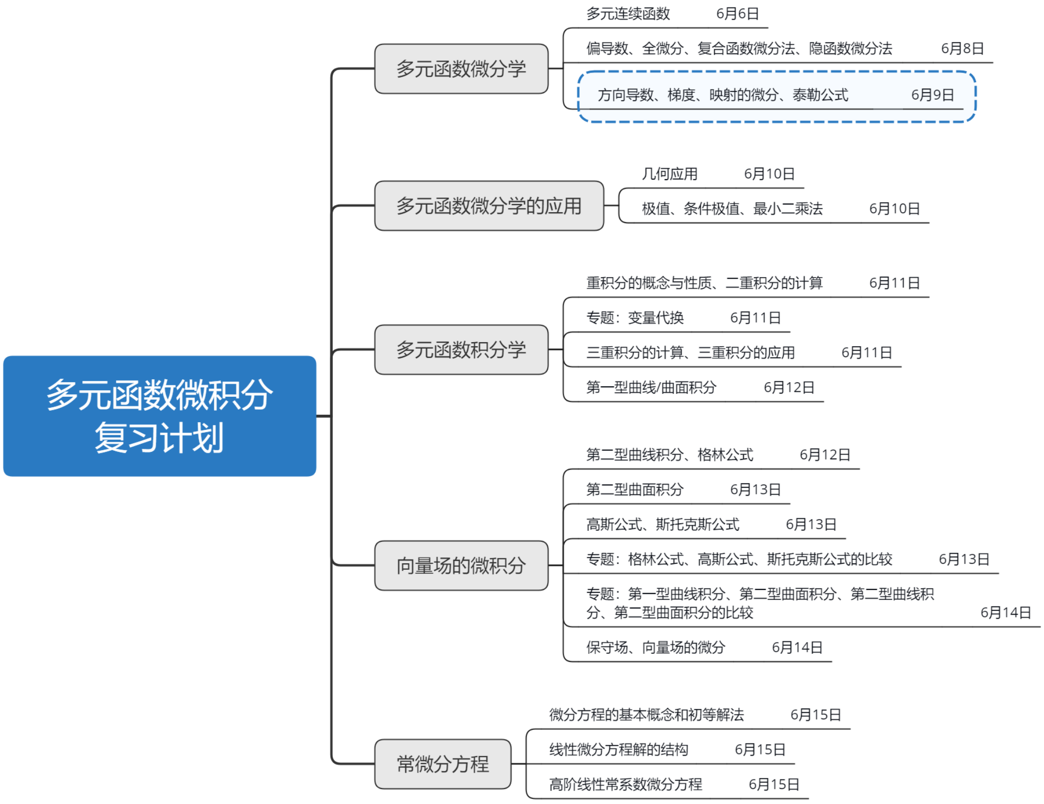
3 方向导数、梯度、映射的微分、泰勒公式

3.1 复习计划



3.2 知识结构

- 多元函数微分学
  - 多元连续函数
  - 偏导数
  - 全微分
  - 复合函数微分法
  - 隐函数微分法
  - 方向导数和梯度
    - 多元函数的方向导数
      - 方向导数的概念
        - $\boldsymbol{l}=(\cos\alpha,\sin\alpha)$, $\frac{\partial f(a,b)}{\partial \boldsymbol{l}}=\lim\limits_{t\to 0}\frac{f(a+t\cos\alpha,b+t\sin\alpha)-f(a,b)}{t}$
        - $g(t)=f(a+t\cos\alpha,b+t\sin\alpha)$, $\frac{\partial f(a,b)}{\partial \boldsymbol{l}}=g'(0)$
        - 偏导数是特殊的方向导数
      - 方向导数的计算
        - $f(x,y)$在点(a,b)处可微$\Rightarrow \frac{\partial f(x,y)}{\partial \boldsymbol{l}}=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\sin\alpha$
          - $\boldsymbol{l}=(\cos\alpha,\sin\alpha)$ 是单位向量
    - 梯度
      - 梯度向量的概念
        - $\mathrm{grad} f(a,b)=\frac{\partial f(a,b)}{\partial \boldsymbol{l}_0}\boldsymbol{l}_0$，其中$\boldsymbol{l}_0$满足$\frac{\partial f(a,b)}{\partial \boldsymbol{l}_0}=\max\limits_{\|\boldsymbol{l}\|=1}\{\frac{\partial f(a,b)}{\partial \boldsymbol{l}}\}$
          - 梯度向量的方向：取到最大的方向导数
          - 梯度向量的长度：方向导数的最大值
      - 梯度向量的计算
        - $f(x,y)$在点(a,b)处可微$\Rightarrow \mathrm{grad} f(a,b)=(\frac{\partial f(a,b)}{\partial x},\frac{\partial f(a,b)}{\partial y})$
          - $f(x,y)$在点(a,b)处可微$\Rightarrow \frac{\partial f(a,b)}{\partial \boldsymbol{l}}=\mathrm{grad} f(a,b)\cdot\boldsymbol{l}$
          - 梯度向量与等值线垂直
  - 映射及其微分
    - $\mathbb{R}^n\to\mathbb{R}^m$的映射
      - 定义
      - 连续映射
    - 映射的微分
      - 概念
      - 映射的Jacobi矩阵
    - 复合映射的微分法
    - 一元向量值函数($\mathbb{R}^1\to\mathbb{R}^m$的映射)的导数与积分
      - 一元向量值函数的概念　$\boldsymbol{r}=\boldsymbol{f}(t), t\in\mathbb{R}, \boldsymbol{r}\in\mathbb{R}^n$
      - 一元向量值函数的导数
        - $\boldsymbol{f}'(t)=\begin{pmatrix} x'(t)\\ y'(t)\\ z'(t)\end{pmatrix}$
        - $\mathrm{d}\boldsymbol{f}(t)=\begin{pmatrix} x'(t)\mathrm{d}t\\ y'(t)\mathrm{d}t\\ z'(t)\mathrm{d}t\end{pmatrix}$
      - 求导法则
        - $(c\boldsymbol{u}(t))'=c\boldsymbol{u}'(t)$
        - $(\boldsymbol{u}(t)+\boldsymbol{v}(t))'=\boldsymbol{u}'(t)+\boldsymbol{v}'(t)$
        - $(\lambda(t)\boldsymbol{u}(t))'=\lambda'(t)\boldsymbol{u}(t)+\lambda(t)\boldsymbol{u}'(t)$
        - $(\boldsymbol{u}(t)\cdot\boldsymbol{v}(t))'=\boldsymbol{u}'(t)\cdot\boldsymbol{v}(t)+\boldsymbol{u}(t)\cdot\boldsymbol{v}'(t)$
        - $(\boldsymbol{u}(t)\times\boldsymbol{v}(t))'=\boldsymbol{u}'(t)\times\boldsymbol{v}(t)+\boldsymbol{u}(t)\times\boldsymbol{v}'(t)$
      - 向量值函数的积分　$\int_a^b \boldsymbol{f}(t)\mathrm{d}t=\begin{pmatrix}\int_a^b x(t)\mathrm{d}t\\ \int_a^b y(t)\mathrm{d}t\\ \int_a^b z(t)\mathrm{d}t\end{pmatrix}$
  - 多元函数的泰勒公式
    - 二元函数的n阶带有 Peano型余项的泰勒公式
      $$\begin{aligned}f(x,y)&=f(a,b)\\ &+[\tfrac{\partial}{\partial x}(x-a)+\tfrac{\partial}{\partial y}(y-b)]f(a,b)\\ &+\tfrac{1}{2}[\tfrac{\partial}{\partial x}(x-a)+\tfrac{\partial}{\partial y}(y-b)]^2 f(a,b)\\ &+\cdots\\ &+\tfrac{1}{n!}[\tfrac{\partial}{\partial x}(x-a)+\tfrac{\partial}{\partial y}(y-b)]^n f(a,b)\\ &+o((\sqrt{(x-a)^2+(y-b)^2})^n)\end{aligned}$$
    - 二元函数的n阶带有Lagrange型余项的泰勒公式
      $$\begin{aligned}f(x,y)&=f(a,b)\\ &+[\tfrac{\partial}{\partial x}(x-a)+\tfrac{\partial}{\partial y}(y-b)]f(a,b)\\ &+\tfrac{1}{2}[\tfrac{\partial}{\partial x}(x-a)+\tfrac{\partial}{\partial y}(y-b)]^2 f(a,b)\\ &+\cdots\\ &+\tfrac{1}{n!}[\tfrac{\partial}{\partial x}(x-a)+\tfrac{\partial}{\partial y}(y-b)]^n f(a,b)\\ &+\tfrac{1}{(n+1)!}[\tfrac{\partial}{\partial x}(x-a)+\tfrac{\partial}{\partial y}(y-b)]^{n+1} f(a+\theta\Delta x,b+\theta\Delta y),\\ &0<\theta<1\end{aligned}$$

3.3 重要知识

3.4 习题分类与解题思路

1. 方向导数与梯度

   (a) 方向导数的计算，可用定义或者梯度进行计算. 当用梯度计算时，须注意在所求点上函数应可微，且应把方向向量单位化.

   【如习题10.4中的5.】

   (b) 考查梯度的几何意义.

   具体可见【习题10.4中的6.】

2. 一元向量值函数的导数与积分

   (a) 考查向量值函数的四则运算法则.

   【如习题11.1中的1., 2.】

   (b) 求曲线的切向量或切线方程，即一元向量值函数的导数.

   【如习题11.1中的2., 3., 7., 8.】

   (c) 求一元向量值函数的积分.

   【如习题11.1中的4., 9.】

   (d) 已知一元向量值函数的导函数和定解条件，求原函数.

   【如习题11.1中的5.】

3. 多元函数的泰勒公式

   (a) 利用带佩亚诺余项的泰勒公式做近似计算.

   【如习题10.6中的1.】

   (b) 将函数展开成泰勒多项式.

   【如习题10.6中的2.】

   (c) 将函数展开成带拉格朗日余项的泰勒多项式.

   【如习题10.6中的3.】

【第10章补充题的题目有一些综合性，大家可做一下积累.】

3.5 习题10.4解答

5. 求$z=\ln(\mathrm{e}^{-x}+\frac{x^2}{y})$在点$(1,1)$处沿$\boldsymbol{v}=(a,b)^T(a\neq 0)$的方向导数.

   解：$\because \frac{\partial z}{\partial x}=\frac{-\mathrm{e}^{-x}+\frac{2x}{y}}{\mathrm{e}^{-x}+\frac{x^2}{y}}, \frac{\partial z}{\partial y}=\frac{-\frac{x^2}{y^2}}{\mathrm{e}^{-x}+\frac{x^2}{y}}$,

   $\therefore \mathrm{grad}z(1,1)=(\frac{\partial z(1,1)}{\partial x},\frac{\partial z(1,1)}{\partial y})=(\frac{-\mathrm{e}^{-1}+2}{\mathrm{e}^{-1}+1},\frac{-1}{\mathrm{e}^{-1}+1})=(\frac{2\mathrm{e}-1}{\mathrm{e}+1},-\frac{\mathrm{e}}{\mathrm{e}+1})$,

   $\because \frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$在点$(1,1)$及其附近存在且在点$(1,1)$处连续,

   $\therefore f(x,y)$在点$(1,1)$处可微,

   $\therefore \frac{\partial z(1,1)}{\partial \boldsymbol{v}}=\mathrm{grad}z(1,1)\cdot\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}=(\frac{2\mathrm{e}-1}{\mathrm{e}+1},-\frac{\mathrm{e}}{\mathrm{e}+1})\cdot\frac{1}{\sqrt{a^2+b^2}}(a,b)^T=\frac{1}{\sqrt{a^2+b^2}}(\frac{2a\mathrm{e}-a-b\mathrm{e}}{\mathrm{e}+1})$.

6. 已知$f(x,y)=x^2-xy+y^2$.

   (1)当$\boldsymbol{v}$分别为何向量时，方向导数$\frac{\partial f(1,1)}{\partial \boldsymbol{v}}$会取到最大值、最小值和零值？并求出其最大值和最小值.

   (2)试求$\mathrm{grad}f(1,1)$，并说明其方向与大小的意义.

   解：(1)$\because \frac{\partial f(x,y)}{\partial x}=2x-y,\frac{\partial f(x,y)}{\partial y}=2y-x$,

   $\therefore \mathrm{grad}f(1,1)=(\frac{\partial f(1,1)}{\partial x},\frac{\partial f(1,1)}{\partial y})=(1,1)$,

   $\because \frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$在点$(1,1)$及其附近存在且在点$(1,1)$处连续,

   $\therefore f(x,y)$在点$(1,1)$处可微,

   $\therefore \frac{\partial f(1,1)}{\partial \boldsymbol{v}}=\mathrm{grad}f(1,1)\cdot\boldsymbol{v}=\|\mathrm{grad}f(1,1)\|\|\boldsymbol{v}\|\cos\theta=\|\mathrm{grad}f(1,1)\|\cos\theta$，

   当$\boldsymbol{v}$与梯度向量的夹角$\theta=0$即$\boldsymbol{v}=\frac{1}{\sqrt{2}}(1,1)$时，方向导数$\frac{\partial f(1,1)}{\partial \boldsymbol{v}}$取得最大值$\|\mathrm{grad}f(1,1)\|=\sqrt{2}$；

   当$\boldsymbol{v}$与梯度向量的夹角$\theta=\pi$即$\boldsymbol{v}=-\frac{1}{\sqrt{2}}(1,1)$时，方向导数$\frac{\partial f(1,1)}{\partial \boldsymbol{v}}$取得最小值$-\|\mathrm{grad}f(1,1)\|=-\sqrt{2}$；

   当$\boldsymbol{v}$与梯度向量的夹角$\theta=\frac{\pi}{2}$即$\boldsymbol{v}=\frac{1}{\sqrt{2}}(-1,1)$或$\frac{1}{\sqrt{2}}(1,-1)$时，方向导数$\frac{\partial f(1,1)}{\partial \boldsymbol{v}}=0$.

   (2)$\mathrm{grad}f(1,1)=(1,1)$，其方向表示方向导数最大的方向，其大小为方向导数的最大值.

3.6 习题10.6解答

1. 写出$f(x,y)=x^y$在点$(1,1)$带佩亚诺余项的三阶泰勒公式，由此计算$1.1^{1.02}$.

   解：$f(1,1)=1$,

   $\frac{\partial f}{\partial x}=yx^{y-1},\frac{\partial f}{\partial y}=x^y\ln x$,

   $\frac{\partial^2 f}{\partial x^2}=y(y-1)x^{y-2},\frac{\partial^2 f}{\partial x\partial y}=x^{y-1}+yx^{y-1}\ln x,\frac{\partial^2 f}{\partial y^2}=x^y(\ln x)^2$,

$\frac{\partial^3 f}{\partial x^3} = y(y-1)(y-2)x^{y-3}, \frac{\partial^3 f}{\partial x \partial y^2} = yx^{y-1}(\ln x)^2,$
$\frac{\partial^3}{\partial y \partial x^2} = (2y-1)x^{y-2} + y(y-1)x^{y-2}\ln x = [(2y-1) + y(y-1)\ln x]x^{y-2},$
$\frac{\partial^3 f}{\partial y^3} = x^y(\ln x)^3,$

$\therefore$

$$\begin{aligned}
&f(x,y)\\
=&f(1,1) + [\frac{\partial f(1,1)}{\partial x}(x-1) + \frac{\partial f(1,1)}{\partial y}(y-1)]\\
&+ \frac{1}{2}[\frac{\partial^2 f(1,1)}{\partial x^2}(x-1)^2 + 2\frac{\partial^2 f(1,1)}{\partial x \partial y}(x-1)(y-1) + \frac{\partial^2 f(1,1)}{\partial y^2}(y-1)^2]\\
&+ \frac{1}{3}[\frac{\partial^3 f(1,1)}{\partial x^3}(x-1)^3 + 3\frac{\partial^3 f(1,1)}{\partial x \partial y^2}(x-1)(y-1)^2 + 3\frac{\partial^3 f(1,1)}{\partial y \partial x^2}(x-1)^2(y-1)\\
&\qquad + \frac{\partial^3 f(1,1)}{\partial y^3}(y-1)^3] + o[(\sqrt{(x-1)^2+(y-1)^2})^3]\\
=&1 + (x-1) + \frac{1}{2}[2(x-1)(y-1)] + \frac{1}{3}[3(x-1)^2(y-1)] + o[(\sqrt{(x-1)^2+(y-1)^2})^3]\\
=&x + (x-1)(y-1) + (x-1)^2(y-1) + o[(\sqrt{(x-1)^2+(y-1)^2})^3]
\end{aligned}$$

$\therefore 1.1^{1.02} = f(1.1, 1.02) \approx 1 + 0.1 + 0.1 \times 0.02 + 0.1^2 \times 0.02 = 1.1022.$

2. 证明当$|x|, |y|$充分小时，有$\frac{\cos x}{\cos y} \approx 1 - \frac{1}{2}(x^2 - y^2).$

证明：记$f(x,y) = \frac{\cos x}{\cos y}$,

$f(0,0) = 1,$

$$\frac{\partial f}{\partial x} = -\frac{\sin x}{\cos y}, \frac{\partial f}{\partial y} = \frac{\cos x \sin y}{\cos^2 y},$$
$$\frac{\partial^2 f}{\partial x^2} = -\frac{\cos x}{\cos y}, \frac{\partial^2 f}{\partial x \partial y} = -\frac{\sin x \sin y}{\cos^2 y},$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\cos x \cos y \cos^2 y + \cos x \sin y 2\cos y \sin y}{\cos^4 y} = \frac{\cos x \cos^2 y + 2\cos x \sin^2 y}{\cos^4 y},$$

$\therefore$

$$\begin{aligned}
f(x,y) =&f(0,0) + [\frac{\partial f(0,0)}{\partial x}x + \frac{\partial f(0,0)}{\partial y}y] + \frac{1}{2}[\frac{\partial^2 f(0,0)}{\partial x^2}x^2 + 2\frac{\partial^2 f(0,0)}{\partial x \partial y}xy + \frac{\partial^2 f(0,0)}{\partial y^2}y^2]\\
&+ o[(\sqrt{x^2+y^2})^2]\\
=&1 + (0+0) + \frac{1}{2}(-x^2 + 0 + y^2) + o[(\sqrt{x^2+y^2})^2]\\
=&1 - \frac{1}{2}(x^2 - y^2) + o[(\sqrt{x^2+y^2})^2],
\end{aligned}$$

$\therefore$当$|x|, |y|$充分小时，有$\frac{\cos x}{\cos y} \approx 1 - \frac{1}{2}(x^2 - y^2).$

3. 写出$f(x,y)=\sqrt{1+y^2}\cos x$在点$(0,1)$的一阶泰勒多项式及拉格朗日余项.

解：$f(0,1)=\sqrt{2}$,

$$\frac{\partial f}{\partial x}=-\sqrt{1+y^2}\sin x,\frac{\partial f}{\partial y}=\frac{2y\cos x}{2\sqrt{1+y^2}}=\frac{y\cos x}{\sqrt{1+y^2}},$$
$$\frac{\partial^2 f}{\partial x^2}=-\sqrt{1+y^2}\cos x,\frac{\partial^2 f}{\partial x\partial y}=\frac{-2y\sin x}{2\sqrt{1+y^2}}=\frac{-y\sin x}{\sqrt{1+y^2}},$$
$$\frac{\partial^2 f}{\partial y^2}=\frac{\cos x\sqrt{1+y^2}-y\cos x\frac{2y}{2\sqrt{1+y^2}}}{1+y^2}=\frac{\cos x}{(1+y^2)^{\frac{3}{2}}},$$

$\therefore$

$$\begin{aligned}
f(x,y)=&f(0,1)+[\frac{\partial f(0,1)}{\partial x}x+\frac{\partial f}{\partial y}(y-1)]\\
&+\frac{1}{2}[\frac{\partial^2 f(\theta x,1+\theta(y-1))}{\partial x^2}x^2+2\frac{\partial^2 f(\theta x,1+\theta(y-1))}{\partial x\partial y}x(y-1)\\
&\qquad+\frac{\partial^2 f(\theta x,1+\theta(y-1))}{\partial y^2}(y-1)^2]\\
=&\sqrt{2}+\frac{\sqrt{2}}{2}(y-1)\\
&+\frac{1}{2}\Big(-\sqrt{1+[1+\theta(y-1)]^2}\cos(\theta x)x^2-\frac{2[1+\theta(y-1)]\sin\theta x}{\sqrt{1+[1+\theta(y-1)]^2}}x(y-1)\\
&\qquad+\frac{\cos\theta x}{\{1+[1+\theta(y-1)]^2\}^{\frac{3}{2}}}(y-1)^2\Big)\\
=&\sqrt{2}+\frac{\sqrt{2}}{2}(y-1)\\
&+\frac{1}{2}\Big\{-x^2\sqrt{1+(1+\theta(y-1))^2}\cos\theta x-2x(y-1)\frac{1+\theta(y-1)}{\sqrt{1+(1+\theta(y-1))^2}}\sin\theta x\\
&\qquad+(y-1)^2\frac{\cos\theta x}{\{1+(1+\theta(y-1))^2\}^{\frac{3}{2}}}\Big\},0<\theta<1.
\end{aligned}$$

3.7 第10章补充题

1. 设$f(x,y)$是定义在整个平面上的连续函数，当$x^2+y^2\to+\infty$时，$f(x,y)\to+\infty$. 求证存在(x_0,y_0)，使

$$f(x_0,y_0)=\min\left\{f(x,y)\;\middle|\;(x,y)\in\mathbb{R}^2\right\}.$$

证明：$\because$当$x^2+y^2\to+\infty$时，$f(x,y)\to+\infty$,

$\therefore$对于$f(0,0),\exists N>0,s.t.f(x,y)>f(0,0),x^2+y^2>N^2$,

$\because$在有界闭区域$D=\{(x,y)\mid x^2+y^2\le N^2\}$内部$f(x,y)$连续，

$\therefore\exists(x_0,y_0)\in D,s.t.f(x_0,y_0)\le f(x,y),(x,y)\in D$，此时$f(x_0,y_0)\le f(0,0)$，

$\therefore f(x_0,y_0)\le f(0,0)<f(x,y), x^2+y^2>N^2,$

$\therefore f(x_0,y_0)\le f(x,y), (x,y)\in\mathbb{R}^2,$

$\therefore$存在(x_0,y_0)，使$f(x_0,y_0)=\min\{f(x,y)\mid (x,y)\in\mathbb{R}^2\}$.

2. 设$f(x,y)$是定义在整个平面上的连续函数，$f(0,0)=0$，且当$(x,y)\neq(0,0)$时，$f(x,y)>0$，又设对于任意的(x,y)和任意实数c，都有

$$f(cx,cy)=c^2f(x,y).$$

求证存在正数a,b，使得对于任意的(x,y)，都有

$$a(x^2+y^2)\le f(x,y)\le b(x^2+y^2).$$

证明：$\because f(x,y)$是定义在整个平面上的连续函数，

$\therefore f(x,y)$在有界闭区域$D=\{(x,y)\mid 0.5<x^2+y^2\le 1.5\}$上连续，

$\because$当$(x,y)\neq(0,0)$时，$f(x,y)>0$，

$\therefore \exists b\ge a>0, s.t. a\le f(x,y)\le b, (x,y)\in D,$

$\therefore$当$(x,y)\in D^*=\{(x,y)\mid x^2+y^2=1\}\subset D$时，$a\le f(x,y)\le b$,

$\because f(x,y)=f(\sqrt{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}},\sqrt{x^2+y^2}\frac{y}{x^2+y^2})=(x^2+y^2)f(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}),$

又$\because a\le f(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}})\le b,$

$\therefore a(x^2+y^2)\le f(x,y)\le b(x^2+y^2).$

3. 若对于任意实数t，函数$f(x,y,z)$满足$f(tx,ty,tz)=t^kf(x,y,z)$，则称$f(x,y,z)$为k次齐次函数. 试证k次齐次函数$f(x,y,z)$满足方程

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=kf(x,y,z).$$

证明：方法1：等式$f(tx,ty,tz)=t^kf(x,y,z)$两边分别对t求偏导得

$$x\frac{\partial f(tx,ty,tz)}{\partial x}+y\frac{\partial f(tx,ty,tz)}{\partial y}+z\frac{\partial f(tx,ty,tz)}{\partial z}=kt^{k-1}f(x,y,z),$$

令$t=1$得

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=kf(x,y,z).$$

方法2：$\because f(tx,ty,tz)=t^kf(x,y,z),$

$\therefore$

$$\frac{\partial f(x,y,z)}{\partial x}=\frac{\partial}{\partial x}\frac{f(tx,ty,tz)}{t^k}=\frac{1}{t^k}\frac{\partial f(tx,ty,tz)}{\partial x}t=\frac{1}{t^{k-1}}\frac{\partial f(tx,ty,tz)}{\partial x}, \tag{1a}$$

$$\frac{\partial f(x,y,z)}{\partial y}=\frac{\partial}{\partial y}\frac{f(tx,ty,tz)}{t^k}=\frac{1}{t^k}\frac{\partial f(tx,ty,tz)}{\partial y}t=\frac{1}{t^{k-1}}\frac{\partial f(tx,ty,tz)}{\partial y}, \tag{1b}$$
$$\frac{\partial f(x,y,z)}{\partial z}=\frac{\partial}{\partial z}\frac{f(tx,ty,tz)}{t^k}=\frac{1}{t^k}\frac{\partial f(tx,ty,tz)}{\partial z}t=\frac{1}{t^{k-1}}\frac{\partial f(tx,ty,tz)}{\partial z}, \tag{1c}$$

方程$f(tx,ty,tz)=t^kf(x,y,z)$两边分别对t求导

$$x\frac{\partial f(tx,ty,tz)}{\partial x}+y\frac{\partial f(tx,ty,tz)}{\partial y}+z\frac{\partial f(tx,ty,tz)}{\partial z}=kt^{k-1}f(x,y,z),$$

将式 (1a)-(1c)代入上式

$$xt^{k-1}\frac{\partial f(x,y,z)}{\partial x}+yt^{k-1}\frac{\partial f(x,y,z)}{\partial y}+zt^{k-1}\frac{\partial f(x,y,z)}{\partial z}=kt^{k-1}f(x,y,z),$$

即

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=kf(x,y,z).$$

4. 设F为三元可微函数，$u=u(x,y,z)$是由方程$F(u^2-x^2,u^2-y^2,u^2-z^2)=0$确定的隐函数. 求证

$$\frac{u'_x}{x}+\frac{u'_y}{y}+\frac{u'_z}{z}=\frac{1}{u}.$$

证明：方程$F(u^2-x^2,u^2-y^2,u^2-z^2)=0$两边分别对$x$求偏导

$$F'_1(2u\frac{\partial u}{\partial x}-2x)+F'_2 2u\frac{\partial u}{\partial x}+F'_3 2u\frac{\partial u}{\partial x}=0$$

得

$$\frac{\partial u}{\partial x}=\frac{xF'_1}{u(F'_1+F'_2+F'_3)}$$

同理

$$\frac{\partial u}{\partial y}=\frac{yF'_2}{u(F'_1+F'_2+F'_3)},$$
$$\frac{\partial u}{\partial z}=\frac{zF'_3}{u(F'_1+F'_2+F'_3)},$$

$\therefore$

$$\frac{u'_x}{x}+\frac{u'_y}{y}+\frac{u'_z}{z}=\frac{F'_1+F'_2+F'_3}{u(F'_1+F'_2+F'_3)}=\frac{1}{u}.$$

5. 求方程$\frac{\partial^2 z}{\partial x\partial y}=x+y$满足条件$z(x,0)=x,z(0,y)=y^2$的解$z(z,y)$.

解：方法1：$\because\ \frac{\partial^2 z}{\partial x\partial y}=x+y$,

$\therefore\ \frac{\partial z}{\partial y}=\int_0^x(x+y)\mathrm{d}x+\varphi_0(y)=\frac{x^2}{2}+xy+\varphi_0(y)$,

$\because\ z(0,y)=y^2$,

$\therefore\ \frac{z(0,y)}{\partial y}=2y=\varphi_0(y)$,

$\therefore z(x,y)=\int_0^y[\frac{x^2}{2}+xy+2y]\mathrm{d}y+\psi(x)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+\psi(x),$

$\because z(x,0)=x=\psi(x),$

$\therefore z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x.$

方法2：$\because \frac{\partial^2 z}{\partial x\partial y}=x+y,$

$\therefore \int\frac{\partial^2 z}{\partial x\partial y}\mathrm{d}x=\int(x+y)\mathrm{d}x=\frac{1}{2}x^2+xy+C_1(y)=\frac{\partial z}{\partial y}+C_1(y),$

$\therefore$可设$\frac{\partial z}{\partial y}=\frac{1}{2}x^2+xy+C(y)$，其中$C(y)$是与$x$无关的$y$的函数，

$\because z(0,y)=y^2,$

$\therefore \frac{\partial z(0,y)}{\partial y}=2y=C(y),$

$\therefore \frac{\partial z}{\partial y}=\frac{1}{2}x^2+xy+2y,$

$\therefore \int\frac{\partial z}{\partial y}\mathrm{d}y=\int(\frac{1}{2}x^2+xy+2y)\mathrm{d}y=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+C_3(x)=z(x,y)+C_4(x),$

$\therefore$可设$z(x,y)=\int(\frac{1}{2}x^2+xy+2y)\mathrm{d}y+C^*(x)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+C^*(x)$，其中$C^*(x)$是与$y$无关的$x$的函数，

$\because z(x,0)=x=C^*(x),$

$\therefore z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x.$

方法3：$\because \frac{\partial^2 z}{\partial x\partial y}=x+y,$

$\therefore \int\frac{\partial^2 z}{\partial x\partial y}\mathrm{d}x=\int(x+y)\mathrm{d}x=\frac{1}{2}x^2+xy+C_1(y)=\frac{\partial z}{\partial y}+C_2(y),$

$\therefore$可设$\frac{\partial z}{\partial y}=\frac{1}{2}x^2+xy+C(y)$，其中$C(y)$是与$x$无关的$y$的函数，

$\therefore \int\frac{\partial z}{\partial y}\mathrm{d}y=\frac{1}{2}x^2y+\frac{1}{2}xy^2+\int C(y)\mathrm{d}y=z(x,y)+C_3(x),$

$\therefore$可设$z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+F(y)+C^*(x)$，其中$F(y)$是$C(y)$的一个与$x$无关的原函数，$C^*(x)$是与$y$无关的$x$的函数，

$\because z(0,y)=y^2,z(x,0)=x,$

$\therefore F(y)+C^*(0)=y^2,F(0)+C^*(x)=x,$

$\therefore F(y)=y^2-C^*(0),C^*(x)=x-F(0)$，且$F(0)+C^*(0)=0,$

$\therefore z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x-[F(0)+C^*(0)]=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x.$

6. 设$z=f(x,y)$处处可微，a,b不全等于零. 求证满足方程$bz'_x=az'_y$的充分条件是存在一元函数$g(u)$，使得$z=f(x,y)=g(ax+by)$.

证明：$\because$存在一元函数$g(u)$，使得$z=f(x,y)=g(ax+by),$

$\therefore$对于任意常数C，z在直线$ax+by=C$上恒等于常数，

$\therefore z=f(x,y)$在直线$ax+by=C$上任意一点处沿该直线方向的方向导数均等于零，

由a,b不全为零知直线$ax+by=C$的方向向量可表示为$(-b,a)$，

又$\because z=f(x,y)$处处可微，

$\therefore z=f(x,y)$在直线$ax+by=C$上的每一点处沿$(-b,a)$方向的方向导数

$$\frac{\partial z}{\partial \boldsymbol{l}}=\mathrm{grad}z\cdot\frac{1}{a^2+b^2}(-b,a)=(z'_x,z'_y)\cdot\frac{1}{a^2+b^2}(-b,a)=\frac{1}{\sqrt{a^2+b^2}}(-bz'_x+az'_y)=0,$$

$\therefore$在直线$ax+by=C$上的每一点处$bz'_x=az'_y$，由C的任意性知$bz'_x=az'_y$处处成立.

7. 设D为包含原点$O(0,0)$的一个圆域. $f(x,y)$在D中处处有连续偏导数，并且满足$xf'_x+yf'_y=0$. 求证$f(x,y)$在D中恒等于某个常数.

证明：$\because f(x,y)$在D中处处有连续偏导数，

$\therefore f(x,y)$在D中处处可微，

$\therefore f(x,y)$在点$(x,y)\in D((x,y)\neq(0,0))$处由原点$(0,0)$指向点$(x,y)$方向的方向导数

$$\frac{\partial f(x,y)}{\partial \boldsymbol{v}}=\mathrm{grad}f(x,y)\cdot\frac{1}{\sqrt{x^2+y^2}}(x,y)=\frac{1}{\sqrt{x^2+y^2}}(f'_x,f'_y)\cdot(x,y)=\frac{xf'_x+yf'_y}{\sqrt{x^2+y^2}}=0,$$

$\therefore f(x,y)$在D中由原点出发且不含原点的每一条射线上任一点处沿该射线方向的方向导数均为0,

$\therefore f(x,y)$在D中由原点出发且不含原点的每一条射线上均为常数，

$\because f(x,y)$在D中处处可微，故处处连续，故在原点$(0,0)$处连续，

$\therefore f(x,y)$在D中由原点出发的每一条射线上均等于$f(0,0)$，

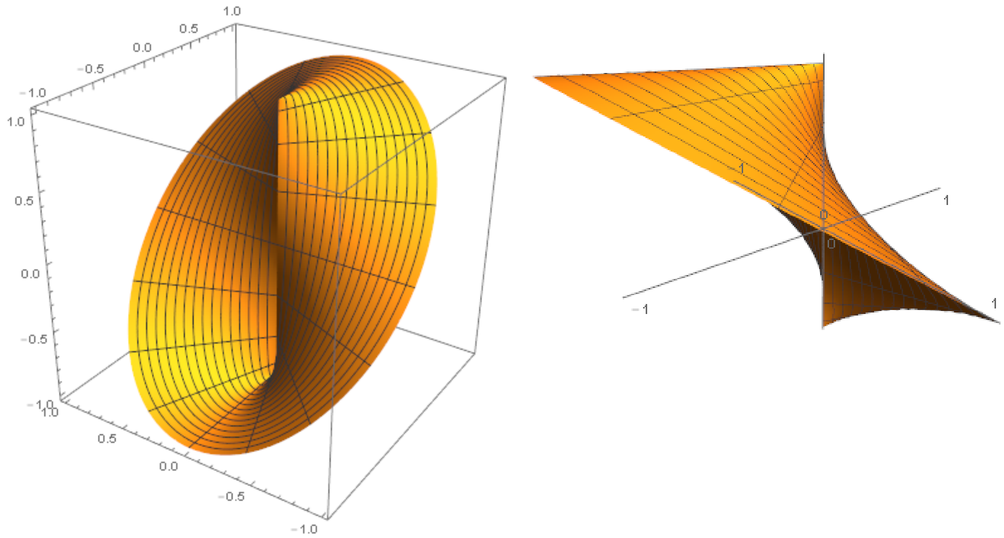
$\therefore f(x,y)=f(0,0),(x,y)\in D.$

【注意:】如果区域D不包含原点，但仍有$f(x,y)$在D中处处有连续偏导数，并且满足$xf'_x+yf'_y=0$，则$f(x,y)$在D中不一定恒等于常数，比如函数

$$f(x,y)=\begin{cases}\cos(\arccos\frac{x}{\sqrt{x^2+y^2}}), & y\geqslant 0,\\ \cos(2\pi-\arccos\frac{x}{\sqrt{x^2+y^2}}), & y<0.\end{cases}$$

该函数在极坐标下的方程是$f(r\cos\theta,r\sin\theta)=\cos\theta$. 函数$f(x,y)$在$D$中处处有连续偏导数，且在由原点出发的每一条射线$\theta=C$上均为常数$\cos C$，故满足$xf'_x+yf'_y=0$，但$f(x,y)\not\equiv const.$

函数$f(x,y),(x,y)\in\{(x,y)\mid 0<x^2+y^2\leqslant 1\}$的图形如图 1所示.



(a)　　(b)

图 1: 函数$f(r\cos\theta, r\sin\theta)=\cos\theta, (r,\theta)\in\{(r,\theta)\mid 0<r\leqslant 1, 0\leqslant\theta\leqslant 2\pi\}$的图形

3.8　习题11.1解答

1. 设$\boldsymbol{u}(t),\boldsymbol{v}(t)$是可导的向量值函数，$\lambda(t)$为可导数值函数，求证：

   (1)$\frac{\mathrm{d}}{\mathrm{d}t}(\lambda(t)\boldsymbol{u}(t))=\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}\boldsymbol{u}(t)+\lambda(t)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t)$；

   (2)$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}(t)\cdot\boldsymbol{v}(t))=(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t))\cdot\boldsymbol{v}(t)+\boldsymbol{u}(t)\cdot(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(t))$.

   证明：(1)设$\boldsymbol{u}(t)=(u_1(t),u_2(t),u_3(t))$，则$\lambda(t)\boldsymbol{u}(t)=(\lambda(t)u_1(t),\lambda(t)u_2(t),\lambda(t)u_3(t))$，

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\lambda(t)\boldsymbol{u}(t)) &= (\mathrm{d}t[\lambda(t)u_1(t)]',[\lambda(t)u_2(t)]',[\lambda(t)u_3(t)]')\\
&=(\lambda'(t)u_1(t)+\lambda(t)u_1'(t),\lambda'(t)u_2(t)+\lambda(t)u_2'(t),\lambda'(t)u_3(t)+\lambda(t)u_3'(t))\\
&=(\lambda'(t)u_1(t),\lambda'(t)u_2(t),\lambda'(t)u_3(t))+(\lambda(t)u_1'(t),\lambda(t)u_2'(t),\lambda(t)u_3'(t))\\
&=\lambda'(t)(u_1(t),u_2(t),u_3(t))+\lambda(t)(u_1'(t),u_2'(t),u_3'(t))\\
&=\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}\boldsymbol{u}(t)+\lambda(t)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t).
\end{aligned}
$$

   (2)设$\boldsymbol{u}(t)=(u_1(t),u_2(t),u_3(t)),\boldsymbol{v}(t)=(v_1(t),v_2(t),v_3(t))$，

   则$\boldsymbol{u}(t)\cdot\boldsymbol{v}(t)=u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t)$，

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}(t)\cdot\boldsymbol{v}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}[u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t)]\\
&=u_1'(t)v_1(t)+u_1(t)v_1'(t)+u_2'(t)v_2(t)+u_2(t)v_2'(t)+u_3'(t)v_3(t)+u_3(t)v_3'(t)\\
&=u_1'(t)v_1(t)+u_2'(t)v_2(t)+u_3'(t)v_3(t)+u_1(t)v_1'(t)+u_2(t)v_2'(t)+u_3(t)v_3'(t)\\
&=(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t))\cdot\boldsymbol{v}(t)+\boldsymbol{u}(t)\cdot(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(t)).
\end{aligned}
$$

2. 求下列曲线在指定点的单位切向量：

(1)$\boldsymbol{r}(t)=(\mathrm{e}^{2t},\mathrm{e}^{-2t},t\mathrm{e}^{2t}),t=0$；

(2)$\boldsymbol{r}(t)=t\boldsymbol{i}+2\sin t\boldsymbol{j}+3\cos t\boldsymbol{k},t=\frac{\pi}{6}$.

解：(1)单位切向量$\boldsymbol{t}=\frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}=\frac{(2\mathrm{e}^{2t},-2\mathrm{e}^{-2t},(1+2t)\mathrm{e}^{2t})}{\|(2\mathrm{e}^{2t},-2\mathrm{e}^{-2t},(1+2t)\mathrm{e}^{2t})\|}\Big|_{t=0}=\frac{(2,-2,1)}{\sqrt{4+4+1}}=(\frac{2}{3},-\frac{2}{3},\frac{1}{3})$.

(2)单位切向量$\boldsymbol{t}=\frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}=\frac{\boldsymbol{i}+2\cos t\boldsymbol{j}-3\sin t\boldsymbol{k}}{\|\boldsymbol{i}+2\cos t\boldsymbol{j}-3\sin t\boldsymbol{k}\|}\Big|_{t=\frac{\pi}{6}}=\frac{\boldsymbol{i}+\sqrt{3}\boldsymbol{j}-\frac{3}{2}\boldsymbol{k}}{\sqrt{1+3+\frac{9}{4}}}=\frac{2}{5}\boldsymbol{i}+\frac{2\sqrt{3}}{5}\boldsymbol{j}-\frac{3}{5}\boldsymbol{k}$.

3. 求下列曲线在指定点的切线方程：

(1)$\boldsymbol{r}(t)=(1+2t,1+t-t^2,1-t+t^2-t^3),M(1,1,1)$；

(2)$\boldsymbol{r}(t)=\sin(\pi t)\boldsymbol{i}+\sqrt{t}\boldsymbol{j}+\cos(\pi t)\boldsymbol{k},M(0,1,-1)$.

解：(1)在$M(1,1,1)$点处$t=0$，切向量$\boldsymbol{t}=\boldsymbol{r}'(0)=(2,1-2t,-1+2t-3t^2)_{t=0}=(2,1,-1)$，则切线方程为

$$\frac{x-1}{2}=y-1=-(z-1).$$

(2)在$M(0,1,-1)$点处$t=1$，切向量$\boldsymbol{t}=\boldsymbol{r}'(1)=\pi\cos(\pi t)\boldsymbol{i}+\frac{1}{2\sqrt{t}}\boldsymbol{j}-\pi\sin(\pi t)\boldsymbol{k}\Big|_{t=1}=-\pi\boldsymbol{i}+\frac{1}{2}\boldsymbol{j}$，则切线方程为$\begin{cases}\frac{x}{-\pi}=2(y-1),\\ z=-1,\end{cases}$ 即$\begin{cases}x+2\pi y=2\pi,\\ z=-1.\end{cases}$

4. 求下列向量值函数的积分：

(1)$\int_0^{\frac{\pi}{4}}[\cos(2t)\boldsymbol{i}+\sin(2t)\boldsymbol{j}+t\sin t\boldsymbol{k}]\mathrm{d}t$；

(2)$\int_1^4(\sqrt{t}\boldsymbol{i}+t\mathrm{e}^{-t}\boldsymbol{j}+\frac{1}{t^2}\boldsymbol{k})\mathrm{d}t$.

解：(1)$\int_0^{\frac{\pi}{4}}[\cos(2t)\boldsymbol{i}+\sin(2t)\boldsymbol{j}+t\sin t\boldsymbol{k}]\mathrm{d}t=\int_0^{\frac{\pi}{4}}\cos(2t)\mathrm{d}t\boldsymbol{i}+\int_0^{\frac{\pi}{4}}\sin(2t)\mathrm{d}t\boldsymbol{j}+\int_0^{\frac{\pi}{4}}t\sin t\mathrm{d}t\boldsymbol{k}$,

$\because\int_0^{\frac{\pi}{4}}\cos(2t)\mathrm{d}t=\frac{1}{2}\sin(2t)\Big|_0^{\frac{\pi}{4}}=\frac{1}{2}$，$\int_0^{\frac{\pi}{4}}\sin(2t)\mathrm{d}t=-\frac{1}{2}\cos(2t)\Big|_0^{\frac{\pi}{4}}=\frac{1}{2}$,

$\int_0^{\frac{\pi}{4}}t\sin t\mathrm{d}t=-\int_0^{\frac{\pi}{4}}t\mathrm{d}\cos t=-t\cos t\Big|_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}\cos t\mathrm{d}t=-\frac{\sqrt{2}\pi}{8}+\sin t\Big|_0^{\frac{\pi}{4}}=-\frac{\sqrt{2}\pi}{8}+\frac{\sqrt{2}}{2}$,

$\therefore\int_0^{\frac{\pi}{4}}[\cos(2t)\boldsymbol{i}+\sin(2t)\boldsymbol{j}+t\sin t\boldsymbol{k}]\mathrm{d}t=\frac{1}{2}\boldsymbol{i}+\frac{1}{2}\boldsymbol{j}+(-\frac{\sqrt{2}\pi}{8}+\frac{\sqrt{2}}{2})\boldsymbol{k}$.

(2)$\int_1^4(\sqrt{t}\boldsymbol{i}+t\mathrm{e}^{-t}\boldsymbol{j}+\frac{1}{t^2}\boldsymbol{k})\mathrm{d}t=\int_1^4\sqrt{t}\mathrm{d}t\boldsymbol{i}+\int_1^4t\mathrm{e}^{-t}\mathrm{d}t\boldsymbol{j}+\int_1^4\frac{1}{t^2}\mathrm{d}t\boldsymbol{k}$,

$\because\int_1^4\sqrt{t}\mathrm{d}t=\frac{1}{1+\frac{1}{2}}t^{\frac{1}{2}+1}\Big|_1^4=\frac{14}{3},\int_1^4t\mathrm{e}^{-t}\mathrm{d}t=-\int_1^4t\mathrm{d}\mathrm{e}^{-t}=-t\mathrm{e}^{-t}\Big|_1^4+\int_1^4\mathrm{e}^{-t}\mathrm{d}t$

$=-4\mathrm{e}^{-4}+\mathrm{e}^{-1}-\mathrm{e}^{-t}\Big|_1^4=-4\mathrm{e}^{-4}+\mathrm{e}^{-1}-\mathrm{e}^{-4}+\mathrm{e}^{-1}=-5\mathrm{e}^{-4}+2\mathrm{e}^{-1}$,

$\int_1^4\frac{1}{t^2}\mathrm{d}t=\frac{1}{-2+1}t^{-2+1}\Big|_1^4=-\frac{1}{4}+1=\frac{3}{4}$,

$\therefore\int_1^4(\sqrt{t}\boldsymbol{i}+t\mathrm{e}^{-t}\boldsymbol{j}+\frac{1}{t^2}\boldsymbol{k})\mathrm{d}t=\frac{14}{3}\boldsymbol{i}+(-5\mathrm{e}^{-4}+2\mathrm{e}^{-1})\boldsymbol{j}+\frac{3}{4}\boldsymbol{k}$.

5. 已知$\boldsymbol{r}'(t),\boldsymbol{r}(0)$，求$\boldsymbol{r}(t)$：

(1)$\boldsymbol{r}'(t)=(t^2,4t^3,-t^2),\boldsymbol{r}(0)=(0,1,0)$；

(2)$\boldsymbol{r}'(t)=\sin t\boldsymbol{i}-\cos t\boldsymbol{j}+2t\boldsymbol{k},\boldsymbol{r}(0)=\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$.

解：(1)方法1：

$$\begin{aligned}\boldsymbol{r}(t)&=\int_0^t \boldsymbol{r}'(t)\mathrm{d}t+\boldsymbol{C}=(\int_0^t t^2\mathrm{d}t+C_1,\int_0^t 4t^3\mathrm{d}t+C_2,\int_0^t(-t^2)\mathrm{d}t+C_3)\\&=(\frac{1}{3}t^3+C_1,t^4+C_2,-\frac{1}{3}t^3+C_3),\end{aligned}$$

$\therefore \boldsymbol{r}(0)=(C_1,C_2,C_3)=(0,1,0)$,

$\therefore \boldsymbol{r}(t)=(\frac{1}{3}t^3,t^4+1,-\frac{1}{3}t^3)$.

方法2：$\because \int t^2\mathrm{d}t=\frac{1}{3}t^3+C,\ \int 4t^3\mathrm{d}t=t^4+C,\ \int(-t^2)\mathrm{d}t=-\frac{1}{3}t^3+C$,

$\therefore r(t)=\int \boldsymbol{r}'(t)\mathrm{d}t=(\frac{1}{3}t^3+C_1,t^4+C_2,-\frac{1}{3}t^3+C_3)$,

$\because \boldsymbol{r}(0)=(0,1,0)$,

$\therefore C_1=0,\ C_2=1,\ C_3=0$,

$\therefore \boldsymbol{r}(t)=(\frac{1}{3}t^3,t^4+1,-\frac{1}{3}t^3)$.

(2)方法1：

$$\begin{aligned}\boldsymbol{r}(t)=&\int_0^t \boldsymbol{r}'(t)\mathrm{d}t+\boldsymbol{C}=(\int_0^t \sin t\mathrm{d}t+C_1)\boldsymbol{i}+(-\int_0^t \cos t\mathrm{d}t+C_2)\boldsymbol{j}+(\int_0^t 2t\mathrm{d}t+C_3)\boldsymbol{k}\\=&(-\cos t+C_1)\boldsymbol{i}+(-\sin t+C_2)\boldsymbol{j}+(t^2+C_3)\boldsymbol{k},\end{aligned}$$

$\because \boldsymbol{r}(0)=(-1+C_1)\boldsymbol{i}+C_2\boldsymbol{j}+C_3\boldsymbol{k}=\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$,

$\therefore C_1=2,C_2=1,C_3=2$,

$\therefore \boldsymbol{r}(t)=(-\cos t+2)\boldsymbol{i}+(-\sin t+1)\boldsymbol{j}+(t^3+2)\boldsymbol{k}$.

方法2：$\because \int \sin t\mathrm{d}t=-\cos t+C,\ \int(-\cos t)\mathrm{d}t=-\sin t+C,\ \int 2t\mathrm{d}t=t^2+C$,

$\therefore \boldsymbol{r}(t)=\int \boldsymbol{r}'(t)\mathrm{d}t=(-\cos t+C_1)\boldsymbol{i}+(-\sin t+C_2)\boldsymbol{j}+(t^2+C_3)\boldsymbol{k}$,

$\because \boldsymbol{r}(0)=\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$,

$\therefore C_1=2,\ C_2=1,\ C_3=2$,

$\therefore \boldsymbol{r}(t)=(-\cos t+2)\boldsymbol{i}+(-\sin t+1)\boldsymbol{j}+(t^3+2)\boldsymbol{k}$.

6. 证明下列等式：

(1)$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}(t)\times\boldsymbol{r}'(t))=\boldsymbol{r}(t)\times\boldsymbol{r}''(t)$；

(2)$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{r}(t)\|=\frac{\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)}{\|\boldsymbol{r}(t)\|}(\boldsymbol{r}(t)\neq\boldsymbol{0})$；

(3)$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{r}(t)\cdot(\boldsymbol{r}'(t)\times\boldsymbol{r}''(t))]=\boldsymbol{r}(t)\cdot[\boldsymbol{r}'(t)\times\boldsymbol{r}'''(t)]$.

证明：(1)$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}(t)\times\boldsymbol{r}'(t))=\boldsymbol{r}'(t)\times\boldsymbol{r}'(t)+\boldsymbol{r}(t)\times\boldsymbol{r}''(t)=\boldsymbol{r}(t)\times\boldsymbol{r}''(t)$.

(2)$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{r}(t)\|=\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)}=\frac{1}{2\sqrt{\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)}}\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)]$

$=\frac{1}{2\sqrt{\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)}}[\boldsymbol{r}'(t)\cdot\boldsymbol{r}(t)+\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)]=\frac{2\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)}{2\sqrt{\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)}}=\frac{\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)}{\|\boldsymbol{r}(t)\|}(\boldsymbol{r}(t)\neq\boldsymbol{0})$.

(3)$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{r}(t)\cdot(\boldsymbol{r}'(t)\times\boldsymbol{r}''(t))]=\boldsymbol{r}'(t)\cdot(\boldsymbol{r}'(t)\times\boldsymbol{r}''(t))+\boldsymbol{r}(t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}'(t)\times\boldsymbol{r}''(t))$
$=\boldsymbol{r}(t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}'(t)\times\boldsymbol{r}''(t))=\boldsymbol{r}(t)\cdot(\boldsymbol{r}''(t)\times\boldsymbol{r}''(t)+\boldsymbol{r}'(t)\times\boldsymbol{r}'''(t))=\boldsymbol{r}(t)\cdot[\boldsymbol{r}'(t)\times\boldsymbol{r}'''(t)]$.

7. 求等速圆周运动$\boldsymbol{r}=R\cos(\omega t)\boldsymbol{i}+R\sin(\omega t)\boldsymbol{j}$在$t$时刻的速度与加速度.

   解：t时刻的速度$\boldsymbol{v}(t)=\boldsymbol{r}'(t)=-R\omega\sin(\omega t)\boldsymbol{i}+R\omega\cos(\omega t)\boldsymbol{j}$,

   t时刻的加速度$\boldsymbol{a}(t)=\boldsymbol{v}'(t)=-R\omega^2\cos(\omega t)\boldsymbol{i}-R\omega^2\sin(\omega t)\boldsymbol{j}$.

8. 已知螺旋线的向量方程为$\boldsymbol{r}=a\cos\theta\boldsymbol{i}+a\sin\theta\boldsymbol{j}+b\theta\boldsymbol{k}(a>0,b>0)$，求在$\theta_0$处的切线方程.

   解：在θ_0处的切向量$\boldsymbol{r}'(\theta_0)=-a\sin\theta_0\boldsymbol{i}+a\cos\theta_0\boldsymbol{j}+b\boldsymbol{k}$，切线方程

$$\frac{x-a\cos\theta_0}{-a\sin\theta_0}=\frac{y-a\sin\theta_0}{a\cos\theta_0}=\frac{z-b\theta_0}{b}.$$

9. 设$\boldsymbol{r}=-a\sin\theta\boldsymbol{i}+a\cos\theta\boldsymbol{j}+b\theta\boldsymbol{k}$，求$\frac{1}{2}\int_0^{2\pi}(\boldsymbol{r}\times\boldsymbol{r}')\mathrm{d}\theta$.

   解：$\boldsymbol{r}'(\theta)=-a\cos\theta\boldsymbol{i}-a\sin\theta\boldsymbol{j}+b\boldsymbol{k}$,

$$\boldsymbol{r}(\theta)\times\boldsymbol{r}'(\theta)=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\-a\sin\theta&a\cos\theta&b\theta\\-a\cos\theta&-a\sin\theta&b\end{vmatrix}=\begin{vmatrix}a\cos\theta&b\theta\\-a\sin\theta&b\end{vmatrix}\boldsymbol{i}+\begin{vmatrix}b\theta&-a\sin\theta\\b&-a\cos\theta\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}-a\sin\theta&a\cos\theta\\-a\cos\theta&-a\sin\theta\end{vmatrix}\boldsymbol{k}$$

   $=(ab\cos\theta+ab\theta\sin\theta)\boldsymbol{i}-(-ab\sin\theta+ab\theta\cos\theta)\boldsymbol{j}+a^2\boldsymbol{k}$,

   $\because\int_0^{2\pi}(ab\cos\theta+ab\theta\sin\theta)\mathrm{d}\theta=ab\sin\theta\big|_0^{2\pi}-\int_0^{2\pi}ab\theta\mathrm{d}\cos\theta$
   $=-ab\theta\cos\theta\big|_0^{2\pi}+\int_0^{2\pi}ab\cos\theta\mathrm{d}\theta=-2\pi ab+ab\sin\theta\big|_0^{2\pi}=-2\pi ab$,
   $\int_0^{2\pi}-(-ab\sin\theta+ab\theta\cos\theta)\mathrm{d}\theta=\int_0^{2\pi}(ab\sin\theta-ab\theta\cos\theta)\mathrm{d}\theta=-ab\cos\theta\big|_0^{2\pi}-ab\int_0^{2\pi}\theta\mathrm{d}\sin\theta$
   $=-ab\theta\sin\theta\big|_0^{2\pi}+ab\int_0^{2\pi}\sin\theta\mathrm{d}\theta=-ab\cos\theta\big|_0^{2\pi}=0$,
   $\int_0^{2\pi}a^2\mathrm{d}\theta=2\pi a^2$,

   $\therefore\frac{1}{2}\int_0^{2\pi}(\boldsymbol{r}\times\boldsymbol{r}')\mathrm{d}\theta=-\pi ab\boldsymbol{i}+\pi a^2\boldsymbol{k}$.