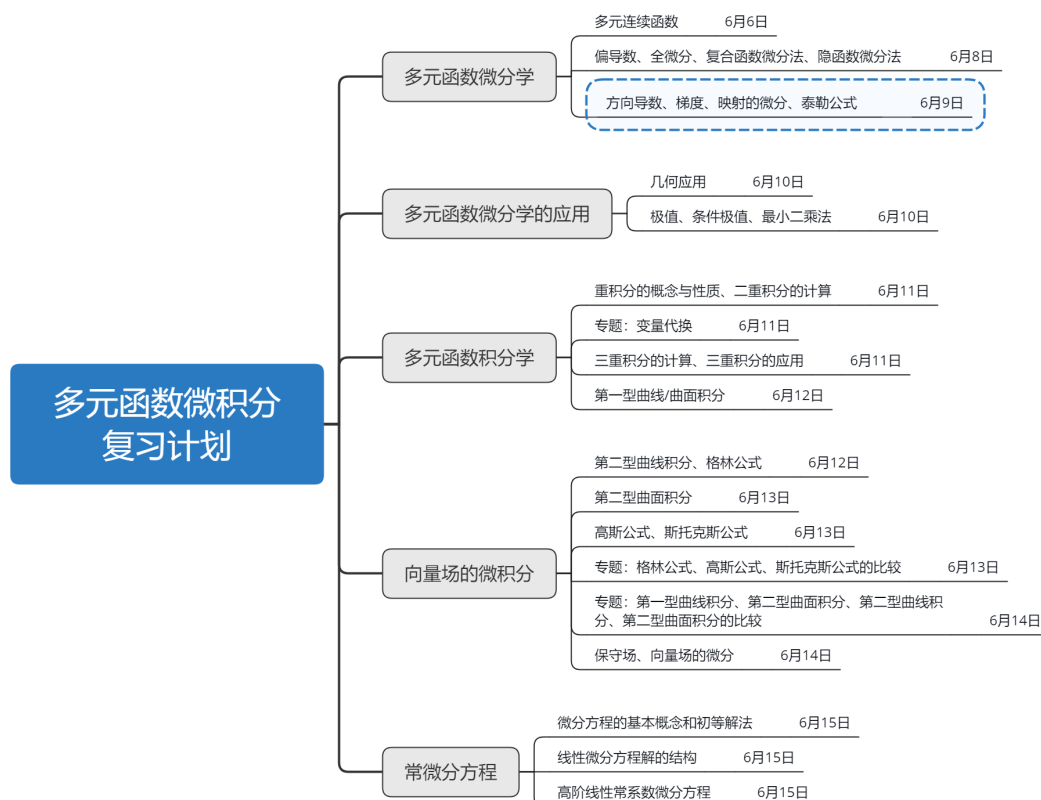
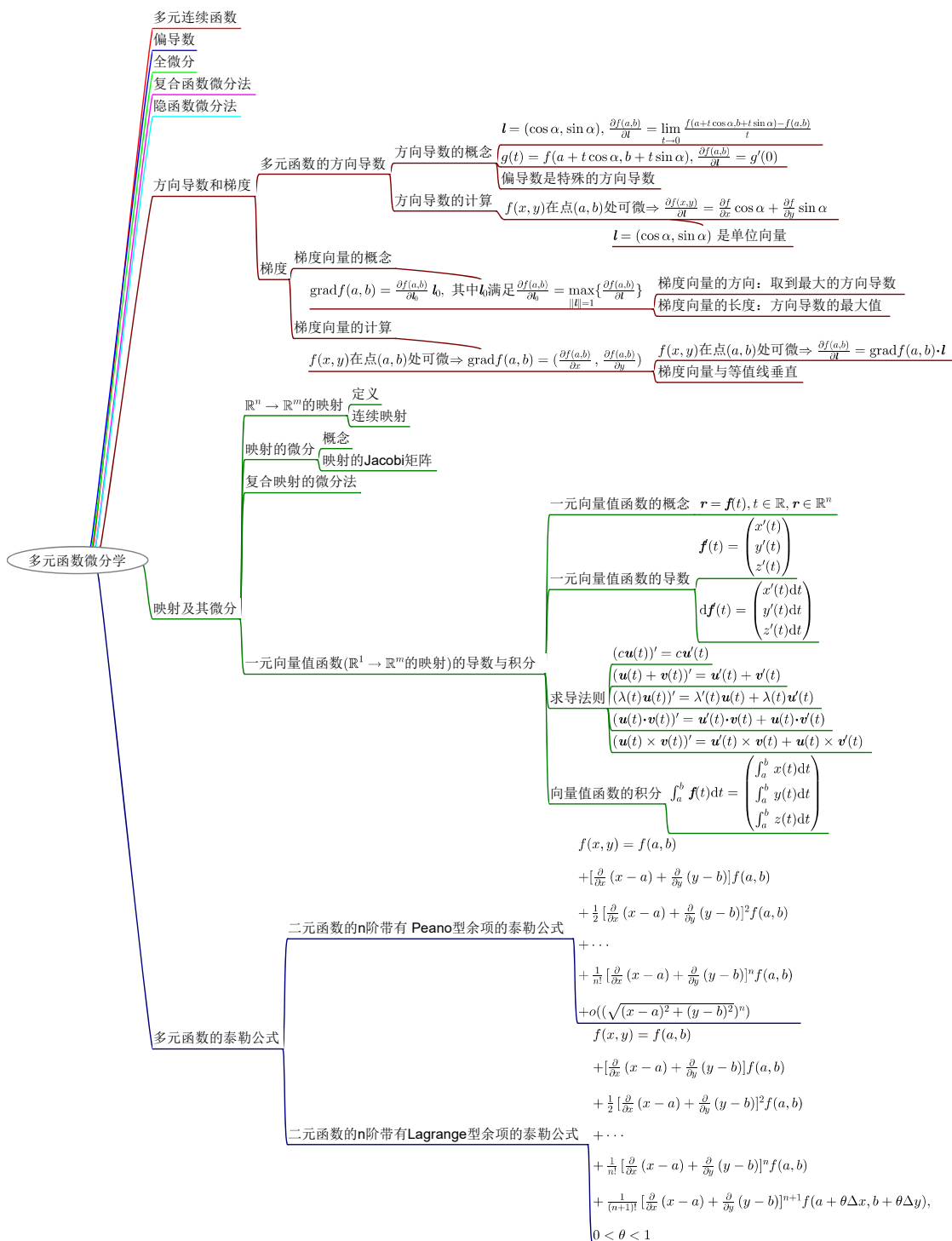


3 方向导数、梯度、映射的微分、泰勒公式

3.1 复习计划



3.2 知识结构



3.3 重要知识

1. 方向导数的几何意义.

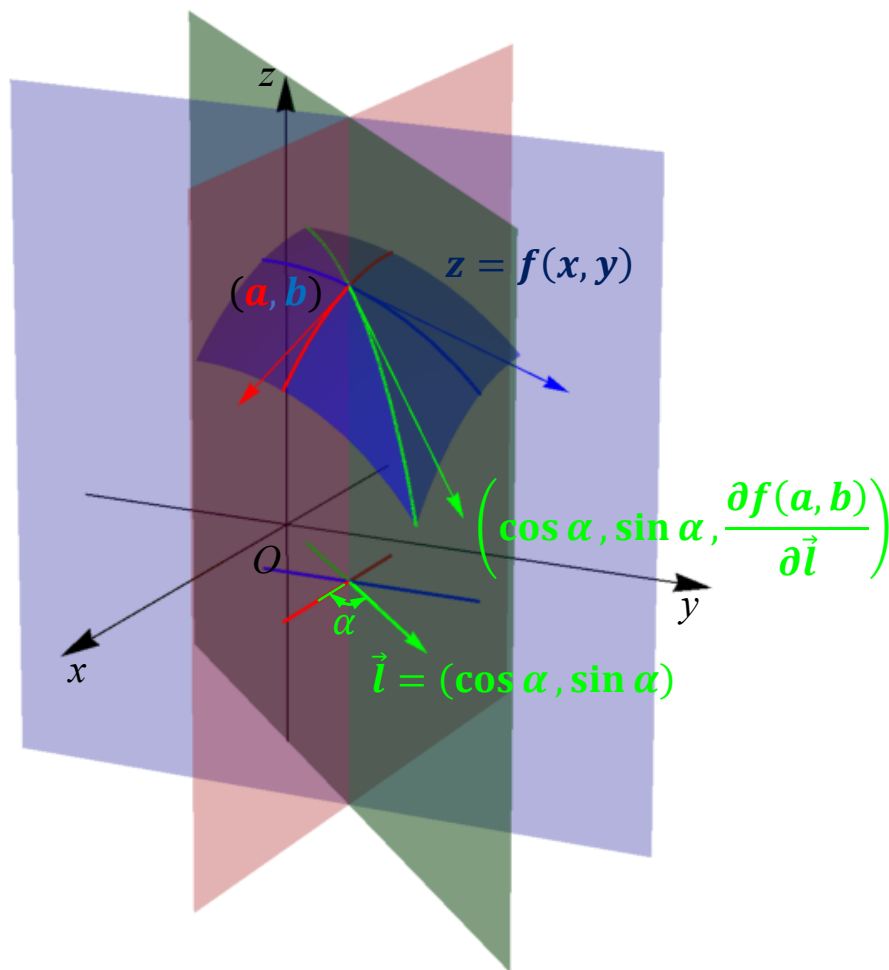


图 1: 方向导数的几何意义

方向导数的几何意义如上图所示，须注意以下几点：

- (a) 二元函数 $f(x, y)$ 在一点处的方向导数 $\frac{\partial f(a,b)}{\partial l}$ 是函数在该点沿确定方向 $\mathbf{l} = (\cos \alpha, \sin \alpha)$ 的变化率；
- (b) 向量 $(\cos \alpha, \sin \alpha, \frac{\partial f(a,b)}{\partial l})$ 是函数 $f(x, y)$ 在点 (a, b) 沿 \mathbf{l} 方向的切向量；
- (c) 所有切向量的集合 $\{\boldsymbol{\tau} \mid \boldsymbol{\tau} = (\cos \alpha, \sin \alpha, \frac{\partial f(a,b)}{\partial l}), \mathbf{l} = (\cos \alpha, \sin \alpha), \alpha \in [0, 2\pi)\}$ 共面；
- (d) 偏导数是沿坐标轴正方向的方向导数，

(e) 当函数 $f(x, y)$ 在点 (a, b) 可微时, $\frac{\partial f(a, b)}{\partial l} = \text{grad} f(a, b) \cdot \mathbf{l} = \frac{\partial f(a, b)}{\partial x} \cos \alpha + \frac{\partial f(a, b)}{\partial y} \sin \alpha$.

2. 梯度的几何意义.

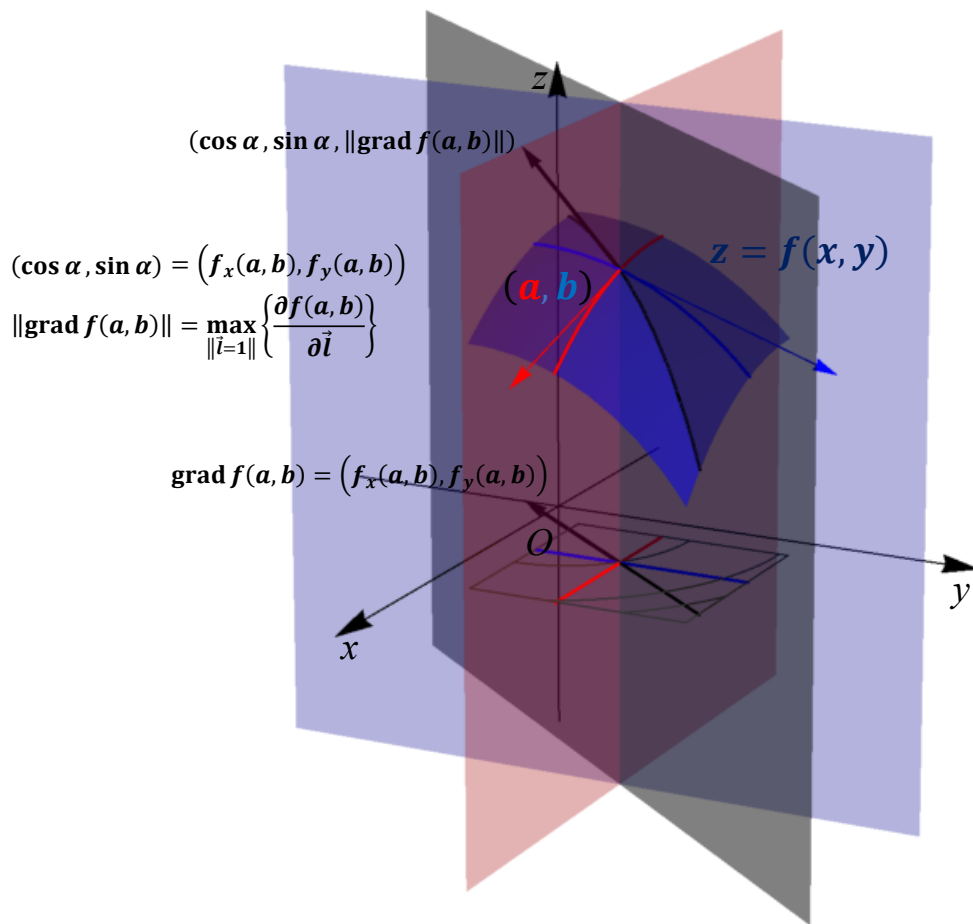


图 2: 梯度的几何意义

梯度的几何意义如上图所示, 须注意以下几点:

- (a) 函数 $f(x, y)$ 在点 (a, b) 处可微时, 梯度 $\text{grad} f(a, b) = (\frac{\partial f(a, b)}{\partial x}, \frac{\partial f(a, b)}{\partial y})$;
- (b) 梯度 $\text{grad} f(a, b)$ 的方向为函数 $f(x, y)$ 在点 (a, b) 处方向导数最大的方向;
- (c) 因在一点 (a, b) 处的所有切向量的集合 $\{\boldsymbol{\tau} \mid \boldsymbol{\tau} = (\cos \alpha, \sin \alpha, \frac{\partial f(a, b)}{\partial l}), \mathbf{l} = (\cos \alpha, \sin \alpha), \alpha \in [0, 2\pi)\}$ 共面, 故最大的方向导数 $\max_{\alpha \in [0, 2\pi)} \{\frac{\partial f(a, b)}{\partial l}\}$ 必为非负;
- (d) 梯度向量的长度为方向导数的最大值, 即 $\|\text{grad} f(a, b)\| = \max_{\alpha \in [0, 2\pi)} \{\frac{\partial f(a, b)}{\partial l}\}$.

3. 可微与方向导数是否存在的关系.

函数 $f(x, y)$ 在点 (a, b) 处可微, 则函数 $f(x, y)$ 在点 (a, b) 处的方向导数存在, 但函数 $f(x, y)$ 在点 (a, b) 处的方向导数存在时函数不一定可微.

反例1: 函数 $f(x, y) = \begin{cases} 1, & y = x^3, x \neq 0, \\ 0, & \text{otherwise,} \end{cases}$ 在点 $(0, 0)$ 沿任意方向的方向导数都存在, 其

值为0. 但函数 $f(x, y)$ 在点 $(0, 0)$ 处不连续, 故不可微. 此时 $\frac{\partial f(a,b)}{\partial x} = 0, \frac{\partial f(a,b)}{\partial y} = 0$, 方向导数 $\frac{\partial f(x,y)}{\partial l} = 0 = \frac{\partial f(a,b)}{\partial x} \cos \alpha + \frac{\partial f(a,b)}{\partial y} \sin \alpha$.

反例2: 已知 $f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0, \end{cases} \quad l = (1, 1), \text{ 求 } \frac{\partial f(0,0)}{\partial l}.$

令 $g(t) = f(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}) = \begin{cases} \frac{1}{2\sqrt{2}}t, & t \neq 0, \\ 0, & t = 0, \end{cases}$ 则 $\frac{\partial f(0,0)}{\partial l} = \frac{1}{2\sqrt{2}}$.

此时 $\frac{\partial f(0,0)}{\partial x} = 0, \frac{\partial f(0,0)}{\partial y} = 0$, 若函数可微, 则 $df(0, 0) = 0$,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\frac{x^2 y}{x^2 + y^2} - 0 - 0}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{(x^2 + y^2)^{\frac{3}{2}}} = 0.$$

但是 $\lim_{\substack{x \rightarrow 0 \\ y=kx}} \frac{kx^3}{(\sqrt{1+k^2})^3 x^3} = \frac{k}{(1+k^2)^{\frac{3}{2}}}$ 不恒等于0.

故 $f(x, y)$ 在点 $(0, 0)$ 不可微.

此时 $\frac{\partial f(0,0)}{\partial l} = \frac{1}{2\sqrt{2}} \neq 0 = \frac{\partial f(a,b)}{\partial x} \cos \alpha + \frac{\partial f(a,b)}{\partial y} \sin \alpha$.

3.4 习题分类与解题思路

1. 方向导数与梯度

(a) 方向导数的计算, 可用定义或者梯度进行计算. 当用梯度计算时, 须注意在所求点上函数应可微, 且应把方向向量单位化.

【如习题10.4中的5.】

(b) 考查梯度的几何意义.

具体可见【习题10.4中的6.】

2. 一元向量值函数的导数与积分

(a) 考查向量值函数的四则运算法则.

【如习题11.1中的1., 2.】

(b) 求曲线的切向量或切线方程, 即一元向量值函数的导数.

【如习题11.1中的2., 3., 7., 8.】

(c) 求一元向量值函数的积分.

【如习题11.1中的4., 9.】

(d) 已知一元向量值函数的导函数和定解条件, 求原函数.

【如习题11.1中的5.】

3. 多元函数的泰勒公式

(a) 利用带佩亚诺余项的泰勒公式做近似计算.

【如习题10.6中的1.】

(b) 将函数展开成泰勒多项式.

【如习题10.6中的2.】

(c) 将函数展开成带拉格朗日余项的泰勒多项式.

【如习题10.6中的3.】

【第10章补充题的题目有一些综合性, 大家可做一些积累.】

3.5 习题10.4解答

5. 求 $z = \ln(e^{-x} + \frac{x^2}{y})$ 在点 $(1, 1)$ 处沿 $\mathbf{v} = (a, b)^T (a \neq 0)$ 的方向导数.

$$\text{解: } \because \frac{\partial z}{\partial x} = \frac{-e^{-x} + \frac{2x}{y}}{e^{-x} + \frac{x^2}{y}}, \frac{\partial z}{\partial y} = \frac{-\frac{x^2}{y^2}}{e^{-x} + \frac{x^2}{y}},$$

$$\therefore \text{grad} z(1, 1) = \left(\frac{\partial z(1, 1)}{\partial x}, \frac{\partial z(1, 1)}{\partial y} \right) = \left(\frac{-e^{-1} + 2}{e^{-1} + 1}, \frac{-1}{e^{-1} + 1} \right) = \left(\frac{2e-1}{e+1}, -\frac{e}{e+1} \right),$$

$\therefore \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ 在点 $(1, 1)$ 及其附近存在且在点 $(1, 1)$ 处连续,

$\therefore f(x, y)$ 在点 $(1, 1)$ 处可微,

$$\therefore \frac{\partial z(1, 1)}{\partial \mathbf{v}} = \text{grad} z(1, 1) \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{2e-1}{e+1}, -\frac{e}{e+1} \right) \cdot \frac{1}{\sqrt{a^2+b^2}}(a, b)^T = \frac{1}{\sqrt{a^2+b^2}} \left(\frac{2ae-a-be}{e+1} \right).$$

6. 已知 $f(x, y) = x^2 - xy + y^2$.

(1) 当 \mathbf{v} 分别为何向量时, 方向导数 $\frac{\partial f(1, 1)}{\partial \mathbf{v}}$ 会取到最大值、最小值和零值? 并求出其最大值和最小值.

(2) 试求 $\text{grad} f(1, 1)$, 并说明其方向与大小的意义.

$$\text{解: } (1) \because \frac{\partial f(x, y)}{\partial x} = 2x - y, \frac{\partial f(x, y)}{\partial y} = 2y - x,$$

$$\therefore \text{grad} f(1, 1) = \left(\frac{\partial f(1, 1)}{\partial x}, \frac{\partial f(1, 1)}{\partial y} \right) = (1, 1),$$

$\therefore \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ 在点 $(1, 1)$ 及其附近存在且在点 $(1, 1)$ 处连续,

$\therefore f(x, y)$ 在点 $(1, 1)$ 处可微,

$$\therefore \frac{\partial f(1,1)}{\partial \mathbf{v}} = \text{grad} f(1,1) \cdot \mathbf{v} = \|\text{grad} f(1,1)\| \|\mathbf{v}\| \cos \theta = \|\text{grad} f(1,1)\| \cos \theta,$$

当 \mathbf{v} 与梯度向量的夹角 $\theta = 0$ 即 $\mathbf{v} = \frac{1}{\sqrt{2}}(1, 1)$ 时, 方向导数 $\frac{\partial f(1,1)}{\partial \mathbf{v}}$ 取得最大值 $\|\text{grad} f(1,1)\| = \sqrt{2}$;

当 \mathbf{v} 与梯度向量的夹角 $\theta = \pi$ 即 $\mathbf{v} = -\frac{1}{\sqrt{2}}(1, 1)$ 时, 方向导数 $\frac{\partial f(1,1)}{\partial \mathbf{v}}$ 取得最小值 $-\|\text{grad} f(1,1)\| = -\sqrt{2}$;

当 \mathbf{v} 与梯度向量的夹角 $\theta = \frac{\pi}{2}$ 即 $\mathbf{v} = \frac{1}{\sqrt{2}}(-1, 1)$ 或 $\frac{1}{\sqrt{2}}(1, -1)$ 时, 方向导数 $\frac{\partial f(1,1)}{\partial \mathbf{v}} = 0$.

(2) $\text{grad} f(1,1) = (1, 1)$, 其方向表示方向导数最大的方向, 其大小为方向导数的最大值.

3.6 习题10.6解答

1. 写出 $f(x, y) = x^y$ 在点 $(1, 1)$ 带佩亚诺余项的三阶泰勒公式, 由此计算 $1.1^{1.02}$.

解: $f(1, 1) = 1$,

$$\frac{\partial f}{\partial x} = yx^{y-1}, \frac{\partial f}{\partial y} = x^y \ln x,$$

$$\frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2}, \frac{\partial^2 f}{\partial x \partial y} = x^{y-1} + yx^{y-1} \ln x, \frac{\partial^2 f}{\partial y^2} = x^y (\ln x)^2,$$

$$\frac{\partial^3 f}{\partial x^3} = y(y-1)(y-2)x^{y-3}, \frac{\partial^3 f}{\partial x^2 \partial y} = yx^{y-1} (\ln x)^2,$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = (2y-1)x^{y-2} + y(y-1)x^{y-2} \ln x = [(2y-1) + y(y-1) \ln x]x^{y-2},$$

$$\frac{\partial^3 f}{\partial y^3} = x^y (\ln x)^3,$$

\therefore

$$\begin{aligned} & f(x, y) \\ &= f(1, 1) + \left[\frac{\partial f(1, 1)}{\partial x}(x-1) + \frac{\partial f(1, 1)}{\partial y}(y-1) \right] \\ &+ \frac{1}{2} \left[\frac{\partial^2 f(1, 1)}{\partial x^2}(x-1)^2 + 2 \frac{\partial^2 f(1, 1)}{\partial x \partial y}(x-1)(y-1) + \frac{\partial^2 f(1, 1)}{\partial y^2}(y-1)^2 \right] \\ &+ \frac{1}{6} \left[\frac{\partial^3 f(1, 1)}{\partial x^3}(x-1)^3 + 3 \frac{\partial^3 f(1, 1)}{\partial x^2 \partial y}(x-1)(y-1)^2 + 3 \frac{\partial^3 f(1, 1)}{\partial y \partial x^2}(x-1)^2(y-1) \right. \\ &\quad \left. + \frac{\partial^3 f(1, 1)}{\partial y^3}(y-1)^3 \right] + o[(\sqrt{(x-1)^2 + (y-1)^2})^3] \\ &= 1 + (x-1) + \frac{1}{2}[2(x-1)(y-1)] + \frac{1}{6}[3(x-1)^2(y-1)] + o[(\sqrt{(x-1)^2 + (y-1)^2})^3] \\ &= x + (x-1)(y-1) + \frac{1}{2}(x-1)^2(y-1) + o[(\sqrt{(x-1)^2 + (y-1)^2})^3] \end{aligned}$$

$$\therefore 1.1^{1.02} = f(1.1, 1.02) \approx 1 + 0.1 + 0.1 \times 0.02 + \frac{1}{2} \times 0.1^2 \times 0.02 = 1.1021.$$

2. 证明当 $|x|, |y|$ 充分小时, 有 $\frac{\cos x}{\cos y} \approx 1 - \frac{1}{2}(x^2 - y^2)$.

证明: 记 $f(x, y) = \frac{\cos x}{\cos y}$,

$$f(0, 0) = 1,$$

$$\frac{\partial f}{\partial x} = -\frac{\sin x}{\cos y}, \quad \frac{\partial f}{\partial y} = \frac{\cos x \sin y}{\cos^2 y},$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\cos x}{\cos y}, \quad \frac{\partial^2 f}{\partial x \partial y} = -\frac{\sin x \sin y}{\cos^2 y},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\cos x \cos y \cos^2 y + \cos x \sin y 2 \cos y \sin y}{\cos^4 y} = \frac{\cos x \cos^2 y + 2 \cos x \sin^2 y}{\cos^3 y},$$

\therefore

$$\begin{aligned} f(x, y) &= f(0, 0) + \left[\frac{\partial f(0, 0)}{\partial x} x + \frac{\partial f(0, 0)}{\partial y} y \right] + \frac{1}{2} \left[\frac{\partial^2 f(0, 0)}{\partial x^2} x^2 + 2 \frac{\partial^2 f(0, 0)}{\partial x \partial y} xy + \frac{\partial^2 f(0, 0)}{\partial y^2} y^2 \right] \\ &\quad + o[(\sqrt{x^2 + y^2})^2] \\ &= 1 + (0 + 0) + \frac{1}{2}(-x^2 + 0 + y^2) + o[(\sqrt{x^2 + y^2})^2] \\ &= 1 - \frac{1}{2}(x^2 - y^2) + o[(\sqrt{x^2 + y^2})^2], \end{aligned}$$

\therefore 当 $|x|, |y|$ 充分小时, 有 $\frac{\cos x}{\cos y} \approx 1 - \frac{1}{2}(x^2 - y^2)$.

3. 写出 $f(x, y) = \sqrt{1 + y^2} \cos x$ 在点 $(0, 1)$ 的一阶泰勒多项式及拉格朗日余项.

解: $f(0, 1) = \sqrt{2}$,

$$\frac{\partial f}{\partial x} = -\sqrt{1 + y^2} \sin x, \quad \frac{\partial f}{\partial y} = \frac{2y \cos x}{2\sqrt{1 + y^2}} = \frac{y \cos x}{\sqrt{1 + y^2}},$$

$$\frac{\partial^2 f}{\partial x^2} = -\sqrt{1 + y^2} \cos x, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{-2y \sin x}{2\sqrt{1 + y^2}} = \frac{-y \sin x}{\sqrt{1 + y^2}},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\cos x \sqrt{1 + y^2} - y \cos x \frac{2y}{2\sqrt{1 + y^2}}}{1 + y^2} = \frac{\cos x}{(1 + y^2)^{\frac{3}{2}}},$$

∴

$$\begin{aligned}
 f(x, y) &= f(0, 1) + \left[\frac{\partial f(0, 1)}{\partial x} x + \frac{\partial f}{\partial y}(y-1) \right] \\
 &\quad + \frac{1}{2} \left[\frac{\partial^2 f(\theta x, 1 + \theta(y-1))}{\partial x^2} x^2 + 2 \frac{\partial^2 f(\theta x, 1 + \theta(y-1))}{\partial x \partial y} x(y-1) \right. \\
 &\quad \left. + \frac{\partial^2 f(\theta x, 1 + \theta(y-1))}{\partial y^2} (y-1)^2 \right] \\
 &= \sqrt{2} + \frac{\sqrt{2}}{2}(y-1) \\
 &\quad + \frac{1}{2} \left(-\sqrt{1 + [1 + \theta(y-1)]^2} \cos(\theta x) x^2 - \frac{2[1 + \theta(y-1)] \sin \theta x}{\sqrt{1 + [1 + \theta(y-1)]^2}} x(y-1) \right. \\
 &\quad \left. + \frac{\cos \theta x}{\{1 + [1 + \theta(y-1)]^2\}^{\frac{3}{2}}} (y-1)^2 \right) \\
 &= \sqrt{2} + \frac{\sqrt{2}}{2}(y-1) \\
 &\quad + \frac{1}{2} \left\{ -x^2 \sqrt{1 + (1 + \theta(y-1))^2} \cos \theta x - 2x(y-1) \frac{1 + \theta(y-1)}{\sqrt{1 + (1 + \theta(y-1))^2}} \sin \theta x \right. \\
 &\quad \left. + (y-1)^2 \frac{\cos \theta x}{\{1 + (1 + \theta(y-1))^2\}^{\frac{3}{2}}} \right\}, 0 < \theta < 1.
 \end{aligned}$$

3.7 第10章补充题

1. 设 $f(x, y)$ 是定义在整个平面上的连续函数, 当 $x^2 + y^2 \rightarrow +\infty$ 时, $f(x, y) \rightarrow +\infty$. 求证存在 (x_0, y_0) , 使

$$f(x_0, y_0) = \min\{f(x, y) \mid (x, y) \in \mathbb{R}^2\}.$$

证明: ∵ 当 $x^2 + y^2 \rightarrow +\infty$ 时, $f(x, y) \rightarrow +\infty$,

∴ 对于 $f(0, 0)$, $\exists N > 0$, s.t. $f(x, y) > f(0, 0)$, $x^2 + y^2 > N^2$,

∴ 在有界闭区域 $D = \{(x, y) \mid x^2 + y^2 \leq N^2\}$ 内部 $f(x, y)$ 连续,

∴ $\exists (x_0, y_0) \in D$, s.t. $f(x_0, y_0) \leq f(x, y)$, $(x, y) \in D$, 此时 $f(x_0, y_0) \leq f(0, 0)$,

∴ $f(x_0, y_0) \leq f(0, 0) < f(x, y)$, $x^2 + y^2 > N^2$,

∴ $f(x_0, y_0) \leq f(x, y)$, $(x, y) \in \mathbb{R}^2$,

∴ 存在 (x_0, y_0) , 使 $f(x_0, y_0) = \min\{f(x, y) \mid (x, y) \in \mathbb{R}^2\}$.

2. 设 $f(x, y)$ 是定义在整个平面上的连续函数, $f(0, 0) = 0$, 且当 $(x, y) \neq (0, 0)$ 时, $f(x, y) > 0$, 又设对于任意的 (x, y) 和任意实数 c , 都有

$$f(cx, cy) = c^2 f(x, y).$$

求证存在正数 a, b , 使得对于任意的 (x, y) , 都有

$$a(x^2 + y^2) \leq f(x, y) \leq b(x^2 + y^2).$$

证明: $\because f(x, y)$ 是定义在整个平面上的连续函数,

$\therefore f(x, y)$ 在有界闭区域 $D = \{(x, y) \mid 0.5 < x^2 + y^2 \leq 1.5\}$ 上连续,

\because 当 $(x, y) \neq (0, 0)$ 时, $f(x, y) > 0$,

$\therefore \exists b \geq a > 0, s.t. a \leq f(x, y) \leq b, (x, y) \in D$,

\therefore 当 $(x, y) \in D^* = \{(x, y) \mid x^2 + y^2 = 1\} \subset D$ 时, $a \leq f(x, y) \leq b$,

$\because f(x, y) = f(\sqrt{x^2 + y^2} \frac{x}{\sqrt{x^2 + y^2}}, \sqrt{x^2 + y^2} \frac{y}{\sqrt{x^2 + y^2}}) = (x^2 + y^2) f(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}})$,

又 $\because a \leq f(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}) \leq b$,

$\therefore a(x^2 + y^2) \leq f(x, y) \leq b(x^2 + y^2)$.

3. 若对于任意实数 t , 函数 $f(x, y, z)$ 满足 $f(tx, ty, tz) = t^k f(x, y, z)$, 则称 $f(x, y, z)$ 为 k 次齐次函数. 试证 k 次齐次函数 $f(x, y, z)$ 满足方程

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = k f(x, y, z).$$

证明: 方法1: 等式 $f(tx, ty, tz) = t^k f(x, y, z)$ 两边分别对 t 求偏导得

$$x \frac{\partial f(tx, ty, tz)}{\partial x} + y \frac{\partial f(tx, ty, tz)}{\partial y} + z \frac{\partial f(tx, ty, tz)}{\partial z} = k t^{k-1} f(x, y, z),$$

令 $t = 1$ 得

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = k f(x, y, z).$$

方法2: $\because f(tx, ty, tz) = t^k f(x, y, z)$,

\therefore

$$\frac{\partial f(x, y, z)}{\partial x} = \frac{\partial}{\partial x} \frac{f(tx, ty, tz)}{t^k} = \frac{1}{t^k} \frac{\partial f(tx, ty, tz)}{\partial x} t = \frac{1}{t^{k-1}} \frac{\partial f(tx, ty, tz)}{\partial x}, \quad (1a)$$

$$\frac{\partial f(x, y, z)}{\partial y} = \frac{\partial}{\partial y} \frac{f(tx, ty, tz)}{t^k} = \frac{1}{t^k} \frac{\partial f(tx, ty, tz)}{\partial y} t = \frac{1}{t^{k-1}} \frac{\partial f(tx, ty, tz)}{\partial y}, \quad (1b)$$

$$\frac{\partial f(x, y, z)}{\partial z} = \frac{\partial}{\partial z} \frac{f(tx, ty, tz)}{t^k} = \frac{1}{t^k} \frac{\partial f(tx, ty, tz)}{\partial z} t = \frac{1}{t^{k-1}} \frac{\partial f(tx, ty, tz)}{\partial z}, \quad (1c)$$

方程 $f(tx, ty, tz) = t^k f(x, y, z)$ 两边分别对 t 求导

$$x \frac{\partial f(tx, ty, tz)}{\partial x} + y \frac{\partial f(tx, ty, tz)}{\partial y} + z \frac{\partial f(tx, ty, tz)}{\partial z} = k t^{k-1} f(x, y, z),$$

将式 (1a)-(1c) 代入上式

$$xt^{k-1}\frac{\partial f(x,y,z)}{\partial x} + yt^{k-1}\frac{\partial f(x,y,z)}{\partial y} + zt^{k-1}\frac{\partial f(x,y,z)}{\partial z} = kt^{k-1}f(x,y,z),$$

即

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = kf(x,y,z).$$

4. 设 F 为三元可微函数, $u = u(x, y, z)$ 是由方程 $F(u^2 - x^2, u^2 - y^2, u^2 - z^2) = 0$ 确定的隐函数. 求证

$$\frac{u'_x}{x} + \frac{u'_y}{y} + \frac{u'_z}{z} = \frac{1}{u}.$$

证明: 方程 $F(u^2 - x^2, u^2 - y^2, u^2 - z^2) = 0$ 两边分别对 x 求偏导

$$F'_1(2u\frac{\partial u}{\partial x} - 2x) + F'_2 2u\frac{\partial u}{\partial x} + F'_3 2u\frac{\partial u}{\partial x} = 0$$

得

$$\frac{\partial u}{\partial x} = \frac{x F'_1}{u(F'_1 + F'_2 + F'_3)}$$

同理

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{y F'_2}{u(F'_1 + F'_2 + F'_3)}, \\ \frac{\partial u}{\partial z} &= \frac{z F'_3}{u(F'_1 + F'_2 + F'_3)},\end{aligned}$$

\therefore

$$\frac{u'_x}{x} + \frac{u'_y}{y} + \frac{u'_z}{z} = \frac{F'_1 + F'_2 + F'_3}{u(F'_1 + F'_2 + F'_3)} = \frac{1}{u}.$$

5. 求方程 $\frac{\partial^2 z}{\partial x \partial y} = x + y$ 满足条件 $z(x, 0) = x, z(0, y) = y^2$ 的解 $z(x, y)$.

解: 方法1: $\because \frac{\partial^2 z}{\partial x \partial y} = x + y,$

$$\therefore \frac{\partial z}{\partial y} = \int_0^x (x + y) dx + \varphi_0(y) = \frac{x^2}{2} + xy + \varphi_0(y),$$

$$\because z(0, y) = y^2,$$

$$\therefore \frac{z(0, y)}{\partial y} = 2y = \varphi_0(y),$$

$$\therefore z(x, y) = \int_0^y [\frac{x^2}{2} + xy + 2y] dy + \psi(x) = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + y^2 + \psi(x),$$

$$\because z(x, 0) = x = \psi(x),$$

$$\therefore z(x, y) = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + y^2 + x.$$

方法2: $\because \frac{\partial^2 z}{\partial x \partial y} = x + y,$

$$\therefore \int \frac{\partial^2 z}{\partial x \partial y} dx = \int (x + y) dx = \frac{1}{2}x^2 + xy + C_1(y) = \frac{\partial z}{\partial y} + C_1(y),$$

\therefore 可设 $\frac{\partial z}{\partial y} = \frac{1}{2}x^2 + xy + C(y)$, 其中 $C(y)$ 是与 x 无关的 y 的函数,

$\therefore z(0, y) = y^2$,

$\therefore \frac{\partial z(0, y)}{\partial y} = 2y = C(y)$,

$\therefore \frac{\partial z}{\partial y} = \frac{1}{2}x^2 + xy + 2y$,

$\therefore \int \frac{\partial z}{\partial y} dy = \int (\frac{1}{2}x^2 + xy + 2y) dy = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + y^2 + C_3(x) = z(x, y) + C_4(x)$,

\therefore 可设 $z(x, y) = \int (\frac{1}{2}x^2 + xy + 2y) dy + C^*(x) = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + y^2 + C^*(x)$, 其中 $C^*(x)$ 是与 y 无关的 x 的函数,

$\therefore z(x, 0) = x = C^*(x)$,

$\therefore z(x, y) = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + y^2 + x$.

方法3: $\therefore \frac{\partial^2 z}{\partial x \partial y} = x + y$,

$\therefore \int \frac{\partial^2 z}{\partial x \partial y} dx = \int (x + y) dx = \frac{1}{2}x^2 + xy + C_1(y) = \frac{\partial z}{\partial y} + C_2(y)$,

\therefore 可设 $\frac{\partial z}{\partial y} = \frac{1}{2}x^2 + xy + C(y)$, 其中 $C(y)$ 是与 x 无关的 y 的函数,

$\therefore \int \frac{\partial z}{\partial y} dy = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + \int C(y) dy = z(x, y) + C_3(x)$,

\therefore 可设 $z(x, y) = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + F(y) + C^*(x)$, 其中 $F(y)$ 是 $C(y)$ 的一个与 x 无关的原函数, $C^*(x)$ 是与 y 无关的 x 的函数,

$\therefore z(0, y) = y^2, z(x, 0) = x$,

$\therefore F(y) + C^*(0) = y^2, F(0) + C^*(x) = x$,

$\therefore F(y) = y^2 - C^*(0), C^*(x) = x - F(0)$, 且 $F(0) + C^*(0) = 0$,

$\therefore z(x, y) = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + y^2 + x - [F(0) + C^*(0)] = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + y^2 + x$.

6. 设 $z = f(x, y)$ 处处可微, a, b 不全等于零. 求证满足方程 $bz'_x = az'_y$ 的充分条件是存在一元函数 $g(u)$, 使得 $z = f(x, y) = g(ax + by)$.

证明: \therefore 存在一元函数 $g(u)$, 使得 $z = f(x, y) = g(ax + by)$,

\therefore 对于任意常数 C , z 在直线 $ax + by = C$ 上恒等于常数,

$\therefore z = f(x, y)$ 在直线 $ax + by = C$ 上任意一点处沿该直线方向的方向导数均等于零,

由 a, b 不全为零知直线 $ax + by = C$ 的方向向量可表示为 $(-b, a)$,

又 $\therefore z = f(x, y)$ 处处可微,

$\therefore z = f(x, y)$ 在直线 $ax + by = C$ 上的每一点处沿 $(-b, a)$ 方向的方向导数

$$\frac{\partial z}{\partial l} = \text{grad} z \cdot \frac{1}{a^2 + b^2}(-b, a) = (z'_x, z'_y) \cdot \frac{1}{a^2 + b^2}(-b, a) = \frac{1}{\sqrt{a^2 + b^2}}(-bz'_x + az'_y) = 0,$$

\therefore 在直线 $ax + by = C$ 上的每一点处 $bz'_x = az'_y$, 由 C 的任意性知 $bz'_x = az'_y$ 处处成立.

7. 设 D 为包含原点 $O(0,0)$ 的一个圆域. $f(x,y)$ 在 D 中处处有连续偏导数, 并且满足 $xf'_x + yf'_y = 0$. 求证 $f(x,y)$ 在 D 中恒等于某个常数.

证明: $\because f(x,y)$ 在 D 中处处有连续偏导数,

$\therefore f(x,y)$ 在 D 中处处可微,

$\therefore f(x,y)$ 在点 $(x,y) \in D((x,y) \neq (0,0))$ 处由原点 $(0,0)$ 指向点 (x,y) 方向的方向导数

$$\frac{\partial f(x,y)}{\partial \mathbf{v}} = \text{grad} f(x,y) \cdot \frac{1}{\sqrt{x^2+y^2}}(x,y) = \frac{1}{\sqrt{x^2+y^2}}(f'_x, f'_y) \cdot (x,y) = \frac{xf'_x + yf'_y}{\sqrt{x^2+y^2}} = 0,$$

$\therefore f(x,y)$ 在 D 中由原点出发且不含原点的每一条射线上任一点处沿该射线方向的方向导数均为0,

$\therefore f(x,y)$ 在 D 中由原点出发且不含原点的每一条射线上均为常数,

$\therefore f(x,y)$ 在 D 中处处可微, 故处处连续, 故在原点 $(0,0)$ 处连续,

$\therefore f(x,y)$ 在 D 中由原点出发的每一条射线上均等于 $f(0,0)$,

$\therefore f(x,y) = f(0,0), (x,y) \in D$.

【注意:】 如果区域 D 不包含原点, 但仍有 $f(x,y)$ 在 D 中处处有连续偏导数, 并且满足 $xf'_x + yf'_y = 0$, 则 $f(x,y)$ 在 D 中不一定恒等于常数, 比如函数

$$f(x,y) = \begin{cases} \cos(\arccos \frac{x}{\sqrt{x^2+y^2}}), & y \geq 0, \\ \cos(2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}}), & y < 0. \end{cases}$$

该函数在极坐标下的方程是 $f(r \cos \theta, r \sin \theta) = \cos \theta$. 函数 $f(x,y)$ 在 D 中处处有连续偏导数, 且在由原点出发的每一条射线 $\theta = C$ 上均为常数 $\cos C$, 故满足 $xf'_x + yf'_y = 0$, 但 $f(x,y) \neq \text{const}$.

函数 $f(x,y), (x,y) \in \{(x,y) \mid 0 < x^2 + y^2 \leq 1\}$ 的图形如图 3所示.

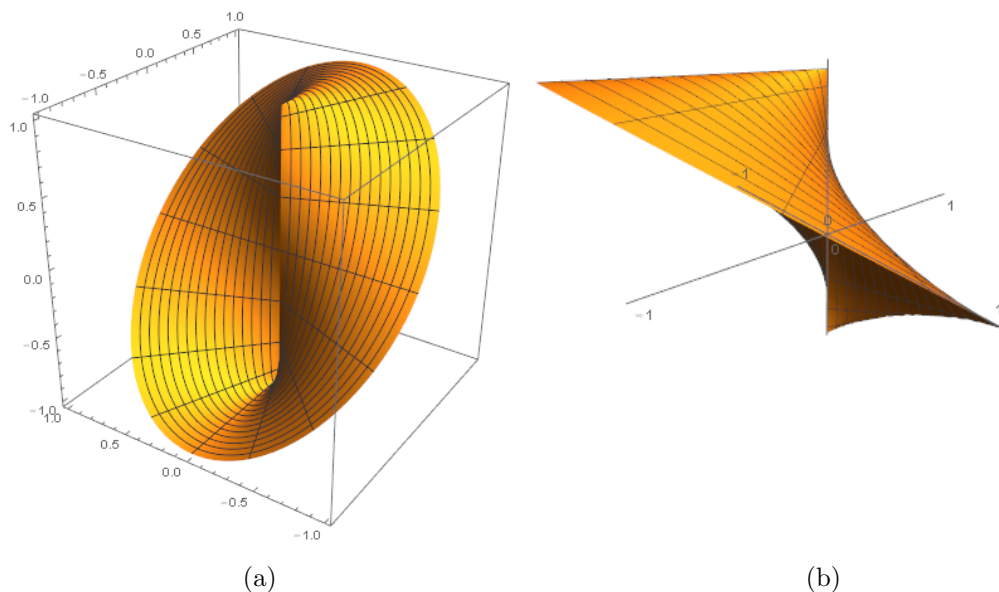


图 3: 函数 $f(r \cos \theta, r \sin \theta) = \cos \theta, (r, \theta) \in \{(r, \theta) \mid 0 < r \leq 1, 0 \leq \theta \leq 2\pi\}$ 的图形

3.8 习题11.1解答

1. 设 $\mathbf{u}(t), \mathbf{v}(t)$ 是可导的向量值函数, $\lambda(t)$ 为可导数值函数, 求证:

$$(1) \frac{d}{dt}(\lambda(t)\mathbf{u}(t)) = \frac{d\lambda(t)}{dt}\mathbf{u}(t) + \lambda(t)\frac{d}{dt}\mathbf{u}(t);$$

$$(2) \frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \left(\frac{d}{dt}\mathbf{u}(t)\right) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \left(\frac{d}{dt}\mathbf{v}(t)\right).$$

证明: (1) 设 $\mathbf{u}(t) = (u_1(t), u_2(t), u_3(t))$, 则 $\lambda(t)\mathbf{u}(t) = (\lambda(t)u_1(t), \lambda(t)u_2(t), \lambda(t)u_3(t))$,

$$\begin{aligned} \frac{d}{dt}(\lambda(t)\mathbf{u}(t)) &= (d[\lambda(t)u_1(t)]', [\lambda(t)u_2(t)]', [\lambda(t)u_3(t)]') \\ &= (\lambda'(t)u_1(t) + \lambda(t)u_1'(t), \lambda'(t)u_2(t) + \lambda(t)u_2'(t), \lambda'(t)u_3(t) + \lambda(t)u_3'(t)) \\ &= (\lambda'(t)u_1(t), \lambda'(t)u_2(t), \lambda'(t)u_3(t)) + (\lambda(t)u_1'(t), \lambda(t)u_2'(t), \lambda(t)u_3'(t)) \\ &= \lambda'(t)(u_1(t), u_2(t), u_3(t)) + \lambda(t)(u_1'(t), u_2'(t), u_3'(t)) \\ &= \frac{d\lambda(t)}{dt}\mathbf{u}(t) + \lambda(t)\frac{d}{dt}\mathbf{u}(t). \end{aligned}$$

(2) 设 $\mathbf{u}(t) = (u_1(t), u_2(t), u_3(t))$, $\mathbf{v}(t) = (v_1(t), v_2(t), v_3(t))$,

则 $\mathbf{u}(t) \cdot \mathbf{v}(t) = u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)$,

$$\begin{aligned} \frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) &= \frac{d}{dt}[u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)] \\ &= u_1'(t)v_1(t) + u_1(t)v_1'(t) + u_2'(t)v_2(t) + u_2(t)v_2'(t) + u_3'(t)v_3(t) + u_3(t)v_3'(t) \\ &= u_1'(t)v_1(t) + u_2'(t)v_2(t) + u_3'(t)v_3(t) + u_1(t)v_1'(t) + u_2(t)v_2'(t) + u_3(t)v_3'(t) \\ &= \left(\frac{d}{dt}\mathbf{u}(t)\right) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \left(\frac{d}{dt}\mathbf{v}(t)\right). \end{aligned}$$

2. 求下列曲线在指定点的单位切向量:

(1) $\mathbf{r}(t) = (e^{2t}, e^{-2t}, te^{2t}), t = 0$;

(2) $\mathbf{r}(t) = t\mathbf{i} + 2\sin t\mathbf{j} + 3\cos t\mathbf{k}, t = \frac{\pi}{6}$.

解: (1) 单位切向量 $\mathbf{t} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{(2e^{2t}, -2e^{-2t}, (1+2t)e^{2t})}{\|(2e^{2t}, -2e^{-2t}, (1+2t)e^{2t})\|} \Big|_{t=0} = \frac{(2, -2, 1)}{\sqrt{4+4+1}} = (\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})$.

(2) 单位切向量 $\mathbf{t} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{i} + 2\cos t\mathbf{j} - 3\sin t\mathbf{k}}{\|\mathbf{i} + 2\cos t\mathbf{j} - 3\sin t\mathbf{k}\|} \Big|_{t=\frac{\pi}{6}} = \frac{\mathbf{i} + \sqrt{3}\mathbf{j} - \frac{3}{2}\mathbf{k}}{\sqrt{1+3+\frac{9}{4}}} = \frac{2}{5}\mathbf{i} + \frac{2\sqrt{3}}{5}\mathbf{j} - \frac{3}{5}\mathbf{k}$.

3. 求下列曲线在指定点的切线方程:

(1) $\mathbf{r}(t) = (1+2t, 1+t-t^2, 1-t+t^2-t^3), M(1, 1, 1)$;

(2) $\mathbf{r}(t) = \sin(\pi t)\mathbf{i} + \sqrt{t}\mathbf{j} + \cos(\pi t)\mathbf{k}, M(0, 1, -1)$.

解: (1) 在 $M(1, 1, 1)$ 点处 $t = 0$, 切向量 $\mathbf{t} = \mathbf{r}'(0) = (2, 1-2t, -1+2t-3t^2)_{t=0} = (2, 1, -1)$, 则切线方程为

$$\frac{x-1}{2} = y-1 = -(z-1).$$

(2) 在 $M(0, 1, -1)$ 点处 $t = 1$, 切向量 $\mathbf{t} = \mathbf{r}'(1) = \pi \cos(\pi t)\mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j} - \pi \sin(\pi t)\mathbf{k} \Big|_{t=1} = -\pi\mathbf{i} + \frac{1}{2}\mathbf{j}$, 则切线方程为
$$\begin{cases} \frac{x}{-\pi} = 2(y-1), \\ z = -1, \end{cases} \quad \text{即} \begin{cases} x + 2\pi y = 2\pi, \\ z = -1. \end{cases}$$

4. 求下列向量值函数的积分:

(1) $\int_0^{\frac{\pi}{4}} [\cos(2t)\mathbf{i} + \sin(2t)\mathbf{j} + t \sin t\mathbf{k}] dt$;

(2) $\int_1^4 (\sqrt{t}\mathbf{i} + te^{-t}\mathbf{j} + \frac{1}{t^2}\mathbf{k}) dt$.

解: (1) $\int_0^{\frac{\pi}{4}} [\cos(2t)\mathbf{i} + \sin(2t)\mathbf{j} + t \sin t\mathbf{k}] dt = \int_0^{\frac{\pi}{4}} \cos(2t) dt \mathbf{i} + \int_0^{\frac{\pi}{4}} \sin(2t) dt \mathbf{j} + \int_0^{\frac{\pi}{4}} t \sin t dt \mathbf{k}$,

$\because \int_0^{\frac{\pi}{4}} \cos(2t) dt = \frac{1}{2} \sin(2t) \Big|_0^{\frac{\pi}{4}} = \frac{1}{2}, \int_0^{\frac{\pi}{4}} \sin(2t) dt = -\frac{1}{2} \cos(2t) \Big|_0^{\frac{\pi}{4}} = \frac{1}{2},$

$\int_0^{\frac{\pi}{4}} t \sin t dt = -\int_0^{\frac{\pi}{4}} t d \cos t = -t \cos t \Big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \cos t dt = -\frac{\sqrt{2}\pi}{8} + \sin t \Big|_0^{\frac{\pi}{4}} = -\frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2},$

$\therefore \int_0^{\frac{\pi}{4}} [\cos(2t)\mathbf{i} + \sin(2t)\mathbf{j} + t \sin t\mathbf{k}] dt = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + (-\frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2})\mathbf{k}.$

(2) $\int_1^4 (\sqrt{t}\mathbf{i} + te^{-t}\mathbf{j} + \frac{1}{t^2}\mathbf{k}) dt = \int_1^4 \sqrt{t} dt \mathbf{i} + \int_1^4 te^{-t} dt \mathbf{j} + \int_1^4 \frac{1}{t^2} dt \mathbf{k},$

$\because \int_1^4 \sqrt{t} dt = \frac{1}{1+\frac{1}{2}} t^{\frac{1}{2}+1} \Big|_1^4 = \frac{14}{3}, \int_1^4 te^{-t} dt = -\int_1^4 t de^{-t} = -te^{-t} \Big|_1^4 + \int_1^4 e^{-t} dt$

$= -4e^{-4} + e^{-1} - e^{-t} \Big|_1^4 = -4e^{-4} + e^{-1} - e^{-4} + e^{-1} = -5e^{-4} + 2e^{-1},$

$\int_1^4 \frac{1}{t^2} dt = \frac{1}{-2+1} t^{-2+1} \Big|_1^4 = -\frac{1}{4} + 1 = \frac{3}{4},$

$\therefore \int_1^4 (\sqrt{t}\mathbf{i} + te^{-t}\mathbf{j} + \frac{1}{t^2}\mathbf{k}) dt = \frac{14}{3}\mathbf{i} + (-5e^{-4} + 2e^{-1})\mathbf{j} + \frac{3}{4}\mathbf{k}.$

5. 已知 $\mathbf{r}'(t), \mathbf{r}(0)$, 求 $\mathbf{r}(t)$:

(1) $\mathbf{r}'(t) = (t^2, 4t^3, -t^2), \mathbf{r}(0) = (0, 1, 0)$;

(2) $\mathbf{r}'(t) = \sin t\mathbf{i} - \cos t\mathbf{j} + 2t\mathbf{k}, \mathbf{r}(0) = \mathbf{i} + \mathbf{j} + 2\mathbf{k}.$

解: (1)方法1:

$$\begin{aligned}\mathbf{r}(t) &= \int_0^t \mathbf{r}'(t)dt + \mathbf{C} = \left(\int_0^t t^2 dt + C_1, \int_0^t 4t^3 dt + C_2, \int_0^t (-t^2)dt + C_3 \right) \\ &= \left(\frac{1}{3}t^3 + C_1, t^4 + C_2, -\frac{1}{3}t^3 + C_3 \right),\end{aligned}$$

$$\therefore \mathbf{r}(0) = (C_1, C_2, C_3) = (0, 1, 0),$$

$$\therefore \mathbf{r}(t) = \left(\frac{1}{3}t^3, t^4 + 1, -\frac{1}{3}t^3 \right).$$

方法2: $\because \int t^2 dt = \frac{1}{3}t^3 + C, \int 4t^3 dt = t^4 + C, \int (-t^2)dt = -\frac{1}{3}t^3 + C,$

$$\therefore \mathbf{r}(t) = \int \mathbf{r}'(t)dt = \left(\frac{1}{3}t^3 + C_1, t^4 + C_2, -\frac{1}{3}t^3 + C_3 \right),$$

$$\therefore \mathbf{r}(0) = (0, 1, 0),$$

$$\therefore C_1 = 0, C_2 = 1, C_3 = 0,$$

$$\therefore \mathbf{r}(t) = \left(\frac{1}{3}t^3, t^4 + 1, -\frac{1}{3}t^3 \right).$$

(2)方法1:

$$\begin{aligned}\mathbf{r}(t) &= \int_0^t \mathbf{r}'(t)dt + \mathbf{C} = \left(\int_0^t \sin t dt + C_1 \right) \mathbf{i} + \left(-\int_0^t \cos t dt + C_2 \right) \mathbf{j} + \left(\int_0^t 2t dt + C_3 \right) \mathbf{k} \\ &= (-\cos t + C_1) \mathbf{i} + (-\sin t + C_2) \mathbf{j} + (t^2 + C_3) \mathbf{k},\end{aligned}$$

$$\therefore \mathbf{r}(0) = (-1 + C_1) \mathbf{i} + C_2 \mathbf{j} + C_3 \mathbf{k} = \mathbf{i} + \mathbf{j} + 2\mathbf{k},$$

$$\therefore C_1 = 2, C_2 = 1, C_3 = 2,$$

$$\therefore \mathbf{r}(t) = (-\cos t + 2) \mathbf{i} + (-\sin t + 1) \mathbf{j} + (t^2 + 2) \mathbf{k}.$$

方法2: $\because \int \sin t dt = -\cos t + C, \int (-\cos t)dt = -\sin t + C, \int 2t dt = t^2 + C,$

$$\therefore \mathbf{r}(t) = \int \mathbf{r}'(t)dt = (-\cos t + C_1) \mathbf{i} + (-\sin t + C_2) \mathbf{j} + (t^2 + C_3) \mathbf{k},$$

$$\therefore \mathbf{r}(0) = \mathbf{i} + \mathbf{j} + 2\mathbf{k},$$

$$\therefore C_1 = 2, C_2 = 1, C_3 = 2,$$

$$\therefore \mathbf{r}(t) = (-\cos t + 2) \mathbf{i} + (-\sin t + 1) \mathbf{j} + (t^2 + 2) \mathbf{k}.$$

6. 证明下列等式:

$$(1) \frac{d}{dt}(\mathbf{r}(t) \times \mathbf{r}'(t)) = \mathbf{r}(t) \times \mathbf{r}''(t);$$

$$(2) \frac{d}{dt} \|\mathbf{r}(t)\| = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|} (\mathbf{r}(t) \neq \mathbf{0});$$

$$(3) \frac{d}{dt} [\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))] = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)].$$

证明: (1) $\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{r}'(t)) = \mathbf{r}'(t) \times \mathbf{r}'(t) + \mathbf{r}(t) \times \mathbf{r}''(t) = \mathbf{r}(t) \times \mathbf{r}''(t).$

$$\begin{aligned}(2) \frac{d}{dt} \|\mathbf{r}(t)\| &= \frac{d}{dt} \sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)} = \frac{1}{2\sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)}} \frac{d}{dt} [\mathbf{r}(t) \cdot \mathbf{r}(t)] \\ &= \frac{1}{2\sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)}} [\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t)] = \frac{2\mathbf{r}(t) \cdot \mathbf{r}'(t)}{2\sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)}} = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|} (\mathbf{r}(t) \neq \mathbf{0}).\end{aligned}$$

$$\begin{aligned}
 (3) \frac{d}{dt}[\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))] &= \mathbf{r}'(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t)) + \mathbf{r}(t) \cdot \frac{d}{dt}(\mathbf{r}'(t) \times \mathbf{r}''(t)) \\
 &= \mathbf{r}(t) \cdot \frac{d}{dt}(\mathbf{r}'(t) \times \mathbf{r}''(t)) = \mathbf{r}(t) \cdot (\mathbf{r}''(t) \times \mathbf{r}'''(t) + \mathbf{r}'(t) \times \mathbf{r}''''(t)) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)].
 \end{aligned}$$

7. 求等速圆周运动 $\mathbf{r} = R \cos(\omega t)\mathbf{i} + R \sin(\omega t)\mathbf{j}$ 在 t 时刻的速度与加速度.

解: t 时刻的速度 $\mathbf{v}(t) = \mathbf{r}'(t) = -R\omega \sin(\omega t)\mathbf{i} + R\omega \cos(\omega t)\mathbf{j}$,

t 时刻的加速度 $\mathbf{a}(t) = \mathbf{v}'(t) = -R\omega^2 \cos(\omega t)\mathbf{i} - R\omega^2 \sin(\omega t)\mathbf{j}$.

8. 已知螺旋线的向量方程为 $\mathbf{r} = a \cos \theta \mathbf{i} + a \sin \theta \mathbf{j} + b\theta \mathbf{k}$ ($a > 0, b > 0$), 求在 θ_0 处的切线方程.

解: 在 θ_0 处的切向量 $\mathbf{r}'(\theta_0) = -a \sin \theta_0 \mathbf{i} + a \cos \theta_0 \mathbf{j} + b\mathbf{k}$, 切线方程

$$\frac{x - a \cos \theta_0}{-a \sin \theta_0} = \frac{y - a \sin \theta_0}{a \cos \theta_0} = \frac{z - b\theta_0}{b}.$$

9. 设 $\mathbf{r} = -a \sin \theta \mathbf{i} + a \cos \theta \mathbf{j} + b\theta \mathbf{k}$, 求 $\frac{1}{2} \int_0^{2\pi} (\mathbf{r} \times \mathbf{r}') d\theta$.

解: $\mathbf{r}'(\theta) = -a \cos \theta \mathbf{i} - a \sin \theta \mathbf{j} + b\mathbf{k}$,

$$\begin{aligned}
 \mathbf{r}(\theta) \times \mathbf{r}'(\theta) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin \theta & a \cos \theta & b\theta \\ -a \cos \theta & -a \sin \theta & b \end{vmatrix} = \begin{vmatrix} a \cos \theta & b\theta \\ -a \sin \theta & b \end{vmatrix} \mathbf{i} + \begin{vmatrix} b\theta & -a \sin \theta \\ b & -a \cos \theta \end{vmatrix} \mathbf{j} + \begin{vmatrix} -a \sin \theta & a \cos \theta \\ -a \cos \theta & -a \sin \theta \end{vmatrix} \mathbf{k} \\
 &= (ab \cos \theta + ab\theta \sin \theta) \mathbf{i} - (-ab \sin \theta + ab\theta \cos \theta) \mathbf{j} + a^2 \mathbf{k},
 \end{aligned}$$

$$\therefore \int_0^{2\pi} (ab \cos \theta + ab\theta \sin \theta) d\theta = ab \sin \theta \Big|_0^{2\pi} - \int_0^{2\pi} ab\theta d \cos \theta$$

$$= -ab\theta \cos \theta \Big|_0^{2\pi} + \int_0^{2\pi} ab \cos \theta d\theta = -2\pi ab + ab \sin \theta \Big|_0^{2\pi} = -2\pi ab,$$

$$\int_0^{2\pi} -(-ab \sin \theta + ab\theta \cos \theta) d\theta = \int_0^{2\pi} (ab \sin \theta - ab\theta \cos \theta) d\theta = -ab \cos \theta \Big|_0^{2\pi} - ab \int_0^{2\pi} \theta d \sin \theta$$

$$= -ab\theta \sin \theta \Big|_0^{2\pi} + ab \int_0^{2\pi} \sin \theta d\theta = -ab \cos \theta \Big|_0^{2\pi} = 0,$$

$$\int_0^{2\pi} a^2 d\theta = 2\pi a^2,$$

$$\therefore \frac{1}{2} \int_0^{2\pi} (\mathbf{r} \times \mathbf{r}') d\theta = -\pi ab \mathbf{i} + \pi a^2 \mathbf{k}.$$