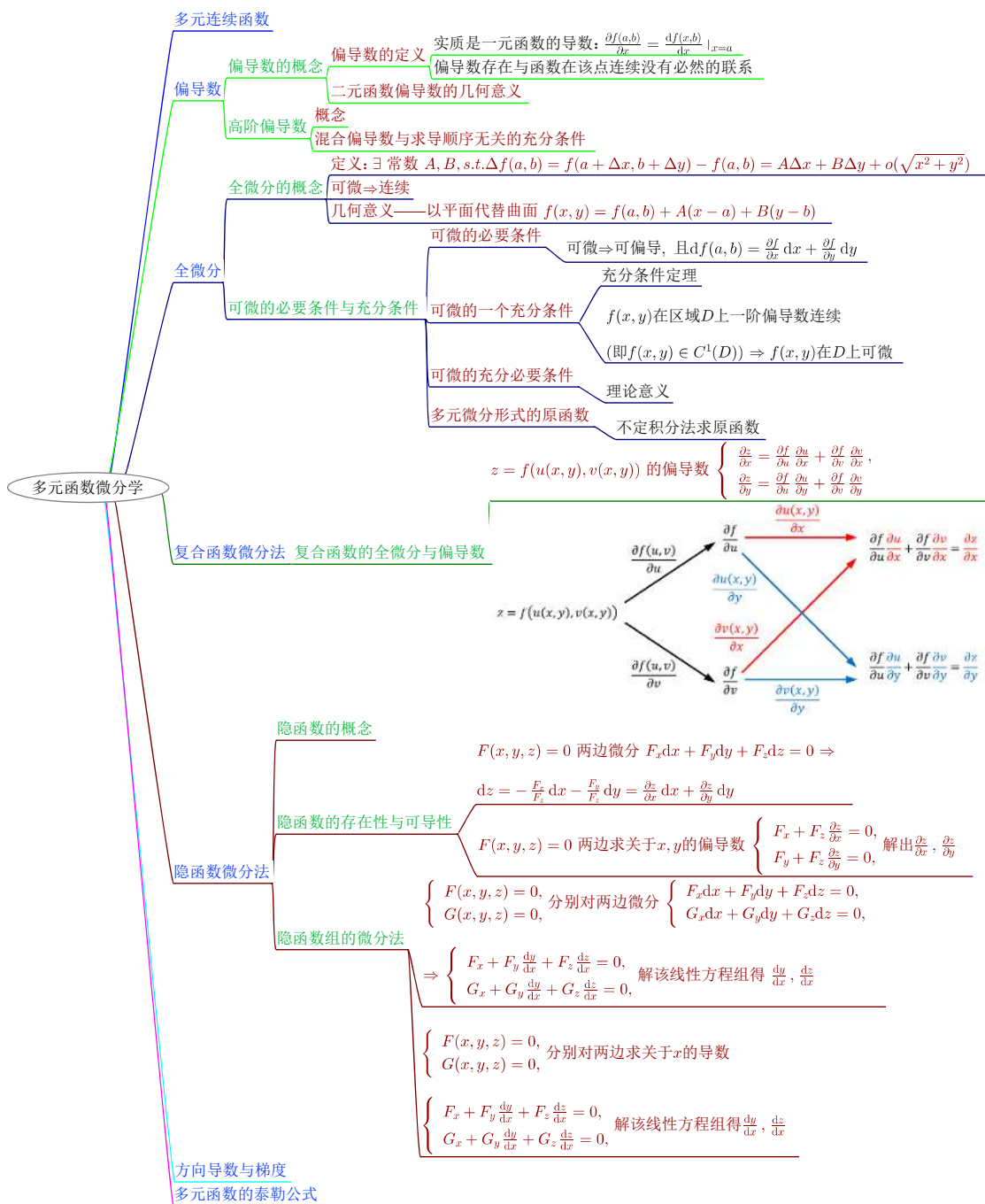


2 偏导数、全微分、复合函数微分法、隐函数微分法

2.1 复习计划



2.2 知识结构



2.3 重要知识

1. 二元函数偏导数的几何意义

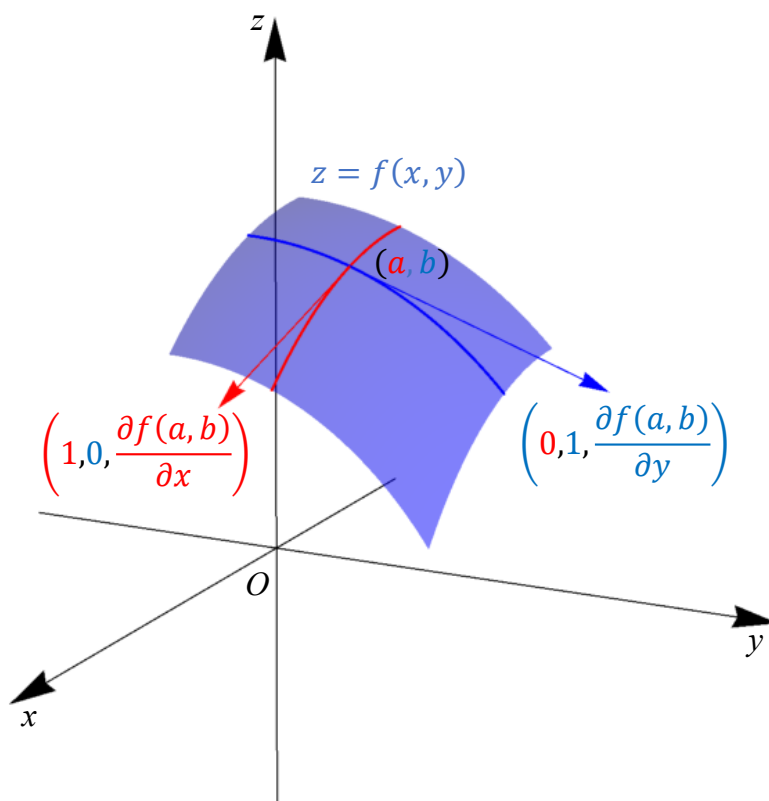


图 1: 二元函数偏导数的几何意义

2. 二元函数在一点连续、偏导数存在、可微及一阶偏导数连续之间的关系:

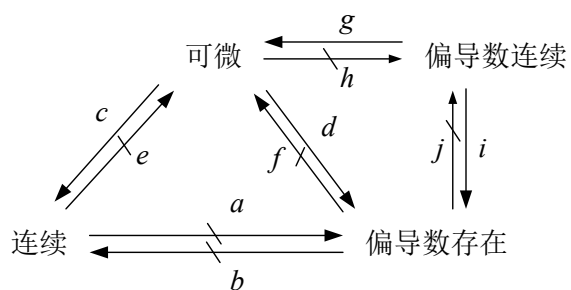


图 2: 二元函数在一点连续、偏导数存在、可微及一阶偏导数连续之间的关系

a: 反例: 函数 $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0), \end{cases}$ 在点 $(0, 0)$ 不连续 (沿直线 $y = kx$ 趋于 $(0, 0)$ 时的极限为 $\frac{k}{1+k^2} \neq \text{const}$), 但偏导数存在:

$$f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0 - 0}{\Delta x} = 0,$$

$$f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0 - 0}{\Delta y} = 0.$$

b: 反例: 函数 $g(x, y) = \sqrt{x^2 + y^2}$ 在点 $(0, 0)$ 连续, 但没有偏导数 $g_x(0, 0), g_y(0, 0)$, 因为:

$$\lim_{x \rightarrow 0+} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0+} \frac{\sqrt{x^2} - 0}{x} = 1 \neq \lim_{x \rightarrow 0-} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0-} \frac{\sqrt{x^2} - 0}{x} = -1,$$

$$\lim_{y \rightarrow 0+} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \rightarrow 0+} \frac{\sqrt{y^2} - 0}{y} = 1 \neq \lim_{y \rightarrow 0-} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \rightarrow 0-} \frac{\sqrt{y^2} - 0}{y} = -1.$$

c. 由可微的定义可知.

d. 可微的必要条件.

e. 由上述b.中的反例, 函数 $g(x, y)$ 在 $(0, 0)$ 处连续, 但在 $(0, 0)$ 处的偏导数不存在, 故函数 $g(x, y)$ 在 $(0, 0)$ 处不可微, 故连续不一定可微.

f. 由上述a.中的反例, 函数 $f(x, y)$ 在 $(0, 0)$ 处偏导数存在, 但不连续, 故不可微, 因此偏导数存在不一定可微.

g. 由可微的充分条件可知.

h. 反例: 函数 $f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0), \end{cases}$ 在 $(0, 0)$ 处

$$f_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \frac{x^2 \sin \frac{1}{x^2}}{x} = 0,$$

$$f_y(0, 0) = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y} = \frac{y^2 \sin \frac{1}{y^2}}{y} = 0,$$

且

$$\begin{aligned} & \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{f(\Delta x, \Delta y) - f(0, 0) - f_x(0, 0)\Delta x - f_y(0, 0)\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \\ &= \lim_{(x, y) \rightarrow (0, 0)} \frac{(x^2 + y^2) \sin \frac{1}{x^2+y^2} - 0 - 0 - 0}{\sqrt{x^2 + y^2}} \\ &= \lim_{(x, y) \rightarrow (0, 0)} \sqrt{x^2 + y^2} \sin \frac{1}{x^2 + y^2} = 0 \end{aligned}$$

故 $f(x, y)$ 在 $(0, 0)$ 处可微. 但

$$f_x(x, y) = \begin{cases} 0, & (x, y) = (0, 0), \\ 2x \sin \frac{1}{x^2+y^2} - \frac{2x}{x^2+y^2} \cos \frac{1}{x^2+y^2}, & (x, y) \neq (0, 0), \end{cases}$$

在 $(0, 0)$ 点不连续.

i. 显然.

j. 如上述h.中的反例.

3. 隐函数求导两种方法等价性的证明

设二元函数 $z = z(x, y)$ 由方程 $F(x, y, z) = 0$ 确定, $F(x, y, z)$ 有连续的偏导数, 求 $z = z(x, y)$ 关于 x, y 的偏导数有以下两种方法:

(1) 将方程 $F(x, y, z) = 0$ 两边分别对 x, y 求偏导数:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0, \quad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} = 0,$$

解得

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z}, \quad \frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}.$$

(2) 将方程 $F(x, y, z) = 0$ 两边求全微分:

$$dF(x, y, z) = F'_x dx + F'_y dy + F'_z dz = 0,$$

整理得

$$dz = -\frac{F'_x}{F'_z} dx - \frac{F'_y}{F'_z} dy,$$

则

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z}, \quad \frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}.$$

以上这两种方法是等价的, 可做如下证明. 因为 $z = z(x, y)$ 是方程 $F(x, y, z) = 0$ 确定的隐函数, 所以将 $z = z(x, y)$ 代入函数 $F = F(x, y, z)$ 得到一个关于 x, y 的常数函数 $f(x, y) = F(x, y, z(x, y)) = 0$ (比如方程 $F(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0$ 确定隐函数 $z(x, y) = \sqrt{R^2 - x^2 - y^2}$, 将 $z(x, y)$ 代入 $F(x, y, z) = x^2 + y^2 + z^2 - R^2$ 得到 $f(x, y) = F(x, y, z(x, y)) = x^2 + y^2 + (\sqrt{R^2 - x^2 - y^2})^2 - R^2 \equiv 0$, 为一恒等于0的常数函数), 该常数函数关于 x, y 的两个偏导数都等于0, 即

$$\frac{\partial f}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} = 0,$$

可据此求出 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$, 即方法(1).

因为 $f(x, y) = F(x, y, z(x, y)) = 0$ 是一个常数函数, 所以对该函数关于 x, y 求全微分等于 0, 即

$$df(x, y) = \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} \right) dx + \left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} \right) dy = 0dx + 0dy = 0$$

将该式做如下整理:

$$\begin{aligned} 0 = df(x, y) &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \left(\frac{\partial F}{\partial z} \frac{\partial z}{\partial x} dx + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} dy \right) \\ &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} \left(\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \right) \\ &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz \\ &= dF(x, y, z) \end{aligned} \quad (1)$$

可得到

$$dz = -\frac{F'_x}{F'_z} dx - \frac{F'_y}{F'_z} dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy,$$

可据此求出 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$, 即方法(2). 式(1)的推导过程即是全微分形式不变性的证明过程.

2.4 习题分类与解题思路

1. 偏导数.

(a) 第一类题目: 二元函数在一点连续、偏导数存在、可微及一阶偏导数连续之间的关系.

解题思路见上述2.3小节中的2.

【如习题10.2中的1.】

(b) 求偏导数和高阶偏导数, 有以下几种方法:

i. 利用一元函数的求导法则.

【如习题10.2中的3.(1)/(2)/(3)/(4)/(5)/(6)/(7).】

ii. 对于特殊点处直接利用偏导数的定义求解.

【如习题10.2中的2.】

2. 全微分

(a) 求全微分.

【如习题10.3中的1.】

(b) 证明函数在一点不可微.

可采取如下思路:

第一步 先判断函数在该点是否连续, 若不连续则不可微.

第二步 若连续, 判断函数在该点的偏导数是否存在, 若不存在, 则不可微.

【如习题10.3中的2.(1)】

第三步 若函数在该点的偏导数存在, 则可假设函数可微, 从而求出该点的全微分

值 $df(x_0, y_0) = f_x \Delta x + f_y \Delta y$, 计算极限 $\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - df(x_0, y_0)}{\sqrt{\Delta x^2 + \Delta y^2}}$ 是否为0, 若为0则函数可微, 若不为0则函数不可微.

【如习题10.3中的2.(2)】

(c) 证明函数可微.

i. 利用可微的充分条件.

【如习题10.3中的3.】

ii. 利用可微的定义. 可采用和“(b)证明函数不可微”相同的思路.

(d) 利用函数微分做近似计算.

【如习题10.3中的4.】

(e) 求全微分式的原函数. 可用不定积分法.

【如习题10.3中的6.】

【其他类型的题目: 习题10.3中的5.】

3. 复合函数的导数.

(a) 利用复合函数求导的链式法则求偏导数.

【如习题10.4中的1.(1)/(2)/(3)/(4)/(5)/(6), 2., 3.】

(b) 考查高阶混合偏导数与求导顺序无关的条件.

【如习题10.4中的4.(该题需要特殊的技巧)】

4. 隐函数的导数.

(a) 求隐函数的偏导数.

有以下两种方法:

i. 方程两边求偏导.

ii. 方程两边求全微分.

具体可见【习题10.5中的2., 3., 5.】

(b) 求隐函数组的偏导数.

i. 方程组两边求偏导.

ii. 方程组两边求全微分.

具体可见【习题10.5中的1., 6.】

【其他类型的题目：习题10.5中的4.】

【习题10.5中3.的方法2是一个特殊的技巧，大家可以积累一下.】

2.5 习题10.2解答

1. 若 $f(x, y)$ 在点 (x, y) 处连续，能否推出 $f(x, y)$ 在点 (x, y) 的两个偏导数存在？若 $f(x, y)$ 在点 (x, y) 的两个偏导数都存在，能否推出 $f(x, y)$ 在点 (x, y) 处连续？

解：(1)不能. 如函数 $f(x, y) = \sqrt{x^2 + y^2}$ 在原点连续，但是下列两个极限都不存在：

$$\lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{x^2}}{x},$$

$$\lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{\sqrt{y^2}}{y}.$$

所以在原点 $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ 都不存在。

(2)不能. 如函数 $f(x, y) = \begin{cases} 1, & y = x^2, x > 0, \\ 0, & \text{其他.} \end{cases}$ 因为 $f(x, 0) \equiv 0, f(0, y) \equiv 0$ ，故 $f(x, y)$ 在原点的两个偏导数 $\frac{\partial f(0, 0)}{\partial x} = 0, \frac{\partial f(0, 0)}{\partial y} = 0$. 但 $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ 不存在（参见教材例10.1.2），所以该函数在原点不连续。

2. 设 $z = \sqrt{|xy|}$, 求 $\frac{\partial z}{\partial x}$.

$$\text{解: } \because z = \sqrt{|xy|} = \sqrt{|y|}\sqrt{|x|} = \begin{cases} \sqrt{|y|}\sqrt{x}, & x \geq 0, \\ \sqrt{|y|}\sqrt{-x}, & x < 0. \end{cases}$$

$$\therefore \text{当 } x > 0 \text{ 时, } \frac{\partial z}{\partial x} = \frac{\sqrt{|y|}}{2\sqrt{x}},$$

$$\text{当 } x < 0 \text{ 时, } \frac{\partial z}{\partial x} = -\frac{\sqrt{|y|}}{2\sqrt{-x}},$$

$$\text{当 } x = 0 \text{ 时 } \lim_{x \rightarrow 0^-} \frac{\sqrt{|xy|}}{x} = \lim_{x \rightarrow 0^-} -\frac{\sqrt{-x|y|}}{-x} = \lim_{x \rightarrow 0^-} -\frac{\sqrt{|y|}}{\sqrt{-x}} = \begin{cases} -\infty, & y \neq 0, \\ 0, & y = 0 \end{cases},$$

$$\lim_{x \rightarrow 0^+} \frac{\sqrt{|xy|}}{x} = \lim_{x \rightarrow 0^+} \frac{\sqrt{x|y|}}{x} = \lim_{x \rightarrow 0^+} \frac{\sqrt{|y|}}{\sqrt{x}} = \begin{cases} +\infty, & y \neq 0, \\ 0, & y = 0 \end{cases}.$$

$$\therefore \frac{\partial z}{\partial x} = \begin{cases} \frac{\sqrt{|y|}}{2\sqrt{x}}, & x > 0, \\ \text{不存在}, & x = 0 \text{ 且 } y \neq 0 \\ 0, & x = 0, y = 0 \\ -\frac{\sqrt{|y|}}{2\sqrt{-x}}, & x < 0. \end{cases}$$

3. 求下列偏导数:

(1) $z = \frac{x+y}{x-y}$, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$;

(2) $f(x, y) = \arctan \frac{y}{x}$, 求 $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$;

(3) $z = \cos \frac{y}{x} \sin \frac{x}{y}$, 求 $\frac{\partial z(2, \pi)}{\partial x}, \frac{\partial z(2, \pi)}{\partial y}$;

(4) $z = \arcsin \sqrt{\frac{x}{y}} + \frac{1}{xy} e^{\frac{y}{x}}$, 求 $\frac{\partial z(1, 2)}{\partial x}, \frac{\partial z(1, 2)}{\partial y}$;

(5) $z = \ln(\sqrt{x} + \sqrt{y})$, 求 $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$;

(6) $z = \frac{x-y}{x+y} \ln \frac{y}{x}$, 求 $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$;

(7) $u = \sqrt{x^2 + y^2 + z^2}$, 求 $(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2$.

解: (1) $\frac{\partial z}{\partial x} = \frac{x-y-(x+y)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$, $\frac{\partial z}{\partial y} = \frac{x-y+(x+y)}{(x-y)^2} = \frac{2x}{(x-y)^2}$.

(2) $\frac{\partial f}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} (x \neq 0)$.

(3) $\frac{\partial z}{\partial x} = -(-\frac{y}{x^2}) \sin \frac{y}{x} \sin \frac{x}{y} + \frac{1}{y} \cos \frac{y}{x} \cos \frac{x}{y} = \frac{y}{x^2} \sin \frac{y}{x} \sin \frac{x}{y} + \frac{1}{y} \cos \frac{y}{x} \cos \frac{x}{y}$,
 $\frac{\partial z}{\partial y} = -\frac{1}{x} \sin \frac{y}{x} \sin \frac{x}{y} - \frac{x}{y^2} \cos \frac{y}{x} \cos \frac{x}{y}$,

$\therefore \frac{\partial z(2, \pi)}{\partial x} = \frac{\pi}{4} \sin \frac{\pi}{2} \sin \frac{2}{\pi} + \frac{1}{\pi} \cos \frac{\pi}{2} \cos \frac{2}{\pi} = \frac{\pi}{4} \sin \frac{2}{\pi}$,
 $\frac{\partial z(2, \pi)}{\partial y} = -\frac{1}{2} \sin \frac{\pi}{2} \sin \frac{2}{\pi} - \frac{2}{\pi^2} \cos \frac{\pi}{2} \cos \frac{2}{\pi} = -\frac{1}{2} \sin \frac{2}{\pi}$.

(4) $\therefore \frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-\frac{x}{y}}} \cdot \frac{1}{2\sqrt{xy}} - \frac{1}{x^2 y} e^{\frac{y}{x}} + \frac{1}{xy} e^{\frac{y}{x}} (-\frac{y}{x^2}) = \frac{1}{2\sqrt{xy-x^2}} - (\frac{1}{x^2 y} + \frac{1}{x^3}) e^{\frac{y}{x}}$,
 $\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-\frac{x}{y}}} (-\frac{1}{2} \sqrt{\frac{x}{y^3}}) - \frac{1}{xy^2} e^{\frac{y}{x}} + \frac{1}{xy} e^{\frac{y}{x}} \frac{1}{x} = -\frac{1}{2} \sqrt{\frac{x}{y^3-xy^2}} + (\frac{1}{x^2 y} - \frac{1}{xy^2}) e^{\frac{y}{x}}$,

$\therefore \frac{\partial z(1, 2)}{\partial x} = \frac{1}{2\sqrt{2-1}} - (\frac{1}{2} + 1) e^2 = \frac{1}{2} - \frac{3}{2} e^2$, $\frac{\partial z(1, 2)}{\partial y} = -\frac{1}{2} \sqrt{\frac{1}{8-4}} + (\frac{1}{2} - \frac{1}{4}) e^2 = -\frac{1}{4} + \frac{1}{4} e^2$.

(5) $\therefore \frac{\partial z}{\partial x} = \frac{1}{\sqrt{x+\sqrt{y}}} \cdot \frac{1}{2\sqrt{x}}$, $\frac{\partial z}{\partial y} = \frac{1}{\sqrt{x+\sqrt{y}}} \cdot \frac{1}{2\sqrt{y}}$,

$\therefore x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{\sqrt{x}}{2(\sqrt{x+\sqrt{y}})} + \frac{\sqrt{y}}{2(\sqrt{x+\sqrt{y}})} = \frac{1}{2}$.

(6) $\therefore \frac{\partial z}{\partial x} = \frac{x+y-(x-y)}{(x+y)^2} \ln \frac{y}{x} + \frac{x-y}{x+y} \cdot \frac{x}{y} (-\frac{y}{x^2}) = \frac{2y}{(x+y)^2} \ln \frac{y}{x} - \frac{1}{x} \frac{x-y}{x+y}$,
 $\frac{\partial z}{\partial y} = \frac{-(x+y)-(x-y)}{(x+y)^2} \ln \frac{y}{x} + \frac{x-y}{x+y} \cdot \frac{x}{y} \frac{1}{x} = \frac{-2x}{(x+y)^2} \ln \frac{y}{x} + \frac{1}{y} \frac{x-y}{x+y}$,

$\therefore x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{2xy}{(x+y)^2} \ln \frac{y}{x} - \frac{x-y}{x+y} + \frac{-2xy}{(x+y)^2} \ln \frac{y}{x} + \frac{x-y}{x+y} = 0$.

(7) $\therefore \frac{\partial u}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}$, $\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$, $\frac{\partial u}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$ (后两式可根据 x, y, z 的对称性得到),

$\therefore (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2 = \frac{x^2+y^2+z^2}{x^2+y^2+z^2} = 1$.

4. 求下列高阶导数:

$$(1) z = x + y + \frac{1}{xy}, \text{ 求 } \frac{\partial^2 z(1,1)}{\partial x \partial y};$$

$$(2) z = y^{\ln x}, \text{ 求 } \frac{\partial^2 z}{\partial x \partial y};$$

$$(3) z = \ln(x + \sqrt{x^2 + y^2}), \text{ 求 } \frac{\partial^2 z}{\partial x \partial y};$$

$$(4) z = \ln(\sqrt{(x-a)^2 + (y-b)^2}), \text{ 求 } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2};$$

$$(5) u = \sqrt{x^2 + y^2 + z^2}, \text{ 求 } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2};$$

$$(6) z = \sin(xy), \text{ 求 } \frac{\partial^3 z}{\partial x \partial y^2};$$

$$(7) f(x, y, z) = xy^2 + yz^2 + zx^2, \text{ 求 } \frac{\partial^2 f(0,0,1)}{\partial x^2}, \frac{\partial^2 f(1,0,2)}{\partial x \partial z}, \frac{\partial^2 f(0,-1,0)}{\partial y \partial z}, \frac{\partial^3 f(2,0,1)}{\partial x \partial z^2}.$$

$$\text{解: (1)} \because \frac{\partial z}{\partial y} = 1 - \frac{1}{xy^2}, \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{x^2 y^2}, \therefore \frac{\partial^2 z(1,1)}{\partial x \partial y} = 1.$$

$$(2) \frac{\partial z}{\partial y} = y^{\ln x - 1} \ln x, \frac{\partial^2 z}{\partial x \partial y} = y^{\ln x - 1} \ln y \frac{1}{x} \ln x + y^{\ln x - 1} \frac{1}{x} = \frac{y^{\ln x}}{xy} (\ln y \ln x + 1).$$

$$(3) \frac{\partial z}{\partial y} = \frac{\frac{2y}{2\sqrt{x^2+y^2}}}{x + \sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}(x + \sqrt{x^2+y^2})},$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{-y[\frac{2x}{2\sqrt{x^2+y^2}}(x + \sqrt{x^2+y^2}) + \sqrt{x^2+y^2}(1 + \frac{2x}{2\sqrt{x^2+y^2}})]}{(x^2+y^2)(x + \sqrt{x^2+y^2})^2} = \frac{-y(\frac{x}{\sqrt{x^2+y^2}} + 1)(x + \sqrt{x^2+y^2})}{(x^2+y^2)(x + \sqrt{x^2+y^2})^2} \\ &= \frac{-y\frac{1}{\sqrt{x^2+y^2}}(x + \sqrt{x^2+y^2})^2}{(x^2+y^2)(x + \sqrt{x^2+y^2})^2} = -\frac{y}{(x^2+y^2)^{\frac{3}{2}}}. \end{aligned}$$

$$(4) \frac{\partial z}{\partial x} = \frac{1}{\sqrt{(x-a)^2 + (y-b)^2}} \frac{2(x-a)}{2\sqrt{(x-a)^2 + (y-b)^2}} = \frac{x-a}{(x-a)^2 + (y-b)^2},$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{(x-a)^2 + (y-b)^2}} \frac{2(y-b)}{2\sqrt{(x-a)^2 + (y-b)^2}} = \frac{y-b}{(x-a)^2 + (y-b)^2},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{(x-a)^2 + (y-b)^2 - (x-a)2(x-a)}{[(x-a)^2 + (y-b)^2]^2} = \frac{(y-b)^2 - (x-a)^2}{[(x-a)^2 + (y-b)^2]^2},$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{(x-a)^2 + (y-b)^2 - (y-b)2(y-b)}{[(x-a)^2 + (y-b)^2]^2} = \frac{(x-a)^2 - (y-b)^2}{[(x-a)^2 + (y-b)^2]^2},$$

$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{(y-b)^2 - (x-a)^2}{[(x-a)^2 + (y-b)^2]^2} + \frac{(x-a)^2 - (y-b)^2}{[(x-a)^2 + (y-b)^2]^2} = 0.$$

$$(5) \frac{\partial u}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{\partial^2 u}{\partial x^2} = \frac{\sqrt{x^2+y^2+z^2} - x \frac{2x}{2\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

$$\text{根据 } x, y, z \text{ 的对称性可得 } \frac{\partial^2 u}{\partial y^2} = \frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{2}{\sqrt{x^2+y^2+z^2}}.$$

$$(6) \frac{\partial z}{\partial y} = x \cos(xy), \frac{\partial^2 z}{\partial y^2} = -x^2 \sin(xy), \frac{\partial^3 z}{\partial x \partial y^2} = -2x \sin(xy) - x^2 y \cos(xy).$$

$$(7) \because \frac{\partial f}{\partial x} = y^2 + 2zx, \frac{\partial^2 f}{\partial x^2} = 2z, \therefore \frac{\partial^2 f(0,0,1)}{\partial x^2} = 2,$$

$$\therefore \frac{\partial f}{\partial z} = 2yz + x^2, \frac{\partial^2 f}{\partial x \partial z} = 2x, \therefore \frac{\partial^2 f(1,0,2)}{\partial x \partial z} = 2,$$

$$\therefore \frac{\partial^2 f}{\partial y \partial z} = 2z, \therefore \frac{\partial^2 f(0,-1,0)}{\partial y \partial z} = 0,$$

$$\therefore \frac{\partial^2 f}{\partial z^2} = 2y, \frac{\partial^3 f}{\partial x \partial z^2} = 0, \therefore \frac{\partial^3 f(2,0,1)}{\partial x \partial z^2} = 0.$$

2.6 习题10.3解答

1. 求下列函数在指定点的全微分:

(1) $z = \arctan \frac{x+y}{x-y}$, 在任意点 (x, y) ;

(2) $z = \ln \sqrt{1+x^2+y^2}$, 在点 $(1, 1)$;

(3) $z = e^{-(\frac{y}{x} - \frac{x}{y})}$, 在点 $(1, -1)$;

(4) $z = \arctan \frac{x}{1+y^2}$, 求 $dz(1, 1)$;

(5) $u = (\frac{x}{y})^z$, 在任一点 (x, y, z) .

解: (1) $\frac{\partial z}{\partial x} = \frac{1}{1+(\frac{x+y}{x-y})^2} \frac{x-y-(x+y)}{(x-y)^2} = \frac{-2y}{(x+y)^2+(x-y)^2} = \frac{-y}{x^2+y^2}$,

$\frac{\partial z}{\partial y} = \frac{1}{1+(\frac{x+y}{x-y})^2} \frac{x-y+(x+y)}{(x-y)^2} = \frac{x}{x^2+y^2}$,

当 $x \neq y$ 时, $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ 均连续, 故函数 z 在任意点 (x, y) 可微,

$dz(x, y) = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{-y dx + x dy}{x^2+y^2}$.

(2) $\frac{\partial z}{\partial x} = \frac{\frac{1}{2} \ln(1+x^2+y^2)}{\frac{\partial}{\partial x} \ln(1+x^2+y^2)} = \frac{1}{2} \frac{2x}{1+x^2+y^2} = \frac{x}{1+x^2+y^2}$, $\frac{\partial z}{\partial y} = \frac{y}{1+x^2+y^2}$,

因为 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ 在点 $(1, 1)$ 及其附近存在且在点 $(1, 1)$ 处连续, 故函数 z 在点 $(1, 1)$ 处可微,

$dz(1, 1) = \frac{\partial z(1, 1)}{\partial x} dx + \frac{\partial z(1, 1)}{\partial y} dy = \frac{1}{3}(dx + dy)$.

(3) $\frac{\partial z}{\partial x} = e^{-(\frac{y}{x} - \frac{x}{y})} [-(\frac{y}{x^2} - \frac{1}{y})] = e^{-(\frac{y}{x} - \frac{x}{y})} (\frac{y}{x^2} + \frac{1}{y})$, $\frac{\partial z}{\partial y} = e^{-(\frac{y}{x} - \frac{x}{y})} (-\frac{1}{x} - \frac{x}{y^2})$,

因为 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ 在点 $(1, -1)$ 及其附近存在且在点 $(1, -1)$ 处连续, 故函数 z 在点 $(1, -1)$ 可微,

$dz(1, -1) = \frac{\partial z(1, -1)}{\partial x} dx + \frac{\partial z(1, -1)}{\partial y} dy = -2dx - 2dy$.

(4) $\frac{\partial z}{\partial x} = \frac{1}{1+(\frac{x}{1+y^2})^2} \frac{1}{1+y^2} = \frac{1+y^2}{x^2+(1+y^2)^2}$, $\frac{\partial z}{\partial y} = \frac{1}{1+(\frac{x}{1+y^2})^2} \frac{-2xy}{(1+y^2)^2} = \frac{-2xy}{x^2+(1+y^2)^2}$,

因为 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ 在点 $(1, 1)$ 及其附近存在且在点 $(1, 1)$ 处连续, 故函数 z 在点 $(1, 1)$ 可微,

$dz(1, 1) = \frac{\partial z(1, 1)}{\partial x} dx + \frac{\partial z(1, 1)}{\partial y} dy = \frac{2}{5}dx - \frac{2}{5}dy$.

(5) $\frac{\partial u}{\partial x} = \frac{z}{y} (\frac{x}{y})^{z-1}$, $\frac{\partial u}{\partial y} = -\frac{z}{x} (\frac{y}{x})^{-z-1}$, $\frac{\partial u}{\partial z} = (\frac{x}{y})^z \ln \frac{x}{y}$,

当 $xy > 0$ 时, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ 均连续, 故函数 z 在任一点 (x, y, z) 可微,

$dz = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = \frac{z}{y} (\frac{x}{y})^{z-1} dx - \frac{z}{x} (\frac{y}{x})^{-z-1} dy + (\frac{x}{y})^z \ln \frac{x}{y} dz$.

2. 试证明下列函数在 $(0, 0)$ 点不可微:

(1) $f(x, y) = \sqrt{x} \cos y$;

(2) $f(x, y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$

解: (1) 假设 $f(x, y)$ 在点 $(0, 0)$ 处可微, 则 $\frac{\partial f(0, 0)}{\partial x}$ 存在,

$\therefore f(x, y) = \sqrt{x} \cos y$,

$\therefore x \geq 0$,

$$\because \lim_{x \rightarrow 0^+} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x \rightarrow 0^+} \frac{\sqrt{x}-0}{x-0} = +\infty,$$

$\therefore \frac{\partial f}{\partial x}$ 在点(0,0)不存在, 故函数 $f(x,y)$ 在点(0,0)不可微.

$$(2) \because \left| \frac{2xy}{\sqrt{x^2+y^2}} - f(0,0) \right| = \frac{|2xy|}{\sqrt{x^2+y^2}} \leq \frac{2|x||y|}{|x|} = 2|y|,$$

$$\text{又} \because \lim_{(x,y) \rightarrow (0,0)} 2|y| = 0,$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{\sqrt{x^2+y^2}} = f(0,0),$$

$\therefore f(x,y)$ 在点(0,0)处连续,

$$\because f(x,0) = 0, f(0,y) = 0$$

$$\therefore \frac{\partial f(0,0)}{\partial x} = 0, \frac{\partial f(0,0)}{\partial y} = 0$$

\therefore 若 $f(x,y)$ 在点(0,0)处可微, 则 $df(0,0) = 0$, 且

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y)-f(0,0)-df(0,0)}{\sqrt{x^2+y^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{x^2+y^2} = 0,$$

$$\because \lim_{\substack{x \rightarrow 0 \\ y=x}} \frac{2xy}{x^2+y^2} = \lim_{x \rightarrow 0} \frac{2x^2}{x^2+x^2} = 1 \neq 0,$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y)-f(0,0)-df(0,0)}{\sqrt{x^2+y^2}} \neq 0, \text{ 矛盾,}$$

\therefore 函数 $f(x,y)$ 在点(0,0)不可微.

3. 已知函数 $g(x), h(x)$ 分别在区间 $[x_0, x_1]$ 与 $[y_0, y_1]$ 上连续, 试证函数

$$f(x,y) = \int_{x_0}^x g(s)ds \int_{y_0}^y h(t)dt$$

在点 (x,y) 可微, 其中 $(x,y) \in D = \{(x,y) \mid x_0 \leq x \leq x_1, y_0 \leq y \leq y_1\}$.

证明: $\because g(x), h(x)$ 分别在区间 $[x_0, x_1]$ 与 $[y_0, y_1]$ 上连续,

$\therefore \frac{\partial f(x,y)}{\partial x} = g(x) \int_{y_0}^y h(t)dt, \frac{\partial f(x,y)}{\partial y} = h(y) \int_{x_0}^x g(s)ds$ 且 $\int_{x_0}^x g(s)ds$ 和 $\int_{y_0}^y h(t)dt$ 分别在区间 $[x_0, x_1]$ 与 $[y_0, y_1]$ 上连续,

$\therefore \frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}$ 在 D 中任意一点 (x,y) 连续,

\therefore 函数 $f(x,y)$ 在 D 中任意一点 (x,y) 可微.

4. 用函数微分计算下列数值的近似值:

$$(1) \sqrt{1.02^2 + 1.97^2}; \quad (2) 0.97^{1.05}.$$

解: (1) 令 $f(x,y) = \sqrt{x^2 + y^2}$,

$\therefore \frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}, \frac{\partial f}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$ 在点(1,2)及其附近存在且在点(1,2)处连续, 故 $f(x,y)$ 在点(1,2)可微,

$$\begin{aligned}\therefore \sqrt{1.02^2 + 1.97^2} &= f(1.02, 1.97) \approx f(1, 2) + 0.02 \frac{\partial f(1, 2)}{\partial x} + (-0.03) \frac{\partial f(1, 2)}{\partial y} \\ &= \sqrt{5} + 0.02 \times \frac{1}{\sqrt{5}} - 0.03 \times \frac{2}{\sqrt{5}} = \frac{5-0.04}{\sqrt{5}} \approx 2.2182.\end{aligned}$$

$$(2) \text{ 令 } f(x, y) = x^y,$$

$$\therefore \frac{\partial f}{\partial x} = yx^{y-1}, \frac{\partial f}{\partial y} = x^y \ln x \text{ 在点 } (1, 1) \text{ 及其附近存在且在点 } (1, 1) \text{ 处连续},$$

$$\begin{aligned}\therefore 0.97^{1.05} &= f(0.97, 1.05) \approx f(1, 1) + (-0.03) \frac{\partial f(1, 1)}{\partial x} + 0.05 \frac{\partial f(1, 1)}{\partial y} \\ &= 1 - 0.03 \times 1 + 0.05 \times 0 = 0.97.\end{aligned}$$

5. 设二元函数 $z(x, y)$ 满足方程 $\frac{\partial^2 z}{\partial x \partial y} = x + y$, 并且 $z(x, 0) = x, z(0, y) = y^2$. 试求 $z(x, y)$.

$$\text{解: 方法1: } \because \frac{\partial^2 z}{\partial x \partial y} = x + y,$$

$$\therefore \int \frac{\partial^2 z}{\partial x \partial y} dx = \int (x + y) dx = \frac{1}{2}x^2 + xy + C_1(y) = \frac{\partial z}{\partial y} + C_1(y),$$

$$\therefore \text{可设 } \frac{\partial z}{\partial y} = \frac{1}{2}x^2 + xy + C(y), \text{ 其中 } C(y) \text{ 是与 } x \text{ 无关的 } y \text{ 的函数},$$

$$\because z(0, y) = y^2,$$

$$\therefore \frac{\partial z(0, y)}{\partial y} = 2y = C(y),$$

$$\therefore \frac{\partial z}{\partial y} = \frac{1}{2}x^2 + xy + 2y,$$

$$\therefore \int \frac{\partial z}{\partial y} dy = \int (\frac{1}{2}x^2 + xy + 2y) dy = \frac{1}{2}x^2 y + \frac{1}{2}xy^2 + y^2 + C_3(x) = z(x, y) + C_4(x),$$

$$\therefore \text{可设 } z(x, y) = \int (\frac{1}{2}x^2 + xy + 2y) dy + C^*(x) = \frac{1}{2}x^2 y + \frac{1}{2}xy^2 + y^2 + C^*(x), \text{ 其中 } C^*(x) \text{ 是与 } y \text{ 无关的 } x \text{ 的函数},$$

$$\because z(x, 0) = x = C^*(x),$$

$$\therefore z(x, y) = \frac{1}{2}x^2 y + \frac{1}{2}xy^2 + y^2 + x.$$

$$\text{方法2: } \because \frac{\partial^2 z}{\partial x \partial y} = x + y,$$

$$\therefore \int \frac{\partial^2 z}{\partial x \partial y} dx = \int (x + y) dx = \frac{1}{2}x^2 + xy + C_1(y) = \frac{\partial z}{\partial y} + C_2(y),$$

$$\therefore \text{可设 } \frac{\partial z}{\partial y} = \frac{1}{2}x^2 + xy + C(y), \text{ 其中 } C(y) \text{ 是与 } x \text{ 无关的 } y \text{ 的函数},$$

$$\therefore \int \frac{\partial z}{\partial y} dy = \frac{1}{2}x^2 y + \frac{1}{2}xy^2 + \int C(y) dy = z(x, y) + C_3(x),$$

$$\therefore \text{可设 } z(x, y) = \frac{1}{2}x^2 y + \frac{1}{2}xy^2 + F(y) + C^*(x), \text{ 其中 } F(y) \text{ 是 } C(y) \text{ 的一个与 } x \text{ 无关的原函数}, \\ C^*(x) \text{ 是与 } y \text{ 无关的 } x \text{ 的函数},$$

$$\because z(0, y) = y^2, z(x, 0) = x,$$

$$\therefore F(y) + C^*(0) = y^2, F(0) + C^*(x) = x,$$

$$\therefore F(y) = y^2 - C^*(0), C^*(x) = x - F(0), \text{ 且 } F(0) + C^*(0) = 0,$$

$$\therefore z(x, y) = \frac{1}{2}x^2 y + \frac{1}{2}xy^2 + y^2 + x - [F(0) + C^*(0)] = \frac{1}{2}x^2 y + \frac{1}{2}xy^2 + y^2 + x.$$

6. 求 $y^2 e^{x+y}(dx + dy) + 2ye^{x+y}dy$ 的原函数.

解: 方法1: 设 $f(x, y)$ 是 $y^2 e^{x+y}(dx + dy) + 2ye^{x+y}dy = y^2 e^{x+y}dx + (y^2 + 2y)e^{x+y}dy$ 的原函数, 则 $\frac{\partial f}{\partial x} = y^2 e^{x+y}$,

$$\therefore \int \frac{\partial f}{\partial x} dx = \int y^2 e^{x+y} dx = y^2 e^y e^x + C_1(y) = f(x, y) + C_2(y),$$

$$\therefore f(x, y) = y^2 e^y e^x + C(y),$$

$$\therefore \frac{\partial f}{\partial y} = 2ye^y e^x + y^2 e^y e^x + C'(y) = (y^2 + 2y)e^{x+y} + C'(y),$$

$$\text{又} \because \frac{\partial f}{\partial y} = (y^2 + 2y)e^{x+y},$$

$$\therefore C'(y) = 0,$$

$$\therefore C(y) = C,$$

$$\therefore f(x, y) = y^2 e^{x+y} + C.$$

方法2: 设 $f(x, y)$ 是 $y^2 e^{x+y}(dx + dy) + 2ye^{x+y}dy = y^2 e^{x+y}dx + (y^2 + 2y)e^{x+y}dy$ 的原函数, 则 $\frac{\partial f}{\partial y} = (y^2 + 2y)e^{x+y}$,

$$\begin{aligned} \therefore \int \frac{\partial f}{\partial y} dy &= \int (y^2 + 2y)e^{x+y} dy = e^x \int (y^2 + 2y) de^y = e^x [(y^2 + 2y)e^y - \int e^y d(y^2 + 2y)] \\ &= e^x [(y^2 + 2y)e^y - \int e^y (2y + 2) dy] = e^x [(y^2 + 2y)e^y - \int (2y + 2) de^y] \\ &= e^x [(y^2 + 2y)e^y - (2y + 2)e^y + \int e^y d(2y + 2)] = e^x [(y^2 + 2y)e^y - (2y + 2)e^y + 2 \int e^y dy] \\ &= e^x [(y^2 + 2y)e^y - (2y + 2)e^y + 2e^y] = y^2 e^{x+y} + C_1(x) = f(x, y) + C_2(x), \end{aligned}$$

$$\therefore f(x, y) = y^2 e^{x+y} + C(x),$$

$$\therefore \frac{\partial f}{\partial x} = y^2 e^{x+y} + C'(x),$$

$$\text{又} \because \frac{\partial f}{\partial x} = y^2 e^{x+y},$$

$$\therefore C'(x) = 0,$$

$$\therefore C(x) = C,$$

$$\therefore f(x, y) = y^2 e^{x+y} + C.$$

2.7 习题10.4解答

1. 求下列复合函数的偏导数:

(1) $z = xy + xf(u)$, $u = \frac{y}{x}$, 其中 f 为 C^1 类函数, 求 $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$;

(2) $z = f(u, v)$, $u = x$, $v = \frac{x}{y}$, 其中 f 为 C^2 类函数, 求 $\frac{\partial^2 z}{\partial y^2}$;

(3) $z = xf(\frac{y}{x}) + yg(\frac{x}{y})$, 其中 f, g 为 C^2 类函数, 求 $\frac{\partial^2 z}{\partial x \partial y}$;

(4) $z = \frac{y}{f(x^2 - y^2)}$, 其中 f 为可微函数, 求 $\frac{1}{x} \frac{\partial z}{\partial x} + \frac{1}{y} \frac{\partial z}{\partial y}$;

(5) $u = f(x, xy, xyz)$, 其中 f 为可微函数, 求 $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$;

(6) $z = e^{x-2y}$, $x = \sin t$, $y = t^3$, 求 $\frac{dz}{dt}$.

解: (1) $\because z = xy + xf(u) = xy + xf(\frac{y}{x})$,

$$\therefore \frac{\partial z}{\partial x} = y + f(\frac{y}{x}) + xf'(\frac{y}{x})(-\frac{y}{x^2}) = y + f(\frac{y}{x}) - \frac{y}{x}f'(\frac{y}{x}), \quad \frac{\partial z}{\partial y} = x + xf'(\frac{y}{x})\frac{1}{x} = x + f'(\frac{y}{x}),$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x[y + f(\frac{y}{x}) - \frac{y}{x}f'(\frac{y}{x})] + y[x + f'(\frac{y}{x})] = 2xy + xf(\frac{y}{x}).$$

$$(2) \frac{\partial z}{\partial y} = \frac{\partial f(u,v)}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f(u,v)}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial f(u,v)}{\partial v}(-\frac{x}{y^2}),$$

$$\begin{aligned} \frac{\partial^2 z}{\partial y^2} &= -\frac{x}{y^2} [\frac{\partial^2 f(u,v)}{\partial u \partial v} \frac{\partial u}{\partial y} + \frac{\partial^2 f(u,v)}{\partial v^2} \frac{\partial v}{\partial y}] + \frac{\partial f(u,v)}{\partial v} \frac{2x}{y^3} = -\frac{x}{y^2} [\frac{\partial^2 f(u,v)}{\partial v^2}(-\frac{x}{y^2})] + \frac{\partial f(u,v)}{\partial v} \frac{2x}{y^3} \\ &= \frac{x^2}{y^4} \frac{\partial^2 f(u,v)}{\partial v^2} + \frac{2x}{y^3} \frac{\partial f(u,v)}{\partial v}. \end{aligned}$$

$$(3) \frac{\partial z}{\partial y} = xf'(\frac{y}{x})\frac{1}{x} + g(\frac{x}{y}) + yg'(\frac{x}{y})(-\frac{x}{y^2}) = f'(\frac{y}{x}) + g(\frac{x}{y}) - \frac{x}{y}g'(\frac{x}{y}),$$

$$\frac{\partial^2 z}{\partial x \partial y} = f''(\frac{y}{x})(-\frac{y}{x^2}) + g'(\frac{x}{y})\frac{1}{y} - \frac{1}{y}g'(\frac{x}{y}) - \frac{x}{y}g''(\frac{x}{y})\frac{1}{y} = -\frac{y}{x^2}f''(\frac{y}{x}) - \frac{x}{y^2}g''(\frac{x}{y}).$$

$$(4) \frac{\partial z}{\partial x} = \frac{-yf'(x^2-y^2)2x}{[f(x^2-y^2)]^2} = \frac{-2xyf'(x^2-y^2)}{[f(x^2-y^2)]^2}, \quad \frac{\partial z}{\partial y} = \frac{f(x^2-y^2)-yf'(x^2-y^2)(-2y)}{[f(x^2-y^2)]^2} = \frac{f(x^2-y^2)+2y^2f'(x^2-y^2)}{[f(x^2-y^2)]^2},$$

$$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{1}{x} \frac{-2xyf'(x^2-y^2)}{[f(x^2-y^2)]^2} + \frac{1}{y} \frac{f(x^2-y^2)+2y^2f'(x^2-y^2)}{[f(x^2-y^2)]^2} = \frac{f(x^2-y^2)}{y[f(x^2-y^2)]^2} = \frac{1}{yf(x^2-y^2)}.$$

$$(5) \frac{\partial u}{\partial x} = f'_1 + f'_2 y + f'_3 y z = f'_1 + y f'_2 + y z f'_3, \quad \frac{\partial u}{\partial y} = x f'_2 + x z f'_3, \quad \frac{\partial u}{\partial z} = x y f'_3.$$

$$(6) \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = e^{x-2y} \cos t + e^{x-2y}(-2)3t^2 = e^{x-2y}(\cos t - 6t^2) \\ = e^{\sin t - 2t^3}(\cos t - 6t^2).$$

2. 已知 $z = f(x + y^2)$, 其中函数 f 二阶可导, 试求 $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$.

$$\text{解: } \frac{\partial z}{\partial x} = f'(x^2 + y^2)2x, \quad \frac{\partial^2 z}{\partial x^2} = 2f'(x^2 + y^2) + 2xf''(x^2 + y^2)2x \\ = 2f'(x^2 + y^2) + 4x^2f''(x^2 + y^2),$$

$$\frac{\partial z}{\partial y} = f'(x^2 + y^2)2y, \quad \frac{\partial^2 z}{\partial y^2} = 2f'(x^2 + y^2) + 2yf''(x^2 + y^2)2y = 2f'(x^2 + y^2) + 4y^2f''(x^2 + y^2),$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2yf''(x^2 + y^2)2x = 4xyf''(x^2 + y^2).$$

3. 设 $z = yf(x^2y, \frac{y}{x})$, 其中 f 具有连续的二阶偏导数, 求 z''_{xx}, z''_{xy} .

$$\text{解: } z'_x = y[f'_1(x^2y, \frac{y}{x})2xy + f'_2(x^2y, \frac{y}{x})(-\frac{y}{x^2})] = 2xy^2f'_1(x^2y, \frac{y}{x}) - \frac{y^2}{x^2}f'_2(x^2y, \frac{y}{x}),$$

$$\begin{aligned} z''_{xx} &= 2y^2f'_1 + 2xy^2[f''_{11}2xy + f''_{12}(-\frac{y}{x^2})] - (-\frac{2y^2}{x^3})f'_2 - \frac{y^2}{x^2}[f''_{21}2xy + f''_{22}(-\frac{y}{x^2})] \\ &= 2y^2f'_1 + \frac{2y^2}{x^3}f'_2 + 4x^2y^3f''_{11} - \frac{4y^3}{x}f''_{12} + \frac{y^3}{x^4}f''_{22} \\ &= 2y^2f'_1 + \frac{2y^2}{x^3}f'_2 + 4x^2y^3f''_{11} - \frac{4y^3}{x}f''_{12} + \frac{y^3}{x^4}f''_{22}, \end{aligned}$$

$$\begin{aligned} z''_{xy} &= 4xyf'_1 + 2xy^2[f''_{11}x^2 + f''_{12}\frac{1}{x}] - \frac{2y}{x^2}f'_2 - \frac{y^2}{x^2}[f''_{21}x^2 + f''_{22}\frac{1}{x}] \\ &= 4xyf'_1 - \frac{2y}{x^2}f'_2 + 2x^3y^2f''_{11} + y^2f''_{12} - \frac{y^2}{x^3}f''_{22} \\ &= 4xyf'_1 - \frac{2y}{x^2}f'_2 + 2x^3y^2f''_{11} + y^2f''_{12} - \frac{y^2}{x^3}f''_{22}. \end{aligned}$$

4. 设函数 f, g 有连续导数, 令 $u = yf(\frac{x}{y}) + xg(\frac{y}{x})$, 求 $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y}$.

解: 【该做法应加上 f, g 有二阶连续导数的条件:】

$$\frac{\partial u}{\partial x} = yf'(\frac{x}{y})\frac{1}{y} + g(\frac{y}{x}) + xg'(\frac{y}{x})(-\frac{y}{x^2}) = f'(\frac{x}{y}) + g(\frac{y}{x}) - \frac{y}{x}g'(\frac{y}{x}),$$

$$\frac{\partial^2 u}{\partial x^2} = f''(\frac{x}{y})\frac{1}{y} + g'(\frac{y}{x})(-\frac{y}{x^2}) - (-\frac{y}{x^2})g'(\frac{y}{x}) - \frac{y}{x}g''(\frac{y}{x})(-\frac{y}{x^2}) = \frac{1}{y}f''(\frac{x}{y}) + \frac{y^2}{x^3}g''(\frac{y}{x}),$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = f''\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) + g'\left(\frac{y}{x}\right)\frac{1}{x} - \frac{1}{x}g'\left(\frac{y}{x}\right) - \frac{y}{x}g''\left(\frac{y}{x}\right)\frac{1}{x} = -\frac{x}{y^2}f''\left(\frac{x}{y}\right) - \frac{y}{x^2}g''\left(\frac{y}{x}\right),$$

$$\therefore x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = x\left[\frac{1}{y}f''\left(\frac{x}{y}\right) + \frac{y^2}{x^3}g''\left(\frac{y}{x}\right)\right] + y\left[-\frac{x}{y^2}f''\left(\frac{x}{y}\right) - \frac{y}{x^2}g''\left(\frac{y}{x}\right)\right] = 0.$$

【正确做法:】 $\because f, g$ 有连续导数,

$$\therefore u = yf\left(\frac{x}{y}\right) + xg\left(\frac{y}{x}\right) \in C^1(\mathbb{R}^2 \setminus \{(x, y) | x = 0 \text{ 或 } y = 0\}),$$

\therefore

$$\frac{\partial u}{\partial x} = yf'\left(\frac{x}{y}\right)\frac{1}{y} + g\left(\frac{y}{x}\right) + xg'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = g\left(\frac{y}{x}\right) + f'\left(\frac{x}{y}\right) - \frac{y}{x}g'\left(\frac{y}{x}\right),$$

$$\frac{\partial u}{\partial y} = f\left(\frac{x}{y}\right) + yf'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) + xg'\left(\frac{y}{x}\right)\frac{1}{x} = f\left(\frac{x}{y}\right) - \frac{x}{y}f'\left(\frac{x}{y}\right) + g'\left(\frac{y}{x}\right),$$

\therefore

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xg\left(\frac{y}{x}\right) + yf\left(\frac{x}{y}\right) = u \in C^1,$$

\therefore

$$\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) - \frac{\partial u}{\partial x} = x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = 0.$$

2.8 习题10.5解答

1. 设 $y = y(x), z = z(x)$ 是由方程 $z = xf(x+y)$ 和 $F(x, y, z) = 0$ 所确定的函数, 其中 f 和 F 分别具有连续导数和偏导数, 求 $\frac{dz}{dx}$.

解: 方法1: 将 $z = xf(x+y), F(x, y, z) = 0$ 两边分别对 x 求导:

$$\frac{dz}{dx} = f(x+y) + xf'(x+y)\left[1 + \frac{dy}{dx}\right],$$

$$F'_x + F'_y \frac{dy}{dx} + F'_z \frac{dz}{dx} = 0$$

由该方程组解得

$$\frac{dz}{dx} = \frac{[f(x+y) + xf'(x+y)]F'_y - xf'(x+y)F'_x}{F'_y + xf'(x+y)F'_z}.$$

方法2: 将 $xf(x+y) - z = 0, F(x, y, z) = 0$ 两边求全微分:

$$[f(x+y) + xf'(x+y)]dx + xf'(x+y)dy - dz = 0,$$

$$F'_x dx + F'_y dy + F'_z dz = 0,$$

因为 $y = y(x), z = z(x)$, 将以上方程两边分别除以 dx 得

$$[f(x+y) + xf'(x+y)] + xf'(x+y)\frac{dy}{dx} - \frac{dz}{dx} = 0,$$

$$F'_x + F'_y \frac{dy}{dx} + F'_z \frac{dz}{dx} = 0,$$

由该方程组解得

$$\frac{dz}{dx} = \frac{(f + xf')F'_y - xf'F'_x}{F'_y + xf'F'_z}.$$

2. 设由方程 $F(\frac{x}{z}, \frac{z}{y}) = 0$ 可以确定隐函数 $z = z(x, y)$, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

解: 方法1: 将 $F(\frac{x}{z}, \frac{z}{y}) = 0$ 两边分别对 x, y 求偏导:

$$\begin{aligned} F'_1(\frac{1}{z} + \frac{-x}{z^2} \frac{\partial z}{\partial x}) + F'_2 \frac{1}{y} \frac{\partial z}{\partial x} &= 0, \\ F'_1 \frac{-x}{z^2} \frac{\partial z}{\partial y} + F'_2(\frac{1}{y} \frac{\partial z}{\partial y} + \frac{-z}{y^2}) &= 0, \end{aligned}$$

\therefore

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\frac{1}{z} F'_1}{\frac{x}{z^2} F'_1 - \frac{1}{y} F'_2} = \frac{F'_1}{\frac{x}{z} F'_1 - \frac{z}{y} F'_2}, \\ \frac{\partial z}{\partial y} &= \frac{\frac{z}{y^2} F'_2}{\frac{1}{y} F'_2 - \frac{x}{z^2} F'_1} = \frac{\frac{z^2}{y^2} F'_2}{\frac{z}{y} F'_2 - \frac{x}{z} F'_1}. \end{aligned}$$

方法2: 将方程 $F(\frac{x}{z}, \frac{z}{y}) = 0$ 两边求全微分:

$$F'_1 \frac{1}{z} dx + F'_2(-\frac{z}{y^2}) dy + [F'_1(-\frac{x}{z^2}) + F'_2 \frac{1}{y}] dz = 0,$$

即

$$dz = -\frac{F'_1 \frac{1}{z}}{F'_1(-\frac{x}{z^2}) + F'_2 \frac{1}{y}} dx - \frac{F'_2(-\frac{z}{y^2})}{F'_1(-\frac{x}{z^2}) + F'_2 \frac{1}{y}} dy,$$

\therefore

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{-F'_1 \frac{1}{z}}{F'_1(-\frac{x}{z^2}) + F'_2 \frac{1}{y}} = \frac{F'_1}{\frac{x}{z} F'_1 - \frac{z}{y} F'_2}, \\ \frac{\partial z}{\partial y} &= \frac{-F'_2(-\frac{z}{y^2})}{F'_1(-\frac{x}{z^2}) + F'_2 \frac{1}{y}} = \frac{\frac{z^2}{y^2} F'_2}{\frac{z}{y} F'_2 - \frac{x}{z} F'_1}. \end{aligned}$$

3. 证明: 方程 $F(x + \frac{z}{y}, y + \frac{z}{x}) = 0$ 所确定的隐函数 $z = z(x, y)$ 满足方程

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z - xy.$$

证明: 方法1:

\therefore

$$dF(x + \frac{z}{y}, y + \frac{z}{x}) = [F'_1 + F'_2(-\frac{z}{x^2})] dx + [F'_1(\frac{-z}{y^2}) + F'_2] dy + (F'_1 \frac{1}{y} + F'_2 \frac{1}{x}) dz = 0,$$

\therefore

$$dz = -\frac{F'_1 + F'_2(-\frac{z}{x^2})}{F'_1 \frac{1}{y} + F'_2 \frac{1}{x}} dx - \frac{F'_1(\frac{-z}{y^2}) + F'_2}{F'_1 \frac{1}{y} + F'_2 \frac{1}{x}} dy,$$

∴

$$\frac{\partial z}{\partial x} = -\frac{F'_1 + F'_2(-\frac{z}{x^2})}{F'_1\frac{1}{y} + F'_2\frac{1}{x}}, \quad \frac{\partial z}{\partial y} = -\frac{F'_1(\frac{-z}{y^2}) + F'_2}{F'_1\frac{1}{y} + F'_2\frac{1}{x}},$$

∴

$$\begin{aligned} x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} &= -x\frac{F'_1 + F'_2(-\frac{z}{x^2})}{F'_1\frac{1}{y} + F'_2\frac{1}{x}} - y\frac{F'_1(\frac{-z}{y^2}) + F'_2}{F'_1\frac{1}{y} + F'_2\frac{1}{x}} = -\frac{x F'_1 + F'_2(-\frac{z}{x}) + F'_1(\frac{-z}{y}) + y F'_2}{F'_1\frac{1}{y} + F'_2\frac{1}{x}} \\ &= -\frac{(x - \frac{z}{y})F'_1 + (y - \frac{z}{x})F'_2}{F'_1\frac{1}{y} + F'_2\frac{1}{x}} = -(xy - z)\frac{\frac{1}{y}F'_1 + \frac{1}{x}F'_2}{F'_1\frac{1}{y} + F'_2\frac{1}{x}} \\ &= z - xy. \end{aligned}$$

方法2: 将方程 $F(x + \frac{z}{y}, y + \frac{z}{x}) = 0$ 两边分别对 x, y 求偏导:

$$\begin{cases} F'_1(1 + \frac{1}{y}\frac{\partial z}{\partial x}) + F'_2(\frac{-z}{x^2} + \frac{1}{x}\frac{\partial z}{\partial x}) = 0, \\ F'_1(\frac{-z}{y^2} + \frac{1}{y}\frac{\partial z}{\partial y}) + F'_2(1 + \frac{1}{x}\frac{\partial z}{\partial y}) = 0, \end{cases}$$

这是一个关于 F'_1, F'_2 的齐次方程组, 要使该方程组有非零解, 则必须

$$\begin{vmatrix} 1 + \frac{1}{y}\frac{\partial z}{\partial x} & \frac{-z}{x^2} + \frac{1}{x}\frac{\partial z}{\partial x} \\ \frac{-z}{y^2} + \frac{1}{y}\frac{\partial z}{\partial y} & 1 + \frac{1}{x}\frac{\partial z}{\partial y} \end{vmatrix} = 0,$$

即

$$(1 + \frac{1}{y}\frac{\partial z}{\partial x})(1 + \frac{1}{x}\frac{\partial z}{\partial y}) - (\frac{-z}{x^2} + \frac{1}{x}\frac{\partial z}{\partial x})(\frac{-z}{y^2} + \frac{1}{y}\frac{\partial z}{\partial y}) = 0,$$

整理得

$$\begin{aligned} &1 + \frac{1}{y}\frac{\partial z}{\partial x} + \frac{1}{x}\frac{\partial z}{\partial y} - \frac{z^2}{x^2y^2} + \frac{z}{x^2y}\frac{\partial z}{\partial y} + \frac{z}{xy^2}\frac{\partial z}{\partial x} \\ &= 1 - \frac{z^2}{x^2y^2} + \frac{xy + z}{xy^2}\frac{\partial z}{\partial x} + \frac{xy + z}{x^2y}\frac{\partial z}{\partial y} \\ &= \frac{(xy + z)(xy - z)}{x^2y^2} + \frac{xy + z}{xy^2}\frac{\partial z}{\partial x} + \frac{xy + z}{x^2y}\frac{\partial z}{\partial y} \\ &= \frac{xy + z}{x^2y^2}[(xy - z) + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}] \\ &= 0, \end{aligned}$$

∴ $xy + z = 0$ 或 $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z - xy$, ∵ $xy + z = 0$ 也满足 $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z - xy$,

故 $z = z(x, y)$ 满足方程 $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z - xy$.

4. 设 $z = f(u)$, 且 $u = u(x, y)$ 满足 $u = \varphi(u) + \int_y^x p(t)dt$ (其中 f 可导, $\varphi \in C^1$, 且 $\varphi'(u) \neq 1, p \in C$). 求证: $p(y)\frac{\partial z}{\partial x} + p(x)\frac{\partial z}{\partial y} = 0$.

证明:

$$\frac{\partial z}{\partial x} = f'(u) \frac{\partial u}{\partial x}, \frac{\partial z}{\partial y} = f'(u) \frac{\partial u}{\partial y},$$

\therefore

$$u = \varphi(u) + \int_y^x p(t) dt = \varphi(u) + \int_0^x p(t) dt + \int_y^0 p(t) dt,$$

\therefore

$$\frac{\partial u}{\partial x} = \varphi'(u) \frac{\partial u}{\partial x} + p(x), \frac{\partial u}{\partial y} = \varphi'(u) \frac{\partial u}{\partial y} - p(y), (*)$$

$\because \varphi'(u) \neq 1,$

\therefore

$$\frac{\partial u}{\partial x} = \frac{p(x)}{1 - \varphi'(u)}, \frac{\partial u}{\partial y} = \frac{-p(y)}{1 - \varphi'(u)},$$

\therefore

$$\begin{aligned} \frac{\partial z}{\partial x} &= f'(u) \frac{\partial u}{\partial x} = f'(u) \frac{p(x)}{1 - \varphi'(u)}, \\ \frac{\partial z}{\partial y} &= f'(u) \frac{\partial u}{\partial y} = f'(u) \frac{-p(y)}{1 - \varphi'(u)}, \end{aligned}$$

\therefore

$$\begin{aligned} & p(y) \frac{\partial z}{\partial x} + p(x) \frac{\partial z}{\partial y} \\ &= p(y) f'(u) \frac{p(x)}{1 - \varphi'(u)} + p(x) f'(u) \frac{-p(y)}{1 - \varphi'(u)} \\ &= 0. \end{aligned}$$

5. 已知方程 $F(x+y, y+z) = 1$ 确定了隐函数 $z = z(x, y)$, 其中 F 具有连续的二阶偏导数, 求 $\frac{\partial^2 z}{\partial y \partial x}$.

解: 方法1: 将方程 $F(x+y, y+z) = 1$ 两边对 x 求偏导:

$$F'_1(x+y, y+z) + F'_2(x+y, y+z) \frac{\partial z}{\partial x} = 0,$$

得

$$\frac{\partial z}{\partial x} = -\frac{F'_1(x+y, y+z)}{F'_2(x+y, y+z)},$$

\therefore

$$\frac{\partial^2 z}{\partial y \partial x} = -\frac{[F''_{11} + F''_{12}(1 + \frac{\partial z}{\partial y})]F'_2 - F'_1[F''_{21} + F''_{22}(1 + \frac{\partial z}{\partial y})]}{(F'_2)^2},$$

方程 $F(x+y, y+z) = 1$ 两边对 y 求偏导:

$$F'_1(x+y, y+z) + F'_2(x+y, y+z)(1 + \frac{\partial z}{\partial y}) = 0,$$

得

$$1 + \frac{\partial z}{\partial y} = -\frac{F'_1(x+y, y+z)}{F'_2(x+y, y+z)},$$

\therefore

$$\begin{aligned}\frac{\partial^2 z}{\partial y \partial x} &= -\frac{[F''_{11} + F''_{12}(-\frac{F'_1}{F'_2})]F'_2 - F'_1[F''_{21} + F''_{22}(-\frac{F'_1}{F'_2})]}{(F'_2)^2} \\ &= -\frac{(F'_2)^2 F''_{11} - F'_1 F'_2 F''_{12} - F'_1 F'_2 F''_{21} + (F'_1)^2 F''_{22}}{(F'_2)^3} \\ &= \frac{-(F'_2)^2 F''_{11} + 2F'_1 F'_2 F''_{12} - (F'_1)^2 F''_{22}}{(F'_2)^3}.\end{aligned}$$

方法2: 方程 $F(x+y, y+z) = 1$ 两边求全微分:

$$dF(x+y, y+z) = F'_1(x+y, y+z)dx + [F'_1(x+y, y+z) + F'_2(x+y, y+z)]dy + F'_2(x+y, y+z)dz = 0,$$

即

$$\begin{aligned}dz &= -\frac{F'_1(x+y, y+z)}{F'_2(x+y, y+z)}dx - \frac{F'_1(x+y, y+z) + F'_2(x+y, y+z)}{F'_2(x+y, y+z)}dz \\ &= -\frac{F'_1(x+y, y+z)}{F'_2(x+y, y+z)}dx - [1 + \frac{F'_1(x+y, y+z)}{F'_2(x+y, y+z)}]dz,\end{aligned}$$

\therefore

$$\frac{\partial z}{\partial x} = -\frac{F'_1(x+y, y+z)}{F'_2(x+y, y+z)},$$

\therefore

$$\frac{\partial^2 z}{\partial y \partial x} = -\frac{[F''_{11} + F''_{12}(1 + \frac{\partial z}{\partial y})]F'_2 - F'_1[F''_{21} + F''_{22}(1 + \frac{\partial z}{\partial y})]}{(F'_2)^2},$$

\therefore

$$\frac{\partial z}{\partial y} = -1 - \frac{F'_1(x+y, y+z)}{F'_2(x+y, y+z)},$$

\therefore

$$\begin{aligned}\frac{\partial^2 z}{\partial y \partial x} &= -\frac{[F''_{11} + F''_{12}(-\frac{F'_1}{F'_2})]F'_2 - F'_1[F''_{21} + F''_{22}(-\frac{F'_1}{F'_2})]}{(F'_2)^2} \\ &= -\frac{(F'_2)^2 F''_{11} - F'_1 F'_2 F''_{12} - F'_1 F'_2 F''_{21} + (F'_1)^2 F''_{22}}{(F'_2)^3} \\ &= \frac{-(F'_2)^2 F''_{11} + 2F'_1 F'_2 F''_{12} - (F'_1)^2 F''_{22}}{(F'_2)^3}.\end{aligned}$$

6. 设方程组 $\begin{cases} x^2 + y^2 + z^2 = 3x, \\ 2x - 3y + 5z = 4, \end{cases}$ 确定 y 与 z 是 x 的函数, 求 $\frac{dy}{dx}, \frac{dz}{dx}$.

解: 方法1: 将方程组的两个方程两边分别对 x 求导:

$$\begin{cases} 2x + 2y \frac{dy}{dx} + 2z \frac{dz}{dx} = 3, \\ 2 - 3 \frac{dy}{dx} + 5 \frac{dz}{dx} = 0, \end{cases}$$

$$\text{可解得 } \frac{dy}{dx} = \frac{\begin{vmatrix} 3-2x & 2z \\ -2 & 5 \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ -3 & 5 \end{vmatrix}} = \frac{15-10x+4z}{10y+6z}, \frac{dz}{dy} = \frac{\begin{vmatrix} 2y & 3-2x \\ -3 & -2 \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ -3 & 5 \end{vmatrix}} = \frac{-4y+9-6x}{10y+6z}.$$

方法2: 将方程组的两个方程两边分别求全微分:

$$\begin{cases} 2dx + 2ydy + 2zdz = 3, \\ 2dx - 3dy + 5dz = 0, \end{cases}$$

$\because y$ 与 z 是 x 的函数

$$\therefore \begin{cases} 2x + 2y \frac{dy}{dx} + 2z \frac{dz}{dx} = 3, \\ 2 - 3 \frac{dy}{dx} + 5 \frac{dz}{dx} = 0, \end{cases}$$

$$\text{可解得 } \frac{dy}{dx} = \frac{\begin{vmatrix} 3-2x & 2z \\ -2 & 5 \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ -3 & 5 \end{vmatrix}} = \frac{15-10x+4z}{10y+6z}, \frac{dz}{dy} = \frac{\begin{vmatrix} 2y & 3-2x \\ -3 & -2 \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ -3 & 5 \end{vmatrix}} = \frac{-4y+9-6x}{10y+6z}.$$