25C 第13章补充题

25C.1 第13章补充题解答

1. 设D为$y=x,y=4x,xy=1,xy=4$所围成的区域，F是一元连续可微函数，$f=F'$.
求证：$\oint_{\partial D}\frac{F(xy)}{y}\mathrm{d}y=\ln 2\int_1^4 f(v)\mathrm{d}v$.
证明：$\because D$是第一象限内部的区域，x轴不穿过D，且$F\in C^1$,
$\therefore \frac{F(xy)}{y}\in C^1(D)$,
$\therefore \oint_{\partial D}\frac{F(xy)}{y}\mathrm{d}y=\iint\limits_D\frac{\partial}{\partial x}[\frac{F(xy)}{y}-\frac{\partial 0}{\partial y}]\mathrm{d}x\mathrm{d}y=\iint\limits_D F'(xy)\mathrm{d}x\mathrm{d}y$,
设$\begin{cases}u=\frac{y}{x},\\ v=xy,\end{cases}$ 则$D=\{(u,v)\mid 1\leqslant u\leqslant 4,1\leqslant v\leqslant 4\}$,
$\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}=\begin{vmatrix}-\frac{y}{x^2} & \frac{1}{x}\\ y & x\end{vmatrix}=-\frac{y}{x}-\frac{y}{x}=-2u$,
$\therefore \iint\limits_D F'(xy)\mathrm{d}x\mathrm{d}y=\iint\limits_D f(xy)\mathrm{d}x\mathrm{d}y=\iint\limits_D f(v)\frac{1}{|\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}|}\mathrm{d}u\mathrm{d}v=\iint\limits_D f(v)\frac{1}{2u}\mathrm{d}u\mathrm{d}v=\int_1^4\frac{1}{2u}\mathrm{d}u\int_1^4 f(v)\mathrm{d}v$
$=\frac{1}{2}\ln u\Big|_1^4\int_1^4 f(v)\mathrm{d}v=\ln 2\int_1^4 f(v)\mathrm{d}v$.

2. 设D是平面上的有界区域，函数$u(x,y)$与$v(x,y)$在$\bar{D}$上存在二阶连续偏导数.
求证:$\oint_{\partial D}\begin{vmatrix}\frac{\partial u}{\partial \boldsymbol{n}} & \frac{\partial v}{\partial \boldsymbol{n}}\\ u & v\end{vmatrix}\mathrm{d}l=\iint\limits_D\begin{vmatrix}\Delta u & \Delta v\\ u & v\end{vmatrix}\mathrm{d}\sigma$.
证明：$\because u(x,y),v(x,y)\in C^2(D)$,
方法1：$\therefore \frac{\partial u}{\partial \boldsymbol{n}}=\boldsymbol{n}\cdot\mathrm{grad}u,\frac{\partial v}{\partial \boldsymbol{n}}=\boldsymbol{n}\cdot\mathrm{grad}v$,
$\therefore \oint_{\partial D}\begin{vmatrix}\boldsymbol{n}\cdot\mathrm{grad}u & \boldsymbol{n}\cdot\mathrm{grad}v\\ u & v\end{vmatrix}\mathrm{d}l=\oint_{\partial D}[v(\boldsymbol{n}\cdot\mathrm{grad}u)-u(\boldsymbol{n}\cdot\mathrm{grad}v)]\mathrm{d}l$
$=\oint_{\partial D}(v\mathrm{grad}u-u\mathrm{grad}v)\cdot\boldsymbol{n}\mathrm{d}l=\oint_{\partial D}(v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x},v\frac{\partial u}{\partial y}-u\frac{\partial v}{\partial y})\cdot\boldsymbol{n}\mathrm{d}l$
$=\iint\limits_D[\frac{\partial}{\partial x}(v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x})+\frac{\partial}{\partial y}(v\frac{\partial u}{\partial y}-u\frac{\partial v}{\partial y})]\mathrm{d}\sigma$
$=\iint\limits_D[\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+v\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}-u\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}-u\frac{\partial^2 v}{\partial y^2}]\mathrm{d}\sigma$
$=\iint\limits_D(v\frac{\partial^2 u}{\partial x^2}-u\frac{\partial^2 v}{\partial x^2}+v\frac{\partial^2 u}{\partial y^2}-u\frac{\partial^2 v}{\partial y^2})\mathrm{d}\sigma=\iint\limits_D[v(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})-u(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2})]\mathrm{d}\sigma$
$=\iint\limits_D[v\Delta u-u\Delta v]\mathrm{d}\sigma=\iint\limits_D\begin{vmatrix}\Delta u & \Delta v\\ u & v\end{vmatrix}\mathrm{d}\sigma$.
方法2：$\therefore \frac{\partial u}{\partial \boldsymbol{n}}=\boldsymbol{\nabla}u\cdot\boldsymbol{n},\frac{\partial v}{\partial \boldsymbol{n}}=\boldsymbol{\nabla}v\cdot\boldsymbol{n}$,
$\therefore \oint_{\partial D}\begin{vmatrix}\frac{\partial u}{\partial \boldsymbol{n}} & \frac{\partial v}{\partial \boldsymbol{n}}\\ u & v\end{vmatrix}\mathrm{d}l=\oint_{\partial D}(v\frac{\partial u}{\partial \boldsymbol{n}}-u\frac{\partial v}{\partial \boldsymbol{n}})\mathrm{d}l=\oint_{\partial D}(v\boldsymbol{\nabla}u\cdot\boldsymbol{n}-u\boldsymbol{\nabla}v\cdot\boldsymbol{n})\mathrm{d}l$
$=\oint_{\partial D}(v\boldsymbol{\nabla}u-u\boldsymbol{\nabla}v)\cdot\boldsymbol{n}\mathrm{d}l=\iint\limits_D\boldsymbol{\nabla}\cdot(v\boldsymbol{\nabla}u-u\boldsymbol{\nabla}v)\mathrm{d}\sigma$

$$= \iint_D (\nabla v\cdot\nabla u + v\nabla\cdot\nabla u - \nabla u\cdot\nabla v - u\nabla\cdot\nabla v)\mathrm{d}\sigma$$
$$= \iint_D (v\nabla\cdot\nabla u - u\nabla\cdot\nabla v)\mathrm{d}\sigma = \iint_D (v\Delta u - u\Delta v)\mathrm{d}\sigma = \iint_D \begin{vmatrix}\Delta u & \Delta v\\ u & v\end{vmatrix}\mathrm{d}\sigma.$$

3. 计算$I=\oiint_S \frac{\cos\widehat{\boldsymbol{r}\boldsymbol{n}}}{r^2}\mathrm{d}S$，其中$S$为任意光滑闭曲面，$\boldsymbol{n}$为$S$的外单位法向量，$M_0(x_0,y_0,z_0)$是$S$内的一个确定点，$\boldsymbol{r}$是连接$M_0(x_0,y_0,z_0)$和$S$上点$M(x,y,z)$的向量，$r$是$\boldsymbol{r}$的长度.

解：$I=\oiint_S \frac{\cos\widehat{\boldsymbol{r}\boldsymbol{n}}}{r^2}\mathrm{d}S=\oiint_S \frac{\boldsymbol{r}\cdot\boldsymbol{n}}{r^3}\mathrm{d}S,$

取$S_1: x^2+y^2+z^2=a^2$, 其中a足够小，使得S_1完全包含在S内，S_1的外侧为正，设S和S_{1-}围成的区域为Ω, 设S_1围成的区域为Ω_1,

$\because \frac{\boldsymbol{r}}{r^3}=\frac{1}{r^3}(x-x_0,y-y_0,z-z_0)\in C^1(\Omega),\ r=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2},$

又$\because \frac{\partial r}{\partial x}=\frac{2(x-x_0)}{2\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}=\frac{x-x_0}{r}, \frac{\partial r}{\partial y}=\frac{y-y_0}{r}, \frac{\partial =}{\partial z}\frac{z-z_0}{r},$

$\therefore \oiint_S \frac{\boldsymbol{r}\cdot\boldsymbol{n}}{r^3}\mathrm{d}S+\oiint_{S_{1-}}\frac{\boldsymbol{r}\cdot\boldsymbol{n}}{r^3}\mathrm{d}S$

$=\oiint_{S+S_1-}\frac{1}{r^3}(x-x_0,y-y_0,z-z_0)\cdot\boldsymbol{n}\mathrm{d}S$

$=\iiint_\Omega[\frac{\partial}{\partial x}(\frac{x-x_0}{r^3})+\frac{\partial}{\partial y}(\frac{y-y_0}{r^3})+\frac{\partial}{\partial z}(\frac{z-z_0}{r^3})]\mathrm{d}x\mathrm{d}y\mathrm{d}z$

$=\iiint_\Omega[\frac{r^3-(x-x_0)3r^2\frac{x-x_0}{r}}{r^6}+\frac{r^3-(y-y_0)3r^2\frac{y-y_0}{r}}{r^6}+\frac{r^3-(z-z_0)3r^2\frac{z-z_0}{r}}{r^6}]\mathrm{d}x\mathrm{d}y\mathrm{d}z$

$=\iiint_\Omega[\frac{r^2-3(x-x_0)^2}{r^5}+\frac{r^2-3(y-y_0)^2}{r^5}+\frac{r^2-3(z-z_0)^2}{r^5}]\mathrm{d}x\mathrm{d}y\mathrm{d}z$

$=\iiint_\Omega[\frac{(y-y_0)^2+(z-z_0)^2-2(x-x_0)^2}{r^5}+\frac{(z-z_0)^2+(x-x_0)^2-2(y-y_0)^2}{r^5}+\frac{(x-x_0)^2+(y-y_0)^2-2(z-z_0)^2}{r^5}]\mathrm{d}x\mathrm{d}y\mathrm{d}z$

$=\iiint_\Omega 0\mathrm{d}x\mathrm{d}y\mathrm{d}z=0,$

$\therefore I=\oiint_S\frac{\boldsymbol{r}\cdot\boldsymbol{n}}{r^3}\mathrm{d}S=-\oiint_{S_{1-}}\frac{\boldsymbol{r}\cdot\boldsymbol{n}}{r^3}\mathrm{d}S=\oiint_{S_1}\frac{\boldsymbol{r}\cdot\boldsymbol{n}}{r^3}\mathrm{d}S=\oiint_{S_1}\frac{\boldsymbol{r}\cdot\boldsymbol{n}}{a^3}\mathrm{d}S=\frac{1}{a^3}\oiint_{S_1}\boldsymbol{r}\cdot\boldsymbol{n}\mathrm{d}S$

$=\frac{1}{a^3}\iiint_{\Omega_1}\nabla\cdot\boldsymbol{r}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\frac{1}{a^3}\iiint_{\Omega_1}[\frac{\partial}{\partial x}(x-x_0)+\frac{\partial(y-y_0)}{\partial y}+\frac{\partial(z-z_0)}{\partial z}]\mathrm{d}x\mathrm{d}y\mathrm{d}z=\frac{1}{a^3}\iiint_{\Omega_1}3\mathrm{d}x\mathrm{d}y\mathrm{d}z$

$=\frac{3}{a^3}\frac{4}{3}\pi a^3=4\pi.$

4. 设$\Omega\subset\mathbb{R}^3$为有界区域，其边界$\partial\Omega$为逐片光滑的闭曲面，$\boldsymbol{n}$是$\partial\Omega$的外单位法向量. 函数$u$和$v$在$\Omega$中有连续偏导数[1]. 求证：

(1)$\oiint_{\partial\Omega}\frac{\partial u}{\partial\boldsymbol{n}}\mathrm{d}S=\iiint_\Omega\Delta u\mathrm{d}V;$

(2)$\oiint_{\partial\Omega}u\frac{\partial u}{\partial\boldsymbol{n}}\mathrm{d}S=\iiint_\Omega[(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2+(\frac{\partial u}{\partial z})^2]\mathrm{d}V+\iiint_\Omega u\Delta u\mathrm{d}V;$

(3)$\oiint_{\partial\Omega}(u\frac{\partial v}{\partial\boldsymbol{n}}-v\frac{\partial u}{\partial\boldsymbol{n}})\mathrm{d}S=\iiint_\Omega(u\Delta v-v\Delta u)\mathrm{d}V.$

证明：(1)$\because u,v\in C^2(\Omega),$

[1]这里应是二阶连续偏导数

方法1：$\because \underset{\partial\Omega}{\oiint} \frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}S = \underset{\partial\Omega}{\oiint} \mathrm{grad}u \cdot \boldsymbol{n}\mathrm{d}S = \underset{\partial\Omega}{\oiint}(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}) \cdot \boldsymbol{n}\mathrm{d}S = \underset{\Omega}{\iiint}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})\mathrm{d}V$
$= \underset{\Omega}{\iiint} \Delta u\mathrm{d}V.$

方法2：$\because \underset{\partial\Omega}{\oiint} \frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}S = \underset{\partial\Omega}{\oiint} \boldsymbol{\nabla} u \cdot \boldsymbol{n}\mathrm{d}S = \underset{\Omega}{\iiint} \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u\mathrm{d}V = \underset{\Omega}{\iiint} \Delta u\mathrm{d}V.$

(2)$\because u, v \in C^2(\Omega)$,

方法1：$\because \underset{\partial\Omega}{\oiint} u\frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}S = \underset{\partial\Omega}{\oiint} u\mathrm{grad}u \cdot \boldsymbol{n}\mathrm{d}S = \underset{\partial\Omega}{\oiint}(u\frac{\partial u}{\partial x}, u\frac{\partial u}{\partial y}, u\frac{\partial u}{\partial z}) \cdot \boldsymbol{n}\mathrm{d}S$
$= \underset{\Omega}{\iiint}[\frac{\partial}{\partial x}(u\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(u\frac{\partial u}{\partial y}) + \frac{\partial}{\partial z}(u\frac{\partial u}{\partial z})]\mathrm{d}V = \underset{\Omega}{\iiint}[(\frac{\partial u}{\partial x})^2 + u\frac{\partial^2 u}{\partial x^2} + (\frac{\partial u}{\partial y})^2 + u\frac{\partial^2 u}{\partial y^2} + (\frac{\partial u}{\partial z})^2 + u\frac{\partial^2 u}{\partial z^2}]\mathrm{d}V$
$= \underset{\Omega}{\iiint}[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2]\mathrm{d}V + \underset{\Omega}{\iiint}[u\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^2 u}{\partial y^2} + u\frac{\partial^2 u}{\partial z^2}]\mathrm{d}V$
$= \underset{\Omega}{\iiint}[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2]\mathrm{d}V + \underset{\Omega}{\iiint} u\Delta u\mathrm{d}V.$

方法2：$\because \underset{\partial\Omega}{\oiint} u\frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}S = \underset{\partial\Omega}{\oiint} u\boldsymbol{\nabla} u \cdot \boldsymbol{n}\mathrm{d}S = \underset{\Omega}{\iiint} \boldsymbol{\nabla} \cdot (u\boldsymbol{\nabla} u)\mathrm{d}V$
$= \underset{\Omega}{\iiint}(\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u + u\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u)\mathrm{d}V = \underset{\Omega}{\iiint}[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2]\mathrm{d}V + \underset{\Omega}{\iiint} u\Delta u\mathrm{d}V.$

(3)$\underset{\partial\Omega}{\oiint}(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}})\mathrm{d}S = \underset{\partial\Omega}{\oiint}(u\mathrm{grad}v \cdot \boldsymbol{n} - v\mathrm{grad}u \cdot \boldsymbol{n})\mathrm{d}S = \underset{\partial\Omega}{\oiint}(u\mathrm{grad}v - v\mathrm{grad}u) \cdot \boldsymbol{n}\mathrm{d}S$
$= \underset{\partial\Omega}{\oiint}(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}, u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}, u\frac{\partial v}{\partial z} - v\frac{\partial u}{\partial z}) \cdot \boldsymbol{n}\mathrm{d}S$
$= \underset{\Omega}{\iiint}[\frac{\partial}{\partial x}(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}) + \frac{\partial}{\partial z}(u\frac{\partial v}{\partial z} - v\frac{\partial u}{\partial z})]\mathrm{d}V$
$= \underset{\Omega}{\iiint}[(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} - v\frac{\partial^2 u}{\partial x^2}) + (\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - v\frac{\partial^2 u}{\partial y^2})$
$+ (\frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + u\frac{\partial^2 v}{\partial z^2} - \frac{\partial v}{\partial z}\frac{\partial u}{\partial z} - v\frac{\partial^2 u}{\partial z^2})]\mathrm{d}V$
$= \underset{\Omega}{\iiint}[(u\frac{\partial^2 v}{\partial x^2} - v\frac{\partial^2 u}{\partial x^2}) + (u\frac{\partial^2 v}{\partial y^2} - v\frac{\partial^2 u}{\partial y^2}) + (u\frac{\partial^2 v}{\partial z^2} - v\frac{\partial^2 u}{\partial z^2})]\mathrm{d}V$
$= \underset{\Omega}{\iiint}[u(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}) - v(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})]\mathrm{d}V$
$= \underset{\Omega}{\iiint}(u\Delta v - v\Delta u)\mathrm{d}V.$

方法2：$\because \underset{\partial\Omega}{\oiint}(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}})\mathrm{d}S = \underset{\partial\Omega}{\oiint}(u\boldsymbol{\nabla} v \cdot \boldsymbol{n} - v\boldsymbol{\nabla} u \cdot \boldsymbol{n})\mathrm{d}S = \underset{\partial\Omega}{\oiint}(u\boldsymbol{\nabla} v - v\boldsymbol{\nabla} u) \cdot \boldsymbol{n}\mathrm{d}S$
$= \underset{\Omega}{\iiint} \boldsymbol{\nabla} \cdot (u\boldsymbol{\nabla} v - v\boldsymbol{\nabla} u)\mathrm{d}V = \underset{\Omega}{\iiint}(\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + u\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} v - \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} u - v\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u)\mathrm{d}V$
$= \underset{\Omega}{\iiint}(u\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} v - v\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u)\mathrm{d}V = \underset{\Omega}{\iiint}(u\Delta v - v\Delta u)\mathrm{d}V.$

5. 设D为平面区域，$u(x, y)$在D上有二阶连续偏导数. 求证下列命题等价：

(1)$u(x, y)$在D上是调和函数，即$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$;

(2)对于D内任意一条圆周L，如果L包围的区域完全属于D，则有$\oint_L \frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}l = 0$.

证明：(1) $\Rightarrow$ (2):

$\because u(x, y) \in C^2(D)$,

$\therefore$对于D内任意一条圆周L，如果L包围的区域D_L完全属于D，

则$\oint_L \frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}l = \oint_L \boldsymbol{\nabla} u \cdot \boldsymbol{n}\mathrm{d}l = \underset{D_L}{\iint} \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u\mathrm{d}\sigma = \underset{D_L}{\iint} \Delta u\mathrm{d}\sigma = 0.$

$(2) \Rightarrow (1)$:

方法1：$\forall (x,y) \in D$取D内包围(x,y)的圆周L，使得L包围的区域D_L完全属于D,

则$\oint_L \frac{\partial u}{\partial \boldsymbol{n}} \mathrm{d}l = 0$,

$\because u(x,y) \in C^2(D)$,

$\therefore \Delta u \in C(D)$, 且$\iint\limits_{D_L} \Delta u \mathrm{d}\sigma = \iint\limits_{D_L} \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u \mathrm{d}\sigma = \oint_L \boldsymbol{\nabla} u \cdot \boldsymbol{n} \mathrm{d}l = \oint_L \frac{\partial u}{\partial \boldsymbol{n}} \mathrm{d}l = 0$,

根据积分中值定理$\exists (\xi,\eta) \in D_L \subset D, s.t. \iint\limits_{D_L} \Delta u \mathrm{d}\sigma = \Delta u(\xi,\eta) A(D_L) = 0$, 其中$A(D_L)$为区域$D_L$的面积，

$\therefore \lim\limits_{D_L \to (x,y)} \frac{\iint\limits_{D_L} \Delta u \mathrm{d}\sigma}{A(D_L)} = \lim\limits_{(\xi,\eta)\to(x,y)} \Delta u(\xi,\eta) = \Delta u(x,y) = 0.$

方法2：反证. 若$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$在$D$上不等于零，则至少存在一点$M_0$使得

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\Big|_{M_0} \neq 0.$$

不妨设$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\Big|_{M_0} > 0$. 由连续性知，存在位于$D$内的、以$M_0$为中心的一个闭圆域$U$, 在$U$上处处有$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) > 0$, 从而

$$\iint\limits_U \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \mathrm{d}x\mathrm{d}y > 0.$$

用∂U表示U的正向边界(这是D内的一个圆周)，于是由格林公式得到

$$\oint_{\partial U} \frac{\partial u}{\partial \boldsymbol{n}} \mathrm{d}l = \iint\limits_U \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \mathrm{d}x\mathrm{d}y > 0.$$

这与假设矛盾.

6. 设Ω为光滑曲面S围成的有界闭区域，u是Ω上的调和函数($\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \equiv 0$). 求证：

$$\oiint\limits_S \left[\frac{1}{r}\frac{\partial u}{\partial \boldsymbol{n}} - u\frac{\partial(\frac{1}{r})}{\partial \boldsymbol{n}}\right] \mathrm{d}S = -\oiint\limits_{S_\delta} \left[\frac{1}{r}\frac{\partial u}{\partial \boldsymbol{n}} - u\frac{\partial(\frac{1}{r})}{\partial \boldsymbol{n}}\right] \mathrm{d}S,$$

其中$M_0(x_0,y_0,z_0)$是Ω内一点，S_δ是以$M_0(x_0,y_0,z_0)$为中心，δ为半径的球面，且$S_\delta \subset \Omega$; $\boldsymbol{r} = (x-x_0)\boldsymbol{i} + (y-y_0)\boldsymbol{j} + (z-z_0)\boldsymbol{k}, r$是$\boldsymbol{r}$的长度；$\boldsymbol{n}$是$S$的外单位法向量、$S_\delta$的内单位法向量.

证明：设Ω_1为S内S_δ外的区域，在Ω_1上

$$\frac{1}{r}\frac{\partial u}{\partial \boldsymbol{n}} - u\frac{\partial(\frac{1}{r})}{\partial \boldsymbol{n}} = \frac{1}{r}\boldsymbol{\nabla} u \cdot \boldsymbol{n} - u\boldsymbol{\nabla}\left(\frac{1}{r}\right) \cdot \boldsymbol{n} = \left[\frac{1}{r}\boldsymbol{\nabla} u - u\boldsymbol{\nabla}\left(\frac{1}{r}\right)\right] \cdot \boldsymbol{n},$$

$\therefore$根据高斯公式

$$
\begin{aligned}
&\oiint\limits_{S+S_\delta}[\frac{1}{r}\boldsymbol{\nabla}u-u\boldsymbol{\nabla}(\frac{1}{r})]\cdot\boldsymbol{n}\mathrm{d}S\\
=&\iiint\limits_{\Omega_1}\boldsymbol{\nabla}\cdot[\frac{1}{r}\boldsymbol{\nabla}u-u\boldsymbol{\nabla}(\frac{1}{r})]\mathrm{d}V\\
=&\iiint\limits_{\Omega_1}[\boldsymbol{\nabla}(\frac{1}{r})\cdot\boldsymbol{\nabla}u+\frac{1}{r}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}u-\boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}(\frac{1}{r})-u\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}(\frac{1}{r})]\mathrm{d}V\\
=&\iiint\limits_{\Omega_1}[\frac{1}{r}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}u-u\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}(\frac{1}{r})]\mathrm{d}V\\
=&\iiint\limits_{\Omega_1}[\frac{1}{r}\Delta u-u\Delta(\frac{1}{r})]\mathrm{d}V\\
=&-\iiint\limits_{\Omega_1}u\Delta(\frac{1}{r})\mathrm{d}V,
\end{aligned}
$$

$\because \frac{\partial}{\partial x}(\frac{1}{r})=\frac{-\frac{\partial r}{\partial x}}{r^2}=\frac{1}{r^2}\frac{-2(x-x_0)}{2\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}=\frac{-(x-x_0)}{r^3}$,

$\frac{\partial}{\partial y}(\frac{1}{r})=\frac{-(y-y_0)}{r^3}$, $\frac{\partial}{\partial z}(\frac{1}{r})=\frac{-(z-z_0)}{r^3}$,

$\frac{\partial^2}{\partial x^2}(\frac{1}{r})=\frac{\partial}{\partial x}[\frac{-(x-x_0)}{r^3}]=\frac{-r^3+(x-x_0)3r^2\frac{\partial r}{\partial x}}{r^6}=\frac{\partial}{\partial x}[\frac{-(x-x_0)}{r^3}]=\frac{-r^3+(x-x_0)3r^2\frac{2(x-x_0)}{2\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}}{r^6}$

$=\frac{\partial}{\partial x}[\frac{-(x-x_0)}{r^3}]=\frac{-r^3+(x-x_0)3r^2\frac{(x-x_0)}{r}}{r^6}=\frac{-r^2+3(x-x_0)^2}{r^5}=\frac{2(x-x_0)^2-(y-y_0)^2-(z-z_0)^2}{r^5}$,

$\frac{\partial^2}{\partial y^2}(\frac{1}{r})=\frac{2(y-y_0)^2-(z-z_0)^2-(x-x_0)^2}{r^5}$, $\frac{\partial^2}{\partial z^2}(\frac{1}{r})=\frac{2(z-z_0)^2-(x-x_0)^2-(y-y_0)^2}{r^5}$,

$\therefore \Delta(\frac{1}{r})=\frac{\partial^2}{\partial x^2}(\frac{1}{r})+\frac{\partial^2}{\partial y^2}(\frac{1}{r})+\frac{\partial^2}{\partial z^2}(\frac{1}{r})$

$=\frac{2(x-x_0)^2-(y-y_0)^2-(z-z_0)^2}{r^5}+\frac{2(y-y_0)^2-(z-z_0)^2-(x-x_0)^2}{r^5}+\frac{2(z-z_0)^2-(x-x_0)^2-(y-y_0)^2}{r^5}=0$,

$\therefore$

$$\oiint\limits_{S+S_\delta}[\frac{1}{r}\boldsymbol{\nabla}u-u\boldsymbol{\nabla}(\frac{1}{r})]\cdot\boldsymbol{n}\mathrm{d}S=-\iiint\limits_{\Omega_1}u\Delta(\frac{1}{r})\mathrm{d}V=0,$$

$\therefore$

$$\oiint\limits_{S+S_\delta}[\frac{1}{r}\frac{\partial u}{\partial\boldsymbol{n}}-u\frac{\partial(\frac{1}{r})}{\partial\boldsymbol{n}}]\cdot\boldsymbol{n}\mathrm{d}S=0,$$

$\therefore$

$$\oiint\limits_{S}[\frac{1}{r}\frac{\partial u}{\partial\boldsymbol{n}}-u\frac{\partial(\frac{1}{r})}{\partial\boldsymbol{n}}]\mathrm{d}S=-\oiint\limits_{S_\delta}[\frac{1}{r}\frac{\partial u}{\partial\boldsymbol{n}}-u\frac{\partial(\frac{1}{r})}{\partial\boldsymbol{n}}]\mathrm{d}S.$$

7. 设$\Omega\subset\mathbb{R}^3$为有界区域，其边界$\partial\Omega$逐片光滑，$\boldsymbol{n}$是$S$的外单位法向量. f在Ω内调和，在$\partial\Omega$上有连续的偏导数，并且在闭区域$\bar{\Omega}$上连续. 求证：

(1)$\oiint\limits_{\partial\Omega}\frac{\partial f}{\partial\boldsymbol{n}}\mathrm{d}S=0$;

(2)$\oiint\limits_{\partial\Omega}f\frac{\partial f}{\partial\boldsymbol{n}}\mathrm{d}S=\iiint\limits_{\Omega}\|\boldsymbol{\nabla}f\|^2\mathrm{d}V$($\|\boldsymbol{\nabla}f\|$是向量$\boldsymbol{\nabla}f$的长度);

(3)若当$(x,y,z)\in\partial\Omega$时，$f(x,y,z)\equiv0$, 求证在Ω内$f(x,y,z)\equiv0$.

证明：(1)$\because f$在Ω内调和，即$\Delta f(x,y,z)=0,(x,y,z)\in\Omega$,

$\therefore \oiint\limits_{\partial\Omega}\frac{\partial f}{\partial \boldsymbol{n}}\mathrm{d}S=\oiint\limits_{\partial\Omega}\boldsymbol{\nabla} f\cdot\boldsymbol{n}\mathrm{d}S=\iiint\limits_{\Omega}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f\mathrm{d}V=\iiint\limits_{\Omega}\Delta f\mathrm{d}V=\iiint\limits_{\Omega}0\mathrm{d}V=0.$

(2)$\oiint\limits_{\partial\Omega}f\frac{\partial f}{\partial \boldsymbol{n}}\mathrm{d}S=\oiint\limits_{\partial\Omega}f\boldsymbol{\nabla} f\cdot\boldsymbol{n}\mathrm{d}S=\iiint\limits_{\Omega}\boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla} f)\mathrm{d}V=\iiint\limits_{\Omega}(\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} f+f\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f)\mathrm{d}V$
$=\iiint\limits_{\Omega}(\|\boldsymbol{\nabla} f\|^2+f\Delta f)\mathrm{d}V=\iiint\limits_{\Omega}\|\boldsymbol{\nabla} f\|^2\mathrm{d}V.$

(3)$\because$当$(x,y,z)\in\partial\Omega$时，$f(x,y,z)\equiv 0$,

$\therefore \iiint\limits_{\Omega}\|\boldsymbol{\nabla} f\|^2\mathrm{d}V=\oiint\limits_{\partial\Omega}f\frac{\partial f}{\partial \boldsymbol{n}}\mathrm{d}S=\oiint\limits_{\partial\Omega}0\mathrm{d}S=0,$

$\because \|\boldsymbol{\nabla} f\|^2\geqslant 0,$

$\therefore \|\boldsymbol{\nabla} f\|^2=0,\ \boldsymbol{\nabla} f=\boldsymbol{0},$

$\therefore f(x,y,z)=Const,(x,y,z)\in\Omega,$

$\because f$在闭区域$\bar{\Omega}$上连续,

$\therefore f(x,y,z)=0,(x,y,z)\in\Omega.$

8. 设$M_0(x_0,y_0,z_0)$为空间一确定点，S是点M_0之外的一张逐片光滑曲面. 从点M_0出发作射线，假定每一条这样的射线与曲面最多相交于一点，则所有与曲面S相交的射线构成一个锥体Λ. 以点M_0为中心，以任意正数a为半径做球，并设该球面含于锥体Λ内部的部分面积为S_a. 定义曲面S关于点M_0的立体角为$\Omega_S=\frac{S_a}{a^2}$.
(1)若S是以点M_0为中心，以任意正数a为半径的球面，求Ω_S;
(2)令$\boldsymbol{v}=\frac{\boldsymbol{r}}{r^3}$, 求证：$\Omega_S=\iint\limits_{S}\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}S$. 其中$\boldsymbol{r}=\overrightarrow{M_0M},r=\|\boldsymbol{r}\|$.

解：(1)$\Omega_S=\frac{S_a}{a^2}=\frac{4\pi a^2}{a^2}=4\pi$.

(2)证明：记半径为a的球面含于锥体Λ内部的部分为S_a, 不妨设S_a包含在锥体Λ的由顶点M_0到曲面S之间的部分,设S和S_a与锥体Λ侧面围成的区域为Ω, 锥体Λ的侧面在S和S_a之间的部分记为S_1, 易知在S_1上$\boldsymbol{v}\cdot\boldsymbol{n}=0$, 其中$\boldsymbol{n}$为区域$\Omega$边界上的外向单位法向量，

$\because \boldsymbol{v}=\frac{\boldsymbol{r}}{r^2}\in C^1(\Omega),$

$\therefore \iint\limits_{S}\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}S+\iint\limits_{S_a}\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}S=\iint\limits_{S}\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}S+\iint\limits_{S_a}\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}S+\iint\limits_{S_1}\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}S$
$=\oiint\limits_{S+S_a+S_1}\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}S=\iiint\limits_{\Omega}\boldsymbol{\nabla}\cdot\boldsymbol{v}\mathrm{d}V=\iiint\limits_{\Omega}[\frac{\partial}{\partial x}(\frac{x-x_0}{r^3})+\frac{\partial}{\partial y}(\frac{y-y_0}{r^3})+\frac{\partial}{\partial z}(\frac{z-z_0}{r^3})]\mathrm{d}V$
$=\iiint\limits_{\Omega}[\frac{r^3-(x-x_0)3r^2\frac{\partial r}{\partial x}}{r^6}+\frac{r^3-(y-y_0)3r^2\frac{\partial r}{\partial y}}{r^6}+\frac{r^3-(z-z_0)3r^2\frac{\partial r}{\partial z}}{r^6}]\mathrm{d}V$
$=\iiint\limits_{\Omega}[\frac{r^3-(x-x_0)3r^2\frac{x-x_0}{r}}{r^6}+\frac{r^3-(y-y_0)3r^2\frac{y-y_0}{r}}{r^6}+\frac{r^3-(z-z_0)3r^2\frac{z-z_0}{r}}{r^6}]\mathrm{d}V$
$=\iiint\limits_{\Omega}[\frac{r^2-3(x-x_0)^2}{r^5}+\frac{r^2-3(y-y_0)^2}{r^5}+\frac{r^2-3(z-z_0)^2}{r^5}]\mathrm{d}V$
$=\iiint\limits_{\Omega}[\frac{(y-y_0)^2+(z-z_0)^2-2(x-x_0)^2}{r^5}+\frac{(z-z_0)^2+(x-x_0)^2-2(y-y_0)^2}{r^5}+\frac{(x-x_0)^2+(y-y_0)^2-2(z-z_0)^2}{r^5}]\mathrm{d}V$
$=\iiint\limits_{\Omega}0\mathrm{d}V=0,$

$\therefore \iint\limits_S \boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}S = -\iint\limits_{S_a} \boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}S = -\iint\limits_{S_a} \frac{\boldsymbol{r}}{r^3}\cdot\boldsymbol{n}\mathrm{d}S = -\iint\limits_{S_a} \frac{\boldsymbol{r}}{a^3}\cdot\boldsymbol{n}\mathrm{d}S = -\frac{1}{a^3}\iint\limits_{S_a} \boldsymbol{r}\cdot\boldsymbol{n}\mathrm{d}S,$

$\because$球面S_a上$\boldsymbol{n} = -(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}) = -\frac{1}{a}\boldsymbol{r},$

$\therefore \iint\limits_S \boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}S = -\frac{1}{a^3}\iint\limits_{S_a} \boldsymbol{r}\cdot\boldsymbol{n}\mathrm{d}S = \frac{1}{a^3}\frac{1}{a}\iint\limits_{S_a} \boldsymbol{r}\cdot\boldsymbol{r}\mathrm{d}S = \frac{1}{a^3}\frac{1}{a}\iint\limits_{S_a}[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]\mathrm{d}S$
$= \frac{1}{a^3}\frac{1}{a}\iint\limits_{S_a} a^2\mathrm{d}S = \frac{1}{a^2}\iint\limits_{S_a}\mathrm{d}S = \frac{S_a}{a^2} = \Omega_S.$