28C 第14章补充题

28C.1 第14章补充题解答

1. 设$f(x)$在$[0,+\infty)$连续，且$\lim\limits_{x\to+\infty}f(x)=b$. 求证：
(1)若$a>0$, 则方程$y'+ay=f(x)$的每个解$y(x)$都满足$\lim\limits_{x\to+\infty}y(x)=\frac{b}{a}$;
(2)若$a<0$, 则方程$y'+ay=f(x)$只有一个解$y_0(x)$满足$\lim\limits_{x\to+\infty}y_0(x)=\frac{b}{a}$.

证明：$y'+ay=f(x)$的解为

$$y=\mathrm{e}^{-\int_0^x a\mathrm{d}t}[\int_0^x f(t)\mathrm{e}^{\int_0^t a\mathrm{d}u}\mathrm{d}t+C]=\mathrm{e}^{-ax}[\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C]=\frac{\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C}{\mathrm{e}^{ax}},$$

(1)当$a>0$时

$$\lim_{x\to+\infty}y(x)=\lim_{x\to+\infty}\frac{\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C}{\mathrm{e}^{ax}}=\lim_{x\to+\infty}\frac{f(x)\mathrm{e}^{ax}}{a\mathrm{e}^{ax}}=\lim_{x\to+\infty}\frac{f(x)}{a}=\frac{b}{a}.$$

(2)当$a<0$时，$\lim\limits_{x\to+\infty}\mathrm{e}^{ax}=0$, 要使极限$\lim\limits_{x\to+\infty}y(x)=\lim\limits_{x\to+\infty}\frac{\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C}{\mathrm{e}^{ax}}$存在，则

$$\lim_{x\to+\infty}[\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C]=0,$$

即$C=-\int_0^{+\infty}f(t)\mathrm{e}^{at}\mathrm{d}x$,

此时

$$\lim_{x\to+\infty}y(x)=\lim_{x\to+\infty}\frac{\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C}{\mathrm{e}^{ax}}=\lim_{x\to+\infty}\frac{f(x)\mathrm{e}^{ax}}{a\mathrm{e}^{ax}}=\lim_{x\to+\infty}\frac{f(x)}{a}=\frac{b}{a}.$$

故只有一个解$y_0(x)=\mathrm{e}^{-ax}[\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x-\int_0^{+\infty}f(t)\mathrm{e}^{at}\mathrm{d}x]$满足$\lim\limits_{x\to+\infty}y_0(x)=\frac{b}{a}$.

2. 设$f(x)$连续.
(1)求方程$y'+ay=f(x)$满足$y\big|_{x=0}=0$的解$y(x)(a>0)$;
(2)若$|f(x)|\leqslant k$, 求证当$x\geqslant 0$时，有$|y(x)|\leqslant\frac{k}{a}(1-\mathrm{e}^{-ax})$.

解：(1)一阶线性非齐次微分方程$y'+ay=f(x)$的解为

$$y=\mathrm{e}^{-\int_0^x a\mathrm{d}t}[\int_0^x f(t)\mathrm{e}^{\int_0^t a\mathrm{d}u}\mathrm{d}t+C]=\mathrm{e}^{-ax}[\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C]=\frac{\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C}{\mathrm{e}^{ax}},$$

$\because y\big|_{x=0}=\frac{C}{1}=C=0$,

$\therefore$满足$y\big|_{x=0}=0$的解$y(x)=\frac{\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x}{\mathrm{e}^{ax}}$.

(2)$\because|f(x)|\leqslant k$,

$\therefore$当$x\geqslant 0$时

$|y(x)|=\frac{|\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x|}{\mathrm{e}^{ax}}\leqslant\frac{\int_0^x|f(t)|\mathrm{e}^{at}\mathrm{d}t}{\mathrm{e}^{ax}}\leqslant\frac{\int_0^x k\mathrm{e}^{at}\mathrm{d}t}{\mathrm{e}^{ax}}=\frac{k\int_0^x\mathrm{e}^{at}\mathrm{d}t}{\mathrm{e}^{ax}}=\frac{\frac{k}{a}\mathrm{e}^{ax}\big|_0^x}{\mathrm{e}^{ax}}=\frac{\frac{k}{a}(\mathrm{e}^{ax}-1)}{\mathrm{e}^{ax}}=\frac{k}{a}(1-\mathrm{e}^{-ax})$.

3. 设$y_1(x), y_2(x)$是方程$y'' + p(x)y' + q(x)y = 0$的两个解，并且函数$f(x) = \frac{y_2(x)}{y_1(x)}$在某个点$x_0$处取得极值. 问$y_1(x)$和$y_2(x)$能否构成该方程的一个基本解组？

   解：$\because f(x) = \frac{y_2(x)}{y_1(x)}$在点$x_0$处取得极值,

   $\therefore f'(x_0) = \frac{y_2'(x_0)y_1(x_0) - y_1'(x_0)y_2(x_0)}{[y_1(x_0)]^2} = 0$, 即$y_2'(x_0)y_1(x_0) - y_1'(x_0)y_2(x_0) = 0$,

   $\therefore W[y_1, y_2](x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = y_2'(x_0)y_1(x_0) - y_1'(x_0)y_2(x_0) = 0$,

   $\therefore y_1(x), y_2(x)$线性相关，故$y_1(x)$和$y_2(x)$不能构成该方程的一个基本解组.

4. 已知$f(x)$二阶连续可导，并且对于xOy平面上每一条逐段光滑的有向曲线L都有

$$\oint_L [f'(x) + 6f(x) + 4\mathrm{e}^{-x}]y\mathrm{d}x + f'(x)\mathrm{d}y = 0.$$

   试求$f(x)$.

   解：$\because f(x) \in C^2(\mathbb{R})$,

   $\therefore [f'(x) + 6f(x) + 4\mathrm{e}^{-x}]y, f'(x) \in C^2(\mathbb{R}^2)$,

   $\because$对于xOy平面上每一条逐段光滑的有向曲线L都有
   $\oint_L [f'(x) + 6f(x) + 4\mathrm{e}^{-x}]y\mathrm{d}x + f'(x)\mathrm{d}y = 0$,

   $\therefore \frac{\partial}{\partial y}[f'(x) + 6f(x) + 4\mathrm{e}^{-x}]y = f'(x) + 6f(x) + 4\mathrm{e}^{-x} = \frac{\partial}{\partial x}f'(x) = f''(x)$,

   即$f''(x) - f'(x) - 6f(x) = 4\mathrm{e}^{-x}$,

   该二阶常系数线性非齐次微分方程的齐次方程的特征方程为$\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0$，特征根$\lambda_1 = 3, \lambda_2 = -2$, 故齐次方程的通解为$y = C_1\mathrm{e}^{3x} + C_2\mathrm{e}^{-2x}$.

   非齐次方程的自由项$4\mathrm{e}^{-x}$中的$\lambda = -1$, 不是特征方程的解，故可设非齐次方程的特解为$y = a\mathrm{e}^{-x}$, 代入原非齐次方程得$a\mathrm{e}^{-x} + a\mathrm{e}^{-x} - 6a\mathrm{e}^{-x} = -4a\mathrm{e}^{-x} = 4\mathrm{e}^{-x}$,

   $\therefore a = -1$,

   $\therefore f(x) = C_1\mathrm{e}^{3x} + C_2\mathrm{e}^{-2x} - \mathrm{e}^{-x}, C_1, C_2 \in \mathbb{R}$.

5. 假定对于半空间$x > 0$的任意光滑封闭曲面S，有

$$\oiint_S xf(x)\mathrm{d}y \wedge \mathrm{d}z - xyf(x)\mathrm{d}z \wedge \mathrm{d}x + \mathrm{e}^{2x}z\mathrm{d}x \wedge \mathrm{d}y = 0,$$

   其中$f(x)$在$(0, +\infty)$有连续导数，且满足$\lim\limits_{x \to 0_+} f(x) = 1$. 求$f(x)$.

   解：$\because f(x) \in C^1(0, +\infty)$,

   $\therefore xf(x), -xyf(x), \mathrm{e}^{2x}z \in C^1((0, +\infty) \times (-\infty, +\infty) \times (-\infty, +\infty))$,

$\forall(x,y,z)\in(0,+\infty)\times(-\infty,+\infty)\times(-\infty,+\infty)$, 设$S$为包围该点的任意光滑封闭曲面，记$S$围成的区域为$\Omega$, 则

$$\begin{aligned}&\oiint\limits_S xf(x)\mathrm{d}y\wedge\mathrm{d}z-xyf(x)\mathrm{d}z\wedge\mathrm{d}x+\mathrm{e}^{2x}z\mathrm{d}x\wedge\mathrm{d}y\\=&\iiint\limits_\Omega\{\frac{\partial}{\partial x}[xf(x)]+\frac{\partial}{\partial y}[-xyf(x)]+\frac{\partial}{\partial z}(\mathrm{e}^{2x}z)\}\mathrm{d}V\\=&\iiint\limits_\Omega[f(x)+xf'(x)-xf(x)+\mathrm{e}^{2x}]\mathrm{d}V=0,\end{aligned}$$

$\because f(x)+xf'(x)-xf(x)+\mathrm{e}^{2x}$连续，

$\therefore$根据积分中值定理$\exists(\xi,\eta,\zeta)\in\Omega$, 使得

$$\iiint\limits_\Omega[f(x)+xf'(x)-xf(x)+\mathrm{e}^{2x}]\mathrm{d}V=[f(\xi)+\xi f'(\xi)-\xi f(\xi)+\mathrm{e}^{2\xi}]V(\Omega)=0,$$

其中$V(\Omega)$表示区域Ω的体积，

$\therefore$

$$\begin{aligned}f(x)+xf'(x)-xf(x)+\mathrm{e}^{2x}&=\lim_{(\xi,\eta,\zeta)\to(x,y,z)}[f(\xi)+\xi f'(\xi)-\xi f(\xi)+\mathrm{e}^{2\xi}]\\&=\lim_{\Omega\to(x,y,z)}\frac{\iiint\limits_\Omega[f(x)+xf'(x)-xf(x)+\mathrm{e}^{2x}]\mathrm{d}V}{V(\Omega)}\\&=\lim_{\Omega\to(x,y,z)}\frac{0}{V(\Omega)}\\&=0,(x,y,z)\in(0,+\infty)\times(-\infty,+\infty)\times(-\infty,+\infty),\end{aligned}$$

$\therefore f'(x)+\frac{1-x}{x}f(x)=-\frac{1}{x}\mathrm{e}^{2x},x>0,$

$\therefore f(x)=\mathrm{e}^{-\int\frac{1-x}{x}\mathrm{d}x}[\int(-\frac{1}{x}\mathrm{e}^{2x})\mathrm{e}^{\int\frac{1-x}{x}\mathrm{d}x}\mathrm{d}x+C]=\mathrm{e}^{x-\ln x}[\int(-\frac{1}{x}\mathrm{e}^{2x})\mathrm{e}^{\ln x-x}\mathrm{d}x+C]$
$=\frac{1}{x}\mathrm{e}^x[\int(-\frac{1}{x}\mathrm{e}^{2x})\frac{x}{\mathrm{e}^x}\mathrm{d}x+C]=\frac{1}{x}\mathrm{e}^x[\int(-\mathrm{e}^x)\mathrm{d}x+C]=\frac{1}{x}\mathrm{e}^x(-\mathrm{e}^x+C)=\frac{1}{x}(C\mathrm{e}^x-1),$

$\because\lim\limits_{x\to0_+}f(x)=\lim\limits_{x\to0_+}\frac{C\mathrm{e}^x-1}{x}=\lim\limits_{x\to0_+}\frac{C\mathrm{e}^x}{1}=C=1,$

$\therefore f(x)=\frac{1}{x}(\mathrm{e}^x-1).$

6. 设$f(x)$有二阶连续导数，并满足方程$f(x)=\int_0^x f(1-t)\mathrm{d}t+1$, 求$f(x)$.

解：$\because f(x)=\int_0^x f(1-t)\mathrm{d}t+1,$

$\therefore f'(x)=f(1-x)(*),$

$\therefore f'(1-x)=f(x),$

$\therefore$(*)式两边求导得$f''(x)=-f'(1-x)=-f(x),$

即$f''(x)+f(x)=0$, 该齐次线性微分方程的特征方程$\lambda^2+1=0$的根为$\lambda_{1,2}=\pm\mathrm{i}$, 通解为$f(x)=C_1\cos x+C_2\sin x,$

$\because f(0)=1$, 由(*)式得$f'(0)=f(1)$,

$\therefore C_1=1, C_2=C_1\cos 1+C_2\sin 1$,

$\therefore C_2=\frac{\cos 1}{1-\sin 1}$,

$\therefore f(x)=\cos x+\frac{\cos 1}{1-\sin 1}\sin x$.

7. 求级数$x+\frac{1}{1\times 3}x^3+\frac{1}{1\times 3\times 5}x^5+\cdots+\frac{1}{(2n+1)!!}x^{2n+1}+\cdots$的收敛域以及和函数.

解：原级数可以记为$\sum_{n=1}^{\infty}\frac{1}{(2n+1)!!}x^{2n+1}=\sum_{n=1}^{\infty}a_n$,

$\because \forall x\in\mathbb{R}, \lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim\limits_{n\to\infty}\frac{\frac{1}{(2n+3)!!}|x|^{2n+3}}{\frac{1}{(2n+1)!!}|x|^{2n+1}}=\lim\limits_{n\to\infty}\frac{1}{2n+3}|x|^2=0$,

$\therefore$由比值判别法可知，$\sum_{n=1}^{\infty}a_n$对所有x都绝对收敛. 故收敛域为$(-\infty,+\infty)$.

记和函数为$S(x)=\sum_{n=1}^{\infty}\frac{1}{(2n+1)!!}x^{2n+1}$, 则

$$
\begin{aligned}
S'(x)&=[\sum_{n=1}^{\infty}\frac{1}{(2n+1)!!}x^{2n+1}]'=\sum_{n=1}^{\infty}[\frac{1}{(2n+1)!!}x^{2n+1}]'\\
&=1+\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n-1)!!}=1+x\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!!}=1+xS(x),
\end{aligned}
$$

即$S(x)$应满足一阶线性非齐次微分方程

$$S'(x)-xS(x)=1,$$

其通解为$S(x)=\mathrm{e}^{-\int_0^x(-x)\mathrm{d}x}[\int_0^x 1\mathrm{e}^{\int_0^t(-u)\mathrm{d}u}\mathrm{d}t+C]=\mathrm{e}^{\frac{1}{2}x^2}[\int_0^x\mathrm{e}^{-\frac{1}{2}t^2}\mathrm{d}t+C]$,

$\because S(0)=C=0$,

$\therefore S(x)=\mathrm{e}^{\frac{1}{2}x^2}\int_0^x\mathrm{e}^{-\frac{1}{2}t^2}\mathrm{d}t$.

8. 求方程$y''\cos x-2y'\sin x+3y\cos x=\mathrm{e}^x$的通解.

解：令$u(x)=y\cos x$, 则$u'=y'\cos x-y\sin x$, $u''=y''\cos x-y'\sin x-y'\sin x-y\cos x=y''\cos x-2y'\sin x-y\cos x$,

$\therefore u''+4u=\mathrm{e}^x(*)$, 该非齐次线性微分方程对应的齐次方程的特征方程为$\lambda^2+4=0$, 特征根为$\lambda_{1,2}=\pm 2\mathrm{i}$，齐次方程的通解为$u=C_1\cos 2x+C_2\sin 2x$,

非齐次方程(*)的自由项e^x中的$\lambda=1$不是特征方程的根，故可设非齐次方程的特解为$u^*=a\mathrm{e}^x$，代入该非齐次方程得$a\mathrm{e}^x+4a\mathrm{e}^x=5a\mathrm{e}^x=\mathrm{e}^x$, $a=\frac{1}{5}$,

$\therefore$非齐次方程$(*)$的通解为$u=C_1\cos 2x+C_2\sin 2x+\frac{1}{5}\mathrm{e}^x$,

$\therefore$原方程的通解为$y=u\cos x=\cos x(C_1\cos 2x+C_2\sin 2x+\frac{1}{5}\mathrm{e}^x)$.

9. 设$f(x)$是定解问题

$$\begin{cases} y' = x^2 + y^2, \\ y(0) = 0, \end{cases}$$

的解. 试研究函数$f(x)$的增减性和凸凹性, 并求$\lim\limits_{x\to 0}\frac{f(x)}{x^3}$.

解：$\because f(x)$是定解问题$\begin{cases} y' = x^2 + y^2, \\ y(0) = 0, \end{cases}$ 的解,

$\therefore f'(x) = x^2 + [f(x)]^2 \geqslant 0$, 且当$x \neq 0$时$f'(x) > 0$,

$\therefore f(x)$在$(-\infty, +\infty)$上是单调增加的函数.

$\because f''(x) = 2x + 2f(x)f'(x) = 2x + 2f(x)\{x^2 + [f(x)]^2\}$,

$\because f(x)$在$(-\infty, +\infty)$上单调增加，

$\therefore f''(x) = 2x + 2f(x)\{x^2 + [f(x)]^2\}$在$(-\infty, +\infty)$上单调增加,

$\because f(0) = 0, f''(0) = 0$,

$\therefore$当$x > 0$时$f''(x) > 0$, 当$x < 0$时$f''(x) < 0$,

$\therefore f(x)$在$(0, +\infty)$上是下凸函数，在$(-\infty, 0)$上是上凸函数.

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)}{x^3} &= \lim_{x\to 0}\frac{f'(x)}{3x^2} = \lim_{x\to 0}\frac{f''(x)}{6x} = \lim_{x\to 0}\frac{f'''(x)}{6} \\ &= \lim_{x\to 0}\frac{2 + 2[f'(x)]^2 + 2f(x)f''(x)}{6} \\ &= \lim_{x\to 0}\frac{2 + 2\{x^2 + [f(x)]^2\}^2 + 2f(x)f''(x)}{6} = \frac{2}{6} = \frac{1}{3}.\end{aligned}$$