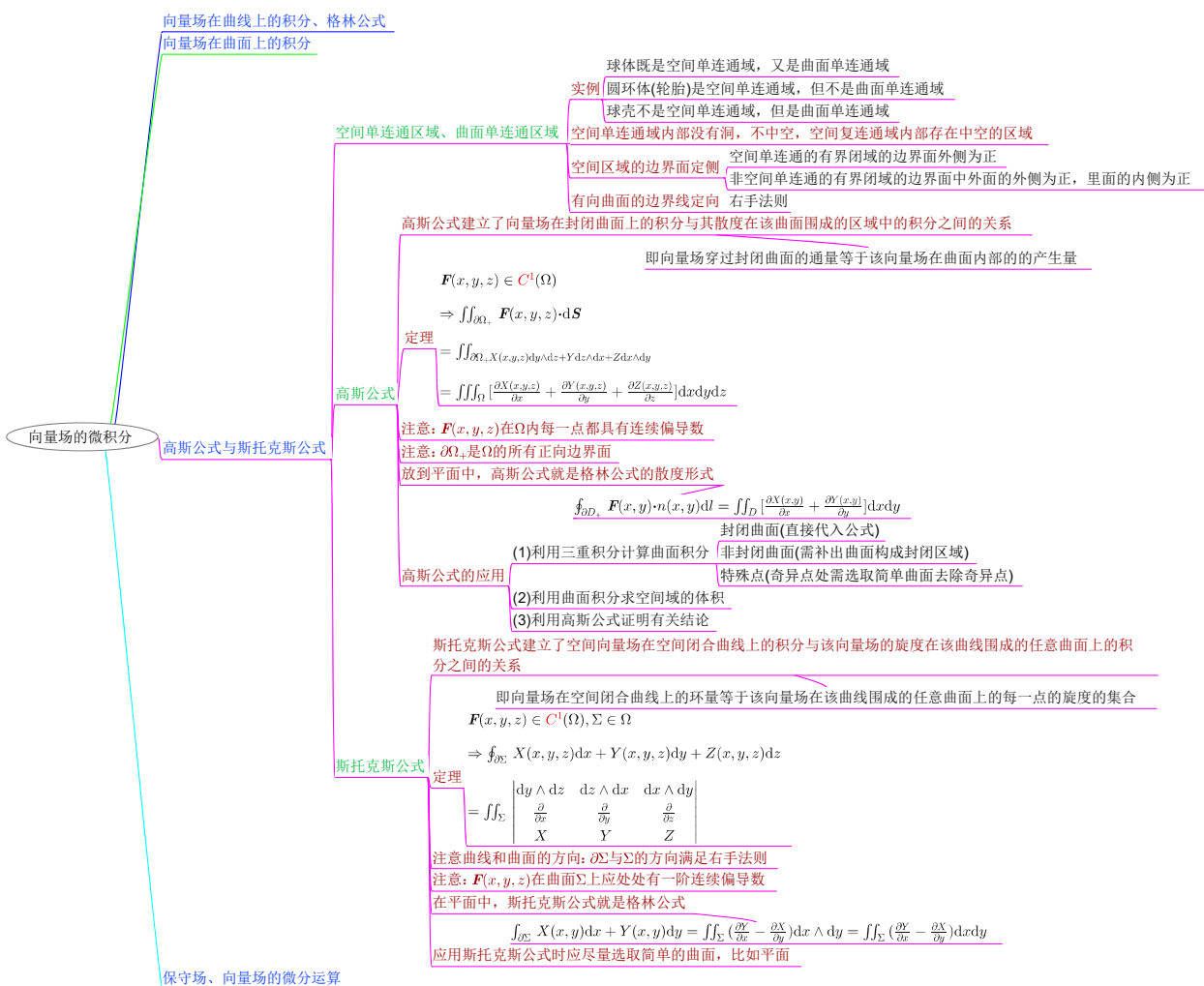


12 高斯公式、斯托克斯公式

12.1 复习计划



12.2 知识结构



12.3 高斯公式与斯托克斯公式

1. 这一部分的习题的图示大家可以看看, 可结合图示理解高斯公式和斯托克斯公式的使用.
2. 可参看后续格林公式、高斯公式和斯托克斯公式的比较专题.
3. 高斯公式

$\Omega \subset \mathbb{R}^3, \partial\Omega$ 逐片光滑,

向量场 $\mathbf{F}(x, y, z) = X(x, y, z)\mathbf{i} + Y(x, y, z)\mathbf{j} + Z(x, y, z)\mathbf{k} \in C^1(\Omega)$,

则

$$\begin{aligned}\iint_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S} &= \iint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} dS = \iint_{\partial\Omega} X dy \wedge dz + Y dz \wedge dx + Z dx \wedge dy \\ &= \iiint_{\Omega} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) dx dy dz \\ &= \iiint_{\Omega} \nabla \cdot \mathbf{F} dx dy dz^1 \\ &= \iiint_{\Omega} \operatorname{div} \mathbf{F} dx dy dz.\end{aligned}$$

【直接代入公式计算的题目: 1.(1)/(2)/(3)/(4)/(6). (第一类题目)】

注意:

(a) 在非封闭曲面上的积分, 可添加简单曲面构造封闭区域, 利用高斯公式计算.

【利用该点计算的题目: 1.(5). (第二类题目)】

(b) $\mathbf{F}(x, y, z) = X(x, y, z)\mathbf{i} + Y(x, y, z)\mathbf{j} + Z(x, y, z)\mathbf{k} \in C^1(\Omega)$, 即 X, Y, Z 有一阶连续偏导数, 也即 X, Y, Z 连续可微.

【考察该点的题目: 1.(7). (第三类题目)】

(c) $\partial\Omega$ 外侧为正, 即在 $\partial\Omega$ 的外侧积分.

【考察该点的题目: 1.(2).】

(d) $dy \wedge dz, dz \wedge dx, dx \wedge dy$ 与 X, Y, Z 的对应关系.

【考察该点的题目: 1.(4)/(6).】

(e) 利用奇偶性和对称性可简化计算.

【考察该点的题目: 1.(2)(奇偶性), (3)/(4)(轮换对称性).】

¹这里的符号 ∇ 可读作 nabla.

4. 斯托克斯公式

$\Omega \subset \mathbb{R}^3, \Sigma \subset \Omega$ 为逐片光滑的有向曲面, $\partial\Sigma$ 逐段光滑,

向量场 $\mathbf{F}(x, y, z) = X(x, y, z)\mathbf{i} + Y(x, y, z)\mathbf{j} + Z(x, y, z)\mathbf{k} \in C^1(\Omega)$,

则

$$\begin{aligned} \oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{l} &= \oint_{\partial\Sigma} \mathbf{F} \cdot \boldsymbol{\tau} d\mathbf{l} = \oint_{\partial\Sigma} X dx + Y dy + Z dz \\ &= \iint_{\Sigma} \begin{vmatrix} dy \wedge dz & dz \wedge dx & dx \wedge dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix} \\ &= \iint_{\Sigma} \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} \right) dy \wedge dz + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dx \wedge dy \\ &= \iint_{\Sigma} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix} \cdot d\mathbf{S} = \iint_{\Sigma} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix} \cdot \mathbf{n} dS \\ &= \iint_{\Sigma} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{\Sigma} (\text{rot} \mathbf{F}) \cdot d\mathbf{S}. \end{aligned}$$

注意:

- (a) Σ 与 $\partial\Sigma$ 的方向符合右手法则. 在用斯托克斯公式进行计算时, 选取的曲面方向应与已知曲线方向满足右手法则.

【如本节习题中2.(1)/(2), 3.(1)/(2)的图形所示(图 8, 9, 10, 11).】

- (b) 曲面 Σ 应尽量简单, 便于求面积或便于计算曲面积分. 如本节习题中的2.(1), 已知曲线 L 是一个大圆, 可直接求出其围成的圆形平面的面积, 故 Σ 取为 L 围成的圆形平面, 根据右手法则, 上侧为正. 本节习题中的2.(2)中, 已知曲线是一个平面曲线, Σ 可取为该平面曲线围成的平面, 根据右手法则, 上侧为正.

【第一类习题: 2.(1)/(2).】

- (c) $\mathbf{F}(x, y, z) = X(x, y, z)\mathbf{i} + Y(x, y, z)\mathbf{j} + Z(x, y, z)\mathbf{k} \in C^1(\Sigma)$, 即 X, Y, Z 有一阶连续偏导数, 也即 X, Y, Z 连续可微.

【考察该点的题目: 3.(1)/(2). (第二类习题)】

【其他类型的题目: 4. (第三类习题)】

12.4 习题13.5解答

1. 用高斯公式计算下列曲面积分:

(1) $\iint_S x^3 dy \wedge dz + y^3 dz \wedge dx + z^3 dx \wedge dy$, 其中 S 为区域 $x^2 + y^2 \leq z, 0 \leq z \leq 4$ 的边界外

侧;

(2) $\iint_S x^2 dy \wedge dz + y^2 dz \wedge dx + z^2 dx \wedge dy$, 其中 S 为 $x^2 + y^2 + z^2 = R^2$ 的内侧;

(3) $\iint_S x^2 dy \wedge dz + y^2 dz \wedge dx + z^2 dx \wedge dy$, 其中 S 为正方体 $\Omega: 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$ 的外表面;

(4) $\iint_S xz dx \wedge dy + xy dy \wedge dz + yz dz \wedge dx$, 其中 S 为平面 $x + y + z = 1$ 与三个坐标面围成的区域边界外侧;

(5) $\iint_S x^2 dy \wedge dz + (z^2 - 2z) dx \wedge dy$, 其中 S 为 $z = \sqrt{x^2 + y^2}$ 被平面 $z = 0$ 和 $z = 1$ 所截部分的外侧;

(6) $\oiint_S (x - y) dx \wedge dy + (y - z) x dy \wedge dz$, 其中 S 为柱面 $x^2 + y^2 = 1$ 与平面 $z = 1, z = 3$ 所围柱体的边界外侧;

(7) $\oiint_S \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$, 其中 S 为椭球面 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

解: (1) 记 S 围成的区域为 Ω , 则 $x^3, y^3, z^3 \in C^1(\Omega)$,

$$\begin{aligned} \therefore \iint_S x^3 dy \wedge dz + y^3 dz \wedge dx + z^3 dx \wedge dy &= \iiint_{\Omega} \left(\frac{\partial x^3}{\partial x} + \frac{\partial y^3}{\partial y} + \frac{\partial z^3}{\partial z} \right) dx dy dz \\ &= \iiint_{\Omega} (3x^2 + 3y^2 + 3z^2) dx dy dz = 3 \int_0^{2\pi} d\theta \int_0^2 dr \int_{r^2}^4 (r^2 + z^2) \cdot r dz = 6\pi \int_0^2 \left(r^3 z + \frac{1}{3} r z^3 \right) \Big|_{r^2}^4 dr \\ &= 6\pi \int_0^2 \left(4r^3 + \frac{64}{3} r - r^5 - \frac{1}{3} r^7 \right) dr = 6\pi \left(r^4 + \frac{32}{3} r^2 - \frac{1}{6} r^6 - \frac{1}{3 \cdot 8} r^8 \right) \Big|_0^2 \\ &= 6\pi \left(16 + \frac{32}{3} \cdot 4 - \frac{1}{6} \cdot 64 - \frac{1}{3} \cdot \frac{1}{8} \cdot 2^8 \right) = 6\pi \left(16 + \frac{64}{3} \right) = (96 + 128)\pi = 224\pi. \end{aligned}$$

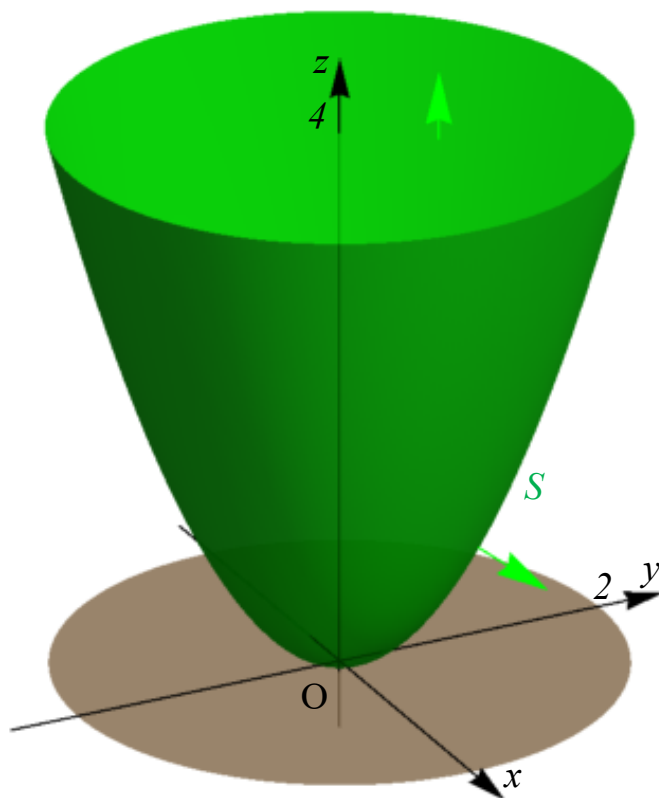


图 1: 习题13.5 1.(1)题图示

(2) 记 Ω 为 S^- 围成的区域,

$$\because x^2, y^2, z^2 \in C^1(\Omega),$$

$$\begin{aligned} \therefore \iint_S x^2 dy \wedge dz + y^2 dz \wedge dx + z^2 dx \wedge dy &= - \iint_{S^-} x^2 dy \wedge dz + y^2 dz \wedge dx + z^2 dx \wedge dy \\ &= - \iiint_{\Omega} \left(\frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z} \right) dx dy dz = - \iiint_{\Omega} (2x + 2y + 2z) dx dy dz = 0. \end{aligned}$$

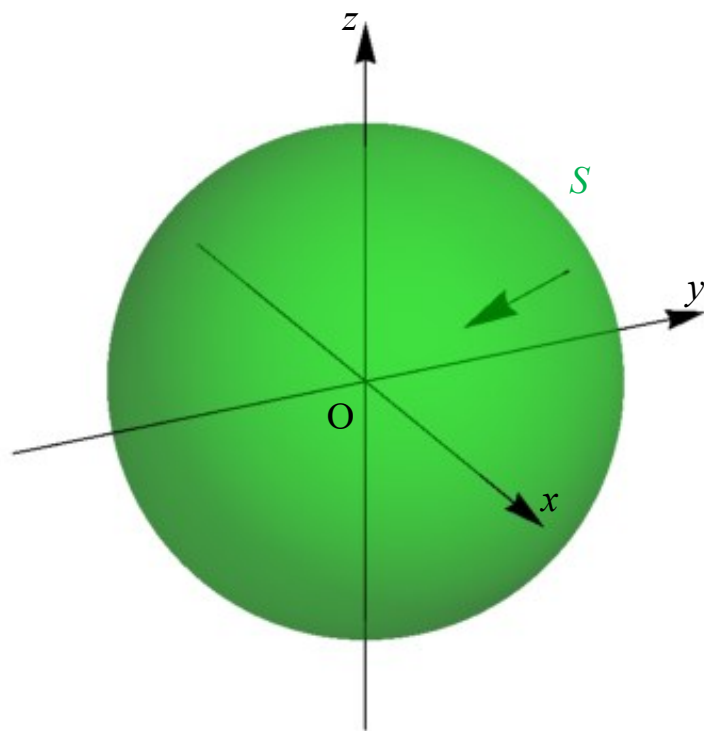


图 2: 习题13.5 1.(2)题图示

$$(3) \because x^2, y^2, z^2 \in C^1(\Omega),$$

$$\begin{aligned} \therefore \iint_S x^2 dy \wedge dz + y^2 dz \wedge dx + z^2 dx \wedge dy &= \iiint_{\Omega} \left(\frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z} \right) dx dy dz \\ &= 2 \iiint_{\Omega} (x + y + z) dx dy dz = 6 \iiint_{\Omega} x dx dy dz = 6 \int_0^1 x dx \int_0^1 dy \int_0^1 dz = 6 \int_0^1 x dx \int_0^1 dy \\ &= 6 \int_0^1 x dx = 3x^2 \Big|_0^1 = 3. \end{aligned}$$

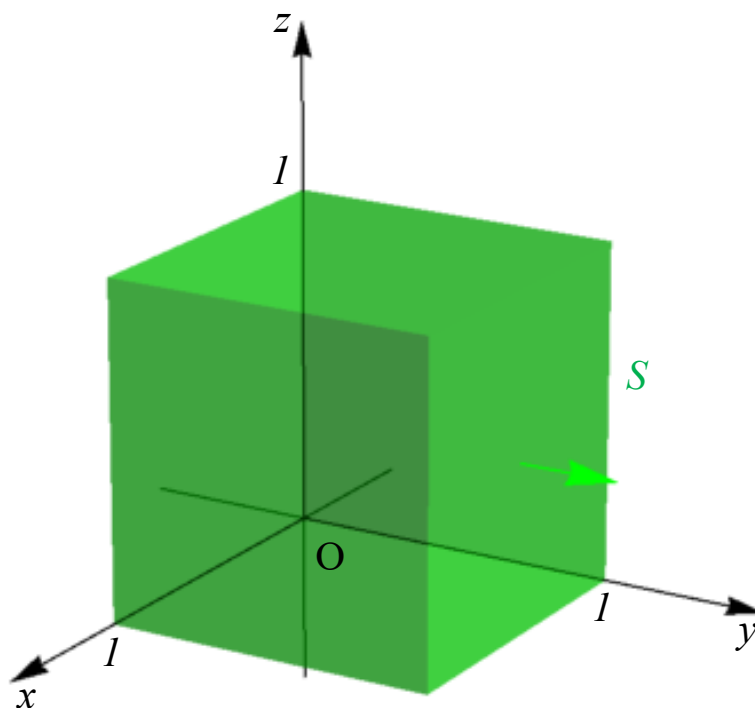


图 3: 习题13.5 1.(3)题图示

(4) 记 S 围成的区域为 Ω , 则 $xz, xy, yz \in C^1(\Omega)$,

$$\begin{aligned}
 \therefore \iint_S xz dx \wedge dz + xy dy \wedge dz + yz dz \wedge dx &= \iiint_{\Omega} \left(\frac{\partial xy}{\partial x} + \frac{\partial yz}{\partial y} + \frac{\partial xz}{\partial z} \right) dx dy dz = \iiint_{\Omega} (y + z + x) dx dy dz \\
 &= 3 \iiint_{\Omega} x dx dy dz = 3 \int_0^1 x dx \int_0^{1-x} dy \int_0^{1-x-y} dz = 3 \int_0^1 x dx \int_0^{1-x} (1-x-y) dy \\
 &= 3 \int_0^1 x \left(y - xy - \frac{1}{2}y^2 \right) \Big|_0^{1-x} dx = 3 \int_0^1 x \left(1-x - \frac{1}{2}y \right) \Big|_0^{1-x} dx \\
 &= 3 \int_0^1 x \left[1-x - \frac{1}{2}(1-x) \right] (1-x) dx = 3 \int_0^1 x \frac{1}{2} (1-x)^2 dx = \frac{3}{2} \int_0^1 x (1-2x+x^2) dx \\
 &= \frac{3}{2} \int_0^1 (x - 2x^2 + x^3) dx = \frac{3}{2} \left(\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 \right) \Big|_0^1 = \frac{3}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{3}{2} \left(\frac{3}{4} - \frac{2}{3} \right) = \frac{3}{2} \cdot \frac{1}{12} = \frac{1}{8}.
 \end{aligned}$$

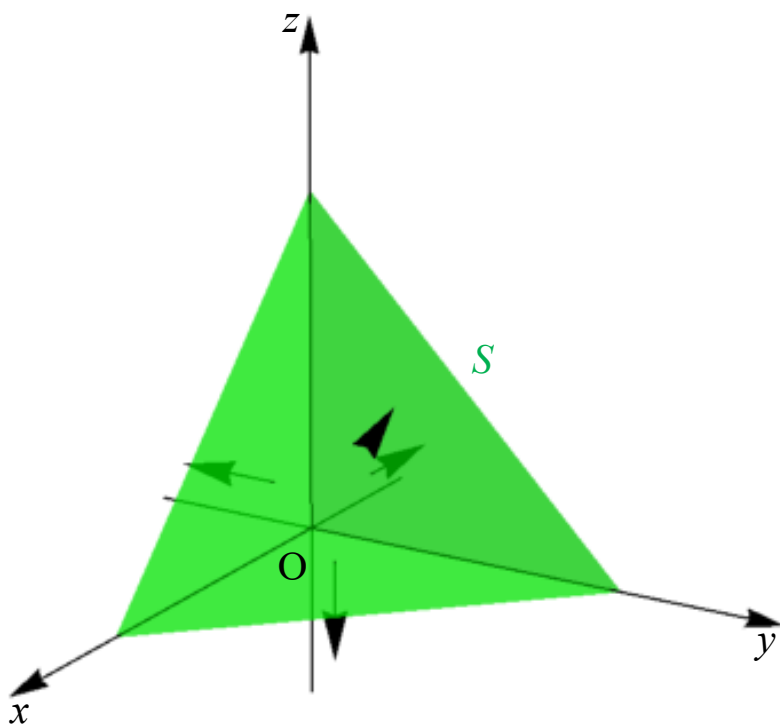


图 4: 习题13.5 1.(4)题图示

(5)取平面 $S_1: z = 1, x^2 + y^2 \leq 1$, 上侧为正, 记 S 与 S_1 围成的区域为 Ω ,

$\because x^2, z^2 - 2z \in C^1(\Omega)$,

$$\begin{aligned} \therefore \iint_{S+S_1} x^2 dy \wedge dz + (z^2 - 2z) dx \wedge dy &= \iiint_{\Omega} \left(\frac{\partial x^2}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial (z^2 - 2z)}{\partial z} \right) dx dy dz \\ &= \iiint_{\Omega} (2x + 2z - 2) dx dy dz, \end{aligned}$$

由对称性可知 $\iiint_{\Omega} 2x dx dy dz = 0$,

$$\begin{aligned} \therefore \text{上式} &= \iiint_{\Omega} (2z - 2) dx dy dz = \int_0^1 (2z - 2) dz \iint_{x^2+y^2 \leq z^2} dx dy = \int_0^1 (2z - 2) \pi z^2 dz \\ &= 2\pi \int_0^1 (z^3 - z^2) dz = 2\pi \left(\frac{1}{4} z^4 - \frac{1}{3} z^3 \right) \Big|_0^1 = -\frac{1}{6} \pi, \end{aligned}$$

\therefore 在 S_1 上 $d\mathbf{S} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) dx dy = (0, 0, 1) dx dy = (dy \wedge dz, dz \wedge dx, dx \wedge dy)$,

$$\therefore \iint_{S_1} x^2 dy \wedge dz + (z^2 - 2z) dx \wedge dy = \iint_{x^2+y^2 \leq 1} [x^2 \cdot 0 + (1^2 - 2 \cdot 1)] dx dy = - \iint_{x^2+y^2 \leq 1} dx dy = -\pi,$$

$$\therefore \iint_S x^2 dy \wedge dz + (z^2 - 2z) dx \wedge dy = -\frac{\pi}{6} - \iint_{S_1} x^2 dy \wedge dz + (z^2 - 2z) dx \wedge dy = -\frac{\pi}{6} + \pi = \frac{5}{6} \pi.$$

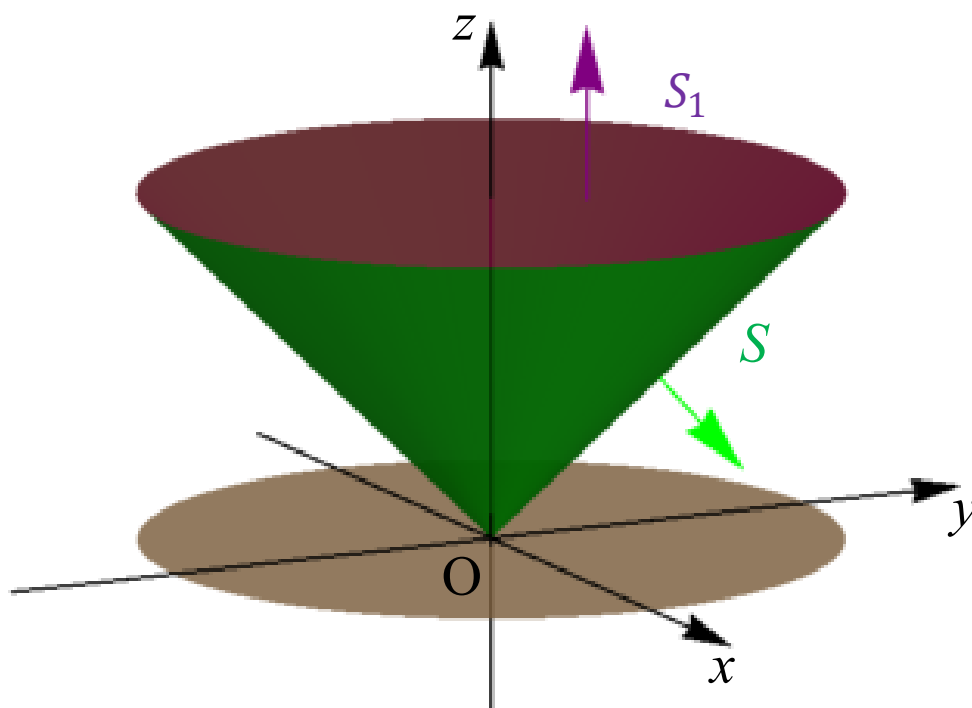


图 5: 习题13.5 1.(5)题图示

(6) 记 S 围成的圆柱体区域为 Ω , 则 $x - y, (y - z)x \in C^1(\Omega)$,

$$\begin{aligned} \therefore \oint_S (x - y)dx \wedge dy + (y - z)xdy \wedge dz &= \iiint_{\Omega} \left[\frac{\partial(y-z)x}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial(x-y)}{\partial z} \right] dx dy dz \\ &= \iiint_{\Omega} (y - z) dx dy dz, \end{aligned}$$

由对称性可知 $\iiint_{\Omega} y dx dy dz = 0$,

$$\therefore \text{上式} = - \iiint_{\Omega} z dx dy dz = - \int_1^3 z dz \iint_{x^2+y^2 \leq 1} dx dy = -\pi \int_1^3 z dz = -\frac{1}{2}\pi z^2 \Big|_1^3 = -4\pi.$$

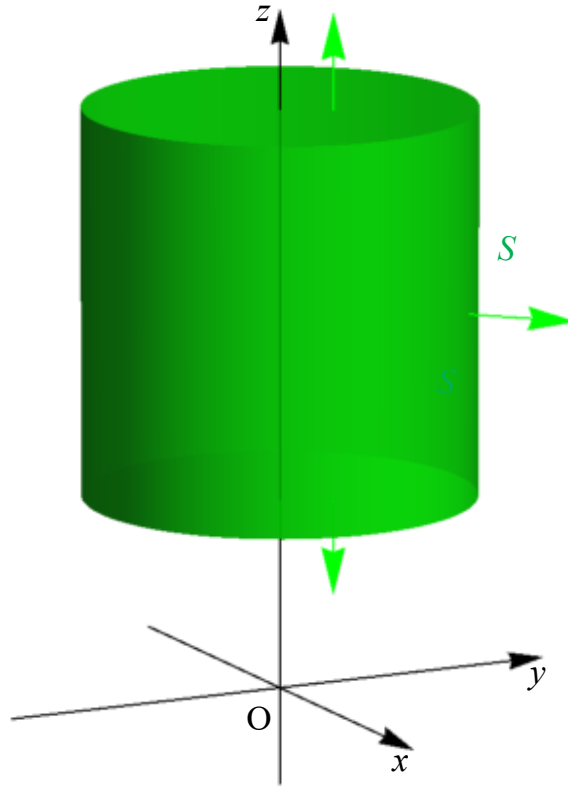


图 6: 习题13.5 1.(6)题图示

(7)取球面 $S_1: x^2 + y^2 + z^2 = r^2, r < \min\{a, b, c\}$, 外侧为正, 设 S 与 S_1^- 围成的区域为 Ω , S_1 围成的区域为 Ω_1 ,

则 $X = \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, Y = \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, Z = \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \in C^1(\Omega)$,

$$\frac{\partial X}{\partial x} = \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - x \cdot \frac{3}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}} \cdot 2x}{(x^2 + y^2 + z^2)^3} = \frac{y^2 + z^2 - 2x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}},$$

$$\frac{\partial Y}{\partial y} = \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - y \cdot \frac{3}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}} \cdot 2y}{(x^2 + y^2 + z^2)^3} = \frac{z^2 + x^2 - 2y^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}},$$

$$\frac{\partial Z}{\partial z} = \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - z \cdot \frac{3}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}} \cdot 2z}{(x^2 + y^2 + z^2)^3} = \frac{x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}},$$

\therefore

$$\begin{aligned} & \oint_{S+S_1^-} \frac{\partial X}{\partial x} dy \wedge dz + \frac{\partial Y}{\partial y} dz \wedge dx + \frac{\partial Z}{\partial z} dx \wedge dy = \iiint_{\Omega} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) dx dy dz \\ &= \iiint_{\Omega} \frac{y^2 + z^2 - 2x^2 + z^2 + x^2 - 2y^2 + x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} dx dy dz = \iiint_{\Omega} 0 dx dy dz = 0, \end{aligned}$$

\therefore

$$\begin{aligned}
 & \oint_S \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = 0 - \oint_{S_1^-} \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \\
 & = \oint_{S_1} \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = \oint_{S_1} \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(r^2)^{\frac{3}{2}}} \\
 & = \frac{1}{r^3} \oint_{S_1} xdy \wedge dz + ydz \wedge dx + zdx \wedge dy = \frac{1}{r^3} \iiint_{\Omega} \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) dx dy dz \\
 & = \frac{3}{r^3} \iiint_{\Omega} dx dy dz = \frac{3}{r^3} \frac{4}{3} \pi r^3 = 4\pi.
 \end{aligned}$$

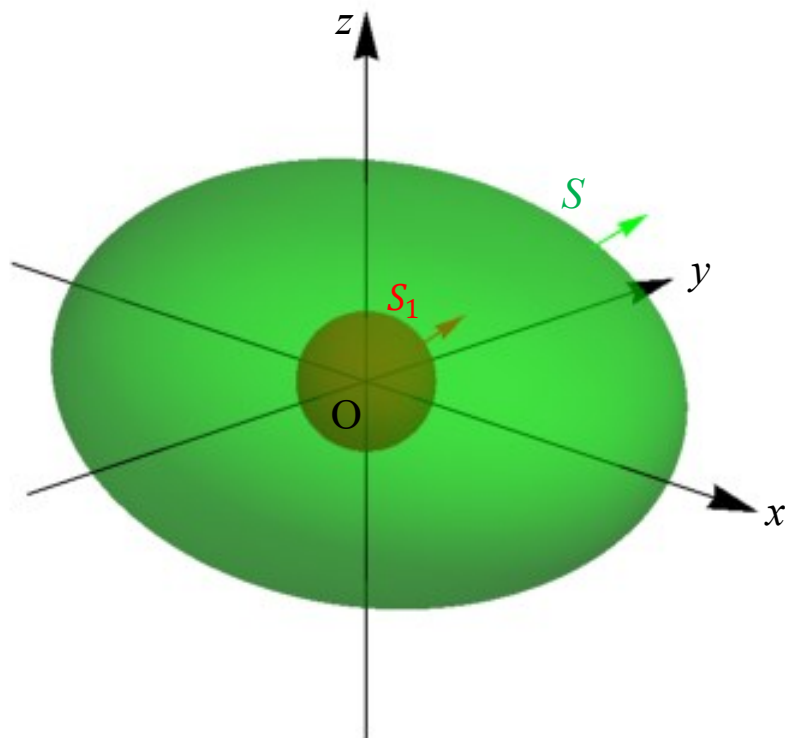


图 7: 习题13.5 1.(7)题图示

2. 用斯托克斯公式计算下列曲线积分:

(1) $\oint_L ydx + zdy + xdz$, 其中 L 是圆周 $\begin{cases} x^2 + y^2 + z^2 = R^2, \\ x + y + z = 0, \end{cases}$ 从 x 轴正向看去为逆时针方向;

(2) $\oint_L (y-x)dx + (z-y)dy + (x-z)dz$, 其中 L 是柱面 $x^2 + y^2 = a^2$ 与平面 $x+z=a$ ($a>0$) 的交线, 从 x 轴正向看去为逆时针方向.

解: (1) 记 Σ 是平面 $x+y+z=0$ 上 L 围成的部分, Σ 与 L 的方向符合右手法则, 记 Σ 的上侧为正,

$\because y, z, x \in C^1(\Sigma)$,

$$\begin{aligned} \therefore \oint_L ydx + zdy + xdz &= \iint_{\Sigma} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} \cdot \mathbf{n} dS = \iint_{\Sigma} \left(\frac{\partial x}{\partial y} - \frac{\partial z}{\partial z}, \frac{\partial z}{\partial y} - \frac{\partial x}{\partial x}, \frac{\partial z}{\partial x} - \frac{\partial y}{\partial y} \right) \cdot \mathbf{n} dS \\ &= \iint_{\Sigma} (0-1, 0-1, 0-1) \cdot \mathbf{n} dS = \iint_{\Sigma} (-1, -1, -1) \cdot \mathbf{n} dS, \\ \because \Sigma \text{ 的单位法向量为 } \mathbf{n} &= (1, 1, 1) \frac{1}{\sqrt{3}}, \\ \therefore \oint_L ydx + zdy + xdz &= \iint_{\Sigma} (-1, -1, -1) \cdot (1, 1, 1) \frac{1}{\sqrt{3}} dS = \frac{1}{\sqrt{3}} \iint_{\Sigma} -3 dS = -\sqrt{3} \pi R^2. \end{aligned}$$

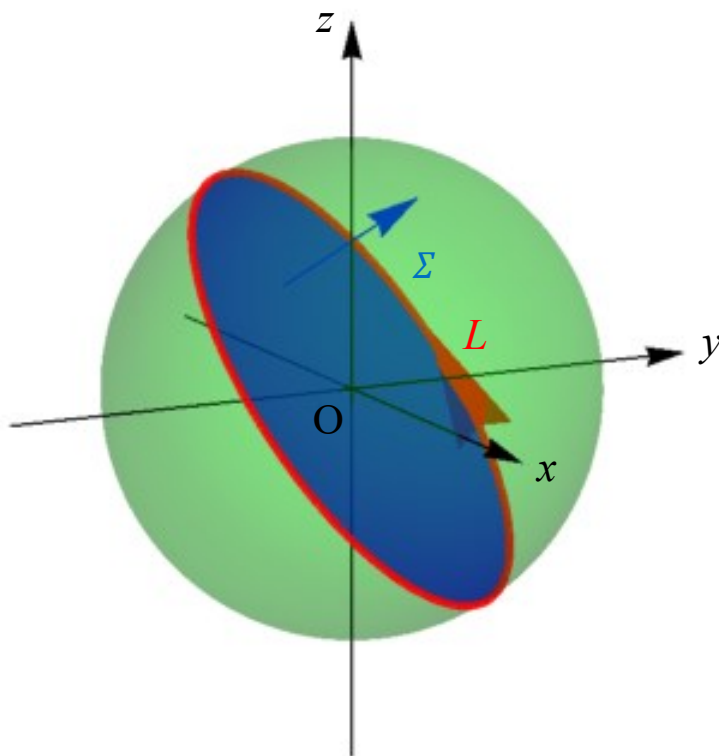


图 8: 习题13.5 2.(1)题图示

(2) 记 Σ 是平面 $x+z=a$ 上 L 围成的部分, Σ 的方向与 L 的方向符合右手法则, 即 Σ 的上

侧为正,

$$\because y-x, z-y, x-z \in C^1(\Sigma),$$

$$\begin{aligned} \therefore \oint_L (y-x)dx + (z-y)dy + (x-z)dz &= \iint_{\Sigma} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-x & z-y & x-z \end{vmatrix} \cdot d\mathbf{S} \\ &= \iint_{\Sigma} \left(\frac{\partial(x-z)}{\partial y} - \frac{\partial(z-y)}{\partial z}, \frac{\partial(y-x)}{\partial z} - \frac{\partial(x-z)}{\partial x}, \frac{\partial(z-y)}{\partial x} - \frac{\partial(y-x)}{\partial y} \right) \cdot d\mathbf{S} = \iint_{\Sigma} (0-1, 0-1, 0-1) \cdot d\mathbf{S}, \end{aligned}$$

\therefore 在 $\Sigma: z=a-x, x^2+y^2 \leq a^2$ 的上侧

$$d\mathbf{S} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) dx dy = (1, 0, 1) dx dy = (dy \wedge dz, dz \wedge dx, dx \wedge dy),$$

$$\therefore \text{上式} = \iint_{x^2+y^2 \leq a^2} (0-1, 0-1, 0-1) \cdot (1, 0, 1) dx dy = -2 \iint_{x^2+y^2 \leq a^2} dx dy = -2\pi a^2.$$

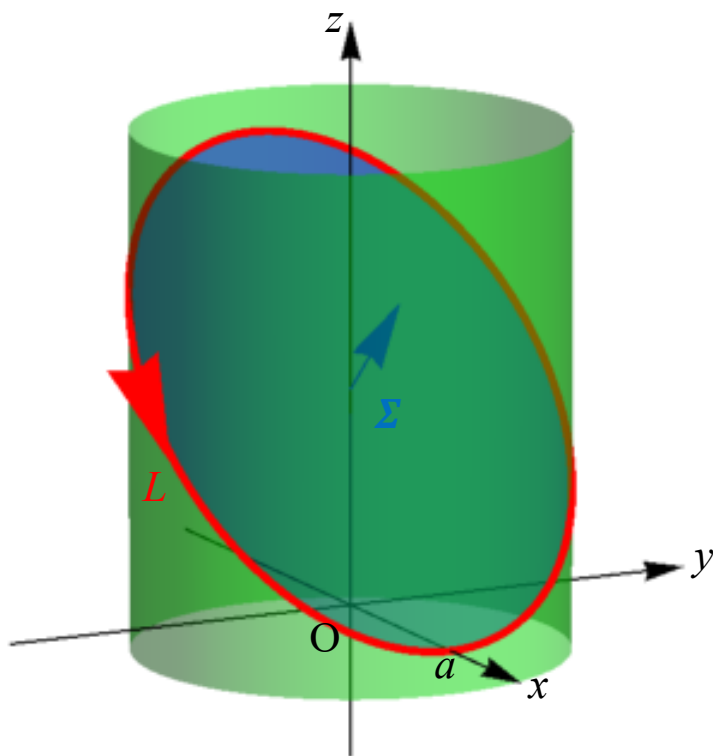


图 9: 习题13.5 2.(2)题图示

3. 计算 $I = \oint_L \frac{-ydx+xdy}{x^2+y^2} + zdz$, 其中 L 是:

- (1) 任意一条既不环绕 z 轴, 也不与 z 轴相交的简单闭曲线;
- (2) 任意一条环绕 z 轴一圈且不与 z 轴相交的简单闭曲线, 从 z 轴正向看去为逆时针方向.

解: (1)取曲面 Σ 为曲线 L 围成的逐片光滑有向曲面, Σ 和 L 的方向符合右手法则, z 轴不穿过 Σ ,

则 $X = \frac{-y}{x^2+y^2}$, $Y = \frac{x}{x^2+y^2}$, $Z = z \in C^1(\Sigma)$,

\therefore

$$\begin{aligned} I &= \oint_L Xdx + Ydy + Zdz = \iint_{\Sigma} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix} \cdot d\mathbf{S} = \iint_{\Sigma} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & z \end{vmatrix} \cdot d\mathbf{S} \\ &= \iint_{\Sigma} \left(\frac{\partial z}{\partial y} - \frac{\partial}{\partial z} \frac{x}{x^2+y^2}, \frac{\partial}{\partial z} \frac{-y}{x^2+y^2} - \frac{\partial z}{\partial x}, \frac{\partial}{\partial x} \frac{x}{x^2+y^2} - \frac{\partial}{\partial y} \frac{-y}{x^2+y^2} \right) \cdot d\mathbf{S} \\ &= \iint_{\Sigma} \left(0 - 0, 0 - 0, \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} - \frac{-(x^2+y^2) + y \cdot 2y}{(x^2+y^2)^2} \right) \cdot d\mathbf{S} \\ &= \iint_{\Sigma} (0, 0, 0) \cdot d\mathbf{S} = 0. \end{aligned}$$

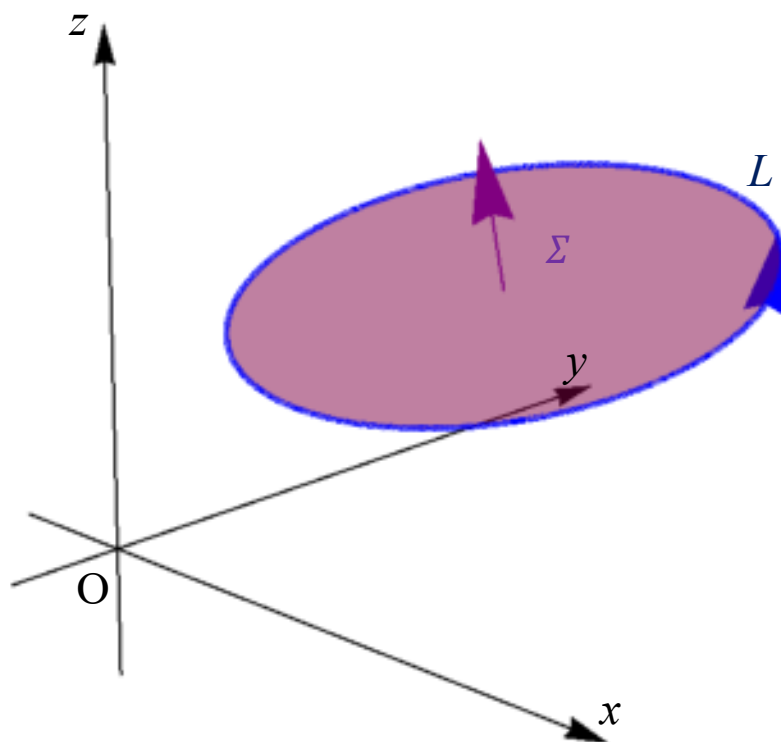


图 10: 习题13.5 3.(1)题图示

(2) 设曲面 Σ 为曲线 L 围成的逐片光滑有向曲面, Σ 与 L 的方向符合右手法则, 即 Σ 的上侧为正, 此时 z 轴穿过 Σ . 取柱面 $x^2 + y^2 = r^2$ 与 Σ 交于曲线 L_1 , r 应足够小使得闭合交线 L_1 全部位于 Σ 上, 设从 z 轴正向看去 L_1 为逆时针方向. 记 Σ_1 为 L_1 围成的逐片光滑正向曲面, Σ_2 为 L 与 L_1^- 围成的逐片光滑正向曲面.

则 $X = \frac{-y}{x^2+y^2}$, $Y = \frac{x}{x^2+y^2}$, $Z = z \in C^1(\Sigma_2)$,

与(1)同理可知 $\oint_{L+L_1^-} Xdx + Ydy + Zdz = 0$,

\therefore

$$\begin{aligned} I &= 0 - \oint_{L_1^-} Xdx + Ydy + Zdz = \oint_{L_1} Xdx + Ydy + Zdz \\ &= \oint_{L_1} \frac{-ydx + xdy}{x^2 + y^2} + zdz = \oint_{L_1} \frac{-ydx + xdy}{r^2} + zdz \\ &= \iint_{\Sigma_1} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{r^2} & \frac{x}{r^2} & z \end{vmatrix} \cdot d\mathbf{S} = \iint_{\Sigma_1} \left(\frac{\partial z}{\partial y} - \frac{\partial}{\partial z} \frac{x}{r^2}, \frac{\partial}{\partial z} \left(-\frac{y}{r^2} \right) - \frac{\partial z}{\partial x}, \frac{\partial}{\partial x} \frac{x}{r^2} - \frac{\partial}{\partial y} \left(-\frac{y}{r^2} \right) \right) \cdot d\mathbf{S} \\ &= \iint_{\Sigma_1} \left(0 - 0, 0 - 0, \frac{1}{r^2} - \left(-\frac{1}{r^2} \right) \right) \cdot d\mathbf{S} = \iint_{\Sigma_1} \left(0, 0, \frac{2}{r^2} \right) \cdot (dy \wedge dz, dz \wedge dx, dx \wedge dy) \\ &= \iint_{\Sigma_1} \frac{2}{r^2} dx \wedge dy = \frac{2}{r^2} \iint_{\Sigma_1} dx \wedge dy, \end{aligned}$$

$\therefore \Sigma_1$ 方程可表示为 $z = f(x, y)$, $x^2 + y^2 \leq r^2$, 且 Σ_1 上侧为正,

$$\therefore I = \frac{2}{r^2} \iint_{\Sigma_1} dx \wedge dy = \frac{2}{r^2} \iint_{x^2+y^2 \leq r^2} dx dy = \frac{2}{r^2} \pi r^2 = 2\pi.$$

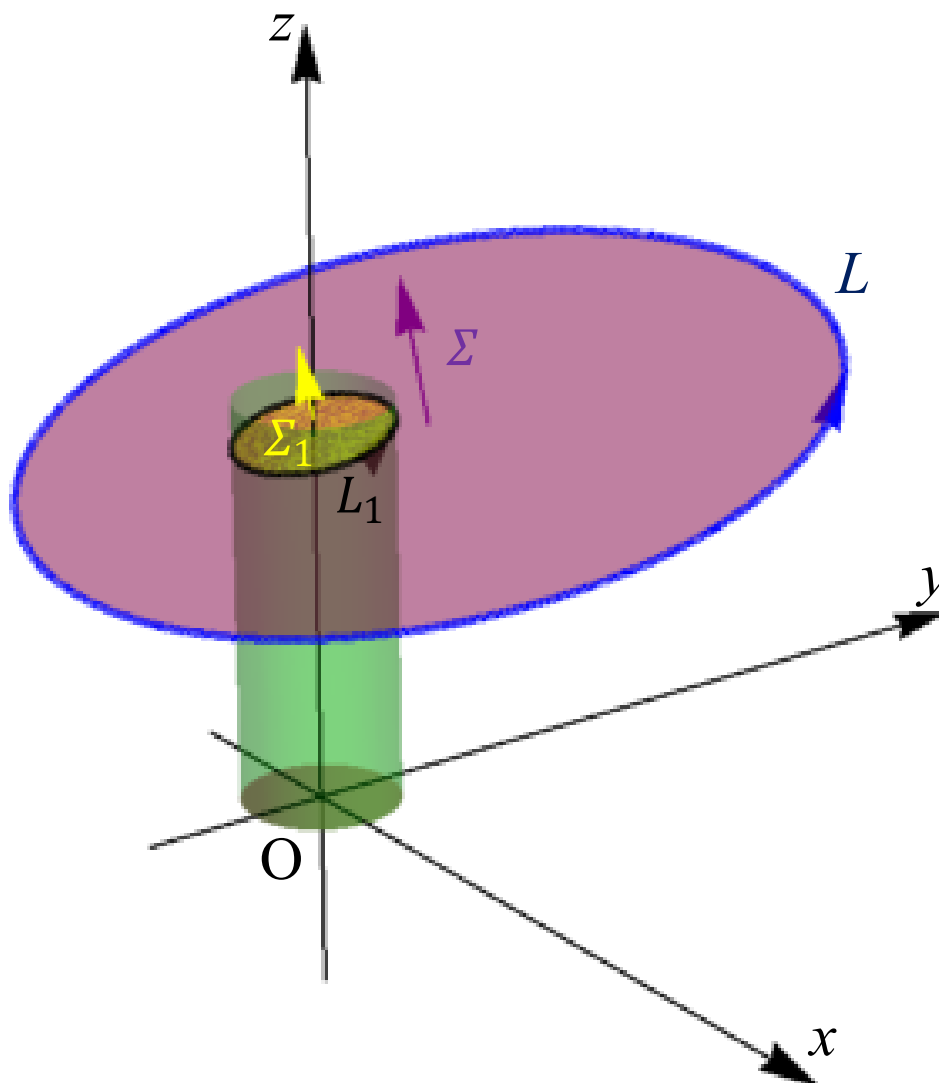


图 11: 习题13.5 3.(2)题图示

4. 证明:

$$\oint_L \begin{vmatrix} dx & dy & dz \\ \cos \alpha & \cos \beta & \cos \gamma \\ x & y & z \end{vmatrix} = 2S,$$

其中 L 是 \mathbb{R}^3 中某个平面上的一条简单逐段光滑闭曲线, $\mathbf{n} = (\cos \alpha, \cos \beta, \cos \gamma)$ 是该平面的单位法向量, L 的方向与 \mathbf{n} 的方向服从右手法则, S 是 L 所围的面积.

证明:

$$\oint_L \begin{vmatrix} dx & dy & dz \\ \cos \alpha & \cos \beta & \cos \gamma \\ x & y & z \end{vmatrix} = \oint_L (z \cos \beta - y \cos \gamma) dx + (x \cos \gamma - z \cos \alpha) dy + (y \cos \alpha - x \cos \beta) dz,$$

记 Σ 是 L 在该平面上围成的部分, Σ 的方向与 \mathbf{n} 的方向一致, 则 Σ 的面积为 S , $z \cos \beta - y \cos \gamma, x \cos \gamma - z \cos \alpha, y \cos \alpha - x \cos \beta \in C^1(\Sigma)$,

\therefore

$$\begin{aligned} \text{上式} &= \iint_{\Sigma} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z \cos \beta - y \cos \gamma & x \cos \gamma - z \cos \alpha & y \cos \alpha - x \cos \beta \end{vmatrix} \\ &= \iint_{\Sigma} \left\{ \left[\frac{\partial(y \cos \alpha - x \cos \beta)}{\partial x} - \frac{\partial(x \cos \gamma - z \cos \alpha)}{\partial z} \right] \mathbf{i} \right. \\ &\quad \left. + \left[\frac{\partial(z \cos \beta - y \cos \gamma)}{\partial z} - \frac{\partial(y \cos \alpha - x \cos \beta)}{\partial x} \right] \mathbf{j} \right. \\ &\quad \left. + \left[\frac{\partial(x \cos \beta - z \cos \alpha)}{\partial x} - \frac{\partial(z \cos \beta - y \cos \gamma)}{\partial y} \right] \mathbf{k} \right\} \cdot \mathbf{n} dS \\ &= \iint_{\Sigma} \{ [\cos \alpha - (-\cos \alpha)] \mathbf{i} + [\cos \beta - (-\cos \beta)] \mathbf{j} + [\cos \gamma - (-\cos \gamma)] \mathbf{k} \} \cdot \mathbf{n} dS \\ &= \iint_{\Sigma} (2 \cos \alpha \mathbf{i} + 2 \cos \beta \mathbf{j} + 2 \cos \gamma \mathbf{k}) \cdot \mathbf{n} dS \\ &= \iint_{\Sigma} (2 \cos \alpha, 2 \cos \beta, 2 \cos \gamma) \cdot (\cos \alpha, \cos \beta, \cos \gamma) dS \\ &= \iint_{\Sigma} (2 \cos^2 \alpha + 2 \cos^2 \beta + 2 \cos^2 \gamma) dS \\ &= 2 \iint_{\Sigma} dS = 2S. \end{aligned}$$