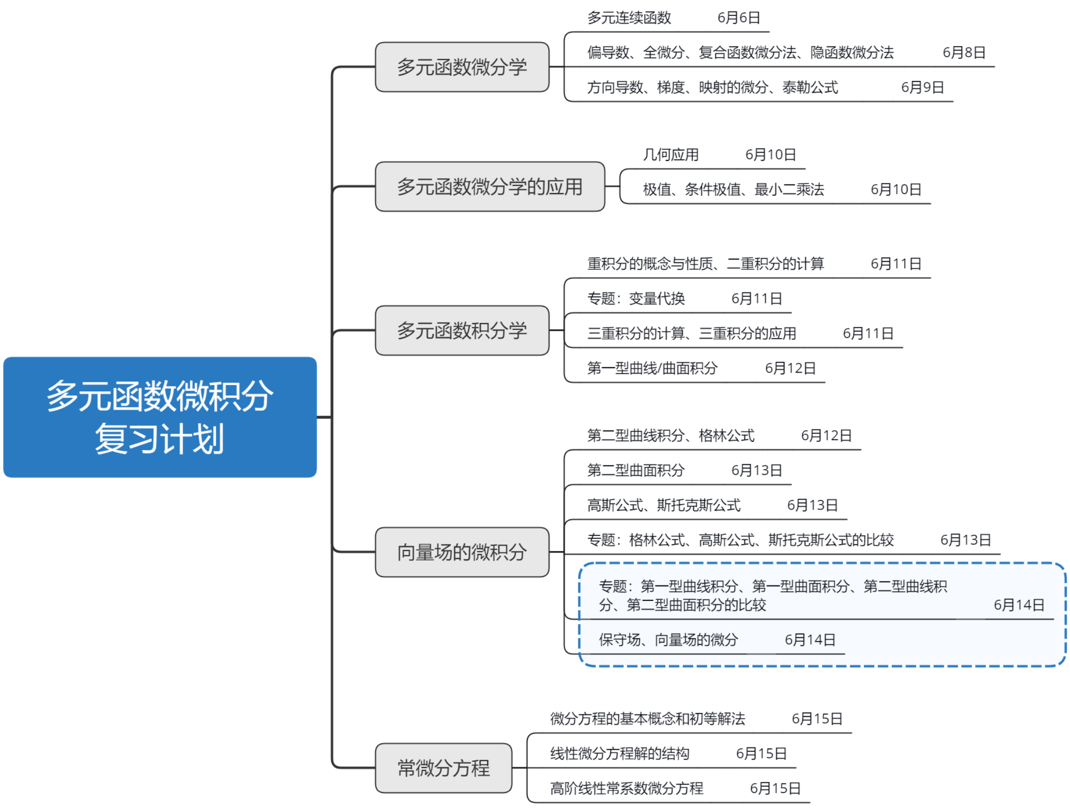
14 保守场、向量场的微分运算

14.1 复习计划

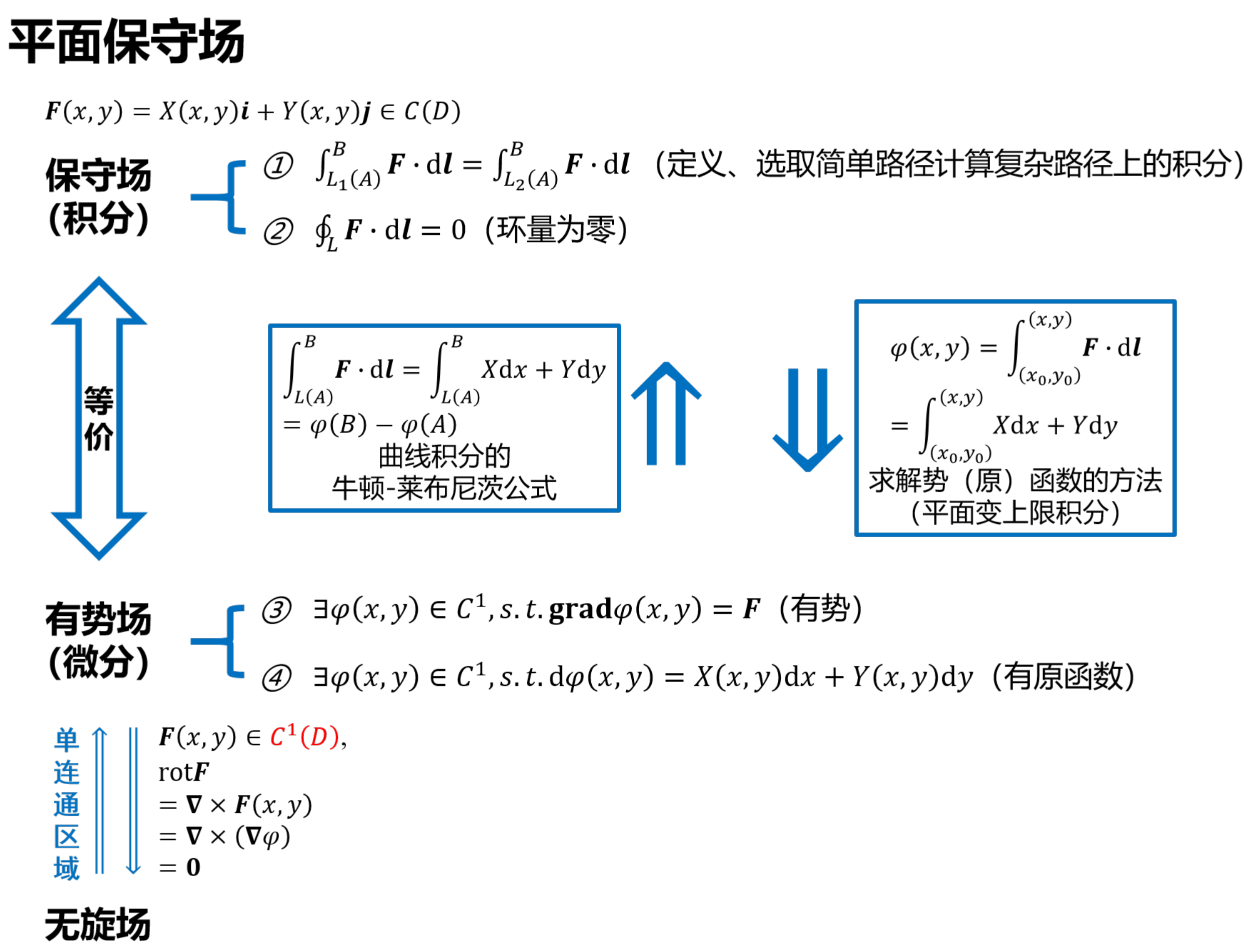


14.2 知识结构

- 向量场的微积分
  - 向量场在曲线上的积分、格林公式
  - 向量场在曲面上的积分
  - 高斯公式与斯托克斯公式
  - 保守场
    - 平面保守场
      - 保守场的几个概念
        - 保守场 $\boldsymbol{F}(x,y)$
          - 曲线积分$\int_{L(A)}^{(B)}\boldsymbol{F}(x,y)\cdot\mathrm{d}\boldsymbol{l}$与路径无关
        - 有势场 $\boldsymbol{F}(x,y)$
          - $\exists\varphi(x,y), s.t.\mathrm{grad}\varphi(x,y)=\boldsymbol{F}(x,y),(x,y)\in D$
          - $\varphi(x,y)$为势函数
          - 可微的有势场是无旋场
            - 即$\mathrm{rot}\boldsymbol{F}=(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y})\boldsymbol{k}=\boldsymbol{0}$
        - 全微分式 $X(x,y)\mathrm{d}x+Y(x,y)\mathrm{d}y$
          - $\exists\varphi(x,y), s.t.\mathrm{d}\varphi(x,y)=X(x,y)\mathrm{d}x+Y(x,y)\mathrm{d}y$
          - $\varphi(x,y)$为原函数
          - $\boldsymbol{F}(x,y)=(X,Y)$有势与$X\mathrm{d}x+Y\mathrm{d}y$是全微分式等价
            - 势函数也是原函数
              - 求原函数的方法
      - 一般区域上保守场的充要条件
        - $\int_{L(A)}^{(B)}\boldsymbol{F}(x,y)\cdot\mathrm{d}\boldsymbol{l}$与路径无关
        - $\oint_L\boldsymbol{F}(x,y)\cdot\mathrm{d}\boldsymbol{l}=0$
        - $\boldsymbol{F}(x,y)=(X,Y)$有势
        - $X\mathrm{d}x+Y\mathrm{d}y$是全微分式
        - 四个条件等价
      - 单连通区域D上保守场的充要条件
        - $\boldsymbol{F}(x,y)=(X,Y)\in C^1(D)$且$\mathrm{rot}\boldsymbol{F}=(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y})\boldsymbol{k}=\boldsymbol{0}$
          - 即$\boldsymbol{F}(x,y)$是D上的无旋场
        - 注意该充要条件只能应用于单连通域
    - 空间保守场
      - 保守场的几个概念
        - 保守场 $\boldsymbol{F}(x,y,z)$
          - 曲线积分$\int_{L(A)}^{(B)}\boldsymbol{F}(x,y,z)\cdot\mathrm{d}\boldsymbol{l}$与路径无关
        - 有势场 $\boldsymbol{F}(x,y,z)$
          - $\exists\varphi(x,y,z), s.t.\mathrm{grad}\varphi(x,y,z)=\boldsymbol{F}(x,y,z),(x,y,z)\in\Omega$
          - $\varphi(x,y,z)$为势函数
          - 可微的有势场是无旋场
            - 即$\mathrm{rot}\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\X&Y&z\end{vmatrix}=\boldsymbol{0}$
        - 全微分式 $X(x,y,z)\mathrm{d}x+Y(x,y,z)\mathrm{d}y+Z(x,y,z)\mathrm{d}z$
          - $\exists\varphi(x,y,z), s.t.\mathrm{d}\varphi(x,y,z)=X(x,y,z)\mathrm{d}x+Y(x,y,z)\mathrm{d}y+Z(x,y,z)\mathrm{d}z$
          - $\varphi(x,y,z)$为原函数
          - $\boldsymbol{F}(x,y,z)=(X,Y)$有势与$X\mathrm{d}x+Y\mathrm{d}y+Z\mathrm{d}z$是全微分式等价
            - 势函数也是原函数
              - 求原函数的方法
                - 曲线积分法
                - 不定积分法
      - 一般区域上保守场的充要条件
        - $\int_{L(A)}^{(B)}\boldsymbol{F}(x,y,z)\cdot\mathrm{d}\boldsymbol{l}$与路径无关
        - $\oint_L\boldsymbol{F}(x,y,z)\cdot\mathrm{d}\boldsymbol{l}=0$
        - $\boldsymbol{F}(x,y,z)=(X,Y,Z)$有势
        - $X\mathrm{d}x+Y\mathrm{d}y+Z\mathrm{d}z$是全微分式
        - 四个条件等价
      - 曲面单连通区域上保守场的充要条件
        - $\boldsymbol{F}(x,y,z)=(X,Y,Z)\in C^1(\Omega)$且$\mathrm{rot}\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\X&Y&Z\end{vmatrix}=\boldsymbol{0}$
          - 即$\boldsymbol{F}(x,y)$是Ω上的无旋场
        - 注意该充要条件只能应用与曲面单连通区域
    - 调和场
      - 调和场的概念
        - 既是有势场又是无源场
        - $\boldsymbol{F}(x,y,z)=\boldsymbol{\nabla}\varphi(x,y,z),\boldsymbol{\nabla}\cdot\boldsymbol{F}(x,y,z)=0$
      - 调合场势函数的性质
        - $\varphi(x,y,z)\in C^2\Rightarrow\Delta\varphi(x,y,z)=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0$
          - $\Delta\varphi(x,y,z)=0$调和方程，其解为调和函数
        - $\varphi(x,y,z)\in C^2(\Omega)$是调和函数，且在$\partial\Omega$上的函数值等于0 $\Rightarrow\varphi(x,y,z)=0,(x,y,z)\in\Omega$
  - 向量场的微分运算
    - 数量场f梯度
      - 二维 $\mathrm{grad}f(x,y)=\boldsymbol{\nabla}f(x,y)=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})$
      - 三维 $\mathrm{grad}f(x,y,z)=\boldsymbol{\nabla}f(x,y,z)=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})$
    - 向量场$\boldsymbol{F}$的散度
      - 二维 $\mathrm{div}\boldsymbol{F}=\boldsymbol{\nabla}\cdot\boldsymbol{F}=\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}$
      - 三维 $\mathrm{div}\boldsymbol{F}=\boldsymbol{\nabla}\cdot\boldsymbol{F}=\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}$
      - $\mathrm{div}\boldsymbol{F}=0$的场为无源场
    - 向量场$\boldsymbol{F}$的旋度
      - 二维 $\mathrm{rot}\boldsymbol{F}=\mathrm{curl}\boldsymbol{F}=\boldsymbol{\nabla}\times\boldsymbol{F}=(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y})\boldsymbol{k}$
      - 三维 $\mathrm{rot}\boldsymbol{F}=\mathrm{curl}\boldsymbol{F}=\boldsymbol{\nabla}\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\X&Y&Z\end{vmatrix}$
      - $\mathrm{rot}\boldsymbol{F}=\boldsymbol{0}$的场为无旋场
    - 散度和旋度的积分定义可参看扈老师的讲课提纲

14.3 保守场

平面保守场的性质如下所示。



说明：

1. 性质①中的简单路径可选为与坐标轴平行的直线或折线路径.

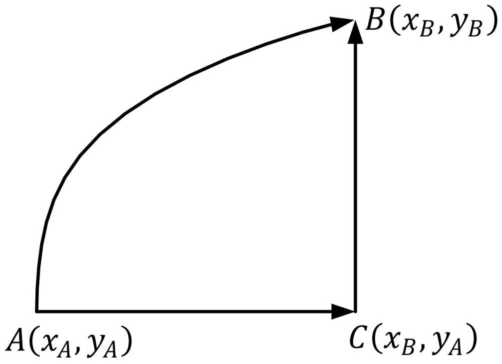


图 1: 与坐标轴平行的折线路径.

在与x轴平行的直线路径AC上，积分$\int_{L(A)}^{C} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{l} = \int_{L(A)}^{C} X(x,y)\mathrm{d}x + Y(x,y)\mathrm{d}y$的被积表达式$X(x,y)\mathrm{d}x + Y(x,y)\mathrm{d}y$中$\mathrm{d}y=0, y=y_A$，故

$$\int_{L(A)}^{C} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{l} = \int_{(x_A,y_A)}^{(x_B,y_A)} X(x,y)\mathrm{d}x + Y(x,y)\mathrm{d}y = \int_{x_A}^{x_B} X(x,y_A)\mathrm{d}x.$$

在与y轴平行的直线路径CB上，积分$\int_{L(C)}^{B} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{l} = \int_{L(C)}^{B} X(x,y)\mathrm{d}x + Y(x,y)\mathrm{d}y$的被积表达式$X(x,y)\mathrm{d}x + Y(x,y)\mathrm{d}y$中$\mathrm{d}x=0, x=x_B$，故

$$\int_{L(C)}^{B} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{l} = \int_{(x_B,y_A)}^{(x_B,y_B)} X(x,y)\mathrm{d}x + Y(x,y)\mathrm{d}y = \int_{y_A}^{y_B} Y(x_B,y)\mathrm{d}y.$$

【考察这一点的习题：1.(1)/(2)/(3)/(4),3.(1)/(2). (第一类题目)】

2. ①,②,③,④相互等价，既是性质定理也是判定定理，知道其中一个即可判定该向量场是保守场，也就可以得到另外三个性质.

3. 保守场是有势场，有势场也是保守场，二者是完全等价的概念. 保守场的两个性质从积分角度描述了向量场，有势场的两个性质从微分角度描述了向量场.

4. 可通过对势函数这样一个数量场的分析来对向量场进行分析，比如可以用引力势能分析引力场，用重力势能分析重力场，用电势能分析电场力的场，用电势分析电场。

5. 势函数与原函数相同.

6. 可微的保守场是无旋场（但无旋场不一定是保守场）.

   【考察这一点的习题：2. (第二类题目)】

7. 平面单连通域上的无旋场是保守场. 常利用平面单连通区域上无旋来判定一个向量场是保守场.

   【考察这一点的习题：1.(1)/(2)/(3)/(4),3.(1)/(2). (第一类题目)】

8. 可微场才可用旋度算子计算旋度. 如一个向量场不可微，则不可用旋度算子计算旋度，难以用旋度为零来判断该向量场是保守场.

   【考察这一点的习题：4. (第三类题目)】

9. 求解全微分式的原函数有两种方法，变上限积分法和不定积分法.

   【考察这一点的习题：3.(1)/(2). (第四类题目)】

10. 空间保守场的性质与平面保守场类似，主要的不同是三维空间中的曲面单连通域（比如球壳是曲面单连通域，圆环体则不是曲面单连通域）上的无旋场是保守场。

14.4 无源场、无旋场

1. 无源场$\mathrm{div}\boldsymbol{F}=\boldsymbol{\nabla}\cdot\boldsymbol{F}=0$.

   【与无源场有关的题目：5,7,8.(1)/(2),9. (第五类题目)】

2. 无旋场$\mathrm{rot}\boldsymbol{F}=\boldsymbol{\nabla}\times\boldsymbol{F}=\boldsymbol{0}$.

   【与无旋场有关的题目：6,10.(1)/(2). (第六类题目)】

   【综合题目：9. (主要考察通量的概念、高斯公式、无源场)】

14.5 向量场的微分运算习题分类

1. 验证三个算子的性质.

   【习题13.1中的1., 2., 4.】

2. 求梯度、散度、旋度.

   【习题13.1中的3., 5., 6.】

14.6 习题13.6解答

1. 利用积分域与路线无关的性质计算下列积分：
   (1)$\int_L(x^3+xy^2)\mathrm{d}x+(y^3+x^2y)\mathrm{d}y$，其中$L$为从$O(0,0)$经$A(1,1)$到$B(2,0)$的折线；
   (2)$\int_L(y+1)\tan x\mathrm{d}x-\ln\cos x\mathrm{d}y$，其中$L$为曲线$x=\cos t,y=2\sin t(0\leqslant t\leqslant\pi)$，顺时针方向；
   (3)$\int_L(\ln\frac{y}{x}-1)\mathrm{d}x+\frac{x}{y}\mathrm{d}y$，其中$L$为由点$A(1,1)$出发到$B(\mathrm{e},3\mathrm{e})$的任何一条不与$x$轴以及$y$轴相交的曲线；
   (4)$\int_L\frac{1+y^2f(xy)}{y}\mathrm{d}x+\frac{x}{y^2}[y^2f(xy)-1]\mathrm{d}y$，其中$L$为由点$A(0,1)$出发到$B(1,2)$的任何一条不与$x$轴相交的曲线，$f$是连续可微的函数.

解：(1)

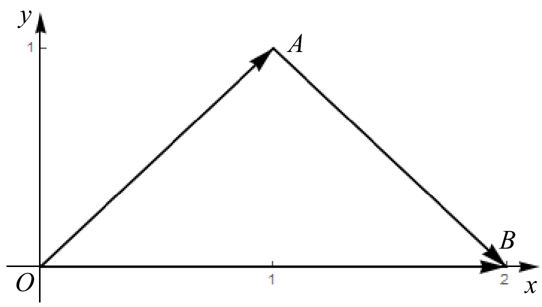


图 2: 习题13.6 1.(1)题图示

令$\begin{cases} X(x,y)=x^3+xy^2, \\ Y(x,y)=y^3+x^2y, \end{cases}$ 则$\boldsymbol{F}(x,y)=(X,Y)\in C^1(\mathbb{R})$,

$\because \mathbb{R}$为单连通域，

且$\mathrm{rot}\boldsymbol{F}=(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y})\boldsymbol{k}=(2xy-2xy)\boldsymbol{k}=\boldsymbol{0}$,

$\therefore \int_L X\mathrm{d}x+Y\mathrm{d}y=\int_{(0,0)}^{(2,0)} X\mathrm{d}x+Y\mathrm{d}y=\int_0^2 x^3\mathrm{d}x=\frac{1}{4}x^4\Big|_0^2=4.$

(2)

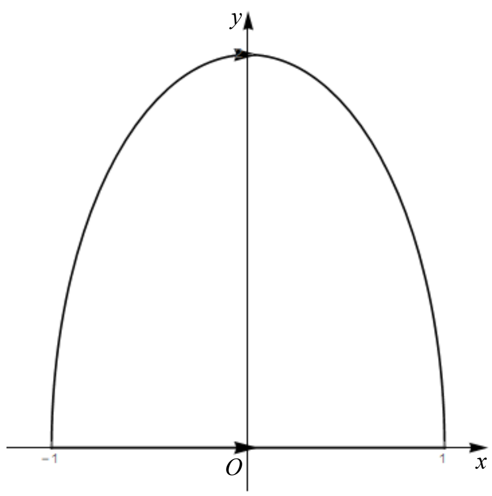


图 3: 习题13.6 1.(2)题图示

曲线L : $\begin{cases} x=\cos t, \\ y=2\sin t, \end{cases}$ $(0\leqslant t\leqslant \pi)$为一椭圆弧$x^2+\frac{y^2}{4}=1, y\geqslant 0$，顺时针的起点

为$(-1,0)$终点为$(1,0)$,

设$D=(-\frac{\pi}{2},\frac{\pi}{2})\times(-\infty,\infty)$，则$L\in D$,

令$\begin{cases}X(x,y)=(y+1)\tan x,\\ Y(x,y)=-\ln\cos x,\end{cases}$ 则$\boldsymbol{F}(x,y)=(X,Y)\in C^1(D)$,

$\because D$是单连通区域，

且$\mathrm{rot}\boldsymbol{F}=(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y})\boldsymbol{k}=(-\frac{-\sin x}{\cos x}-\tan x)\boldsymbol{k}=\boldsymbol{0}$,

$\therefore \int_L X\mathrm{d}x+Y\mathrm{d}y=\int_{(-1,0)}^{(1,0)}X\mathrm{d}x+Y\mathrm{d}y=\int_{-1}^{1}\tan x\mathrm{d}x=0.$

(3)

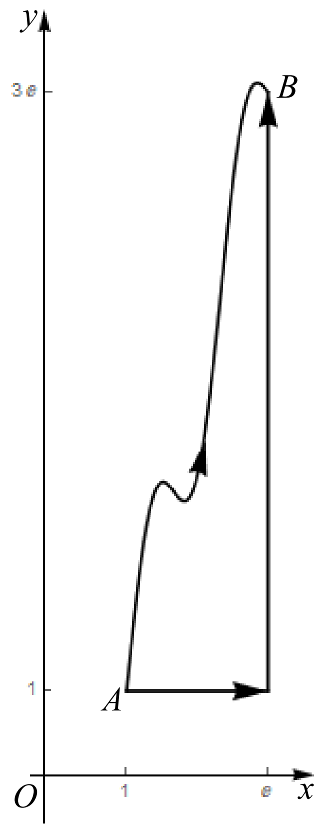


图 4: 习题13.6 1.(3)题图示

设$D=\{(x,y)\mid x>0,y>0\}$，则$L\in D$,

令$\begin{cases}X(x,y)=\ln\frac{y}{x}-1,\\ Y(x,y)=\frac{x}{y},\end{cases}$ 则$\boldsymbol{F}(x,y)=(X,Y)\in C^1(D)$,

$\because D$是单连通域，

且$\mathrm{rot}\boldsymbol{F}=(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y})\boldsymbol{k}=(\frac{1}{y}-\frac{1}{\frac{y}{x}}\frac{1}{x})\boldsymbol{k}=\boldsymbol{0}$,

$\therefore \int_L X\mathrm{d}x+Y\mathrm{d}y=\int_{(1,1)}^{(\mathrm{e},3\mathrm{e})}X\mathrm{d}x+Y\mathrm{d}y=\int_{(1,1)}^{(\mathrm{e},1)}X\mathrm{d}x+Y\mathrm{d}y+\int_{(\mathrm{e},1)}^{(\mathrm{e},3\mathrm{e})}X\mathrm{d}x+Y\mathrm{d}y$
$=\int_1^{\mathrm{e}}(\ln\frac{1}{x}-1)\mathrm{d}x+\int_1^{3\mathrm{e}}\frac{\mathrm{e}}{y}\mathrm{d}y=x(-\ln x-1)\Big|_1^{\mathrm{e}}-\int_1^{\mathrm{e}}x\mathrm{d}(-\ln x-1)+\mathrm{e}\ln y\Big|_1^{3\mathrm{e}}$
$=\mathrm{e}(-1-1)-1\cdot(0-1)+\int_1^{\mathrm{e}}x\frac{1}{x}\mathrm{d}x+\mathrm{e}\ln 3\mathrm{e}-0=-2\mathrm{e}+1+(\mathrm{e}-1)+\mathrm{e}\ln 3+\mathrm{e}=\mathrm{e}\ln 3.$

(4)

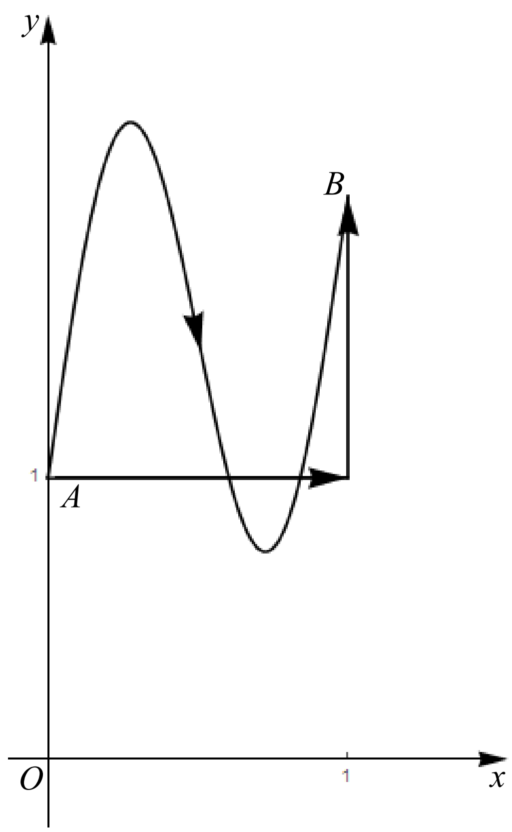


图 5: 习题13.6 1.(4)题图示

设$D=\{(x,y)\mid y>0\}$,

令$\begin{cases}X(x,y)=\frac{1+y^2f(xy)}{y}=\frac{1}{y}+yf(xy),\\ Y(x,y)=\frac{x}{y^2}[y^2f(xy)-1]=xf(xy)-\frac{x}{y^2},\end{cases}$ 则$\boldsymbol{F}(x,y)=(X,Y)\in C^1(D)$,

$\because D$单连通，

且$\mathrm{rot}\boldsymbol{F}=(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y})\boldsymbol{k}=\{f(xy)+xf'(xy)y-\frac{1}{y^2}-[-\frac{1}{y^2}+f(xy)+yf'(xy)x]\}\boldsymbol{k}=\boldsymbol{0}$,

$\therefore \int_L X\mathrm{d}x + Y\mathrm{d}y = \int_{(0,1)}^{(1,2)} X\mathrm{d}x + Y\mathrm{d}y = \int_{(0,1)}^{(1,1)} X\mathrm{d}x + Y\mathrm{d}y + \int_{(1,1)}^{(1,2)} X\mathrm{d}x + Y\mathrm{d}y$
$= \int_0^1 [1 + f(x)]\mathrm{d}x + \int_1^2 [f(y) - \frac{1}{y^2}]\mathrm{d}y = \int_0^1 \mathrm{d}x + \int_0^1 f(x)\mathrm{d}x + \int_1^2 f(y)\mathrm{d}y - \int_1^2 \frac{1}{y^2}\mathrm{d}y$
$= 1 + \int_0^2 f(x)\mathrm{d}x + \frac{1}{y}\Big|_1^2 = \frac{1}{2} + \int_0^2 f(x)\mathrm{d}x.$

2. 确定p的值，使积分$\int_A^B (x^4 + 4xy^p)\mathrm{d}x + (6x^{p-1}y^2 - 5y^4)\mathrm{d}y$与路线无关. 当$A = (0,0), B = (1,2)$时，计算积分的值.

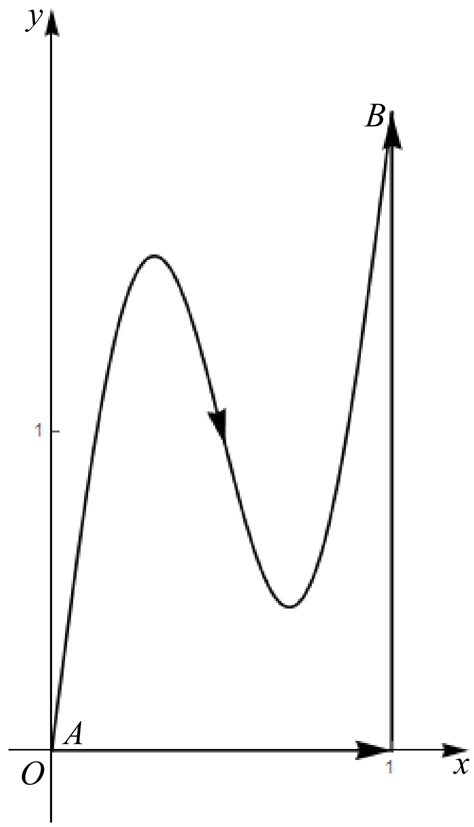


图 6: 习题13.6 2.题图示

解：$\because$积分$\int_A^B (x^4 + 4xy^p)\mathrm{d}x + (6x^{p-1}y^2 - 5y^4)\mathrm{d}y$与路线无关,

令$\begin{cases} X(x,y) = x^4 + 4xy^p, \\ Y(x,y) = 6x^{p-1}y^2 - 5y^4, \end{cases}$ 则$\boldsymbol{F}(x,y) = (X, Y)$是保守场，

$\because \boldsymbol{F}(x,y) \in C^1,$

$\therefore \mathrm{rot}\boldsymbol{F} = (\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y})\boldsymbol{k} = [6(p-1)x^{p-2}y^2 - 4pxy^{p-1}]\boldsymbol{k} = \boldsymbol{0},$

$\therefore p = 3.$

$\because A = (0,0), B = (1,2),$

$\therefore \int_A^B X\mathrm{d}x+Y\mathrm{d}y=\int_{(0,0)}^{(1,2)} X\mathrm{d}x+Y\mathrm{d}y=\int_{(0,0)}^{(1,0)} X\mathrm{d}x+Y\mathrm{d}y+\int_{(1,0)}^{(1,2)} X\mathrm{d}x+Y\mathrm{d}y$
$=\int_0^1 x^4\mathrm{d}x+\int_0^2(6y^2-5y^4)\mathrm{d}y=\frac{1}{5}x^5\big|_0^1+(2y^3-y^5)\big|_0^2=\frac{1}{5}+(2\cdot 8-32)=-\frac{79}{5}.$

3. 判定下列微分形式是否为全微分，若是，求出其原函数：

$(1)(2x\cos y-y^2\sin x)\mathrm{d}x+(2y\cos x-x^2\sin y)\mathrm{d}y;$

$(2)(\mathrm{e}^x\cos y+2xy^2)\mathrm{d}x+(2x^2y-\mathrm{e}^x\sin y)\mathrm{d}y.$

解：(1)令$\begin{cases}X(x,y)=2x\cos y-y^2\sin x,\\ Y(x,y)=2y\cos x-x^2\sin y,\end{cases}$ 则$\boldsymbol{F}(x,y)=(X,Y)\in C^1(\mathbb{R}),$

$\because \mathbb{R}$是单连通区域，

且$\mathrm{rot}\boldsymbol{F}=[\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y})\boldsymbol{k}=(-2y\sin x-2x\sin y-(-2x\sin y-2y\sin x)]\boldsymbol{k}=\mathbf{0},$

$\therefore X\mathrm{d}x+Y\mathrm{d}y$是全微分式，

方法1：原函数$\varphi(x,y)=\int_{(0,0)}^{(x,y)} X(s,t)\mathrm{d}s+Y(s,t)\mathrm{d}t+C$
$=\int_{(0,0)}^{(x,0)} X(s,t)\mathrm{d}s+Y(s,t)\mathrm{d}t+\int_{(x,0)}^{(x,y)} X(s,t)\mathrm{d}s+Y(s,t)\mathrm{d}t+C$
$=\int_0^x 2s\mathrm{d}s+\int_0^y(2t\cos x-x^2\sin t)\mathrm{d}t+C=s^2\big|_0^x+(t^2\cos x+x^2\cos t)\big|_0^y$
$=x^2+(y^2\cos x-x^2\sin y-x^2)+C=y^2\cos x-x^2\sin y+C.$

方法2：设原函数为$\varphi(x,y),$

则$\frac{\partial\varphi(x,y)}{\partial x}=2x\cos y-y^2\sin x,$

$\therefore \varphi(x,y)=x^2\cos y+y^2\cos x+C(y),$

$\therefore \frac{\partial\varphi(x,y)}{\partial y}=-x^2\sin y+2y\cos x+C'(y)=2y\cos x-x^2\sin y,$

$\therefore C'(y)=0,\ C(y)=C,$

$\therefore \varphi(x,y)=x^2\cos y+y^2\cos x+C.$

(2)令$\begin{cases}X(x,y)=\mathrm{e}^x\cos y+2xy^2,\\ Y(x,y)=2x^2y-\mathrm{e}^x\sin y,\end{cases}$ 则$\boldsymbol{F}(x,y)=(X,Y)\in C^1(\mathbb{R}),$

$\because \mathbb{R}$是单连通区域，

且$\mathrm{rot}\boldsymbol{F}=(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y})\boldsymbol{k}=[4xy-\mathrm{e}^x\sin y-(-\mathrm{e}^x\sin y+4xy)]\boldsymbol{k}=\mathbf{0},$

$\therefore X\mathrm{d}x+Y\mathrm{d}y$是全微分式.

方法1：原函数$\varphi(x,y)=\int_{(0,0)}^{(x,y)} X(s,t)\mathrm{d}s+Y(s,t)\mathrm{d}t+C_1$
$=\int_{(0,0)}^{(x,0)} X(s,t)\mathrm{d}s+Y(s,t)\mathrm{d}t+\int_{(x,0)}^{(x,y)} X(s,t)\mathrm{d}s+Y(s,t)\mathrm{d}t+C_1$
$=\int_0^x \mathrm{e}^s\mathrm{d}s+\int_0^y(2x^2t-\mathrm{e}^x\sin t)\mathrm{d}t+C_1=\mathrm{e}^s\big|_0^x+(x^2t^2+\mathrm{e}^x\cos t)\big|_0^y+C_1$
$=\mathrm{e}^x-1+(x^2y^2+\mathrm{e}^x\cos y-\mathrm{e}^x)+C_1=x^2y^2+\mathrm{e}^x\cos y+C.$

方法：设原函数为$\varphi(x,y)$，

则$\frac{\partial\varphi(x,y)}{\partial x}=\mathrm{e}^x\cos y+2xy^2$,

$\therefore\varphi(x,y)=\mathrm{e}^x\cos y+x^2y^2+C(y)$,

$\therefore\frac{\partial\varphi(x,y)}{\partial y}=-\mathrm{e}^x\sin y+2x^2y+C'(y)=2x^2y-\mathrm{e}^x\sin y$,

$\therefore C'(y)=0,C(y)=C$,

$\therefore\varphi(x,y)=\mathrm{e}^x\cos y+x^2y^2+C$.

4. 设$f(u)$连续，L为逐段光滑简单闭曲线，求证：

$$\oint_L f(x^2+y^2)(x\mathrm{d}x+y\mathrm{d}y)=0.$$

证明：令$\varphi(x,y)=\frac{1}{2}\int_0^{x^2+y^2}f(u)\mathrm{d}u$,

$\frac{\partial\varphi}{\partial x}=f(x^2+y^2)x,\frac{\partial\varphi}{\partial y}=f(x^2+y^2)y$,

$\therefore\mathrm{d}\varphi(x,y)=\frac{\partial\varphi}{\partial x}\mathrm{d}x+\frac{\partial\varphi}{\partial y}\mathrm{d}y=f(x^2+y^2)(x\mathrm{d}x+y\mathrm{d}y)$,

$\therefore\oint_L f(x^2+y^2)(x\mathrm{d}x+y\mathrm{d}y)=0$.

5.设一元函数f有连续的导数，计算$\boldsymbol{\nabla}\cdot(f(r)\boldsymbol{r})$，其中

$$\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k},\ r=\sqrt{x^2+y^2+z^2},$$

并说明f满足什么条件时，$f(r)\boldsymbol{r}$为无源场.

解：$\boldsymbol{\nabla}\cdot(f(r)\boldsymbol{r})=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})\cdot(xf(r),yf(r),zf(r))=\frac{\partial[xf(r)]}{\partial x}+\frac{\partial[yf(r)]}{\partial y}+\frac{\partial[zf(r)]}{\partial z}$
$=f(r)+xf'(r)\frac{x}{r}+f(r)+yf'(r)\frac{y}{r}+f(r)+zf'(r)\frac{z}{r}$
$=3f(r)+f'(r)\frac{x^2+y^2+z^2}{r}=3f(r)+rf'(r)$.

$\because f(r)\boldsymbol{r}$无源，

$\therefore 3f(r)+rf'(r)=0$,

当$f(r)\not\equiv 0$时$\frac{\mathrm{d}f(r)}{f(r)}=-\frac{3}{r}\mathrm{d}r$,

$\therefore\ln|f(r)|=-3\ln|r|+C$,

$\therefore f(r)r^3=\pm\mathrm{e}^C$,

$\therefore f(r)r^3=C_0$, C_0,C为任意常数，$f(r)\equiv 0$也满足该式.

5. 设$\boldsymbol{F}=f(r)\boldsymbol{r}$($r$与$\boldsymbol{r}$的意义与上题同)，证明$\mathrm{rot}\boldsymbol{F}=\boldsymbol{0}$.

证明：$\mathrm{rot}\boldsymbol{F}=\boldsymbol{\nabla}\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\xf(r)&yf(r)&zf(r)\end{vmatrix}$
$=(\frac{\partial zf(r)}{\partial y}-\frac{\partial yf(r)}{\partial z},\frac{\partial xf(r)}{\partial z}-\frac{\partial zf(r)}{\partial x},\frac{\partial yf(r)}{\partial x}-\frac{\partial xf(r)}{\partial y})$
$=(zf'(r)\frac{y}{r}-yf'(r)\frac{z}{r},xf'(r)\frac{z}{r}-zf'(r)\frac{x}{r},yf'(r)\frac{y}{r}-xf'(r)\frac{y}{r})$
$=(0,0,0)=\boldsymbol{0}$.

6. 设f有连续的二阶导数，计算$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}f(r))$，其中$r,\boldsymbol{r}$同题5，并说明$f$满足什么条件时，$\nabla f$为无源场.

解：$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}f(r))=\boldsymbol{\nabla}\cdot(f'(r)\frac{x}{r},f'(r)\frac{y}{r},f'(r)\frac{z}{r})=\frac{\partial}{\partial x}[f'(r)\frac{x}{r}]+\frac{\partial}{\partial y}[f'(r)\frac{y}{r}]+\frac{\partial}{\partial z}[f'(r)\frac{z}{r}]$
$=f''(r)\frac{x}{r}\cdot\frac{x}{r}+f'(r)\frac{r-x\frac{x}{r}}{r^2}+f''(r)\frac{y}{r}\cdot\frac{y}{r}+f'(r)\frac{r-y\frac{y}{r}}{r^2}+f''(r)\frac{z}{r}\cdot\frac{z}{r}+f'(r)\frac{r-z\frac{z}{r}}{r^2}$
$=f''(r)\frac{x^2}{r^2}+f'(r)\frac{y^2+z^2}{r^3}+f''(r)\frac{y^2}{r^2}+f'(r)\frac{z^2+x^2}{r^3}+f''(r)\frac{z^2}{r^2}+f'(r)\frac{x^2+y^2}{r^3}$
$=f''(r)\frac{x^2+y^2+z^2}{r^2}+f'(r)\frac{2(x^2+y^2+z^2)}{r^3}=f''(r)+\frac{2}{r}f'(r)$.

$\because\nabla f$为无源场，

$\therefore f''(r)+\frac{2}{r}f'(r)=0$,

当$f'(r)\not\equiv0$时$\frac{\mathrm{d}f'(r)}{f'(r)}=-\frac{2}{r}\mathrm{d}r$,

$\therefore\ln|f'(r)|=-2\ln|r|+C$,

$\therefore f'(r)r^2=\pm\mathrm{e}^C$,

$\therefore f'(r)=\frac{C_0}{r^2}$，$f'(r)\equiv0$也满足该式.

$\therefore f(r)=-\frac{C_0}{r}+C_2=\frac{C_1}{r}+C_2$.

7. 证明下列向量场为无源场：
(1)$\boldsymbol{v}=\boldsymbol{u}_1\times\boldsymbol{u}_2$，其中$\boldsymbol{u}_1,\boldsymbol{u}_2$是无旋场；
(2)$\boldsymbol{v}=\frac{\boldsymbol{r}}{r^3}$，其中$r,\boldsymbol{r}$同题5.

证明：(1)$\mathrm{div}\boldsymbol{v}=\boldsymbol{\nabla}\cdot\boldsymbol{v}=\boldsymbol{\nabla}\cdot(\boldsymbol{u}_1\times\boldsymbol{u}_2)=\boldsymbol{u}_2\cdot(\boldsymbol{\nabla}\times\boldsymbol{u}_1)-\boldsymbol{u}_1\cdot(\boldsymbol{\nabla}\times\boldsymbol{u}_2)$
$=\boldsymbol{u}_2\cdot\boldsymbol{0}-\boldsymbol{u}_1\cdot\boldsymbol{0}=0-0=0$.

注：该公式的证明见习题13.1中的2题.

(2)$\boldsymbol{\nabla}\cdot\boldsymbol{v}=\boldsymbol{\nabla}\cdot(\frac{\boldsymbol{r}}{r^3})=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})\cdot(\frac{x}{r^3},\frac{y}{r^3},\frac{z}{r^3})=\frac{\partial}{\partial x}(\frac{x}{r^3})+\frac{\partial}{\partial y}(\frac{y}{r^3})+\frac{\partial}{\partial z}(\frac{z}{r^3})$
$=\frac{r^3-x3r^3\frac{x}{r}}{r^6}+\frac{r^3-y3r^3\frac{y}{r}}{r^6}+\frac{r^3-z3r^3\frac{z}{r}}{r^6}=\frac{r^2-3x^2}{r^5}+\frac{r^2-3y^2}{r^5}+\frac{r^2-3z^2}{r^5}$
$=\frac{y^2+z^2-2x^2}{r^5}+\frac{z^2+x^2-2y^2}{r^5}+\frac{x^2+y^2-2z^2}{r^5}=0$.

9.求电场$\boldsymbol{v}=\frac{\boldsymbol{r}}{r^3}$穿过包围原点的任意简单光滑闭曲面的电通量，其中$r,\boldsymbol{r}$同题5.

解：设S是包围原点的任意简单光滑闭曲面，S_1是S围成区域中的包围原点的任意简单光滑闭曲面，S_1,S外侧为正，记S,S_1^-围成的区域为Ω,

则$\oiint\limits_S\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}-\oiint\limits_{S_1}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}=\oiint\limits_S\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}+\oiint\limits_{S_1^-}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}=\oiint\limits_{S+S_1^-}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}$,

$\because\Omega$不包含原点，

$\therefore\boldsymbol{v}\in C^1(\Omega)$且由上述题8(2)可知$\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$,

$\therefore\oiint\limits_{S+S_1^-}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}=\iiint\limits_\Omega\boldsymbol{\nabla}\cdot\boldsymbol{v}\mathrm{d}V=\iiint\limits_\Omega0\mathrm{d}V=0$,

$\therefore\oiint\limits_S\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}=\oiint\limits_{S_1}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}$,

$\therefore \boldsymbol{v}=\frac{\boldsymbol{r}}{r^3}$穿过包围原点的任意简单光滑闭曲面的电通量都相等，故可取一个特殊的曲面计算电通量的值.

不妨取$S_1: r=a, a>0$，记Ω_1是S_1围成的区域，

$\therefore \underset{S_1}{\oiint} \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{S}=\underset{S_1}{\oiint} \frac{\boldsymbol{r}}{r^3}\cdot \mathrm{d}\boldsymbol{S}=\underset{S_1}{\oiint} \frac{\boldsymbol{r}}{a^3}\cdot \mathrm{d}\boldsymbol{S}=\frac{1}{a^3}\underset{S_1}{\oiint} \boldsymbol{r}\cdot \mathrm{d}\boldsymbol{S}=\frac{1}{a^3}\underset{\Omega_1}{\iiint} \boldsymbol{\nabla}\cdot \boldsymbol{r}\mathrm{d}V$
$=\frac{1}{a^3}\underset{\Omega_1}{\iiint}(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z})\mathrm{d}V=\frac{1}{a^3}\underset{\Omega_1}{\iiint} 3\mathrm{d}V=\frac{3}{a^3}\underset{\Omega_1}{\iiint}\mathrm{d}V=\frac{3}{a^3}\frac{4}{3}\pi a^3=4\pi.$[1]

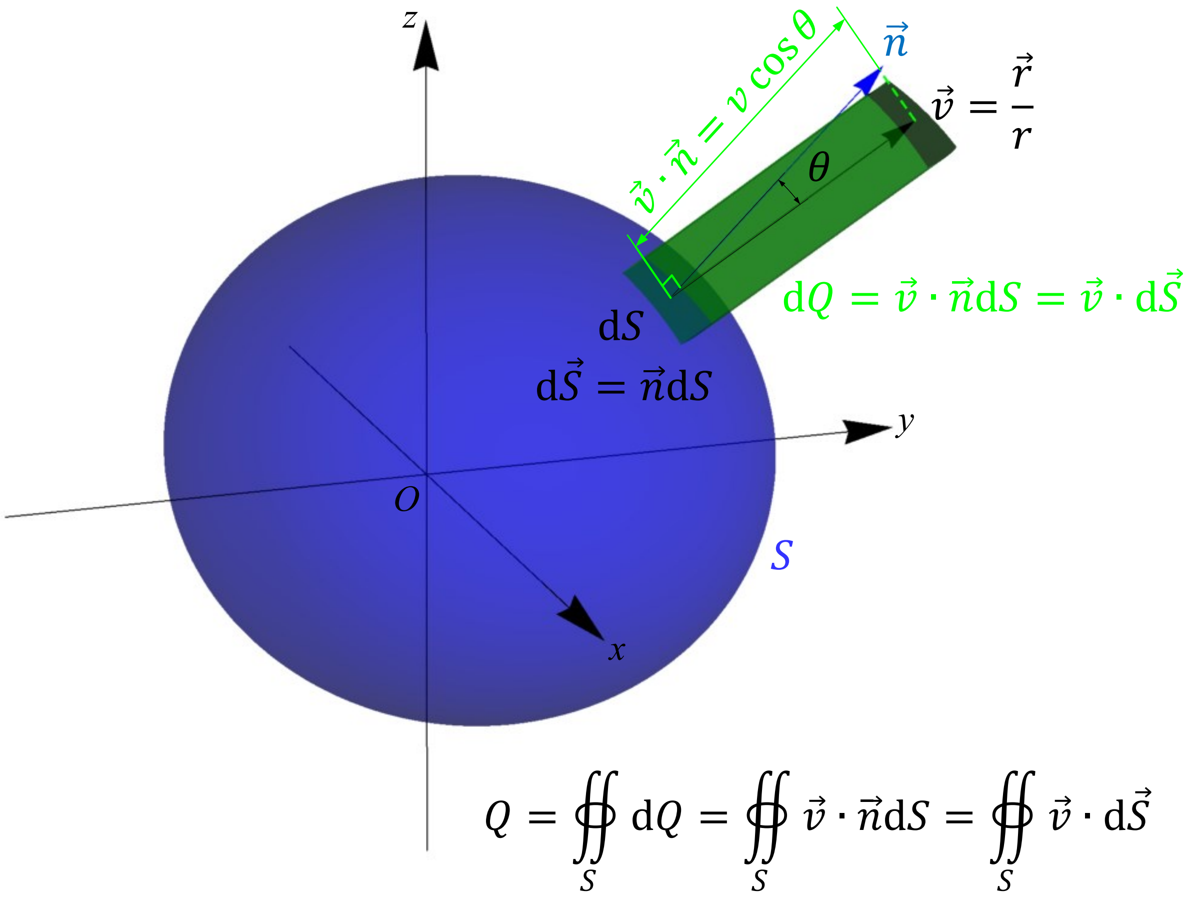
[1]该题给出了真空中点电荷电场的高斯定理的证明. 真空中位于原点的点电荷q产生的电场

$$\boldsymbol{E}=\frac{q}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{r^3}=\frac{q}{4\pi\varepsilon_0}\boldsymbol{v},$$

故该电场穿过包围该电荷的任意简单光滑闭曲面的电通量

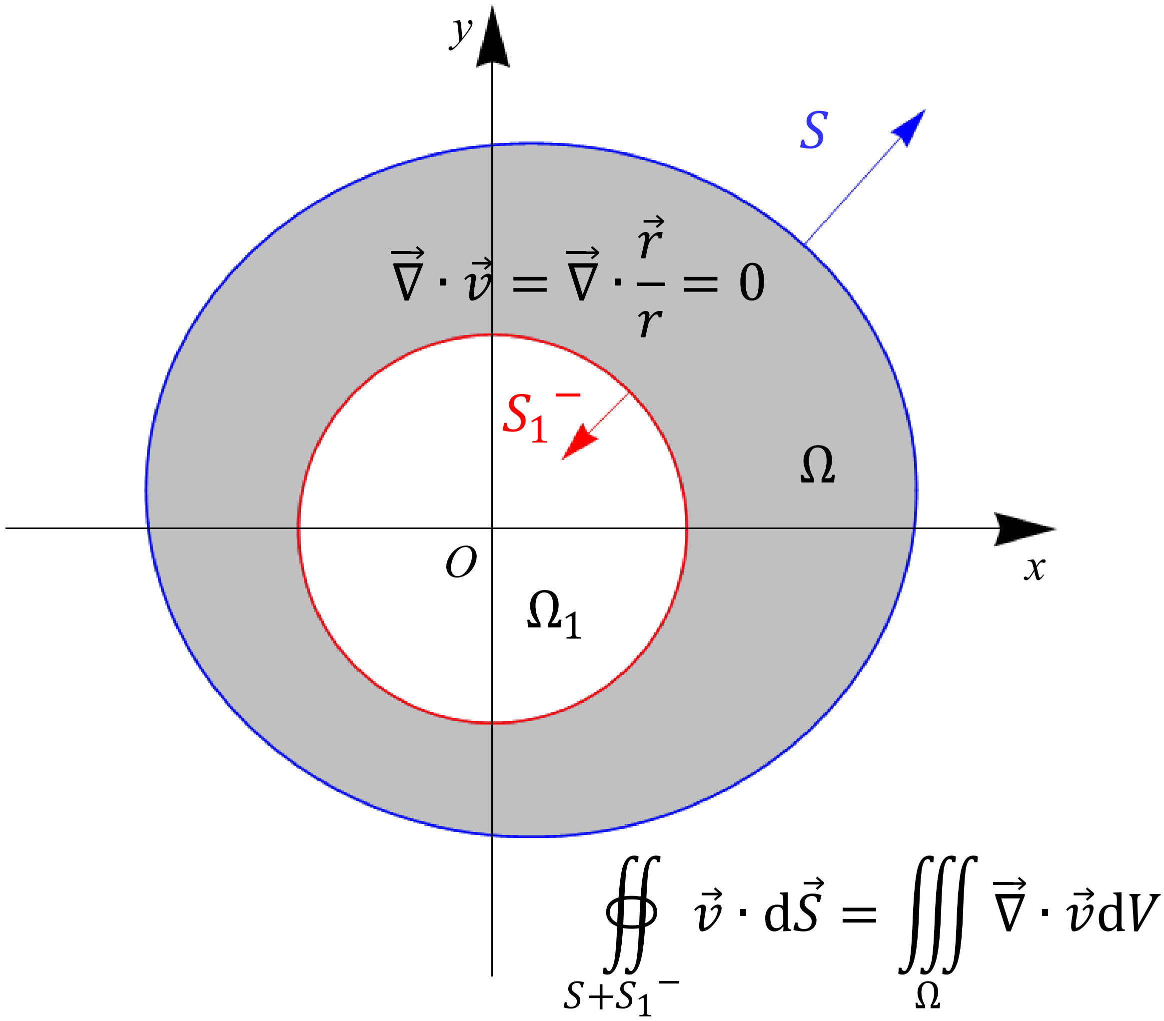
$$\underset{S}{\oiint} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{S}=\frac{q}{4\pi\varepsilon_0}\underset{S}{\oiint} \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{S}=\frac{q}{\varepsilon_0}.$$

即真空中的电荷的电场穿过包围该点电荷的任意简单光滑闭曲面的电通量与该电荷的电量成正比.



z
$\vec{n}$
$\vec{v} = \frac{\vec{r}}{r}$
$\vec{v} \cdot \vec{n} = v\cos\theta$
θ
$\mathrm{d}Q = \vec{v} \cdot \vec{n}\mathrm{d}S = \vec{v} \cdot \mathrm{d}\vec{S}$
$\mathrm{d}S$
$\mathrm{d}\vec{S} = \vec{n}\mathrm{d}S$
y
O
S
x
$Q = \oiint_S \mathrm{d}Q = \oiint_S \vec{v} \cdot \vec{n}\mathrm{d}S = \oiint_S \vec{v} \cdot \mathrm{d}\vec{S}$

(a)



(b)

图 7: 习题13.6 9.题图示

8. 证明下列向量场为无旋场：

(1)$\boldsymbol{v}=(x-x_0)\boldsymbol{i}+(y-y_0)\boldsymbol{j}+(z-z_0)\boldsymbol{k}$;

(2)$\boldsymbol{v}=yz(2x+y+z)\boldsymbol{i}+zx(x+2y+z)\boldsymbol{j}+xy(x+y+2z)\boldsymbol{k}$.

证明：(1)$\mathrm{rot}\boldsymbol{v}=\boldsymbol{\nabla}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x-x_0 & y-y_0 & z-z_0\end{vmatrix}$

$=(\frac{\partial(z-z_0)}{\partial y}-\frac{\partial(y-y_0)}{\partial z},\frac{\partial(x-x_0)}{\partial z}-\frac{\partial(z-z_0)}{\partial x},\frac{\partial(y-y_0)}{\partial x}-\frac{\partial(x-x_0)}{\partial y})=(0,0,0)=\boldsymbol{0}$.

(2)$\boldsymbol{v}=(2xyz+y^2z+yz^2)\boldsymbol{i}+(zx^2+2xyz+xz^2)\boldsymbol{j}+(x^2y+xy^2+2xyz)\boldsymbol{k}$,

$$\begin{aligned}
\mathrm{rot}\boldsymbol{v}=\boldsymbol{\nabla}\times\boldsymbol{v}&=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xyz+y^2z+yz^2 & zx^2+2xyz+xz^2 & x^2y+xy^2+2xyz\end{vmatrix}\\
&=(\frac{\partial(x^2y+xy^2+2xyz)}{\partial y}-\frac{\partial(zx^2+2xyz+xz^2)}{\partial z},\\
&\qquad\frac{\partial(2xyz+y^2z+yz^2)}{\partial z}-\frac{\partial(x^2y+xy^2+2xyz)}{\partial x},\\
&\qquad\qquad\frac{\partial(zx^2+2xyz+xz^2)}{\partial x}-\frac{\partial(2xyz+y^2z+yz^2)}{\partial y})\\
&=(x^2+2xy+2xz-(x^2+2xy+2zx),\\
&\qquad 2xy+y^2+2yz-(2xy+y^2+2yz),\\
&\qquad\qquad 2zx+2yz+z^2-(2zx+2yz+z^2))\\
&=\boldsymbol{0}.
\end{aligned}$$

14.7 习题13.1解答

1. 验证梯度算子$\boldsymbol{\nabla}$的下列性质，其中α,β为任意常数，f,g为任意可微函数：

   (1)$\boldsymbol{\nabla}(\alpha f+\beta g)=\alpha\boldsymbol{\nabla}+\beta\boldsymbol{\nabla}g$;

   (2)$\boldsymbol{\nabla}(fg)=g\boldsymbol{\nabla}f+f\boldsymbol{\nabla}g$;

   (3)$\boldsymbol{\nabla}(\frac{f}{g})=\frac{g\boldsymbol{\nabla}f-f\boldsymbol{\nabla}g}{g^2}$(在$g$不等于零处成立).

   证明：(1)

$$\begin{aligned}
\boldsymbol{\nabla}(\alpha f+\beta g)&=(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})(\alpha f+\beta g)\\
&=\frac{\partial(\alpha f+\beta g)}{\partial x}\boldsymbol{i}+\frac{\partial(\alpha f+\beta g)}{\partial y}\boldsymbol{j}+\frac{\partial(\alpha f+\beta g)}{\partial z}\boldsymbol{k}\\
&=(\alpha\frac{\partial f}{\partial x}+\beta\frac{\partial g}{\partial x})\boldsymbol{i}+(\alpha\frac{\partial f}{\partial y}+\beta\frac{\partial g}{\partial y})\boldsymbol{j}+(\alpha\frac{\partial f}{\partial z}+\beta\frac{\partial g}{\partial z})\boldsymbol{k}\\
&=\alpha(\frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}+\frac{\partial f}{\partial z}\boldsymbol{k})+\beta(\frac{\partial g}{\partial x}\boldsymbol{i}+\frac{\partial g}{\partial y}\boldsymbol{j}+\frac{\partial g}{\partial z}\boldsymbol{k})\\
&=\alpha\boldsymbol{\nabla}f+\beta\boldsymbol{\nabla}g.
\end{aligned}$$

(2)

$$
\begin{aligned}
\boldsymbol{\nabla}(fg) &= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k})(fg) \\
&= \frac{\partial(fg)}{\partial x}\boldsymbol{i} + \frac{\partial(fg)}{\partial y}\boldsymbol{j} + \frac{\partial(fg)}{\partial z}\boldsymbol{k} \\
&= (g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x})\boldsymbol{i} + (g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y})\boldsymbol{j} + (g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z})\boldsymbol{k} \\
&= g(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}) + f(\frac{\partial g}{\partial x}\boldsymbol{i} + \frac{\partial g}{\partial y}\boldsymbol{j} + \frac{\partial g}{\partial z}\boldsymbol{k}) \\
&= g\boldsymbol{\nabla} f + f\boldsymbol{\nabla} g.
\end{aligned}
$$

(3)

$$
\begin{aligned}
\boldsymbol{\nabla}(\frac{f}{g}) &= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k})(\frac{f}{g}) \\
&= \frac{\partial}{\partial x}(\frac{f}{g})\boldsymbol{i} + \frac{\partial}{\partial y}(\frac{f}{g})\boldsymbol{j} + \frac{\partial}{\partial z}(\frac{f}{g})\boldsymbol{k} \\
&= \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\boldsymbol{i} + \frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\boldsymbol{j} + \frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\boldsymbol{k} \\
&= \frac{g(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}) - f(\frac{\partial g}{\partial x}\boldsymbol{i} + \frac{\partial g}{\partial y}\boldsymbol{j} + \frac{\partial g}{\partial z}\boldsymbol{k})}{g^2} \\
&= \frac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2}.
\end{aligned}
$$

2. 验证散度算子的下列性质(其中f为函数，$\boldsymbol{u}, \boldsymbol{v}$是向量场)：

$$
\boldsymbol{\nabla}\cdot(\boldsymbol{u}\times\boldsymbol{v}) = -\boldsymbol{u}\cdot\boldsymbol{\nabla}\times\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\times\boldsymbol{u}.
$$

证明：

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{u}\times\boldsymbol{v}) &= (\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})\cdot\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ u_1(x,y,z) & u_2(x,y,z) & u_3(x,y,z)\\ v_1(x,y,z) & v_2(x,y,z) & v_3(x,y,z)\end{vmatrix}\\
&= (\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})\cdot[(u_2v_3-u_3v_2)\boldsymbol{i}+(u_3v_1-u_1v_3)\boldsymbol{j}+(u_1v_2-u_2v_2)\boldsymbol{k}]\\
&= \frac{\partial(u_2v_3-u_3v_2)}{\partial x}+\frac{\partial(u_3v_1-u_1v_3)}{\partial y}+\frac{\partial(u_1v_2-u_2v_1)}{\partial z}\\
&= \frac{\partial u_2}{\partial x}v_3+u_2\frac{\partial v_3}{\partial x}-\frac{\partial u_3}{\partial x}v_2-u_3\frac{\partial v_2}{\partial x}\\
&+\frac{\partial u_3}{\partial y}v_1+u_3\frac{\partial v_1}{\partial y}-\frac{\partial u_1}{\partial y}v_3-u_1\frac{\partial v_3}{\partial y}\\
&+\frac{\partial u_1}{\partial z}v_2+u_1\frac{\partial v_2}{\partial z}-\frac{\partial u_2}{\partial z}v_1-u_2\frac{\partial v_1}{\partial z}\\
&= u_1(\frac{\partial v_2}{\partial z}-\frac{\partial v_3}{\partial y})+u_2(\frac{\partial v_3}{\partial x}-\frac{\partial v_1}{\partial z})+u_3(\frac{\partial v_1}{\partial y}-\frac{\partial v_2}{\partial x})\\
&+v_1(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z})+v_2(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x})+v_3(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y})\\
&= -u_1(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z})-u_2(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x})-u_3(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y})\\
&\quad+v_1(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z})+v_2(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x})+v_3(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y})\\
&= -(u_1\boldsymbol{i}+u_2\boldsymbol{j}+u_3\boldsymbol{k})\cdot[(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z})\boldsymbol{i}+(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x})\boldsymbol{j}+(\frac{\partial v_3}{\partial x}-\frac{\partial v_1}{\partial y})\boldsymbol{k}]\\
&\quad+(v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k})\cdot[(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z})\boldsymbol{i}+(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x})\boldsymbol{j}+(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y})\boldsymbol{k}]\\
&= -(u_1\boldsymbol{i}+u_2\boldsymbol{j}+u_3\boldsymbol{k})\cdot\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}+(v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k})\cdot\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u_1 & u_2 & u_3\end{vmatrix}\\
&= -\boldsymbol{u}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})+\boldsymbol{v}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u})
\end{aligned}
$$

3. 设$\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}, r=\sqrt{x^2+y^2+z^2}$：

   (1)设$f(u)$为可微函数，求$\boldsymbol{\nabla} f(r)$；

   (2)设$\boldsymbol{F}=f(r)\boldsymbol{r}$，求证$\boldsymbol{\nabla}\times\boldsymbol{F}\equiv\boldsymbol{0}$. 又问当$f$满足什么条件时，$\boldsymbol{\nabla}\cdot\boldsymbol{F}=0$?

解：(1)

$$\begin{aligned}
\boldsymbol{\nabla} f(r) &= \frac{\partial f(r)}{\partial x}\boldsymbol{i} + \frac{\partial f(r)}{\partial y}\boldsymbol{j} + \frac{\partial f(r)}{\partial z}\boldsymbol{k} \\
&= f'(r)\frac{\partial r}{\partial x}\boldsymbol{i} + f'(r)\frac{\partial r}{\partial y}\boldsymbol{j} + f'(r)\frac{\partial r}{\partial z}\boldsymbol{k} = f'(r)(\frac{\partial r}{\partial x}\boldsymbol{i} + \frac{\partial r}{\partial y}\boldsymbol{j} + \frac{\partial r}{\partial z}\boldsymbol{k}) \\
&= f'(r)(\frac{x}{\sqrt{x^2+y^2+z^2}}\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\boldsymbol{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\boldsymbol{k}) \\
&= \frac{f'(r)}{\sqrt{x^2+y^2+z^2}}(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) = \frac{f'(r)}{r}\boldsymbol{r}.
\end{aligned}$$

(2)

i)证明：

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{F} &= \boldsymbol{\nabla} \times [f(r)\boldsymbol{r}] = \boldsymbol{\nabla} f(r) \times \boldsymbol{r} + f(r)\boldsymbol{\nabla} \times \boldsymbol{r} \\
&= \frac{f'(r)}{r}\boldsymbol{r} \times \boldsymbol{r} + f(r)\boldsymbol{\nabla} \times \boldsymbol{r} \\
&= 0 + f(r)\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
&= f(r)[(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z})\boldsymbol{i} + (\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x})\boldsymbol{j} + (\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y})\boldsymbol{k}] \\
&= f(r)(0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}) = 0.
\end{aligned}$$

ii)

$\because$

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{F} &= \boldsymbol{\nabla} \cdot [f(r)\boldsymbol{r}] = \boldsymbol{\nabla} f(r) \cdot \boldsymbol{r} + f(r)\boldsymbol{\nabla} \cdot \boldsymbol{r} = \frac{f'(r)}{r}\boldsymbol{r} \cdot \boldsymbol{r} + f(r)(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}) \\
&= \frac{\mathrm{d}f(r)}{\mathrm{d}r}r + 3f(r) = 0,
\end{aligned}$$

$\therefore$当$f(r) \neq 0$时，$r \neq 0$,

$$\frac{\mathrm{d}f(r)}{f(r)} = -3\frac{\mathrm{d}r}{r},$$

$\therefore$

$$\ln|f(r)| = -3\ln|r| + C_1,$$

$\therefore$

$$f(r) = \frac{C}{r^3}, C\text{为任意常数}.$$

4. 验证旋度算子的下列基本公式：

(1)$\boldsymbol{\nabla}\times(\alpha\boldsymbol{u}+\beta\boldsymbol{v})=\alpha\boldsymbol{\nabla}\times\boldsymbol{u}+\beta\boldsymbol{\nabla}\times\boldsymbol{v}$;

(2)$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f)=\boldsymbol{0}$;

(3)$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})=0$.

证明：(1)

$$\begin{aligned}\boldsymbol{\nabla}\times(\alpha\boldsymbol{u}+\beta\boldsymbol{v})&=\boldsymbol{\nabla}\times(\alpha u_1\boldsymbol{i}+\alpha u_2\boldsymbol{j}+\alpha u_3\boldsymbol{k}+\beta v_1\boldsymbol{i}+\beta v_2\boldsymbol{j}+\beta v_3\boldsymbol{k})\\&=\boldsymbol{\nabla}\times[(\alpha u_1+\beta v_1)\boldsymbol{i}+(\alpha u_2+\beta v_2)\boldsymbol{j}+(\alpha u_3+\beta v_3)\boldsymbol{k}]\\&=\frac{\partial(\alpha u_1+\beta v_1)}{\partial x}+\frac{\partial(\alpha u_2+\beta v_2)}{\partial y}+\frac{\partial(\alpha u_3+\beta v_3)}{\partial z}\\&=\alpha\frac{\partial u_1}{\partial x}+\beta\frac{\partial v_1}{\partial x}+\alpha\frac{\partial u_2}{\partial y}+\beta\frac{\partial v_2}{\partial y}+\alpha\frac{\partial u_3}{\partial z}+\beta\frac{\partial v_3}{\partial z}\\&=\alpha(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z})+\beta(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z})\\&=\alpha\boldsymbol{\nabla}\cdot\boldsymbol{u}+\beta\boldsymbol{\nabla}\cdot\boldsymbol{v}.\end{aligned}$$

(2)当$f\in C^2$时

$$\begin{aligned}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f)=\boldsymbol{\nabla}\times(\frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}+\frac{\partial f}{\partial z}\boldsymbol{k})=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\\frac{\partial f}{\partial x}&\frac{\partial f}{\partial y}&\frac{\partial f}{\partial z}\end{vmatrix}\\=(\frac{\partial^2 f}{\partial x\partial z}-\frac{\partial^2 f}{\partial z\partial x})\boldsymbol{i}+(\frac{\partial^2 f}{\partial z\partial x}-\frac{\partial^2 f}{\partial x\partial z})\boldsymbol{j}+(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x})\boldsymbol{k}\\=0\boldsymbol{i}+0\boldsymbol{j}+0\boldsymbol{k}=\boldsymbol{0}.\end{aligned}$$

(3)当$\boldsymbol{v}\in C^2$时

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})&=(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})\cdot\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\v_1(x,y,z)&v_2(x,y,z)&v_3(x,y,z)\end{vmatrix}\\&=(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})\cdot[(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z})\boldsymbol{i}+(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x})\boldsymbol{j}+(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y})\boldsymbol{k}]\\&=(\frac{\partial^2 v_3}{\partial x\partial y}-\frac{\partial^2 v_2}{\partial x\partial z})+(\frac{\partial^2 v_1}{\partial y\partial z}-\frac{\partial^2 v_3}{\partial y\partial x})+(\frac{\partial^2 v_2}{\partial z\partial x}-\frac{\partial^2 v_1}{\partial z\partial y})\\&=(\frac{\partial^2 v_3}{\partial x\partial y}-\frac{\partial^2 v_3}{\partial y\partial x})+(\frac{\partial^2 v_1}{\partial y\partial z}-\frac{\partial^2 v_1}{\partial z\partial y})+(\frac{\partial^2 v_2}{\partial z\partial x}-\frac{\partial^2 v_2}{\partial x\partial z})\\&=0.\end{aligned}$$

5. 求下列向量场的散度：

(1)$\boldsymbol{v}=xyz(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})$;

(2)$\boldsymbol{v}=(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\times\boldsymbol{c}$;

(3)$\boldsymbol{v}=[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}](x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})$(其中$\boldsymbol{c}$为常值向量).

解：(1)

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{v}&=\boldsymbol{\nabla}(xyz)\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})+xyz\boldsymbol{\nabla}\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\\&=(yz\boldsymbol{i}+xz\boldsymbol{j}+xy\boldsymbol{k})\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})+xyz(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z})\\&=xyz+xyz+xyz+3xyz=6xyz.\end{aligned}$$

(2)

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{v}&=\boldsymbol{c}\cdot[\boldsymbol{\nabla}\times(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})]-(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot(\boldsymbol{\nabla}\times\boldsymbol{c})\\&=\boldsymbol{c}\cdot\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\x&y&z\end{vmatrix}-(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{0}\\&=\boldsymbol{c}\cdot[(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z})\boldsymbol{i}+(\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x})\boldsymbol{j}+(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y})\boldsymbol{k}]-0\\&=\boldsymbol{c}\cdot\boldsymbol{0}-0=0.\end{aligned}$$

(3)记$\boldsymbol{c}=(c_1,c_2,c_3)$

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{v}&=[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}]\boldsymbol{\nabla}\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})+(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{\nabla}[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}]\\&=[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}](\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z})+(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{\nabla}(c_1x+c_2y+c_3z)\\&=3[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}]+(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot(c_1\boldsymbol{i}+c_2\boldsymbol{j}+c_3\boldsymbol{k})\\&=3[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}]+(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}\\&=4(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}.\end{aligned}$$

6. 求下列向量场的旋度：

(1)$\boldsymbol{v}=y^2z\boldsymbol{i}+z^2x\boldsymbol{j}+x^2y\boldsymbol{k}$;

(2)$\boldsymbol{v}=f(\sqrt{x^2+y^2+z^2})\boldsymbol{c}$(其中$\boldsymbol{c}$)为常值向量.

解：(1)

$$\boldsymbol{\nabla}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\y^2z&z^2x&x^2y\end{vmatrix}=(x^2-2xz)\boldsymbol{i}+(y^2-2xy)\boldsymbol{j}+(z^2-2yz)\boldsymbol{k}.$$

(2)

$$
\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{v} &= \boldsymbol{\nabla} \times [f(\sqrt{x^2+y^2+z^2})\boldsymbol{c}] \\
&= \boldsymbol{\nabla} f(\sqrt{x^2+y^2+z^2}) \times \boldsymbol{c} + f(\sqrt{x^2+y^2+z^2})\boldsymbol{\nabla} \times \boldsymbol{c} \\
&= f'(\sqrt{x^2+y^2+z^2})(\frac{x}{\sqrt{x^2+y^2+z^2}}\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\boldsymbol{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\boldsymbol{k}) \times \boldsymbol{c} + \boldsymbol{0} \\
&= \frac{f'(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \times \boldsymbol{c}.
\end{aligned}
$$