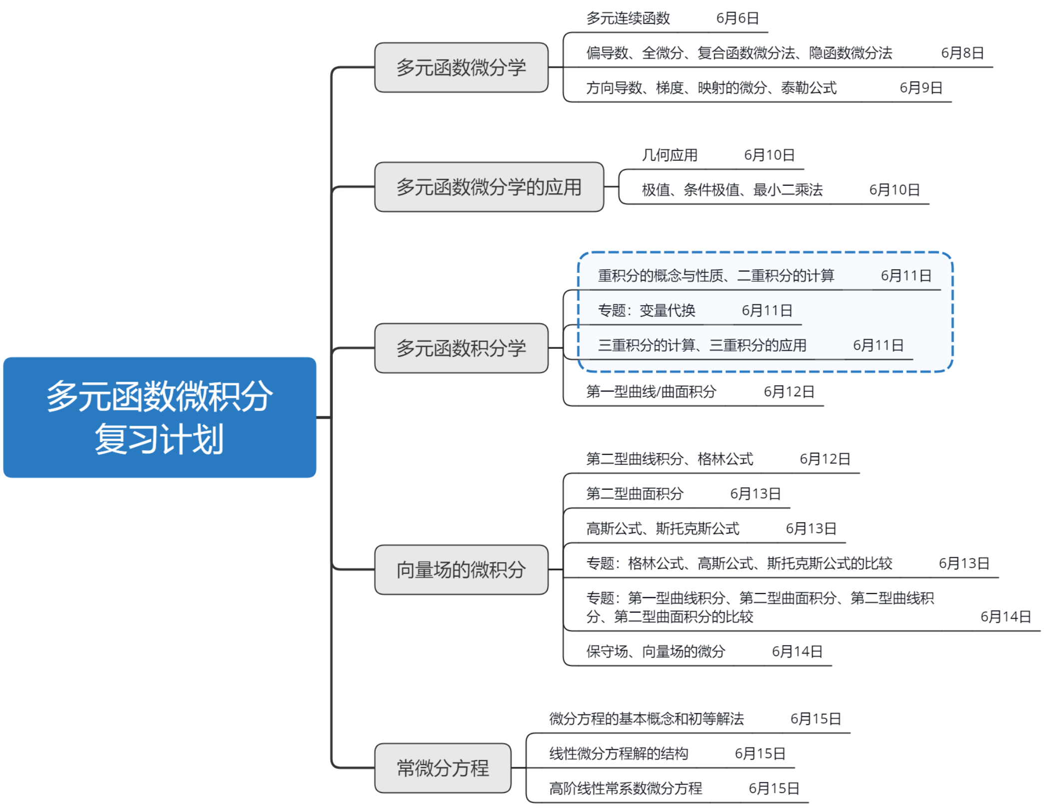
7 专题：变量替换

7.1 复习计划



7.2 习题分类与解题思路

1. 二重积分的变量替换公式：

$$\iint\limits_D f(x,y)\mathrm{d}x\mathrm{d}y=\iint\limits_D f(x(u,v),y(u,v))\left|\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}\right|\mathrm{d}u\mathrm{d}v=\iint\limits_D f(x(u,v),y(u,v))\frac{1}{\left|\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}\right|}\mathrm{d}u\mathrm{d}v.$$

2. 三重积分的变量替换公式：

$$\begin{aligned}\iint\limits_\Omega f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z&=\iint\limits_\Omega f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\mathrm{D}(x,y,z)}{\mathrm{D}(u,v,w)}\right|\mathrm{d}u\mathrm{d}v\\&=\iint\limits_\Omega f(x(u,v,w),y(u,v,w),z(u,v,w))\frac{1}{\left|\frac{\mathrm{D}(u,v,w)}{\mathrm{D}(x,y,z)}\right|}\mathrm{d}u\mathrm{d}v.\end{aligned}$$

3. 变量替换的目的：

(a) 使积分域更简单;

(b) 使被积函数更简单.

4. 变量替换主要有以下几种形式：

(a) 已知$y=kx$，可令$u=\frac{y}{x}$;

(b) 已知$xy=k$，可令$u=xy$;

(c) 已知$x+y=k$，可令$x+y=u$;

(d) 已知$x-y=k$，可令$x-y=u$;

(e) 已知$x^n+y^n=k$，可令$\begin{cases} x=r^{\frac{1}{n}}\cos^{\frac{2}{n}}\theta, \\ y=r^{\frac{1}{n}}\sin^{\frac{2}{n}}\theta. \end{cases}$

5. 解题思路：

第一步 做变量代换，将积分域用新变量表示;

第二步 求解雅可比行列式;

第三步 利用变量替换公式计算重积分.

6. 以下是本章变量替换的习题汇总.

7.3 习题12.3解答

1. 求由$xy=a^2, xy=2a^2, y=x, y=2x$围成的第一象限区域的面积.

解：令$\begin{cases} u=xy, \\ v=\frac{y}{x}, \end{cases}$ 所求区域$D=\{(u,v)\mid a^2\leqslant u\leqslant 2a^2, 1\leqslant v\leqslant 2\}$,

$\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}=\begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix}=\frac{y}{x}+\frac{y}{x}=2v$, $|\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}|=\frac{1}{2v}$,

所求面积$S=\iint\limits_D \mathrm{d}\sigma=\iint\limits_D |\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}|\mathrm{d}u\mathrm{d}v=\int_{a^2}^{2a^2}\mathrm{d}u\int_1^2\frac{1}{2v}\mathrm{d}v=a^2\frac{1}{2}\ln v\Big|_1^2=\frac{\ln 2}{2}a^2$.

2. 计算$I=\iint\limits_D\cos(\frac{x-y}{x+y})\mathrm{d}\sigma, D$由$x+y=1, x=0, y=0$围成.

解：方法1：$\because$区域D关于$y=x$对称，且在关于$y=x$的对称点(x,y)和(y,x)处$\cos(\frac{x-y}{x+y})=\cos(\frac{y-x}{y+x})$,

$\therefore I=\iint\limits_D\cos(\frac{x-y}{x+y})\mathrm{d}\sigma=2\iint\limits_{D_1}\cos(\frac{x-y}{x+y})\mathrm{d}\sigma$，其中区域$D_1$由$x+y=1, y=0, y=x$围成.

令$\begin{cases} u=x+y, \\ v=\frac{y}{x}, \end{cases}$ 则$\begin{cases} x=\frac{u}{1+v}, \\ y=\frac{uv}{1+v}, \end{cases}$ 区域$D_1=\{(u,v)\mid 0\leqslant u\leqslant 1, 0\leqslant v\leqslant 1\}$,

$\frac{D(u,v)}{D(x,y)} = \begin{vmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{x+y}{x^2} = \frac{(1+v)^2}{u}$, $|\frac{D(u,v)}{D(x,y)}| = \frac{u}{(1+v)^2}$,

$\therefore I = 2\iint\limits_{D_1} \cos(\frac{x-y}{x+y})\mathrm{d}\sigma = 2\iint\limits_{D_1} \cos(\frac{1-v}{1+v})\frac{u}{(1+v)^2}\mathrm{d}u\mathrm{d}v = 2\int_0^1 \cos(\frac{1-v}{1+v})\frac{1}{(1+v)^2}\mathrm{d}v \int_0^1 u\mathrm{d}u$

$= \int_0^1 \cos(\frac{1-v}{1+v})\frac{1}{(1+v)^2}\mathrm{d}v = \int_0^1 \cos(-1+\frac{2}{1+v})\frac{1}{(1+v)^2}\mathrm{d}v = -\frac{1}{2}\int_0^1 \cos(-1+\frac{2}{1+v})\mathrm{d}(-1+\frac{2}{1+v})$

$= -\frac{1}{2}\sin(-1+\frac{2}{1+v})\Big|_0^1 = \frac{1}{2}\sin 1.$

注： 如图1所示.

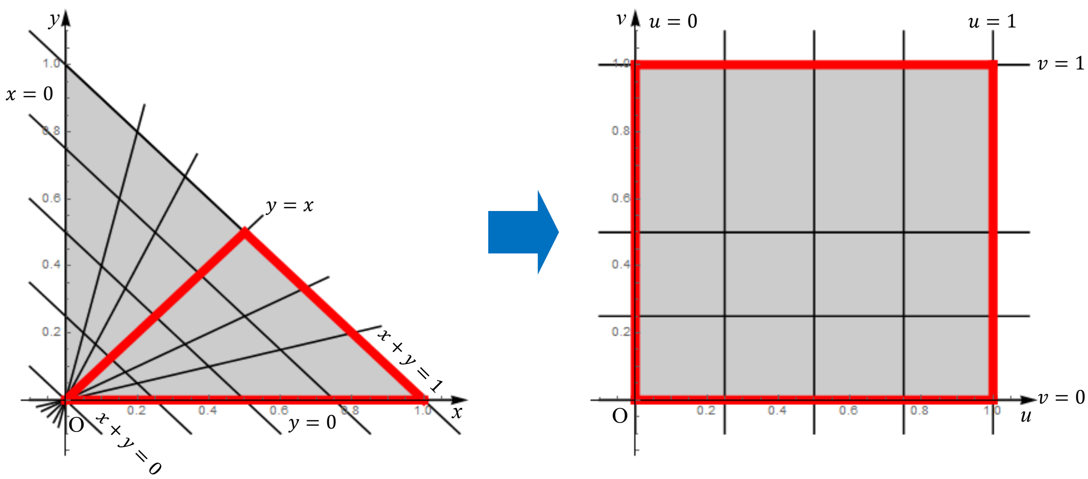


图 1: 习题12.3 2题方法1图示

方法2：令 $\begin{cases} x = r\cos^2\theta, \\ y = r\sin^2\theta, \end{cases}$ 则区域$D = \{(x,y) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant \frac{\pi}{2}\}$,

$\frac{D(x,y)}{D(u,v)} = \begin{vmatrix} \cos^2\theta & -2r\cos\theta\sin\theta \\ \sin^2\theta & 2r\sin\theta\cos\theta \end{vmatrix} = 2r\sin\theta\cos^3\theta + 2r\sin^3\theta\cos\theta = 2r\sin\theta\cos\theta = r\sin 2\theta$,

$\therefore I = \iint\limits_{D} \cos(\frac{r\cos^2\theta - r\sin^2\theta}{r\cos^2\theta + r\sin^2\theta})|\frac{D(x,y)}{D(u,v)}|\mathrm{d}r\mathrm{d}\theta = \iint\limits_{D} \cos(\cos 2\theta)r\sin 2\theta\mathrm{d}r\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \cos(\cos 2\theta)\sin 2\theta\mathrm{d}\theta \int_0^1 r\mathrm{d}r$

$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(\cos 2\theta)\mathrm{d}\cos 2\theta \int_0^1 r\mathrm{d}r = -\frac{1}{2}\sin(\cos 2\theta)\Big|_0^{\frac{\pi}{2}} \frac{1}{2}r^2\Big|_0^1 = -\frac{1}{2}[\sin(-1) - \sin 1]\frac{1}{2}$

$= \frac{1}{2}\sin 1.$

注： 如图2所示.

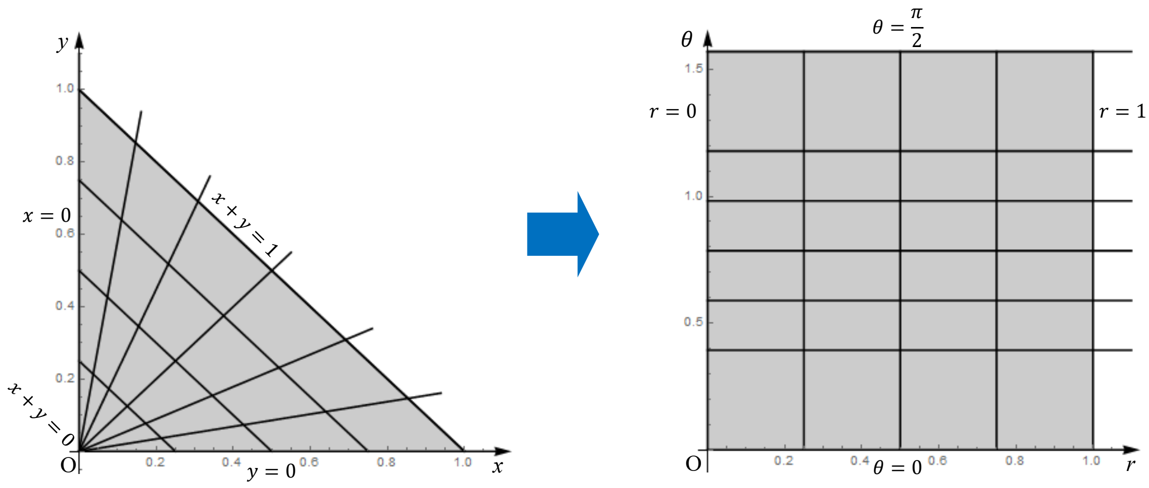


图 2: 习题12.3 2题方法2图示

方法3：令$\begin{cases} u = x - y, \\ v = x + y, \end{cases}$ 则令$\begin{cases} x = \frac{1}{2}(u+v), \\ y = \frac{1}{2}(v-u), \end{cases}$ 区域$D = \{(u,v) \mid 0 \leqslant v \leqslant 1, -v \leqslant u \leqslant v\}$,

$\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$,

$\therefore I = \iint\limits_{D} \cos(\frac{u}{v}) |\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}| \mathrm{d}u\mathrm{d}v = \frac{1}{2} \iint\limits_{D} \cos(\frac{u}{v}) \mathrm{d}u\mathrm{d}v = \frac{1}{2}\int_0^1 \mathrm{d}v \int_{-v}^{v} \cos(\frac{u}{v}) \mathrm{d}u = \frac{1}{2}\int_0^1 \mathrm{d}v \int_{-v}^{v} v\cos(\frac{u}{v}) \mathrm{d}\frac{u}{v}$
$= \frac{1}{2}\int_0^1 \mathrm{d}v [v\sin(\frac{u}{v})]_{-v}^{v} = \frac{1}{2}\int_0^1 [v\sin 1 - v\sin(-1)]\mathrm{d}v = \sin 1 \int_0^1 v\mathrm{d}v = \frac{1}{2}\sin 1.$

注意：因为被积函数是$\cos(\frac{u}{v})$，该函数无关于v初等原函数，故这种变量代换的方法应先积u后积v.

注：如图3所示.

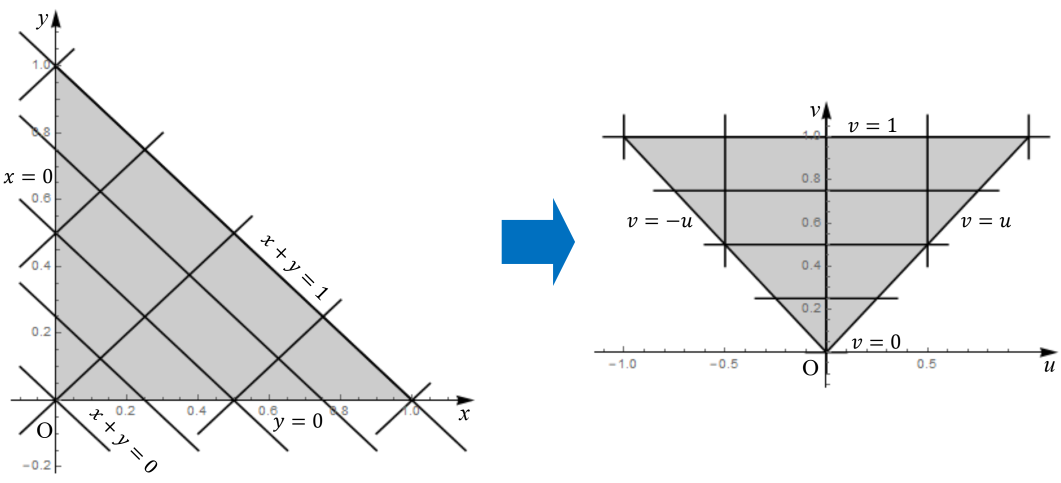


图 3: 习题12.3 2题方法3图示

3. 计算$I=\iint\limits_D(\sqrt{x}+\sqrt{y})\mathrm{d}\sigma, D=\left\{(x,y)\mid\sqrt{x}+\sqrt{y}\leqslant 1\right\}$.

解：方法1：令$\begin{cases}x=r^2\cos^4\theta,\\ y=r^2\sin^4\theta,\end{cases}$ 则区域$D=\left\{(r,\theta)\mid 0\leqslant r\leqslant 1, 0\leqslant\theta\leqslant\frac{\pi}{2}\right\}$,

$\frac{\mathrm{D}(x,y)}{\mathrm{D}(r,\theta)}=\begin{vmatrix}2r\cos^4\theta & -4r^2\cos^3\theta\sin\theta\\ 2r\sin^4\theta & 4r^2\sin^3\theta\cos\theta\end{vmatrix}=8r^3\sin^3\theta\cos^3\theta$,

$\therefore I=\iint\limits_D(\sqrt{x}+\sqrt{y})\mathrm{d}\sigma=\iint\limits_D r|\frac{\mathrm{D}(x,y)}{\mathrm{D}(r,\theta)}|\mathrm{d}u\mathrm{d}v=\iint\limits_D 8r^4\sin^3\theta\cos^3\theta\mathrm{d}u\mathrm{d}v=\int_0^1 r^4\mathrm{d}r\int_0^{\frac{\pi}{2}}2^3\sin^3\theta\cos^3\theta\mathrm{d}\theta$

$=\frac{1}{5}r^5\Big|_0^1\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^3 2\theta\mathrm{d}2\theta=\frac{1}{10}\int_0^{\frac{\pi}{2}}\sin^3 2\theta\mathrm{d}2\theta=\frac{1}{10}\int_0^{\pi}\sin^3\varphi\mathrm{d}\varphi=\frac{2}{10}\int_0^{\frac{\pi}{2}}\sin^3\varphi\mathrm{d}\varphi=\frac{1}{5}\frac{2}{3}=\frac{2}{15}$.

方法2：令$\begin{cases}u=\sqrt{x},\\ v=\sqrt{y},\end{cases}$ 则$\begin{cases}x=u^2,\\ y=v^2,\end{cases}$ 区域$D=\{(u,v)\mid 0\leqslant u\leqslant 1, 0\leqslant v\leqslant 1-u\}$,

$\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}=\begin{vmatrix}2u & 0\\ 0 & 2v\end{vmatrix}=4uv$,

$\therefore I=\iint\limits_D(\sqrt{x}+\sqrt{y})\mathrm{d}\sigma=\iint\limits_D(u+v)|\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}|\mathrm{d}u\mathrm{d}v=\iint\limits_D(u+v)4uv\mathrm{d}u\mathrm{d}v$

$=4\int_0^1\mathrm{d}u\int_0^{1-u}(u^2v+uv^2)\mathrm{d}v=4\int_0^1(\frac{1}{2}u^2v^2+\frac{1}{3}uv^3)\Big|_0^{1-u}\mathrm{d}u$

$=4\int_0^1[\frac{1}{2}u^2(1-u)^2+\frac{1}{3}u(1-u)^3]\mathrm{d}u=4\int_0^1(\frac{1}{2}u^2-u^3+\frac{1}{2}u^4+\frac{1}{3}u-u^2+u^3-\frac{1}{3}u^4)\mathrm{d}u$

$=4\int_0^1(-\frac{1}{2}u^2+\frac{1}{6}u^4+\frac{1}{3}u)\mathrm{d}u=4(-\frac{1}{6}u^3+\frac{1}{30}u^5+\frac{1}{6}u^2)\Big|_0^1=\frac{2}{15}$.

4. 在第1象限中，设D由$xy=1, xy=2, \frac{y}{x}=1$及$\frac{y}{x}=4$围成，试证：

$$\iint\limits_D f(xy)\mathrm{d}\sigma=\ln 2\int_1^2 f(x)\mathrm{d}x.$$

证明：令$\begin{cases} u = xy, \\ v = \frac{y}{x}, \end{cases}$ 则区域$D = \{(u,v) \mid 1 \leqslant u \leqslant 2, 1 \leqslant v \leqslant 4\}$,

$\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{y}{x} + \frac{y}{x} = 2v$, $|\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}| = \frac{1}{2v}$,

$\therefore \iint\limits_D f(xy)\mathrm{d}\sigma = \iint\limits_D f(u)|\frac{\mathrm{D}(x,y)}{\mathrm{D}(u,v)}|\mathrm{d}u\mathrm{d}v = \int_1^2 f(u)\mathrm{d}u \int_1^4 \frac{1}{2v}\mathrm{d}v = \frac{1}{2}\ln v\Big|_1^4 \int_1^2 f(u)\mathrm{d}u$
$= 2\ln 2 \int_1^2 f(u)\mathrm{d}u = \ln 2 \int_1^2 f(x)\mathrm{d}x.$

7.4 习题12.4解答

15 求由平面$a_1x + b_1y + c_1z = \pm h_1, a_2x + b_2y + c_2z = \pm h_2, a_3x + b_3y + c_3z = \pm h_3$所围成平行六面体的体积.

解：令$\begin{cases} u = a_1x + b_1y + c_1z, \\ v = a_2x + b_2y + c_2z, \\ w = a_3x + b_3y + c_3z, \end{cases}$ 不妨设$h_1 > 0, h_2 > 0, h_3 > 0$，则该平行六面体可表示为$\Omega = \{(u,v,w) \mid -h_1 \leqslant u \leqslant h_1, -h_2 \leqslant v \leqslant h_2, -h_3 \leqslant w \leqslant h_3\}$,

$\frac{\mathrm{D}(u,v,w)}{\mathrm{D}(x,y,z)} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \det A$,

当$\det A \neq 0$时，

$\iiint\limits_\Omega \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_\Omega |\frac{\mathrm{D}(x,y,z)}{\mathrm{D}(u,v,w)}|\mathrm{d}u\mathrm{d}v\mathrm{d}w = \iiint\limits_\Omega \frac{1}{|\frac{\mathrm{D}(u,v,w)}{\mathrm{D}(x,y,z)}|}\mathrm{d}u\mathrm{d}v\mathrm{d}w = \iiint\limits_\Omega \frac{1}{|\det A|}\mathrm{d}u\mathrm{d}v\mathrm{d}w$
$= \frac{1}{|\det A|}\iiint\limits_\Omega \mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{2h_1 \cdot 2h_2 \cdot 2h_3}{|\det A|} = \frac{8h_1h_2h_3}{|\det A|}.$

7.5 第12章补充题解答

8 设$D = \{(x,y) \mid |x| + |y| \leqslant 1\}$，将$\iint\limits_D f(x+y)\mathrm{d}x\mathrm{d}y$化为定积分.

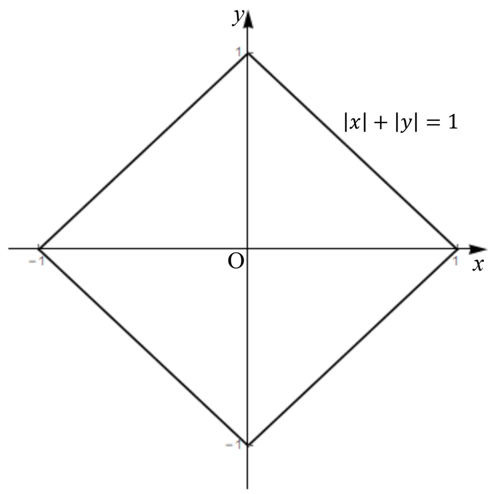


图 4: 第12章补充题 8.题图示

解：方法1：令$\begin{cases} u = x + y, \\ v = x - y, \end{cases}$ 则$D = \{(u, v) \mid -1 \leqslant u \leqslant 1, -1 \leqslant v \leqslant 1\}$,

$\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$,

$\therefore \iint\limits_D f(x+y)\mathrm{d}x\mathrm{d}y = \iint\limits_D f(u)\frac{1}{|\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}|}\mathrm{d}u\mathrm{d}v = \iint\limits_D \frac{1}{2}f(u)\mathrm{d}u\mathrm{d}v = \frac{1}{2}\int_{-1}^1 \mathrm{d}v \int_{-1}^1 f(u)\mathrm{d}u$
$= \int_{-1}^1 f(u)\mathrm{d}u.$

方法2：将重积分化为累次积分，有

$$\iint\limits_{|x|+|y|\leqslant 1} f(x+y)\mathrm{d}x\mathrm{d}y = \int_{-1}^0 \mathrm{d}x \int_{-1-x}^{1+x} f(x+y)\mathrm{d}y + \int_0^1 \mathrm{d}x \int_{-1+x}^{1-x} f(x+y)\mathrm{d}y,$$

令$x + y = u$,

$$\begin{aligned} \text{上式} &= \int_{-1}^0 \mathrm{d}x \int_{-1}^{1+2x} f(u)\mathrm{d}u + \int_0^1 x\mathrm{d}\int_{2x-1}^1 f(u)\mathrm{d}u \\ &= \int_{-1}^1 f(u)\mathrm{d}u \int_{\frac{u-1}{2}}^{\frac{u+1}{2}} \mathrm{d}x = \int_{-1}^1 f(u)\mathrm{d}u. \end{aligned}$$

10 计算下列积分：

(1)$\iiint\limits_\Omega (ax + by + cz)^2\mathrm{d}V$，其中$\Omega = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant R^2\}$；

(2)$\iiint\limits_\Omega (\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2})\mathrm{d}V$，其中$\Omega = \left\{(x, y, z) \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\right\}$.

解：(1)方法1：由对称性可知

$\iiint\limits_{\Omega}(ax+by+cz)^2\mathrm{d}V=\iiint\limits_{\Omega}(a^2x^2+b^2y^2+c^2z^2+2abxy+2acxz+2bcyz)\mathrm{d}V$
$=\iiint\limits_{\Omega}(a^2x^2+b^2y^2+c^2z^2)\mathrm{d}V=(a^2+b^2+c^2)\iiint\limits_{\Omega}z^2\mathrm{d}V$
$=(a^2+b^2+c^2)\iiint\limits_{\Omega}r^2\cos^2\varphi\cdot r^2\sin\varphi\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r$
$=(a^2+b^2+c^2)\int_0^{2\pi}\mathrm{d}\theta\int_0^R r^4\mathrm{d}r\int_0^{\pi}\cos^2\varphi\sin\varphi\mathrm{d}\varphi$
$=\frac{2}{5}\pi R^4(a^2+b^2+c^2)(-\frac{1}{3}\cos^3\varphi)\Big|_0^{\pi}=\frac{4}{15}\pi R^4(a^2+b^2+c^2).$

方法2：$\iiint\limits_{\Omega}(ax+by+cz)^2\mathrm{d}V=\iiint\limits_{\Omega}(a^2x^2+b^2y^2+c^2z^2+2abxy+2acxz+2bcyz)\mathrm{d}V$
$=\iiint\limits_{\Omega}(a^2x^2+b^2y^2+c^2z^2)\mathrm{d}V=(a^2+b^2+c^2)\iiint\limits_{\Omega}z^2\mathrm{d}V=\frac{1}{3}(a^2+b^2+c^2)\iiint\limits_{\Omega}(x^2+y^2+z^2)\mathrm{d}V$
$=\frac{1}{3}(a^2+b^2+c^2)\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^R r^2\cdot r^2\sin\varphi\mathrm{d}r=\frac{2}{3}\pi(a^2+b^2+c^2)\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_0^R r^4\mathrm{d}r$
$=\frac{2}{3}\pi(a^2+b^2+c^2)(-\cos\varphi)\Big|_0^{\pi}\frac{1}{5}r^5\Big|_0^R=\frac{4}{15}\pi R^4(a^2+b^2+c^2).$

(2)令$\begin{cases}x=au,\\ y=bv,\\ z=cw,\end{cases}$ 则$\Omega=\{(u,v,w)\mid u^2+v^2+w^2\leqslant 1\}$,

$\frac{\mathrm{D}(x,y,z)}{\mathrm{D}(u,v,w)}=\begin{vmatrix}a&0&0\\0&b&0\\0&0&c\end{vmatrix}=abc,$

$\therefore\iiint\limits_{\Omega}(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2})\mathrm{d}V=\iiint\limits_{\Omega}(u^2+v^2+w^2)|\frac{\mathrm{D}(u,v,w)}{\mathrm{D}(x,y,z)}|\mathrm{d}u\mathrm{d}v\mathrm{d}w=abc\iiint\limits_{\Omega}(u^2+v^2+w^2)\mathrm{d}u\mathrm{d}v\mathrm{d}w$
$=abc\frac{4}{15}\pi 1^4(1^2+1^2+1^2)=\frac{4}{5}\pi abc.$