}^{t}a_{ij}(s)\mathrm{d}s)_{i=1,\cdots,m;j=1,\cdots,n}$
    - 线性常微分方程组
      - 非齐次方程组$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}=\boldsymbol{A}(t)\boldsymbol{x}(t)+\boldsymbol{f}(t)$
      - 齐次方程组$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}=\boldsymbol{A}(t)\boldsymbol{x}(t)$
      - 常系数线性微分方程组
        - 非齐次方程组$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{f}(t)$
        - 齐次方程组$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}=\boldsymbol{A}\boldsymbol{x}(t)$
    - 线性常微分方程组解的存在唯一性定理
      - $\boldsymbol{A}(t),\boldsymbol{f}(t)\in C(I),t_0\in I$

        $\Rightarrow\forall\boldsymbol{\xi}=(\xi_1,\cdots,\xi_n)^T\in\mathbb{R}^n,s.t.$

        初值问题$\begin{cases}\frac{\boldsymbol{x}(t)}{\mathrm{d}t}=\boldsymbol{A}(t)\boldsymbol{x}(t)+\boldsymbol{f}(t),\\ \boldsymbol{x}(t_0)=\boldsymbol{\xi}\end{cases}$在区间$I$上有唯一解.
    - 向量值函数的线性相关与线性无关
      - $\boldsymbol{\varphi}_1(t)=(\varphi_{11}(t),\cdots,\varphi_{n1}(t))^T,\cdots,\boldsymbol{\varphi}_n(t)=(\varphi_{1n}(t),\cdots,\varphi_{nn}(t))^T$

        是定义在区间I上的n个向量值函数.

        $\exists c_1,c_2,\cdots,c_n$不全为零$s.t.c_1\boldsymbol{\varphi}_1(t)+\cdots+c_n\boldsymbol{\varphi}_n(t)\equiv\boldsymbol{0},t\in I$

        $\Rightarrow$这n个向量值函数在区间I上线性相关．否则为线性无关.
      - 朗斯基行列式 $W(t)=W[\boldsymbol{\varphi}_1,\cdots,\boldsymbol{\varphi}_n](t)=\begin{vmatrix}\varphi_{11}(t)&\cdots&\varphi_{1n}(t)\\ \vdots&\ddots&\vdots\\ \varphi_{n1}(t)&\cdots&\varphi_{nn}(t)\end{vmatrix}$
      - 向量值函数线性相关的必要条件
        - $\boldsymbol{\varphi}_1(t),\cdots,\boldsymbol{\varphi}_2(t)$在区间$I$上线性相关
        - $\Rightarrow W[\boldsymbol{\varphi}_1,\cdots\boldsymbol{\varphi}_n](t)=0$
  - 线性微分方程组解的结构
    - 齐次线性方程组解的结构
      - 齐次方程组$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}=\boldsymbol{A}(t)\boldsymbol{x}(t)$的解集合是一个线性空间
      - n阶齐次线性方程组的解函数组线性相关的充分条件
        - 若向量值函数$\boldsymbol{\varphi}_1(t),\cdots,\boldsymbol{\varphi}_n(t)$都是齐次方程组在区间$I$上的解，

          则$\exists t_0\in I,s.t.W[\boldsymbol{\varphi}_1,\cdots,\boldsymbol{\varphi}_n](t_0)=0\Rightarrow\boldsymbol{\varphi}_1(t),\cdots,\boldsymbol{\varphi}_n(t)$在$I$上线性相关
      - 齐次方程组的解集合是一个n维线性空间
      - 齐次方程组的基本解组和基本解矩阵
        - 基本解组:齐次方程组的任意n个线性无关解

          $\boldsymbol{\varphi}_i(t)=(\varphi_{1i}(t),\cdots,\varphi_{ni}(t))^T,i=1,\cdots,n$
        - 基本解矩阵:由这n个线性无关解为列向量的矩阵

          $\boldsymbol{\phi}(t)=\begin{bmatrix}\varphi_{11}(t)&\cdots&\varphi_{1n}(t)\\ \vdots&\ddots&\vdots\\ \varphi_{n1}(t)&\cdots&\varphi_{nn}(t)\end{bmatrix}$
      - 齐次方程组的通解
        - $\boldsymbol{x}(t)=\boldsymbol{\phi}(t)\boldsymbol{c}$, $\boldsymbol{c}=(c_1,c_2,\cdots,c_n)^T\in\mathbb{R}^n$
        - 若基本解组满足初值条件：

          $\boldsymbol{\varphi}_1(t_0)=(1,0,0,\cdots,0)^T$,

          $\boldsymbol{\varphi}_2(t_0)=(0,1,0,\cdots,0)^T$,

          $\cdots$

          $\boldsymbol{\varphi}_n(t_0)=(0,0,0,\cdots,1)^T$,

          则方程组满足初值条件$\boldsymbol{x}(t_0)=\boldsymbol{\xi}\in\mathbb{R}^n$的解为$\boldsymbol{x}(t)=\boldsymbol{\phi}(t)\boldsymbol{\xi}$.
    - 非齐次线性方程组的解
      - 特解$\boldsymbol{\psi}(t)$：常数变易法
        - $\boldsymbol{\psi}(t)=\boldsymbol{\phi}(t)\int_{t_0}^{t}\boldsymbol{\phi}^{-1}(s)\boldsymbol{f}(s)\mathrm{d}s$，该特解满足初值条件$\boldsymbol{\psi}(t_0)=\boldsymbol{0}$
        - 满足初值条件$\boldsymbol{x}(t_0)=\boldsymbol{\xi}\in\mathbb{R}^n$的解：

          $\boldsymbol{\psi}(t)=\boldsymbol{\phi}(t)\boldsymbol{\phi}^{-1}(t_0)\boldsymbol{\xi}+\boldsymbol{\phi}(t)\int_{t_0}^{t}\boldsymbol{\phi}^{-1}(s)\boldsymbol{f}(s)\mathrm{d}s$
      - 通解：$\boldsymbol{x}(t)=\boldsymbol{\psi}(t)+\boldsymbol{\phi}(t)\boldsymbol{c}$
    - 常系数微分方程组的解
      - 方法1：用指数矩阵$\exp(\boldsymbol{A}t)$表示常系数微分方程组的解
        - 指数矩阵$\exp(\boldsymbol{A}t)$是线性常系数微分方程组$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}=\boldsymbol{A}\boldsymbol{x}(t)$满足初值条件$\boldsymbol{\phi}(0)=\boldsymbol{I}$的基本解矩阵
        - 指数矩阵的求法
          - 第一步：写出系数矩阵$\boldsymbol{A}$
          - 第二步：写出多项式函数$r(\lambda)=a_{n-1}(t)\lambda^{n-1}+\cdots+a_1(t)\lambda+a_0(t)$
          - 第三步：求矩阵$\boldsymbol{A}t$的特征值$\lambda_i,i=1,\cdots,n$
          - 第四步：利用特征值求解多项式函数的系数$a_i(t),i=0,1,\cdots,n-1$
            - 若λ_i是单重特征值则$\mathrm{e}^{\lambda_i}=r(\lambda_i)$;
            - 若λ_i是k重特征值,则

              $\mathrm{e}^{\lambda_i}=r(\lambda_i)$,

              $\mathrm{e}^{\lambda_i}=\frac{\mathrm{d}}{\mathrm{d}\lambda}r(\lambda)|_{\lambda=\lambda_i}$,

              $\cdots$

              $\mathrm{e}^{\lambda_i}=\frac{\mathrm{d}^{k-1}}{\mathrm{d}\lambda^{k-1}}r(\lambda)|_{\lambda=\lambda_i}$;
            - 可得到n个关于$a_i(t),i=0,1,\cdots,n-1$的方程
          - 第五步：$\exp(\boldsymbol{A}t)=a_{n-1}(t)\boldsymbol{A}^{n-1}t^{n-1}+\cdots a_1(t)\boldsymbol{A}t+a_0(t)$
      - 方法2：用系数矩阵的特征值和特征向量表示齐次微分方程组的解
        - 若系数矩阵$\boldsymbol{A}$有n个线性无关的特征向量$\boldsymbol{r}_i\in\mathbb{R}^n(i=1,2,\cdots,n)$,

          且其分别对应于(相同或不同)特征值$\lambda_i(i=1,2,\cdots,n)$，则函数组

          $\boldsymbol{\varphi}_1(t)=\mathrm{e}^{\lambda_1t}\boldsymbol{r}_1,\boldsymbol{\varphi}_2(t)=\mathrm{e}^{\lambda_2t}\boldsymbol{r}_2,\cdots,\boldsymbol{\varphi}_n(t)=\mathrm{e}^{\lambda_nt}\boldsymbol{r}_n$

          是齐次线性微分方程组的一个基本解组.
        - 基本解组的求法
          - 第一步：写出齐次线性方程组的系数矩阵$\boldsymbol{A}$
          - 第二步：求出系数矩阵$\boldsymbol{A}$的特征值$\lambda_i,i=1,2,\cdots,n$
          - 第三步：求出系数矩阵$\boldsymbol{A}$的n个线性无关的特征向量$\boldsymbol{r}_i$
          - 第四步：写出基本解组

            $\boldsymbol{\varphi}_1(t)=\mathrm{e}^{\lambda_1t}\boldsymbol{r}_1,\boldsymbol{\varphi}_2(t)=\mathrm{e}^{\lambda_2t}\boldsymbol{r}_2,\cdots,\boldsymbol{\varphi}_n(t)=\mathrm{e}^{\lambda_nt}\boldsymbol{r}_n$
        - 该方法适用于系数矩阵$\boldsymbol{A}$有n个线性无关的特征向量的情况
        - 利用叠加原理可以求出共轭复特征值对应的两个线性无关实解
      - 方法3：消元法

28.3 齐次线性常系数微分方程组的解法

1. 方法1：用指数矩阵$\exp(\boldsymbol{A}t)$表示线性常系数微分方程组的解. 指数矩阵$\exp(\boldsymbol{A}t)$是线性常系数微分方程组$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}=\boldsymbol{A}\boldsymbol{x}(t)$的基本解矩阵，其求法可参考以下步骤：

第一步 写出齐次线性方程组的系数矩阵$\boldsymbol{A}$;

第二步 写出多项式函数$r(\lambda)=a_{n-1}(t)\lambda^{n-1}+\cdots+a_1(t)\lambda+a_0$;

第三步 求矩阵$\boldsymbol{A}t$的特征值$\lambda_i, i=1,2,\cdots,n$;

第四步 利用特征值求解多项式函数$r(\lambda)$的系数$a_i(t), i=1,2,\cdots,n$：

(1) 对于单重特征值$\lambda_i, a_i(t)$满足$\mathrm{e}^{\lambda_i}=r(\lambda_i)$;

(2) 对于k重特征值$\lambda_i, a_i(t)$满足$\begin{cases}\mathrm{e}^{\lambda_i}=r(\lambda_i),\\ \mathrm{e}^{\lambda_i}=\frac{\mathrm{d}r(\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_i},\\ \cdots\\ \mathrm{e}^{\lambda_i}=\frac{\mathrm{d}r^{k-1}(\lambda)}{\mathrm{d}\lambda^{k-1}}\Big|_{\lambda=\lambda_i}.\end{cases}$

(3) 可据此得到n个关于$a_i(t)$的方程，从而求出$a_i(t), i=1,2,\cdots,n$.

第五步 基本解矩阵

$$\exp(\boldsymbol{A}t)=a_{n-1}(t)\boldsymbol{A}^{n-1}t^{n-1}+\cdots+a_1(t)\boldsymbol{A}t+a_0t.$$

2. 方法2：用系数矩阵的特征值和特征向量表示线性常系数微分方程组的解. 若系数矩阵有n个线性无关的特征向量$\boldsymbol{r}_i\in\mathbb{R}^n(i=1,2,\cdots,n)$, 且其分别对应于(相同或不同)特征值$\lambda_i(i=1,2,\cdots,n)$, 则函数组$\boldsymbol{\varphi}_1(t)=\mathrm{e}^{\lambda_1 t}\boldsymbol{r}_1, \boldsymbol{\varphi}_2(t)=\mathrm{e}^{\lambda_2 t}\boldsymbol{r}_2,\cdots,\boldsymbol{\varphi}_i(t)=\mathrm{e}^{\lambda_n t}\boldsymbol{r}_n(t)$是齐次方程组的一个基本解组. 基本解组的求法可参考以下步骤：

第一步 写出齐次线性方程组的系数矩阵$\boldsymbol{A}$;

第二步 求出系数矩阵$\boldsymbol{A}$的特征值$\lambda_i, i=1,2,\cdots,n$;

第三步 求出系数矩阵$\boldsymbol{A}$的n个线性无关的特征向量$\boldsymbol{r}_i, i=1,2,\cdots,n$;

第四步 写出基本解组

$$\boldsymbol{\varphi}_1(t)=\mathrm{e}^{\lambda_1 t}\boldsymbol{r}_1, \boldsymbol{\varphi}_2(t)=\mathrm{e}^{\lambda_2 t}\boldsymbol{r}_2,\cdots,\boldsymbol{\varphi}_i(t)=\mathrm{e}^{\lambda_n t}\boldsymbol{r}_n(t).$$

注意：

(a) 该方法适用于k重特征值对应有k个线性无关的特征向量的情况，即矩阵$\boldsymbol{A}$有n个线性无关的特征向量. 若系数矩阵$\boldsymbol{A}$有少于n个的线性无关的特征向量，则可采用上述方法1或下述方法3.

【如习题14.5中的1.(3)/(6).】

(b) 复特征值的情况. 设$\lambda_{\pm}=\alpha\pm\beta\mathrm{i}$是系数矩阵$\boldsymbol{A}$的一对共轭复特征值，$\boldsymbol{r}_{\pm}=\boldsymbol{a}\pm\boldsymbol{b}\mathrm{i}$是与其对应的特征向量，则$\begin{cases}\mathrm{e}^{\lambda_+ t}\boldsymbol{r}_+=\mathrm{e}^{(\alpha+\beta\mathrm{i})t}(\boldsymbol{a}+\boldsymbol{b}\mathrm{i}),\\ \mathrm{e}^{\lambda_- t}\boldsymbol{r}_-=\mathrm{e}^{(\alpha-\beta\mathrm{i})t}(\boldsymbol{a}-\boldsymbol{b}\mathrm{i})\end{cases}$是微分方程的两个线性无关解. 利用叠加原理知$\begin{cases}\mathrm{Re}(\mathrm{e}^{\lambda_+ t}\boldsymbol{r}_+)=\mathrm{e}^{\alpha t}(\boldsymbol{a}\cos\beta t-\boldsymbol{b}\sin\beta t),\\ \mathrm{Im}(\mathrm{e}^{\lambda_+ t}\boldsymbol{r}_+)=\mathrm{e}^{\alpha t}(\boldsymbol{b}\cos\beta t+\boldsymbol{a}\sin\beta t)\end{cases}$是微分方程的两个线性无关的实解.

3. 消元法. 适用于二阶齐次线性方程组

【如习题14.5中的1.(1)/(2)/(3)/(4),2.(1)/(4)】

和简单的三阶及以上的线性方程组

【如习题14.5中的2.(2)】.

具体解题步骤可参看上述习题.

4. 三种方法的比较：

(a) 三种方法中消元法较简单且不易出错，如能用消元法，可直接用消元法求解.

(b) 方法2相对于方法1简单且不易出错，但如果系数矩阵线性无关的特征向量的个数少于n则不适用.

(c) 方法1是一种万能的解法，但计算量较大，容易出错. 如果消元法和方法2都不适用，则可用方法1求解.

28.4 非齐次线性常系数微分方程组的解法

非齐次线性常系数微分方程组的求解可参考以下步骤：

第一步 用28.3中的方法求解该非齐次方程组对应的齐次方程组的基本解矩阵$\boldsymbol{\phi}(t)$;

第二步 求$\boldsymbol{\phi}^{-1}(t)$;

第三步 代入公式$\boldsymbol{x}(t)=\boldsymbol{\phi}(t)\boldsymbol{c}+\boldsymbol{\phi}(t)\int_{t_0}^{t}\boldsymbol{\phi}^{-1}(t)\boldsymbol{f}(t)\mathrm{d}t$求解非齐次线性方程组的通解. 若已知初值条件$\boldsymbol{x}(t_0)=\boldsymbol{\xi}$，则可代入公式$\boldsymbol{x}(t)=\boldsymbol{\phi}(t)\boldsymbol{\phi}^{-1}(t_0)\boldsymbol{\xi}+\boldsymbol{\phi}(t)\int_{t_0}^{t}\boldsymbol{\phi}^{-1}(t)\boldsymbol{f}(t)\mathrm{d}t$求解非齐次线性方程组的通解.

28.5 习题14.5解答

1. 求下列方程组满足指定条件的解：

(1)$\begin{cases}\frac{\mathrm{d}x_1}{\mathrm{d}t}=x_1+x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t}=2x_1-4x_2,\end{cases}$ $(x_1(0),x_2(0))=(1,-1)$;

(2)$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 + 2x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = 4x_1 + 3x_2, \end{cases} \quad (x_1(0), x_2(0)) = (1, 0);$

(3)$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 - x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 + 3x_2, \end{cases} \quad (x_1(0), x_2(0)) = (2, 3);$

(4)$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = 4x_1 - 2x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 - 4x_2, \end{cases} \quad (x_1(0), x_2(0)) = (1, 0);$

(5)$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 + x_3, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_3 + x_1, \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1 + x_2, \end{cases} \quad (x_1(0), x_2(0), x_3(0)) = (2, 3, 1);$

(6)$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 - x_1, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = 4x_3 - x_2, \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1 - 4x_3, \end{cases} \quad (x_1(0), x_2(0), x_3(0)) = (2, 3, 1).$

解：(1)方法1：方程组的系数矩阵为$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 2 & -4 \end{bmatrix}$,

$$\exp(\boldsymbol{A}t) = a_1\boldsymbol{A}t + a_0\boldsymbol{I} = \begin{bmatrix} a_1 t + a_0 & a_1 t \\ 2a_1 t & -4a_1 t + a_0 \end{bmatrix},\ r(\lambda t) = a_1\lambda t + a_0,$$

由$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ -2 & \lambda + 4 \end{vmatrix} = (\lambda - 1)(\lambda + 4) - 2 = \lambda^2 + 3\lambda - 6 = 0$得特征值$\lambda_{1,2} = \frac{-3\pm\sqrt{9+4\times 6}}{2} = \frac{-3\pm\sqrt{33}}{2}$, 则$a_1, a_0$满足

$$\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t} = a_1 \cdot (\frac{-3-\sqrt{33}}{2}t) + a_0,\ \mathrm{e}^{\frac{-3+\sqrt{33}}{2}t} = a_1 \cdot (\frac{-3+\sqrt{33}}{2}t) + a_0,$$

则

$$\begin{aligned} a_1 &= \frac{\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t} - \mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}}{\sqrt{33}t}, \\ a_0 &= \frac{\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t} + \mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}}{2} + \frac{3}{2}t\frac{\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t} - \mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}}{\sqrt{33}t} \\ &= \frac{1}{2}(1 + \frac{3}{\sqrt{33}})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t} + \frac{1}{2}(1 - \frac{3}{\sqrt{33}})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}, \end{aligned}$$

$\therefore$

$$\begin{aligned}
a_1t+a_0&=\frac{\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}}{\sqrt{33}t}t+\frac{1}{2}(1+\frac{3}{\sqrt{33}})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+\frac{1}{2}(1-\frac{3}{\sqrt{33}})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}\\
&=\frac{1}{2}(1+\frac{5}{\sqrt{33}})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+\frac{1}{2}(1-\frac{5}{\sqrt{33}})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t},\\
a_1t&=\frac{\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}}{\sqrt{33}t}t=\frac{1}{\sqrt{33}}\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-\frac{1}{\sqrt{33}}\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t},\\
2a_1t&=2\frac{\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}}{\sqrt{33}t}t=\frac{2}{\sqrt{33}}\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-\frac{2}{\sqrt{33}}\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t},\\
-4a_1t+a_0&=-4\frac{\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}}{\sqrt{33}t}t+\frac{1}{2}(1+\frac{3}{\sqrt{33}})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+\frac{1}{2}(1-\frac{3}{\sqrt{33}})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}\\
&=\frac{1}{2}(1-\frac{5}{\sqrt{33}})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+\frac{1}{2}(1+\frac{5}{\sqrt{33}})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t},
\end{aligned}$$

$\therefore$

$$\exp(\boldsymbol{A}t)=\begin{bmatrix}\frac{1}{2}(1+\frac{5}{\sqrt{33}})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+\frac{1}{2}(1-\frac{5}{\sqrt{33}})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t} & \frac{1}{\sqrt{33}}\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-\frac{1}{\sqrt{33}}\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}\\ \frac{2}{\sqrt{33}}\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-\frac{2}{\sqrt{33}}\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t} & \frac{1}{2}(1-\frac{5}{\sqrt{33}})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+\frac{1}{2}(1+\frac{5}{\sqrt{33}})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}\end{bmatrix},$$

方程组的通解为

$$\boldsymbol{x}(t)=\exp(\boldsymbol{A}t)\boldsymbol{c},$$

$\because (x_1(0),x_2(0))=(1,-1)$,

$\therefore \boldsymbol{x}(0)=\exp(At)|_{t=0}\boldsymbol{c}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$,

$\therefore \boldsymbol{c}=\begin{pmatrix}1\\-1\end{pmatrix}$

$\therefore$满足初值条件的特解为

$$\begin{aligned}
\boldsymbol{x}(t)&=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}=\exp(\boldsymbol{A}t)\boldsymbol{c}\\
&=\begin{pmatrix}\frac{1}{2}(1+\frac{5}{\sqrt{33}})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+\frac{1}{2}(1-\frac{5}{\sqrt{33}})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}-[\frac{1}{\sqrt{33}}\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-\frac{1}{\sqrt{33}}\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}]\\ \frac{2}{\sqrt{33}}\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-\frac{2}{\sqrt{33}}\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}-[\frac{1}{2}(1-\frac{5}{\sqrt{33}})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+\frac{1}{2}(1+\frac{5}{\sqrt{33}})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}]\end{pmatrix}\\
&=\begin{pmatrix}\frac{1}{2}(1+\frac{3}{\sqrt{33}})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+\frac{1}{2}(1-\frac{3}{\sqrt{33}})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}\\ -\frac{1}{2}(1-\frac{9}{\sqrt{33}})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-\frac{1}{2}(1+\frac{9}{\sqrt{33}})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}\end{pmatrix}\\
&=\begin{pmatrix}\frac{1}{2}(1+\frac{\sqrt{33}}{11})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+\frac{1}{2}(1-\frac{\sqrt{33}}{11})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}\\ -\frac{1}{2}(1-\frac{3\sqrt{33}}{11})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-\frac{1}{2}(1+\frac{3\sqrt{33}}{11})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}\end{pmatrix}.
\end{aligned}$$

方法2：方程组的系数矩阵为$\boldsymbol{A}=\begin{bmatrix}1&1\\2&-4\end{bmatrix}$,

由$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -1 \\ -2 & \lambda+4 \end{vmatrix} = (\lambda-1)(\lambda+4)-2 = \lambda^2+3\lambda-6=0$得特征值$\lambda_{1,2} = \frac{-3\pm\sqrt{9+4\times6}}{2} = \frac{-3\pm\sqrt{33}}{2}$,

由$(\lambda_1 \boldsymbol{I} - \boldsymbol{A})\boldsymbol{r} = (\frac{-3+\sqrt{33}}{2}\boldsymbol{I} - \boldsymbol{A})\boldsymbol{r} = \begin{bmatrix} \frac{-3+\sqrt{33}-2}{2} & -1 \\ -2 & \frac{-3+\sqrt{33}+8}{2} \end{bmatrix}\boldsymbol{r} = \begin{bmatrix} \frac{-5+\sqrt{33}}{2} & -1 \\ -2 & \frac{5+\sqrt{33}}{2} \end{bmatrix}\boldsymbol{r} = \boldsymbol{0} = \begin{bmatrix} \frac{33-25}{4} & \frac{-5-\sqrt{33}}{2} \\ -2 & \frac{5+\sqrt{33}}{2} \end{bmatrix}\boldsymbol{r} = \begin{bmatrix} 2 & \frac{-5-\sqrt{33}}{2} \\ -2 & \frac{5+\sqrt{33}}{2} \end{bmatrix}\boldsymbol{r} = \begin{bmatrix} 2 & \frac{-5-\sqrt{33}}{2} \\ 0 & 0 \end{bmatrix}\boldsymbol{r}$

得λ_1对应的特征向量是$\boldsymbol{r}_1 = \begin{pmatrix} \frac{5+\sqrt{33}}{2} \\ 2 \end{pmatrix}$,

由$(\lambda_2 \boldsymbol{I} - \boldsymbol{A})\boldsymbol{r} = (\frac{-3-\sqrt{33}}{2}\boldsymbol{I} - \boldsymbol{A})\boldsymbol{r} = \begin{bmatrix} \frac{-3-\sqrt{33}-2}{2} & -1 \\ -2 & \frac{-3-\sqrt{33}+8}{2} \end{bmatrix}\boldsymbol{r} = \begin{bmatrix} \frac{-5-\sqrt{33}}{2} & -1 \\ -2 & \frac{5-\sqrt{33}}{2} \end{bmatrix}\boldsymbol{r} = \boldsymbol{0} = \begin{bmatrix} \frac{25-33}{4} & \frac{5-\sqrt{33}}{2} \\ -2 & \frac{5-\sqrt{33}}{2} \end{bmatrix}\boldsymbol{r} = \begin{bmatrix} -2 & \frac{5-\sqrt{33}}{2} \\ -2 & \frac{5-\sqrt{33}}{2} \end{bmatrix}\boldsymbol{r} = \begin{bmatrix} -2 & \frac{5-\sqrt{33}}{2} \\ 0 & 0 \end{bmatrix}\boldsymbol{r}$

得λ_1对应的特征向量是$\boldsymbol{r}_2 = \begin{pmatrix} \frac{5-\sqrt{33}}{2} \\ 2 \end{pmatrix}$,

方程组的一个基本解组为$\boldsymbol{\varphi}_1(t) = \begin{pmatrix} \frac{5+\sqrt{33}}{2} \\ 2 \end{pmatrix} \mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}$, $\boldsymbol{\varphi}_2(t) = \begin{pmatrix} \frac{5-\sqrt{33}}{2} \\ 2 \end{pmatrix} \mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}$,

通解为$\boldsymbol{x}(t) = c_1 \begin{pmatrix} \frac{5+\sqrt{33}}{2} \\ 2 \end{pmatrix} \mathrm{e}^{\frac{-3+\sqrt{33}}{2}t} + c_2 \begin{pmatrix} \frac{5-\sqrt{33}}{2} \\ 2 \end{pmatrix} \mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}$,

$\because (x_1(0), x_2(0)) = (1, -1)$,

$\therefore \boldsymbol{x}(0) = c_1 \begin{pmatrix} \frac{5+\sqrt{33}}{2} \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} \frac{5-\sqrt{33}}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{5+\sqrt{33}}{2}c_1 + \frac{5-\sqrt{33}}{2}c_2 \\ 2c_1+2c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

解得$c_1 = \frac{-11+3\sqrt{33}}{44}, c_2 = \frac{-11-3\sqrt{33}}{44}$,

故满足初始条件的特解为

$$\begin{aligned}
\boldsymbol{x}(t) &= \frac{-11+3\sqrt{33}}{44} \begin{pmatrix} \frac{5+\sqrt{33}}{2} \\ 2 \end{pmatrix} \mathrm{e}^{\frac{-3+\sqrt{33}}{2}t} + \frac{-11-3\sqrt{33}}{44} \begin{pmatrix} \frac{5-\sqrt{33}}{2} \\ 2 \end{pmatrix} \mathrm{e}^{\frac{-3-\sqrt{33}}{2}t} \\
&= \begin{pmatrix} \frac{-11+3\sqrt{33}}{44}\frac{5+\sqrt{33}}{2}\mathrm{e}^{-\frac{-3+\sqrt{33}}{2}t} + \frac{-11-3\sqrt{33}}{44}\frac{5-\sqrt{33}}{2}\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t} \\ \frac{-11+3\sqrt{33}}{22}\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t} + \frac{-11-3\sqrt{33}}{22}\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}(1+\frac{\sqrt{33}}{11})\mathrm{e}^{-\frac{-3+\sqrt{33}}{2}t} + \frac{1}{2}(1-\frac{\sqrt{33}}{11})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t} \\ -\frac{1}{2}(1-\frac{3\sqrt{33}}{11})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t} - \frac{1}{2}(1+\frac{3\sqrt{33}}{11})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t} \end{pmatrix}.
\end{aligned}$$

方法3：由第一个方程得$x_2 = \frac{\mathrm{d}x_1}{\mathrm{d}t} - x_1$, 两端关于$t$求导得$x_2' = x_1'' - x_1'$.

将以上两个方程代入方程组的第二个方程得

$$x_1''-x_1'=2x_1-4(x_1'-x_1),$$

即

$$x_1''+3x_1'-6x_1=0,$$

此二阶常系数齐次微分方程的特征方程为

$$\lambda^2+3\lambda-6=0,$$

特征值$\lambda_{1,2}=\frac{-3\pm\sqrt{9+24}}{2}=\frac{-3\pm\sqrt{33}}{2}$, 故该二阶常系数齐次微分方程的通解为

$$x_1=c_1\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+c_2\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t},$$

$\therefore$

$$\begin{aligned}x_2&=x_1'-x_1\\&=c_1\frac{-3+\sqrt{33}}{2}\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+c_2\frac{-3-\sqrt{33}}{2}\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}-c_1\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-c_2\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}\\&=c_1\frac{-5+\sqrt{33}}{2}\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+c_2\frac{-5-\sqrt{33}}{2}\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t},\end{aligned}$$

$\because (x_1(0),x_2(0))=(1,-1),$

$\therefore x_1(0)=c_1+c_2=1,\ x_2(0)=c_1\frac{-5+\sqrt{33}}{2}+c_2\frac{-5-\sqrt{33}}{2}=-1,$

$\therefore c_1=\frac{1}{2}(1+\frac{\sqrt{33}}{11}),\ c_2=\frac{1}{2}(1-\frac{\sqrt{33}}{11}),$

$$\begin{aligned}x_1&=\frac{1}{2}(1+\frac{\sqrt{33}}{11})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+\frac{1}{2}(1-\frac{\sqrt{33}}{11})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t},\\x_2&=\frac{1}{2}(1+\frac{\sqrt{33}}{11})\frac{-5+\sqrt{33}}{2}\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}+\frac{1}{2}(1-\frac{\sqrt{33}}{11})\frac{-5-\sqrt{33}}{2}\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}\\&=-\frac{1}{2}(1-\frac{3\sqrt{33}}{11})\mathrm{e}^{\frac{-3+\sqrt{33}}{2}t}-\frac{1}{2}(1+\frac{3\sqrt{33}}{11})\mathrm{e}^{\frac{-3-\sqrt{33}}{2}t}.\end{aligned}$$

(2)方法1：方程组的系数矩阵为$\boldsymbol{A}=\begin{bmatrix}1&2\\4&3\end{bmatrix}$,

$$\exp(\boldsymbol{A}t)=a_1\boldsymbol{A}t+a_0\boldsymbol{I}=\begin{bmatrix}a_1t+a_0&2a_1t\\4a_1t&3a_1t+a_0\end{bmatrix},\ r(\lambda t)=a_1\lambda t+a_0.$$

由$|\lambda\boldsymbol{I}-\boldsymbol{A}|=\begin{vmatrix}\lambda-1&-2\\-4&\lambda-3\end{vmatrix}=(\lambda-1)(\lambda-3)-8=\lambda^2-4\lambda-5=(\lambda+1)(\lambda-5)=0$得$\boldsymbol{A}$的特征值是$\lambda_1=-1,\lambda_2=5,$

则

$$e^{-t}=a_1\cdot(-t)+a_0, e^{5t}=a_1\cdot(5t)+a_0,$$

解这个方程组得到

$$a_1=\frac{e^{5t}-e^{-t}}{6t},\ a_0=e^{-t}+a_1t=e^{-t}+\frac{e^{5t}-e^{-t}}{6t}t=\frac{e^{5t}+5e^{-t}}{6},$$

则

$$\exp(\boldsymbol{A}t)=\begin{bmatrix}\frac{e^{5t}-e^{-t}}{6}+\frac{e^{5t}+5e^{-t}}{6} & \frac{e^{5t}-e^{-t}}{3}\\ \frac{2}{3}(e^{5t}-e^{-t}) & \frac{e^{5t}-e^{-t}}{2}+\frac{e^{5t}+5e^{-t}}{6}\end{bmatrix}=\begin{bmatrix}\frac{e^{5t}+2e^{-t}}{3} & \frac{e^{5t}-e^{-t}}{3}\\ \frac{2e^{5t}-2e^{-t}}{3} & \frac{4e^{-t}+2e^{5t}}{6}\end{bmatrix},$$

方程组的通解是

$$\boldsymbol{x}(t)=\exp(\boldsymbol{A}t)\boldsymbol{c},$$

$\because (x_1(0),x_2(0))=(1,0)$,

$\therefore \boldsymbol{x}(0)=\boldsymbol{I}\boldsymbol{c}=\boldsymbol{c}=(1,0)^T$,

$\therefore$方程组满足初值条件的特解为

$$\boldsymbol{x}(t)=\begin{pmatrix}\frac{e^{5t}+2e^{-t}}{3}\\ \frac{2e^{5t}-2e^{-t}}{3}\end{pmatrix}=\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}.$$

方法2：方程组的系数矩阵为$\boldsymbol{A}=\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$，由$|\lambda\boldsymbol{I}-\boldsymbol{A}|=\begin{vmatrix}\lambda-1 & -2\\ -4 & \lambda-3\end{vmatrix}=(\lambda-1)(\lambda-3)-8=\lambda^2-4\lambda-5=(\lambda+1)(\lambda-5)=0$得$\boldsymbol{A}$的特征值是$\lambda_1=-1,\lambda_2=5$,

解$(\lambda_1\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=(-\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=\begin{bmatrix}-2 & -2\\ -4 & -4\end{bmatrix}\boldsymbol{r}=\boldsymbol{0}=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\boldsymbol{r}$得$\lambda_1$对应的特征向量为$\boldsymbol{r}_1=(-1,1)^T$,

解$(\lambda_2\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=(5\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=\begin{bmatrix}4 & -2\\ -4 & 2\end{bmatrix}\boldsymbol{r}=\boldsymbol{0}=\begin{bmatrix}2 & -1\\ 0 & 0\end{bmatrix}\boldsymbol{r}$得$\lambda_1$对应的特征向量为$\boldsymbol{r}_1=(1,2)^T$,

由此得到方程组的一个基本解组是

$$\boldsymbol{\varphi}_1(t)=e^{-t}\boldsymbol{r}_1,\ \boldsymbol{\varphi}_2(t)=e^{5t}\boldsymbol{r}_2,$$

其通解是

$$\boldsymbol{x}(t)=c_1e^{-t}\boldsymbol{r}_1+c_2e^{5t}\boldsymbol{r}_2,$$

$\because (x_1(0),x_2(0))=(1,0)$,

$\therefore \boldsymbol{x}(0)=c_1(-1,1)^T+c_2(1,2)^T=(-c_1+c_2,c_1+2c_2)^T=(1,0)$,

$\therefore c_1=-\frac{2}{3},\ c_2=\frac{1}{3}$,

∴方程组满足初值条件的特解为

$$\boldsymbol{x}(t)=-\frac{2}{3}\begin{pmatrix}-1\\1\end{pmatrix}\mathrm{e}^{-x}+\frac{1}{3}\begin{pmatrix}1\\2\end{pmatrix}\mathrm{e}^{5x}=\begin{pmatrix}\frac{2}{3}\mathrm{e}^{-x}+\frac{1}{3}\mathrm{e}^{5x}\\-\frac{2}{3}\mathrm{e}^{-x}+\frac{2}{3}\mathrm{e}^{5x}\end{pmatrix}=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}.$$

方法3：由第一个方程得$x_2=\frac{1}{2}x_1'-\frac{1}{2}x_1$, 两端关于$t$求导得$x_2'=\frac{1}{2}x_1''-\frac{1}{2}x_1'$.

将以上两式代入方程组的第二个方程得

$$\frac{1}{2}x_1''-\frac{1}{2}x_1'=4x_1+3(\frac{1}{2}x_1'-\frac{1}{2}x_1),$$

即

$$x_1''-4x_1'-5x_1=0,$$

该二阶线性常系数齐次微分方程的特征方程为$\lambda^2-4\lambda-5=(\lambda+1)(\lambda-5)=0$, 特征根为$\lambda_1=-1,\lambda_2=5$, 通解

$$x_1=c_1\mathrm{e}^{-t}+c_2\mathrm{e}^{5t},$$

∴

$$x_2=\frac{1}{2}x_1'-\frac{1}{2}x_1=-\frac{1}{2}c_1\mathrm{e}^{-t}+\frac{5}{2}c_2\mathrm{e}^{5t}-\frac{1}{2}c_1\mathrm{e}^{-x}-\frac{1}{2}c_2\mathrm{e}^{5t}=-c_1\mathrm{e}^{-t}+2c_2\mathrm{e}^{5t},$$

$\because (x_1(0),x_2(0))=(1,0)$,

$\therefore x_1(0)=c_1+c_2=1, x_2(0)=-c_1+2c_2=0$,

$\therefore c_2=\frac{1}{3}, c_1=\frac{2}{3}$,

∴方程组满足初值条件的特解为

$$x_1(t)=\frac{2}{3}\mathrm{e}^{-t}+\frac{1}{3}\mathrm{e}^{5t},$$
$$x_2(t)=-\frac{2}{3}\mathrm{e}^{-t}+\frac{2}{3}\mathrm{e}^{5t}.$$

(3)方法1：方程组的系数矩阵为$\boldsymbol{A}=\begin{bmatrix}1&-1\\1&3\end{bmatrix}$,

$$\mathrm{e}^{\boldsymbol{A}t}=a_1\boldsymbol{A}t+a_0\boldsymbol{I}=\begin{bmatrix}a_1t+a_0&-a_1t\\a_1t&3a_1t+a_0\end{bmatrix},\ r(\lambda t)=a_1\lambda t+a_0,$$

由$|\lambda\boldsymbol{I}-\boldsymbol{A}|=\begin{vmatrix}\lambda-1&1\\-1&\lambda-3\end{vmatrix}=(\lambda-1)(\lambda-3)+1=\lambda^2-4\lambda+4=(\lambda-2)^2=0$得$\boldsymbol{A}$的特征值$\lambda=2$(二重)，则

$$\mathrm{e}^{2t}=a_1\cdot(2t)+a_0,\ \mathrm{e}^{2t}=a_1,$$

$\therefore a_1=\mathrm{e}^{2t}, a_0=(1-2t)\mathrm{e}^{2t}$,

$\therefore$

$$\mathrm{e}^{\boldsymbol{A}t}=\begin{bmatrix} t\mathrm{e}^{2t}+(1-2t)\mathrm{e}^{2t} & -t\mathrm{e}^{2t} \\ t\mathrm{e}^{2t} & 3t\mathrm{e}^{2t}+(1-2t)\mathrm{e}^{2t}\end{bmatrix}=\begin{bmatrix}(1-t)\mathrm{e}^{2t} & -t\mathrm{e}^{2t} \\ t\mathrm{e}^{2t} & (1+t)\mathrm{e}^{2t}\end{bmatrix},$$

$\therefore$方程组的通解是

$$\boldsymbol{x}(t)=\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{c},$$

$\because (x_1(0),x_2(0))=(2,3)$,

$\therefore \boldsymbol{x}(0)=\boldsymbol{I}\boldsymbol{c}=\boldsymbol{c}=(2,3)^T$,

$\therefore$方程组满足初值条件的特解为

$$\boldsymbol{x}(t)=\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{c}=\begin{bmatrix}(1-t)\mathrm{e}^{2t} & -t\mathrm{e}^{2t} \\ t\mathrm{e}^{2t} & (1+t)\mathrm{e}^{2t}\end{bmatrix}\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}2(1-t)\mathrm{e}^{2t}-3t\mathrm{e}^{2t}\\ 2e\mathrm{e}^{2t}+3(1+t)\mathrm{e}^{2t}\end{pmatrix}=\begin{pmatrix}(2-5t)\mathrm{e}^{2t}\\(3+5t)\mathrm{e}^{2t}\end{pmatrix}.$$

方法2：由第一个方程得$x_2=x_1-x_1'$, 两端关于t求导得$x_2'=x_1'-x_1''$.

将以上两式代入方程组的第二个方程得

$$x_1'-x_1''=x_1+3(x_1-x_1'),$$

即

$$x_1''-4x_1'+4x_1=0,$$

此二阶线性常系数齐次微分方程的特征根为$\lambda_{1,2}=2$, 通解为$x_1=(c_1+c_2t)\mathrm{e}^{2t}$,

$\therefore x_2=(c_1+c_2t)\mathrm{e}^{2t}-(2c_1+2c_2t+c_2)\mathrm{e}^{2t}=(-c_1-c_2-c_2t)\mathrm{e}^{2t}$,

$\because (x_1(0),x_2(0))=(2,3)$,

$\therefore x_1(0)=c_1=2, x_2(0)=-c_1-c_2=3$,

$\therefore c_1=2, c_2=-5$,

$\therefore$方程组满足初值条件的特解为

$$\begin{cases} x_1(t) &=(2-5t)\mathrm{e}^{2t},\\ x_2(t) &=(3+5t)\mathrm{e}^{2t}.\end{cases}$$

【注：】方程组的系数矩阵为$\boldsymbol{A}=\begin{bmatrix}1 & -1\\1 & 3\end{bmatrix}$, 由$|\lambda\boldsymbol{I}-\boldsymbol{A}|=\begin{vmatrix}\lambda-1 & 1\\ -1 & \lambda-3\end{vmatrix}=(\lambda-1)(\lambda-3)+1=\lambda^2-4\lambda+4=(\lambda-2)^2=0$得$\boldsymbol{A}$的特征值$\lambda=2$(二重)，

解$(\lambda\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=(2\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=\begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix}\boldsymbol{r}=\boldsymbol{0}=\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\boldsymbol{r}$得$\lambda$对应的线性无关的特征向量为$\boldsymbol{r}=(1,-1)^T$, 此时二阶方阵$\boldsymbol{A}$只有一个线性无关的特征向量，故此时不适合用系数矩阵的特征值和特征向量表示齐次微分方程组的解.

(4)方法1：方程组的系数矩阵为$\boldsymbol{A}=\begin{bmatrix}4 & -2\\ 1 & -4\end{bmatrix}$,

$$\mathrm{e}^{\boldsymbol{A}t}=a_1\boldsymbol{A}t+a_0\boldsymbol{I}=\begin{bmatrix}4a_1t+a_0 & -2a_1t\\ a_1t & -4a_1t+a_0\end{bmatrix},\ r(\lambda t)=a_1\lambda t+a_0,$$

由$|\lambda\boldsymbol{I}-\boldsymbol{A}|=\begin{vmatrix}\lambda-4 & 2\\ -1 & \lambda+4\end{vmatrix}=(\lambda-4)(\lambda+4)+2=\lambda^2-14=0$得$\boldsymbol{A}$的特征值为$\lambda_{1,2}=\pm\sqrt{14}$, 则

$$\mathrm{e}^{-\sqrt{14}t}=-\sqrt{14}a_1t+a_0,\mathrm{e}^{\sqrt{14}t}=\sqrt{14}a_1t+a_0,$$

$\therefore a_0=\frac{1}{2}(\mathrm{e}^{-\sqrt{14}t}+\mathrm{e}^{\sqrt{14}t}),\ a_1=\frac{\mathrm{e}^{\sqrt{14}t}-\mathrm{e}^{-\sqrt{14}t}}{2\sqrt{14}t}$,

$\therefore$

$$\begin{aligned}\mathrm{e}^{\boldsymbol{A}t}&=\begin{bmatrix}\frac{2}{\sqrt{14}}(\mathrm{e}^{\sqrt{14}t}-\mathrm{e}^{-\sqrt{14}t})+\frac{1}{2}(\mathrm{e}^{\sqrt{14}t}+\mathrm{e}^{-\sqrt{14}t}) & \frac{\mathrm{e}^{-\sqrt{14}t}-\mathrm{e}^{\sqrt{14}t}}{\sqrt{14}}\\ \frac{\mathrm{e}^{\sqrt{14}t}-\mathrm{e}^{-\sqrt{14}t}}{2\sqrt{14}} & -\frac{2}{\sqrt{14}}(\mathrm{e}^{\sqrt{14}t}-\mathrm{e}^{-\sqrt{14}t})+\frac{1}{2}(\mathrm{e}^{-\sqrt{14}t}+\mathrm{e}^{\sqrt{14}t})\end{bmatrix}\\ &=\begin{bmatrix}(\frac{1}{2}+\frac{2}{\sqrt{14}})\mathrm{e}^{\sqrt{14}t}+(\frac{1}{2}-\frac{2}{\sqrt{14}})\mathrm{e}^{-\sqrt{14}t} & \frac{\mathrm{e}^{-\sqrt{14}t}-\mathrm{e}^{\sqrt{14}t}}{\sqrt{14}}\\ \frac{\mathrm{e}^{\sqrt{14}t}-\mathrm{e}^{-\sqrt{14}t}}{2\sqrt{14}} & (\frac{1}{2}-\frac{2}{\sqrt{14}})\mathrm{e}^{\sqrt{14}t}+(\frac{1}{2}+\frac{2}{\sqrt{14}})\mathrm{e}^{-\sqrt{14}t}\end{bmatrix},\end{aligned}$$

方程组的通解为$\boldsymbol{x}(t)=\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{c}$,

$\because (x_1(0),x_2(0))=(1,0)$,

$\therefore \boldsymbol{x}(0)=\boldsymbol{I}\boldsymbol{c}=\boldsymbol{c}=(1,0)^T$,

$\therefore$方程组满足初值条件的特解为

$$\begin{aligned}\boldsymbol{x}(t)&=\begin{bmatrix}(\frac{1}{2}+\frac{2}{\sqrt{14}})\mathrm{e}^{\sqrt{14}t}+(\frac{1}{2}-\frac{2}{\sqrt{14}})\mathrm{e}^{-\sqrt{14}t} & \frac{\mathrm{e}^{-\sqrt{14}t}-\mathrm{e}^{\sqrt{14}t}}{\sqrt{14}}\\ \frac{\mathrm{e}^{\sqrt{14}t}-\mathrm{e}^{-\sqrt{14}t}}{2\sqrt{14}} & (\frac{1}{2}-\frac{2}{\sqrt{14}})\mathrm{e}^{\sqrt{14}t}+(\frac{1}{2}+\frac{2}{\sqrt{14}})\mathrm{e}^{-\sqrt{14}t}\end{bmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\\ &=\begin{pmatrix}(\frac{1}{2}+\frac{2}{\sqrt{14}})\mathrm{e}^{\sqrt{14}t}+(\frac{1}{2}-\frac{2}{\sqrt{14}})\mathrm{e}^{-\sqrt{14}t}\\ \frac{\mathrm{e}^{\sqrt{14}t}-\mathrm{e}^{-\sqrt{14}t}}{2\sqrt{14}}\end{pmatrix}\\ &=\begin{pmatrix}(\frac{1}{2}+\frac{\sqrt{14}}{7})\mathrm{e}^{\sqrt{14}t}+(\frac{1}{2}-\frac{\sqrt{14}}{7})\mathrm{e}^{-\sqrt{14}t}\\ \frac{\sqrt{14}(\mathrm{e}^{\sqrt{14}t}-\mathrm{e}^{-\sqrt{14}t})}{28}\end{pmatrix}.\end{aligned}$$

(2)方程组的系数矩阵为$\boldsymbol{A}=\begin{bmatrix}4 & -2\\ 1 & -4\end{bmatrix}$, 由$|\lambda\boldsymbol{I}-\boldsymbol{A}|=\begin{vmatrix}\lambda-4 & 2\\ -1 & \lambda+4\end{vmatrix}=(\lambda-4)(\lambda+4)+2=\lambda^2-14=0$得$\boldsymbol{A}$的特征值为$\lambda_{1,2}=\pm\sqrt{14}$,

解$(\lambda_1\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=(-\sqrt{14}\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=\begin{bmatrix}-\sqrt{14}-4 & 2\\ -1 & -\sqrt{14}+4\end{bmatrix}\boldsymbol{r}=\boldsymbol{0}=\begin{bmatrix}14-16 & -2\sqrt{14}+8\\ -1 & -\sqrt{14}+4\end{bmatrix}\boldsymbol{r}=\begin{bmatrix}-1 & -\sqrt{14}+4\\ -1 & -\sqrt{14}+4\end{bmatrix}\boldsymbol{r}=\begin{bmatrix}-1 & -\sqrt{14}+4\\ 0 & 0\end{bmatrix}\boldsymbol{r}$得$\lambda_1=-\sqrt{14}$对应的特征向量为$\boldsymbol{r}_1=(-\sqrt{14}+4,1)^T$,

解$(\lambda_2\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=(\sqrt{14}\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=\begin{bmatrix}\sqrt{14}-4 & 2\\ -1 & \sqrt{14}+4\end{bmatrix}\boldsymbol{r}=\boldsymbol{0}=\begin{bmatrix}14-16 & 2\sqrt{14}+8\\ -1 & \sqrt{14}+4\end{bmatrix}\boldsymbol{r}=$ $\begin{bmatrix}-1 & \sqrt{14}+4\\ -1 & \sqrt{14}+4\end{bmatrix}\boldsymbol{r}=\begin{bmatrix}-1 & \sqrt{14}+4\\ 0 & 0\end{bmatrix}\boldsymbol{r}$得$\lambda_1=\sqrt{14}$对应的特征向量为$\boldsymbol{r}_1=(\sqrt{14}+4,1)^T$,
则方程组的一个基本解组是

$$\boldsymbol{\varphi}_1(t)=\mathrm{e}^{-\sqrt{14}t}\boldsymbol{r}_1,\ \boldsymbol{\varphi}_2(t)=\mathrm{e}^{\sqrt{14}t}\boldsymbol{r}_2,$$

通解为

$$\boldsymbol{\varphi}(t)=c_1\mathrm{e}^{-\sqrt{14}t}\boldsymbol{r}_1+c_2\mathrm{e}^{-\sqrt{14}t}\boldsymbol{r}_2,$$

$\because (x_1(0),x_2(0))=(1,0)$,

$\therefore \boldsymbol{\varphi}(0)=c_1\boldsymbol{r}_1+c_2\boldsymbol{r}_2=\begin{pmatrix}(-\sqrt{14}+4)c_1+(\sqrt{14}+4)c_2\\ c_1+c_2\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$,

$\therefore c_1=-\frac{1}{2\sqrt{14}},\ c_2=\frac{1}{2\sqrt{14}}$,

$\therefore$方程组满足初值条件的通解为

$$\begin{aligned}\boldsymbol{x}(t)&=-\frac{1}{2\sqrt{14}}\begin{pmatrix}-\sqrt{14}+4\\1\end{pmatrix}\mathrm{e}^{-\sqrt{14}t}+\frac{1}{2\sqrt{14}}\begin{pmatrix}\sqrt{14}+4\\1\end{pmatrix}\mathrm{e}^{\sqrt{14}t}\\ &=\begin{pmatrix}(\frac{1}{2}-\frac{2}{\sqrt{14}})\mathrm{e}^{-\sqrt{14}t}+(\frac{1}{2}+\frac{2}{\sqrt{14}})\mathrm{e}^{\sqrt{14}t}\\ -\frac{1}{2\sqrt{14}}\mathrm{e}^{-\sqrt{14}t}+\frac{1}{2\sqrt{14}}\mathrm{e}^{\sqrt{14}t}\end{pmatrix}\\ &=\begin{pmatrix}(\frac{1}{2}-\frac{\sqrt{14}}{7})\mathrm{e}^{-\sqrt{14}t}+(\frac{1}{2}+\frac{\sqrt{14}}{7})\mathrm{e}^{\sqrt{14}t}\\ -\frac{\sqrt{14}}{28}\mathrm{e}^{-\sqrt{14}t}+\frac{\sqrt{14}}{28}\mathrm{e}^{\sqrt{14}t}\end{pmatrix}.\end{aligned}$$

方法3：由第一个方程得$x_2=2x_1-\frac{1}{2}x_1'$, 两端关于t求导得$x_2'=2x_1'-\frac{1}{2}x_1''$,
将以上两式代入方程组的第二个方程得

$$2x_1'-\frac{1}{2}x_1''=x_1-4(2x_1-\frac{1}{2}x_1'),$$

即

$$x_1''-14x_1=0,$$

该二阶常系数齐次线性微分方程的特征方程为$\lambda^2-14=0$, 特征根$\lambda_{1,2}=\pm\sqrt{14}$,
故通解

$$x_1(t)=c_1\mathrm{e}^{-\sqrt{14}t}+c_2\mathrm{e}^{\sqrt{14}t},$$

$\therefore$

$$\begin{aligned}x_2(t)&=2x_1-\frac{1}{2}x_1'\\ &=2c_1\mathrm{e}^{-\sqrt{14}t}+2c_2\mathrm{e}^{\sqrt{14}t}-\frac{1}{2}(-\sqrt{14}c_1\mathrm{e}^{-\sqrt{14}t}+\sqrt{14}c_2\mathrm{e}^{\sqrt{14}t})\\ &=c_1(2+\frac{\sqrt{14}}{2})\mathrm{e}^{-\sqrt{14}t}+c_2(2-\frac{\sqrt{14}}{2})\mathrm{e}^{\sqrt{14}},\end{aligned}$$

$\because (x_1(0), x_2(0)) = (1, 0)$,

$\therefore x_1(0) = c_1 + c_2 = 1, x_2(0) = c_1(2+\frac{\sqrt{14}}{2}) + c_2(2-\frac{\sqrt{14}}{2}) = 0$,

$\therefore c_1 = \frac{1}{2} - \frac{2}{\sqrt{14}}$，$c_2 = \frac{1}{2} + \frac{2}{\sqrt{14}}$,

$\therefore$方程组满足初值条件的解是

$$\begin{aligned}
x_1(t) &= (\frac{1}{2} - \frac{2}{\sqrt{14}})\mathrm{e}^{-\sqrt{14}t} + (\frac{1}{2} + \frac{2}{\sqrt{14}})\mathrm{e}^{\sqrt{14}t} \\
&= (\frac{1}{2} - \frac{\sqrt{14}}{7})\mathrm{e}^{-\sqrt{14}t} + (\frac{1}{2} + \frac{\sqrt{14}}{7})\mathrm{e}^{\sqrt{14}t}, \\
x_2(t) &= (\frac{1}{2} - \frac{2}{\sqrt{14}})(2+\frac{\sqrt{14}}{2})\mathrm{e}^{-\sqrt{14}t} + (\frac{1}{2} + \frac{2}{\sqrt{14}})(2-\frac{\sqrt{14}}{2})\mathrm{e}^{\sqrt{14}} \\
&= -\frac{\sqrt{14}}{28}\mathrm{e}^{-\sqrt{14}t} + \frac{\sqrt{14}}{28}\mathrm{e}^{\sqrt{14}t}.
\end{aligned}$$

(5)方法1：方程组的系数矩阵$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$，由$|\lambda\boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} = \lambda^3 + (-1) + (-1) - \lambda - \lambda - \lambda = \lambda^3 - 3\lambda - 2 = \lambda^3 - \lambda - 2\lambda - 2$
$= \lambda(\lambda-1)(\lambda+1) - 2(\lambda+1) = (\lambda+1)(\lambda^2-\lambda-2) = (\lambda+1)(\lambda-2)(\lambda+1)$
$= (\lambda+1)^2(\lambda-2) = 0$得矩阵$\boldsymbol{A}$的特征值$\lambda_{1,2} = -1, \lambda_3 = 2$,

解$(\lambda_{1,2}\boldsymbol{I} - \boldsymbol{A})\boldsymbol{r} = (-\boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\boldsymbol{r} = \boldsymbol{0} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{r}$得$\lambda_{1,2}$对应的两个线性无关的特征向量

$$\boldsymbol{r}_1 = (1, 0, -1)^T, \boldsymbol{r}_2 = (0, 1, -1)^T,$$

解$(\lambda_3\boldsymbol{I} - \boldsymbol{A})\boldsymbol{r} = (2\boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\boldsymbol{r} = \boldsymbol{0} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{r} = \begin{bmatrix} 2 & -1 & -1 \\ -3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{r}$
$= \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{r}$得$\lambda_3$对应的特征向量

$$\boldsymbol{r}_3 = (1, 1, 1)^T,$$

则方程组的一个基本解组是

$$\boldsymbol{\varphi}_1(t) = \mathrm{e}^{\lambda_{1,2}t}\boldsymbol{r}_1,\ \boldsymbol{\varphi}_2(t) = \mathrm{e}^{\lambda_{1,2}t}\boldsymbol{r}_2,\ \boldsymbol{\varphi}_3(t) = \mathrm{e}^{\lambda_3 t}\boldsymbol{r}_3,$$

通解为

$$\boldsymbol{\varphi}(t)=c_1\boldsymbol{\varphi}_1(t)+c_2\boldsymbol{\varphi}_2(t)+\boldsymbol{\varphi}_3(t)=c_1\mathrm{e}^{-t}\begin{pmatrix}1\\0\\-1\end{pmatrix}+c_2\mathrm{e}^{-t}\begin{pmatrix}0\\1\\-1\end{pmatrix}+c_3\mathrm{e}^{2t}\begin{pmatrix}1\\1\\1\end{pmatrix},$$

$\because (x_1(0),x_2(0),x_3(0))=(2,3,1)$,

$\therefore$

$$\boldsymbol{\varphi}(0)=c_1\begin{pmatrix}1\\0\\-1\end{pmatrix}+c_2\begin{pmatrix}0\\1\\-1\end{pmatrix}+c_3\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}c_1+c_3\\c_2+c_3\\-c_1-c_2+c_3\end{pmatrix}=\begin{pmatrix}2\\3\\1\end{pmatrix},$$

$\therefore c_1=0,c_2=1,c_3=2$,

$\therefore$方程组满足初值条件的特解为

$$\boldsymbol{\varphi}(t)=\mathrm{e}^{-t}\begin{pmatrix}0\\1\\-1\end{pmatrix}+2\mathrm{e}^{2t}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}2\mathrm{e}^{2t}\\\mathrm{e}^{-t}+2\mathrm{e}^{2t}\\-\mathrm{e}^{-t}+2\mathrm{e}^{2t}\end{pmatrix}.$$

方法2：该题不适合用指数矩阵$\mathrm{e}^{\boldsymbol{A}t}$表示该常系数线性微分方程组的解，理由如下：

方程组的系数矩阵$\boldsymbol{A}=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$,

$$\mathrm{e}^{\boldsymbol{A}t}=a_2\boldsymbol{A}^2t^2+a_1\boldsymbol{A}t+a_0\boldsymbol{I}=\begin{bmatrix}2a_2t^2+a_0&a_2t^2+a_1t&a_2t^2+a_1t\\a_2t^2+a_1t&2a_2t^2+a_0&a_2t^2+a_1t\\a_2t^2+a_1t&a_2t^2+a_1t&2a_2t^2+a_0\end{bmatrix},\ r(\lambda t)=a_2\cdot(\lambda t)^2+a_1\cdot(\lambda t)+a_0,$$

由$|\lambda\boldsymbol{I}-\boldsymbol{A}|=\begin{vmatrix}\lambda&-1&-1\\-1&\lambda&-1\\-1&-1&\lambda\end{vmatrix}=\lambda^3+(-1)+(-1)-\lambda-\lambda-\lambda=\lambda^3-3\lambda-2=\lambda^3-\lambda-2\lambda-2$

$=\lambda(\lambda-1)(\lambda+1)-2(\lambda+1)=(\lambda+1)(\lambda^2-\lambda-2)=(\lambda+1)(\lambda-2)(\lambda+1)$

$=(\lambda+1)^2(\lambda-2)=0$得矩阵$\boldsymbol{A}$的特征值$\lambda_{1,2}=-1,\lambda_3=2$, 则

$$\begin{cases}\mathrm{e}^{-t}=a_2\cdot(-t)^2+a_1\cdot(-t)+a_0,\\\mathrm{e}^{-t}=2a_2\cdot(-t)+a_1,\\\mathrm{e}^{2t}=a_2\cdot(2t)^2+a_1\cdot(2t)+a_0.\end{cases}$$

可得

$$a_2=\frac{\begin{vmatrix}\mathrm{e}^{-t} & -t & 1\\ \mathrm{e}^{-t} & 1 & 0\\ \mathrm{e}^{2t} & 2t & 1\end{vmatrix}}{\begin{vmatrix}t^2 & -t & 1\\ -2t & 1 & 0\\ 4t^2 & 2t & 1\end{vmatrix}}=\frac{\mathrm{e}^{-t}+0+2t\mathrm{e}^{-t}-\mathrm{e}^{2t}+t\mathrm{e}^{-t}-0}{t^2+0-4t^2-4t^2-0-2t^2}=\frac{(1+3t)\mathrm{e}^{-t}-\mathrm{e}^{2t}}{-9t^2}=\frac{\mathrm{e}^{2t}-(1+3t)\mathrm{e}^{-t}}{9t^2},$$

$$a_1=\frac{\begin{vmatrix}t^2 & \mathrm{e}^{-t} & 1\\ -2t & \mathrm{e}^{-t} & 0\\ 4t^2 & \mathrm{e}^{2t} & 1\end{vmatrix}}{\begin{vmatrix}t^2 & -t & 1\\ -2t & 1 & 0\\ 4t^2 & 2t & 1\end{vmatrix}}=\frac{t^2\mathrm{e}^{-t}+0-2t\mathrm{e}^{2t}-4t^2\mathrm{e}^{-t}-0+2t\mathrm{e}^{-t}}{-9t^2}=\frac{(-3t^2+2t)\mathrm{e}^{-t}-2t\mathrm{e}^{2t}}{-9t^2}$$

$$=\frac{(3t^2-2t)\mathrm{e}^{-t}+2t\mathrm{e}^{2t}}{9t^2}=\frac{(3t-2)\mathrm{e}^{-t}+2\mathrm{e}^{2t}}{9t},$$

$$a_0=\frac{\begin{vmatrix}t^2 & -t & \mathrm{e}^{-t}\\ -2t & 1 & \mathrm{e}^{-t}\\ 4t^2 & 2t & \mathrm{e}^{2t}\end{vmatrix}}{\begin{vmatrix}t^2 & -t & 1\\ -2t & 1 & 0\\ 4t^2 & 2t & 1\end{vmatrix}}=\frac{t^2\mathrm{e}^{2t}-4t^3\mathrm{e}^{-t}-4t^2\mathrm{e}^{-t}-4t^2\mathrm{e}^{-t}-2t^3\mathrm{e}^{-t}-2t^2\mathrm{e}^{2t}}{-9t^2}=\frac{-t^2\mathrm{e}^{2t}-(6t^3+8t^2)\mathrm{e}^{-t}}{-9t^2}$$

$$=\frac{\mathrm{e}^{2t}+(6t+8)\mathrm{e}^{-t}}{9},$$

$\therefore$

$$2a_2t^2+a_0=2t^2\frac{\mathrm{e}^{2t}-(1+3t)\mathrm{e}^{-t}}{9t^2}+\frac{\mathrm{e}^{2t}+(6t+8)\mathrm{e}^{-t}}{9}=\frac{\mathrm{e}^{2t}+2\mathrm{e}^{-t}}{3},$$

$$a_2t^2+a_1t=t^2\frac{\mathrm{e}^{2t}-(1+3t)\mathrm{e}^{-t}}{9t^2}+t\frac{(3t-2)\mathrm{e}^{-t}+2\mathrm{e}^{2t}}{9t}=\frac{\mathrm{e}^{2t}-\mathrm{e}^{-t}}{3},$$

$\therefore$

$$\mathrm{e}^{\boldsymbol{A}t}=\begin{bmatrix}\frac{\mathrm{e}^{2t}+2\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}-\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}-\mathrm{e}^{-t}}{3}\\ \frac{\mathrm{e}^{2t}-\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}+2\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}-\mathrm{e}^{-t}}{3}\\ \frac{\mathrm{e}^{2t}-\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}-\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}+2\mathrm{e}^{-t}}{3}\end{bmatrix}$$

$\therefore$方程组的通解为

$$\boldsymbol{x}(t)=\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{c},$$

$\because (x_1(0),x_2(0),x_3(0))=(2,3,1),$

$\therefore \boldsymbol{x}(0) = \boldsymbol{I}\boldsymbol{c} = \boldsymbol{c} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$,

$\therefore$方程组满足初值条件的特解为

$$\boldsymbol{x}(t) = \begin{bmatrix} \frac{e^{2t}+2e^{-t}}{3} & \frac{e^{2t}-e^{-t}}{3} & \frac{e^{2t}-e^{-t}}{3} \\ \frac{e^{2t}-e^{-t}}{3} & \frac{e^{2t}+2e^{-t}}{3} & \frac{e^{2t}-e^{-t}}{3} \\ \frac{e^{2t}-e^{-t}}{3} & \frac{e^{2t}-e^{-t}}{3} & \frac{e^{2t}+2e^{-t}}{3} \end{bmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{6e^{2t}}{3} \\ \frac{6e^{2t}+3e^{-t}}{3} \\ \frac{6e^{2t}-3e^{-t}}{3} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t}+e^{-t} \\ 2e^{2t}-e^{-t} \end{pmatrix}.$$

(6)方程组的系数矩阵$\boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -4 \end{bmatrix}$,

$$e^{\boldsymbol{A}t} = a_2\boldsymbol{A}^2t^2 + a_1\boldsymbol{A}t + a_0\boldsymbol{I} = a_2t^2 \begin{bmatrix} 1 & -2 & 4 \\ 4 & 1 & -20 \\ -5 & 1 & 16 \end{bmatrix} + a_1t \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -4 \end{bmatrix} + a_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_2t^2 - a_1t + a_0 & -2a_2t^2 + a_1t & 4a_2t^2 \\ 4a_2t^2 & a_2t^2 - a_1t + a_0 & -20a_2t^2 + 4a_1t \\ -5a_2t^2 + a_1t & a_2t^2 & 16a_2t^2 - 4a_1t + a_0 \end{bmatrix},$$

$r(\lambda t) = a_2(\lambda t)^2 + a_1(\lambda t) + a_0$,

由$|\lambda\boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} \lambda+1 & -1 & 0 \\ 0 & \lambda+1 & -4 \\ -1 & 0 & \lambda+4 \end{vmatrix} = (\lambda+1)^2(\lambda+4) - 4 + (\lambda+1) = (\lambda+1)^2(\lambda+4) - 4$
$= (\lambda^2+2\lambda+1)(\lambda+4) - 4 = \lambda^3+2\lambda^2+\lambda+4\lambda^2+8\lambda+4-4 = \lambda^3+6\lambda^2+9\lambda = \lambda(\lambda+3)^2 = 0$得矩阵$\boldsymbol{A}$的特征值$\lambda_1 = 0, \lambda_{2,3} = -3$, 则

$$\begin{cases} e^{0t} = a_2(0t)^2 + a_1(0t) + a_0, \\ e^{-3t} = a_2(-3t)^2 + a_1(-3t) + a_0, \\ e^{-3t} = 2a_2(-3t) + a_1, \end{cases}$$

解得

$$a_0 = 1,$$

$$a_1 = \frac{\begin{vmatrix} 9t^2 & \mathrm{e}^{-3t}-1 \\ -6t & \mathrm{e}^{-3t} \end{vmatrix}}{\begin{vmatrix} 9t^2 & -3t \\ -6t & 1 \end{vmatrix}} = \frac{9t^2\mathrm{e}^{-3t}+6t(\mathrm{e}^{-3t}-1)}{9t^2-18t^2} = \frac{2-(3t+2)\mathrm{e}^{-3t}}{3t},$$

$$a_2 = \frac{\begin{vmatrix} \mathrm{e}^{-3t}-1 & -3t \\ \mathrm{e}^{-3t} & 1 \end{vmatrix}}{\begin{vmatrix} 9t^2 & -3t \\ -6t & 1 \end{vmatrix}} = \frac{\mathrm{e}^{-3t}-1+3t\mathrm{e}^{-3t}}{9t^2-18t^2} = \frac{1-(1+3t)\mathrm{e}^{-3t}}{9t^2},$$

则

$$\begin{aligned}
a_2t^2 - a_1t + a_0 &= \frac{1-(1+3t)\mathrm{e}^{-3t}}{9} - \frac{2-(3t+2)\mathrm{e}^{-3t}}{3} + 1 = \frac{4+(5+6t)\mathrm{e}^{-3t}}{9},\\
-2a_2t^2 + a_1t &= -2t^2\frac{1-(1+3t)\mathrm{e}^{-3t}}{9t^2} + t\frac{2-(3t+2)\mathrm{e}^{-3t}}{3t} = \frac{4-(4+3t)\mathrm{e}^{-3t}}{9},\\
4a_2t^2 &= \frac{4-(4+12t)\mathrm{e}^{-3t}}{9},\\
-20a_2t^2 + 4a_1t &= -20t^2\frac{1-(1+3t)\mathrm{e}^{-3t}}{9t^2} + 4t\frac{2-(3t+2)\mathrm{e}^{-3t}}{3t} = \frac{4-(4-24t)\mathrm{e}^{-3t}}{9},\\
-5a_2t^2 + a_1t &= -5t^2\frac{1-(1+3t)\mathrm{e}^{-3t}}{9t^2} + t\frac{2-(3t+2)\mathrm{e}^{-3t}}{3t} = \frac{1-(1-6t)\mathrm{e}^{-3t}}{9},\\
a_2t^2 &= \frac{1-(1+3t)\mathrm{e}^{-3t}}{9},\\
16a_2t^2 - 4a_1t + a_0 &= 16t^2\frac{1-(1+3t)\mathrm{e}^{-3t}}{9t^2} - 4t\frac{2-(3t+2)\mathrm{e}^{-3t}}{3t} + 1 = \frac{1+(8-12t)\mathrm{e}^{-3t}}{9},
\end{aligned}$$

$\therefore$

$$\mathrm{e}^{\boldsymbol{A}t} = \begin{bmatrix} \frac{4+(5+6t)\mathrm{e}^{-3t}}{9} & \frac{4-(4+3t)\mathrm{e}^{-3t}}{9} & \frac{4-(4+12t)\mathrm{e}^{-3t}}{9} \\ \frac{4-(4+12t)\mathrm{e}^{-3t}}{9} & \frac{4+(5+6t)\mathrm{e}^{-3t}}{9} & \frac{4-(4-24t)\mathrm{e}^{-3t}}{9} \\ \frac{1-(1-6t)\mathrm{e}^{-3t}}{9} & \frac{1-(1+3t)\mathrm{e}^{-3t}}{9} & \frac{1+(8-12t)\mathrm{e}^{-3t}}{9} \end{bmatrix}$$

$\therefore$方程组的通解为

$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{c},$$

$\because (x_1(0), x_2(0), x_3(0)) = (2, 3, 1)$,

$\therefore \boldsymbol{x}(0) = \boldsymbol{I}\boldsymbol{c} = \boldsymbol{c} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix},$

∴方程组满足初值条件的特解为

$$\boldsymbol{x}(t)=\begin{bmatrix}\frac{4+(5+6t)\mathrm{e}^{-3t}}{9} & \frac{4-(4+3t)\mathrm{e}^{-3t}}{9} & \frac{4-(4+12t)\mathrm{e}^{-3t}}{9}\\ \frac{4-(4+12t)\mathrm{e}^{-3t}}{9} & \frac{4+(5+6t)\mathrm{e}^{-3t}}{9} & \frac{4-(4-24t)\mathrm{e}^{-3t}}{9}\\ \frac{1-(1-6t)\mathrm{e}^{-3t}}{9} & \frac{1-(1+3t)\mathrm{e}^{-3t}}{9} & \frac{1+(8-12t)\mathrm{e}^{-3t}}{9}\end{bmatrix}\begin{pmatrix}2\\3\\1\end{pmatrix}=\begin{pmatrix}\frac{24-(-10+12+4-12t+9t+12t)\mathrm{e}^{-3t}}{9}\\ \frac{24-(8-15+4+24t-18t-24t)\mathrm{e}^{-3t}}{9}\\ \frac{6-(2+3-8-12t+9t+12t)\mathrm{e}^{-3t}}{9}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{8-(2+3t)\mathrm{e}^{-3t}}{3}\\ \frac{8-(-1-6t)\mathrm{e}^{-3t}}{3}\\ \frac{2-(-1+3t)\mathrm{e}^{-3t}}{3}\end{pmatrix}=\begin{pmatrix}\frac{8}{3}-(t+\frac{2}{3})\mathrm{e}^{-3t}\\ \frac{8}{3}+(\frac{1}{3}+2t)\mathrm{e}^{-3t}\\ \frac{2}{3}-(t-\frac{1}{3})\mathrm{e}^{-3t}\end{pmatrix}.$$

【注:】该题不适合用特征值和特征向量表示该常系数齐次线性微分方程的解，理由如下：

方程组的系数矩阵$\boldsymbol{A}=\begin{bmatrix}-1&1&0\\0&-1&4\\1&0&-4\end{bmatrix}$,

由$|\lambda\boldsymbol{I}-\boldsymbol{A}|=\begin{vmatrix}\lambda+1&-1&0\\0&\lambda+1&-4\\-1&0&\lambda+4\end{vmatrix}=(\lambda+1)^2(\lambda+4)-4+(\lambda+1)=(\lambda+1)^2(\lambda+4)-4$

$=(\lambda^2+2\lambda+1)(\lambda+4)-4=\lambda^3+2\lambda^2+\lambda+4\lambda^2+8\lambda+4-4=\lambda^3+6\lambda^2+9\lambda=\lambda(\lambda+3)^2=0$得矩阵$\boldsymbol{A}$的特征值$\lambda_1=0,\lambda_{2,3}=-3$,

解$(\lambda_1\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=-\boldsymbol{A}\boldsymbol{r}=\begin{bmatrix}1&-1&0\\0&1&-4\\-1&0&4\end{bmatrix}\boldsymbol{r}=\boldsymbol{0}=\begin{bmatrix}1&-1&0\\0&1&-4\\0&0&0\end{bmatrix}\boldsymbol{r}$得$\lambda_1=0$对应的特征向量为

$$\boldsymbol{r}_1=(4,4,1)^T,$$

解$(\lambda_{2,3}\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=(-3\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=\begin{bmatrix}-2&-1&0\\0&-2&-4\\-1&0&1\end{bmatrix}\boldsymbol{r}=\boldsymbol{0}$得基础解系

$$\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

此时二重特征值$\lambda_{2,3}$只对应一个线性无关的特征向量，故该题不适合用特征值和特征向量表示该常系数齐次线性微分方程的解.

2. 求下列微分方程的通解：

(1)$\begin{cases}\frac{\mathrm{d}x_1}{\mathrm{d}t}=x_1+2x_2-\mathrm{e}^{-t},\\ \frac{\mathrm{d}x_2}{\mathrm{d}t}=4x_1+3x_2+4\mathrm{e}^{-t};\end{cases}$

(2)$\begin{cases}\frac{\mathrm{d}x_1}{\mathrm{d}t}=-x_1-x_2+t^2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t}=-x_2-x_3+2t,\\ \frac{\mathrm{d}x_3}{\mathrm{d}t}=-x_3+t;\end{cases}$

(3)$\begin{cases}\frac{\mathrm{d}x_1}{\mathrm{d}t}=2x_1-x_2+x_3+2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t}=x_1+x_3+1,\\ \frac{\mathrm{d}x_3}{\mathrm{d}t}=-3x_1+x_2-2x_3-3;\end{cases}$

(4)$\begin{cases}4\frac{\mathrm{d}x_1}{\mathrm{d}t}-\frac{\mathrm{d}x_2}{\mathrm{d}t}=-3x_1+\sin t,\\ \frac{\mathrm{d}x_1}{\mathrm{d}t}=-x_2+\cos t.\end{cases}$

解：该线性常系数非齐次方程组对应的齐次方程组为$\begin{cases}\frac{\mathrm{d}x_1}{\mathrm{d}t}=x_1+2x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t}=4x_1+3x_2,\end{cases}$

由第一个方程得$x_2=\frac{1}{2}x_1'-\frac{1}{2}x_1$, 两端求关于$t$的导数得$x_2'=\frac{1}{2}x_1''-\frac{1}{2}x_1'$, 将这两式代入第二个方程得

$$\frac{1}{2}x_1''-\frac{1}{2}x_1'=4x_1+3(\frac{1}{2}x_1'-\frac{1}{2}x_1),$$

即

$$x_1''-4x_1'-5x_1=0,$$

该线性常系数齐次常微分方程的特征方程为$\lambda^2-4\lambda-5=(\lambda-5)(\lambda+1)=0$, 特征值为$\lambda_1=5,\lambda_2=-1$, 通解为

$$x_1(t)=c_1\mathrm{e}^{5t}+c_2\mathrm{e}^{-t},$$

则

$$\begin{aligned}x_2(t)&=\frac{1}{2}x_1'-\frac{1}{2}x_1\\&=\frac{1}{2}[5c_1\mathrm{e}^{5t}-c_2\mathrm{e}^{-t}-c_1\mathrm{e}^{5t}-c_2\mathrm{e}^{-t}]\\&=2c_1\mathrm{e}^{5t}-c_2\mathrm{e}^{-t},\end{aligned}$$

即

$$\boldsymbol{x}(t)=c_1\begin{pmatrix}\mathrm{e}^{5t}\\2\mathrm{e}^{5t}\end{pmatrix}+c_2\begin{pmatrix}\mathrm{e}^{-t}\\-\mathrm{e}^{-t}\end{pmatrix}=c_1\boldsymbol{\varphi}_1(t)+c_2\boldsymbol{\varphi}_2(t),$$

$\because$

$$W[\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2](t)=\begin{vmatrix}\mathrm{e}^{5t}&\mathrm{e}^{-t}\\2\mathrm{e}^{5t}&-\mathrm{e}^{-t}\end{vmatrix}=-\mathrm{e}^{4t}-2\mathrm{e}^{4t}=-3\mathrm{e}^{4t}\not\equiv 0,$$

$\therefore$

$$\boldsymbol{\phi}(t)=\begin{bmatrix}\mathrm{e}^{5t}&\mathrm{e}^{-t}\\2\mathrm{e}^{5t}&-\mathrm{e}^{-t}\end{bmatrix}$$

齐次线性方程组的一个基本解矩阵，其逆矩阵

$$\boldsymbol{\phi}^{-1}(t)=\frac{1}{-3\mathrm{e}^{4t}}\begin{bmatrix}-\mathrm{e}^{-t} & -\mathrm{e}^{-t}\\ -2\mathrm{e}^{5t} & \mathrm{e}^{5t}\end{bmatrix}$$

则非齐次方程的通解为

$$\begin{aligned}
\boldsymbol{x}(t)&=\boldsymbol{\phi}(t)\boldsymbol{c}+\boldsymbol{\phi}(t)\int_0^t\boldsymbol{\phi}^{-1}(s)\boldsymbol{f}(s)\mathrm{d}s\\
&=\boldsymbol{\phi}(t)\boldsymbol{c}+\boldsymbol{\phi}(t)\int_0^t\frac{1}{-3\mathrm{e}^{4s}}\begin{bmatrix}-\mathrm{e}^{-s} & -\mathrm{e}^{-s}\\ -2\mathrm{e}^{5s} & \mathrm{e}^{5s}\end{bmatrix}\begin{pmatrix}-\mathrm{e}^{-s}\\ 4\mathrm{e}^{-s}\end{pmatrix}\mathrm{d}s\\
&=\boldsymbol{\phi}(t)\boldsymbol{c}+\boldsymbol{\phi}(t)\int_0^t\frac{1}{-3\mathrm{e}^{4s}}\begin{pmatrix}\mathrm{e}^{-2s}-4\mathrm{e}^{-2s}\\ 2\mathrm{e}^{4s}+4\mathrm{e}^{4s}\end{pmatrix}\mathrm{d}s\\
&=\boldsymbol{\phi}(t)\boldsymbol{c}+\boldsymbol{\phi}(t)\int_0^t\begin{pmatrix}\mathrm{e}^{-6s}\\ -2\end{pmatrix}\mathrm{d}s\\
&=\boldsymbol{\phi}(t)\boldsymbol{c}+\boldsymbol{\phi}(t)\begin{pmatrix}\frac{1}{-6}(\mathrm{e}^{-6t}-1)\\ -2t\end{pmatrix}\\
&=\begin{bmatrix}\mathrm{e}^{5t} & \mathrm{e}^{-t}\\ 2\mathrm{e}^{5t} & -\mathrm{e}^{-t}\end{bmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}+\begin{bmatrix}\mathrm{e}^{5t} & \mathrm{e}^{-t}\\ 2\mathrm{e}^{5t} & -\mathrm{e}^{-t}\end{bmatrix}\begin{pmatrix}\frac{1}{-6}(\mathrm{e}^{-6t}-1)\\ -2t\end{pmatrix}\\
&=\begin{pmatrix}c_1\mathrm{e}^{5t}+c_2\mathrm{e}^{-t}\\ 2c_1\mathrm{e}^{5t}-c_2\mathrm{e}^{-t}\end{pmatrix}+\begin{pmatrix}-\frac{1}{6}\mathrm{e}^{-t}+\frac{1}{6}\mathrm{e}^{5t}-2t\mathrm{e}^{-t}\\ -\frac{1}{3}\mathrm{e}^{-t}+\frac{1}{3}\mathrm{e}^{5t}+2t\mathrm{e}^{-t}\end{pmatrix}\\
&=\begin{pmatrix}(c_1+\frac{1}{6})\mathrm{e}^{5t}+(c_2-\frac{1}{6})\mathrm{e}^{-t}-2t\mathrm{e}^{-t}\\ 2(c_1+\frac{1}{6})\mathrm{e}^{5t}-(c_2+\frac{1}{3})\mathrm{e}^{-t}+2t\mathrm{e}^{-t}\end{pmatrix}\\
&=\begin{pmatrix}c_2\mathrm{e}^{5t}+c_1\mathrm{e}^{-t}-2t\mathrm{e}^{-t}\\ 2c_2\mathrm{e}^{5t}-(c_1+\frac{1}{2})\mathrm{e}^{-t}+2t\mathrm{e}^{-t}\end{pmatrix}=\begin{pmatrix}c_1\mathrm{e}^{-t}+c_2\mathrm{e}^{5t}-2t\mathrm{e}^{-t}\\ 2c_2\mathrm{e}^{5t}-\frac{1}{2}(1+2c_1)\mathrm{e}^{-t}+2t\mathrm{e}^{-t}\end{pmatrix}.
\end{aligned}$$

(2)该常系数非齐次线性微分方程组对应的齐次方程组为$\begin{cases}\frac{\mathrm{d}x_1}{\mathrm{d}t}=-x_1-x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t}=-x_2-x_3,\\ \frac{\mathrm{d}x_3}{\mathrm{d}t}=-x_3,\end{cases}$

由第三个方程得$\frac{\mathrm{d}x_3}{x_3}=-\mathrm{d}t$, 即$\ln|x_3|=-t+C$, 故

$$x_3=c_1\mathrm{e}^{-t},$$

代入第二个方程得$\frac{\mathrm{d}x_2}{\mathrm{d}t}=-x_2-c_1\mathrm{e}^{-t}$, 即

$$x_2'+x_2=-c_1\mathrm{e}^{-t},$$

该非齐次线性微分方程的通解为

$$x_2=\mathrm{e}^{-\int\mathrm{d}t}(\int-c_1\mathrm{e}^{-t}\mathrm{e}^{\int\mathrm{d}t}+c_2)=\mathrm{e}^{-t}(-c_1\int\mathrm{e}^{-t}\mathrm{e}^{t}\mathrm{d}t+c_2)=\mathrm{e}^{-t}(-c_1t+c_2),$$

代入第一个方程得$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1 - \mathrm{e}^{-t}(-c_1t + c_2)$, 即

$$x_1' + x_1 = \mathrm{e}^{-t}(c_1t - c_2),$$

该非齐次线性微分方程的通解为

$$\begin{aligned} x_1 &= \mathrm{e}^{-\int \mathrm{d}t}[\int \mathrm{e}^{-t}(c_1t - c_2)\mathrm{e}^{\int \mathrm{d}t}\mathrm{d}t + c_3] \\ &= \mathrm{e}^{-t}[\int \mathrm{e}^{-t}(c_1t - c_2)\mathrm{e}^{t}\mathrm{d}t + c_3] \\ &= \mathrm{e}^{-t}[\int (c_1t - c_2)\mathrm{d}t + c_3] \\ &= \mathrm{e}^{-t}[\frac{1}{2}c_1t^2 - c_2t + c_3], \end{aligned}$$

则

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = c_1\begin{pmatrix} \frac{1}{2}t^2\mathrm{e}^{-t} \\ -t\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{pmatrix} + c_2\begin{pmatrix} -t\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \\ 0 \end{pmatrix} + c_3\begin{pmatrix} \mathrm{e}^{-t} \\ 0 \\ 0 \end{pmatrix} = c_1\boldsymbol{\varphi}_1(t) + c_2\boldsymbol{\varphi}_2(t) + c_3\boldsymbol{\varphi}_3(t),$$

$\because$

$$W[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3](t) = \begin{vmatrix} \frac{1}{2}t^2\mathrm{e}^{-t} & -t\mathrm{e}^{-t} & \mathrm{e}^{-t} \\ -t\mathrm{e}^{-t} & \mathrm{e}^{-t} & 0 \\ \mathrm{e}^{-t} & 0 & 0 \end{vmatrix} = (-1)^4\mathrm{e}^{-t}(-\mathrm{e}^{-2t}) = -\mathrm{e}^{-3t} \not\equiv 0,$$

$\therefore$

$$\boldsymbol{\phi}(t) = \begin{bmatrix} \frac{1}{2}t^2\mathrm{e}^{-t} & -t\mathrm{e}^{-t} & \mathrm{e}^{-t} \\ -t\mathrm{e}^{-t} & \mathrm{e}^{-t} & 0 \\ \mathrm{e}^{-t} & 0 & 0 \end{bmatrix}$$

是齐次方程组的一个基本解矩阵，其逆矩阵为

$$\boldsymbol{\phi}^{-1}(t) = \frac{1}{-\mathrm{e}^{-3t}}\begin{bmatrix} 0 & -0 & -\mathrm{e}^{-2t} \\ -0 & -\mathrm{e}^{-2t} & -t\mathrm{e}^{-2t} \\ -\mathrm{e}^{-2t} & -t\mathrm{e}^{-2t} & \frac{1}{2}t^2\mathrm{e}^{-2t} - t^2\mathrm{e}^{-2t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mathrm{e}^{t} \\ 0 & \mathrm{e}^{t} & t\mathrm{e}^{t} \\ \mathrm{e}^{t} & t\mathrm{e}^{t} & \frac{1}{2}t^2\mathrm{e}^{t} \end{bmatrix},$$

故原非齐次方程的通解为

$$\begin{aligned} \boldsymbol{x}(t) &= \boldsymbol{\phi}(t)\boldsymbol{c} + \boldsymbol{\phi}(t)\int_0^t \boldsymbol{\phi}^{-1}(s)\boldsymbol{f}(s)\mathrm{d}s \\ &= \boldsymbol{\phi}(t)\boldsymbol{c} + \boldsymbol{\phi}(t)\int_0^t \begin{bmatrix} 0 & 0 & \mathrm{e}^{s} \\ 0 & \mathrm{e}^{s} & s\mathrm{e}^{s} \\ \mathrm{e}^{s} & s\mathrm{e}^{s} & \frac{1}{2}s^2\mathrm{e}^{s} \end{bmatrix}\begin{pmatrix} s^2 \\ 2s \\ s \end{pmatrix}\mathrm{d}s \\ &= \boldsymbol{\phi}(t)\boldsymbol{c} + \boldsymbol{\phi}(t)\int_0^t \begin{pmatrix} s\mathrm{e}^{s} \\ 2s\mathrm{e}^{s} + s^2\mathrm{e}^{s} \\ s^2\mathrm{e}^{s} + 2s^2\mathrm{e}^{s} + \frac{1}{2}s^3\mathrm{e}^{s} \end{pmatrix}\mathrm{d}s = \boldsymbol{\phi}(t)\boldsymbol{c} + \boldsymbol{\phi}(t)\int_0^t \begin{pmatrix} s\mathrm{e}^{s} \\ (2s + s^2)\mathrm{e}^{s} \\ (3s^2 + \frac{1}{2}s^3)\mathrm{e}^{s} \end{pmatrix}\mathrm{d}s \end{aligned}$$

$\because$

$$\begin{aligned}
\int_0^t s\mathrm{e}^s\mathrm{d}s &= s\mathrm{e}^s\Big|_0^t - \int_0^t \mathrm{e}^s\mathrm{d}s = t\mathrm{e}^t - \mathrm{e}^t + 1,\\
\int_0^t (2s+s^2)\mathrm{e}^s\mathrm{d}s &= (2s+s^2)\mathrm{e}^s\Big|_0^t - \int_0^t \mathrm{e}^s(2+2s)\mathrm{d}s\\
&= (2t+t^2)\mathrm{e}^t - (2+2s)\mathrm{e}^s\Big|_0^t + \int_0^t \mathrm{e}^s 2\mathrm{d}s\\
&= (2t+t^2)\mathrm{e}^t - (2+2t)\mathrm{e}^t + 2 + 2\mathrm{e}^t - 2 = t^2\mathrm{e}^t,\\
\int_0^t (3s^2+\frac{1}{2}s^3)\mathrm{e}^s\mathrm{d}s &= (3s^2+\frac{1}{2}s^3)\mathrm{e}^s\Big|_0^t - \int_0^t \mathrm{e}^s(6s+\frac{3}{2}s^2)\mathrm{d}s\\
&= (3t^2+\frac{1}{2}t^3)\mathrm{e}^t - (6s+\frac{3}{2}s^2)\mathrm{e}^s\Big|_0^t + \int_0^t \mathrm{e}^s(6+3s)\mathrm{d}s\\
&= (3t^2+\frac{1}{2}t^3)\mathrm{e}^t - (6t+\frac{3}{2}t^2)\mathrm{e}^t + (6+3s)\mathrm{e}^s\Big|_0^t - \int_0^t \mathrm{e}^s 3\mathrm{d}s\\
&= (3t^2+\frac{1}{2}t^3)\mathrm{e}^t - (6t+\frac{3}{2}t^2)\mathrm{e}^t + (6+3t)\mathrm{e}^t - 6 - 3\mathrm{e}^t + 3\\
&= \frac{1}{2}t^3\mathrm{e}^t + \frac{3}{2}t^2\mathrm{e}^t - 3t\mathrm{e}^t + 3\mathrm{e}^t - 3,
\end{aligned}$$

$\therefore$

$$\begin{aligned}
\boldsymbol{x}(t) &= \boldsymbol{\phi}(t)\boldsymbol{c} + \begin{bmatrix} \frac{1}{2}t^2\mathrm{e}^{-t} & -t\mathrm{e}^{-t} & \mathrm{e}^{-t}\\ -t\mathrm{e}^{-t} & \mathrm{e}^{-t} & 0\\ \mathrm{e}^{-t} & 0 & 0 \end{bmatrix} \begin{pmatrix} t\mathrm{e}^t - \mathrm{e}^t + 1\\ t^2\mathrm{e}^t\\ \frac{1}{2}t^3\mathrm{e}^t + \frac{3}{2}t^2\mathrm{e}^t - 3t\mathrm{e}^t + 3\mathrm{e}^t - 3 \end{pmatrix}\\
&= \boldsymbol{\phi}(t)\boldsymbol{c} + \begin{pmatrix} \frac{1}{2}t^3 - \frac{1}{2}t^2 + \frac{1}{2}t^2\mathrm{e}^{-t} - t^3 + \frac{1}{2}t^3 + \frac{3}{2}t^2 - 3t + 3 - 3\mathrm{e}^{-t}\\ -t^2 + t - t\mathrm{e}^{-t} + t^2 + 0\\ t - 1 + \mathrm{e}^{-t} \end{pmatrix}\\
&= \boldsymbol{\phi}(t)\boldsymbol{c} + \begin{pmatrix} \frac{1}{2}t^2\mathrm{e}^{-t} - 3\mathrm{e}^{-t} + t^2 - 3t + 3\\ t - t\mathrm{e}^{-t}\\ t + \mathrm{e}^{-t} - 1 \end{pmatrix}\\
&= \begin{bmatrix} \frac{1}{2}t^2\mathrm{e}^{-t} & -t\mathrm{e}^{-t} & \mathrm{e}^{-t}\\ -t\mathrm{e}^{-t} & \mathrm{e}^{-t} & 0\\ \mathrm{e}^{-t} & 0 & 0 \end{bmatrix} \begin{pmatrix} c_1\\ c_2\\ c_3 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}t^2\mathrm{e}^{-t} - 3\mathrm{e}^{-t} + t^2 - 3t + 3\\ t - t\mathrm{e}^{-t}\\ t + \mathrm{e}^{-t} - 1 \end{pmatrix}\\
&= \begin{pmatrix} (c_1+1)\frac{1}{2}t^2\mathrm{e}^{-t} - c_2 t\mathrm{e}^{-t} + (c_3-3)\mathrm{e}^{-t} + t^2 - 3t + 3\\ -(c_1+1)t\mathrm{e}^{-t} + c_2\mathrm{e}^{-t} + t\\ (c_1+1)\mathrm{e}^{-t} + t - 1 \end{pmatrix}.
\end{aligned}$$

【注:】此题书中答案有误.

(3)原方程组对应的齐次线性方程组为$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = 2x_1 - x_2 + x_3,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 + x_3,\\ \frac{\mathrm{d}x_3}{\mathrm{d}t} = -3x_1 + x_2 - 2x_3, \end{cases}$ 系数矩阵$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 1\\ 1 & 0 & 1\\ -3 & 1 & -2 \end{bmatrix}$,

由$|\lambda\boldsymbol{I}-\boldsymbol{A}|=\begin{vmatrix}\lambda-2&1&-1\\-1&\lambda&-1\\3&-1&\lambda+2\end{vmatrix}=(\lambda-2)(\lambda+2)\lambda-1-3+3\lambda-(\lambda-2)+(\lambda+2)=\lambda(\lambda^2-1)$
$=\lambda(\lambda-1)(\lambda+1)=0$得矩阵$\boldsymbol{A}$的特征值$\lambda_1=-1,\lambda_2=0,\lambda_3=1$,

解$(\lambda_1\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=\begin{bmatrix}-3&1&-1\\-1&-1&-1\\3&-1&1\end{bmatrix}\boldsymbol{r}=\boldsymbol{0}=\begin{bmatrix}-3&1&-1\\-1&-1&-1\\0&0&0\end{bmatrix}\boldsymbol{r}=\begin{bmatrix}-3&1&-1\\0&2&1\\0&0&0\end{bmatrix}\boldsymbol{r}$得$\lambda_1$对应的特征向量

$$\boldsymbol{r}_1=(1,1,-2)^T,$$

解$(\lambda_2\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=\begin{bmatrix}-2&1&-1\\-1&0&-1\\3&-1&2\end{bmatrix}\boldsymbol{r}=\boldsymbol{0}=\begin{bmatrix}-2&1&-1\\-1&0&-1\\0&0&0\end{bmatrix}\boldsymbol{r}=\begin{bmatrix}-2&1&-1\\1&0&1\\0&0&0\end{bmatrix}\boldsymbol{r}$得$\lambda_2$对应的特征向量

$$\boldsymbol{r}_2=(1,1,-1)^T,$$

解$(\lambda_3\boldsymbol{I}-\boldsymbol{A})\boldsymbol{r}=\begin{bmatrix}-1&1&-1\\-1&1&-1\\3&-1&3\end{bmatrix}\boldsymbol{r}=\boldsymbol{0}=\begin{bmatrix}0&0&0\\-1&1&-1\\3&-1&3\end{bmatrix}\boldsymbol{r}=\begin{bmatrix}0&0&0\\-1&1&-1\\0&2&0\end{bmatrix}\boldsymbol{r}$得$\lambda_2$对应的特征向量

$$\boldsymbol{r}_3=(1,0,-1)^T,$$

则线性方程组的一个基本解组为

$$\boldsymbol{\varphi}_1(t)=\mathrm{e}^{\lambda_1 t}\boldsymbol{r}_1=\begin{pmatrix}\mathrm{e}^{-t}\\\mathrm{e}^{-t}\\-2\mathrm{e}^{-t}\end{pmatrix},\boldsymbol{\varphi}_2(t)=\mathrm{e}^{\lambda_2 t}\boldsymbol{r}_2=\begin{pmatrix}1\\1\\-1\end{pmatrix},\boldsymbol{\varphi}_3(t)=\mathrm{e}^{\lambda_3 t}\boldsymbol{r}_3=\begin{pmatrix}\mathrm{e}^{t}\\0\\-\mathrm{e}^{t}\end{pmatrix},$$

基本解矩阵

$$\boldsymbol{\phi}(t)=\begin{bmatrix}\mathrm{e}^{-t}&1&\mathrm{e}^{t}\\\mathrm{e}^{-t}&1&0\\-2\mathrm{e}^{-t}&-1&-\mathrm{e}^{t}\end{bmatrix},$$

$$|\boldsymbol{\phi}(t)|=W[\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2,\boldsymbol{\varphi}_3](t)=\begin{vmatrix}\mathrm{e}^{-t}&1&\mathrm{e}^{t}\\\mathrm{e}^{-t}&1&0\\-2\mathrm{e}^{-t}&-1&-\mathrm{e}^{t}\end{vmatrix}=-1-1+0+2-0+1=1,$$

基本解矩阵的逆矩阵

$$\boldsymbol{\phi}^{-1}(t)=\frac{1}{|\boldsymbol{\phi}(t)|}\begin{bmatrix}-\mathrm{e}^{t}&-0&-\mathrm{e}^{t}\\-(-1)&-1+2&-(-1)\\-\mathrm{e}^{-t}+2\mathrm{e}^{-t}&-(-\mathrm{e}^{-t}+2\mathrm{e}^{-t})&0\end{bmatrix}=\begin{bmatrix}-\mathrm{e}^{t}&0&-\mathrm{e}^{t}\\1&1&1\\\mathrm{e}^{-t}&-\mathrm{e}^{-t}&0\end{bmatrix}$$

则非齐次方程的通解为

$$\begin{aligned}
\boldsymbol{x}(t) &= \boldsymbol{\phi}(t)\boldsymbol{c} + \boldsymbol{\phi}(t)\int_0^t \boldsymbol{\phi}^{-1}(s)\boldsymbol{f}(s)\mathrm{d}s \\
&= \boldsymbol{\phi}(t)\boldsymbol{c} + \boldsymbol{\phi}(t)\int_0^t \begin{bmatrix} -\mathrm{e}^s & 0 & -\mathrm{e}^s \\ 1 & 1 & 1 \\ \mathrm{e}^{-s} & -\mathrm{e}^{-s} & 0 \end{bmatrix} \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \mathrm{d}s \\
&= \boldsymbol{\phi}(t)\boldsymbol{c} + \boldsymbol{\phi}(t)\int_0^t \begin{pmatrix} \mathrm{e}^s \\ 0 \\ \mathrm{e}^{-s} \end{pmatrix} \mathrm{d}s \\
&= \boldsymbol{\phi}(t)\boldsymbol{c} + \boldsymbol{\phi}(t) \begin{pmatrix} \mathrm{e}^t - 1 \\ 0 \\ -\mathrm{e}^{-t} + 1 \end{pmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-t} & 1 & \mathrm{e}^t \\ \mathrm{e}^{-t} & 1 & 0 \\ -2\mathrm{e}^{-t} & -1 & -\mathrm{e}^t \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{bmatrix} \mathrm{e}^{-t} & 1 & \mathrm{e}^t \\ \mathrm{e}^{-t} & 1 & 0 \\ -2\mathrm{e}^{-t} & -1 & -\mathrm{e}^t \end{bmatrix} \begin{pmatrix} \mathrm{e}^t - 1 \\ 0 \\ -\mathrm{e}^{-t} + 1 \end{pmatrix} \\
&= \begin{pmatrix} c_1\mathrm{e}^{-t} + c_2 + c_3\mathrm{e}^t + 1 - \mathrm{e}^{-t} + 0 - 1 + \mathrm{e}^t \\ c_1\mathrm{e}^{-t} + c_2 + 0 + 1 - \mathrm{e}^{-t} + 0 + 0 \\ -2c_1\mathrm{e}^{-t} - c_2 - c_3\mathrm{e}^t - 2 + 2\mathrm{e}^{-t} + 0 + 1 - \mathrm{e}^t \end{pmatrix} \\
&= \begin{pmatrix} (c_1 - 1)\mathrm{e}^{-t} + c_2 + (c_3 + 1)\mathrm{e}^t \\ (c_1 - 1)\mathrm{e}^{-t} + c_2 + 1 \\ -2(c_1 - 1)\mathrm{e}^{-t} - (c_2 + 1) - (c_3 + 1)\mathrm{e}^t \end{pmatrix} \\
&= \begin{pmatrix} c_1 - 1 + c_2\mathrm{e}^{-t} - c_3\mathrm{e}^t \\ c_1 + c_2\mathrm{e}^{-t} \\ -c_1 - 2c_2\mathrm{e}^{-t} + c_3\mathrm{e}^t \end{pmatrix}.
\end{aligned}$$

(4)该方程组可化为$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = -x_2 + \cos t, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = 3x_1 - 4x_2 + 4\cos t - \sin t, \end{cases}$ 对应的齐次方程组为$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = -x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = 3x_1 - 4x_2, \end{cases}$ 由第一个方程得$x_2 = -x_1'$, 两端求导得$x_2' = -x_1''$, 将以上两式代入第二个方程得

$$-x_1'' = 3x_1 - 4(-x_1'),$$

即

$$x_1'' + 4x_1' + 3x_1 = 0,$$

该齐次线性微分方程的特征方程为$\lambda^2+4\lambda+3 = (\lambda+1)(\lambda+3) = 0$, 特征根$\lambda_1 = -1, \lambda_2 = -3$, 通解

$$x_1 = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{-3t},$$

则

$$x_2 = -x_1' = c_1 \mathrm{e}^{-t} + 3c_2 \mathrm{e}^{-3t},$$

故齐次方程组的通解为

$$\boldsymbol{x}(t) = c_1 \begin{pmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{pmatrix} + c_2 \begin{pmatrix} \mathrm{e}^{-3t} \\ 3\mathrm{e}^{-3t} \end{pmatrix} = c_1 \boldsymbol{\varphi}_1(t) + c_2 \boldsymbol{\varphi}_2(t),$$

由

$$W[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2](t) = \begin{vmatrix} \mathrm{e}^{-t} & \mathrm{e}^{-3t} \\ \mathrm{e}^{-t} & 3\mathrm{e}^{-3t} \end{vmatrix} = 3\mathrm{e}^{-4t} - \mathrm{e}^{-4t} = 2\mathrm{e}^{-4t} \not\equiv 0,$$

故齐次方程组的一个基本解矩阵为

$$\boldsymbol{\phi}(t) = \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^{-3t} \\ \mathrm{e}^{-t} & 3\mathrm{e}^{-3t} \end{bmatrix},$$

其逆矩阵

$$\boldsymbol{\phi}^{-1}(t) = \frac{1}{2\mathrm{e}^{-4t}} \begin{bmatrix} 3\mathrm{e}^{-3t} & -\mathrm{e}^{-3t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{-t} \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\mathrm{e}^{t} & -\frac{1}{2}\mathrm{e}^{t} \\ -\frac{1}{2}\mathrm{e}^{3t} & \frac{1}{2}\mathrm{e}^{3t} \end{bmatrix},$$

则非齐次方程组的通解

$$\begin{aligned} \boldsymbol{x}(t) &= \boldsymbol{\phi}(t)\boldsymbol{c} + \boldsymbol{\phi}(t) \int_0^t \boldsymbol{\phi}^{-1}(s) \boldsymbol{f}(t) \mathrm{d}s \\ &= \boldsymbol{\phi}(t)\boldsymbol{c} + \boldsymbol{\phi}(t) \int_0^t \begin{bmatrix} \frac{3}{2}\mathrm{e}^{s} & -\frac{1}{2}\mathrm{e}^{s} \\ -\frac{1}{2}\mathrm{e}^{3s} & \frac{1}{2}\mathrm{e}^{3s} \end{bmatrix} \begin{pmatrix} \cos s \\ 4\cos s - \sin s \end{pmatrix} \mathrm{d}s = \\ &= \boldsymbol{\phi}(t)\boldsymbol{c} + \boldsymbol{\phi}(t) \int_0^t \begin{pmatrix} -\frac{1}{2}\mathrm{e}^{s}\cos s + \frac{1}{2}\mathrm{e}^{s}\sin s \\ \frac{3}{2}\mathrm{e}^{3s}\cos s - \frac{1}{2}\mathrm{e}^{3s}\sin s \end{pmatrix} \mathrm{d}s, \end{aligned}$$

$\because$

$$\begin{aligned} \int_0^t \frac{1}{2}\mathrm{e}^{s}\cos s\mathrm{d}s &= \frac{1}{2}\int_0^t \mathrm{e}^{s}\mathrm{d}\sin s = \frac{1}{2}\mathrm{e}^{s}\sin s\Big|_0^t - \frac{1}{2}\int_0^t \mathrm{e}^{s}\sin s\mathrm{d}s \\ &= \frac{1}{2}\mathrm{e}^{t}\sin t + \frac{1}{2}\mathrm{e}^{s}\cos s\Big|_0^t - \frac{1}{2}\int_0^t \mathrm{e}^{s}\cos s\mathrm{d}s = \\ &= \frac{1}{2}\mathrm{e}^{t}\sin t + \frac{1}{2}\mathrm{e}^{t}\cos t - \frac{1}{2} - \frac{1}{2}\int_0^t \mathrm{e}^{s}\cos s\mathrm{d}s \\ &= \frac{1}{4}\mathrm{e}^{t}\sin t + \frac{1}{4}\mathrm{e}^{t}\cos t - \frac{1}{4}, \\ \int_0^t \frac{1}{2}\mathrm{e}^{s}\sin s\mathrm{d}s &= \frac{1}{2}\int_0^t \sin s\mathrm{d}\mathrm{e}^{s} = \frac{1}{2}\mathrm{e}^{s}\sin s\Big|_0^t - \frac{1}{2}\int_0^t \mathrm{e}^{s}\cos s\mathrm{d}s \\ &= \frac{1}{2}\mathrm{e}^{t}\sin t - \frac{1}{2}\mathrm{e}^{s}\cos s\Big|_0^t - \frac{1}{2}\int_0^t \mathrm{e}^{s}\sin s\mathrm{d}s \\ &= \frac{1}{2}\mathrm{e}^{t}\sin t - \frac{1}{2}\mathrm{e}^{t}\cos t + \frac{1}{2} - \frac{1}{2}\int_0^t \mathrm{e}^{s}\sin s\mathrm{d}s \\ &= \frac{1}{4}\mathrm{e}^{t}\sin t - \frac{1}{4}\mathrm{e}^{t}\cos t + \frac{1}{4}, \end{aligned}$$

$$
\begin{aligned}
\int_0^t \frac{3}{2}\mathrm{e}^{3s}\cos s\mathrm{d}s &= \frac{3}{2}\int_0^t \mathrm{e}^{3s}\mathrm{d}\sin s = \frac{3}{2}\mathrm{e}^{3s}\sin s\Big|_0^t - \frac{9}{2}\int_0^t \mathrm{e}^{3s}\sin s\mathrm{d}s \\
&= \frac{3}{2}\mathrm{e}^{3t}\sin t + \frac{9}{2}\int_0^t \mathrm{e}^{3s}\mathrm{d}\cos s \\
&= \frac{3}{2}\mathrm{e}^{3t}\sin t + \frac{9}{2}\mathrm{e}^{3s}\cos s\Big|_0^t - \frac{27}{2}\int_0^t \mathrm{e}^{3s}\cos s\mathrm{d}s \\
&= \frac{3}{2}\mathrm{e}^{3t}\sin t + \frac{9}{2}\mathrm{e}^{3t}\cos t - \frac{9}{2} - \frac{27}{2}\int_0^t \mathrm{e}^{3s}\cos s\mathrm{d}s \\
&= \frac{3}{20}\mathrm{e}^{3t}\sin t + \frac{9}{20}\mathrm{e}^{3t}\cos t - \frac{9}{20},
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t \frac{1}{2}\mathrm{e}^{3s}\sin s\mathrm{d}s &= -\frac{1}{2}\int_0^t \mathrm{e}^{3s}\mathrm{d}\cos s = -\frac{1}{2}\mathrm{e}^{3s}\cos s\Big|_0^t + \frac{3}{2}\int_0^t \mathrm{e}^{3s}\cos s\mathrm{d}s \\
&= -\frac{1}{2}\mathrm{e}^{3t}\cos t + \frac{1}{2} + \frac{3}{2}\mathrm{e}^{3s}\sin s\Big|_0^t - \frac{9}{2}\int_0^t \mathrm{e}^{3s}\sin s\mathrm{d}s \\
&= -\frac{1}{2}\mathrm{e}^{3t}\cos t + \frac{1}{2} + \frac{3}{2}\mathrm{e}^{3t}\sin t - \frac{9}{2}\int_0^t \mathrm{e}^{3s}\sin s\mathrm{d}s \\
&= -\frac{1}{20}\mathrm{e}^{3t}\cos t + \frac{1}{20} + \frac{3}{20}\mathrm{e}^{3t}\sin t,
\end{aligned}
$$

$\therefore$

$$
\begin{aligned}
\boldsymbol{x}(t) &= \boldsymbol{\phi}(t)\boldsymbol{c} + \boldsymbol{\phi}(t)\begin{pmatrix} -(\frac{1}{4}\mathrm{e}^t\sin t + \frac{1}{4}\mathrm{e}^t\cos t - \frac{1}{4}) + \frac{1}{4}\mathrm{e}^t\sin t - \frac{1}{4}\mathrm{e}^t\cos t + \frac{1}{4} \\ \frac{3}{20}\mathrm{e}^{3t}\sin t + \frac{9}{20}\mathrm{e}^{3t}\cos t - \frac{9}{20} - (-\frac{1}{20}\mathrm{e}^{3t}\cos t + \frac{1}{20} + \frac{3}{20}\mathrm{e}^{3t}\sin t) \end{pmatrix} \\
&= \boldsymbol{\phi}(t)\boldsymbol{c} + \boldsymbol{\phi}(t)\begin{pmatrix} -\frac{1}{2}\mathrm{e}^t\cos t + \frac{1}{2} \\ \frac{1}{2}\mathrm{e}^{3t}\cos t - \frac{1}{2} \end{pmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^{-3t} \\ \mathrm{e}^{-t} & 3\mathrm{e}^{-3t} \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^{-3t} \\ \mathrm{e}^{-t} & 3\mathrm{e}^{-3t} \end{bmatrix}\begin{pmatrix} -\frac{1}{2}\mathrm{e}^t\cos t + \frac{1}{2} \\ \frac{1}{2}\mathrm{e}^{3t}\cos t - \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{3t} - \frac{1}{2}\cos t + \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{2}\cos t - \frac{1}{2}\mathrm{e}^{-3t} \\ c_1\mathrm{e}^{-t} + 3c_2\mathrm{e}^{3t} - \frac{1}{2}\cos t + \frac{1}{2}\mathrm{e}^{-t} + \frac{3}{2}\cos t - \frac{3}{2}\mathrm{e}^{-3t} \end{pmatrix} \\
&= \begin{pmatrix} (c_1 + \frac{1}{2})\mathrm{e}^{-t} + (c_2 - \frac{1}{2})\mathrm{e}^{3t} \\ (c_1 + \frac{1}{2})\mathrm{e}^{-t} + 3(c_2 - \frac{1}{2})\mathrm{e}^{3t} + \cos t \end{pmatrix} \\
&= \begin{pmatrix} c_1\mathrm{e}^{-t} + \frac{1}{3}c_2\mathrm{e}^{3t} \\ c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{3t} + \cos t \end{pmatrix}.
\end{aligned}
$$