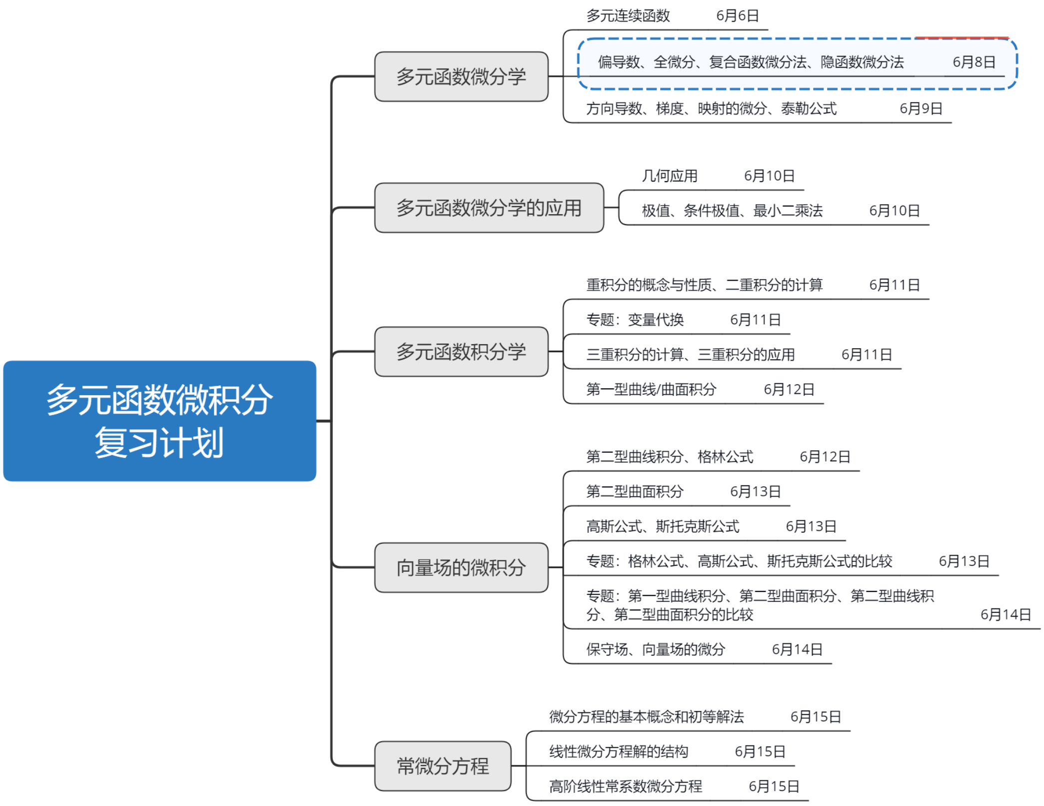
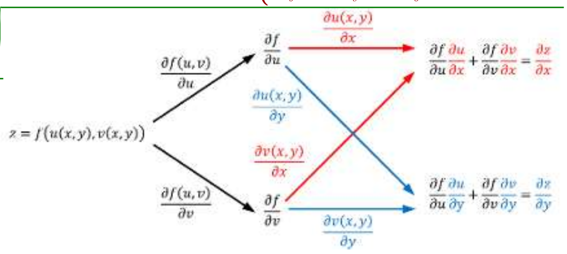
2 偏导数、全微分、复合函数微分法、隐函数微分法

2.1 复习计划



2.2 知识结构

- 多元函数微分学
  - 多元连续函数
  - 偏导数
    - 偏导数的概念
      - 偏导数的定义
        - 实质是一元函数的导数：$\frac{\partial f(a,b)}{\partial x}=\frac{\mathrm{d}f(x,b)}{\mathrm{d}x}\mid_{x=a}$
        - 偏导数存在与函数在该点连续没有必然的联系
      - 二元函数偏导数的几何意义
    - 高阶偏导数
      - 概念
      - 混合偏导数与求导顺序无关的充分条件
  - 全微分
    - 全微分的概念
      - 定义：$\exists$ 常数 $A,B,s.t.\Delta f(a,b)=f(a+\Delta x,b+\Delta y)-f(a,b)=A\Delta x+B\Delta y+o(\sqrt{x^2+y^2})$
      - 可微$\Rightarrow$连续
      - 几何意义——以平面代替曲面 $f(x,y)=f(a,b)+A(x-a)+B(y-b)$
    - 可微的必要条件与充分条件
      - 可微的必要条件
        - 可微$\Rightarrow$可偏导，且$\mathrm{d}f(a,b)=\frac{\partial f}{\partial x}\,\mathrm{d}x+\frac{\partial f}{\partial y}\,\mathrm{d}y$
      - 可微的一个充分条件
        - 充分条件定理
        - $f(x,y)$在区域D上一阶偏导数连续
        - (即$f(x,y)\in C^1(D)$) $\Rightarrow f(x,y)$在D上可微
      - 可微的充分必要条件
        - 理论意义
      - 多元微分形式的原函数
        - 不定积分法求原函数
  - 复合函数微分法
    - 复合函数的全微分与偏导数
      - $z=f(u(x,y),v(x,y))$ 的偏导数 $\begin{cases}\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x},\\ \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\end{cases}$



  - 隐函数微分法
    - 隐函数的概念
    - 隐函数的存在性与可导性
      - $F(x,y,z)=0$ 两边微分 $F_x\mathrm{d}x+F_y\mathrm{d}y+F_z\mathrm{d}z=0\Rightarrow$
        $\mathrm{d}z=-\frac{F_x}{F_z}\,\mathrm{d}x-\frac{F_y}{F_z}\,\mathrm{d}y=\frac{\partial z}{\partial x}\,\mathrm{d}x+\frac{\partial z}{\partial y}\,\mathrm{d}y$
      - $F(x,y,z)=0$ 两边求关于x,y的偏导数 $\begin{cases}F_x+F_z\frac{\partial z}{\partial x}=0,\\ F_y+F_z\frac{\partial z}{\partial y}=0,\end{cases}$ 解出$\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$
    - 隐函数组的微分法
      - $\begin{cases}F(x,y,z)=0,\\ G(x,y,z)=0,\end{cases}$ 分别对两边微分 $\begin{cases}F_x\mathrm{d}x+F_y\mathrm{d}y+F_z\mathrm{d}z=0,\\ G_x\mathrm{d}x+G_y\mathrm{d}y+G_z\mathrm{d}z=0,\end{cases}$
        $\Rightarrow\begin{cases}F_x+F_y\frac{\mathrm{d}y}{\mathrm{d}x}+F_z\frac{\mathrm{d}z}{\mathrm{d}x}=0,\\ G_x+G_y\frac{\mathrm{d}y}{\mathrm{d}x}+G_z\frac{\mathrm{d}z}{\mathrm{d}x}=0,\end{cases}$ 解该线性方程组得 $\frac{\mathrm{d}y}{\mathrm{d}x}$，$\frac{\mathrm{d}z}{\mathrm{d}x}$
      - $\begin{cases}F(x,y,z)=0,\\ G(x,y,z)=0,\end{cases}$ 分别对两边求关于x的导数
        $\begin{cases}F_x+F_y\frac{\mathrm{d}y}{\mathrm{d}x}+F_z\frac{\mathrm{d}z}{\mathrm{d}x}=0,\\ G_x+G_y\frac{\mathrm{d}y}{\mathrm{d}x}+G_z\frac{\mathrm{d}z}{\mathrm{d}x}=0,\end{cases}$ 解该线性方程组得$\frac{\mathrm{d}y}{\mathrm{d}x}$，$\frac{\mathrm{d}z}{\mathrm{d}x}$
  - 方向导数与梯度
  - 多元函数的泰勒公式

2.3 重要知识

1. 二元函数偏导数的几何意义

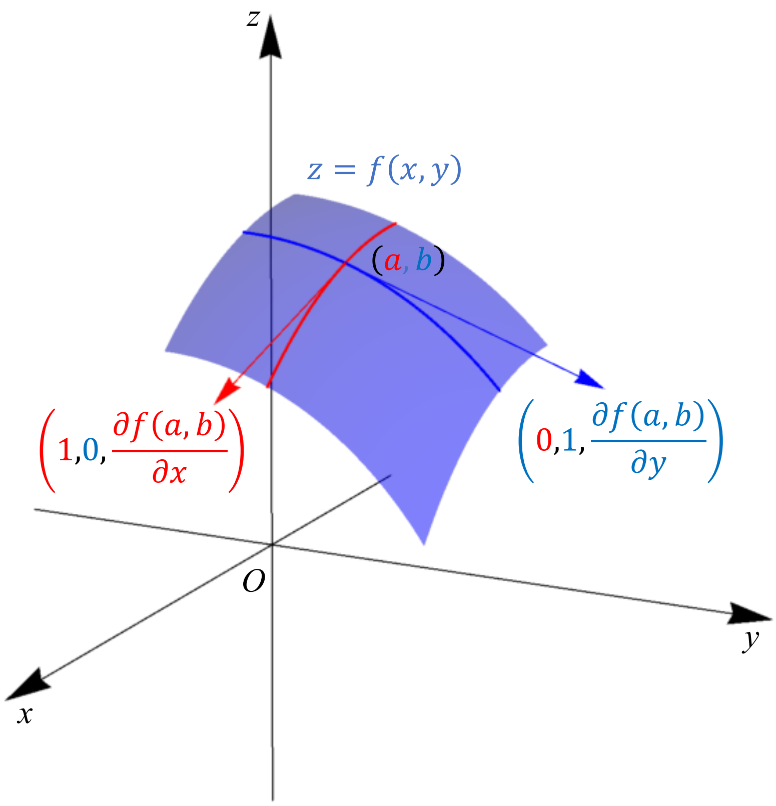


图 1: 二元函数偏导数的几何意义

2. 二元函数在一点连续、偏导数存在、可微及一阶偏导数连续之间的关系：

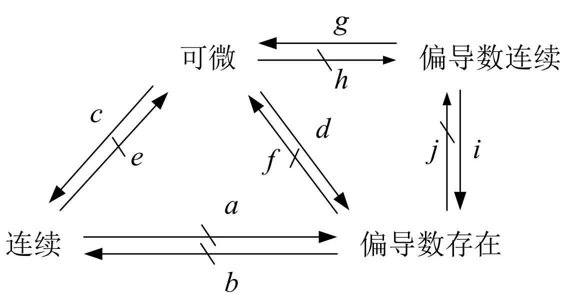


图 2: 二元函数在一点连续、偏导数存在、可微及一阶偏导数连续之间的关系

a: 反例：函数$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0),\end{cases}$在点$(0,0)$不连续（沿直线$y=kx$趋于$(0,0)$时的极限为$\frac{k}{1+k^2}\neq\mathrm{const}$），但偏导数存在：

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0,$$
$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0-0}{\Delta y}=0.$$

b: 反例：函数$g(x,y)=\sqrt{x^2+y^2}$在点$(0,0)$连续，但没有偏导数$g_x(0,0),g_y(0,0)$，因为：

$$\lim_{x\to 0+}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0+}\frac{\sqrt{x^2}-0}{x}=1\neq\lim_{x\to 0-}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0-}\frac{\sqrt{x^2}-0}{x}=-1,$$
$$\lim_{y\to 0+}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0+}\frac{\sqrt{y^2}-0}{y}=1\neq\lim_{y\to 0-}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0-}\frac{\sqrt{y^2}-0}{y}=-1.$$

c. 由可微的定义可知.

d. 可微的必要条件.

e. 由上述b.中的反例，函数$g(x,y)$在$(0,0)$处连续，但在$(0,0)$处的偏导数不存在，故函数$g(x,y)$在$(0,0)$处不可微，故连续不一定可微.

f. 由上述a.中的反例，函数$f(x,y)$在$(0,0)$处偏导数存在，但不连续，故不可微，因此偏导数存在不一定可微.

g. 由可微的充分条件可知.

h. 反例：函数$f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0),\end{cases}$在$(0,0)$处

$$f_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\frac{x^2\sin\frac{1}{x^2}}{x}=0,$$
$$f_y(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\frac{y^2\sin\frac{1}{y^2}}{y}=0,$$

且

$$\begin{aligned}&\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}\\
=&\lim_{(x,y)\to(0,0)}\frac{(x^2+y^2)\sin\frac{1}{x^2+y^2}-0-0-0}{\sqrt{x^2+y^2}}\\
=&\lim_{(x,y)\to(0,0)}\sqrt{x^2+y^2}\sin\frac{1}{x^2+y^2}=0\end{aligned}$$

故$f(x,y)$在$(0,0)$处可微. 但

$$f_x(x,y)=\begin{cases}0, & (x,y)=(0,0),\\ 2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}, & (x,y)\neq(0,0),\end{cases}$$

在$(0,0)$点不连续.

i. 显然.

j. 如上述h.中的反例.

3. 隐函数求导两种方法等价性的证明

设二元函数$z=z(x,y)$由方程$F(x,y,z)=0$确定，$F(x,y,z)$有连续的偏导数，求$z=z(x,y)$关于x,y的偏导数有以下两种方法：

(1) 将方程$F(x,y,z)=0$两边分别对x,y求偏导数：

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0,\ \frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}=0,$$

解得

$$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'},\ \frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}.$$

(2) 将方程$F(x,y,z)=0$两边求全微分：

$$\mathrm{d}F(x,y,z)=F_x'\mathrm{d}x+F_y'\mathrm{d}y+F_z'\mathrm{d}z=0,$$

整理得

$$\mathrm{d}z=-\frac{F_x'}{F_z'}\mathrm{d}x-\frac{F_y'}{F_z'}\mathrm{d}y,$$

则

$$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'},\ \frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}.$$

以上这两种方法是等价的，可做如下证明。因为$z=z(x,y)$是方程$F(x,y,z)=0$确定的隐函数，所以将$z=z(x,y)$代入函数$F=F(x,y,z)$得到一个关于x,y的常数函数$f(x,y)=F(x,y,z(x,y))=0$（比如方程$F(x,y,z)=x^2+y^2+z^2-R^2=0$确定隐函数$z(x,y)=\sqrt{R^2-x^2-y^2}$，将$z(x,y)$代入$F(x,y,z)=x^2+y^2+z^2-R^2$得到$f(x,y)=F(x,y,z(x,y))=x^2+y^2+(\sqrt{R^2-x^2-y^2})^2-R^2\equiv 0$，为一恒等于0的常数函数），该常数函数关于$x,y$的两个偏导数都等于0，即

$$\frac{\partial f}{\partial x}=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0,\ \frac{\partial f}{\partial y}=\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}=0,$$

可据此求出$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$，即方法(1).

因为$f(x,y) = F(x,y,z(x,y)) = 0$是一个常数函数，所以对该函关于$x,y$求全微分等于0，即

$$\mathrm{d}f(x,y) = (\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x})\mathrm{d}x + (\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y})\mathrm{d}y = 0\mathrm{d}x + 0\mathrm{d}y = 0$$

将该式做如下整理：

$$\begin{aligned} 0 = \mathrm{d}f(x,y) &= \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + (\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\mathrm{d}y) \\ &= \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + \frac{\partial F}{\partial z}(\frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y) \\ &= \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + \frac{\partial F}{\partial z}\mathrm{d}z \\ &= \mathrm{d}F(x,y,z) \end{aligned} \tag{1}$$

可得到

$$\mathrm{d}z = -\frac{F'_x}{F'_z}\mathrm{d}x - \frac{F'_y}{F'_z}\mathrm{d}y = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y,$$

可据此求出$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$，即方法(2). 式 (1)的推导过程即是全微分形式不变性的证明过程.

2.4 习题分类与解题思路

1. 偏导数.

   (a) 第一类题目：二元函数在一点连续、偏导数存在、可微及一阶偏导数连续之间的关系.

   解题思路见上述2.3小节中的2.

   【如习题10.2中的1.】

   (b) 求偏导数和高阶偏导数，有以下几种方法：

      i. 利用一元函数的求导法则.

      【如习题10.2中的3.(1)/(2)/(3)/(4)/(5)/(6)/(7).】

      ii. 对于特殊点处直接利用偏导数的定义求解.

      【如习题10.2中的2.】

2. 全微分

   (a) 求全微分.

   【如习题10.3中的1.】

   (b) 证明函数在一点不可微.

   可采取如下思路：

第一步 先判断函数在该点是否连续，若不连续则不可微.

第二步 若连续，判断函数在该点的偏导数是否存在，若不存在，则不可微.
【如习题10.3中的2.(1)】

第三步 若函数在该点的偏导数存在，则可假设函数可微，从而求出该点的全微分值$\mathrm{d}f(x_0,y_0)=f_x\Delta x+f_y\Delta y$，计算极限$\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\frac{f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)-\mathrm{d}f(x_0,y_0)}{\sqrt{\Delta x^2+\Delta y^2}}$是否为0，若为0则函数可微，若不为0则函数不可微.
【如习题10.3中的2.(2)】

(c) 证明函数可微.

i. 利用可微的充分条件.
【如习题10.3中的3.】

ii. 利用可微的定义. 可采用和“(b)证明函数不可微”相同的思路.

(d) 利用函数微分做近似计算.
【如习题10.3中的4.】

(e) 求全微分式的原函数. 可用不定积分法.
【如习题10.3中的6.】

【其他类型的题目：习题10.3中的5.】

3. 复合函数的导数.

(a) 利用复合函数求导的链式法则求偏导数.
【如习题10.4中的1.(1)/(2)/(3)/(4)/(5)/(6), 2., 3.】

(b) 考查高阶混合偏导数与求导顺序无关的条件.
【如习题10.4中的4.(该题需要特殊的技巧)】

4. 隐函数的导数.

(a) 求隐函数的偏导数.
有以下两种方法：

i. 方程两边求偏导.

ii. 方程两边求全微分.

具体可见【习题10.5中的2., 3., 5.】

(b) 求隐函数组的偏导数.

i. 方程组两边求偏导.

ii. 方程组两边求全微分.

具体可见【习题10.5中的1., 6.】

【其他类型的题目：习题10.5中的4.】

【习题10.5中3.的方法2是一个特殊的技巧，大家可以积累一下.】

2.5 习题10.2解答

1. 若$f(x,y)$在点(x,y)处连续，能否推出$f(x,y)$在点(x,y)的两个偏导数存在？若$f(x,y)$在点(x,y)的两个偏导数都存在，能否推出$f(x,y)$在点(x,y)处连续？

   解：(1)不能. 如函数$f(x,y)=\sqrt{x^2+y^2}$在原点连续，但是下列两个极限都不存在：

   $$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{\sqrt{x^2}}{x},$$
   $$\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0}\frac{\sqrt{y^2}}{y}.$$

   所以在原点$\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$都不存在。

   (2)不能. 如函数$f(x,y)=\begin{cases}1, & y=x^2, x>0,\\ 0, & \text{其他}.\end{cases}$ 因为$f(x,0)\equiv 0, f(0,y)\equiv 0$，故$f(x,y)$在原点的两个偏导数$\frac{\partial f(0,0)}{\partial x}=0, \frac{\partial f(0,0)}{\partial y}=0$. 但$\lim\limits_{(x,y)\to(0,0)}f(x,y)$不存在（参见教材例10.1.2），所以该函数在原点不连续.

2. 设$z=\sqrt{|xy|}$,求$\frac{\partial z}{\partial x}$.

   解：$\because z=\sqrt{|xy|}=\sqrt{|y|}\sqrt{|x|}=\begin{cases}\sqrt{|y|}\sqrt{x}, & x\geq 0,\\ \sqrt{|y|}\sqrt{-x}, & x<0.\end{cases}$

   $\therefore$当$x>0$时，$\frac{\partial z}{\partial x}=\frac{\sqrt{|y|}}{2\sqrt{x}}$,

   当$x<0$时，$\frac{\partial z}{\partial x}=-\frac{\sqrt{|y|}}{2\sqrt{-x}}$,

   当$x=0$时$\lim\limits_{x\to 0^-}\frac{\sqrt{|xy|}}{x}=\lim\limits_{x\to 0^-}-\frac{\sqrt{-x|y|}}{-x}=\lim\limits_{x\to 0^-}-\frac{\sqrt{|y|}}{\sqrt{-x}}=\begin{cases}-\infty, & y\neq 0\\ 0, & y=0\end{cases}$,

   $\lim\limits_{x\to 0^+}\frac{\sqrt{|xy|}}{x}=\lim\limits_{x\to 0^+}\frac{\sqrt{x|y|}}{x}=\lim\limits_{x\to 0^+}\frac{\sqrt{|y|}}{\sqrt{x}}=\begin{cases}+\infty, & y\neq 0\\ 0, & y=0\end{cases}$.

$$\therefore \frac{\partial z}{\partial x}=\begin{cases}\frac{\sqrt{|y|}}{2\sqrt{x}}, & x>0,\\ \text{不存在}, & x=0\text{且}y\neq 0\\ 0, & x=0,y=0\\ -\frac{\sqrt{|y|}}{2\sqrt{-x}}, & x<0.\end{cases}$$

3. 求下列偏导数：

(1)$z=\frac{x+y}{x-y}$，求$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$；

(2)$f(x,y)=\arctan\frac{y}{x}$，求$\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$；

(3)$z=\cos\frac{y}{x}\sin\frac{x}{y}$，求$\frac{\partial z(2,\pi)}{\partial x},\frac{\partial z(2,\pi)}{\partial y}$；

(4)$z=\arcsin\sqrt{\frac{x}{y}}+\frac{1}{xy}\mathrm{e}^{\frac{y}{x}}$，求$\frac{\partial z(1,2)}{\partial x},\frac{\partial z(1,2)}{\partial y}$；

(5)$z=\ln(\sqrt{x}+\sqrt{y})$，求$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$；

(6)$z=\frac{x-y}{x+y}\ln\frac{y}{x}$，求$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$；

(7)$u=\sqrt{x^2+y^2+z^2}$，求$(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2+(\frac{\partial u}{\partial z})^2$.

解：(1)$\frac{\partial z}{\partial x}=\frac{x-y-(x+y)}{(x-y)^2}=\frac{-2y}{(x-y)^2},\frac{\partial z}{\partial y}=\frac{x-y+(x+y)}{(x-y)^2}=\frac{2x}{(x-y)^2}$.

(2)$\frac{\partial f}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\frac{-y}{x^2}=\frac{-y}{x^2+y^2},\frac{\partial f}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x}=\frac{x}{x^2+y^2}(x\neq 0)$.

(3)$\frac{\partial z}{\partial x}=-(-\frac{y}{x^2})\sin\frac{y}{x}\sin\frac{x}{y}+\frac{1}{y}\cos\frac{y}{x}\cos\frac{x}{y}=\frac{y}{x^2}\sin\frac{y}{x}\sin\frac{x}{y}+\frac{1}{y}\cos\frac{y}{x}\cos\frac{x}{y}$,

$\frac{\partial z}{\partial y}=-\frac{1}{x}\sin\frac{y}{x}\sin\frac{x}{y}-\frac{x}{y^2}\cos\frac{y}{x}\cos\frac{x}{y}$,

$\therefore\frac{\partial z(2,\pi)}{\partial x}=\frac{\pi}{4}\sin\frac{\pi}{2}\sin\frac{2}{\pi}+\frac{1}{\pi}\cos\frac{\pi}{2}\cos\frac{2}{\pi}=\frac{\pi}{4}\sin\frac{2}{\pi}$,

$\frac{\partial z(2,\pi)}{\partial y}=-\frac{1}{2}\sin\frac{\pi}{2}\sin\frac{2}{\pi}-\frac{2}{\pi^2}\cos\frac{\pi}{2}\cos\frac{2}{\pi}=-\frac{1}{2}\sin\frac{2}{\pi}$.

(4)$\because\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\frac{x}{y}}}\frac{1}{2\sqrt{xy}}-\frac{1}{x^2y}\mathrm{e}^{\frac{y}{x}}+\frac{1}{xy}\mathrm{e}^{\frac{y}{x}}(-\frac{y}{x^2})=\frac{1}{2\sqrt{xy-x^2}}-(\frac{1}{x^2y}+\frac{1}{x^3})\mathrm{e}^{\frac{y}{x}}$,

$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\frac{x}{y}}}(-\frac{1}{2}\sqrt{\frac{x}{y^3}})-\frac{1}{xy^2}\mathrm{e}^{\frac{y}{x}}+\frac{1}{xy}\mathrm{e}^{\frac{y}{x}}\frac{1}{x}=-\frac{1}{2}\sqrt{\frac{x}{y^3-xy^2}}+(\frac{1}{x^2y}-\frac{1}{xy^2})\mathrm{e}^{\frac{y}{x}}$,

$\therefore\frac{\partial z(1,2)}{\partial x}=\frac{1}{2\sqrt{2-1}}-(\frac{1}{2}+1)\mathrm{e}^2=\frac{1}{2}-\frac{3}{2}\mathrm{e}^2,\frac{\partial z(1,2)}{\partial y}=-\frac{1}{2}\sqrt{\frac{1}{8-4}}+(\frac{1}{2}-\frac{1}{4})\mathrm{e}^2=-\frac{1}{4}+\frac{1}{4}\mathrm{e}^2$.

(5)$\because\frac{\partial z}{\partial x}=\frac{1}{\sqrt{x}+\sqrt{y}}\frac{1}{2\sqrt{x}},\frac{\partial z}{\partial y}=\frac{1}{\sqrt{x}+\sqrt{y}}\frac{1}{2\sqrt{y}}$,

$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{\sqrt{x}}{2(\sqrt{x}+\sqrt{y})}+\frac{\sqrt{y}}{2(\sqrt{x}+\sqrt{y})}=\frac{1}{2}$.

(6)$\because\frac{\partial z}{\partial x}=\frac{x+y-(x-y)}{(x+y)^2}\ln\frac{y}{x}+\frac{x-y}{x+y}\frac{x}{y}(-\frac{y}{x^2})=\frac{2y}{(x+y)^2}\ln\frac{y}{x}-\frac{1}{x}\frac{x-y}{x+y}$,

$\frac{\partial z}{\partial y}=\frac{-(x+y)-(x-y)}{(x+y)^2}\ln\frac{y}{x}+\frac{x-y}{x+y}\frac{x}{y}\frac{1}{x}=\frac{-2x}{(x+y)^2}\ln\frac{y}{x}+\frac{1}{y}\frac{x-y}{x+y}$,

$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{2xy}{(x+y)^2}\ln\frac{y}{x}-\frac{x-y}{x+y}+\frac{-2xy}{(x+y)^2}\ln\frac{y}{x}+\frac{x-y}{x+y}=0$.

(7)$\because\frac{\partial u}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2+z^2}}=\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{\partial u}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{\partial u}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}$(后两式可根据$x,y,z$的对称性得到),

$\therefore(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2+(\frac{\partial u}{\partial z})^2=\frac{x^2+y^2+z^2}{x^2+y^2+z^2}=1$.

4. 求下列高阶导数：

(1)$z=x+y+\frac{1}{xy}$，求$\frac{\partial^2 z(1,1)}{\partial x\partial y}$；

(2)$z=y^{\ln x}$，求$\frac{\partial^2 z}{\partial x\partial y}$；

(3)$z=\ln(x+\sqrt{x^2+y^2})$，求$\frac{\partial^2 z}{\partial x\partial y}$；

(4)$z=\ln(\sqrt{(x-a)^2+(y-b)^2})$，求$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$；

(5)$u=\sqrt{x^2+y^2+z^2}$，求$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$；

(6)$z=\sin(xy)$，求$\frac{\partial^3 z}{\partial x\partial y^2}$；

(7)$f(x,y,z)=xy^2+yz^2+zx^2$，求$\frac{\partial^2 f(0,0,1)}{\partial x^2}$，$\frac{\partial^2 f(1,0,2)}{\partial x\partial z}$，$\frac{\partial^2 f(0,-1,0)}{\partial y\partial z}$，$\frac{\partial^3 f(2,0,1)}{\partial x\partial z^2}$.

解：(1)$\because \frac{\partial z}{\partial y}=1-\frac{1}{xy^2}, \frac{\partial^2 z}{\partial x\partial y}=\frac{1}{x^2y^2}, \therefore \frac{\partial^2 z(1,1)}{\partial x\partial y}=1.$

(2)$\frac{\partial z}{\partial y}=y^{\ln x-1}\ln x, \frac{\partial^2 z}{\partial x\partial y}=y^{\ln x-1}\ln y\frac{1}{x}\ln x+y^{\ln x-1}\frac{1}{x}=\frac{y^{\ln x}}{xy}(\ln y\ln x+1).$

(3)$\frac{\partial z}{\partial y}=\frac{\frac{2y}{2\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}=\frac{y}{\sqrt{x^2+y^2}(x+\sqrt{x^2+y^2})},$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{-y[\frac{2x}{2\sqrt{x^2+y^2}}(x+\sqrt{x^2+y^2})+\sqrt{x^2+y^2}(1+\frac{2x}{2\sqrt{x^2+y^2}})]}{(x^2+y^2)(x+\sqrt{x^2+y^2})^2}=\frac{-y(\frac{x}{\sqrt{x^2+y^2}}+1)(x+\sqrt{x^2+y^2})}{(x^2+y^2)(x+\sqrt{x^2+y^2})^2}$$

$$=\frac{-y\frac{1}{\sqrt{x^2+y^2}}(x+\sqrt{x^2+y^2})^2}{(x^2+y^2)(x+\sqrt{x^2+y^2})^2}=-\frac{y}{(x^2+y^2)^{\frac{3}{2}}}.$$

(4)$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{(x-a)^2+(y-b)^2}}\frac{2(x-a)}{2\sqrt{(x-a)^2+(y-b)^2}}=\frac{x-a}{(x-a)^2+(y-b)^2},$

$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{(x-a)^2+(y-b)^2}}\frac{2(y-b)}{2\sqrt{(x-a)^2+(y-b)^2}}=\frac{y-b}{(x-a)^2+(y-b)^2},$

$\frac{\partial^2 z}{\partial x^2}=\frac{(x-a)^2+(y-b)^2-(x-a)2(x-a)}{[(x-a)^2+(y-b)^2]^2}=\frac{(y-b)^2-(x-a)^2}{[(x-a)^2+(y-b)^2]^2},$

$\frac{\partial^2 z}{\partial y^2}=\frac{(x-a)^2+(y-b)^2-(y-b)2(y-b)}{[(x-a)^2+(y-b)^2]^2}=\frac{(x-a)^2-(y-b)^2}{[(x-a)^2+(y-b)^2]^2},$

$\therefore \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{(y-b)^2-(x-a)^2}{[(x-a)^2+(y-b)^2]^2}+\frac{(x-a)^2-(y-b)^2}{[(x-a)^2+(y-b)^2]^2}=0.$

(5)$\frac{\partial u}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2+z^2}}=\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{\partial^2 u}{\partial x^2}=\frac{\sqrt{x^2+y^2+z^2}-x\frac{2x}{2\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}=\frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},$

根据x,y,z的对称性可得$\frac{\partial^2 u}{\partial y^2}=\frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{\partial^2 u}{\partial z^2}=\frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},$

$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\frac{2}{\sqrt{x^2+y^2+z^2}}.$

(6)$\frac{\partial z}{\partial y}=x\cos(xy), \frac{\partial^2 z}{\partial y^2}=-x^2\sin(xy), \frac{\partial^3 z}{\partial x\partial y^2}=-2x\sin(xy)-x^2y\cos(xy).$

(7)$\because \frac{\partial f}{\partial x}=y^2+2zx, \frac{\partial^2 f}{\partial x^2}=2z, \therefore \frac{\partial^2 f(0,0,1)}{\partial x^2}=2,$

$\because \frac{\partial f}{\partial z}=2yz+x^2, \frac{\partial^2 f}{\partial x\partial z}=2x, \therefore \frac{\partial^2 f(1,0,2)}{\partial x\partial z}=2,$

$\because \frac{\partial^2 f}{\partial y\partial z}=2z, \therefore \frac{\partial^2 f(0,-1,0)}{\partial y\partial z}=0,$

$\because \frac{\partial^2 f}{\partial z^2}=2y, \frac{\partial^3 f}{\partial x\partial z^2}=0, \therefore \frac{\partial^3 f(2,0,1)}{\partial x\partial z^2}=0.$

2.6 习题10.3解答

1. 求下列函数在指定点的全微分：

   (1)$z=\arctan\frac{x+y}{x-y}$，在任意点(x,y)；

   (2)$z=\ln\sqrt{1+x^2+y^2}$，在点$(1,1)$；

   (3)$z=\mathrm{e}^{-(\frac{y}{x}-\frac{x}{y})}$，在点$(1,-1)$；

   (4)$z=\arctan\frac{x}{1+y^2}$，求$\mathrm{d}z(1,1)$；

   (5)$u=(\frac{x}{y})^z$，在任一点(x,y,z).

   解：(1)$\frac{\partial z}{\partial x}=\frac{1}{1+(\frac{x+y}{x-y})^2}\frac{x-y-(x+y)}{(x-y)^2}=\frac{-2y}{(x+y)^2+(x-y)^2}=\frac{-y}{x^2+y^2}$,

   $\frac{\partial z}{\partial y}=\frac{1}{1+(\frac{x+y}{x-y})^2}\frac{x-y+(x+y)}{(x-y)^2}=\frac{x}{x^2+y^2}$,

   当$x\neq y$时，$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$均连续，故函数$z$在任意点$(x,y)$可微，

   $\mathrm{d}z(x,y)=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=\frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}$.

   (2)$\frac{\partial z}{\partial x}=\frac{\partial\frac{1}{2}\ln(1+x^2+y^2)}{\partial x}=\frac{1}{2}\frac{2x}{1+x^2+y^2}=\frac{x}{1+x^2+y^2},\frac{\partial z}{\partial y}=\frac{y}{1+x^2+y^2}$,

   因为$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$在点$(1,1)$及其附近存在且在点$(1,1)$处连续，故函数$z$在点$(1,1)$处可微，

   $\mathrm{d}z(1,1)=\frac{\partial z(1,1)}{\partial x}\mathrm{d}x+\frac{\partial z(1,1)}{\partial y}\mathrm{d}y=\frac{1}{3}(\mathrm{d}x+\mathrm{d}y)$.

   (3)$\frac{\partial z}{\partial x}=\mathrm{e}^{-(\frac{y}{x}-\frac{x}{y})}[-(-\frac{y}{x^2}-\frac{1}{y})]=\mathrm{e}^{-(\frac{y}{x}-\frac{x}{y})}(\frac{y}{x^2}+\frac{1}{y})],\frac{\partial z}{\partial y}=\mathrm{e}^{-(\frac{y}{x}-\frac{x}{y})}(-\frac{1}{x}-\frac{x}{y^2})$,

   因为$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$在点$(1,-1)$及其附近存在且在点$(1,-1)$处连续，故函数$z$在点$(1,-1)$可微，

   $\mathrm{d}z(1,-1)=\frac{\partial z(1,-1)}{\partial x}\mathrm{d}x+\frac{\partial z(1,-1)}{\partial y}\mathrm{d}y=-2\mathrm{d}x-2\mathrm{d}y$.

   (4)$\frac{\partial z}{\partial x}=\frac{1}{1+(\frac{x}{1+y^2})^2}\frac{1}{1+y^2}=\frac{1+y^2}{x^2+(1+y^2)^2},\frac{\partial z}{\partial x}=\frac{1}{1+(\frac{x}{1+y^2})^2}\frac{-2xy}{(1+y^2)^2}=\frac{-2xy}{x^2+(1+y^2)^2}$,

   因为$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$在点$(1,1)$及其附近存在且在点$(1,1)$处连续，故函数$z$在点$(1,1)$可微，

   $\mathrm{d}z(1,1)=\frac{\partial z(1,1)}{\partial x}\mathrm{d}x+\frac{\partial z(1,1)}{\partial y}\mathrm{d}y=\frac{2}{5}\mathrm{d}x-\frac{2}{5}\mathrm{d}y$.

   (5)$\frac{\partial u}{\partial x}=\frac{z}{y}(\frac{x}{y})^{z-1},\frac{\partial u}{\partial y}=-\frac{z}{x}(\frac{y}{x})^{-z-1},\frac{\partial u}{\partial z}=(\frac{x}{y})^z\ln\frac{x}{y}$,

   当$xy>0$时，$\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$均连续，故函数$z$在任一点$(x,y,z)$可微，

   $\mathrm{d}z=\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y+\frac{\partial u}{\partial z}\mathrm{d}z=\frac{z}{y}(\frac{x}{y})^{z-1}\mathrm{d}x-\frac{z}{x}(\frac{y}{x})^{-z-1}\mathrm{d}y+(\frac{x}{y})^z\ln\frac{x}{y}\mathrm{d}z$.

2. 试证明下列函数在$(0,0)$点不可微：

   (1)$f(x,y)=\sqrt{x}\cos y$；

   (2)$f(x,y)=\begin{cases}\frac{2xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$

   解：(1) 假设$f(x,y)$在点$(0,0)$处可微，则$\frac{\partial f(0,0)}{\partial x}$存在,

   $\because f(x,y)=\sqrt{x}\cos y$,

   $\therefore x\geq 0$,

$\because \lim\limits_{x\to0^+}\frac{f(x,0)-f(0,0)}{x-0}=\lim\limits_{x\to0^+}\frac{\sqrt{x}-0}{x-0}=+\infty$,

$\therefore \frac{\partial f}{\partial x}$在点$(0,0)$不存在，故函数$f(x,y)$在点$(0,0)$不可微.

(2) $\because |\frac{2xy}{\sqrt{x^2+y^2}}-f(0,0)|=\frac{|2xy|}{\sqrt{x^2+y^2}}\le\frac{2|x||y|}{|x|}=2|y|$,

又$\because \lim\limits_{(x,y)\to(0,0)}2|y|=0$,

$\therefore \lim\limits_{(x,y)\to(0,0)}\frac{2xy}{\sqrt{x^2+y^2}}=f(0,0)$,

$\therefore f(x,y)$在点$(0,0)$处连续,

$\because f(x,0)=0, f(0,y)=0$

$\therefore \frac{\partial f(0,0)}{\partial x}=0, \frac{\partial f(0,0)}{\partial y}=0$

$\therefore$若$f(x,y)$在点$(0,0)$处可微，则$\mathrm{d}f(0,0)=0$，且

$\lim\limits_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-\mathrm{d}f(0,0)}{\sqrt{x^2+y^2}}=\lim\limits_{(x,y)\to(0,0)}\frac{2xy}{x^2+y^2}=0$,

$\because \lim\limits_{\substack{x\to0\\y=x}}\frac{2xy}{x^2+y^2}=\lim\limits_{x\to0}\frac{2x^2}{x^2+x^2}=1\neq0$,

$\therefore \lim\limits_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-\mathrm{d}f(0,0)}{\sqrt{x^2+y^2}}\neq0$，矛盾，

$\therefore$函数$f(x,y)$在点$(0,0)$不可微.

3. 已知函数$g(x),h(x)$分别在区间$[x_0,x_1]$与$[y_0,y_1]$上连续，试证函数

$$f(x,y)=\int_{x_0}^{x}g(s)\mathrm{d}s\int_{y_0}^{y}h(t)\mathrm{d}t$$

在点(x,y)可微，其中$(x,y)\in D=\{(x,y)\mid x_0\le x\le x_1, y_0\le y\le y_1\}$.

证明：$\because g(x),h(x)$分别在区间$[x_0,x_1]$与$[y_0,y_1]$上连续,

$\therefore \frac{\partial f(x,y)}{\partial x}=g(x)\int_{y_0}^{y}h(t)\mathrm{d}t, \frac{\partial f(x,y)}{\partial y}=h(y)\int_{x_0}^{x}g(s)\mathrm{d}s$且$\int_{x_0}^{x}g(s)\mathrm{d}s$和$\int_{y_0}^{y}h(t)\mathrm{d}t$分别在区间$[x_0,x_1]$与$[y_0,y_1]$上连续,

$\therefore \frac{\partial f(x,y)}{\partial x},\frac{\partial f(x,y)}{\partial y}$在$D$中任意一点$(x,y)$连续,

$\therefore$函数$f(x,y)$在D中任意一点(x,y)可微.

4. 用函数微分计算下列数值的近似值：

(1)$\sqrt{1.02^2+1.97^2}$；　　(2)$0.97^{1.05}$.

解：(1)令$f(x,y)=\sqrt{x^2+y^2}$,

$\because \frac{\partial f}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x}{\sqrt{x^2+y^2}}, \frac{\partial f}{\partial y}=\frac{2y}{2\sqrt{x^2+y^2}}=\frac{y}{\sqrt{x^2+y^2}}$在点$(1,2)$及其附近存在且在点$(1,2)$处连续，故$f(x,y)$在点$(1,2)$可微，

$\therefore \sqrt{1.02^2+1.97^2}=f(1.02,1.97)\approx f(1,2)+0.02\frac{\partial f(1,2)}{\partial x}+(-0.03)\frac{\partial f(1,2)}{\partial y}$
$=\sqrt{5}+0.02\times\frac{1}{\sqrt{5}}-0.03\times\frac{2}{\sqrt{5}}=\frac{5-0.04}{\sqrt{5}}\approx 2.2182.$

(2)令$f(x,y)=x^y$,

$\because \frac{\partial f}{\partial x}=yx^{y-1},\frac{\partial f}{\partial y}=x^y\ln x$在点$(1,1)$及其附近存在且在点$(1,1)$处连续，

$\therefore 0.97^{1.05}=f(0.97,1.05)\approx f(1,1)+(-0.03)\frac{\partial f(1,1)}{\partial x}+0.05\frac{\partial f(1,1)}{\partial y}$
$=1-0.03\times 1+0.05\times 0=0.97.$

5. 设二元函数$z(x,y)$满足方程$\frac{\partial^2 z}{\partial x\partial y}=x+y$，并且$z(x,0)=x,z(0,y)=y^2$. 试求$z(x,y)$.

解：方法1：$\because \frac{\partial^2 z}{\partial x\partial y}=x+y$,

$\therefore \int\frac{\partial^2 z}{\partial x\partial y}\mathrm{d}x=\int(x+y)\mathrm{d}x=\frac{1}{2}x^2+xy+C_1(y)=\frac{\partial z}{\partial y}+C_1(y)$,

$\therefore$可设$\frac{\partial z}{\partial y}=\frac{1}{2}x^2+xy+C(y)$，其中$C(y)$是与$x$无关的$y$的函数，

$\because z(0,y)=y^2$,

$\therefore \frac{\partial z(0,y)}{\partial y}=2y=C(y)$,

$\therefore \frac{\partial z}{\partial y}=\frac{1}{2}x^2+xy+2y$,

$\therefore \int\frac{\partial z}{\partial y}\mathrm{d}y=\int(\frac{1}{2}x^2+xy+2y)\mathrm{d}y=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+C_3(x)=z(x,y)+C_4(x)$,

$\therefore$可设$z(x,y)=\int(\frac{1}{2}x^2+xy+2y)\mathrm{d}y+C^*(x)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+C^*(x)$，其中$C^*(x)$是与$y$无关的$x$的函数，

$\because z(x,0)=x=C^*(x)$,

$\therefore z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x.$

方法2：$\because \frac{\partial^2 z}{\partial x\partial y}=x+y$,

$\therefore \int\frac{\partial^2 z}{\partial x\partial y}\mathrm{d}x=\int(x+y)\mathrm{d}x=\frac{1}{2}x^2+xy+C_1(y)=\frac{\partial z}{\partial y}+C_2(y)$,

$\therefore$可设$\frac{\partial z}{\partial y}=\frac{1}{2}x^2+xy+C(y)$，其中$C(y)$是与$x$无关的$y$的函数，

$\therefore \int\frac{\partial z}{\partial y}\mathrm{d}y=\frac{1}{2}x^2y+\frac{1}{2}xy^2+\int C(y)\mathrm{d}y=z(x,y)+C_3(x)$,

$\therefore$可设$z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+F(y)+C^*(x)$，其中$F(y)$是$C(y)$的一个与$x$无关的原函数，$C^*(x)$是与$y$无关的$x$的函数，

$\because z(0,y)=y^2,z(x,0)=x$,

$\therefore F(y)+C^*(0)=y^2,F(0)+C^*(x)=x$,

$\therefore F(y)=y^2-C^*(0),C^*(x)=x-F(0)$，且$F(0)+C^*(0)=0$,

$\therefore z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x-[F(0)+C^*(0)]=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x.$

6. 求$y^2\mathrm{e}^{x+y}(\mathrm{d}x+\mathrm{d}y)+2y\mathrm{e}^{x+y}\mathrm{d}y$的原函数.

   解：方法1：设$f(x,y)$是$y^2\mathrm{e}^{x+y}(\mathrm{d}x+\mathrm{d}y)+2y\mathrm{e}^{x+y}\mathrm{d}y=y^2\mathrm{e}^{x+y}\mathrm{d}x+(y^2+2y)\mathrm{e}^{x+y}\mathrm{d}y$的原函数，则$\frac{\partial f}{\partial x}=y^2\mathrm{e}^{x+y}$,

   $\therefore \int\frac{\partial f}{\partial x}\mathrm{d}x=\int y^2\mathrm{e}^{x+y}\mathrm{d}x=y^2\mathrm{e}^y\mathrm{e}^x+C_1(y)=f(x,y)+C_2(y)$,

   $\therefore f(x,y)=y^2\mathrm{e}^y\mathrm{e}^x+C(y)$,

   $\therefore \frac{\partial f}{\partial y}=2y\mathrm{e}^y\mathrm{e}^x+y^2\mathrm{e}^y\mathrm{e}^x+C'(y)=(y^2+2y)\mathrm{e}^{x+y}+C'(y)$,

   又$\because \frac{\partial f}{\partial y}=(y^2+2y)\mathrm{e}^{x+y}$,

   $\therefore C'(y)=0$,

   $\therefore C(y)=C$,

   $\therefore f(x,y)=y^2\mathrm{e}^{x+y}+C$.

   方法2：设$f(x,y)$是$y^2\mathrm{e}^{x+y}(\mathrm{d}x+\mathrm{d}y)+2y\mathrm{e}^{x+y}\mathrm{d}y=y^2\mathrm{e}^{x+y}\mathrm{d}x+(y^2+2y)\mathrm{e}^{x+y}\mathrm{d}y$的原函数，则$\frac{\partial f}{\partial y}=(y^2+2y)\mathrm{e}^{x+y}$,

   $\therefore \int\frac{\partial f}{\partial y}\mathrm{d}y=\int(y^2+2y)\mathrm{e}^{x+y}\mathrm{d}y=\mathrm{e}^x\int(y^2+2y)\mathrm{d}\mathrm{e}^y=\mathrm{e}^x[(y^2+2y)\mathrm{e}^y-\int\mathrm{e}^y\mathrm{d}(y^2+2y)]$
   $=\mathrm{e}^x[(y^2+2y)\mathrm{e}^y-\int\mathrm{e}^y(2y+2)\mathrm{d}y]=\mathrm{e}^x[(y^2+2y)\mathrm{e}^y-\int(2y+2)\mathrm{d}\mathrm{e}^y]$
   $=\mathrm{e}^x[(y^2+2y)\mathrm{e}^y-(2y+2)\mathrm{e}^y+\int\mathrm{e}^y\mathrm{d}(2y+2)]=\mathrm{e}^x[(y^2+2y)\mathrm{e}^y-(2y+2)\mathrm{e}^y+2\int\mathrm{e}^y\mathrm{d}y]$
   $=\mathrm{e}^x[(y^2+2y)\mathrm{e}^y-(2y+2)\mathrm{e}^y+2\mathrm{e}^y]=y^2\mathrm{e}^{x+y}+C_1(x)=f(x,y)+C_2(x)$,

   $\therefore f(x,y)=y^2\mathrm{e}^{x+y}+C(x)$,

   $\therefore \frac{\partial f}{\partial x}=y^2\mathrm{e}^{x+y}+C'(x)$,

   又$\because \frac{\partial f}{\partial x}=y^2\mathrm{e}^{x+y}$,

   $\therefore C'(x)=0$,

   $\therefore C(x)=C$,

   $\therefore f(x,y)=y^2\mathrm{e}^{x+y}+C$.

2.7 习题10.4解答

1. 求下列复合函数的偏导数：

   (1)$z=xy+xf(u),u=\frac{y}{x}$，其中f为C^1类函数，求$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$；
   (2)$z=f(u,v),u=x,v=\frac{x}{y}$，其中$f$为$C^2$类函数，求$\frac{\partial^2 z}{\partial y^2}$；
   (3)$z=xf(\frac{y}{x})+yg(\frac{x}{y})$，其中$f,g$为$C^2$类函数，求$\frac{\partial^2 z}{\partial x\partial y}$；
   (4)$z=\frac{y}{f(x^2-y^2)}$，其中f为可微函数，求$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}$；
   (5)$u=f(x,xy,xyz)$，其中f为可微函数，求$\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$；
   (6)$z=\mathrm{e}^{x-2y},x=\sin t,y=t^3$，求$\frac{\mathrm{d}z}{\mathrm{d}t}$.

解：(1)$\because z=xy+xf(u)=xy+xf(\frac{y}{x})$,

$\therefore \frac{\partial z}{\partial x}=y+f(\frac{y}{x})+xf'(\frac{y}{x})(-\frac{y}{x^2})=y+f(\frac{y}{x})-\frac{y}{x}f'(\frac{y}{x}), \frac{\partial z}{\partial y}=x+xf'(\frac{y}{x})\frac{1}{x}=x+f'(\frac{y}{x})$,

$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x[y+f(\frac{y}{x})-\frac{y}{x}f'(\frac{y}{x})]+y[x+f'(\frac{y}{x})]=2xy+xf(\frac{y}{x})$.

(2)$\frac{\partial z}{\partial y}=\frac{\partial f(u,v)}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f(u,v)}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial f(u,v)}{\partial v}(-\frac{x}{y^2})$,

$\frac{\partial^2 z}{\partial y^2}=-\frac{x}{y^2}[\frac{\partial^2 f(u,v)}{\partial u\partial v}\frac{\partial u}{\partial y}+\frac{\partial^2 f(u,v)}{\partial v^2}\frac{\partial v}{\partial y}]+\frac{\partial f(u,v)}{\partial v}\frac{2x}{y^3}=-\frac{x}{y^2}[\frac{\partial^2 f(u,v)}{\partial v^2}(-\frac{x}{y^2})]+\frac{\partial f(u,v)}{\partial v}\frac{2x}{y^3}$
$=\frac{x^2}{y^4}\frac{\partial^2 f(u,v)}{\partial v^2}+\frac{2x}{y^3}\frac{\partial f(u,v)}{\partial v}$.

(3)$\frac{\partial z}{\partial y}=xf'(\frac{y}{x})\frac{1}{x}+g(\frac{x}{y})+yg'(\frac{x}{y})(-\frac{x}{y^2})=f'(\frac{y}{x})+g(\frac{x}{y})-\frac{x}{y}g'(\frac{x}{y})$,

$\frac{\partial^2 z}{\partial x\partial y}=f''(\frac{y}{x})(-\frac{y}{x^2})+g'(\frac{x}{y})\frac{1}{y}-\frac{1}{y}g'(\frac{x}{y})-\frac{x}{y}g''(\frac{x}{y})\frac{1}{y}=-\frac{y}{x^2}f''(\frac{y}{x})-\frac{x}{y^2}g''(\frac{x}{y})$.

(4)$\frac{\partial z}{\partial x}=\frac{-yf'(x^2-y^2)2x}{[f(x^2-y^2)]^2}=\frac{-2xyf'(x^2-y^2)}{[f(x^2-y^2)]^2}, \frac{\partial z}{\partial y}=\frac{f(x^2-y^2)-yf'(x^2-y^2)(-2y)}{[f(x^2-y^2)]^2}=\frac{f(x^2-y^2)+2y^2f'(x^2-y^2)}{[f(x^2-y^2)]^2}$,

$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{1}{x}\frac{-2xyf'(x^2-y^2)}{[f(x^2-y^2)]^2}+\frac{1}{y}\frac{f(x^2-y^2)+2y^2f'(x^2-y^2)}{[f(x^2-y^2)]^2}=\frac{f(x^2-y^2)}{y[f(x^2-y^2)]^2}=\frac{1}{yf(x^2-y^2)}$.

(5)$\frac{\partial u}{\partial x}=f_1'+f_2'y+f_3'yz=f_1'+yf_2'+yzf_3', \frac{\partial u}{\partial y}=xf_2'+xzf_3', \frac{\partial u}{\partial z}=xyf_3'$.

(6)$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^{x-2y}\cos t+\mathrm{e}^{x-2y}(-2)3t^2=\mathrm{e}^{x-2y}(\cos t-6t^2)$
$=\mathrm{e}^{\sin t-2t^3}(\cos t-6t^2)$.

2. 已知$z=f(x+y^2)$，其中函数f二阶可导，试求$\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial y^2},\frac{\partial^2 z}{\partial x\partial y}$.

解：$\frac{\partial z}{\partial x}=f'(x^2+y^2)2x, \frac{\partial^2 z}{\partial x^2}=2f'(x^2+y^2)+2xf''(x^2+y^2)2x$
$=2f'(x^2+y^2)+4x^2f''(x^2+y^2)$,

$\frac{\partial z}{\partial y}=f'(x^2+y^2)2y, \frac{\partial^2 z}{\partial y^2}=2f'(x^2+y^2)+2yf''(x^2+y^2)2y=2f'(x^2+y^2)+4y^2f''(x^2+y^2)$,

$\frac{\partial^2 z}{\partial x\partial y}=2yf''(x^2+y^2)2x=4xyf''(x^2+y^2)$.

3. 设$z=yf(x^2y,\frac{y}{x})$，其中f具有连续的二阶偏导数，求z''_{xx},z''_{xy}.

解：$z'_x=y[f_1'(x^2y,\frac{y}{x})2xy+f_2'(x^2y,\frac{y}{x})(-\frac{y}{x^2})]=2xy^2f_1'(x^2y,\frac{y}{x})-\frac{y^2}{x^2}f_2'(x^2y,\frac{y}{x})$,
$z''_{xx}=2y^2f_1'+2xy^2[f''_{11}2xy+f''_{12}(-\frac{y}{x^2})]-(-\frac{2y^2}{x^3})f_2'-\frac{y^2}{x^2}[f''_{21}2xy+f''_{22}(-\frac{y}{x^2})]$
$=2y^2f_1'+\frac{2y^2}{x^3}f_2'+4x^2y^3f''_{11}-\frac{4y^3}{x}f''_{12}+\frac{y^3}{x^4}f''_{22}$
$=2y^2f_1'+\frac{2y^2}{x^3}f_2'+4x^2y^3f''_{11}-\frac{4y^3}{x}f''_{12}+\frac{y^3}{x^4}f''_{22}$,

$z''_{xy}=4xyf_1'+2xy^2[f''_{11}x^2+f''_{12}\frac{1}{x}]-\frac{2y}{x^2}f_2'-\frac{y^2}{x^2}[f''_{21}x^2+f''_{22}\frac{1}{x}]$
$=4xyf_1'-\frac{2y}{x^2}f_2'+2x^3y^2f''_{11}+y^2f''_{12}-\frac{y^2}{x^3}f''_{22}$
$=4xyf_1'-\frac{2y}{x^2}f_2'+2x^3y^2f''_{11}+y^2f''_{12}-\frac{y^2}{x^3}f''_{22}$.

4. 设函数f,g有连续导数，令$u=yf(\frac{x}{y})+xg(\frac{y}{x})$，求$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}$.

解：**【该做法应加上f,g有二阶连续导数的条件：】**

$\frac{\partial u}{\partial x}=yf'(\frac{x}{y})\frac{1}{y}+g(\frac{y}{x})+xg'(\frac{y}{x})(-\frac{y}{x^2})=f'(\frac{x}{y})+g(\frac{y}{x})-\frac{y}{x}g'(\frac{y}{x})$,
$\frac{\partial^2 u}{\partial x^2}=f''(\frac{x}{y})\frac{1}{y}+g'(\frac{y}{x})(-\frac{y}{x^2})-(-\frac{y}{x^2})g'(\frac{y}{x})-\frac{y}{x}g''(\frac{y}{x})(-\frac{y}{x^2})=\frac{1}{y}f''(\frac{x}{y})+\frac{y^2}{x^3}g''(\frac{y}{x})$,

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = f''(\frac{x}{y})(-\frac{x}{y^2}) + g'(\frac{y}{x})\frac{1}{x} - \frac{1}{x}g'(\frac{y}{x}) - \frac{y}{x}g''(\frac{y}{x})\frac{1}{x} = -\frac{x}{y^2}f''(\frac{x}{y}) - \frac{y}{x^2}g''(\frac{y}{x})$,

$\therefore x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = x[\frac{1}{y}f''(\frac{x}{y}) + \frac{y^2}{x^3}g''(\frac{y}{x})] + y[-\frac{x}{y^2}f''(\frac{x}{y}) - \frac{y}{x^2}g''(\frac{y}{x})] = 0.$

【正确做法：】$\because f, g$有连续导数，

$\therefore u = yf(\frac{x}{y}) + xg(\frac{y}{x})) \in C^1(\mathbb{R}^2\backslash\{(x,y)|x=0或y=0\})$,

$\therefore$

$$\frac{\partial u}{\partial x} = yf'(\frac{x}{y})\frac{1}{y} + g(\frac{y}{x}) + xg'(\frac{y}{x})(-\frac{y}{x^2}) = g(\frac{y}{x}) + f'(\frac{x}{y}) - \frac{y}{x}g'(\frac{y}{x}),$$
$$\frac{\partial u}{\partial y} = f(\frac{x}{y}) + yf'(\frac{x}{y})(-\frac{x}{y^2}) + xg'(\frac{y}{x})\frac{1}{x} = f(\frac{x}{y}) - \frac{x}{y}f'(\frac{x}{y}) + g'(\frac{y}{x}),$$

$\therefore$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xg(\frac{y}{x}) + yf(\frac{x}{y}) = u \in C^1,$$

$\therefore$

$$\frac{\partial}{\partial x}(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}) - \frac{\partial u}{\partial x} = x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = 0.$$

2.8 习题10.5解答

1. 设$y = y(x), z = z(x)$是由方程$z = xf(x+y)$和$F(x,y,z) = 0$所确定的函数，其中f和F分别具有连续导数和偏导数，求$\frac{\mathrm{d}z}{\mathrm{d}x}$.

   解：方法1：将$z = xf(x+y), F(x,y,z) = 0$两边分别对$x$求导：

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(x+y) + xf'(x+y)[1 + \frac{\mathrm{d}y}{\mathrm{d}x}],$$
$$F'_x + F'_y\frac{\mathrm{d}y}{\mathrm{d}x} + F'_z\frac{\mathrm{d}z}{\mathrm{d}x} = 0$$

   由该方程组解得

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{[f(x+y) + xf'(x+y)]F'_y - xf'(x+y)F'_x}{F'_y + xf'(x+y)F'_z}.$$

   方法2：将$xf(x+y) - z = 0, F(x,y,z) = 0$两边求全微分：

$$[f(x+y) + xf'(x+y)]\mathrm{d}x + xf'(x+y)\mathrm{d}y - \mathrm{d}z = 0,$$
$$F'_x\mathrm{d}x + F'_y\mathrm{d}y + F'_z\mathrm{d}z = 0,$$

   因为$y = y(x), z = z(x)$，将以上方程两边分别除以$\mathrm{d}x$得

$$[f(x+y) + xf'(x+y)] + xf'(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\mathrm{d}z}{\mathrm{d}x} = 0,$$
$$F'_x + F'_y\frac{\mathrm{d}y}{\mathrm{d}x} + F'_z\frac{\mathrm{d}z}{\mathrm{d}x} = 0,$$

   由该方程组解得

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{(f + xf')F'_y - xf'F'_x}{F'_y + xf'F'_z}.$$

2. 设由方程$F(\frac{x}{z},\frac{z}{y})=0$可以确定隐函数$z=z(x,y)$，求$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

解：方法1：将$F(\frac{x}{z},\frac{z}{y})=0$两边分别对$x,y$求偏导：

$$F_1'(\frac{1}{z}+\frac{-x}{z^2}\frac{\partial z}{\partial x})+F_2'\frac{1}{y}\frac{\partial z}{\partial x}=0,$$
$$F_1'\frac{-x}{z^2}\frac{\partial z}{\partial y}+F_2'(\frac{1}{y}\frac{\partial z}{\partial y}+\frac{-z}{y^2})=0,$$

$\therefore$

$$\frac{\partial z}{\partial x}=\frac{\frac{1}{z}F_1'}{\frac{x}{z^2}F_1'-\frac{1}{y}F_2'}=\frac{F_1'}{\frac{x}{z}F_1'-\frac{z}{y}F_2'},$$
$$\frac{\partial z}{\partial y}=\frac{\frac{z}{y^2}F_2'}{\frac{1}{y}F_2'-\frac{x}{z^2}F_1'}=\frac{\frac{z^2}{y^2}F_2'}{\frac{z}{y}F_2'-\frac{x}{z}F_1'}.$$

方法2：将方程$F(\frac{x}{z},\frac{z}{y})=0$两边求全微分：

$$F_1'\frac{1}{z}\mathrm{d}x+F_2'(-\frac{z}{y^2})\mathrm{d}y+[F_1'(-\frac{x}{z^2})+F_2'\frac{1}{y}]\mathrm{d}z=0,$$

即

$$\mathrm{d}z=-\frac{F_1'\frac{1}{z}}{F_1'(-\frac{x}{z^2})+F_2'\frac{1}{y}}\mathrm{d}x-\frac{F_2'(-\frac{z}{y^2})}{F_1'(-\frac{x}{z^2})+F_2'\frac{1}{y}}\mathrm{d}y,$$

$\therefore$

$$\frac{\partial z}{\partial x}=\frac{-F_1'\frac{1}{z}}{F_1'(-\frac{x}{z^2})+F_2'\frac{1}{y}}=\frac{F_1'}{\frac{x}{z}F_1'-\frac{z}{y}F_2'},$$
$$\frac{\partial z}{\partial y}=\frac{-F_2'(-\frac{z}{y^2})}{F_1'(-\frac{x}{z^2})+F_2'\frac{1}{y}}=\frac{\frac{z^2}{y^2}F_2'}{\frac{z}{y}F_2'-\frac{x}{z}F_1'}.$$

3. 证明：方程$F(x+\frac{z}{y},y+\frac{z}{x})=0$所确定的隐函数$z=z(x,y)$满足方程

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy.$$

证明：方法1：

$\because$

$$\mathrm{d}F(x+\frac{z}{y},y+\frac{z}{x})=[F_1'+F_2'(-\frac{z}{x^2})]\mathrm{d}x+[F_1'(\frac{-z}{y^2})+F_2']\mathrm{d}y+(F_1'\frac{1}{y}+F_2'\frac{1}{x})\mathrm{d}z=0,$$

$\therefore$

$$\mathrm{d}z=\frac{F_1'+F_2'(-\frac{z}{x^2})}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}\mathrm{d}x+\frac{F_1'(\frac{-z}{y^2})+F_2'}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}\mathrm{d}y,$$

$\therefore$

$$\frac{\partial z}{\partial x}=\frac{F_1'+F_2'(-\frac{z}{x^2})}{F_1'\frac{1}{y}+F_2'\frac{1}{x}},\frac{\partial z}{\partial y}=\frac{F_1'(\frac{-z}{y^2})+F_2'}{F_1'\frac{1}{y}+F_2'\frac{1}{x}},$$

$\therefore$

$$\begin{aligned}x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}&=x\frac{F_1'+F_2'(-\frac{z}{x^2})}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}+y\frac{F_1'(\frac{-z}{y^2})+F_2'}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}=\frac{xF_1'+F_2'(-\frac{z}{x})+F_1'(\frac{-z}{y})+yF_2'}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}\\&=\frac{(x-\frac{z}{y})F_1'+(y-\frac{z}{x})F_2'}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}=(xy-z)\frac{\frac{1}{y}F_1'+\frac{1}{x}F_2'}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}\\&=xy-z.\end{aligned}$$

方法2：将方程$F(x+\frac{z}{y},y+\frac{z}{x})=0$两边分别对$x,y$求偏导：

$$\begin{cases}F_1'(1+\frac{1}{y}\frac{\partial z}{\partial x})+F_2'(\frac{-z}{x^2}+\frac{1}{x}\frac{\partial z}{\partial x})=0,\\F_1'(\frac{-z}{y^2}+\frac{1}{y}\frac{\partial z}{\partial y})+F_2'(1+\frac{1}{x}\frac{\partial z}{\partial y})=0,\end{cases}$$

这是一个关于F_1',F_2'的齐次方程组，要使该方程组有非零解，则必须

$$\begin{vmatrix}1+\frac{1}{y}\frac{\partial z}{\partial x} & \frac{-z}{x^2}+\frac{1}{x}\frac{\partial z}{\partial x}\\ \frac{-z}{y^2}+\frac{1}{y}\frac{\partial z}{\partial y} & 1+\frac{1}{x}\frac{\partial z}{\partial y}\end{vmatrix}=0,$$

即

$$(1+\frac{1}{y}\frac{\partial z}{\partial x})(1+\frac{1}{x}\frac{\partial z}{\partial y})-(\frac{-z}{x^2}+\frac{1}{x}\frac{\partial z}{\partial x})(\frac{-z}{y^2}+\frac{1}{y}\frac{\partial z}{\partial y})=0,$$

整理得

$$\begin{aligned}&1+\frac{1}{y}\frac{\partial z}{\partial x}+\frac{1}{x}\frac{\partial z}{\partial y}-\frac{z^2}{x^2y^2}+\frac{z}{x^2y}\frac{\partial z}{\partial y}+\frac{z}{xy^2}\frac{\partial z}{\partial x}\\=&1-\frac{z^2}{x^2y^2}+\frac{xy+z}{xy^2}\frac{\partial z}{\partial x}+\frac{xy+z}{x^2y}\frac{\partial z}{\partial y}\\=&\frac{(xy+z)(xy-z)}{x^2y^2}+\frac{xy+z}{xy^2}\frac{\partial z}{\partial x}+\frac{xy+z}{x^2y}\frac{\partial z}{\partial y}\\=&\frac{xy+z}{x^2y^2}[(xy-z)+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}]\\=&0,\end{aligned}$$

$\therefore xy+z=0$或$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy$, $\because xy+z=0$也满足$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy$,

故$z=z(x,y)$满足方程$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy$.

4. 设$z=f(u)$，且$u=u(x,y)$满足$u=\varphi(u)+\int_y^x p(t)\mathrm{d}t$(其中$f$可导，$\varphi\in C^1$，且$\varphi'(u)\neq 1,p\in C$). 求证：$p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}=0$.

证明：
$$\frac{\partial z}{\partial x}=f'(u)\frac{\partial u}{\partial x},\frac{\partial z}{\partial y}=f'(u)\frac{\partial u}{\partial y},$$
$\because$
$$u=\varphi(u)+\int_y^x p(t)\mathrm{d}t=\varphi(u)+\int_0^x p(t)\mathrm{d}t+\int_y^0 p(t)\mathrm{d}t,$$
$\therefore$
$$\frac{\partial u}{\partial x}=\varphi'(u)\frac{\partial u}{\partial x}+p(x),\frac{\partial u}{\partial y}=\varphi'(u)\frac{\partial u}{\partial y}-p(y),(*)$$
$\because \varphi'(u)\neq 1,$

$\therefore$
$$\frac{\partial u}{\partial x}=\frac{p(x)}{1-\varphi'(u)},\frac{\partial u}{\partial y}=\frac{-p(y)}{1-\varphi'(u)},$$
$\therefore$
$$\frac{\partial z}{\partial x}=f'(u)\frac{\partial u}{\partial x}=f'(u)\frac{p(x)}{1-\varphi'(u)},$$
$$\frac{\partial z}{\partial y}=f'(u)\frac{\partial u}{\partial y}=f'(u)\frac{-p(y)}{1-\varphi'(u)},$$
$\therefore$
$$\begin{aligned}&p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}\\=&p(y)f'(u)\frac{p(x)}{1-\varphi'(u)}+p(x)f'(u)\frac{-p(y)}{1-\varphi'(u)}\\=&0.\end{aligned}$$

5. 已知方程$F(x+y,y+z)=1$确定了隐函数$z=z(x,y)$，其中F具有连续的二阶偏导数，求$\frac{\partial^2 z}{\partial y\partial x}$.

解：方法1：将方程$F(x+y,y+z)=1$两边对x求偏导：
$$F_1'(x+y,y+z)+F_2'(x+y,y+z)\frac{\partial z}{\partial x}=0,$$
得
$$\frac{\partial z}{\partial x}=-\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)},$$
$\therefore$
$$\frac{\partial^2 z}{\partial y\partial x}=-\frac{[F_{11}''+F_{12}''(1+\frac{\partial z}{\partial y})]F_2'-F_1'[F_{21}''+F_{22}''(1+\frac{\partial z}{\partial y})]}{(F_2')^2},$$
方程$F(x+y,y+z)=1$两边对y求偏导：
$$F_1'(x+y,y+z)+F_2'(x+y,y+z)(1+\frac{\partial z}{\partial y})=0,$$

得

$$1+\frac{\partial z}{\partial y}=-\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)},$$

∴

$$\begin{aligned}\frac{\partial^2 z}{\partial y\partial x}&=-\frac{[F_{11}''+F_{12}''(-\frac{F_1'}{F_2'})]F_2'-F_1'[F_{21}''+F_{22}''(-\frac{F_1'}{F_2'})]}{(F_2')^2}\\&=-\frac{(F_2')^2F_{11}''-F_1'F_2'F_{12}''-F_1'F_2'F_{21}''+(F_1')^2F_{22}''}{(F_2')^3}\\&=\frac{-(F_2')^2F_{11}''+2F_1'F_2'F_{12}''-(F_1')^2F_{22}''}{(F_2')^3}.\end{aligned}$$

方法2：方程$F(x+y,y+z)=1$两边求全微分：

$$\mathrm{d}F(x+y,y+z)=F_1'(x+y,y+z)\mathrm{d}x+[F_1'(x+y,y+z)+F_2'(x+y,y+z)]\mathrm{d}y+F_2'(x+y,y+z)\mathrm{d}z=0,$$

即

$$\begin{aligned}\mathrm{d}z&=-\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)}\mathrm{d}x-\frac{F_1'(x+y,y+z)+F_2'(x+y,y+z)}{F_2'(x+y,y+z)}\mathrm{d}z\\&=-\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)}\mathrm{d}x-[1+\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)}]\mathrm{d}z,\end{aligned}$$

∴

$$\frac{\partial z}{\partial x}=-\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)},$$

∴

$$\frac{\partial^2 z}{\partial y\partial x}=-\frac{[F_{11}''+F_{12}''(1+\frac{\partial z}{\partial y})]F_2'-F_1'[F_{21}''+F_{22}''(1+\frac{\partial z}{\partial y})]}{(F_2')^2},$$

∵

$$\frac{\partial z}{\partial y}=-1-\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)},$$

∴

$$\begin{aligned}\frac{\partial^2 z}{\partial y\partial x}&=-\frac{[F_{11}''+F_{12}''(-\frac{F_1'}{F_2'})]F_2'-F_1'[F_{21}''+F_{22}''(-\frac{F_1'}{F_2'})]}{(F_2')^2}\\&=-\frac{(F_2')^2F_{11}''-F_1'F_2'F_{12}''-F_1'F_2'F_{21}''+(F_1')^2F_{22}''}{(F_2')^3}\\&=\frac{-(F_2')^2F_{11}''+2F_1'F_2'F_{12}''-(F_1')^2F_{22}''}{(F_2')^3}.\end{aligned}$$

6. 设方程组$\begin{cases}x^2+y^2+z^2=3x,\\2x-3y+5z=4,\end{cases}$ 确定y与z是x的函数，求$\frac{\mathrm{d}y}{\mathrm{d}x},\frac{\mathrm{d}z}{\mathrm{d}x}$.

解：方法1：将方程组的两个方程两边分别对x求导：

$$\begin{cases} 2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}+2z\frac{\mathrm{d}z}{\mathrm{d}x}=3, \\ 2-3\frac{\mathrm{d}y}{\mathrm{d}x}+5\frac{\mathrm{d}z}{\mathrm{d}x}=0, \end{cases}$$

可解得 $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\begin{vmatrix} 3-2x & 2z \\ -2 & 5 \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ -3 & 5 \end{vmatrix}}=\frac{15-10x+4z}{10y+6z}$, $\frac{\mathrm{d}z}{\mathrm{d}y}=\frac{\begin{vmatrix} 2y & 3-2x \\ -3 & -2 \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ -3 & 5 \end{vmatrix}}=\frac{-4y+9-6x}{10y+6z}$.

方法2：将方程组的两个方程两边分别求全微分：

$$\begin{cases} 2\mathrm{d}x+2y\mathrm{d}y+2z\mathrm{d}z=3, \\ 2\mathrm{d}x-3\mathrm{d}y+5\mathrm{d}z=0, \end{cases}$$

$\because y$与z是x的函数

$$\therefore \begin{cases} 2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}+2z\frac{\mathrm{d}z}{\mathrm{d}x}=3, \\ 2-3\frac{\mathrm{d}y}{\mathrm{d}x}+5\frac{\mathrm{d}z}{\mathrm{d}x}=0, \end{cases}$$

可解得 $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\begin{vmatrix} 3-2x & 2z \\ -2 & 5 \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ -3 & 5 \end{vmatrix}}=\frac{15-10x+4z}{10y+6z}$, $\frac{\mathrm{d}z}{\mathrm{d}y}=\frac{\begin{vmatrix} 2y & 3-2x \\ -3 & -2 \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ -3 & 5 \end{vmatrix}}=\frac{-4y+9-6x}{10y+6z}$.