14 多元函数微分学(1)

14.1 有关说明

- 基础习题课时间：周一第六大节, 地点: 新水325
- 基础习题课公共答疑时间：周二下午三点半到五点，地点：理科楼B203
- 基础习题课助教：赵东阳，电话：18811708556，邮箱：dy-zhao14@mails.tsinghua.edu.cn，微信: 18811708556
- 基础习题课的教学目标:
  - 使同学掌握课程基本内容
  - 使同学掌握常见问题的一般解法
  - 使同学学会正确地书写解答过程
- 基础习题课的主要内容：课后习题
- 其他要求和说明:
  - 每次课前签到（只是为了统计，不是为了考勤）
  - 上课的同学可以加入基础习题课微信群
  - 基础习题课的题目比较基础，注重一般解法和解答过程的书写，如果觉得题目简单可有选择地自愿参加
  - 这是我做微积分B基础习题课助教的第二个学期，这个学期的情况可能与上个学期不同，我仍然处在一个探索的阶段，有些问题可能会解释不清楚，有些环节的安排可能会不太合理，有些方面可能会考虑不到. 大家如果觉得听不懂，或者觉得什么环节安排得不合理，或者觉得我有什么地方没有考虑到，可以直接指出
  - 有问题可随时在微信上给我留言，也可以随时给我打电话，发短信，发邮件

14.2 知识结构

第10章多元函数微分学

10.1 多元连续函数

10.1.1 多元函数概念

10.1.2 二元函数的图形和等值线

14.3 习题10.1解答

1. 求下列二元函数的定义域.

   (1)$f(x,y)=\sqrt{x}\ln(x+y)$； (2)$f(x,y)=\ln(y-x^2)$；

   (3)$f(x,y)=\frac{e^{\frac{x}{y}}}{x-y}$； (4)$f(x,y)=\arcsin\frac{x}{y}$.

   解：(1)由$x\geq 0, x+y>0$得该函数的定义域为$\{(x,y)\mid x\geq 0且x+y>0\}$.

   (2)由$y-x^2>0$得该函数的定义域为$\{(x,y)\mid y>x^2\}$.

   (3)由$y\neq 0$且$x-y^2\neq 0$得该函数的定义域为$\{(x,y)\mid x\neq y^2且y\neq 0\}$.

   (4)由$y\neq 0$且$-1\leq\frac{x}{y}\leq 1$得该函数的定义域为$\left\{(x,y)\;\middle|\; y\neq 0且-1\leq\frac{x}{y}\leq 1\right\}$.

2. 下列函数在$(0,0)$点的极限是否存在？若存在请求其值.

   (1)$f(x,y)=\frac{x+y}{|x|+|y|}$； (2)$f(x,y)=\frac{x^2+y^2}{|x|+|y|}$；

   (3)$f(x,y)=\frac{\sin(x^2+y^2)}{x^2+y^2}$； (4)$f(x,y)=\frac{1-\cos(xy)}{x^2+y^2}$.

   解：(1)当点(x,y)在第一象限沿直线$y=x$趋于$(0,0)$时$\lim\limits_{\substack{x\to 0^+\\ y\to 0^+}}f(x,y)=\lim\limits_{x\to 0^+}\frac{2x}{2|x|}=1$，

   当点(x,y)在第二象限沿直线$y=-x$趋于$(0,0)$时$\lim\limits_{\substack{x\to 0^-\\ y\to 0^+}}f(x,y)=\lim\limits_{x\to 0^-}\frac{x-x}{2|x|}=0\neq 1$，

故该函数在$(0,0)$点处的极限不存在.

(2)$\because \forall (x,y)\neq(0,0), 0<x^2+y^2\leq x^2+y^2+2|x||y|=(|x|+|y|)^2\leq 2(x^2+y^2)$

$\therefore \left|\frac{x^2+y^2}{|x|+|y|}-0\right|\leq \frac{(|x|+|y|)^2}{|x|+|y|}\leq |x|+|y|\leq \sqrt{2(x^2+y^2)}$

$\therefore \forall \varepsilon>0$可取$\delta=\frac{1}{\sqrt{2}}\varepsilon$，当$d((x,y),(0,0))=\sqrt{x^2+y^2}<\delta=\frac{1}{\sqrt{2}}\varepsilon$时

$\left|\frac{x^2+y^2}{|x|+|y|}-0\right|\leq \sqrt{2(x^2+y^2)}<\varepsilon$

$\therefore \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=0.$

(3)方法1：$\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{\sin(x^2+y^2)}{x^2+y^2}=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^2+y^2}{x^2+y^2}=1.$

方法2：$\because \forall (x,y)\neq(0,0)$(不妨设$0<x^2+y^2<\frac{\pi}{2}$)$, 0<\sin(x^2+y^2)<x^2+y^2<\tan(x^2+y^2)$

$\therefore 1<\frac{x^2+y^2}{\sin(x^2+y^2)}<\frac{1}{\cos(x^2+y^2)}$，即$\cos(x^2+y^2)<\frac{\sin(x^2+y^2)}{x^2+y^2}<1$

$\because \lim\limits_{\substack{x\to 0\\ y\to 0}}\cos(x^2+y^2)=1$

$\therefore \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=1.$

(4)方法1：$\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{1-\cos(xy)}{x^2+y^2}=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{2\sin^2(xy)}{x^2+y^2}=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{2(xy)^2}{x^2+y^2}=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{1}{\frac{1}{x^2}+\frac{1}{y^2}}=0.$

方法2：$\because \left|\frac{1-\cos(xy)}{x^2+y^2}-0\right|=\frac{2\sin^2(xy)}{x^2+y^2}\leq\frac{2(xy)^2}{x^2+y^2}\leq\frac{2[\frac{1}{2}(x^2+y^2)]^2}{x^2+y^2}=\frac{1}{2}(x^2+y^2),$

$\therefore \forall\varepsilon>0$取$\boldsymbol{\delta=\sqrt{2\varepsilon}}$[1]，当$d((x,y),(0,0))=\sqrt{x^2+y^2}<\delta=\sqrt{2\varepsilon}$时$\left|\frac{1-\cos(xy)}{x^2+y^2}-0\right|$
$\leq\frac{1}{2}(x^2+y^2)<\varepsilon,$

$\therefore \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=0.$

3. 设P_0是$\mathbb{R}^2$中一个确定点，在$\mathbb{R}^2$上定义函数$f(P)=d(P,P_0)$，求证这是一个连续函数.

证明：方法1：$\because \forall Q\in\mathbb{R}^2, |d(P,P_0)-d(Q,P_0)|<d(P,Q)$

$\therefore \forall\varepsilon>0$取$\delta=\varepsilon$，当$d(P,Q)<\delta=\varepsilon$时$|d(P,P_0)-d(Q,P_0)|<d(P,Q)<\varepsilon$

$\therefore \lim\limits_{P\to Q} f(P)=f(Q)$

$\therefore$函数$f(P)$是一个连续函数.

方法2：$\because \forall Q\in\mathbb{R}^2, d(Q,P_0)-d(Q,P)\leq d(P,P_0)\leq d(Q,P_0)+d(Q,P)$

当$P\to Q$时$\lim\limits_{P\to Q} d(Q,P)=0$，根据夹逼定理$\lim\limits_{P\to Q} d(P,P_0)=d(Q,P_0)$，即$\lim\limits_{P\to Q} f(P)=f(Q)$

$\therefore$函数$f(P)$是一个连续函数.

[1]这里由2ε改成了$\sqrt{2\varepsilon}$.

14.4 习题10.2解答

1. 若$f(x,y)$在点(x,y)处连续，能否推出$f(x,y)$在点(x,y)的两个偏导数存在？若$f(x,y)$在点(x,y)的两个偏导数都存在，能否推出$f(x,y)$在点(x,y)处连续？

   解：(1)不能. 如函数$f(x,y)=\sqrt{x^2+y^2}$在原点连续，但是下列两个极限都不存在：

   $$\begin{aligned}\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}&=\lim_{x\to 0}\frac{\sqrt{x^2}}{x},\\ \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}&=\lim_{y\to 0}\frac{\sqrt{y^2}}{y}.\end{aligned}$$

   所以在原点$\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$都不存在。

   (2)不能. 如函数$f(x,y)=\begin{cases}1, & y=x^2,x>0,\\ 0, & 其他.\end{cases}$ 因为$f(x,0)\equiv 0, f(0,y)\equiv 0$，故$f(x,y)$在原点的两个偏导数$\frac{\partial f(0,0)}{\partial x}=0,\frac{\partial f(0,0)}{\partial y}=0$. 但$\lim\limits_{(x,y)\to(0,0)}f(x,y)$不存在（参见教材例10.1.2），所以该函数在原点不连续.

2. 设$z=\sqrt{|xy|}$,求$\frac{\partial z}{\partial x}$.

   解：$\because z=\sqrt{|xy|}=\sqrt{|y|}\sqrt{|x|}=\begin{cases}\sqrt{|y|}\sqrt{x}, & x\geq 0,\\ \sqrt{|y|}\sqrt{-x}, & x<0.\end{cases}$

   $\therefore$当$x>0$时，$\frac{\partial z}{\partial x}=\frac{\sqrt{|y|}}{2\sqrt{x}}$,

   当$x<0$时，$\frac{\partial z}{\partial x}=-\frac{\sqrt{|y|}}{2\sqrt{-x}}$,

   当$x=0$时$\lim\limits_{x\to 0^-}\frac{\sqrt{|xy|}}{x}=\lim\limits_{x\to 0^-}-\frac{\sqrt{-x|y|}}{-x}=\lim\limits_{x\to 0^-}-\frac{\sqrt{|y|}}{\sqrt{-x}}=\begin{cases}-\infty, & y\neq 0\\ 0, & y=0\end{cases}$,

   $\lim\limits_{x\to 0^+}\frac{\sqrt{|xy|}}{x}=\lim\limits_{x\to 0^+}\frac{x|y|}{x}=\lim\limits_{x\to 0^+}\frac{\sqrt{|y|}}{\sqrt{x}}=\begin{cases}+\infty, & y\neq 0\\ 0, & y=0\end{cases}$.

   $$\therefore \frac{\partial z}{\partial x}=\begin{cases}\frac{\sqrt{|y|}}{2\sqrt{x}}, & x>0,\\ 不存在, & x=0且y\neq 0\\ 0, & x=0,y=0\\ -\frac{\sqrt{|y|}}{2\sqrt{-x}}, & x<0.\end{cases}$$

3. 求下列偏导数：

   (1)$z=\frac{x+y}{x-y}$，求$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$；

   (2)$f(x,y)=\arctan\frac{y}{x}$，求$\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$；

$(3)z=\cos\frac{y}{x}\sin\frac{x}{y}$，求$\frac{\partial z(2,\pi)}{\partial x},\frac{\partial z(2,\pi)}{\partial y}$；
$(4)z=\arcsin\sqrt{\frac{x}{y}}+\frac{1}{xy}e^{\frac{y}{x}}$，求$\frac{\partial z(1,2)}{\partial x},\frac{\partial z(1,2)}{\partial y}$；
$(5)z=\ln(\sqrt{x}+\sqrt{y})$，求$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$；
$(6)z=\frac{x-y}{x+y}\ln\frac{y}{x}$，求$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$；
$(7)u=\sqrt{x^2+y^2+z^2}$，求$(\frac{\partial u}{\partial x})^{+}(\frac{\partial u}{\partial y})^2+(\frac{\partial u}{\partial z})^2$.

解：$(1)\frac{\partial z}{\partial x}=\frac{x-y-(x+y)}{(x-y)^2}=\frac{-2y}{(x-y)^2},\frac{\partial z}{\partial y}=\frac{x-y+(x+y)}{(x-y)^2}=\frac{2x}{(x-y)^2}$.

$(2)\frac{\partial f}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\frac{-y}{x^2}=\frac{-y}{x^2+y^2},\frac{\partial f}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x}=\frac{x}{x^2+y^2}(x\neq 0)$.

$(3)\frac{\partial z}{\partial x}=-(-\frac{y}{x^2})\sin\frac{y}{x}\sin\frac{x}{y}+\frac{1}{y}\cos\frac{y}{x}\cos\frac{x}{y}=\frac{y}{x^2}\sin\frac{y}{x}\sin\frac{x}{y}+\frac{1}{y}\cos\frac{y}{x}\cos\frac{x}{y}$,
$\frac{\partial z}{\partial y}=-\frac{1}{x}\sin\frac{y}{x}\sin\frac{x}{y}-\frac{x}{y^2}\cos\frac{y}{x}\cos\frac{x}{y}$,

$\therefore\frac{\partial z(2,\pi)}{\partial x}=\frac{\pi}{4}\sin\frac{\pi}{2}\sin\frac{2}{\pi}+\frac{1}{\pi}\cos\frac{\pi}{2}\cos\frac{2}{\pi}=\frac{\pi}{4}\sin\frac{2}{\pi}$,
$\frac{\partial z(2,\pi)}{\partial y}=-\frac{1}{2}\sin\frac{\pi}{2}\sin\frac{2}{\pi}-\frac{2}{\pi^2}\cos\frac{\pi}{2}\cos 2\pi=-\frac{1}{2}\sin\pi 2$.

$(4)\because\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\frac{x}{y}}}\frac{1}{2\sqrt{xy}}-\frac{1}{x^2y}e^{\frac{y}{x}}+\frac{1}{xy}e^{\frac{y}{x}}(-\frac{y}{x^2})=\frac{1}{2\sqrt{xy-x^2}}-(\frac{1}{x^2y}+\frac{1}{x^3})e^{\frac{y}{x}}$,
$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\frac{x}{y}}}(-\frac{1}{2}\sqrt{\frac{x}{y^3}})-\frac{1}{xy^2}e^{\frac{y}{x}}+\frac{1}{xy}e^{\frac{y}{x}}\frac{1}{x}=-\frac{1}{2}\sqrt{\frac{x}{y^3-xy^2}}+(\frac{1}{x^2y}-\frac{1}{xy^2})e^{\frac{y}{x}}$,

$\therefore\frac{\partial z(1,2)}{\partial x}=\frac{1}{2\sqrt{2-1}}-(\frac{1}{2}+1)e^2=\frac{1}{2}-\frac{3}{2}e^2,\frac{\partial z(1,2)}{\partial y}=-\frac{1}{2}\sqrt{\frac{1}{8-4}}+(\frac{1}{2}-\frac{1}{4})e^2=-\frac{1}{4}+\frac{1}{4}e^2$.

$(5)\because\frac{\partial z}{\partial x}=\frac{1}{\sqrt{x}+\sqrt{y}}\frac{1}{2\sqrt{x}},\frac{\partial z}{\partial y}=\frac{1}{\sqrt{x}+\sqrt{y}}\frac{1}{2\sqrt{y}}$,

$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{\sqrt{x}}{2(\sqrt{x}+\sqrt{y})}+\frac{\sqrt{y}}{2(\sqrt{x}+\sqrt{y})}=\frac{1}{2}$.

$(6)\because\frac{\partial z}{\partial x}=\frac{x+y-(x-y)}{(x+y)^2}\ln\frac{y}{x}+\frac{x-y}{x+y}\frac{x}{y}(-\frac{y}{x^2})=\frac{2y}{(x+y)^2}\ln\frac{y}{x}-\frac{1}{x}\frac{x-y}{x+y}$,
$\frac{\partial z}{\partial y}=\frac{-(x+y)-(x-y)}{(x+y)^2}\ln\frac{y}{x}+\frac{x-y}{x+y}\frac{x}{y}\frac{1}{x}=\frac{-2x}{(x+y)^2}\ln\frac{y}{x}+\frac{1}{y}\frac{x-y}{x+y}$,

$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{2xy}{(x+y)^2}\ln\frac{y}{x}-\frac{x-y}{x+y}+\frac{-2xy}{(x+y)^2}\ln\frac{y}{x}+\frac{x-y}{x+y}=0$.

$(7)\because\frac{\partial u}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2+z^2}}=\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{\partial u}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{\partial u}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}$(后两式可根据$x,y,z$的对称性得到),

$\therefore(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2+(\frac{\partial u}{\partial z})^2=\frac{x^2+y^2+z^2}{x^2+y^2+z^2}=1$.

4. 求下列高阶导数：

$(1)z=x+y+\frac{1}{xy}$，求$\frac{\partial^2 z(1,1)}{\partial x\partial y}$；
$(2)z=y^{\ln x}$，求$\frac{\partial^2 z}{\partial x\partial y}$；
$(3)z=\ln(x+\sqrt{x^2+y^2})$，求$\frac{\partial^2 z}{\partial x\partial y}$；
$(4)z=\ln(\sqrt{(x-a)^2+(y-b)^2})$，求$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$；
$(5)u=\sqrt{x^2+y^2+z^2}$，求$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$；
$(6)z=\sin(xy)$，求$\frac{\partial^3 z}{\partial x\partial y^2}$；
$(7)f(x,y,z)=xy^2+yz^2+zx^2$，求$\frac{\partial^2 f(0,0,1)}{\partial x^2},\frac{\partial^2 f(1,0,2)}{\partial x\partial z},\frac{\partial^2 f(0,-1,0)}{\partial y\partial z},\frac{\partial^3 f(2,0,1)}{\partial x\partial z^2}$.

解：$(1)\because\frac{\partial z}{\partial y}=1-\frac{1}{xy^2},\frac{\partial^2 z}{\partial x\partial y}=\frac{1}{x^2y^2},\therefore\frac{\partial^2 z(1,1)}{\partial x\partial y}=1$.

(2)$\frac{\partial z}{\partial y}=y^{\ln x-1}\ln x,\frac{\partial^2 z}{\partial x\partial y}=y^{\ln x-1}\ln y\frac{1}{x}\ln x+y^{\ln x-1}\frac{1}{x}=\frac{y^{\ln x}}{xy}(\ln y\ln x+1).$

(3)$\frac{\partial z}{\partial y}=\frac{\frac{2y}{2\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}=\frac{y}{\sqrt{x^2+y^2}(x+\sqrt{x^2+y^2})},$

$\frac{\partial^2 z}{\partial x\partial y}=\frac{-y[\frac{2x}{2\sqrt{x^2+y^2}}(x+\sqrt{x^2+y^2})+\sqrt{x^2+y^2}(1+\frac{2x}{2\sqrt{x^2+y^2}})]}{(x^2+y^2)(x+\sqrt{x^2+y^2})^2}=\frac{-y(\frac{x}{\sqrt{x^2+y^2}}+1)(x+\sqrt{x^2+y^2})}{(x^2+y^2)(x+\sqrt{x^2+y^2})^2}$

$=\frac{-y\frac{1}{\sqrt{x^2+y^2}}(x+\sqrt{x^2+y^2})^2}{(x^2+y^2)(x+\sqrt{x^2+y^2})^2}=-\frac{y}{(x^2+y^2)^{\frac{3}{2}}}.$

(4)$\frac{\partial z}{\partial x}=\frac{2(x-a)}{2\sqrt{(x-a)^2+(y-b)^2}}=\frac{x-a}{\sqrt{(x-a)^2+(y-b)^2}},\frac{\partial z}{\partial y}=\frac{2(y-b)}{2\sqrt{(x-a)^2+(y-b)^2}}=\frac{y-b}{\sqrt{(x-a)^2+(y-b)^2}},$

$\frac{\partial^2 z}{\partial x^2}=\frac{\sqrt{(x-a)^2+(y-b)^2}-(x-a)\frac{2(x-a)}{2\sqrt{(x-a)^2+(y-b)^2}}}{(x-a)^2+(y-b)^2}=\frac{(y-b)^2}{[(x-a)^2+(y-b)^2]^{\frac{3}{2}}},$

$\frac{\partial^2 z}{\partial y^2}=\frac{\sqrt{(x-a)^2+(y-b)^2}-(y-b)\frac{2(y-b)}{2\sqrt{(x-a)^2+(y-b)^2}}}{(x-a)^2+(y-b)^2}=\frac{(x-a)^2}{[(x-a)^2+(y-b)^2]^{\frac{3}{2}}},$

$\therefore\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{(x-a)^2+(y-b)^2}{[(x-a)^2+(y-b)^2]^{\frac{3}{2}}}=\frac{1}{\sqrt{(x-a)^2+(y-b)^2}}.$

(5)$\frac{\partial u}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2+z^2}}=\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{\partial^2 u}{\partial x^2}=\frac{\sqrt{x^2+y^2+z^2}-x\frac{2x}{2\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}=\frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},$

根据x,y,z的对称性可得$\frac{\partial^2 u}{\partial y^2}=\frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},\frac{\partial^2 u}{\partial z^2}=\frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},$

$\therefore\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\frac{2}{\sqrt{x^2+y^2+z^2}}.$

(6)$\frac{\partial z}{\partial y}=x\cos(xy),\frac{\partial^2 z}{\partial y^2}=-x^2\sin(xy),\frac{\partial^3 z}{\partial x\partial y^2}=-2x\sin(xy)-x^2y\cos(xy).$

(7)$\because\frac{\partial f}{\partial x}=y^2+2zx,\frac{\partial^2 f}{\partial x^2}=2z,\therefore\frac{\partial^2 f(0,0,1)}{\partial x^2}=2,$

$\because\frac{\partial f}{\partial z}=2yz+x^2,\frac{\partial^2 f}{\partial x\partial z}=2x,\therefore\frac{\partial^2 f(1,0,2)}{\partial x\partial z}=2,$

$\because\frac{\partial^2 f}{\partial y\partial z}=2z,\therefore\frac{\partial^2 f(0,-1,0)}{\partial y\partial z}=0,$

$\because\frac{\partial^2 f}{\partial z^2}=2y,\frac{\partial^3 f}{\partial x\partial z^2}=0,\therefore\frac{\partial^3 f(2,0,1)}{\partial x\partial z^2}=0.$

14.5 习题10.3解答

1. 求下列函数在指定点的全微分：

(1)$z=\arctan\frac{x+y}{x-y}$，在任意点(x,y)；

(2)$z=\ln\sqrt{1+x^2+y^2}$，在点$(1,1)$；

(3)$z=\mathrm{e}^{-(\frac{y}{x}-\frac{x}{y})}$，在点$(1,-1)$；

(4)$z=\arctan\frac{x}{1+y^2}$，求$\mathrm{d}z(1,1)$；

(5)$u=(\frac{x}{y})^z$，在任一点(x,y,z).

解：(1)$\frac{\partial z}{\partial x}=\frac{1}{1+(\frac{x+y}{x-y})^2}\frac{x-y-(x+y)}{(x-y)^2}=\frac{-2y}{(x+y)^2+(x-y)^2}=\frac{-y}{x^2+y^2},$

$\frac{\partial z}{\partial y}=\frac{1}{1+(\frac{x+y}{x-y})^2}\frac{x-y+(x+y)}{(x-y)^2}=\frac{x}{x^2+y^2},$

当$x\neq y$时，$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$均连续，故函数$z$在任意点$(x,y)$可微，

$\mathrm{d}z(x,y)=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=\frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}$.

(2)$\frac{\partial z}{\partial x}=\frac{\partial\frac{1}{2}\ln(1+x^2+y^2)}{\partial x}=\frac{1}{2}\frac{2x}{1+x^2+y^2}=\frac{x}{1+x^2+y^2},\frac{\partial z}{\partial y}=\frac{y}{1+x^2+y^2}$,

因为$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$在点$(1,1)$**及其附近存在且在点**$(1,1)$[2]处连续，故函数$z$在点$(1,1)$处可微，

$\mathrm{d}z(1,1)=\frac{\partial z(1,1)}{\partial x}\mathrm{d}x+\frac{\partial z(1,1)}{\partial y}\mathrm{d}y=\frac{1}{3}(\mathrm{d}x+\mathrm{d}y)$.

(3)$\frac{\partial z}{\partial x}=\mathrm{e}^{-(\frac{y}{x}-\frac{x}{y})}[-(-\frac{y}{x^2}-\frac{1}{y})]=\mathrm{e}^{-(\frac{y}{x}-\frac{x}{y})}(\frac{y}{x^2}+\frac{1}{y})],\frac{\partial z}{\partial y}=\mathrm{e}^{-(\frac{y}{x}-\frac{x}{y})}(-\frac{1}{x}-\frac{x}{y^2})$,

因为$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$在点$(1,-1)$**及其附近存在且在点**$(1,-1)$处连续，故函数$z$在点$(1,-1)$可微，

$\mathrm{d}z(1,-1)=\frac{\partial z(1,-1)}{\partial x}\mathrm{d}x+\frac{\partial z(1,-1)}{\partial y}\mathrm{d}y=-2\mathrm{d}x-2\mathrm{d}y$.

(4)$\frac{\partial z}{\partial x}=\frac{1}{1+(\frac{x}{1+y^2})^2}\frac{1}{1+y^2}=\frac{1+y^2}{x^2+(1+y^2)^2},\frac{\partial z}{\partial x}=\frac{1}{1+(\frac{x}{1+y^2})^2}\frac{-2xy}{(1+y^2)^2}=\frac{-2xy}{x^2+(1+y^2)^2}$,

因为$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$在点$(1,1)$**及其附近存在且在点**$(1,1)$处连续，故函数$z$在点$(1,1)$可微，

$\mathrm{d}z(1,1)=\frac{\partial z(1,1)}{\partial x}\mathrm{d}x+\frac{\partial z(1,1)}{\partial y}\mathrm{d}y=\frac{2}{5}\mathrm{d}x-\frac{2}{5}\mathrm{d}y$.

(5)$\frac{\partial u}{\partial x}=\frac{z}{y}(\frac{x}{y})^{z-1},\frac{\partial u}{\partial y}=-\frac{z}{x}(\frac{y}{x})^{-z-1},\frac{\partial u}{\partial z}=(\frac{x}{y})^z\ln\frac{x}{y}$,

当$xy>0$时，$\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$均连续，故函数$z$在任一点$(x,y,z)$可微，

$\mathrm{d}z=\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y+\frac{\partial u}{\partial z}\mathrm{d}z=\frac{z}{y}(\frac{x}{y})^{z-1}\mathrm{d}x-\frac{z}{x}(\frac{y}{x})^{-z-1}\mathrm{d}y+(\frac{x}{y})^z\ln\frac{x}{y}\mathrm{d}z$.

2. 试证明下列函数在$(0,0)$点不可微：

(1)$f(x,y)=\sqrt{x}\cos y$;

$$(2)f(x,y)=\begin{cases}\frac{2xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

解：(1) **【不太好的做法：】**$\because\frac{\partial f}{\partial x}=\frac{\cos y}{2\sqrt{x}},\frac{\partial f}{\partial y}=-\sqrt{x}\sin y$,

$\therefore\frac{\partial f}{\partial x}$在点$(0,0)$不存在，故函数$f(x,y)$在点$(0,0)$不可微.

【比较好的做法：】假设$f(x,y)$在点$(0,0)$处可微，则$\frac{\partial f(0,0)}{\partial x}$存在,

$\because f(x,y)=\sqrt{x}\cos y$,

$\therefore x\geq 0$,

$\because\lim\limits_{x\to0^+}\frac{f(x,0)-f(0,0)}{x-0}=\lim\limits_{x\to0^+}\frac{\sqrt{x}-0}{x-0}=+\infty$,

$\therefore\frac{\partial f}{\partial x}$在点$(0,0)$不存在，故函数$f(x,y)$在点$(0,0)$不可微.

(2) **【不太好的做法：】**$\because f(x,0)=0,f(0,y)=0$,

$\therefore\frac{\partial f(0,0)}{\partial x}=0,\frac{\partial f(0,0)}{\partial y}=0$,

[2]这里及后文中类似的标红语句为增加的内容. 根据可微的充分条件，若函数$f(x,y)$在点(x_0,y_0)及其附近的偏导数存在，且$\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$在点$(x_0,y_0)$处连续，则$f(x,y)$在点$(x_0,y_0)$处可微. 利用可微的充分条件判断函数在一点处可微应增加偏导数在该点及其附近存在的条件.

$\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\frac{f(0+\Delta x,0+\Delta y)-f(0,0)-[\frac{\partial f(0,0)}{\partial x}\Delta x+\frac{\partial f(0,0)}{\partial y}\Delta y]}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$
$=\lim\limits_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-[\frac{\partial f(0,0)}{\partial x}x+\frac{\partial f(0,0)}{\partial y}y]}{\sqrt{x^2+y^2}}$
$=\lim\limits_{(x,y)\to(0,0)}\frac{f(x,y)}{\sqrt{x^2+y^2}}=\lim\limits_{(x,y)\to(0,0)}\frac{2xy}{x^2+y^2},(*)$

将$y=kx$代入上式得$\lim\limits_{x\to0}\frac{2kx^2}{x^2+k^2x^2}=\frac{2k}{1+k^2}$，该极限随$k$的取值不同而变化，故极限(*)不存在，故函数$f(x,y)$在点$(0,0)$不可微.

【比较好的做法:】$\because\left|\frac{2xy}{\sqrt{x^2+y^2}}-f(0,0)\right|=\frac{|2xy|}{\sqrt{x^2+y^2}}\le\frac{2|x||y|}{|x|}=2|y|$,

又$\because\lim\limits_{(x,y)\to(0,0)}2|y|=0$,

$\therefore\lim\limits_{(x,y)\to(0,0)}\frac{2xy}{\sqrt{x^2+y^2}}=f(0,0)$,

$\therefore f(x,y)$在点$(0,0)$处连续,

$\because f(x,0)=0, f(0,y)=0$

$\therefore\frac{\partial f(0,0)}{\partial x}=0,\frac{\partial f(0,0)}{\partial y}=0$

$\therefore$若$f(x,y)$在点$(0,0)$处可微，则$\mathrm{d}f(0,0)=0$，且

$\lim\limits_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-\mathrm{d}f(0,0)}{\sqrt{x^2+y^2}}=\lim\limits_{(x,y)\to(0,0)}\frac{2xy}{x^2+y^2}=0$,

$\because\lim\limits_{\substack{x\to0\\y=x}}\frac{2xy}{x^2+y^2}=\lim\limits_{x\to0}\frac{2x^2}{x^2+x^2}=1\ne0$,

$\therefore\lim\limits_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-\mathrm{d}f(0,0)}{\sqrt{x^2+y^2}}\ne0$，矛盾，

$\therefore$函数$f(x,y)$在点$(0,0)$不可微.

3. 已知函数$g(x),h(x)$分别在区间$[x_0,x_1]$与$[y_0,y_1]$上连续，试证函数

$$f(x,y)=\int_{x_0}^{x}g(s)\mathrm{d}s\int_{y_0}^{y}h(t)\mathrm{d}t$$

在点(x,y)可微，其中$(x,y)\in D=\{(x,y)\mid x_0\le x\le x_1,y_0\le y\le y_1\}$.

证明：方法1：$\because g(x),h(x)$分别在区间$[x_0,x_1]$与$[y_0,y_1]$上连续,

$\therefore\frac{\partial f(x,y)}{\partial x}=g(x)\int_{y_0}^{y}h(t)\mathrm{d}t,\frac{\partial f(x,y)}{\partial y}=h(y)\int_{x_0}^{x}g(s)\mathrm{d}s$且$\int_{x_0}^{x}g(s)\mathrm{d}s$和$\int_{y_0}^{y}h(t)\mathrm{d}t$分别在区间$[x_0,x_1]$与$[y_0,y_1]$上连续,

$\therefore\frac{\partial f(x,y)}{\partial x},\frac{\partial f(x,y)}{\partial y}$在$D$中任意一点$(x,y)$连续,

$\therefore$函数$f(x,y)$在D中任意一点(x,y)可微.

方法2：$\because g(x),h(x)$分别在区间$[x_0,x_1]$与$[y_0,y_1]$上连续

$\therefore \frac{\partial f(x,y)}{\partial x} = g(x)\int_{y_0}^{y} h(t)\mathrm{d}t, \frac{\partial f(x,y)}{\partial y} = h(y)\int_{x_0}^{x} g(s)\mathrm{d}s$

$$\begin{aligned}
&\therefore f(x+\Delta x, y+\Delta y) - f(x,y) - [\frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y] \\
=&\int_{x_0}^{x+\Delta x} g(s)\mathrm{d}s \int_{y_0}^{y+\Delta y} h(t)\mathrm{d}t - \int_{x_0}^{x} g(s)\mathrm{d}s \int_{y_0}^{y} h(t)\mathrm{d}t - [g(x)\Delta x \int_{y_0}^{y} h(t)\mathrm{d}t + h(y)\Delta y \int_{x_0}^{x} g(s)\mathrm{d}s] \\
=&[\int_{x_0}^{x} g(s)\mathrm{d}s + \int_{x}^{x+\Delta x} g(s)\mathrm{d}s][\int_{y_0}^{y} h(t)\mathrm{d}t + \int_{y}^{y+\Delta y} h(t)\mathrm{d}t] - \int_{x_0}^{x} g(s)\mathrm{d}s \int_{y_0}^{y} h(t)\mathrm{d}t \\
&\qquad - [g(x)\Delta x \int_{y_0}^{y} h(t)\mathrm{d}t + h(y)\Delta y \int_{x_0}^{x} g(s)\mathrm{d}s] \\
=&\int_{x_0}^{x} g(s)\mathrm{d}s \int_{y}^{y+\Delta y} h(t)\mathrm{d}t + \int_{y_0}^{y} h(t)\mathrm{d}t \int_{x}^{x+\Delta x} g(s)\mathrm{d}s + \int_{x}^{x+\Delta x} g(s)\mathrm{d}s \int_{y}^{y+\Delta y} h(t)\mathrm{d}t \\
&\qquad - [g(x)\Delta x \int_{y_0}^{y} h(t)\mathrm{d}t + h(y)\Delta y \int_{x_0}^{x} g(s)\mathrm{d}s] \\
=&h(y+\lambda_2\Delta y)\Delta y \int_{x_0}^{x} g(s)\mathrm{d}s + g(x+\lambda_1\Delta x)\Delta x \int_{y_0}^{y} h(t)\mathrm{d}t + g(x+\lambda_1\Delta x)h(y+\lambda_2\Delta y)\Delta x\Delta y \\
&\qquad - [g(x)\Delta x \int_{y_0}^{y} h(t)\mathrm{d}t + h(y)\Delta y \int_{x_0}^{x} g(s)\mathrm{d}s] \\
=&[g(x+\lambda_1\Delta x) - g(x)]\Delta x \int_{y_0}^{y} h(t)\mathrm{d}t + [h(y+\lambda_2\Delta y) - h(y)]\Delta y \int_{x_0}^{x} g(s)\mathrm{d}s \\
&\qquad + g(x+\lambda_1\Delta x)h(y+\lambda_2\Delta y)\Delta x\Delta y, 0<\lambda_{1,2}<1
\end{aligned}$$

$\therefore$

$$\begin{aligned}
&\left|\frac{f(x+\Delta x, y+\Delta y) - f(x,y) - [\frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y]}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\right| \\
=&\Big|\frac{[g(x+\lambda_1\Delta x) - g(x)]\Delta x \int_{y_0}^{y} h(t)\mathrm{d}t}{\sqrt{(\Delta x)^2+(\Delta y)^2}} + \frac{[h(y+\lambda_2\Delta y) - h(y)]\Delta y \int_{x_0}^{x} g(s)\mathrm{d}s}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \\
&\qquad + \frac{g(x+\lambda_1\Delta x)h(y+\lambda_2\Delta y)\Delta x\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\Big| \\
\le&\frac{|g(x+\lambda_1\Delta x) - g(x)||\Delta x||\int_{y_0}^{y} h(t)\mathrm{d}t|}{\sqrt{(\Delta x)^2+(\Delta y)^2}} + \frac{|h(y+\lambda_2\Delta y) - h(y)||\Delta y||\int_{x_0}^{x} g(s)\mathrm{d}s|}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \\
&\qquad + \frac{|g(x+\lambda_1\Delta x)||h(y+\lambda_2\Delta y)||\Delta x||\Delta y|}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \\
\le&\frac{|g(x+\lambda_1\Delta x) - g(x)||\Delta x||\int_{y_0}^{y} h(t)\mathrm{d}t|}{|\Delta x|} + \frac{|h(y+\lambda_2\Delta y) - h(y)||\Delta y||\int_{x_0}^{x} g(s)\mathrm{d}s|}{|\Delta y|} \\
&\qquad + \frac{|g(x+\lambda_1\Delta x)||h(y+\lambda_2\Delta y)||\Delta x||\Delta y|}{|\Delta x|} \\
=&|g(x+\lambda_1\Delta x) - g(x)||\int_{y_0}^{y} h(t)\mathrm{d}t| + |h(y+\lambda_2\Delta y) - h(y)||\int_{x_0}^{x} g(s)\mathrm{d}s| \\
&\qquad + |g(x+\lambda_1\Delta x)||h(y+\lambda_2\Delta y)||\Delta y|
\end{aligned}$$

$\because$

$$\lim_{(\Delta x,\Delta y)\to(0,0)}[\left|g(x+\lambda_1\Delta x)-g(x)\right|\left|\int_{y_0}^{y}h(t)\mathrm{d}t\right|+\left|h(y+\lambda_2\Delta y)-h(y)\right|\left|\int_{x_0}^{x}g(s)\mathrm{d}s\right| \\ +|g(x+\lambda_1\Delta x)||h(y+\lambda_2\Delta y)||\Delta y|]$$

$$=\lim_{\substack{\Delta x\to0\\ \Delta y\to0}}[\left|g(x+\lambda_1\Delta x)-g(x)\right|\left|\int_{y_0}^{y}h(t)\mathrm{d}t\right|+\left|h(y+\lambda_2\Delta y)-h(y)\right|\left|\int_{x_0}^{x}g(s)\mathrm{d}s\right| \\ +|g(x+\lambda_1\Delta x)||h(y+\lambda_2\Delta y)||\Delta y|]=0$$

$\therefore$

$$\frac{f(x+\Delta x,y+\Delta y)-f(x,y)-[\frac{\partial f(x,y)}{\partial x}\Delta x+\frac{\partial f(x,y)}{\partial y}\Delta y]}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\to0,(\Delta x,\Delta y)\to(0,0)$$

$\therefore$函数$f(x,y)$在点(x,y)可微.

4. 用函数微分计算下列数值的近似值：

$(1)\sqrt{1.02^2+1.97^2}$；　　$(2)0.97^{1.05}$.

解：(1)令$f(x,y)=\sqrt{x^2+y^2}$,

$\because\ \frac{\partial f}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x}{\sqrt{x^2+y^2}},\frac{\partial f}{\partial y}=\frac{2y}{2\sqrt{x^2+y^2}}=\frac{y}{\sqrt{x^2+y^2}}$在点$(1,2)$及其附近存在且在点$(1,2)$处连续，故$f(x,y)$在点$(1,2)$可微，

$\therefore\sqrt{1.02^2+1.97^2}=f(1.02,1.97)\approx f(1,2)+0.02\frac{\partial f(1,2)}{\partial x}+(-0.03)\frac{\partial f(1,2)}{\partial y}$

$=\sqrt{5}+0.02\times\frac{1}{\sqrt5}-0.03\times\frac{2}{\sqrt5}=\frac{5-0.04}{\sqrt5}\approx2.2182.$

(2)令$f(x,y)=x^y$,

$\because\ \frac{\partial f}{\partial x}=yx^{y-1},\frac{\partial f}{\partial y}=x^y\ln x$在点$(1,1)$及其附近存在且在点$(1,1)$处连续，

$\therefore 0.97^{1.05}=f(0.97,1.05)\approx f(1,1)+(-0.03)\frac{\partial f(1,1)}{\partial x}+0.05\frac{\partial f(1,1)}{\partial y}$

$=1-0.03\times1+0.05\times0=0.97.$

5. 设二元函数$z(x,y)$满足方程$\frac{\partial^2 z}{\partial x\partial y}=x+y$，并且$z(x,0)=x,z(0,y)=y^2$. 试求$z(x,y)$.

解：方法1：$\because\ \frac{\partial^2 z}{\partial x\partial y}=x+y,$

$\therefore\int\frac{\partial^2 z}{\partial x\partial y}\mathrm{d}x=\int(x+y)\mathrm{d}x=\frac12x^2+xy+C_1(y)=\frac{\partial z}{\partial y}+C_1(y),$

$\therefore$可设$\frac{\partial z}{\partial y}=\frac12x^2+xy+C(y)$，其中$C(y)$是与$x$无关的$y$的函数，

$\because z(0,y)=y^2,$

$\therefore\ \frac{\partial z(0,y)}{\partial y}=2y=C(y),$

$\therefore\ \frac{\partial z}{\partial y}=\frac12x^2+xy+2y,$

$\therefore\int\frac{\partial z}{\partial y}\mathrm{d}y=\int(\frac12x^2+xy+2y)\mathrm{d}y=\frac12x^2y+\frac12xy^2+y^2+C_3(x)=z(x,y)+C_4(x),$

$\therefore$可设$z(x,y)=\int(\frac{1}{2}x^2+xy+2y)\mathrm{d}y+C^*(x)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+C^*(x)$，其中$C^*(x)$是与$y$无关的$x$的函数，

$\because z(x,0)=x=C^*(x)$,

$\therefore z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x.$

方法2：$\because \frac{\partial^2 z}{\partial x\partial y}=x+y$,

$\therefore \int\frac{\partial^2 z}{\partial x\partial y}\mathrm{d}x=\int(x+y)\mathrm{d}x=\frac{1}{2}x^2+xy+C_1(y)=\frac{\partial z}{\partial y}+C_2(y)$,

$\therefore$可设$\frac{\partial z}{\partial y}=\frac{1}{2}x^2+xy+C(y)$，其中$C(y)$是与$x$无关的$y$的函数，

$\therefore \int\frac{\partial z}{\partial y}\mathrm{d}y=\frac{1}{2}x^2y+\frac{1}{2}xy^2+\int C(y)\mathrm{d}y=z(x,y)+C_3(x)$,

$\therefore$可设$z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+F(y)+C^*(x)$，其中$F(y)$是$C(y)$的一个与$x$无关的原函数，$C^*(x)$是与$y$无关的$x$的函数，

$\because z(0,y)=y^2, z(x,0)=x$,

$\therefore F(y)+C^*(0)=y^2, F(0)+C^*(x)=x$,

$\therefore F(y)=y^2-C^*(0), C^*(x)=x-F(0)$，且$F(0)+C^*(0)=0$,

$\therefore z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x-[F(0)+C^*(0)]=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x.$

6. 求$y^2\mathrm{e}^{x+y}(\mathrm{d}x+\mathrm{d}y)+2y\mathrm{e}^{x+y}\mathrm{d}y$的原函数.

解：方法1：设$f(x,y)$是$y^2\mathrm{e}^{x+y}(\mathrm{d}x+\mathrm{d}y)+2y\mathrm{e}^{x+y}\mathrm{d}y=y^2\mathrm{e}^{x+y}\mathrm{d}x+(y^2+2y)\mathrm{e}^{x+y}\mathrm{d}y$的原函数，则$\frac{\partial f}{\partial x}=y^2\mathrm{e}^{x+y}$,

$\therefore \int\frac{\partial f}{\partial x}\mathrm{d}x=\int y^2\mathrm{e}^{x+y}\mathrm{d}x=y^2\mathrm{e}^y\mathrm{e}^x+C_1(y)=f(x,y)+C_2(y)$,

$\therefore f(x,y)=y^2\mathrm{e}^y\mathrm{e}^x+C(y)$,

$\therefore \frac{\partial f}{\partial y}=2y\mathrm{e}^y\mathrm{e}^x+y^2\mathrm{e}^y\mathrm{e}^x+C'(y)=(y^2+2y)\mathrm{e}^{x+y}+C'(y)$,

又$\because \frac{\partial f}{\partial y}=(y^2+2y)\mathrm{e}^{x+y}$,

$\therefore C'(y)=0$,

$\therefore C(y)=C$,

$\therefore f(x,y)=y^2\mathrm{e}^{x+y}+C.$

方法2：设$f(x,y)$是$y^2\mathrm{e}^{x+y}(\mathrm{d}x+\mathrm{d}y)+2y\mathrm{e}^{x+y}\mathrm{d}y=y^2\mathrm{e}^{x+y}\mathrm{d}x+(y^2+2y)\mathrm{e}^{x+y}\mathrm{d}y$的原函数，则$\frac{\partial f}{\partial y}=(y^2+2y)\mathrm{e}^{x+y}$,

$\therefore \int\frac{\partial f}{\partial y}\mathrm{d}y=\int(y^2+2y)\mathrm{e}^{x+y}\mathrm{d}y=\mathrm{e}^x\int(y^2+2y)\mathrm{d}\mathrm{e}^y=\mathrm{e}^x[(y^2+2y)\mathrm{e}^y-\int\mathrm{e}^y\mathrm{d}(y^2+2y)]$
$=\mathrm{e}^x[(y^2+2y)\mathrm{e}^y-\int\mathrm{e}^y(2y+2)\mathrm{d}y]=\mathrm{e}^x[(y^2+2y)\mathrm{e}^y-\int(2y+2)\mathrm{d}\mathrm{e}^y]$
$=\mathrm{e}^x[(y^2+2y)\mathrm{e}^y-(2y+2)\mathrm{e}^y+\int\mathrm{e}^y\mathrm{d}(2y+2)]=\mathrm{e}^x[(y^2+2y)\mathrm{e}^y-(2y+2)\mathrm{e}^y+2\int\mathrm{e}^y\mathrm{d}y]$
$=\mathrm{e}^x[(y^2+2y)\mathrm{e}^y-(2y+2)\mathrm{e}^y+2\mathrm{e}^y]=y^2\mathrm{e}^{x+y}+C_1(x)=f(x,y)+C_2(x)$,

$\therefore f(x,y) = y^2\mathrm{e}^{x+y} + C(x),$

$\therefore \frac{\partial f}{\partial x} = y^2\mathrm{e}^{x+y} + C'(x),$

又$\because \frac{\partial f}{\partial x} = y^2\mathrm{e}^{x+y},$

$\therefore C'(x) = 0,$

$\therefore C(x) = C,$

$\therefore f(x,y) = y^2\mathrm{e}^{x+y} + C.$