22 第二型曲线积分、格林公式

22.1 知识结构

第13章向量场的微积分

13.2 第二型曲线积分

13.2.1 有向曲线

13.2.2 向量场在有向曲线的积分概念

13.2.3 第二型曲线积分的计算

13.3 格林公式

22.2 习题13.2解答

1. 计算$\int_L (x+y)\mathrm{d}x-(x-y)\mathrm{d}y$，其中$L$为椭圆$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$的上半周，逆时针方向为正.

解：$\int_L (x+y)\mathrm{d}x-(x-y)\mathrm{d}y=\int_0^\pi (a\cos\theta+b\sin\theta)\mathrm{d}(a\cos\theta)-(a\cos\theta-b\sin\theta)\mathrm{d}(b\sin\theta)$
$=\int_0^\pi (-a^2\sin\theta\cos\theta-ab\sin^2\theta-ab\cos^2\theta+b^2\sin\theta\cos\theta)\mathrm{d}\theta$
$=(b^2-a^2)\int_0^\pi \sin\theta\cos\theta\mathrm{d}\theta-ab\int_0^\pi \mathrm{d}\theta=0-\pi ab=-\pi ab.$

2. 计算$\int_L y^2\mathrm{d}x-x^2\mathrm{d}y$，其中$L$为抛物线$y=x^2$自$x=-1$到$x=1$的一段.

解：$\int_L y^2\mathrm{d}x-x^2\mathrm{d}y=\int_{-1}^1 (x^2)^2\mathrm{d}x-x^2\mathrm{d}(x^2)=\int_{-1}^1 x^4\mathrm{d}x-2x^3\mathrm{d}x=\int_{-1}^1 x^4-2x^3\mathrm{d}\mathrm{d}x$
$=(\frac{1}{5}x^5-\frac{2}{4}x^4)\big|_{-1}^1=\frac{2}{5}.$

3. 计算$\oint_L \frac{x\mathrm{d}y-y\mathrm{d}x}{x^2+y^2}$，其中$L$为圆周$x^2+y^2=a^2$(逆时针方向为正).

解：$\oint_L \frac{x\mathrm{d}y-y\mathrm{d}x}{x^2+y^2}=\frac{1}{a^2}\oint_L x\mathrm{d}y-y\mathrm{d}x=\frac{1}{a^2}\int_0^{2\pi} a\cos\theta\mathrm{d}(a\sin\theta)-a\sin\theta\mathrm{d}(a\cos\theta)$
$=\frac{1}{a^2}\int_0^{2\pi} a\cos\theta(a\cos\theta)\mathrm{d}\theta-a\sin\theta(-a\sin\theta)\mathrm{d}\theta=\frac{1}{a^2}\int_0^{2\pi}(a^2\cos^2\theta+a^2\sin\theta)\mathrm{d}\theta=2\pi.$

4. 计算$\oint_L \frac{y\mathrm{d}y-x\mathrm{d}x}{x^2+y^2}$，其中$L$为圆周$x^2+y^2=a^2$(逆时针方向为正).

解：$\oint_L \frac{y\mathrm{d}y-x\mathrm{d}x}{x^2+y^2}=\frac{1}{a^2}\oint_L y\mathrm{d}y-x\mathrm{d}x=\frac{1}{a^2}\int_0^{2\pi} a\sin\theta\mathrm{d}(a\sin\theta)-a\cos\theta\mathrm{d}(a\cos\theta)$
$=\frac{1}{a^2}\int_0^{2\pi} a\sin\theta(a\cos\theta)\mathrm{d}\theta-a\cos\theta(-a\sin\theta)\mathrm{d}\theta=\frac{1}{a^2}\int_0^{2\pi}(a^2\sin\theta\cos\theta+a^2\sin\theta\cos\theta)\mathrm{d}\theta$
$=\int_0^{2\pi}\sin\theta\cos\theta\mathrm{d}\theta=\int_0^{2\pi}\sin\theta\mathrm{d}\sin\theta=\frac{1}{2}\sin^2\theta\big|_0^{2\pi}=0.$

5. 计算$\int_L \frac{\mathrm{d}x+\mathrm{d}y}{|x|+|y|}$，其中$L$为由$A(0,-1)$到$B(1,0)$再到$C(0,1)$的折线.

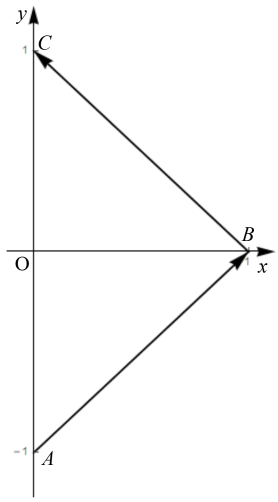

图 1: 习题13.2 5.题图示

解：AB段的方程为$\frac{x}{1}+\frac{y}{-1}=1$，即$y=x-1, x:0\to 1$，$BC$段的方程为$\frac{x}{1}+\frac{y}{1}=1$，即$y=1-x, x:1\to 0,$

易知折线ABC满足方程$|x|+|y|=1,$

$\therefore \int_L \frac{\mathrm{d}x+\mathrm{d}y}{|x|+|y|}=\int_L \mathrm{d}x+\mathrm{d}y=(\int_{AB}+\int_{BC})(\mathrm{d}x+\mathrm{d}y)=\int_{AB}\mathrm{d}x+\mathrm{d}y+\int_{BC}\mathrm{d}x+\mathrm{d}y$
$=\int_0^1 \mathrm{d}x+d(x-1)+\int_1^0 \mathrm{d}x+\mathrm{d}(1-x)=\int_0^1 \mathrm{d}x+\mathrm{d}x+\int_1^0 \mathrm{d}x-\mathrm{d}x=2\int_0^1 \mathrm{d}x+0=2.$

6. 计算$\int_L (x^2+y^2)\mathrm{d}x+(x^2-y^2)\mathrm{d}y$，其中$L$为$y=1-|1-x|, x\in[0,2]$，曲线正向为$x$增长的方向.

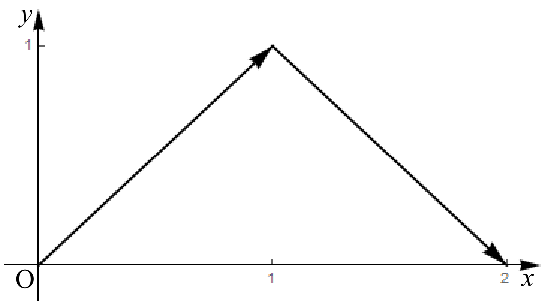

图 2: 习题13.2 6.题图示

解：$y=1-|1-x|=\begin{cases} x, & 0\leqslant x<1, \\ 2-x, & 1\leqslant x\leqslant 2, \end{cases}$

$\therefore \int_L (x^2+y^2)\mathrm{d}x+(x^2-y^2)\mathrm{d}y$
$= \int_0^1 (x^2+x^2)\mathrm{d}x+(x^2-x^2)\mathrm{d}x+\int_1^2 [x^2+(2-x)^2]\mathrm{d}x+[x^2-(2-x)^2]\mathrm{d}(2-x)$
$= \int_0^1 2x^2\mathrm{d}x+\int_1^2 [x^2+4-4x+x^2]\mathrm{d}x-[x^2-4+4x-x^2]\mathrm{d}x$
$= \frac{2}{3}x^3\Big|_0^1+\int_1^2 [x^2+4-4x+x^2-x^2+4-4x+x^2]\mathrm{d}x$
$= \frac{2}{3}+\int_1^2 [2x^2-8x+8]\mathrm{d}x$
$= \frac{2}{3}+(\frac{2}{3}x^3-4x^2+8x)\Big|_1^2=\frac{2}{3}+\frac{2}{3}\cdot 7-4\cdot 3+8=\frac{4}{3}.$

7. 计算$\oint_L xyz\mathrm{d}z$，其中L为$x^2+y^2+z^2=1, z=y$，由z轴正向看去为逆时针方向.

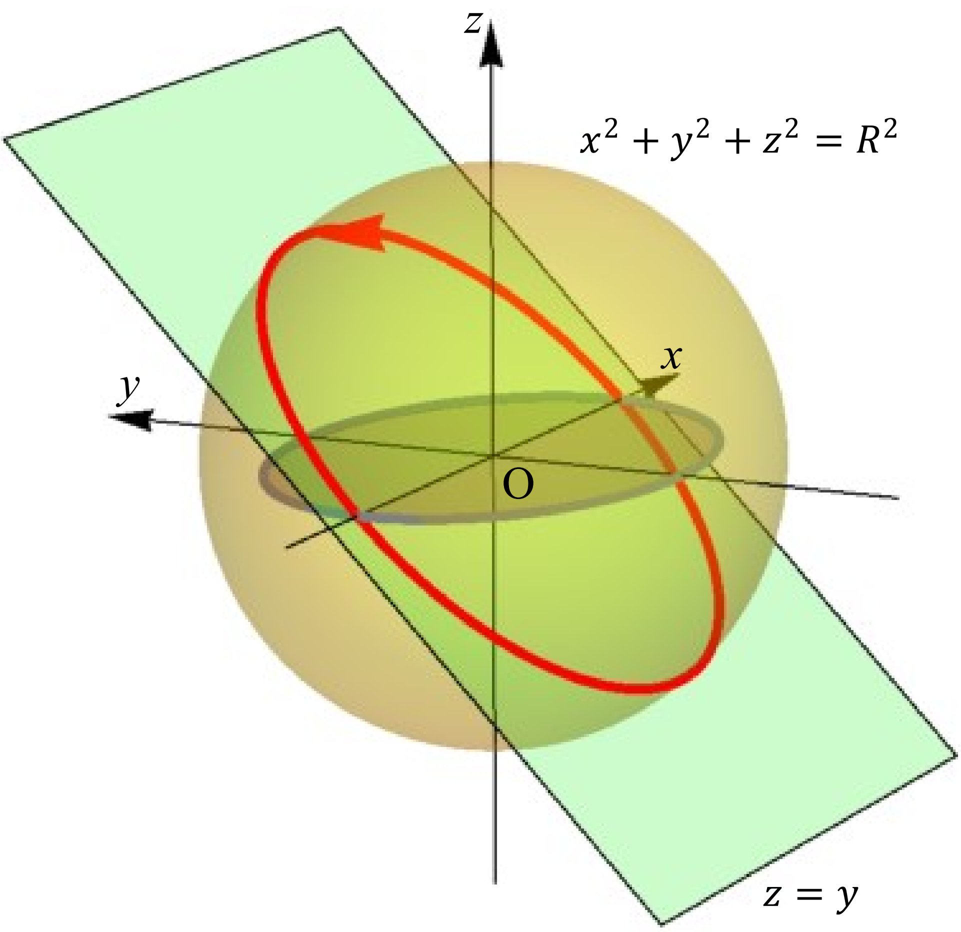

图 3: 习题13.2 7.题图示

解：由$\begin{cases} x^2+y^2+z^2=1, \\ z=y, \end{cases}$ 得L所在的投影柱面为$x^2+2y^2=1$，故L的方程可表示为$\begin{cases} x^2+2y^2=1, \\ z=y, \end{cases}$ 故可令$L:\begin{cases} x=\cos\theta, \\ y=\frac{1}{\sqrt{2}}\sin\theta, \quad \theta: 0\to 2\pi, \\ z=\frac{1}{\sqrt{2}}\sin\theta, \end{cases}$

$\therefore \oint_L xyz\mathrm{d}z = \int_0^{2\pi}\cos\theta\frac{1}{\sqrt{2}}\sin\theta\cdot\frac{1}{\sqrt{2}}\sin\theta\mathrm{d}\frac{1}{\sqrt{2}}\sin\theta = \int_0^{2\pi}\cos\theta\frac{1}{\sqrt{2}}\sin\theta\cdot\frac{1}{\sqrt{2}}\sin\theta\cdot\frac{1}{\sqrt{2}}\cos\theta\mathrm{d}\theta$
$= \frac{1}{2\sqrt{2}}\int_0^{2\pi}\sin^2\theta\cos^2\theta\mathrm{d}\theta = \frac{1}{2\sqrt{2}}\cdot 4\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\mathrm{d}\theta = \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\sin^2 2\theta\mathrm{d}\theta$
$= \frac{1}{2\sqrt{2}}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 2\theta\mathrm{d}(2\theta) = \frac{1}{2\sqrt{2}}\cdot\frac{1}{2}\int_0^{\pi}\sin^2\varphi\mathrm{d}\varphi = \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\sin^2\varphi\mathrm{d}\varphi = \frac{1}{2\sqrt{2}}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{8\sqrt{2}}.$

8. 计算$\int_L \frac{\mathrm{d}x+\mathrm{d}y}{|x|+|y|}$，其中$L$为$|x|+|y|=1$，逆时针方向为正.

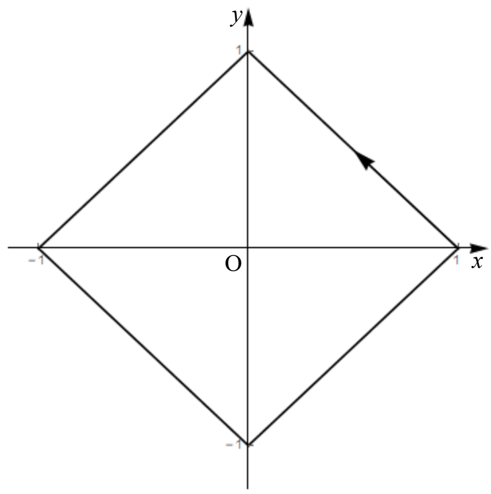

图 4: 习题13.2 8.题图示

解：L的方程$|x|+|y|=1$等价于$y=\begin{cases} 1-x, & 0\leqslant x<1, \\ 1+x, & -1\leqslant x<0, \\ -1-x, & -1<x\leqslant 0, \\ x-1, & 0<x\leqslant 1, \end{cases}$

$\therefore \int_L \frac{\mathrm{d}x+\mathrm{d}y}{|x|+|y|} = \int_L \mathrm{d}x+\mathrm{d}y$
$= \int_0^1 \mathrm{d}x+\mathrm{d}(1-x) + \int_0^{-1}\mathrm{d}x+\mathrm{d}(1+x) + \int_{-1}^0 \mathrm{d}x+\mathrm{d}(-1-x) + \int_0^1 \mathrm{d}x+\mathrm{d}(x-1)$
$= \int_0^1 \mathrm{d}x-\mathrm{d}x + \int_0^{-1}\mathrm{d}x+\mathrm{d}x + \int_{-1}^0\mathrm{d}x-\mathrm{d}x + \int_0^1\mathrm{d}x+\mathrm{d}x$
$= 0+2\int_0^{-1}\mathrm{d}x + 0 + 2\int_0^1\mathrm{d}x = -2+2=0.$

9. 计算$\oint_L (y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z$，其中$L$为$x^2+y^2+z^2=1$在第一卦限与三个坐标面的交线，方向是$A(1,0,0)\to B(0,1,0)\to C(0,0,1)\to A(1,0,0)$.

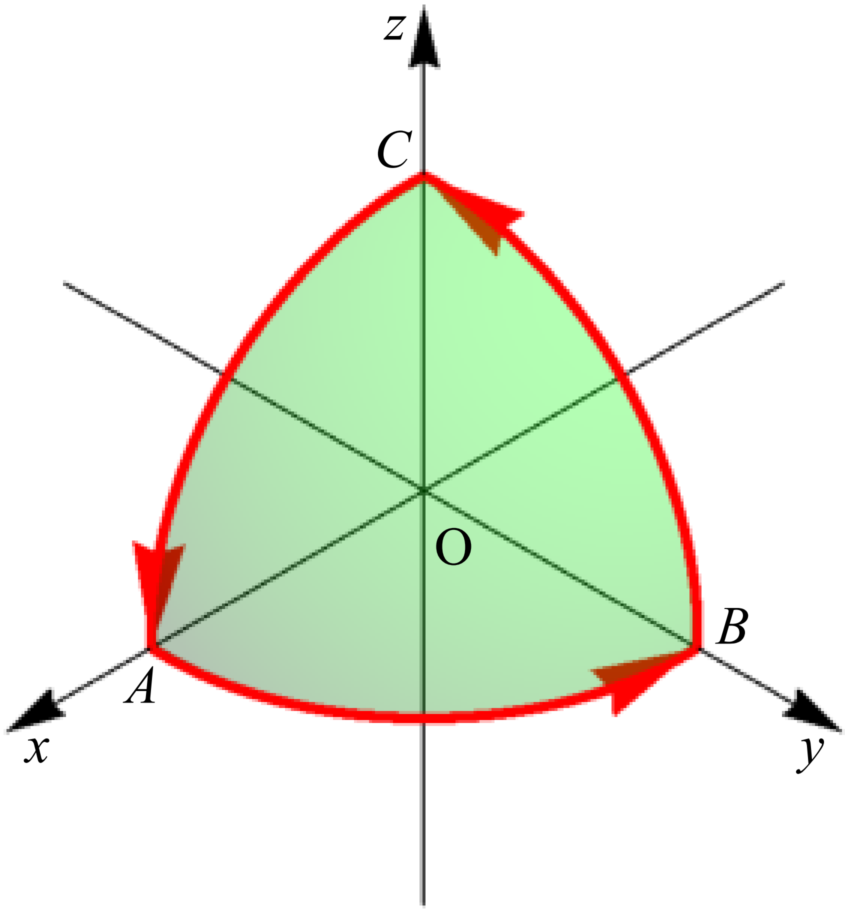

图 5: 习题13.2 9.题图示

解：$\oint_L (y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z$

$=(\int_{\widehat{AB}}+\int_{\widehat{BC}}+\int_{\widehat{CA}})[(y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z]$

$=\int_{\widehat{AB}}(y^2-0^2)\mathrm{d}x+(0^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}0$

$+\int_{\widehat{BC}}(y^2-z^2)\mathrm{d}0+(z^2-0^2)\mathrm{d}y+(0^2+y^2)\mathrm{d}z$

$+\int_{\widehat{CA}}(0^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}0+(x^2-0^2)\mathrm{d}z$

$$= \int_{\overset{\frown}{AB}} y^2\mathrm{d}x - x^2\mathrm{d}y + \int_{\overset{\frown}{BC}} z^2\mathrm{d}y - y^2\mathrm{d}z + \int_{\overset{\frown}{CA}} x^2\mathrm{d}z - z^2\mathrm{d}x$$

$$\Bigg(= \int\limits_{\substack{x^2+y^2=1,z=0\\x\geqslant0,y\geqslant0}} y^2\mathrm{d}x - x^2\mathrm{d}y + \int\limits_{\substack{y^2+z^2=1,x=0\\y\geqslant0,z\geqslant0}} z^2\mathrm{d}y - y^2\mathrm{d}z + \int\limits_{\substack{z^2+x^2=1,y=0\\z\geqslant0,x\geqslant0}} x^2\mathrm{d}z - z^2\mathrm{d}x$$

$$= {}^{1}3 \int\limits_{\substack{x^2+y^2=1,z=0\\x\geqslant0,y\geqslant0}} y^2\mathrm{d}x - x^2\mathrm{d}y \Bigg)$$

$$= 3\int_{\overset{\frown}{AB}} y^2\mathrm{d}x - x^2\mathrm{d}y$$

$$= 3\int_0^{\frac{\pi}{2}} \sin^2\theta\mathrm{d}\cos\theta - \cos^2\theta\mathrm{d}\sin\theta = 3\int_0^{\frac{\pi}{2}} -\sin^2\theta\sin\theta\mathrm{d}\theta - \cos^2\theta\cos\theta\mathrm{d}\theta$$

$$= -3\int_0^{\frac{\pi}{2}} \sin^3\theta\mathrm{d}\theta - 3\int_0^{\frac{\pi}{2}} \cos^3\theta\mathrm{d}\theta = -3\cdot\frac{2}{3} - 3\cdot\frac{2}{3} = -4.$$

10. 设平面上的力场$\boldsymbol{F}(x,y)$指向原点，每点处力的大小与该点到原点的距离成正比，比例系数为k. 设一单位质量的质点沿曲线$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$从点$A(a,0)$到$B(0,b)$，求力场$\boldsymbol{F}(x,y)$所做的功.

解：因$\boldsymbol{F}(x,y)$指向原点，故可设$\boldsymbol{F}(x,y)=-\lambda(x,y), \lambda>0$,

$\because$每点处力的大小$|\boldsymbol{F}(x,y)|$与该点到原点的距离成正比，比例系数为k,

$\therefore |\boldsymbol{F}(x,y)| = \lambda\sqrt{x^2+y^2} = k\sqrt{x^2+y^2}$,

$\therefore \lambda = k, \boldsymbol{F}(x,y) = -k(x,y) = (-kx,-ky)$,

[1]这里的依据是轮换对称性，即按照$x\to y, y\to z, z\to x$可将等号左边第一项变换成第二项，因表达式的值与数学符号无关，故第一项和第二项相等. 同理，按照$x\to y, y\to z, z\to x$可将等号左边第三项变成第一项，故第三项和第一项相等. 变换顺序可用图 6表示.

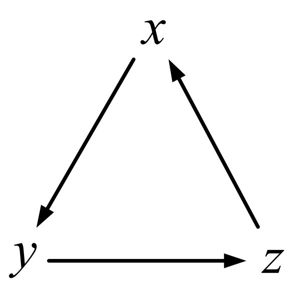

图 6: 轮换对称性的变换顺序

$\therefore$力场$\boldsymbol{F}(x,y)$所做的功

$$\begin{aligned}
W &= \int_{L(A)}^{B} \boldsymbol{F}(x,y)\cdot \mathrm{d}\boldsymbol{l} = \int_{L(A)}^{B} (-kx,-ky)\cdot(\mathrm{d}x,\mathrm{d}y) = \int_{L(A)}^{B} -kx\mathrm{d}x - ky\mathrm{d}y \\
&= \int_0^{\frac{\pi}{2}} -ka\cos\theta\mathrm{d}(a\cos\theta) - kb\sin\theta\mathrm{d}(b\sin\theta) \\
&= \int_0^{\frac{\pi}{2}} -ka\cos\theta(-a\sin\theta)\mathrm{d}\theta - kb\sin\theta(b\cos\theta)\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} (ka^2\sin\theta\cos\theta - kb^2\sin\theta\cos\theta)\mathrm{d}\theta \\
&= k(a^2-b^2)\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\mathrm{d}\theta = k(a^2-b^2)\int_0^{\frac{\pi}{2}} \sin\theta\mathrm{d}\sin\theta \\
&= k(a^2-b^2)\frac{1}{2}\sin^2\theta\Big|_0^{\frac{\pi}{2}} = \frac{1}{2}k(a^2-b^2).
\end{aligned}$$

11. 计算$\int_L y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}x$，其中$L$是螺旋线$x=a\cos t, y=a\sin t, z=bt(0\leqslant t\leqslant 2\pi)$，正向为$t$增加的方向.

解：$\int_L y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z = \int_0^{2\pi} a\sin t\mathrm{d}(a\cos t) + bt\mathrm{d}(a\sin t) + a\cos t\mathrm{d}(bt)$
$= \int_0^{2\pi} a\sin t(-a\sin t)\mathrm{d}t + bta\cos t\mathrm{d}t + ab\cos t\mathrm{d}t$
$= -a^2\int_0^{2\pi}\sin^2 t\mathrm{d}t + ab\int_0^{2\pi} t\cos t\mathrm{d}t + ab\int_0^{2\pi}\cos t\mathrm{d}t$
$= -4a^2\int_0^{\frac{\pi}{2}}\sin^2 t\mathrm{d}t + ab\int_0^{2\pi} t\mathrm{d}\sin t + 0$
$= -4a^2\cdot\frac{1}{2}\cdot\frac{\pi}{2} + ab(t\sin t\big|_0^{2\pi} - \int_0^{2\pi}\sin t\mathrm{d}t) = -\pi a^2 + (0-0) = -\pi a^2.$

22.3 习题13.3解答

1. 用格林公式计算下列积分：

(1)$\oint_L xy^2\mathrm{d}y - x^2y\mathrm{d}x$，其中$L$为圆周$x^2+y^2=a^2$，逆时针方向；
(2)$\oint_L (3x+y)\mathrm{d}y - (x-y)\mathrm{d}x$，其中$L$为圆周$(x-1)^2+(y-4)^2=9$；
(3)$\oint_L (x^2+y^2)\mathrm{d}x + (y^2-x^2)\mathrm{d}y$，其中$L$是区域$D$的边界正向(逆时针)，区域$D$由直线$y=0, x=1, y=x$围成；
(4)$\oint_L (2xy-x^2)\mathrm{d}x + (x+y^2)\mathrm{d}y$，其中$L$是区域$D$的边界正向，区域$D$由曲线$x=y^2, y=x^2$围成；
(5)$\oint_L (x+y)\mathrm{d}x + xy\mathrm{d}y$，其中$L$为椭圆周$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$，正向；
(6)$\oint_L \sqrt{x^2+y^2}\mathrm{d}x + [x + y\ln(x+\sqrt{x^2+y^2})]\mathrm{d}y$，其中$L$为圆周$(x-2)^2+y^2=1$，逆时针方向；
(7)$\oint_L \mathrm{e}^x[(1-\cos y)\mathrm{d}x - (y-\sin y)\mathrm{d}y]$，其中$L$为$D=\{(x,y)\mid 0\leqslant x\leqslant\pi, 0\leqslant y\leqslant\sin x\}$的正向边界.

解：(1)设D为以L为边界的有界闭域，

$\because \frac{\partial(xy^2)}{\partial x} = y^2$，$\frac{\partial(-x^2y)}{\partial y} = -x^2$均在$D$上连续，

$\therefore \oint_L xy^2\mathrm{d}y - x^2y\mathrm{d}x = \iint\limits_D [\frac{\partial(xy^2)}{\partial x} - \frac{\partial(-x^2y)}{\partial y}]\mathrm{d}x\mathrm{d}y = \iint\limits_D (y^2+x^2)\mathrm{d}x\mathrm{d}y = \int_0^{2\pi}\mathrm{d}\theta\int_0^a r^2\cdot r\mathrm{d}r$
$= 2\pi\cdot\frac{1}{4}r^4\Big|_0^a = \frac{\pi}{2}a^4.$

(2)设D为以L为边界的有界闭域，

$\because \frac{\partial(3x+y)}{\partial x} = 3,\ \frac{\partial[-(x-y)]}{\partial y} = 1$均在$D$上连续，

$\therefore \oint_L(3x+y)\mathrm{d}y - (x-y)\mathrm{d}x = \iint\limits_D \{\frac{\partial(3x+y)}{\partial x} - \frac{\partial[-(x-y)]}{\partial y}\}\mathrm{d}xy = \iint\limits_D (3-1)\mathrm{d}xy = 2\iint\limits_D \mathrm{d}xy$
$= 2\pi\cdot 9 = 18\pi.$

(3)$\because \frac{\partial(y^2-x^2)}{\partial x} = -2x,\ \frac{\partial(x^2+y^2)}{\partial y} = 2y$均在$D$上连续，

$\therefore \oint_L(x^2+y^2)\mathrm{d}x + (y^2-x^2)\mathrm{d}y = \iint\limits_D [\frac{\partial(y^2-x^2)}{\partial x} - \frac{\partial(x^2+y^2)}{\partial y}]\mathrm{d}x\mathrm{d}y = \iint\limits_D (-2x-2y)\mathrm{d}x\mathrm{d}y$
$= \int_0^1\mathrm{d}x\int_0^x(-2x-2y)\mathrm{d}y = \int_0^1(-2xy-y^2)\Big|_0^x\mathrm{d}x = \int_0^1 -2x^2-x^2\mathrm{d}x = -3x^3\Big|_0^1 = -3.$

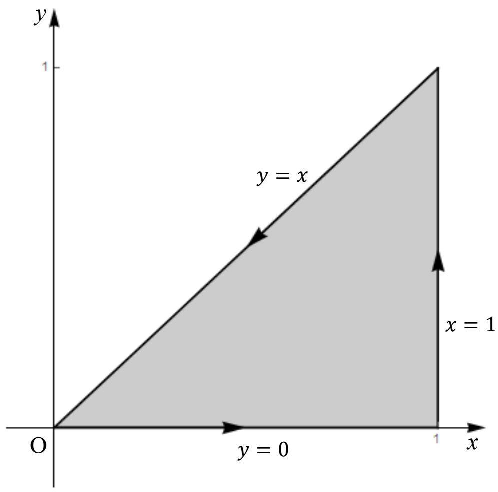

图 7: 习题13.3 1.(3)题图示

(4)$\because \frac{\partial(x+y^2)}{\partial x} = 1,\ \frac{\partial(2xy-x^2)}{\partial y} = 2x$均在$D$上连续，

$\therefore \oint_L(2xy-x^2)\mathrm{d}x + (x+y^2)\mathrm{d}y = \iint\limits_D [\frac{\partial(x+y^2)}{\partial x} - \frac{\partial(2xy-x^2)}{\partial y}]\mathrm{d}x\mathrm{d}y = \iint\limits_D (1-2x)\mathrm{d}x\mathrm{d}y$
$= \int_0^1\mathrm{d}x\int_{x^2}^{\sqrt{x}}(1-2x)\mathrm{d}y = \int_0^1(y-2xy)\Big|_{x^2}^{\sqrt{x}}\mathrm{d}x = \int_0^1(\sqrt{x}-x^2-2x^{\frac{3}{2}}+2x^3)\mathrm{d}x$
$= (\frac{1}{1+\frac{1}{2}}x^{1+\frac{1}{2}} - \frac{1}{3}x^3 - \frac{2}{1+\frac{3}{2}}x^{\frac{3}{2}+1} + \frac{2}{4}x^4)\Big|_0^1 = \frac{2}{3} - \frac{1}{3} - \frac{4}{5} + \frac{1}{2} = \frac{1}{30}.$

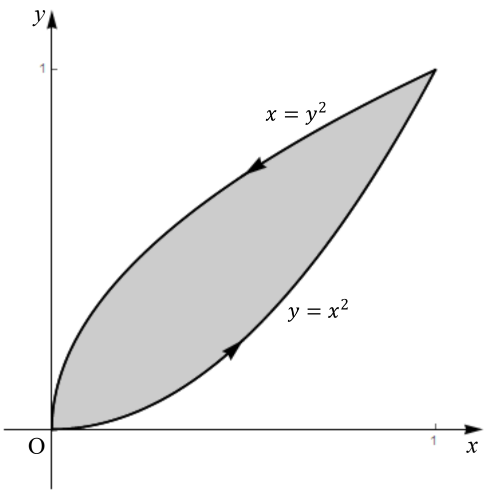

图 8: 习题13.3 1.(4)题图示

(5)设D为以L为边界的有界闭域，

$\because \frac{\partial(xy)}{\partial x}=y$，$\frac{\partial(x+y)}{\partial y}=1$均在$D$上连续，

$\therefore \oint_L(x+y)\mathrm{d}x+xy\mathrm{d}y=\iint\limits_D[\frac{\partial(xy)}{\partial x}-\frac{\partial(x+y)}{\partial y}]\mathrm{d}x\mathrm{d}y=\iint\limits_D(y-1)\mathrm{d}x\mathrm{d}y$

$=\int_0^{2\pi}\mathrm{d}\theta\int_0^1(br\sin\theta-1)\cdot abr\mathrm{d}r=ab^2\int_0^{2\pi}\sin\theta\mathrm{d}\theta\int_0^1 r^2\mathrm{d}r-ab\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r\mathrm{d}r$

$=0-ab2\pi\cdot\frac{1}{2}=-\pi ab.$

(6)设D为以L为边界的有界闭域，

$\because \frac{\partial[x+y\ln(x+\sqrt{x^2+y^2})]}{\partial x}=1+y\frac{1+\frac{x}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}=1+y\frac{\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}=1+\frac{y}{\sqrt{x^2+y^2}}$，$\frac{\partial\sqrt{x^2+y^2}}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$均在$D$上连续，

$\therefore \oint_L\sqrt{x^2+y^2}\mathrm{d}x+[x+y\ln(x+\sqrt{x^2+y^2})]\mathrm{d}y=\iint\limits_D\{\frac{\partial[x+y\ln(x+\sqrt{x^2+y^2})]}{\partial x}-\frac{\partial\sqrt{x^2+y^2}}{\partial y}\}\mathrm{d}x\mathrm{d}y$

$=\iint\limits_D(1+\frac{y}{\sqrt{x^2+y^2}}-\frac{y}{\sqrt{x^2+y^2}})\mathrm{d}x\mathrm{d}y=\iint\limits_D\mathrm{d}x\mathrm{d}y=\pi.$

(7)$\because \frac{\partial[-\mathrm{e}^x(y-\sin y)]}{\partial x}=-\mathrm{e}^x(y-\sin y)$，$\frac{\partial[\mathrm{e}^x(1-\cos y)]}{\partial y}=\mathrm{e}^x\sin y$均在$D$上连续，

$\therefore \oint_L\mathrm{e}^x[(1-\cos y)\mathrm{d}x-(y-\sin y)\mathrm{d}y]=\iint\limits_D\{\frac{\partial[-\mathrm{e}^x(y-\sin y)]}{\partial x}-\frac{\partial[\mathrm{e}^x(1-\cos y)]}{\partial y}\}\mathrm{d}x\mathrm{d}y$

$=\iint\limits_D[-\mathrm{e}^x(y-\sin y)-\mathrm{e}^x\sin y]\mathrm{d}x\mathrm{d}y=\iint\limits_D-\mathrm{e}^x y\mathrm{d}x\mathrm{d}y=-\int_0^{\pi}\mathrm{d}x\int_0^{\sin x}\mathrm{e}^x y\mathrm{d}y=-\int_0^{\pi}\mathrm{e}^x\frac{1}{2}y^2\Big|_0^{\sin x}\mathrm{d}x$

$=-\frac{1}{2}\int_0^{\pi}\mathrm{e}^x\sin^2 x\mathrm{d}x=-\frac{1}{2}\int_0^{\pi}\sin^2 x\mathrm{d}\mathrm{e}^x=-\frac{1}{2}\mathrm{e}^x\sin^2 x\Big|_0^{\pi}+\frac{1}{2}\int_0^{\pi}\mathrm{e}^x\mathrm{d}\sin^2 x=\frac{1}{2}\int_0^{\pi}\mathrm{e}^x 2\sin x\cos x\mathrm{d}x$

$=\frac{1}{2}\int_0^{\pi}\mathrm{e}^x\sin 2x\mathrm{d}x=\frac{1}{2}\int_0^{\pi}\sin 2x\mathrm{d}\mathrm{e}^x=\frac{1}{2}\mathrm{e}^x\sin 2x\Big|_0^{\pi}-\frac{1}{2}\int_0^{\pi}\mathrm{e}^x\mathrm{d}\sin 2x=-\frac{1}{2}\int_0^{\pi}\mathrm{e}^x 2\cos 2x\mathrm{d}x$

$=-\int_0^{\pi}\cos 2x\mathrm{d}\mathrm{e}^x=-\mathrm{e}^x\cos 2x\Big|_0^{\pi}+\int_0^{\pi}\mathrm{e}^x\mathrm{d}\cos 2x=-\mathrm{e}^{\pi}+1-2\int_0^{\pi}\mathrm{e}^x\sin 2x\mathrm{d}x$

$=\frac{1}{2}\frac{1}{\frac{1}{2}+2}(-\mathrm{e}^{\pi}+1)=\frac{1}{5}(1-\mathrm{e}^{\pi}).$

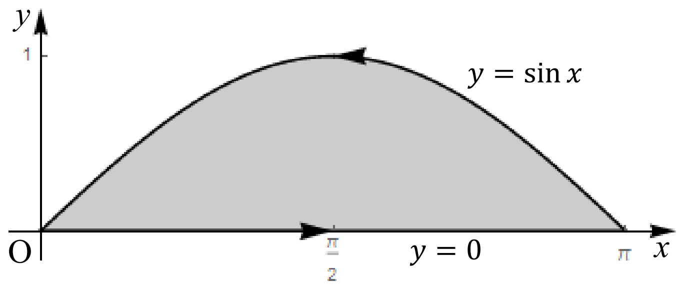

图 9: 习题13.3 1.(7)题图示

2. 计算$I = \oint_L \frac{(x+y)\mathrm{d}x-(x-y)\mathrm{d}y}{x^2+y^2}$，其中$L$：

(1)$D = \{(x,y) \mid r^2 \leqslant x^2+y^2 \leqslant R^2\}\,(0<r<R)$；

(2)$D = \left\{(x,y) \;\middle|\; \frac{x^2}{a^2}+\frac{y^2}{b^2}\right\}$的正向边界.

解：令$Y(x,y) = -\frac{x-y}{x^2+y^2}, X(x,y) = \frac{x+y}{x^2+y^2}$，则

$$\frac{\partial Y(x,y)}{\partial x} = \frac{-(x^2+y^2)+(x-y)\cdot 2x}{(x^2+y^2)^2} = \frac{x^2-y^2-2xy}{(x^2+y^2)^2},$$
$$\frac{\partial X(x,y)}{\partial y} = \frac{x^2+y^2-(x+y)\cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2-2xy}{(x^2+y^2)^2}.$$

(1)$\because \frac{\partial Y(x,y)}{\partial x}, \frac{\partial X(x,y)}{\partial y}$在$D = \{(x,y) \mid r^2 \leqslant x^2+y^2 \leqslant R^2\}\,(0<r<R)$上连续，

$\therefore$

$$I = \iint\limits_D [\frac{\partial Y(x,y)}{\partial x} - \frac{\partial X(x,y)}{\partial y}]\mathrm{d}x\mathrm{d}y = 0.$$

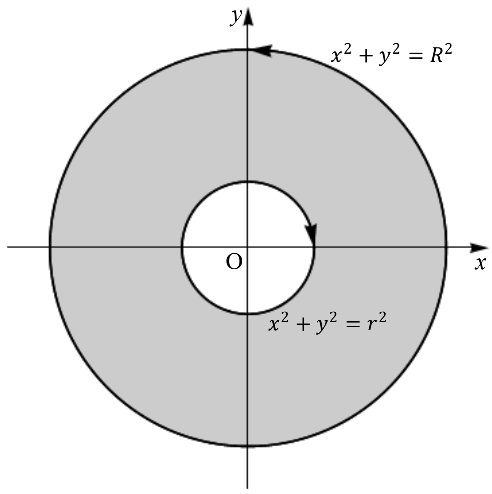

图 10: 习题13.3 2.(1)题图示

(2)取正向圆周$L_1: x^2+y^2=r^2, r<\min\{a,b\}$，记其围成的区域为$D_1$，则$\frac{\partial Y(x,y)}{\partial x}, \frac{\partial X(x,y)}{\partial y}$在$L$和$L_{1-}$围成的区域$D^*$内连续,

$\therefore$

$$\oint_{L+L_{1-}} \frac{(x+y)\mathrm{d}x-(x-y)\mathrm{d}y}{x^2+y^2} = \iint\limits_{D^*}[\frac{\partial Y(x,y)}{\partial x} - \frac{\partial X(x,y)}{\partial y}]\mathrm{d}x\mathrm{d}y = 0,$$

$\therefore$

$$(\oint_L + \oint_{L_{1-}})\frac{(x+y)\mathrm{d}x-(x-y)\mathrm{d}y}{x^2+y^2} = I + \oint_{L_{1-}} \frac{(x+y)\mathrm{d}x-(x-y)\mathrm{d}y}{x^2+y^2} = 0,$$

$\therefore$

$$\begin{aligned}
I &= -\oint_{L_{1-}} \frac{(x+y)\mathrm{d}x-(x-y)\mathrm{d}y}{x^2+y^2} = \oint_{L_{1+}} \frac{(x+y)\mathrm{d}x-(x-y)\mathrm{d}y}{x^2+y^2} \\
&= \frac{1}{r^2}\oint_{L_{1+}}(x+y)\mathrm{d}x-(x-y)\mathrm{d}y \\
&= \frac{1}{r^2}\iint\limits_{D_1}[\frac{\partial(-x+y)}{\partial x} - \frac{\partial(x+y)}{\partial y}]\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{r^2}\iint\limits_{D_1} -2\mathrm{d}x\mathrm{d}y = \frac{1}{r^2}(-2)\pi r^2 = -2\pi.
\end{aligned}$$

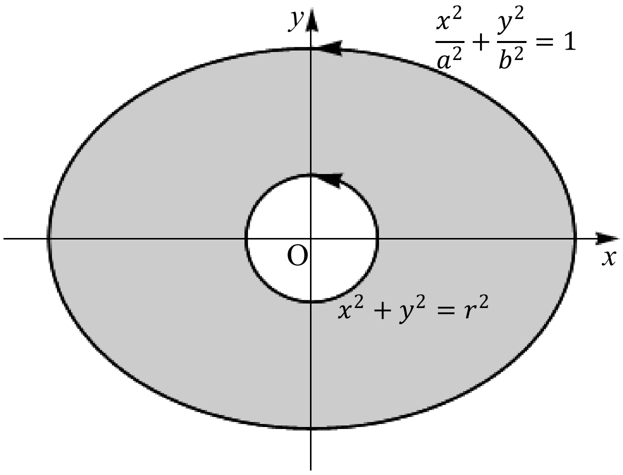

图 11: 习题13.3 2.(2)题图示

3. 设D为平面区域，∂D为逐段光滑曲线，$(\bar{x},\bar{y})$是D的形心，D的面积等于$\sigma(D)$. 试证：

   (1)$\int_{\partial D} x^2\mathrm{d}y = 2\sigma(D)\bar{x}$；

   (2)$\int_{\partial D} xy\mathrm{d}y = \sigma(D)\bar{y}$.

   解：(1)$\because x^2, 0 \in C^1(D)$,

   $\therefore \int_{\partial D} x^2\mathrm{d}y = \int_{\partial D} 0\mathrm{d}x + x^2\mathrm{d}y = \iint\limits_D [\frac{\partial(x^2)}{\partial x} - \frac{\partial 0}{\partial y}]\mathrm{d}x\mathrm{d}y = \iint\limits_D 2x\mathrm{d}x\mathrm{d}y = 2\iint\limits_D x\mathrm{d}x\mathrm{d}y = 2\sigma(D)\bar{x}.$

   (2)$\because xy, 0 \in C^1(D)$,

   $\therefore \int_{\partial D} xy\mathrm{d}y = \int_{\partial D} 0\mathrm{d}x + xy\mathrm{d}y = \iint\limits_D [\frac{\partial(xy)}{\partial x} - \frac{\partial 0}{\partial y}]\mathrm{d}x\mathrm{d}y = \iint\limits_D y\mathrm{d}x\mathrm{d}y = \sigma(D)\bar{y}.$

4. 设D为平面区域，∂D为逐段光滑曲线，$f \in C^2(\bar{D})$，求证：

$$\oint_{\partial D} \frac{\partial f}{\partial \boldsymbol{n}}\mathrm{d}l = \iint\limits_D (\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2})\mathrm{d}x\mathrm{d}y.$$

   证明：$\because f \in C^2(\bar{D})$,

   $\therefore f \in C^1(\bar{D})$,

   $\therefore f$在$\bar{D}$上可微，

   $\therefore \frac{\partial f}{\partial \boldsymbol{n}} = \mathrm{grad} f(x,y) \cdot \boldsymbol{n} = (\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}) \cdot \boldsymbol{n}$,

$\therefore$

$$\begin{aligned}\oint_{\partial D}\frac{\partial f}{\partial \boldsymbol{n}}\mathrm{d}l &= \oint_{\partial D}\mathrm{grad}f(x,y)\cdot\boldsymbol{n}\mathrm{d}l = \oint_{\partial D}(\frac{\partial f(x,y)}{\partial x},\frac{\partial f(x,y)}{\partial y})\cdot\boldsymbol{n}\mathrm{d}l\\ &= \iint\limits_D[\frac{\partial}{\partial x}(\frac{\partial f}{\partial x})+\frac{\partial}{\partial y}(\frac{\partial f}{\partial y})]\mathrm{d}x\mathrm{d}y\\ &= \iint\limits_D(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2})\mathrm{d}x\mathrm{d}y.\end{aligned}$$