11 第二型曲面积分

11.1 复习计划

11.2 知识结构

- 向量场的微积分
  - 向量值函数在曲线上的积分、格林公式
  - 向量值函数在曲面上的积分
    - 有向曲面
      - 双侧曲面 莫比乌斯带
      - 有向曲面 选定了正向法向量的双侧曲面
      - 正向单位法向量的求法
        - 曲面方程是显示方程
          - $\Sigma : z = f(x, y) \in C^1$

            $\Rightarrow \boldsymbol{n}=''\pm'' \frac{1}{sqrt1+(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2} (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1)$
            - 上正下负
          - $\Sigma : x = g(y, z) \in C^1$，前正后负
          - $\Sigma : y = g(z, x) \in C^1$，右正左负
        - 曲面方程是参数方程
          - $\Sigma : \begin{cases} x = x(u, v), \\ y = y(u, v), x(u, v), y(u, v), z(u, v) \in C^1, \\ z = z(u, v), \end{cases}$

            $A = \frac{\mathrm{D}(y,z)}{\mathrm{D}(u,v)}, B = \frac{\mathrm{D}(z,x)}{\mathrm{D}(u,v)}, C = \frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}$

            $\Rightarrow \boldsymbol{n}=''\pm'' \frac{1}{A^2+B^2+C^2} (A, B, C)$
            - 注意正负号的选取
    - 第二型曲面积分的概念
      - 物理背景 定常流$\boldsymbol{v} = (X, Y, Z)$穿过某双侧曲面的流体体积
        - 分割、取点、求和、取极限
      - 第二型曲面积分的定义
        - 分割、取点、求和、取极限

          $\iint_\Sigma \boldsymbol{F}(x, y, z)\cdot\mathrm{d}\boldsymbol{S} = \lim\limits_{\lambda\to 0} \sum_{k=1}^m [\boldsymbol{F}(\xi_k, \eta_k, \zeta_k)\cdot\boldsymbol{n}(\xi_k, \eta_k, \zeta_k)]\Delta S_k$
        - $\mathrm{d}\boldsymbol{S} = \boldsymbol{n}\mathrm{d}S$为有向面积元素
        - 第一型曲面积分和第二型曲面积分的关系
          - $\iint_\Sigma \boldsymbol{F}(x, y, z)\cdot\mathrm{d}\boldsymbol{S} = \iint_\Sigma [\boldsymbol{F}(x, y, z)\cdot\boldsymbol{n}(x, y, z)]\mathrm{d}S$
        - 记$\boldsymbol{n} = (\cos\alpha, \cos\beta, \cos\gamma)$,

          $\Rightarrow \mathrm{d}\boldsymbol{S} = \boldsymbol{n}\mathrm{d}S = (\cos\alpha, \cos\beta, \cos\gamma)\mathrm{d}S = (\cos\alpha\mathrm{d}S, \cos\beta\mathrm{d}S, \cos\gamma\mathrm{d}S)$

          $= (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y)$
        - 分量形式 $\iint_\Sigma \boldsymbol{F}(x, y, z)\cdot\mathrm{d}\boldsymbol{S} = \iint_\Sigma X(x, y, z)\mathrm{d}y \wedge \mathrm{d}z + Y(x, y, z)\mathrm{d}z \wedge \mathrm{d}x + Z(x, y, z)\mathrm{d}x \wedge \mathrm{d}y$
      - 第二型曲面积分的性质
        - 有向性
          - $\iint_{\Sigma^-} \boldsymbol{F}(x, y, z)\cdot\mathrm{d}\boldsymbol{S} = -\iint_\Sigma \boldsymbol{F}(x, y, z)\cdot\mathrm{d}\boldsymbol{S}$
        - 区域可加性 $\iint_\Sigma \boldsymbol{F}(x, y, z)\cdot\mathrm{d}\boldsymbol{S} = \iint_{\Sigma_1} \boldsymbol{F}(x, y, z)\cdot\mathrm{d}\boldsymbol{S} + \iint_{\Sigma_2} \boldsymbol{F}(x, y, z)\cdot\mathrm{d}\boldsymbol{S}$
      - 第二型曲面积分的计算
        - 曲面方程是显式方程
          - $\Sigma : z = f(x, y), (x, y) \in D, \boldsymbol{F}(x, y, z) \in C(\Sigma)$

            $\Rightarrow \iint_\Sigma \boldsymbol{F}(x, y, z)\cdot\mathrm{d}\boldsymbol{S}$

            $= \iint_\Sigma X(x, y, z)\mathrm{d}y \wedge \mathrm{d}z + Y(x, y, z)\mathrm{d}z \wedge \mathrm{d}x + Z(x, y, z)\mathrm{d}x \wedge \mathrm{d}y$

            $=''\pm'' \iint_D [-\frac{\partial f(x,y)}{\partial x} X(x, y, f(x, y)) - \frac{\partial f(x,y)}{\partial y} Y(x, y, f(x, y)) + Z(x, y, f(x, y))]\mathrm{d}x\mathrm{d}y$
          - $\Sigma : y = g(z, x)$ 右正左负
          - $\Sigma : x = h(y, z)$ 前正后负
          - 平行于坐标轴的直线与Σ最多只有一个交点时
            - $\iint_\Sigma \boldsymbol{F}(x, y, z)\cdot\mathrm{d}\boldsymbol{S} = \iint_\Sigma X(x, y, z)\mathrm{d}y \wedge \mathrm{d}z + Y\mathrm{d}z \wedge \mathrm{d}x + Z\mathrm{d}x \wedge \mathrm{d}y$

              $=''\pm'' \iint_{D_{yz}} X(h(y, z), y, z)\mathrm{d}y\mathrm{d}z+''\pm'' \iint_{D_{zx}} Y(x, g(z, x), z)\mathrm{d}z\mathrm{d}x+''\pm'' \iint_{D_{xy}} Z(x, y, f(x, y))\mathrm{d}x\mathrm{d}y$
        - 曲面方程是参数方程
          - $\Sigma : \begin{cases} x = x(u, v), \\ y = y(u, v), (u, v) \in D, F(x, y, z) \in C(\Sigma), \\ z = z(u, v), \end{cases}$

            $\Rightarrow \iint_\Sigma \boldsymbol{F}(x, y, z)\cdot\mathrm{d}\boldsymbol{S}$

            $= \iint_\Sigma X(x, y, z)\mathrm{d}y \wedge \mathrm{d}z + Y\mathrm{d}z \wedge \mathrm{d}x + Z\mathrm{d}x \wedge \mathrm{d}y$

            $=''\pm'' \iint_D [X(x(u, v), y(u, v), z(u, v))A + YB + ZC]\mathrm{d}u\mathrm{d}v$

            其中$A = \frac{\mathrm{D}(y,z)}{\mathrm{D}(u,v)}, B = \frac{\mathrm{D}(z,x)}{\mathrm{D}(u,v)}, C = \frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}$
  - 高斯公式与斯托克斯公式
  - 保守场
  - 向量场的微分运算

11.3 通量的图示

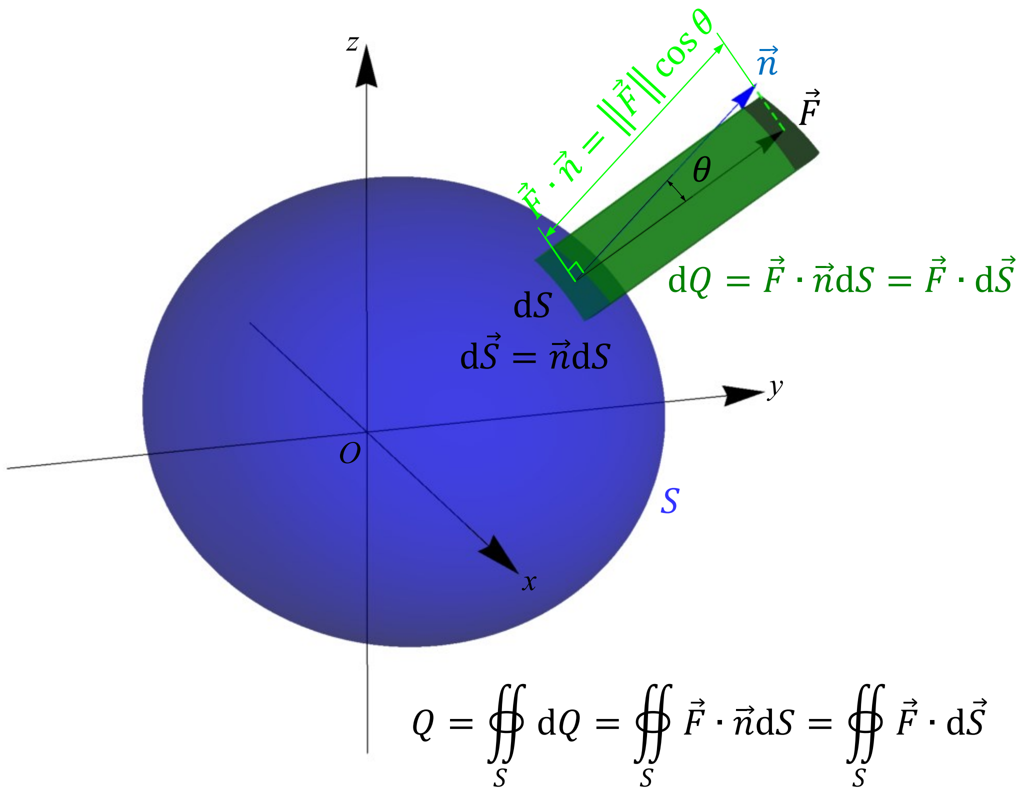

图 1: 通量的图示

11.4 第二型曲面积分的计算

第二型曲面积分$\iint\limits_S \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S}$中：

$\boldsymbol{F}(x,y,z)=(X(x,y,z),Y(x,y,z),Z(x,y,z))\in C(S)$,为空间向量场；

$$\begin{aligned}\mathrm{d}\boldsymbol{S}=\boldsymbol{n}\mathrm{d}S&=(\cos\alpha,\cos\beta,\cos\gamma)\mathrm{d}S=(\cos\alpha\mathrm{d}S,\cos\beta\mathrm{d}S,\cos\gamma\mathrm{d}S)\\&=(\mathrm{d}y\wedge\mathrm{d}z,\mathrm{d}z\wedge\mathrm{d}x,\mathrm{d}x\wedge\mathrm{d}y),\text{为有向面积元素；}\end{aligned}$$

有向曲面S可有以下几种形式：

1. 显示方程

   (a) $S:z=f(x,y),(x,y)\in D_{xy}$. 此时

$$\boldsymbol{n}=\pm(-\frac{\partial f}{\partial x},-\frac{\partial f}{\partial y},1)\frac{1}{\sqrt{(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2+1}},\ \mathrm{d}S=\sqrt{(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2+1}\mathrm{d}x\mathrm{d}y$$

$$\mathrm{d}\boldsymbol{S}=\boldsymbol{n}\mathrm{d}S=\pm(-\frac{\partial f}{\partial x},-\frac{\partial f}{\partial y},1)\mathrm{d}x\mathrm{d}y=(\mathrm{d}y\wedge\mathrm{d}z,\mathrm{d}z\wedge\mathrm{d}x,\mathrm{d}x\wedge\mathrm{d}y).$$

若在曲面S的$+z$侧做积分取+号，若在曲面S的$-z$侧做积分取−号.

故$\iint\limits_S \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S}$

$$=\iint\limits_S X(x,y,z)\mathrm{d}y\wedge\mathrm{d}z+Y(x,y,z)\mathrm{d}z\wedge\mathrm{d}x+Z(x,y,z)\mathrm{d}x\wedge\mathrm{d}y$$

$$=\pm\iint\limits_S[-X(x,y,f(x,y))\frac{\partial f}{\partial x}-Y(x,y,f(x,y))\frac{\partial f}{\partial y}+Z(x,y,f(x,y))]\mathrm{d}x\mathrm{d}y$$

【利用该公式进行计算的习题：8.(第三类题目)】

注意此时$\mathrm{d}x\wedge\mathrm{d}y=\pm\mathrm{d}x\mathrm{d}y$，在曲面$+z$侧积分，取+号，在曲面$-z$侧积分，取−号.

【利用这一点进行计算的习题：1,2,3,4,5.(第一类题目) 7,10.(第二类题目)】

(b) $S: x=g(y,z),(y,x)\in D_{yz}$. 可进行类似的推导.

注意此时$\mathrm{d}y\wedge\mathrm{d}z=\pm\mathrm{d}y\mathrm{d}z$，在曲面$+x$侧积分，取+号，在曲面$-x$侧积分，取−号.

(c) $S: y=g(z,x),(z,x)\in D_{zx}$. 可进行类似的推导.

注意此时$\mathrm{d}z\wedge\mathrm{d}x=\pm\mathrm{d}z\mathrm{d}x$，在曲面$+y$侧积分，取+号，在曲面$-y$侧积分，取−号.

2. 参数方程

$$S:\begin{cases}x=x(u,v),\\ y=y(u,v), & (u,v)\in D_{uv}.\ \text{此时}\\ z=z(u,v),\end{cases}$$

$$\boldsymbol{n}=\pm(A,B,C)\frac{1}{\sqrt{A^2+B^2+C^2}},\ \mathrm{d}S=\sqrt{A^2+B^2+C^2}\mathrm{d}u\mathrm{d}v,$$

$$\mathrm{d}\boldsymbol{S}=\boldsymbol{n}\mathrm{d}S='\pm'(A,B,C)\mathrm{d}S=(\mathrm{d}y\wedge\mathrm{d}z,\mathrm{d}z\wedge\mathrm{d}x,\mathrm{d}x\wedge\mathrm{d}y),$$

其中$A=\dfrac{\mathrm{D}(y,z)}{\mathrm{D}(u,v)},\ B=\dfrac{\mathrm{D}(z,x)}{\mathrm{D}(u,v)},\ C=\dfrac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}$.

若在曲面的$+x$侧积分，则选取±号使得$\pm A>0$；若在曲面的下侧积分，则选取±号使得$\pm A<0$；若在曲面的$+y$侧积分，则选取±号使得$\pm B>0$；若在曲面的$-y$侧积分，则选取±号使得$\pm B<0$；若在曲面的$+z$侧积分，则选取±号使得$\pm C>0$；若在曲面的$-z$侧积分，则选取±号使得$\pm C<0$.

故$\iint\limits_S \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S}$

$$=\iint\limits_S X(x,y,z)\mathrm{d}y\wedge\mathrm{d}z+Y(x,y,z)\mathrm{d}z\wedge\mathrm{d}x+Z(x,y,z)\mathrm{d}x\wedge\mathrm{d}y$$

$$=\pm\iint\limits_S[X(x(u,v),y(u,v),z(u,v))A+YB+ZC]\mathrm{d}u\mathrm{d}v.$$

【利用该公式进行计算的习题：9,6.(第五类题目)】

【其他类型的习题：9,11,6.(第四类题目)】

11.5 习题13.4解答

计算下列第二型曲面积分：

1. $\iint\limits_S (x^2+y^2)\mathrm{d}x\wedge\mathrm{d}y$, S为$x^2+y^2\leqslant 1, z=0$的下侧.

   解：$\iint\limits_S (x^2+y^2)\mathrm{d}x\wedge\mathrm{d}y = \iint\limits_{x^2+y^2\leqslant 1}(x^2+y^2)(-\mathrm{d}x\mathrm{d}y) = -\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r^2\cdot r\mathrm{d}r = -2\pi\cdot\frac{1}{4}r^4\Big|_0^1$
   $= -\frac{\pi}{2}$.

2. $\iint\limits_S z\mathrm{d}x\wedge\mathrm{d}y$, S为$x^2+y^2+z^2=R^2$上半部分的下侧.

   解：$\iint\limits_S z\mathrm{d}x\wedge\mathrm{d}y = \iint\limits_{x^2+y^2\leqslant R^2}\sqrt{R^2-x^2-y^2}(-\mathrm{d}x\mathrm{d}y) = -\int_0^{2\pi}\mathrm{d}\theta\int_0^R\sqrt{R^2-r^2}\cdot r\mathrm{d}r$
   $= -2\pi(-\frac{1}{2})\frac{1}{1+\frac{1}{2}}(R^2-r^2)^{\frac{1}{2}+1}\Big|_0^R = -\frac{2}{3}\pi R^3$.

3. $\iint\limits_S xz^2\mathrm{d}x\wedge\mathrm{d}y$, S为$x^2+y^2+z^2=1$第一卦限部分的外侧.

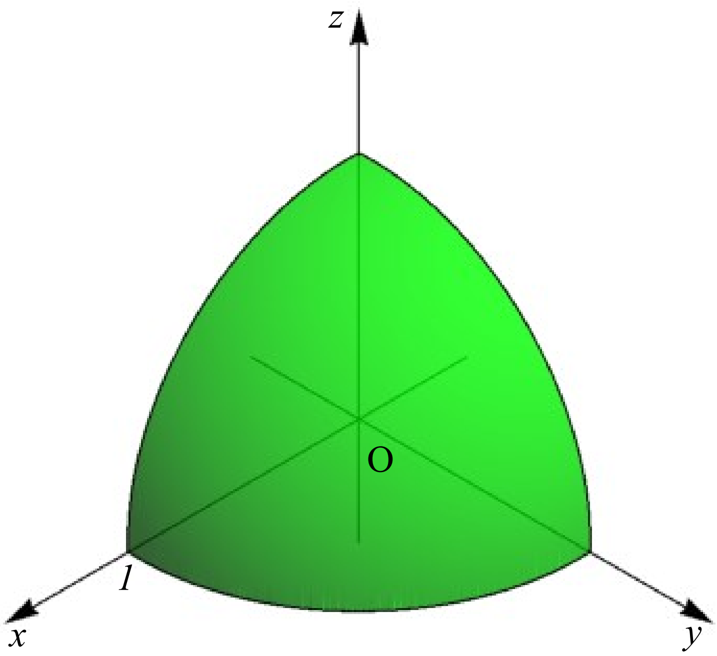

图 2: 习题13.4 3.题图示

解：曲面S的方程可以表示为$z=\sqrt{1-x^2-y^2}, x^2+y^2\leqslant 1, x\geqslant 0, y\geqslant 0$,

$\therefore \iint\limits_S xz^2\mathrm{d}x\wedge\mathrm{d}y = \iint\limits_{\substack{x^2+y^2\leqslant 1,\\ x\geqslant 0, y\geqslant 0}} x(1-x^2-y^2)\mathrm{d}x\mathrm{d}y = \int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^1 r\cos\theta(1-r^2)\cdot r\mathrm{d}r$

$=\int_0^{\frac{\pi}{2}}\cos\theta\mathrm{d}\theta\int_0^1 r^2(1-r^2)\mathrm{d}r = 1\cdot(\frac{1}{3}r^2-\frac{1}{5}r^5)\Big|_0^1=\frac{2}{15}$.

4. $\iint\limits_S z^2\mathrm{d}x\wedge\mathrm{d}y, S$为$z=\sqrt{a^2-x^2-y^2}(a>0)$被圆柱面$x^2+y^2=ax$所截部分的上侧.

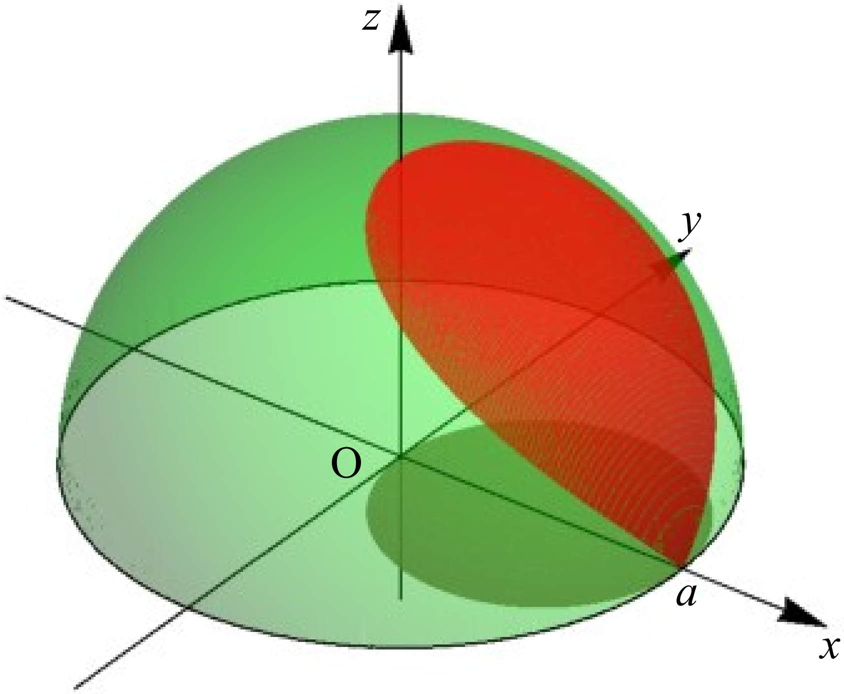

图 3: 习题13.4 4.题图示

解：$\iint\limits_S z^2 \mathrm{d}x\wedge\mathrm{d}y = \iint\limits_{x^2+y^2\leqslant ax}(a^2-x^2-y^2)\mathrm{d}x\mathrm{d}y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{a\cos\theta}(a^2-r^2)\cdot r\mathrm{d}r$
$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\frac{1}{2}a^2r^2-\frac{1}{4}r^4)\big|_0^{a\cos\theta}\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\frac{1}{2}a^4\cos^2\theta-\frac{1}{4}a^4\cos^4\theta)\mathrm{d}\theta = 2\int_0^{\frac{\pi}{2}}(\frac{1}{2}a^4\cos^2\theta-\frac{1}{4}a^4\cos^4\theta)\mathrm{d}\theta$
$=2(\frac{1}{2}a^4\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{4}a^4\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2})=\frac{5}{32}\pi a^4.$

5. $\iint\limits_S 2y\mathrm{d}z\wedge\mathrm{d}x$，其中$S$为锥面$y=\sqrt{x^2+z^2}$介于$y=1$和$y=2$之间的部分外侧.

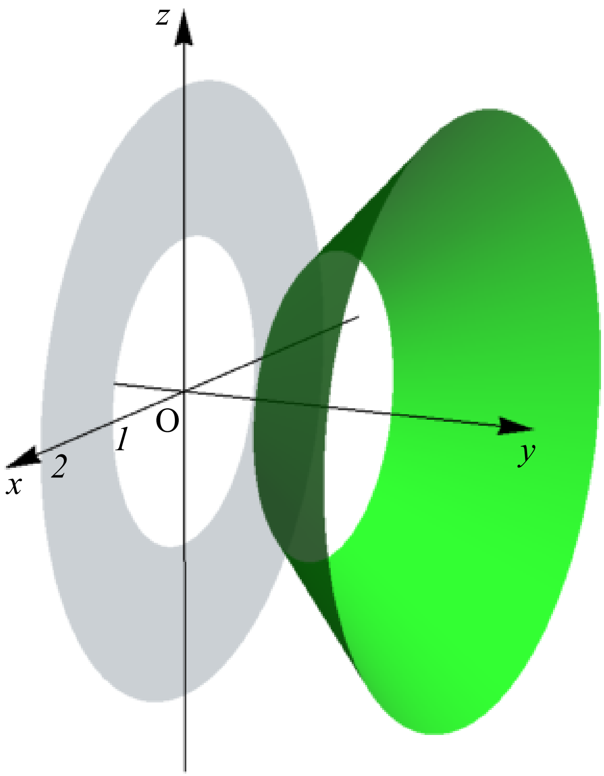

图 4: 习题13.4 5.题图示

解：$\iint\limits_S 2y\mathrm{d}z\wedge\mathrm{d}x = \iint\limits_{1\leqslant x^2+z^2\leqslant 4} 2\sqrt{x^2+z^2}(-\mathrm{d}x\mathrm{d}z) = -\int_0^{2\pi}\mathrm{d}\theta\int_1^2 2r\cdot r\mathrm{d}r = -2\pi\frac{2}{3}r^3\Big|_1^2$
$= -\frac{28}{3}\pi.$

6. $\iint\limits_S x\mathrm{d}y\wedge\mathrm{d}z + z\mathrm{d}x\wedge\mathrm{d}y, S$为$x^2+y^2=a^2$在第一卦限中介于$z=0, z=h(h>0)$之间的部分，外侧为正.

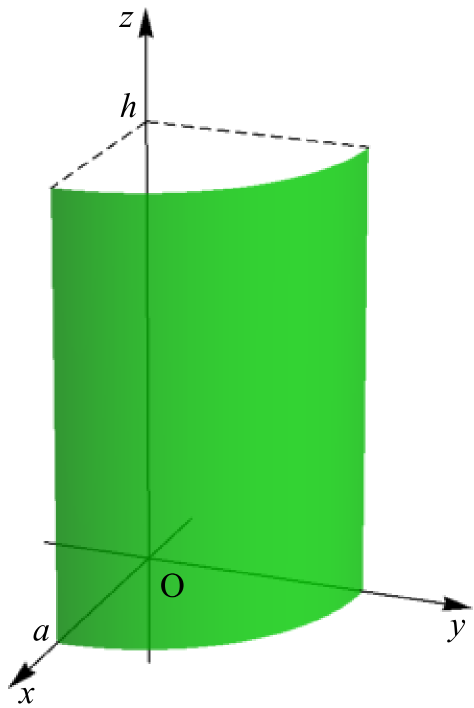

图 5: 习题13.4 6.题图示

解：方法1：令$\begin{cases} x = a\cos\theta, \\ y = a\sin\theta, \\ z = z, \end{cases}$ $\quad (\theta, z) \in D = \left\{(\theta, z) \,\middle|\, 0 \leqslant \theta \leqslant \frac{\pi}{2}, 0 \leqslant z \leqslant h\right\}$,

则$A = \frac{\mathrm{D}(y,z)}{\mathrm{D}(\theta,z)} = \begin{vmatrix} a\cos\theta & 0 \\ 0 & 1 \end{vmatrix} = a\cos\theta$, $B = \frac{\mathrm{D}(z,x)}{\mathrm{D}(\theta,z)} = \begin{vmatrix} 0 & 1 \\ -a\sin\theta & 0 \end{vmatrix} = a\sin\theta$,

$C = \frac{\mathrm{D}(x,y)}{\mathrm{D}(\theta,z)} = \begin{vmatrix} -a\sin\theta & 0 \\ a\cos\theta & 0 \end{vmatrix} = 0$,

$\therefore \mathrm{d}\boldsymbol{S} = (A, B, C)\mathrm{d}\theta\mathrm{d}z = (a\cos\theta, a\sin\theta, 0)\mathrm{d}\theta\mathrm{d}z = (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y)$,

$\therefore \iint\limits_S x\mathrm{d}y \wedge \mathrm{d}z + z\mathrm{d}x \wedge \mathrm{d}y = \iint\limits_D a\cos\theta \cdot a\cos\theta\mathrm{d}\theta\mathrm{d}z + z \cdot 0 = a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta\mathrm{d}\theta \int_0^h \mathrm{d}z = \frac{\pi}{4}a^2 h.$

方法2：曲面S的正向单位法向量可以表示为$\boldsymbol{n} = \frac{1}{a}(x, y, 0)$,

$\therefore \mathrm{d}\boldsymbol{S} = \boldsymbol{n}\mathrm{d}S = \frac{1}{a}(x, y, 0)\mathrm{d}S = (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y)$,

$\therefore \iint\limits_S x\mathrm{d}y \wedge \mathrm{d}z + z\mathrm{d}x \wedge \mathrm{d}y = \iint\limits_S x \cdot \frac{1}{a}x\mathrm{d}S + z \cdot 0 = \frac{1}{a}\iint\limits_S x^2\mathrm{d}S$,

$\because$曲面S关于平面$y=x$对称,

$\therefore$上式$=\frac{1}{a}\frac{1}{2}\iint\limits_S(x^2+y^2)\mathrm{d}S=\frac{1}{2a}\iint\limits_S a^2\mathrm{d}S=\frac{a^2}{2a}\iint\limits_S\mathrm{d}S=\frac{a}{2}\cdot\frac{1}{4}\cdot 2\pi a\cdot h=\frac{\pi}{4}a^2h.$

7. $\iint\limits_S z\mathrm{d}x\wedge\mathrm{d}y+\mathrm{d}y\wedge\mathrm{d}z$, S为平面$x+y-z=1$在第五卦限中的部分下侧.

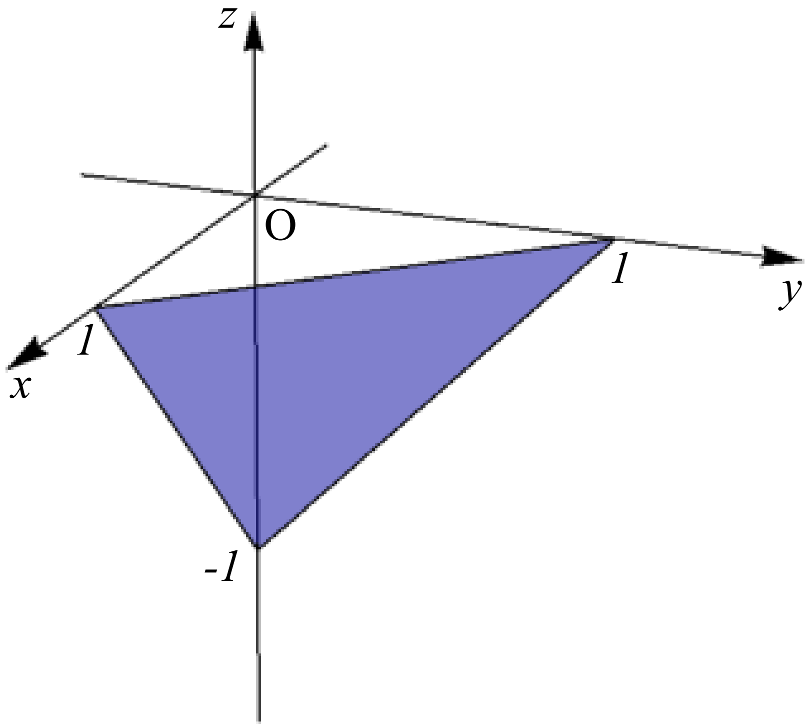

图 6: 习题13.4 7.题图示

解：方法1：$\iint\limits_S z\mathrm{d}x\wedge\mathrm{d}y+\mathrm{d}y\wedge\mathrm{d}z=\iint\limits_S z\mathrm{d}x\wedge\mathrm{d}y+\iint\limits_S\mathrm{d}y\wedge\mathrm{d}z$

$=\iint\limits_{D_{xy}}(x+y-1)(-\mathrm{d}x\mathrm{d}y)+\iint\limits_{D_{yz}}\mathrm{d}y\mathrm{d}z=\iint\limits_{D_{xy}}(1-x-y)\mathrm{d}x\mathrm{d}y+\iint\limits_{D_{yz}}\mathrm{d}y\mathrm{d}z,$

其中D_{xy}与D_{yz}为S在xOy平面和yOz平面上的投影，D_{xy}关于$y=x$对称，

$\therefore$上式$=\iint\limits_{D_{xy}}\mathrm{d}x\mathrm{d}y-2\iint\limits_{D_{xy}}x\mathrm{d}x\mathrm{d}y+\iint\limits_{D_{yz}}\mathrm{d}y\mathrm{d}z=\frac{1}{2}-2\int_0^1 x\mathrm{d}x\int_0^{1-x}\mathrm{d}y+\frac{1}{2}$

$=1-2\int_0^1 x(1-x)\mathrm{d}x=1-2(\frac{1}{2}x^2-\frac{1}{3}x^3)\big|_0^1=\frac{2}{3}.$

方法2：平面S的方程可表示为$z=x+y-1, (x,y)\in D=\{(x,y)\mid 0\leqslant x\leqslant 1, 0\leqslant y\leqslant 1-x\}$,

$\therefore\ \mathrm{d}\boldsymbol{S}=-(-\frac{\partial z}{\partial x},-\frac{\partial z}{\partial y},1)\mathrm{d}x\mathrm{d}y=(1,1,-1)\mathrm{d}x\mathrm{d}y=(\mathrm{d}y\wedge\mathrm{d}z,\mathrm{d}z\wedge\mathrm{d}x,\mathrm{d}x\wedge\mathrm{d}y),$

$\therefore \iint\limits_{S} z\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}z = \iint\limits_{D} [(x+y-1)(-\mathrm{d}x\mathrm{d}y) + \mathrm{d}x\mathrm{d}y] = \iint\limits_{D} (2-x-y)\mathrm{d}x\mathrm{d}y,$

$\because$区域D关于$y=x$对称，

$\therefore$上式$= 2\iint\limits_{D} \mathrm{d}x\mathrm{d}y - 2\iint\limits_{D} x\mathrm{d}x\mathrm{d}y = 2\cdot\frac{1}{2} - 2\int_0^1 x\mathrm{d}x\int_0^{1-x}\mathrm{d}y = 1 - 2\int_0^1 x(1-x)\mathrm{d}x$

$= 1 - 2(\frac{1}{2}x^2 - \frac{1}{3}x^3)\Big|_0^1 = \frac{2}{3}.$

8. $\iint\limits_{S} (y-z)\mathrm{d}y\wedge\mathrm{d}z + (z-x)\mathrm{d}z\wedge\mathrm{d}x + (x-y)\mathrm{d}x\wedge\mathrm{d}y$, S为$y=\sqrt{x^2+z^2}$, $y=h(h>0)$所围区域的表面外侧.

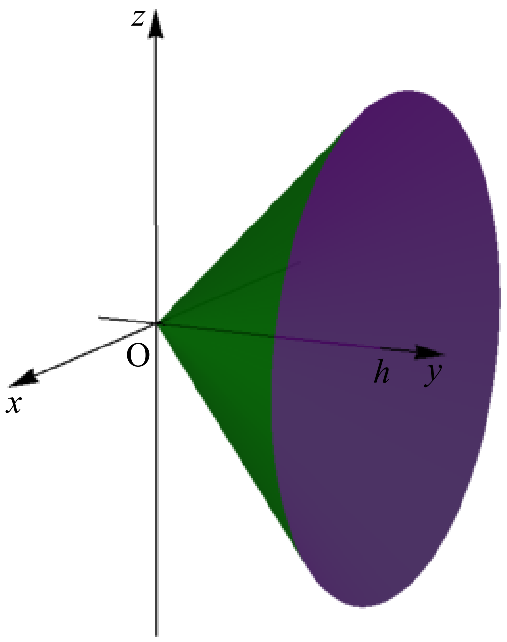

图 7: 习题13.4 8.题图示

解：S由锥面$S_1: y=\sqrt{x^2+z^2}, x^2+y^2\leqslant h^2$的左侧和平面$S_2: y=h, x^2+y^2\leqslant h^2$的右侧组成，

在锥面S_1的左侧$\mathrm{d}\boldsymbol{S} = -(-\frac{\partial y}{\partial x}, 1, -\frac{\partial y}{\partial z})\mathrm{d}z\mathrm{d}x = (\frac{x}{\sqrt{x^2+z^2}}, -1, \frac{z}{\sqrt{x^2+z^2}})\mathrm{d}z\mathrm{d}x$

$= (\mathrm{d}y\wedge\mathrm{d}z, \mathrm{d}z\wedge\mathrm{d}x, \mathrm{d}x\wedge\mathrm{d}y),$

$\therefore \iint\limits_{S_1} (y-z)\mathrm{d}y\wedge\mathrm{d}z + (z-x)\mathrm{d}z\wedge\mathrm{d}x + (x-y)\mathrm{d}x\wedge\mathrm{d}y$

$= \iint\limits_{x^2+z^2\leqslant h^2} (\sqrt{x^2+z^2}-z)\frac{x}{\sqrt{x^2+z^2}}\mathrm{d}z\mathrm{d}x-(z-x)\mathrm{d}z\mathrm{d}x+(x-\sqrt{x^2+z^2})\frac{z}{\sqrt{x^2+z^2}}\mathrm{d}z\mathrm{d}x$

$= \iint\limits_{x^2+z^2\leqslant h^2} (2x-2z)\mathrm{d}z\mathrm{d}x,$

由对称性可知上式$=0$,

在平面S_2的右侧$\mathrm{d}\boldsymbol{S}=(-\frac{\partial y}{\partial x},1,-\frac{\partial y}{\partial z})\mathrm{d}z\mathrm{d}x=(0,1,0)\mathrm{d}z\mathrm{d}x=(\mathrm{d}y\wedge\mathrm{d}z,\mathrm{d}z\wedge\mathrm{d}x,\mathrm{d}x\wedge\mathrm{d}y)$,

$\therefore \iint\limits_{S_2}(y-z)\mathrm{d}y\wedge\mathrm{d}z+(z-x)\mathrm{d}z\wedge\mathrm{d}x+(x-y)\mathrm{d}x\wedge\mathrm{d}y=\iint\limits_{x^2+z^2\leqslant h^2} 0+(z-x)+0\mathrm{d}z\mathrm{d}x,$

由对称性可知上式$=0$,

$\therefore \iint\limits_{S}(y-z)\mathrm{d}y\wedge\mathrm{d}z+(z-x)\mathrm{d}z\wedge\mathrm{d}x+(x-y)\mathrm{d}x\wedge\mathrm{d}y$

$=(\iint\limits_{S_1}+\iint\limits_{S_2})(y-z)\mathrm{d}y\wedge\mathrm{d}z+(z-x)\mathrm{d}z\wedge\mathrm{d}x+(x-y)\mathrm{d}x\wedge\mathrm{d}y$

$=0+0=0.$

9. $\oiint\limits_{S} x\mathrm{d}y\wedge\mathrm{d}z+y\mathrm{d}z\wedge\mathrm{d}x+z\mathrm{d}x\wedge\mathrm{d}y, S$为$x^2+y^2+z^2=R^2$的外侧.

解：球面S的外向单位法向量为$\boldsymbol{n}=\frac{1}{R}(x,y,z)$,

$\therefore \mathrm{d}\boldsymbol{S}=\frac{1}{R}(x,y,z)\mathrm{d}S,$

$\therefore \oiint\limits_{S} x\mathrm{d}y\wedge\mathrm{d}z+y\mathrm{d}z\wedge\mathrm{d}x+z\mathrm{d}x\wedge\mathrm{d}y=\oiint\limits_{S}(x\cdot\frac{x}{R}+y\cdot\frac{y}{R}+z\cdot\frac{z}{R})\mathrm{d}S=\oiint\limits_{S}\frac{x^2+y^2+z^2}{R}\mathrm{d}S$

$=R\oiint\limits_{S}\mathrm{d}S=4\pi R^3.$

10. $\oiint\limits_{S} yz\mathrm{d}y\wedge\mathrm{d}z+zx\mathrm{d}z\wedge\mathrm{d}x+xy\mathrm{d}x\wedge\mathrm{d}y, S$为区域$\begin{cases} x+y+z\leqslant a, a>0 \\ x\geqslant 0, y\geqslant 0, z\geqslant 0 \end{cases}$ 表面的外侧.

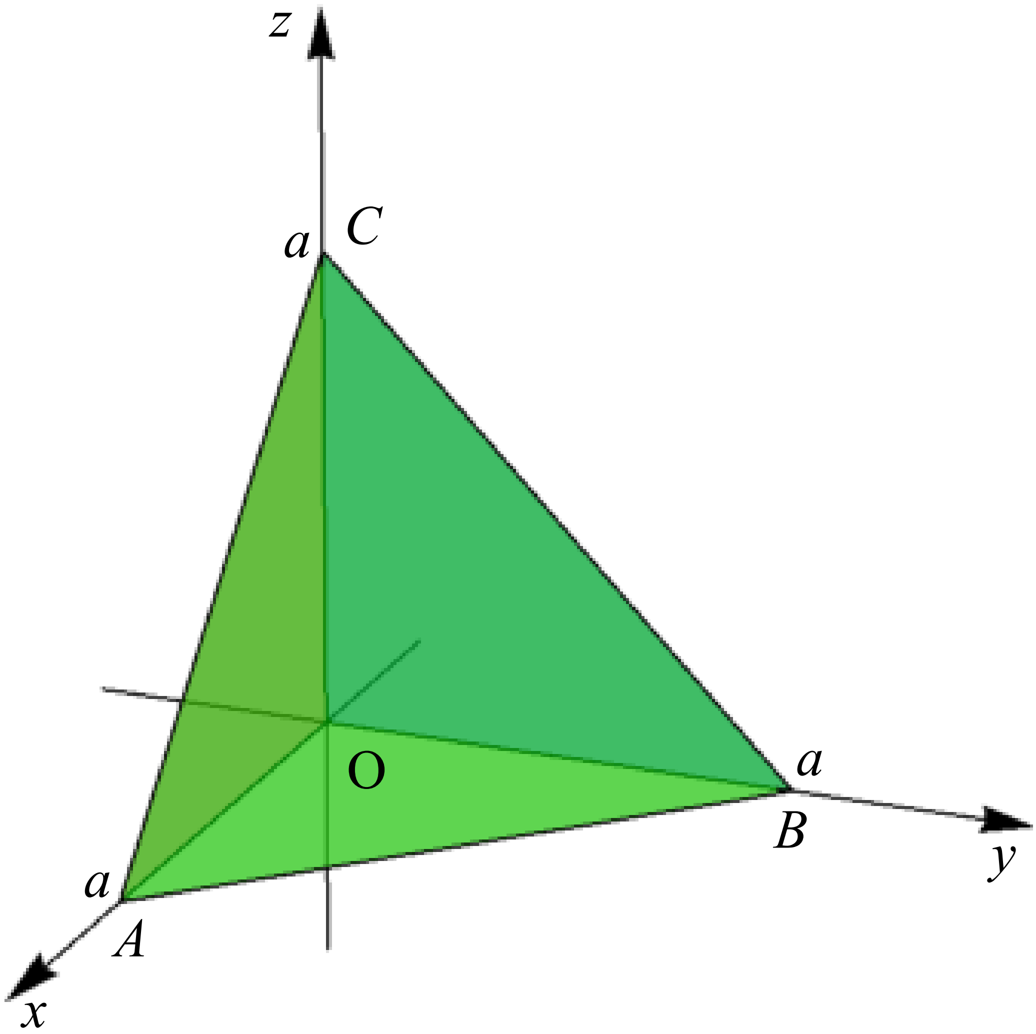

图 8: 习题13.4 10.题图示

解：如图 8所示，S由四个平面ABC, OBC, OCA, OAB组成，

$$\therefore \oiint\limits_{S} yz\mathrm{d}y\wedge\mathrm{d}z+zx\mathrm{d}z\wedge\mathrm{d}x+xy\mathrm{d}x\wedge\mathrm{d}y$$
$$=(\iint\limits_{OBC}+\iint\limits_{OCA}+\iint\limits_{OAB}+\iint\limits_{ABC})yz\mathrm{d}y\wedge\mathrm{d}z+zx\mathrm{d}z\wedge\mathrm{d}x+xy\mathrm{d}x\wedge\mathrm{d}y,$$

$\because$在平面OBC的后侧$x=0$,

$$\therefore \iint\limits_{OBC} yz\mathrm{d}y\wedge\mathrm{d}z+zx\mathrm{d}z\wedge\mathrm{d}x+xy\mathrm{d}x\wedge\mathrm{d}y=\iint\limits_{OBC} yz\mathrm{d}y\wedge\mathrm{d}z=-\iint\limits_{OBC} yz\mathrm{d}y\mathrm{d}z,$$

这里的OBC同时表示yOz平面上积分域，

同理，$\iint\limits_{OCA} yz\mathrm{d}y\wedge\mathrm{d}z+zx\mathrm{d}z\wedge\mathrm{d}x+xy\mathrm{d}x\wedge\mathrm{d}y=\iint\limits_{OCA} zx\mathrm{d}z\wedge\mathrm{d}x=-\iint\limits_{OCA} zx\mathrm{d}z\mathrm{d}x,$

$\iint\limits_{OAB} yz\mathrm{d}y\wedge\mathrm{d}z+zx\mathrm{d}z\wedge\mathrm{d}x+xy\mathrm{d}x\wedge\mathrm{d}y=\iint\limits_{OAB} xy\mathrm{d}x\wedge\mathrm{d}y=-\iint\limits_{OAB} xy\mathrm{d}x\mathrm{d}y,$

$\because$平面ABC的上侧可表示为$x=a-y-z,(y,z)\in OBC,$

$\therefore \iint\limits_{ABC} yz\mathrm{d}y\wedge\mathrm{d}z=\iint\limits_{OBC} yz\mathrm{d}y\mathrm{d}z,$

同理，$\iint\limits_{ABC} zx\mathrm{d}z\wedge\mathrm{d}x=\iint\limits_{OCA} zx\mathrm{d}z\mathrm{d}x,\ \iint\limits_{ABC} xy\mathrm{d}x\wedge\mathrm{d}y=\iint\limits_{OAB} xy\mathrm{d}x\mathrm{d}y,$

$\therefore \iint\limits_{ABC} yz\mathrm{d}y\wedge\mathrm{d}z+zx\mathrm{d}z\wedge\mathrm{d}x+xy\mathrm{d}x\wedge\mathrm{d}y=\iint\limits_{ABC} yz\mathrm{d}y\wedge\mathrm{d}z+\iint\limits_{ABC} zx\mathrm{d}z\wedge\mathrm{d}x+\iint\limits_{ABC} xy\mathrm{d}x\wedge\mathrm{d}y$
$=\iint\limits_{ABC} yz\mathrm{d}y\mathrm{d}z+\iint\limits_{ABC} zx\mathrm{d}z\mathrm{d}x+\iint\limits_{ABC} xy\mathrm{d}x\mathrm{d}y,$

$\therefore \oiint\limits_{S} yz\mathrm{d}y\wedge\mathrm{d}z+zx\mathrm{d}z\wedge\mathrm{d}x+xy\mathrm{d}x\wedge\mathrm{d}y=-\iint\limits_{OBC} yz\mathrm{d}y\mathrm{d}z-\iint\limits_{OCA} zx\mathrm{d}z\mathrm{d}x-\iint\limits_{OAB} xy\mathrm{d}x\mathrm{d}y$
$+\iint\limits_{ABC} yz\mathrm{d}y\mathrm{d}z+\iint\limits_{ABC} zx\mathrm{d}z\mathrm{d}x+\iint\limits_{ABC} xy\mathrm{d}x\mathrm{d}y$
$=0.$

11. $\oiint\limits_{S} \frac{x}{r^3}\mathrm{d}y\wedge\mathrm{d}z+\frac{y}{r^3}\mathrm{d}y\wedge\mathrm{d}x+\frac{z}{r^3}\mathrm{d}x\wedge\mathrm{d}y$，其中$r=\sqrt{x^2+y^2+z^2},S$为球面$x^2+y^2+z^2=a^2$的外侧.

解：球面S的外向单位法向量可表示为$\boldsymbol{n}=\frac{1}{a}(x,y,z),$

$\therefore \mathrm{d}\boldsymbol{S}=\boldsymbol{n}\mathrm{d}S=\frac{1}{a}(x,y,z)\mathrm{d}S=(\mathrm{d}y\wedge\mathrm{d}z,\mathrm{d}y\wedge\mathrm{d}z,\mathrm{d}x\wedge\mathrm{d}y),$

$\therefore \oiint\limits_{S} \frac{x}{r^3}\mathrm{d}y\wedge\mathrm{d}z+\frac{y}{r^3}\mathrm{d}y\wedge\mathrm{d}x+\frac{z}{r^3}\mathrm{d}x\wedge\mathrm{d}y=\oiint\limits_{S} \frac{x}{r^3}\cdot\frac{x}{a}\mathrm{d}S+\frac{y}{r^3}\cdot\frac{y}{a}\mathrm{d}S+\frac{z}{r^3}\cdot\frac{z}{a}\mathrm{d}S=\oiint\limits_{S} \frac{x^2+y^2+z^2}{ar^3}\mathrm{d}S$
$=\oiint\limits_{S} \frac{a^2}{aa^3}\mathrm{d}S=\frac{1}{a^2}\oiint\limits_{S}\mathrm{d}S=\frac{1}{a^2}\cdot 4\pi a^2=4\pi.$