24 高斯公式与斯托克斯公式

24.1 知识结构

第13章向量场的微积分

13.5 高斯公式与斯托克斯公式

　　13.5.1 高斯公式

　　13.5.2 斯托克斯公式

24.2 高斯公式与斯托克斯公式

1. 高斯公式

　　$\Omega \subset \mathbb{R}^3, \partial\Omega$逐片光滑，

　　向量场$\boldsymbol{F}(x,y,z) = X(x,y,z)\boldsymbol{i} + Y(x,y,z)\boldsymbol{j} + Z(x,y,z)\boldsymbol{k} \in C^1(\Omega)$,

　　则

$$
\begin{aligned}
\oiint_{\partial\Omega} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \oiint_{\partial\Omega} \boldsymbol{F} \cdot \boldsymbol{n}\mathrm{d}S &= \oiint_{\partial\Omega} X\mathrm{d}y \wedge \mathrm{d}z + Y\mathrm{d}z \wedge \mathrm{d}x + Z\mathrm{d}x \wedge \mathrm{d}y \\
&= \iiint_{\Omega} (\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \iiint_{\Omega} \boldsymbol{\nabla} \cdot \boldsymbol{F}\mathrm{d}x\mathrm{d}y\mathrm{d}z^1 \\
&= \iiint_{\Omega} \mathrm{div}\boldsymbol{F}\mathrm{d}x\mathrm{d}y\mathrm{d}z.
\end{aligned}
$$

【直接代入公式计算的题目：1.(1)/(2)/(3)/(4)/(6). (第一类题目)】

注意：

(a) 在非封闭曲面上的积分，可添加简单曲面构造封闭区域，利用高斯公式计算.

【利用该点计算的题目：1.(5). (第二类题目)】

(b) $\boldsymbol{F}(x,y,z) = X(x,y,z)\boldsymbol{i} + Y(x,y,z)\boldsymbol{j} + Z(x,y,z)\boldsymbol{k} \in C^1(\Omega)$，即$X,Y,Z$有一阶连续偏导数，也即$X,Y,Z$连续可微.

【考察该点的题目：1.(7). (第三类题目)】

(c) $\partial\Omega$外侧为正，即在$\partial\Omega$的外侧积分.

【考察该点的题目：1.(2).】

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[1]这里的符号∇可读作nabla.

(d) $\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y$ 与 X, Y, Z 的对应关系.

【考察该点的题目：1.(4)/(6).】

(e) 利用奇偶性和对称性可简化计算.

【考察该点的题目：1.(2)(奇偶性),(3)/(4)(轮换对称性).】

2. 斯托克斯公式

Σ 为逐片光滑的有向曲面，$\partial\Sigma$ 逐段光滑，

向量场 $\boldsymbol{F}(x, y, z) = X(x, y, z)\boldsymbol{i} + Y(x, y, z)\boldsymbol{j} + Z(x, y, z)\boldsymbol{k} \in C^1(\Sigma)$,

则

$$\oint_{\partial\Sigma} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l} = \oint_{\partial\Sigma} \boldsymbol{F} \cdot \boldsymbol{\tau} \mathrm{d}l = \oint_{\partial\Sigma} X\mathrm{d} + Y\mathrm{d}y + Z\mathrm{d}z$$

$$= \iint_{\Sigma} \begin{vmatrix} \mathrm{d}y \wedge \mathrm{d}z & \mathrm{d}z \wedge \mathrm{d}x & \mathrm{d}x \wedge \mathrm{d}y \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix}$$

$$= \iint_{\Sigma} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix} \cdot \mathrm{d}\boldsymbol{S} = \iint_{\Sigma} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix} \cdot \boldsymbol{n}\mathrm{d}S$$

$$= \iint_{\Sigma} (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S} = \iint_{\Sigma} (\mathrm{rot}\boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S}.$$

注意：

(a) Σ 与 $\partial\Sigma$ 的方向符合右手法则. 在用斯托克斯公式进行计算时，选取的曲面方向应与已知曲线方向满足右手法则.

【如本节习题中2.(1)/(2),3.(1)/(2)的图形所示(图 8, 9, 10, 11).】

(b) 曲面Σ应尽量简单，便于求面积或便于计算曲面积分. 如本节习题中的2.(1)，已知曲线L是一个大圆，可直接求出其围成的圆形平面的面积，故Σ取为L围成的圆形平面，根据右手法则，上侧为正. 本节习题中的2.(2)中，已知曲线是一个平面曲线，Σ可取为该平面曲线围成的平面，根据右手法则，上侧为正.

【第一类习题：2.(1)/(2).】

(c) $\boldsymbol{F}(x, y, z) = X(x, y, z)\boldsymbol{i} + Y(x, y, z)\boldsymbol{j} + Z(x, y, z)\boldsymbol{k} \in C^1(\Sigma)$，即$X, Y, Z$有一阶连续偏导数，也即$X, Y, Z$连续可微.

【考察该点的题目：3.(1)/(2). (第二类习题)】

【其他类型的题目：4. (第三类习题)】

24.3　习题13.5解答

1. 用高斯公式计算下列曲面积分：

(1)$\iint\limits_{S} x^3\mathrm{d}y \wedge \mathrm{d}z + y^3\mathrm{d}z \wedge \mathrm{d}x + z^3\mathrm{d}x \wedge \mathrm{d}y$，其中$S$为区域$x^2 + y^2 \leqslant z, 0 \leqslant z \leqslant 4$的边界外侧；

(2)$\iint\limits_{S} x^2\mathrm{d}y \wedge \mathrm{d}z + y^2\mathrm{d}z \wedge \mathrm{d}x + z^2\mathrm{d}x \wedge \mathrm{d}y$，其中$S$为$x^2 + y^2 + z^2 = R^2$的内侧；

(3)$\iint\limits_{S} x^2\mathrm{d}y \wedge \mathrm{d}z + y^2\mathrm{d}z \wedge \mathrm{d}x + z^2\mathrm{d}x \wedge \mathrm{d}y$，其中$S$为正方体$\Omega: 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 1$的外表面；

(4)$\iint\limits_{S} xz\mathrm{d}x \wedge \mathrm{d}y + xy\mathrm{d}y \wedge \mathrm{d}z + yz\mathrm{d}z \wedge \mathrm{d}x$，其中$S$为平面$x + y + z = 1$与三个坐标面围成的区域边界外侧；

(5)$\iint\limits_{S} x^2\mathrm{d}y \wedge \mathrm{d}z + (z^2 - 2z)\mathrm{d}x \wedge \mathrm{d}y$，其中$S$为$z = \sqrt{x^2 + y^2}$被平面$z = 0$和$z = 1$所截部分的外侧；

(6)$\oiint\limits_{S}(x - y)\mathrm{d}x \wedge \mathrm{d}y + (y - z)x\mathrm{d}y \wedge \mathrm{d}z$，其中$S$为柱面$x^2 + y^2 = 1$与平面$z = 1, z = 3$所围柱体的边界外侧；

(7)$\oiint\limits_{S} \dfrac{x\mathrm{d}y\wedge\mathrm{d}z + y\mathrm{d}z\wedge\mathrm{d}x + z\mathrm{d}x\wedge\mathrm{d}y}{(x^2+y^2+z^2)^{\frac{3}{2}}}$，其中$S$为椭球面$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

解：(1)记S围成的区域为Ω，则$x^3, y^3, z^3 \in C^1(\Omega)$，

$\therefore \iint\limits_{S} x^3\mathrm{d}y \wedge \mathrm{d}z + y^3\mathrm{d}z \wedge \mathrm{d}x + z^3\mathrm{d}x \wedge \mathrm{d}y = \iiint\limits_{\Omega}(\frac{\partial x^3}{\partial x} + \frac{\partial y^3}{\partial y} + \frac{\partial z^3}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z$

$= \iiint\limits_{\Omega}(3x^2 + 3y^2 + 3z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z = 3\int_0^{2\pi}\mathrm{d}\theta\int_0^2\mathrm{d}r\int_{r^2}^4(r^2 + z^2)\cdot r\mathrm{d}z = 6\pi\int_0^2(r^3z + \frac{1}{3}rz^3)\big|_{r^2}^4\mathrm{d}r$

$= 6\pi\int_0^2(4r^3 + \frac{64}{3}r - r^5 - \frac{1}{3}r^7)\mathrm{d}r = 6\pi(r^4 + \frac{32}{3}r^2 - \frac{1}{6}r^6 - \frac{1}{3}\frac{1}{8}r^8)\big|_0^2$

$= 6\pi(16 + \frac{32}{3}\cdot 4 - \frac{1}{6}\cdot 64 - \frac{1}{3}\cdot\frac{1}{8}\cdot 2^8) = 6\pi(16 + \frac{64}{3}) = (96 + 128)\pi = 224\pi.$

图 1: 习题13.5 1.(1)题图示

(2)记Ω为S^-围成的区域,
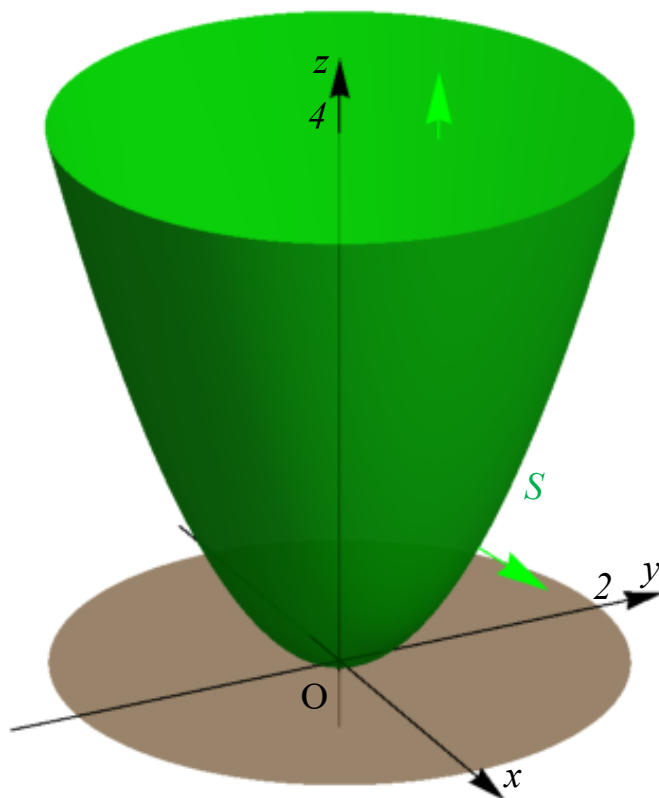
$\because x^2, y^2, z^2 \in C^1(\Omega),$

$\therefore \iint\limits_{S} x^2 \mathrm{d}y \wedge \mathrm{d}z + y^2 \mathrm{d}z \wedge \mathrm{d}x + z^2 \mathrm{d}x \wedge \mathrm{d}y = -\iint\limits_{S^-} x^2 \mathrm{d}y \wedge \mathrm{d}z + y^2 \mathrm{d}z \wedge \mathrm{d}x + z^2 \mathrm{d}x \wedge \mathrm{d}y$

$= -\iiint\limits_{\Omega} (\frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z}) \mathrm{d}x \mathrm{d}y \mathrm{d}z = -\iiint\limits_{\Omega} (2x + 2y + 2z) \mathrm{d}x \mathrm{d}y \mathrm{d}z = 0.$
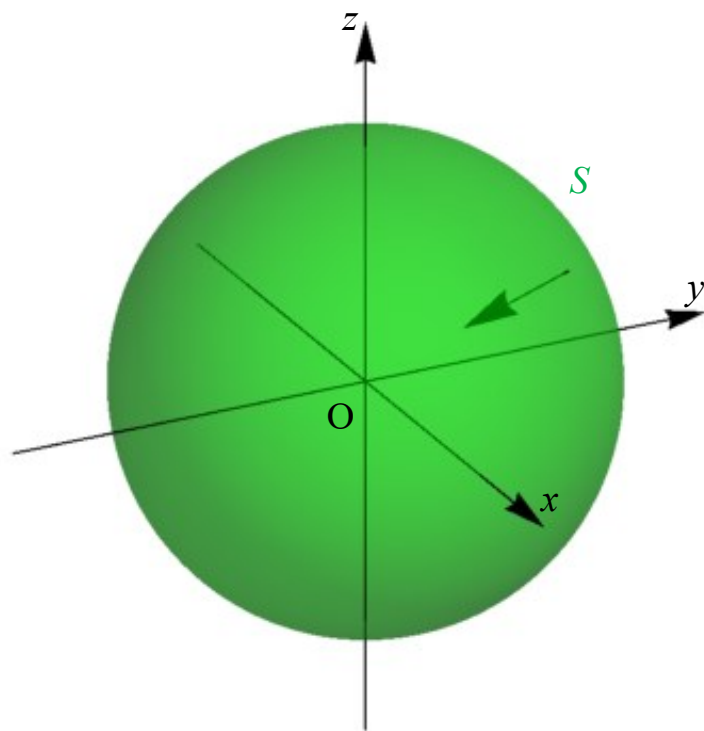
图 2: 习题13.5 1.(2)题图示

$(3)\because x^2, y^2, z^2 \in C^1(\Omega),$

$\therefore \iint\limits_{S} x^2 \mathrm{d}y \wedge \mathrm{d}z + y^2 \mathrm{d}z \wedge \mathrm{d}x + z^2 \mathrm{d}x \wedge \mathrm{d}y = \iiint\limits_{\Omega} (\frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z}) \mathrm{d}x \mathrm{d}y \mathrm{d}z$

$= 2 \iiint\limits_{\Omega} (x+y+z) \mathrm{d}x \mathrm{d}y \mathrm{d}z = 6 \iiint\limits_{\Omega} x \mathrm{d}x \mathrm{d}y \mathrm{d}z = 6 \int_0^1 x \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z = 6 \int_0^1 x \mathrm{d}x \int_0^1 \mathrm{d}y$

$= 6 \int_0^1 x \mathrm{d}x = 3x^2 \big|_0^1 = 3.$
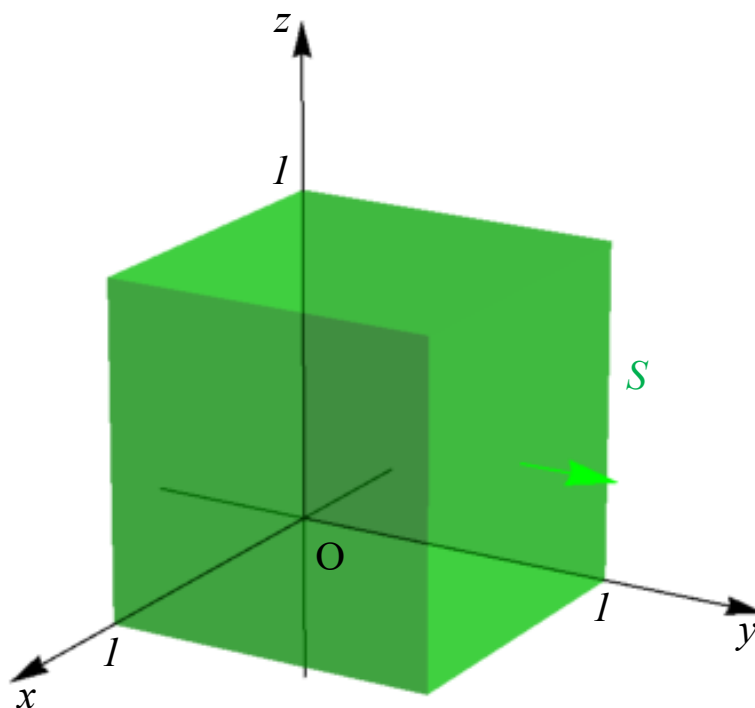
图 3: 习题13.5 1.(3)题图示

(4)记S围成的区域为Ω，则$xz, xy, yz \in C^1(\Omega)$,

$\therefore \iint\limits_{S} xz\mathrm{d}x\wedge\mathrm{d}z+xy\mathrm{d}y\wedge\mathrm{d}z+yz\mathrm{d}z\wedge\mathrm{d}x = \iiint\limits_{\Omega}(\frac{\partial xy}{\partial x}+\frac{\partial yz}{\partial y}+\frac{\partial xz}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{\Omega}(y+z+x)\mathrm{d}x\mathrm{d}y\mathrm{d}z$

$= 3\iiint\limits_{\Omega} x\mathrm{d}x\mathrm{d}y\mathrm{d}z = 3\int_0^1 x\mathrm{d}x \int_0^{1-x}\mathrm{d}y \int_0^{1-x-y}\mathrm{d}z = 3\int_0^1 x\mathrm{d}x \int_0^{1-x}(1-x-y)\mathrm{d}y$

$= 3\int_0^1 x(y-xy-\frac{1}{2}y^2)\big|_0^{1-x}\mathrm{d}x = 3\int_0^1 x(1-x-\frac{1}{2}y)y\big|_0^{1-x}\mathrm{d}x$

$= 3\int_0^1 x[1-x-\frac{1}{2}(1-x)](1-x)\mathrm{d}x = 3\int_0^1 x\frac{1}{2}(1-x)^2\mathrm{d}x = \frac{3}{2}\int_0^1 x(1-2x+x^2)\mathrm{d}x$

$= \frac{3}{2}\int_0^1 (x-2x^2+x^3)\mathrm{d}x = \frac{3}{2}(\frac{1}{2}x^2-\frac{2}{3}x^3+\frac{1}{4}x^4)\big|_0^1 = \frac{3}{2}(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}) = \frac{3}{2}(\frac{3}{4}-\frac{2}{3}) = \frac{3}{2}\cdot\frac{1}{12} = \frac{1}{8}.$
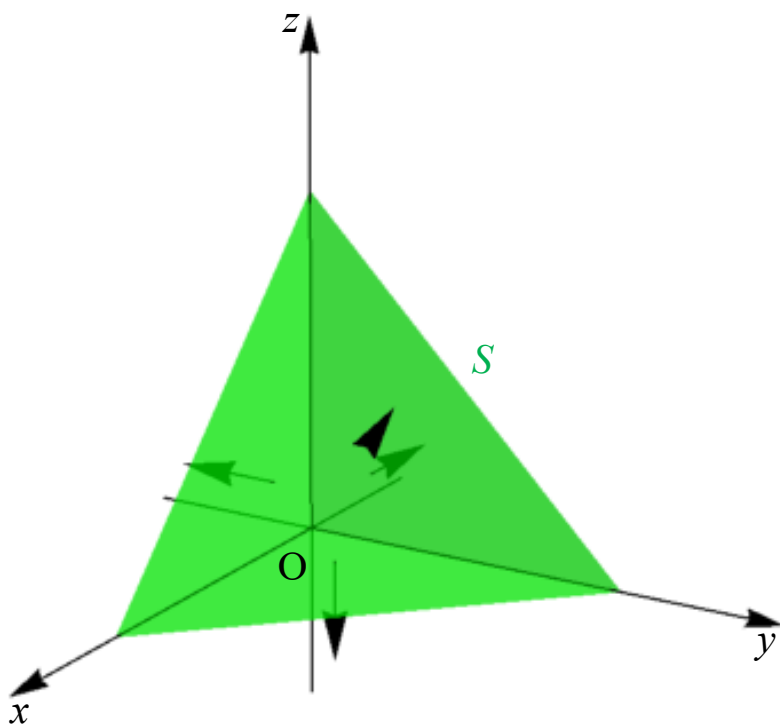
图 4: 习题13.5 1.(4)题图示

(5)取平面$S_1 : z = 1, x^2 + y^2 \leqslant 1$，上侧为正，记$S$与$S_1$围成的区域为$\Omega$,

$\because x^2, z^2 - 2z \in C^1(\Omega)$,

$\therefore \iint\limits_{S+S_1} x^2\mathrm{d}y \wedge \mathrm{d}z + (z^2 - 2z)\mathrm{d}x \wedge \mathrm{d}y = \iiint\limits_{\Omega} (\frac{\partial x^2}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial(z^2-2z)}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z$

$= \iiint\limits_{\Omega}(2x + 2z - 2)\mathrm{d}x\mathrm{d}y\mathrm{d}z$,

由对称性可知$\iiint\limits_{\Omega} 2x\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$,

$\therefore$上式$= \iiint\limits_{\Omega}(2z - 2)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 (2z-2)\mathrm{d}z \iint\limits_{x^2+y^2\leqslant z^2}\mathrm{d}x\mathrm{d}y = \int_0^1 (2z-2)\pi z^2 \mathrm{d}z$

$= 2\pi \int_0^1 (z^3 - z^2)\mathrm{d}z = 2\pi(\frac{1}{4}z^4 - \frac{1}{3}z^3)\big|_0^1 = -\frac{1}{6}\pi$,

$\because$在S_1上$\mathrm{d}\boldsymbol{S} = (-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1)\mathrm{d}x\mathrm{d}y = (0, 0, 1)\mathrm{d}x\mathrm{d}y = (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y)$,

$\therefore \iint\limits_{S_1} x^2\mathrm{d}y \wedge \mathrm{d}z + (z^2 - 2z)\mathrm{d}x \wedge \mathrm{d}y = \iint\limits_{x^2+y^2\leqslant 1}[x^2 \cdot 0 + (1^2 - 2\cdot 1)]\mathrm{d}x\mathrm{d}y = -\iint\limits_{x^2+y^2\leqslant 1}\mathrm{d}x\mathrm{d}y = -\pi$,

$\therefore \iint\limits_{S} x^2\mathrm{d}y \wedge \mathrm{d}z + (z^2 - 2z)\mathrm{d}x \wedge \mathrm{d}y = -\frac{\pi}{6} - \iint\limits_{S_1} x^2\mathrm{d}y \wedge \mathrm{d}z + (z^2 - 2z)\mathrm{d}x \wedge \mathrm{d}y = -\frac{\pi}{6} + \pi = \frac{5}{6}\pi$.
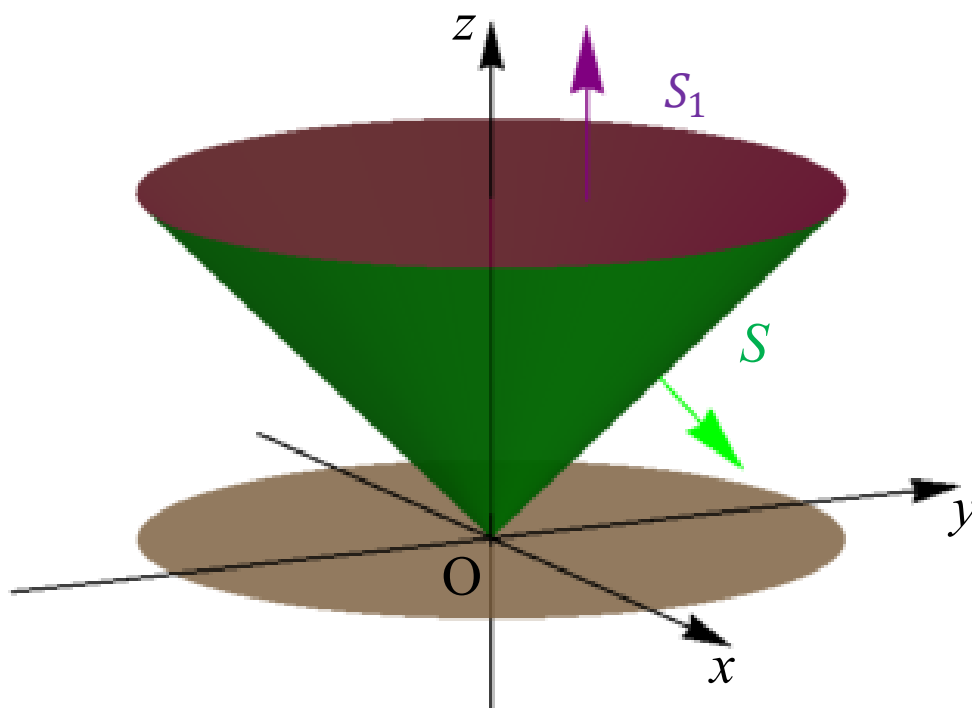
图 5: 习题13.5 1.(5)题图示

(6)记S围成的圆柱体区域为Ω，则$x-y,(y-z)x \in C^1(\Omega)$,

$\therefore \oiint\limits_{S}(x-y)\mathrm{d}x \wedge \mathrm{d}y + (y-z)x\mathrm{d}y \wedge \mathrm{d}z = \iiint\limits_{\Omega}[\frac{\partial(y-z)x}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial(x-y)}{\partial z}]\mathrm{d}x\mathrm{d}y\mathrm{d}z$
$= \iiint\limits_{\Omega}(y-z)\mathrm{d}x\mathrm{d}y\mathrm{d}z,$

由对称性可知$\iiint\limits_{\Omega} y\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0,$

$\therefore$上式$= -\iiint\limits_{\Omega} z\mathrm{d}x\mathrm{d}y\mathrm{d}z = -\int_1^3 z\mathrm{d}z \iint\limits_{x^2+y^2 \leqslant 1} \mathrm{d}x\mathrm{d}y = -\pi\int_1^3 z\mathrm{d}z = -\frac{1}{2}\pi z^2\big|_1^3 = -4\pi.$
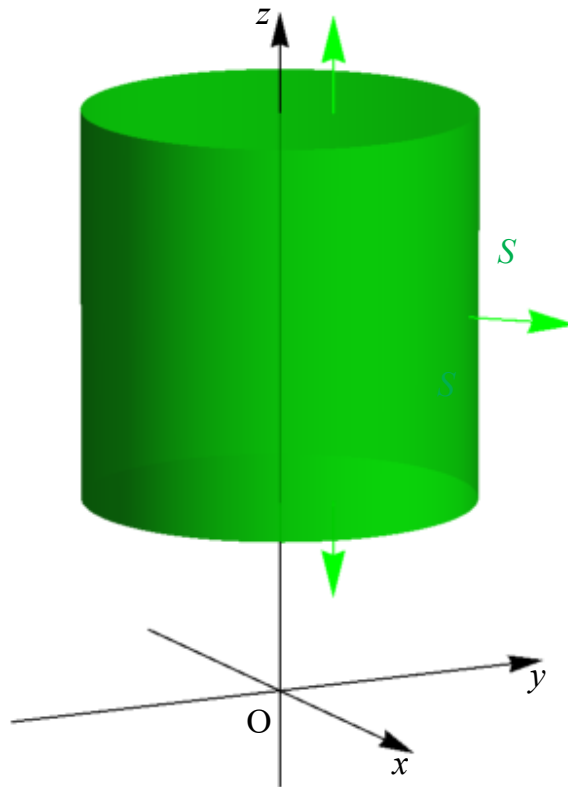
图 6: 习题13.5 1.(6)题图示

(7)取球面S_1: $x^2+y^2+z^2=r^2, r<\min\{a,b,c\}$，外侧为正，设$S$与$S_1^-$围成的区域为$\Omega$，$S_1$围成的区域为$\Omega_1$，

则$X=\dfrac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$, $Y=\dfrac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}$, $Z=\dfrac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\in C^1(\Omega)$,

$$\frac{\partial X}{\partial x}=\frac{(x^2+y^2+z^2)^{\frac{3}{2}}-x\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2x}{(x^2+y^2+z^2)^3}=\frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}},$$

$$\frac{\partial Y}{\partial y}=\frac{(x^2+y^2+z^2)^{\frac{3}{2}}-y\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2y}{(x^2+y^2+z^2)^3}=\frac{z^2+x^2-2y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}},$$

$$\frac{\partial Z}{\partial z}=\frac{(x^2+y^2+z^2)^{\frac{3}{2}}-z\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2z}{(x^2+y^2+z^2)^3}=\frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}},$$

$\therefore$

$$\oiint\limits_{S+S_1^-}\frac{\partial X}{\partial x}\mathrm{d}y\wedge\mathrm{d}z+\frac{\partial Y}{\partial y}\mathrm{d}z\wedge\mathrm{d}x+\frac{\partial Z}{\partial z}\mathrm{d}x\wedge\mathrm{d}y=\iiint\limits_{\Omega}(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$=\iiint\limits_{\Omega}\frac{y^2+z^2-2x^2+z^2+x^2-2y^2+x^2+y^2-2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint\limits_{\Omega}0\mathrm{d}x\mathrm{d}y\mathrm{d}z=0,$$

∴

$$\iint\limits_{S} \frac{x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = 0 - \iint\limits_{S_1^-} \frac{x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$= \iint\limits_{S_1} \frac{x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = \iint\limits_{S_1} \frac{x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y}{(r^2)^{\frac{3}{2}}}$$

$$= \frac{1}{r^3} \iint\limits_{S_1} x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y = \frac{1}{r^3} \iiint\limits_{\Omega} (\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \frac{3}{r^3} \iiint\limits_{\Omega} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{3}{r^3} \frac{4}{3}\pi r^3 = 4\pi.$$
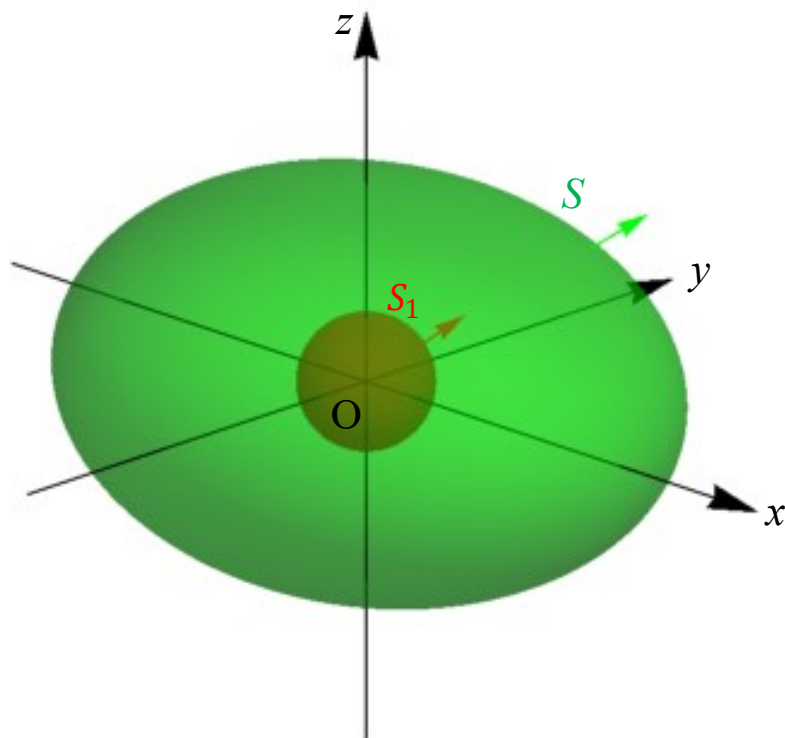
图 7: 习题13.5 1.(7)题图示

2. 用斯托克斯公式计算下列曲线积分：

(1)$\oint_L y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z$，其中$L$是圆周 $\begin{cases} x^2 + y^2 + z^2 = R^2, \\ x + y + z = 0, \end{cases}$ 从x轴正向看去为逆时针方向；

(2)$\oint_L (y-x)\mathrm{d}x+(z-y)\mathrm{d}y+(x-z)\mathrm{d}z$，其中$L$是柱面$x^2+y^2=a^2$与平面$x+z=a(a>0)$的交线，从$x$轴正向看去为逆时针方向.

解：(1)记Σ是平面$x+y+z=0$上L围成的部分，Σ与L的方向符合右手法则，记Σ的上侧为正，

$\because y,z,x \in C^1(\Sigma)$,

$\therefore \oint_L y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z = \iint\limits_{\Sigma} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} \cdot \boldsymbol{n}\mathrm{d}S = \iint\limits_{\Sigma}(\frac{\partial x}{\partial y} - \frac{\partial z}{\partial z}, \frac{\partial z}{\partial y} - \frac{\partial x}{\partial x}, \frac{\partial z}{\partial x} - \frac{\partial y}{\partial y}) \cdot \boldsymbol{n}\mathrm{d}S$

$= \iint\limits_{\Sigma}(0-1, 0-1, 0-1) \cdot \boldsymbol{n}\mathrm{d}S = \iint\limits_{\Sigma}(-1,-1,-1) \cdot \boldsymbol{n}\mathrm{d}S$,

$\because \Sigma$的单位法向量为$\boldsymbol{n} = (1,1,1)\frac{1}{\sqrt{3}}$,

$\therefore \oint_L y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z = \iint\limits_{\Sigma}(-1,-1,-1) \cdot (1,1,1)\frac{1}{\sqrt{3}}\mathrm{d}S = \frac{1}{\sqrt{3}}\iint\limits_{\Sigma}-3\mathrm{d}S = -\sqrt{3}\pi R^2$.
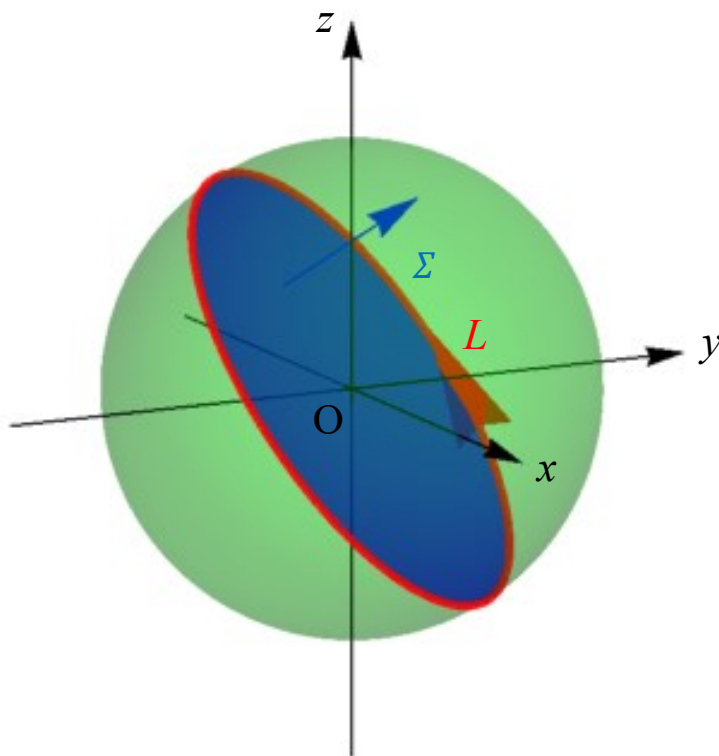
图 8: 习题13.5 2.(1)题图示

(2)记Σ是平面$x+z=a$上L围成的部分，Σ的方向与L的方向符合右手法则，即Σ的上

侧为正，

$\because y - x, z - y, x - z \in C^1(\Sigma),$

$\therefore \oint_L (y-x)\mathrm{d}x + (z-y)\mathrm{d}y + (x-z)\mathrm{d}z = \iint\limits_{\Sigma} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-x & z-y & x-z \end{vmatrix} \cdot \mathrm{d}\boldsymbol{S}$

$= \iint\limits_{\Sigma} (\frac{\partial(x-z)}{\partial y} - \frac{\partial(z-y)}{\partial z}, \frac{\partial(y-x)}{\partial z} - \frac{\partial(x-z)}{\partial x}, \frac{\partial(z-y)}{\partial x} - \frac{\partial(y-x)}{\partial y}) \cdot \mathrm{d}\boldsymbol{S} = \iint\limits_{\Sigma} (0-1, 0-1, 0-1) \cdot \mathrm{d}\boldsymbol{S},$

$\because$ 在 $\Sigma: z = a - x, x^2 + y^2 \leqslant a^2$ 的上侧

$\mathrm{d}\boldsymbol{S} = (-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1)\mathrm{d}x\mathrm{d}y = (1, 0, 1)\mathrm{d}x\mathrm{d}y = (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y),$

$\therefore$ 上式 $= \iint\limits_{x^2+y^2\leqslant a^2} (0-1, 0-1, 0-1) \cdot (1, 0, 1)\mathrm{d}x\mathrm{d}y = -2 \iint\limits_{x^2+y^2\leqslant a^2} \mathrm{d}x\mathrm{d}y = -2\pi a^2.$
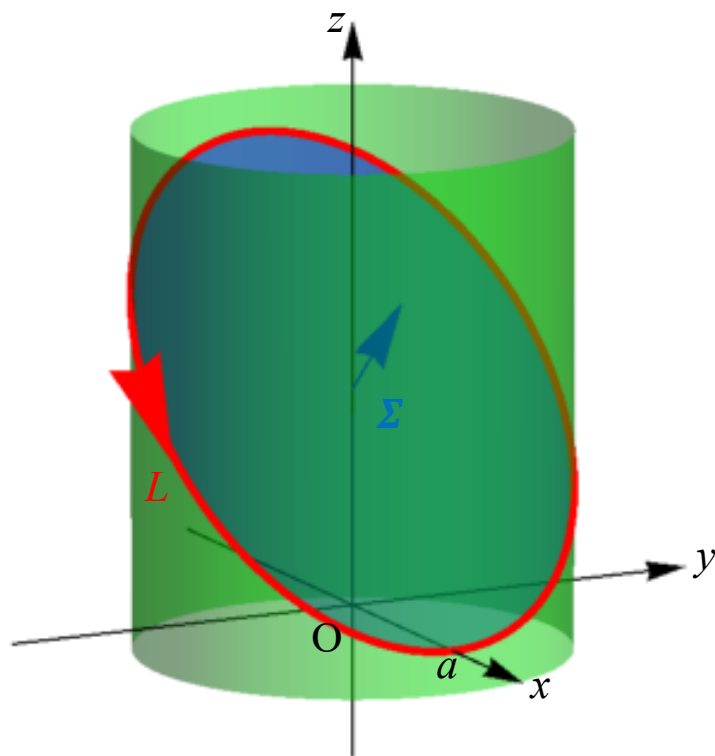
图 9: 习题13.5 2.(2)题图示

3. 计算 $I = \oint_L \frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2} + z\mathrm{d}z$，其中 L 是：

(1)任意一条既不环绕 z 轴，也不与 z 轴相交的简单闭曲线；

(2)任意一条环绕 z 轴一圈且不与 z 轴相交的简单闭曲线，从 z 轴正向看去为逆时针方向.

解：(1)取曲面Σ为曲线L围成的逐片光滑有向曲面，Σ和L的方向符合右手法则，z轴不穿过Σ,

则$X = \frac{-y}{x^2+y^2}$, $Y = \frac{x}{x^2+y^2}$, $Z = z \in C^1(\Sigma)$,

$\therefore$

$$I = \oint_L X\mathrm{d}x + Y\mathrm{d}y + Z\mathrm{d}z = \iint_\Sigma \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix} \cdot \mathrm{d}\boldsymbol{S} = \iint_\Sigma \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & z \end{vmatrix} \cdot \mathrm{d}\boldsymbol{S}$$

$$= \iint_\Sigma \left(\frac{\partial z}{\partial y} - \frac{\partial}{\partial z}\frac{x}{x^2+y^2}, \frac{\partial}{\partial z}\frac{-y}{x^2+y^2} - \frac{\partial z}{\partial x}, \frac{\partial}{\partial x}\frac{x}{x^2+y^2} - \frac{\partial}{\partial y}\frac{-y}{x^2+y^2}\right) \cdot \mathrm{d}\boldsymbol{S}$$

$$= \iint_\Sigma \left(0 - 0, 0 - 0, \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} - \frac{-(x^2+y^2) + y \cdot 2y}{(x^2+y^2)^2}\right) \cdot \mathrm{d}\boldsymbol{S}$$

$$= \iint_\Sigma (0, 0, 0) \cdot \mathrm{d}\boldsymbol{S} = 0.$$
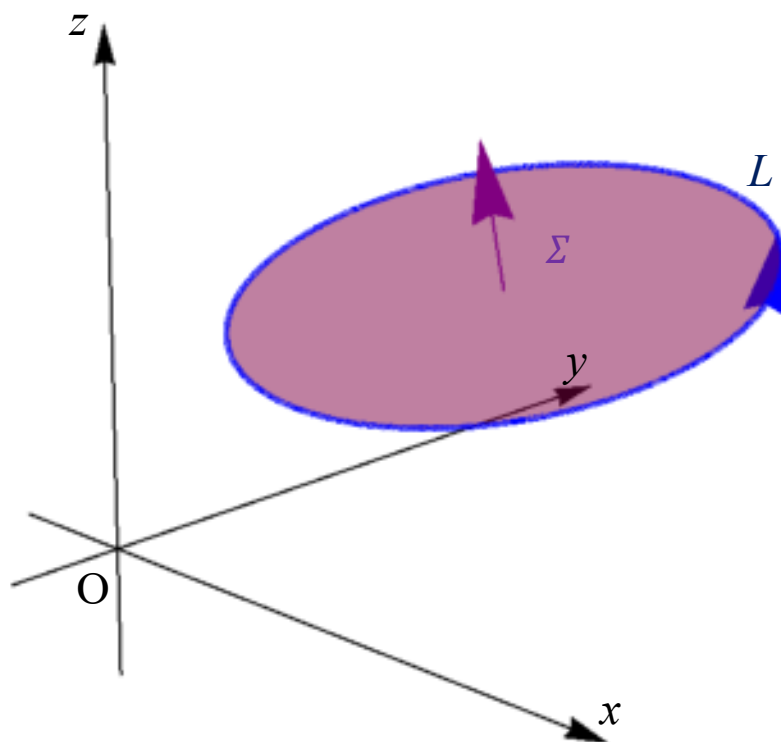
图 10: 习题13.5 3.(1)题图示

(2)设曲面Σ为曲线L围成的逐片光滑有向曲面，Σ与L的方向符合右手法则，即Σ的上侧为正，此时z轴穿过Σ. 取柱面$x^2 + y^2 = r^2$与Σ交于曲线L_1，r应足够小使得闭合交线L_1全部位于Σ上，设从z轴正向看去L_1为逆时针方向. 记Σ_1为L_1围成的逐片光滑正向曲面，Σ_2为L与L_1^-围成的逐片光滑正向曲面.

则$X = \frac{-y}{x^2+y^2}$, $Y = \frac{x}{x^2+y^2}$, $Z = z \in C^1(\Sigma_2)$,

与(1)同理可知$\oint_{L+L_1^-} X\mathrm{d}x + Y\mathrm{d}y + Z\mathrm{d}z = 0$,

$\therefore$

$$
\begin{aligned}
I &= 0 - \oint_{L_1^-} X\mathrm{d}x + Y\mathrm{d}y + Z\mathrm{d}z = \oint_{L_1} X\mathrm{d}x + Y\mathrm{d}y + Z\mathrm{d}z \\
&= \oint_{L_1} \frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2} + z\mathrm{d}z = \oint_{L_1} \frac{-y\mathrm{d}x + x\mathrm{d}y}{r^2} + z\mathrm{d}z \\
&= \iint_{\Sigma_1} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{r^2} & \frac{x}{r^2} & z \end{vmatrix} \cdot \mathrm{d}\boldsymbol{S} = \iint_{\Sigma_1} (\frac{\partial z}{\partial y} - \frac{\partial}{\partial z}\frac{x}{r^2}, \frac{\partial}{\partial z}(-\frac{y}{r^2}) - \frac{\partial z}{\partial x}, \frac{\partial}{\partial x}\frac{x}{r^2} - \frac{\partial}{\partial y}(-\frac{y}{r^2})) \cdot \mathrm{d}\boldsymbol{S} \\
&= \iint_{\Sigma_1} (0-0, 0-0, \frac{1}{r^2} - (-\frac{1}{r^2})) \cdot \mathrm{d}\boldsymbol{S} = \iint_{\Sigma_1} (0, 0, \frac{2}{r^2}) \cdot (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y) \\
&= \iint_{\Sigma_1} \frac{2}{r^2}\mathrm{d}x \wedge \mathrm{d}y = \frac{2}{r^2} \iint_{\Sigma_1} \mathrm{d}x \wedge \mathrm{d}y,
\end{aligned}
$$

$\because \Sigma_1$方程可表示为$z = f(x,y), x^2 + y^2 \leqslant r^2$，且$\Sigma_1$上侧为正，

$\therefore I = \frac{2}{r^2} \iint_{\Sigma_1} \mathrm{d}x \wedge \mathrm{d}y = \frac{2}{r^2} \iint_{x^2+y^2 \leqslant r^2} \mathrm{d}x\mathrm{d}y = \frac{2}{r^2}\pi r^2 = 2\pi.$
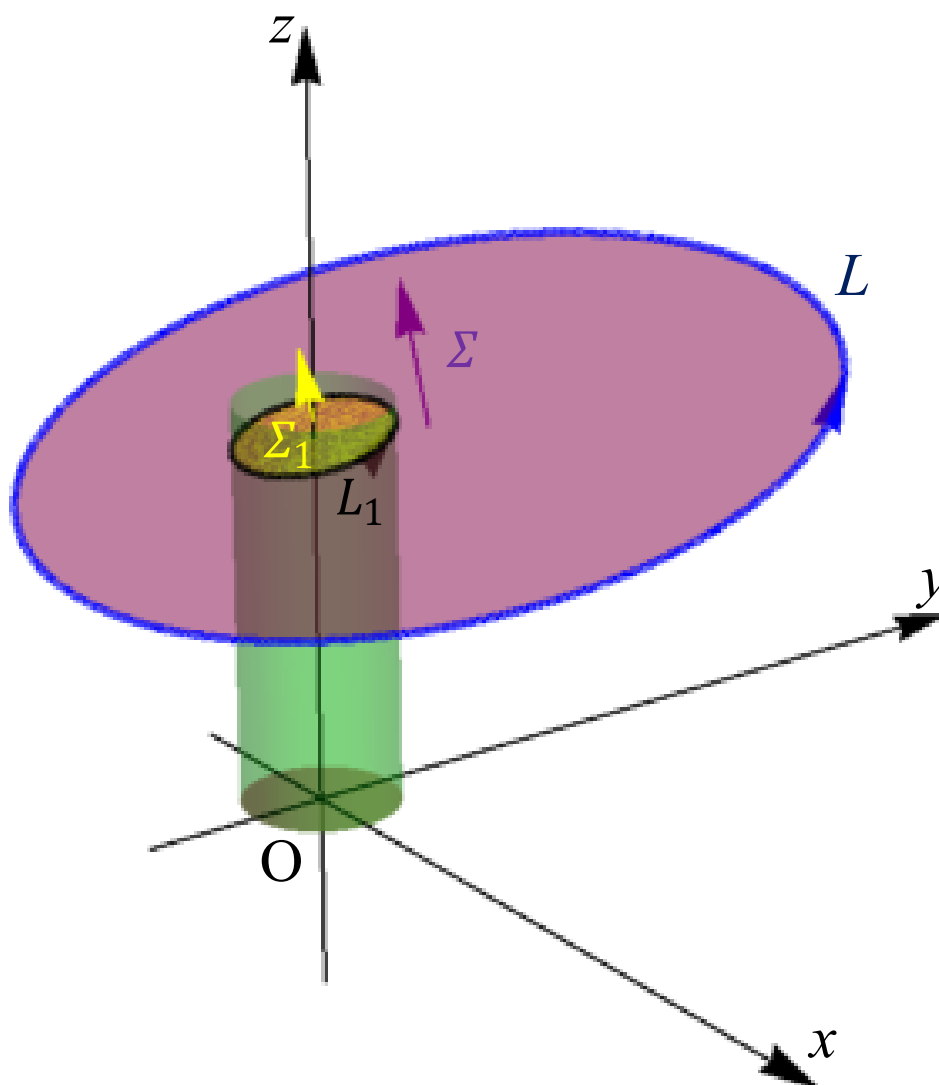
图 11: 习题13.5 3.(2)题图示

4. 证明：

$$
\oint_L \begin{vmatrix} \mathrm{d}x & \mathrm{d}y & \mathrm{d}z \\ \cos\alpha & \cos\beta & \cos\gamma \\ x & y & z \end{vmatrix} = 2S,
$$

其中L是$\mathbb{R}^3$中某个平面上的一条简单逐段光滑闭曲线，$\boldsymbol{n} = (\cos\alpha, \cos\beta, \cos\gamma)$是该平面的单位法向量，$L$的方向与$\boldsymbol{n}$的方向服从右手法则，$S$是$L$所围的面积.

证明：

$$\oint_L \begin{vmatrix} \mathrm{d}x & \mathrm{d}y & \mathrm{d}z \\ \cos\alpha & \cos\beta & \cos\gamma \\ x & y & z \end{vmatrix} = \oint_L (z\cos\beta - y\cos\gamma)\mathrm{d}z + (x\cos\gamma - z\cos\alpha)\mathrm{d}y + (y\cos\alpha - x\cos\gamma)\mathrm{d}z,$$

记 Σ 是 L 在该平面上围成的部分，Σ 的方向与 $\boldsymbol{n}$ 的方向一致，则 Σ 的面积为 S，$z\cos\beta - y\cos\gamma, x\cos\gamma - z\cos\alpha, y\cos\alpha - x\cos\gamma \in C^1(\Sigma)$,

$\therefore$

$$上式 = \iint_\Sigma \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\cos\beta - y\cos\gamma & x\cos\gamma - z\cos\alpha & y\cos\alpha - x\cos\gamma \end{vmatrix}$$

$$= \iint_\Sigma \left\{ \left[\frac{\partial(y\cos\alpha - x\cos\gamma)}{\partial x} - \frac{\partial(x\cos\gamma - z\cos\alpha)}{\partial z} \right]\boldsymbol{i} \right.$$
$$+ \left[\frac{\partial(z\cos\beta - y\cos\gamma)}{\partial z} - \frac{\partial(y\cos\alpha - x\cos\gamma)}{\partial x} \right]\boldsymbol{j}$$
$$\left. + \left[\frac{\partial(x\cos\gamma - z\cos\alpha)}{\partial x} - \frac{\partial(z\cos\beta - y\cos\gamma)}{\partial y} \right]\boldsymbol{k} \right\} \cdot \boldsymbol{n}\,\mathrm{d}S$$

$$= \iint_\Sigma \left\{ [\cos\alpha - (-\cos\alpha)]\boldsymbol{i} + [\cos\beta - (-\cos\gamma)]\boldsymbol{j} + [\cos\gamma - (-\cos\gamma)]\boldsymbol{k} \right\} \cdot \boldsymbol{n}\,\mathrm{d}S$$

$$= \iint_\Sigma (2\cos\alpha\,\boldsymbol{i} + 2\cos\beta\,\boldsymbol{j} + 2\cos\gamma\,\boldsymbol{k}) \cdot \boldsymbol{n}\,\mathrm{d}S$$

$$= \iint_\Sigma (2\cos\alpha, 2\cos\beta, 2\cos\gamma) \cdot (\cos\alpha, \cos\beta, \cos\gamma)\,\mathrm{d}S$$

$$= \iint_\Sigma (2\cos^2\alpha + 2\cos^2\beta + 2\cos^2\gamma)\,\mathrm{d}S$$

$$= 2\iint_\Sigma \mathrm{d}S = 2S.$$