18 重积分的概念和性质、二重积分的计算

18.1 知识结构

第12章重积分

12.1 二重积分的概念和性质

12.1.1 引例(二重积分的几何意义)

- 曲顶柱体的体积
- 质量非均匀分布的平板质量

12.1.2 二重积分的概念

12.1.3 二重积分的性质

- 线性性质
- 区域可加性
- 保序性
- 绝对值函数的可积性
- 估值定理
- 积分中值定理
- 区域对称性

12.2 二重积分的计算

12.2.1 用直角坐标计算二重积分

12.2.2 用极坐标计算二重积分

12.3 二重积分的变量代换

18.2 习题12.1解答

1. 利用重积分的几何意义求下列积分值：

(1)$\iint\limits_D \sqrt{R^2-x^2-y^2}\mathrm{d}\sigma, D=\{(x,y)\mid x^2+y^2\leqslant R^2\}$;

(2)$\iint\limits_D 2\mathrm{d}\sigma, D=\{(x,y)\mid x+y\leqslant 1, y-x\leqslant 1, y\geqslant 0\}$.

解：(1)$\iint\limits_D \sqrt{R^2-x^2-y^2}\mathrm{d}\sigma$的大小是曲面$z=\sqrt{R^2-x^2-y^2}$与平面$z=0$围成区域的体积，即等于半径为$a$的半球的体积，故

$$\iint\limits_D \sqrt{R^2-x^2-y^2}\mathrm{d}\sigma=\frac{2}{3}\pi R^3.$$

(2)区域D的图形如图 1所示，

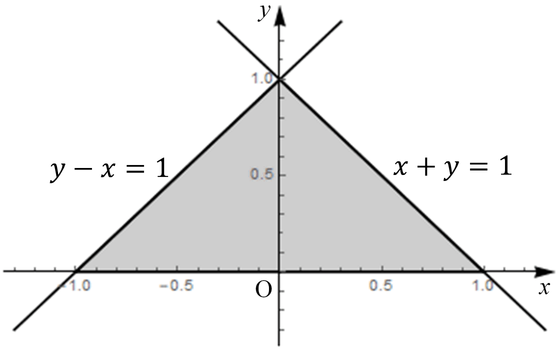

图 1: 习题12.1 1.(2)题图示

积分$\iint\limits_D 2\mathrm{d}\sigma$表示以区域$D$为底，高为2的棱柱体的体积，故

$$\iint\limits_D 2\mathrm{d}\sigma = 2\times(\frac{1}{2}\times 1\times 2) = 2.$$

2. 利用重积分的性质估计下列积分值：

(1)$\iint\limits_D (1+y)x\mathrm{d}\sigma, D=\{(x,y)\mid x^2+y^2\leqslant 1, x\geqslant 0, y\geqslant 0\}$;

(2)$\iint\limits_D (x^2+y^2)\mathrm{d}\sigma, D=\{(x,y)\mid 2x\leqslant x^2+y^2\leqslant 4x\}$.

解：(1)令$x=r\cos\theta, y=r\sin\theta$，则$D=\{(x,y)\mid x^2+y^2\leqslant 1, x\geqslant 0, y\geqslant 0\}$
$=\left\{(r,\theta)\;\middle|\; 0\leqslant r\leqslant 1, 0\leqslant\theta\leqslant\frac{\pi}{2}\right\}$,

$\because$当$0\leqslant r\leqslant 1, 0\leqslant\theta\leqslant\frac{\pi}{2}$时，$0\leqslant\cos\theta\leqslant 1, 0\leqslant\sin 2\theta\leqslant 1$,

$\therefore (1+y)x=(1+r\sin\theta)r\cos\theta = r\cos\theta + r^2\sin\theta\cos\theta = r\cos\theta+\frac{1}{2}r^2\sin 2\theta\in[0,\frac{3}{2}]$,

$\because \iint\limits_D \mathrm{d}\sigma = \frac{\pi}{4}$,

$\therefore 0\leqslant\iint\limits_D (1+y)x\mathrm{d}\sigma\leqslant\frac{3}{2}\iint\limits_D \mathrm{d}\sigma = \frac{3}{2}\cdot\frac{\pi}{4}=\frac{3}{8}\pi$，即$\iint\limits_D (1+y)x\mathrm{d}\sigma\in[0,\frac{3}{8}\pi]$.

(2)方法1：$\because 2x\leqslant x^2+y^2\leqslant 4x \Leftrightarrow \begin{cases} x^2+y^2\geqslant 2x, \\ x^2+y^2\leqslant 4x, \end{cases} \Leftrightarrow \begin{cases} (x-1)^2+y^2\geqslant 1, \\ (x-2)^2+y^2\leqslant 4, \end{cases}$

$\therefore$区域D如图 2所示，

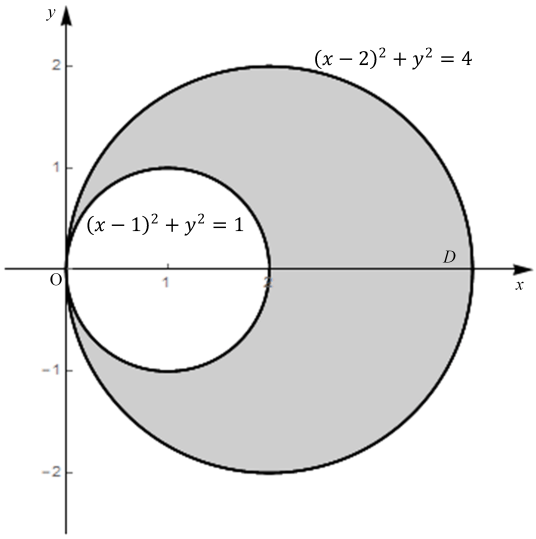

图 2: 习题12.1 2.(2)题图示

由图 2可知$0 \leqslant x^2+y^2 \leqslant 4^2+0^2=16$,

$\because \iint\limits_D \mathrm{d}\sigma = 2^2\pi - 1^2\pi = 3\pi$,

$\therefore 0 \leqslant \iint\limits_D (x^2+y^2)\mathrm{d}\sigma \leqslant 16\iint\limits_D \mathrm{d}\sigma = 48\pi$.

方法2：令$x=r\cos\theta, y=r\sin\theta$，则$D=\{(x,y)\mid 2x \leqslant x^2+y^2 \leqslant 4x\}$

$=\{(r,\theta)\mid 2r\cos\theta \leqslant r^2 \leqslant 4r\cos\theta\}=\{(r,\theta)\mid 2\cos\theta \leqslant r \leqslant 4\cos\theta, 0 \leqslant \theta \leqslant 2\pi\}$,

$\therefore 0 \leqslant 2\cos^2\theta \leqslant r^2 \leqslant 16\cos^2\theta \leqslant 16$，即$0 \leqslant x^2+y^2 \leqslant 4^2+0^2=16$,

$\because \iint\limits_D \mathrm{d}\sigma = 2^2\pi - 1^2\pi = 3\pi$,

$\therefore 0 \leqslant \iint\limits_D (x^2+y^2)\mathrm{d}\sigma \leqslant 16\iint\limits_D \mathrm{d}\sigma = 48\pi$.

3. 比较下列各组积分值的大小：

(1)$\iint\limits_D (x+y)^2\mathrm{d}\sigma$与$\iint\limits_D (x+y)^3\mathrm{d}\sigma$，其中$D=\{(x,y)\mid (x-2)^2+(y-2)^2 \leqslant 2\}$;

(2)$\iint\limits_D \ln(x+y)\mathrm{d}\sigma$与$\iint\limits_D xy\mathrm{d}\sigma$，其中$D$由直线$x=0, y=0, x+y=\frac{1}{2}$及$x+y=1$围成.

解：(1)区域D的图形如图 3所示，

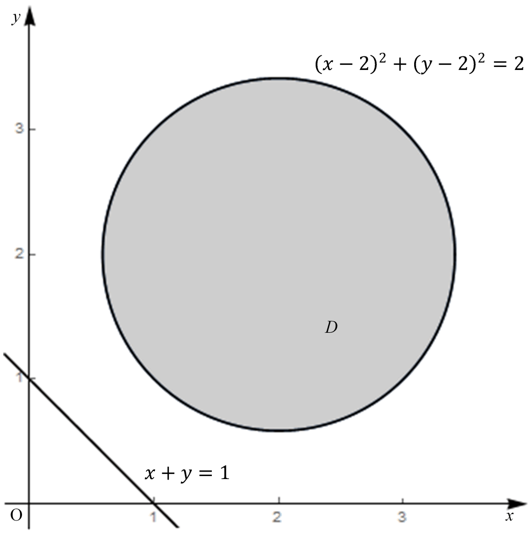

图 3: 习题12.1 3.(1)题图示

由图 3可知在区域D上$x+y>1$，则$(x+y)^2<(x+y)^3$,

$\therefore \iint\limits_D (x+y)^2 \mathrm{d}\sigma < \iint\limits_D (x+y)^3 \mathrm{d}\sigma.$

(2)区域D的图形如图 4所示，

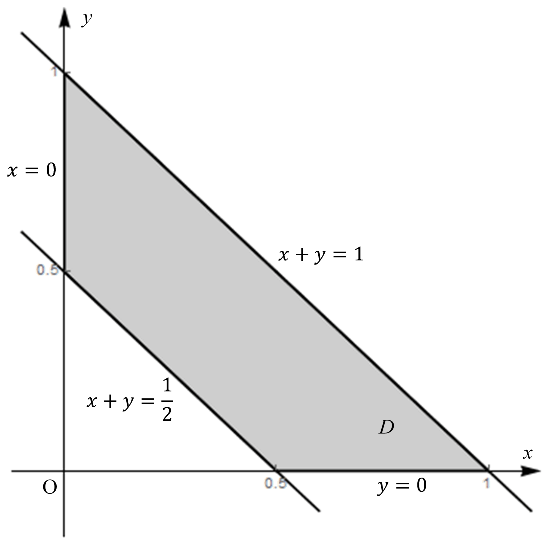

图 4: 习题12.1 3.(2)题图示

由图 3可知在区域D上$0<x+y\leqslant 1$，则$\ln(x+y)\leqslant 0\leqslant xy$,

$\therefore \iint\limits_D \ln(x+y)\mathrm{d}\sigma < 0 < \iint\limits_D xy\mathrm{d}\sigma.$

4. 设$D \subset \mathbb{R}^2$是一有界闭域，$f(x,y) \in C(D)$且非负，试证：若$\iint\limits_D f(x,y)\mathrm{d}\sigma = 0$，则$f(x,y) \equiv 0, \forall (x,y) \in D.$

证明：假设$f(x,y)$不恒为0，

$\because f(x,y) \in C(D)$且非负，

$\therefore \exists P(x_0,y_0) \in D$满足$f(x_0,y_0) > 0$，且存在$P(x_0,y_0)$的一个邻域$N(P,\delta)$使得$f(x,y) > \frac{1}{2}f(x_0,y_0) > 0$，

$\therefore \iint\limits_D f(x,y)\mathrm{d}\sigma = \iint\limits_{N(P,\delta)} f(x,y)\mathrm{d}\sigma + \iint\limits_{D\setminus N(P,\delta)} f(x,y)\mathrm{d}\sigma \geqslant \iint\limits_{N(P,\delta)} f(x,y)\mathrm{d}\sigma + 0 > 0$，这与$\iint\limits_D f(x,y)\mathrm{d}\sigma = 0$矛盾，

$\therefore$假设不成立，

$\therefore f(x,y) \equiv 0, \forall (x,y) \in D.$

5. 证明：若$f(x,y) \in C(D), g(x,y) \in R(D)$且不变号，则$\exists (\xi,\eta) \in D$使得

$$\iint\limits_D f(x,y)g(x,y)\mathrm{d}\sigma = f(\xi,\eta)\iint\limits_D g(x,y)\mathrm{d}\sigma.$$

证明：$\because f(x,y) \in C(D)$,

$\therefore \exists P,Q \in D, s.t.\ f(P) = m, f(Q) = M,$ 且$m \leqslant f(x,y) \leqslant M, \forall (x,y) \in D,$

$\because g(x,y) \in R(D)$且不变号，不妨设$g(x,y) \geqslant 0,$

$\therefore mg(x,y) \leqslant f(x,y)g(x,y) \leqslant Mg(x,y),$

$\therefore m\iint\limits_D g(x,y)\mathrm{d}\sigma \leqslant \iint\limits_D f(x,y)g(x,y)\mathrm{d}\sigma \leqslant M\iint\limits_D g(x,y)\mathrm{d}\sigma,$

$\therefore$

i)当$\iint\limits_D g(x,y)\mathrm{d}\sigma = 0$时$\iint\limits_D f(x,y)g(x,y)\mathrm{d}\sigma = 0,$故$\iint\limits_D f(x,y)g(x,y)\mathrm{d}\sigma = f(\xi,\eta)\iint\limits_D g(x,y)\mathrm{d}\sigma$成立；

ii)当$\iint\limits_D g(x,y)\mathrm{d}\sigma \neq 0$时$m \leqslant \frac{\iint\limits_D f(x,y)g(x,y)\mathrm{d}\sigma}{\iint\limits_D g(x,y)\mathrm{d}\sigma} \leqslant M$，由连续函数的介值定理知$\exists (\xi,\eta) \in D$满足

$$f(\xi,\eta) = \frac{\iint\limits_D f(x,y)g(x,y)\mathrm{d}\sigma}{\iint\limits_D g(x,y)\mathrm{d}\sigma},$$

即

$$\iint\limits_D f(x,y)g(x,y)\mathrm{d}\sigma = f(\xi,\eta)\iint\limits_D g(x,y)\mathrm{d}\sigma.$$

6. 利用性质7的结论计算下列积分（其中区域D为圆盘$x^2+y^2\leqslant R^2$）:

   (1)$\iint\limits_D y\sqrt{R^2-x^2}\mathrm{d}\sigma$;　(2)$\iint\limits_D y^3x^2\mathrm{d}\sigma$;

   (3)$\iint\limits_D x^5\sqrt{R^2-y^2}\mathrm{d}\sigma$;　(4)$\iint\limits_D x^my^n\mathrm{d}\sigma$.

   解：(1)因为区域D关于x轴对称，被积函数$f(x,y)=y\sqrt{R^2-x^2}=-f(x,-y)$关于$y$是奇函数，故$\iint\limits_D y\sqrt{R^2-x^2}\mathrm{d}\sigma=0$.

   (2)因为区域D关于x轴对称，被积函数$f(x,y)=y^3x^2=-f(x,-y)$关于y是奇函数，故$\iint\limits_D y^3x^2\mathrm{d}\sigma=0$.

   (3)因为区域D关于y轴对称，被积函数$f(x,y)=x^5\sqrt{R^2-y^2}=-f(-x,y)$关于$x$是奇函数，故$\iint\limits_D x^5\sqrt{R^2-y^2}\mathrm{d}\sigma=0$.

   (4)区域D关于x轴和y轴均对称，

   i)当m与n都是偶数时,$f(x,y)=x^my^n=f(-x,y)=f(x,-y)$关于$x$和$y$均是偶函数,故$\iint\limits_D x^my^n\mathrm{d}\sigma=4\iint\limits_{D_1}x^my^n\mathrm{d}\sigma$,其中$D_1=\{(x,y)\mid x^2+y^2\leqslant R^2,x\geqslant 0,y\geqslant 0\}$;

   ii)当m与n都是奇数时,$f(x,y)=x^my^n=-f(-x,y)$关于x是奇函数,$\iint\limits_D x^my^n\mathrm{d}\sigma=0$;

   iii)当m是奇数n是偶数时,$f(x,y)=x^my^n=-f(-x,y)$关于x是奇函数,$\iint\limits_D x^my^n\mathrm{d}\sigma=0$;

   iv)当m是偶数n是奇数时,$f(x,y)=x^my^n=-f(x,-y)$关于y是奇函数,$\iint\limits_D x^my^n\mathrm{d}\sigma=0$.

   综上所述,当m,n均为偶数时,$\iint\limits_D x^my^n\mathrm{d}\sigma=4\iint\limits_{D_1}x^my^n\mathrm{d}\sigma$,其中$D_1$为区域$D$落在第一象限的部分;当$m,n$中至少有一个奇数时,$\iint\limits_D x^my^n\mathrm{d}\sigma=0$.

18.3　习题12.2解答

1. 计算下列二重积分：

   (1)$\iint\limits_D\cos(x+y)\mathrm{d}\sigma,D$是由$x=0,y=\pi$和$y=x$围成的区域；

   (2)$\iint\limits_D xy\ln(1+x^2+y^2)\mathrm{d}\sigma,D$是由$y=x^3,y=1$和$x=-1$围成的区域；

   (3)$\iint\limits_D\sin(x+y)\mathrm{d}\sigma$，其中$D$由直线$x=0,y=x,y=\pi$围成；

   (4)$\iint\limits_D|x^2-y|\mathrm{d}\sigma,D=\{(x,y)\mid 0\leqslant x,y\leqslant 1\}$；

   (5)$\iint\limits_D\frac{x\sin y}{y}\mathrm{d}\sigma$，其中$D$由$y=x,y=x^2$围成.

   解：(1)方法1：$\iint\limits_D\cos(x+y)\mathrm{d}\sigma=\int_0^\pi\mathrm{d}x\int_x^\pi\cos(x+y)\mathrm{d}y=\int_0^\pi\sin(x+y)\big|_x^\pi\mathrm{d}x$

   $=\int_0^\pi[\sin(x+\pi)-\sin 2x]\mathrm{d}x=\int_0^\pi(-\sin x-\sin 2x)\mathrm{d}x=\cos x\big|_0^\pi+\frac{1}{2}\cos 2x\big|_0^\pi$

   $=-1-1+\frac{1}{2}(1-1)=-2.$

方法2：$\iint\limits_{D}\cos(x+y)\mathrm{d}\sigma=\int_0^{\pi}\mathrm{d}y\int_0^{y}\cos(x+y)\mathrm{d}x=\int_0^{\pi}\sin(x+y)\big|_0^{y}\mathrm{d}y$

$=\int_0^{\pi}(\sin 2y-\sin x)\mathrm{d}y=-\frac{1}{2}\cos 2y\big|_0^{\pi}+\cos x\big|_0^{\pi}=-\frac{1}{2}(1-1)+(-1-1)=-2.$

(2)区域D如图 5所示，

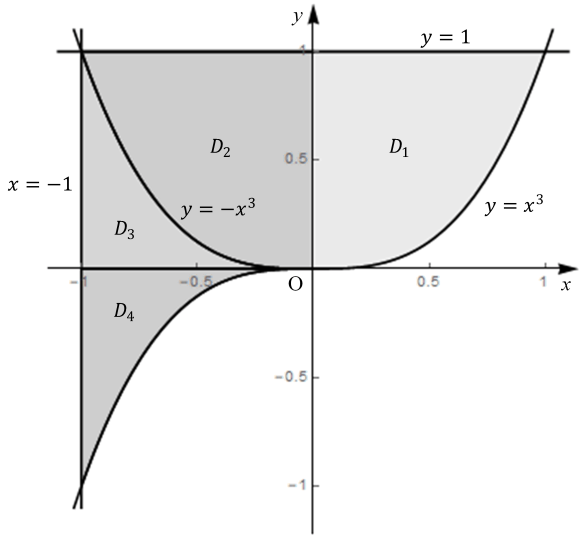

图 5: 习题12.2 1.(2)题图示

可将区域D划分为图中的四个区域，其中D_1与D_2关于y轴对称，D_3与D_4关于x轴对称，被积函数$f(x,y)=xy\ln(1+x^2+y^2)$关于x和y均为奇函数，

$\therefore \iint\limits_{D_1}x\mathrm{d}y\ln(1+x^2+y^2)\mathrm{d}\sigma+\iint\limits_{D_2}x\mathrm{d}y\ln(1+x^2+y^2)\mathrm{d}\sigma=0,$

$\iint\limits_{D_3}xy\ln(1+x^2+y^2)\mathrm{d}\sigma+\iint\limits_{D_4}xy\ln(1+x^2+y^2)\mathrm{d}\sigma=0,$

$\therefore \iint\limits_{D}xy\ln(1+x^2+y^2)\mathrm{d}\sigma$

$=\iint\limits_{D_1}xy\ln(1+x^2+y^2)\mathrm{d}\sigma+\iint\limits_{D_2}xy\ln(1+x^2+y^2)\mathrm{d}\sigma+\iint\limits_{D_3}xy\ln(1+x^2+y^2)\mathrm{d}\sigma$

$+\iint\limits_{D_4}xy\ln(1+x^2+y^2)\mathrm{d}\sigma=0.$

(3)方法1：$\iint\limits_{D}\sin(x+y)\mathrm{d}\sigma=\int_0^{\pi}\mathrm{d}y\int_0^{y}\sin(x+y)\mathrm{d}x=\int_0^{\pi}-\cos(x+y)\big|_0^{y}\mathrm{d}y$

$=\int_0^{\pi}(-\cos y+\cos 2y)\mathrm{d}y=(-\sin y+\frac{1}{2}\sin 2y)\big|_0^{\pi}=0.$

方法2：$\iint\limits_{D}\sin(x+y)\mathrm{d}\sigma=\int_0^{\pi}\mathrm{d}x\int_x^{\pi}\sin(x+y)\mathrm{d}x=\int_0^{\pi}-\cos(x+y)\big|_x^{\pi}\mathrm{d}x$

$=\int_0^{\pi}[-\cos(\pi+x)+\cos 2x]\mathrm{d}x=\int_0^{\pi}(\cos x+\cos 2x)\mathrm{d}x=(\sin x+\frac{1}{2}\sin 2x)\big|_0^{\pi}=0.$

(4)将积分域D划分为以下两个区域$D_1=\{(x,y)\mid 0\leqslant x\leqslant 1,0\leqslant y\leqslant x^2\}$

和$D_2=\{(x,y)\mid 0\leqslant x\leqslant 1,x^2\leqslant y\leqslant 1\}$,

则$\iint\limits_{D}|x^2-y|\mathrm{d}\sigma=\iint\limits_{D_1}(x^2-y)\mathrm{d}\sigma+\iint\limits_{D_2}(y-x^2)\mathrm{d}\sigma=\int_0^1\mathrm{d}x\int_0^{x^2}(x^2-y)\mathrm{d}y+\int_0^1\mathrm{d}x\int_{x^2}^1(y-x^2)\mathrm{d}y$
$=\int_1^0(x^2y-\frac{1}{2}y^2)\Big|_0^{x^2}\mathrm{d}x+\int_0^1(\frac{1}{2}y^2-x^2y)\Big|_{x^2}^1\mathrm{d}x=\int_1^0(x^4-\frac{1}{2}x^4)\mathrm{d}x+\int_0^1(\frac{1}{2}-x^2-\frac{1}{2}x^4+x^4)\mathrm{d}x$
$=\int_0^1(2x^4-x^4-x^2+\frac{1}{2})\mathrm{d}x=\int_0^1(x^4-x^2+\frac{1}{2})\mathrm{d}x=(\frac{1}{5}x^5-\frac{1}{3}x^3+\frac{1}{2}x)\Big|_0^1=\frac{1}{5}-\frac{1}{3}+\frac{1}{2}=\frac{11}{30}.$

(5)$\iint\limits_{D}\frac{x\sin y}{y}\mathrm{d}\sigma=\int_0^1\frac{\sin y}{y}\mathrm{d}y\int_y^{\sqrt{y}}x\mathrm{d}x=\int_0^1\frac{\sin y}{y}\frac{1}{2}x^2\Big|_y^{\sqrt{y}}\mathrm{d}y=\int_0^1\frac{\sin y}{y}\frac{1}{2}(y-y^2)\mathrm{d}y$
$=\frac{1}{2}\int_0^1(\sin y-y\sin y)\mathrm{d}y=\frac{1}{2}(-\cos y\Big|_0^1+y\cos y\Big|_0^1-\int_0^1\cos y\mathrm{d}y)$
$=\frac{1}{2}(1-\cos 1+\cos 1-\sin y\Big|_0^1)=\frac{1}{2}(1-\sin 1).$

注意：该题应先对x积分后对y积分，因$\frac{\sin y}{y}$无初等原函数，故不能先对y积分.

2. 计算下列二重积分：

(1)$\iint\limits_{D}\sin\sqrt{x^2+y^2}\mathrm{d}\sigma, D=\{(x,y)\mid \pi^2\leqslant x^2+y^2\leqslant 4\pi^2\};$
(2)$\iint\limits_{D}\frac{1}{1+x^2+y^2}\mathrm{d}\sigma, D=\{(x,y)\mid x^2+y^2\leqslant 1\};$
(3)$\iint\limits_{D}\arctan\frac{y}{x}\mathrm{d}\sigma, D=\{(x,y)\mid 1\leqslant x^2+y^2\leqslant 4, x\geq 0, y\geq 0\};$
(4)$\iint\limits_{D}|x^2+y^2-4|\mathrm{d}\sigma, D=\{(x,y)\mid x^2+y^2\leqslant 16\}.$

解：(1)$\iint\limits_{D}\sin\sqrt{x^2+y^2}\mathrm{d}\sigma=\int_0^{2\pi}\mathrm{d}\theta\int_\pi^{2\pi}r\sin r\mathrm{d}r=2\pi(-r\cos r\Big|_\pi^{2\pi}+\int_\pi^{2\pi}\cos r\mathrm{d}r)$
$=2\pi(-2\pi-\pi+\sin r\Big|_\pi^{2\pi})=-6\pi^2.$

(2)$\iint\limits_{D}\frac{1}{1+x^2+y^2}\mathrm{d}\sigma=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\frac{r}{1+r^2}\mathrm{d}r=\pi\ln(1+r^2)\Big|_0^1=\pi\ln 2.$

(3)$\iint\limits_{D}\arctan\frac{y}{x}\mathrm{d}\sigma=\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_1^2 r\arctan(\frac{r\sin\theta}{r\cos\theta})\mathrm{d}r=\int_0^{\frac{\pi}{2}}\theta\mathrm{d}\theta\int_1^2 r\mathrm{d}r=(\frac{1}{2}\theta^2\Big|_0^{\frac{\pi}{2}})(\frac{1}{2}r^2\Big|_1^2)=\frac{3\pi^2}{16}.$

(4)$\iint\limits_{D}|x^2+y^2-4|\mathrm{d}\sigma=\int_0^{2\pi}\mathrm{d}\theta\int_0^4|r^2-4|r\mathrm{d}r=\int_0^{2\pi}\mathrm{d}\theta[\int_0^2(4r-r^3)\mathrm{d}r+\int_2^4(r^3-4r)\mathrm{d}r]$
$=2\pi[(2r^2-\frac{1}{4}r^4)\Big|_0^2+(\frac{1}{4}r^4-2r^2)\Big|_2^4]=2\pi(8-4+64-32-4+8)=80\pi.$

3. 改变下列累次积分中的积分顺序，并给出相应重积分的积分域的集合表示：

(1)$\int_0^1\mathrm{d}y\int_0^y f(x,y)\mathrm{d}x;$ (2)$\int_{-1}^1\mathrm{d}x\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}f(x,y)\mathrm{d}y;$
(3)$\int_0^a\mathrm{d}x\int_{a-x}^{\sqrt{a^2-x^2}}f(x,y)\mathrm{d}y;$ (4)$\int_1^{\mathrm{e}}\mathrm{d}x\int_0^{\ln x}f(x,y)\mathrm{d}y.$

解：(1)积分域$D=\{(x,y)\mid 0\leqslant y\leqslant 1, 0\leqslant x\leqslant y\}=\{(x,y)\mid 0\leqslant x\leqslant 1, x\leqslant y\leqslant 1\},$

则$\int_0^1\mathrm{d}y\int_0^y f(x,y)\mathrm{d}x=\int_0^1\mathrm{d}x\int_x^1 f(x,y)\mathrm{d}y.$

(2)积分域$D=\left\{(x,y)\;\middle|\;-1\leqslant x\leqslant 1, -\sqrt{1-x^2}\leqslant y\leqslant\sqrt{1-x^2}\right\}$
$=\left\{(x,y)\;\middle|\;-1\leqslant y\leqslant 1, -\sqrt{1-y^2}\leqslant x\leqslant\sqrt{1-y^2}\right\}=\{(x,y)\mid x^2+y^2\leq 1\},$

则$\int_{-1}^1\mathrm{d}x\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}f(x,y)\mathrm{d}y=\int_{-1}^1\mathrm{d}y\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}f(x,y)\mathrm{d}x.$

(3)积分域$D=\left\{(x,y)\;\middle|\;0\leqslant x\leqslant a, a-x\leqslant y\leqslant\sqrt{a^2-x^2}\right\}$
$=\left\{(x,y)\;\middle|\;0\leqslant y\leqslant a, a-y\leqslant x\leqslant\sqrt{a^2-y^2}\right\}$
$=\{(x,y)\mid$ 直线$x+y=a$与圆$x^2+y^2=a^2$在第一象限围成的部分$\},$

则$\int_0^a \mathrm{d}x \int_{a-x}^{\sqrt{a^2-x^2}} f(x,y)\mathrm{d}y = \int_0^a \mathrm{d}y \int_{a-y}^{\sqrt{a^2-y^2}} f(x,y)\mathrm{d}x.$

(4)积分域$D=\{(x,y)\mid 1\leqslant x\leqslant \mathrm{e}, 0\leqslant y\leqslant \ln x\}=\{(x,y)\mid 0\leqslant y\leqslant 1, \mathrm{e}^y\leqslant x\leqslant \mathrm{e}\}$,

则$\int_1^{\mathrm{e}} \mathrm{d}x \int_0^{\ln x} f(x,y)\mathrm{d}y = \int_0^1 \mathrm{d}y \int_{\mathrm{e}^y}^{\mathrm{e}} f(x,y)\mathrm{d}x.$

4. 将下列累次积分交换积分顺序：

(1)$\int_0^a \mathrm{d}x \int_x^{\sqrt{2ax-x^2}} f(x,y)\mathrm{d}y$; (2)$\int_{-6}^2 \mathrm{d}x \int_{\frac{1}{4}x^2-1}^{2-x} f(x,y)\mathrm{d}y.$

解：(1)积分域$D=\left\{(x,y)\,\middle|\, 0\leqslant x\leqslant a, x\leqslant y\leqslant \sqrt{2ax-x^2}\right\}$
$=\left\{(x,y)\,\middle|\, 0\leqslant y\leqslant a, a-\sqrt{a^2-y^2}\leqslant x\leqslant y\right\}$,

则$\int_0^a \mathrm{d}x \int_x^{\sqrt{2ax-x^2}} f(x,y)\mathrm{d}y = \int_0^a \mathrm{d}y \int_{a-\sqrt{a^2-y^2}}^{y} f(x,y)\mathrm{d}x.$

(2)积分域$D=\left\{(x,y)\,\middle|\, -6\leqslant x\leqslant 2, \frac{1}{4}x^2-1\leqslant y\leqslant 2-x\right\}$
$=\left\{(x,y)\,\middle|\, 0\leqslant y\leqslant 8, -2\sqrt{1+y}\leqslant x\leqslant 2-y\right\}\cup\left\{(x,y)\,\middle|\, -1\leqslant y\leqslant 0, -2\sqrt{1+y}\leqslant x\leqslant 2\sqrt{1+y}\right\}$,

则$\int_{-6}^2 \mathrm{d}x \int_{\frac{1}{4}x^2-1}^{2-x} f(x,y)\mathrm{d}y = \int_0^8 \mathrm{d}y \int_{-2\sqrt{1+y}}^{2-y} f(x,y)\mathrm{d}x + \int_{-1}^0 \mathrm{d}y \int_{-2\sqrt{1+y}}^{2\sqrt{1+y}} f(x,y)\mathrm{d}x.$

5. 已知函数f连续且$f>0$，试求$\iint\limits_D \frac{af(x)+bf(y)}{f(x)+f(y)}\mathrm{d}\sigma$的值，其中$D=\{(x,y)\mid x^2+y^2\leqslant R^2\}$.

解：$\because$积分域D关于$y=x$对称，且在关于$y=x$的对称点(x,y)和(y,x)处
$\frac{f(x)}{f(x)+f(y)}=\frac{f(y)}{f(x)+f(y)}$,

$\therefore \iint\limits_D \frac{f(x)}{f(x)+f(y)}\mathrm{d}\sigma = \iint\limits_D \frac{f(y)}{f(x)+f(y)}\mathrm{d}\sigma$,

$\because \iint\limits_D \frac{f(x)}{f(x)+f(y)}\mathrm{d}\sigma + \iint\limits_D \frac{f(y)}{f(x)+f(y)}\mathrm{d}\sigma = \iint\limits_D \frac{f(x)+f(y)}{f(x)+f(y)}\mathrm{d}\sigma = \iint\limits_D \sigma\mathrm{d} = \pi R^2$,

$\therefore \iint\limits_D \frac{f(x)}{f(x)+f(y)}\mathrm{d}\sigma = \iint\limits_D \frac{f(y)}{f(x)+f(y)}\mathrm{d}\sigma = \frac{\pi}{2}R^2$,

$\therefore \iint\limits_D \frac{af(x)+bf(y)}{f(x)+f(y)}\mathrm{d}\sigma = a\iint\limits_D \frac{f(x)}{f(x)+f(y)}\mathrm{d}\sigma + b\iint\limits_D \frac{f(y)}{f(x)+f(y)}\mathrm{d}\sigma = \frac{a+b}{2}\pi R^2.$

18.4 习题12.3解答

1. 求由$xy=a^2, xy=2a^2, y=x, y=2x$围成的第一象限区域的面积.

解：令$\begin{cases} u=xy, \\ v=\frac{y}{x}, \end{cases}$ 所求区域$D=\{(u,v)\mid a^2\leqslant u\leqslant 2a^2, 1\leqslant v\leqslant 2\}$,

$\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}=\begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x}\end{vmatrix}=\frac{y}{x}+\frac{y}{x}=2v,\ \left|\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}\right|=\frac{1}{2v}$,

所求面积$S=\iint\limits_D \mathrm{d}\sigma = \iint\limits_D \left|\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}\right|\mathrm{d}u\mathrm{d}v = \int_{a^2}^{2a^2}\mathrm{d}u\int_1^2 \frac{1}{2v}\mathrm{d}v = a^2\frac{1}{2}\ln v\Big|_1^2 = \frac{\ln 2}{2}a^2.$

2. 计算$I=\iint\limits_D \cos(\frac{x-y}{x+y})\mathrm{d}\sigma, D$由$x+y=1, x=0, y=0$围成.

解：方法1：$\because$区域D关于$y=x$对称，且在关于$y=x$的对称点(x,y)和(y,x)处$\cos(\frac{x-y}{x+y})=\cos(\frac{y-x}{y+x})$,

$\therefore I=\iint\limits_{D}\cos(\frac{x-y}{x+y})\mathrm{d}\sigma=2\iint\limits_{D_1}\cos(\frac{x-y}{x+y})\mathrm{d}\sigma$，其中区域$D_1$由$x+y=1,y=0,y=x$围成.

令$\begin{cases}u=x+y,\\ v=\frac{y}{x},\end{cases}$ 则$\begin{cases}x=\frac{u}{1+v},\\ y=\frac{uv}{1+v},\end{cases}$ 区域$D_1=\{(u,v)\mid 0\leqslant u\leqslant 1,0\leqslant v\leqslant 1\}$,

$\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}=\begin{vmatrix}1 & 1\\ -\frac{y}{x^2} & \frac{1}{x}\end{vmatrix}=\frac{x+y}{x^2}=\frac{(1+v)^2}{u}$, $|\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}|=\frac{u}{(1+v)^2}$,

$\therefore I=2\iint\limits_{D_1}\cos(\frac{x-y}{x+y})\mathrm{d}\sigma=2\iint\limits_{D_1}\cos(\frac{1-v}{1+v})\frac{u}{(1+v)^2}\mathrm{d}u\mathrm{d}v=2\int_0^1\cos(\frac{1-v}{1+v})\frac{1}{(1+v)^2}\mathrm{d}v\int_0^1 u\mathrm{d}u$
$=\int_0^1\cos(\frac{1-v}{1+v})\frac{1}{(1+v)^2}\mathrm{d}v=\int_0^1\cos(-1+\frac{2}{1+v})\frac{1}{(1+v)^2}\mathrm{d}v=-\frac{1}{2}\int_0^1\cos(-1+\frac{2}{1+v})\mathrm{d}(-1+\frac{2}{1+v})$
$=-\frac{1}{2}\sin(-1+\frac{2}{1+v})\big|_0^1=\frac{1}{2}\sin 1.$

注： 如图6所示.[1]

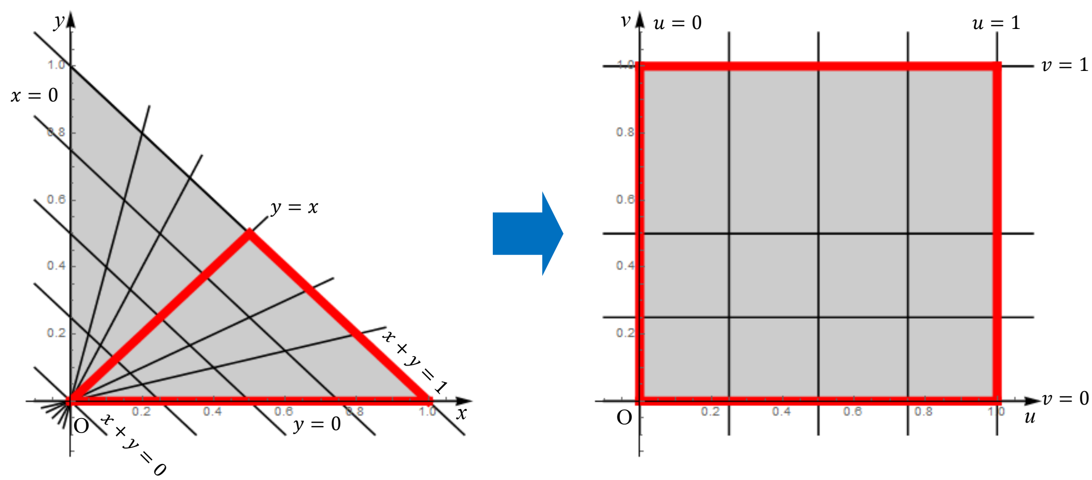

图 6: 习题12.3 2题方法1图示

方法2：令$\begin{cases}x=r\cos^2\theta,\\ y=r\sin^2\theta,\end{cases}$ 则区域$D=\left\{(x,y)\,\middle|\,0\leqslant r\leqslant 1,0\leqslant\theta\leqslant\frac{\pi}{2}\right\}$,

$\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}=\begin{vmatrix}\cos^2\theta & -2r\cos\theta\sin\theta\\ \sin^2\theta & 2r\sin\theta\cos\theta\end{vmatrix}=2r\sin\theta\cos^3\theta+2r\sin^2\theta\cos\theta=2r\sin\theta\cos\theta=r\sin 2\theta,$

$\therefore I=\iint\limits_{D}\cos(\frac{r\cos^2\theta-r\sin^2\theta}{r\cos^2\theta+r\sin^2\theta})|\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}|\mathrm{d}r\mathrm{d}\theta=\iint\limits_{D}\cos(\cos 2\theta)r\sin 2\theta\mathrm{d}r\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\cos(\cos 2\theta)\sin 2\theta\mathrm{d}\theta\int_0^1 r\mathrm{d}r$

[1]这是修订版增加的内容.

$= -\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos(\cos 2\theta)\mathrm{d}\cos 2\theta \int_0^1 r\mathrm{d}r = -\frac{1}{2}\sin(\cos 2\theta)\Big|_0^{\frac{\pi}{2}} \frac{1}{2}r^2\Big|_0^1 = -\frac{1}{2}[\sin(-1)-\sin 1]\frac{1}{2}$
$= \frac{1}{2}\sin 1.$

注： 如图7所示.[2]

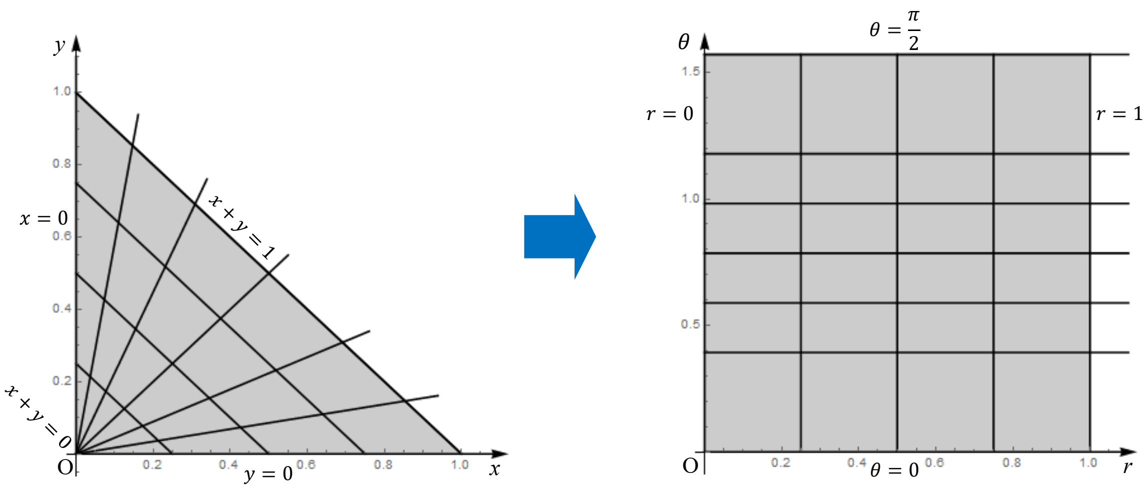

图 7: 习题12.3 2题方法2图示

方法3：令$\begin{cases} u = x - y, \\ v = x + y, \end{cases}$ 则令$\begin{cases} x = \frac{1}{2}(u+v), \\ y = \frac{1}{2}(v-u), \end{cases}$ 区域$D = \{(u,v) \mid 0 \leqslant v \leqslant 1, -v \leqslant u \leqslant v\}$,

$\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2},$

$\therefore I = \iint\limits_D \cos(\frac{u}{v})|\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}|\mathrm{d}u\mathrm{d}v = \frac{1}{2}\iint\limits_D \cos(\frac{u}{v})\mathrm{d}u\mathrm{d}v = \frac{1}{2}\int_0^1 \mathrm{d}v \int_{-v}^{v}\cos(\frac{u}{v})\mathrm{d}u = \frac{1}{2}\int_0^1 \mathrm{d}v \int_{-v}^{v} v\cos(\frac{u}{v})\mathrm{d}\frac{u}{v}$
$= \frac{1}{2}\int_0^1 \mathrm{d}v[v\sin(\frac{u}{v})]_{-v}^{v} = \frac{1}{2}\int_0^1 [v\sin 1 - v\sin(-1)]\mathrm{d}v = \sin 1\int_0^1 v\mathrm{d}v = \frac{1}{2}\sin 1.$

注意：因为被积函数是$\cos(\frac{u}{v})$，该函数无关于v初等原函数，故这种变量代换的方法应先积u后积v.

注： 如图8所示.[3]

[2]这是修订版增加的内容.

[3]这是修订版增加的内容.

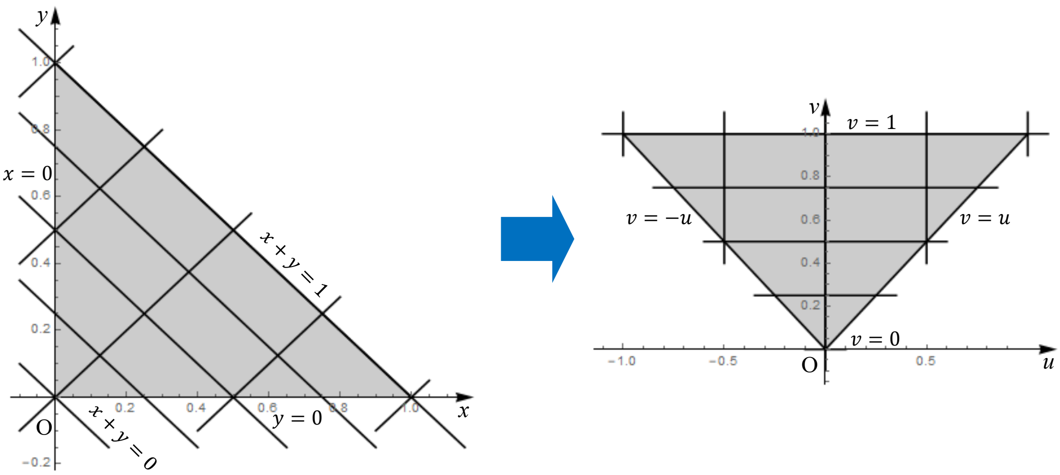

图 8: 习题12.3 2题方法3图示

3. 计算$I=\iint\limits_{D}(\sqrt{x}+\sqrt{y})\mathrm{d}\sigma, D=\left\{(x,y)\ \middle|\ \sqrt{x}+\sqrt{y}\leqslant 1\right\}$.

解：方法1：令$\begin{cases}x=r^2\cos^4\theta,\\ y=r^2\sin^4\theta,\end{cases}$ 则区域$D=\left\{(r,\theta)\ \middle|\ 0\leqslant r\leqslant 1, 0\leqslant\theta\leqslant\frac{\pi}{2}\right\}$,

$$\frac{\mathrm{D}(x,y)}{\mathrm{D}(r,\theta)}=\begin{vmatrix}2r\cos^4\theta & -4r^2\cos^3\theta\sin\theta\\ 2r\sin^4\theta & 4r^2\sin^3\theta\cos\theta\end{vmatrix}=8r^3\sin^3\theta\cos^3\theta,$$

$$\therefore I=\iint\limits_{D}(\sqrt{x}+\sqrt{y})\mathrm{d}\sigma=\iint\limits_{D}r\left|\frac{\mathrm{D}(x,y)}{\mathrm{D}(r,\theta)}\right|\mathrm{d}u\mathrm{d}v=\iint\limits_{D}8r^4\sin^3\theta\cos^3\theta\mathrm{d}u\mathrm{d}v=\int_0^1 r^4\mathrm{d}r\int_0^{\frac{\pi}{2}}2^3\sin^3\theta\cos^3\theta\mathrm{d}\theta$$

$$=\frac{1}{5}r^5\Big|_0^1\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^3 2\theta\mathrm{d}2\theta=\frac{1}{10}\int_0^{\frac{\pi}{2}}\sin^3 2\theta\mathrm{d}2\theta=\frac{1}{10}\int_0^{\pi}\sin^3\varphi\mathrm{d}\varphi=\frac{2}{10}\int_0^{\frac{\pi}{2}}\sin^3\varphi\mathrm{d}\varphi=\frac{1}{5}\frac{2}{3}=\frac{2}{15}.$$

方法2[4]：令$\begin{cases}u=\sqrt{x},\\ v=\sqrt{y},\end{cases}$ 则$\begin{cases}x=u^2,\\ y=v^2,\end{cases}$ 区域$D=\{(u,v)\mid 0\leqslant u\leqslant 1, 0\leqslant v\leqslant 1-u\}$,

$$\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}=\begin{vmatrix}2u & 0\\ 0 & 2v\end{vmatrix}=4uv,$$

$$\therefore I=\iint\limits_{D}(\sqrt{x}+\sqrt{y})\mathrm{d}\sigma=\iint\limits_{D}(u+v)\left|\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}\right|\mathrm{d}u\mathrm{d}v=\iint\limits_{D}(u+v)4uv\mathrm{d}u\mathrm{d}v$$

$$=4\int_0^1\mathrm{d}u\int_0^{1-u}(u^2v+uv^2)\mathrm{d}v=4\int_0^1\left(\frac{1}{2}u^2v^2+\frac{1}{3}uv^3\right)\Big|_0^{1-u}\mathrm{d}u$$

$$=4\int_0^1\left[\frac{1}{2}u^2(1-u)^2+\frac{1}{3}u(1-u)^3\right]\mathrm{d}u=4\int_0^1\left(\frac{1}{2}u^2-u^3+\frac{1}{2}u^4+\frac{1}{3}u-u^2+u^3-\frac{1}{3}u^4\right)\mathrm{d}u$$

$$=4\int_0^1\left(-\frac{1}{2}u^2+\frac{1}{6}u^4+\frac{1}{3}u\right)\mathrm{d}u=4\left(-\frac{1}{6}u^3+\frac{1}{30}u^5+\frac{1}{6}u^2\right)\Big|_0^1=\frac{2}{15}.$$

[4]这是修订版增加的内容.

4. 在第1象限中，设D由$xy=1, xy=2, \frac{y}{x}=1$及$\frac{y}{x}=4$围成，试证：

$$\iint\limits_D f(xy)\mathrm{d}\sigma = \ln 2\int_1^2 f(x)\mathrm{d}x.$$

证明：令$\begin{cases} u=xy, \\ v=\frac{y}{x}, \end{cases}$ 则区域$D=\{(u,v)\mid 1\leqslant u\leqslant 2, 1\leqslant v\leqslant 4\}$,

$\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{y}{x}+\frac{y}{x}=2v$, $|\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}| = \frac{1}{2v}$,

$\therefore \iint\limits_D f(xy)\mathrm{d}\sigma = \iint\limits_D f(u)|\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}|\mathrm{d}u\mathrm{d}v = \int_1^2 f(u)\mathrm{d}u\int_1^4 \frac{1}{2v}\mathrm{d}v = \frac{1}{2}\ln v\Big|_1^4\int_1^2 f(u)\mathrm{d}u$
$= 2\ln 2\int_1^2 f(u)\mathrm{d}u = \ln 2\int_1^2 f(x)\mathrm{d}x.$