21 含参变量的积分、向量场的微分运算

21.1 知识结构

第12章重积分

12.7 含参变量的积分

12.7.1 引言

12.7.2 含参变量的定积分

12.7.3 含参变量的广义积分

第13章向量场的微积分

13.1 向量场的微分运算

(a) 数量场的梯度算子

(b) 向量场的散度算子

(c) 向量场的旋度算子

21.2 习题12.7解答

1. 求下列含参变量积分的导数：

$(1) f(x)=\int_0^{\pi}\sin(xy)\mathrm{d}y$; $\quad (2) f(x)=\int_0^{x}\sin(xy)\mathrm{d}y$;

$(3) f(x)=\int_0^{1}\frac{x\mathrm{d}y}{\sqrt{1-x^2y^2}}$; $\quad (4) f(x)=\int_0^{x}f(y+x,y-x)\mathrm{d}y$.

解：$(1) f'(x)=\int_0^{\pi}\frac{\partial}{\partial x}\sin(xy)\mathrm{d}y=\int_0^{\pi}y\cos(xy)\mathrm{d}y=\frac{1}{x}\int_0^1 y\cos(xy)\mathrm{d}(xy)=\frac{1}{x}\int_0^{\pi}y\mathrm{d}\sin(xy)$

$=\frac{y}{x}\sin(xy)\Big|_0^{\pi}-\frac{1}{x}\int_0^{\pi}\sin(xy)\mathrm{d}y=\frac{\pi\sin(\pi x)}{x}+\frac{1}{x^2}\cos(xy)\Big|_0^{\pi}=\frac{\pi\sin(\pi x)}{x}+\frac{\cos(\pi x)}{x^2}-\frac{1}{x^2}$.

$(2) f'(x)=\int_0^{x}\frac{\partial}{\partial x}\sin(xy)\mathrm{d}y+\sin(x\cdot x)\cdot\frac{\mathrm{d}x}{\mathrm{d}x}-\sin(x\cdot 0)\cdot\frac{\mathrm{d}0}{\mathrm{d}x}=\int_0^{x}y\cos(xy)\mathrm{d}y-\sin(x^2)$

$=\frac{1}{x}\int_0^{x}y\mathrm{d}\sin(xy)+\sin(x^2)=\frac{y}{x}\sin(xy)\Big|_0^{x}-\frac{1}{x}\int_0^{x}\sin(xy)\mathrm{d}y+\sin(x^2)$

$=\sin(x^2)-\frac{1}{x^2}\int_0^{x}\sin(xy)\mathrm{d}(xy)+\sin(x^2)=2\sin(x^2)+\frac{1}{x^2}\cos(xy)\Big|_0^{x}$

$=2\sin(x^2)+\frac{\cos(x^2)}{x^2}-\frac{1}{x^2}$.

(3)方法1：$f'(x)=\int_0^{1}\frac{\partial}{\partial x}[\frac{x}{\sqrt{1-x^2y^2}}]\mathrm{d}y=\int_0^{1}\frac{\sqrt{1-x^2y^2}-x\frac{-2xy^2}{2\sqrt{1-x^2y^2}}}{1-x^2y^2}\mathrm{d}y=\int_0^1\frac{1}{(1-x^2y^2)^{\frac{3}{2}}}\mathrm{d}y$

$=\frac{1}{x}\int_0^1\frac{1}{(1-x^2y^2)^{\frac{3}{2}}}\mathrm{d}(xy)\xlongequal{xy=\sin\theta}\frac{1}{x}\int_0^{\arcsin x}\frac{1}{\cos^3\theta}\mathrm{d}\sin\theta=\frac{1}{x}\int_0^{\arcsin x}\frac{\cos\theta}{\cos^3\theta}\mathrm{d}\theta=\frac{1}{x}\int_0^{\arcsin x}\sec^2\theta\mathrm{d}\theta$

$=\frac{1}{x}\tan\theta\Big|_0^{\arcsin x}=\frac{1}{x}\frac{x}{\sqrt{1-x^2}}=\frac{1}{\sqrt{1-x^2}}$.

方法2：$\because f(x)=\int_0^1\frac{x\mathrm{d}y}{\sqrt{1-x^2y^2}}=\int_0^1\frac{\mathrm{d}(xy)}{\sqrt{1-x^2y^2}}=\arcsin(xy)\Big|_0^1=\arcsin x$,

$\therefore f'(x)=\frac{1}{\sqrt{1-x^2}}$.

(4)$f'(x)=\int_0^x \frac{\partial}{\partial x}f(y+x,y-x)\mathrm{d}y+f(x+x,x-x)\frac{\mathrm{d}x}{\mathrm{d}x}-f(0+x,0-x)\frac{\mathrm{d}0}{\mathrm{d}x}$
$=\int_0^x[f_1'(y+x,y-x)-f_2'(y+x,y-x)]\mathrm{d}y+f(2x,0)$.

2. 设$f(x)$是$[0,1]$上的连续函数，对$x\in[0,1]$，令$F(x)=\int_0^x f(t)(x-t)^{n-1}\mathrm{d}t$，求$F^{(n)}(x)$.

解：$F'(x)=\int_0^x \frac{\partial}{\partial x}[f(t)(x-t)^{n-1}]\mathrm{d}t+f(x)(x-x)^{n-1}\frac{\mathrm{d}x}{\mathrm{d}x}-f(0)(x-0)\frac{\mathrm{d}0}{\mathrm{d}x}$
$=(n-1)\int_0^x f(t)(x-t)^{n-2}\mathrm{d}t$,

$F''(x)=(n-1)\int_0^x \frac{\partial}{\partial x}[f(t)(x-t)^{n-2}]\mathrm{d}t+(n-1)f(x)(x-x)^{n-2}\frac{\mathrm{d}x}{\mathrm{d}x}-(n-1)f(0)(x-0)^{n-2}\frac{\mathrm{d}0}{\mathrm{d}x}$
$=(n-1)(n-2)\int_0^x f(t)(x-t)^{n-3}\mathrm{d}t$,

$\cdots$

$F^{(n-1)}(x)=(n-1)(n-2)\cdots[n-(n-1)]\int_0^x f(t)(x-t)^{n-(n-1)-1}\mathrm{d}t=(n-1)!\int_0^x f(t)\mathrm{d}t$,

$F^{(n)}(x)=(n-1)!f(x)$.

3. 设$f(y)=\int_0^1(x-1)x^y\ln^{-1}x\mathrm{d}x$，求$f'(y)$和$\lim\limits_{y\to+\infty}f(y)$，并证明$f(y)=\ln\frac{2+y}{1+y}(y>-1)$.

证明：$f'(y)=\int_0^1\frac{\partial}{\partial y}[(x-1)x^y\ln^{-1}x]\mathrm{d}x=\int_0^1(x-1)x^y\ln x\ln^{-1}x\mathrm{d}x=\int_0^1(x-1)x^y\mathrm{d}x$
$=\int_0^1(x^{y+1}-x^y)\mathrm{d}x=(\frac{1}{y+2}x^{y+2}-\frac{1}{y+1}x^{y+1})\big|_0^1=\frac{1}{y+2}-\frac{1}{y+1}$,
$\lim\limits_{y\to+\infty}f(y)=\int_0^1\lim\limits_{y\to+\infty}(x-1)x^y\ln^{-1}x\mathrm{d}x=\int_0^1 0\mathrm{d}x=0$,
$\therefore f(y)=\int_{+\infty}^y f'(t)\mathrm{d}t=\int_{+\infty}^y(\frac{1}{t+2}-\frac{1}{t+1})\mathrm{d}t=\ln\frac{t+2}{t+1}\big|_{+\infty}^y=\ln\frac{y+2}{y+1}-\lim\limits_{y\to+\infty}\ln(1+\frac{1}{t+1})$
$=\ln\frac{y+2}{y+1}$.

21.3 第12章补充题解答

1. 设$f(u)$是连续函数，求证：

$$\int_a^b\mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{n-1}}f(x_n)\mathrm{d}x_n=\frac{1}{(n-1)!}\int_a^b(b-x)^{n-1}f(x)\mathrm{d}x.$$

证明：当$n=1$时$\int_a^b f(x_1)\mathrm{d}x_1=\frac{1}{(1-1)!}\int_a^b(b-x)^{1-1}f(x)\mathrm{d}x$，命题成立，

当$n=2$时$\int_a^b\mathrm{d}x_1\int_a^{x_1}f(x_2)\mathrm{d}x_2=\int_a^b\mathrm{d}x_2\int_{x_2}^b f(x_2)\mathrm{d}x_1=\int_a^b(b-x_2)f(x_2)\mathrm{d}x_2$
$=\int_a^b(b-x)f(x)\mathrm{d}x$，命题成立，

假设当$n=k$时命题成立，即

$$\int_a^b\mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{k-1}}f(x_k)\mathrm{d}x_k=\frac{1}{(k-1)!}\int_a^b(b-x)^{k-1}f(x)\mathrm{d}x.$$

则当$n=k+1$时，令$F(x)=\int_a^x f(x_{k+1})\mathrm{d}x_{k+1}$

$$\begin{aligned}
&\int_a^b \mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{k-1}}\mathrm{d}x_k\int_a^{x_k}f(x_{k+1})\mathrm{d}x_{k+1}\\
=&\int_a^b \mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{k-1}}F(x_k)\mathrm{d}x_k\\
=&\frac{1}{(k-1)!}\int_a^b(b-x)^{k-1}F(x)\mathrm{d}x\\
=&\frac{1}{(k-1)!}\int_a^b(b-x)^{k-1}[\int_a^x f(x_{k+1})\mathrm{d}x_{k+1}]\mathrm{d}x\\
=&\frac{1}{(k-1)!}\int_a^b f(x_{k+1})[\int_{x_{k+1}}^b(b-x)^{k-1}\mathrm{d}x]\mathrm{d}x_{k+1}\\
=&\frac{1}{(k-1)!}\int_a^b f(x_{k+1})[-\frac{1}{k-1+1}(b-x)^{k-1+1}\Big|_{x_{k+1}}^b]\mathrm{d}x_{k+1}\\
=&\frac{1}{(k-1)!}\int_a^b f(x_{k+1})\frac{1}{k}(b-x_{k+1})^k\mathrm{d}x_{k+1}\\
=&\frac{1}{k!}\int_a^b f(x)(b-x)^k\mathrm{d}x,
\end{aligned}$$

故当$n=k+1$时命题也成立. 证毕.

2. 求椭圆$\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1$(质量均匀)绕直线$y=kx$的转动惯量，并说明$k$为何值时转动惯量最大.

   解：设$D=\{(x,y)\mid \frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1\}$，$\forall(x,y)\in D$到直线$y=kx$的距离为$d=\frac{|kx-y|}{\sqrt{1+k^2}}$，设椭圆的密度为$\mu$,

   则已知椭圆绕直线$y=kx$的转动惯量

   $J=\iint\limits_D d^2\mu\mathrm{d}x\mathrm{d}y=\iint\limits_D\frac{(kx-y)^2}{1+k^2}\mu\mathrm{d}x\mathrm{d}y$
   $=\frac{\mu}{1+k^2}\iint\limits_D(kar\cos\theta-br\sin\theta)^2\cdot abr\mathrm{d}r\mathrm{d}\theta=\frac{ab\mu}{1+k^2}\int_0^1 r^3\mathrm{d}r\int_0^{2\pi}(ka\cos\theta-b\sin\theta)^2\mathrm{d}\theta$
   $=\frac{ab\mu}{4(1+k^2)}\int_0^{2\pi}(k^2a^2\cos^2\theta+b^2\sin^2\theta-2kab\sin\theta\cos\theta)\mathrm{d}\theta$
   $=\frac{ab\mu}{4(1+k^2)}(k^2a^2\int_0^{2\pi}\cos^2\theta\mathrm{d}\theta+b^2\int_0^{2\pi}\sin^2\theta\mathrm{d}\theta-2ab\int_0^{2\pi}\sin\theta\cos\theta\mathrm{d}\theta)$
   $=\frac{ab\mu}{4(1+k^2)}(k^2a^24\cdot\frac{\pi}{4}+b^24\cdot\frac{\pi}{4}-0)=\frac{\pi ab\mu}{4(1+k^2)}(k^2a^2+b^2)=\frac{\pi}{4}ab\mu\frac{k^2a^2+b^2}{1+k^2}=\frac{\pi}{4}ab\mu(a^2+\frac{b^2-a^2}{1+k^2})$,

   i)若$a>b>0$，则$k=\infty$时转动惯量J取得最大值；

   ii)若$b>a>0$，则$k=0$时转动惯量J取得最大值；

   iii)若$a=b>0$，则转动惯量J恒为常数.

3. 设有半径为R，高为H的正圆锥体(质量均匀)，试求：
   (1)该圆锥体对位于其顶点处质量为m的质点的引力；
   (2)该圆锥体关于它的中心轴的转动惯量.

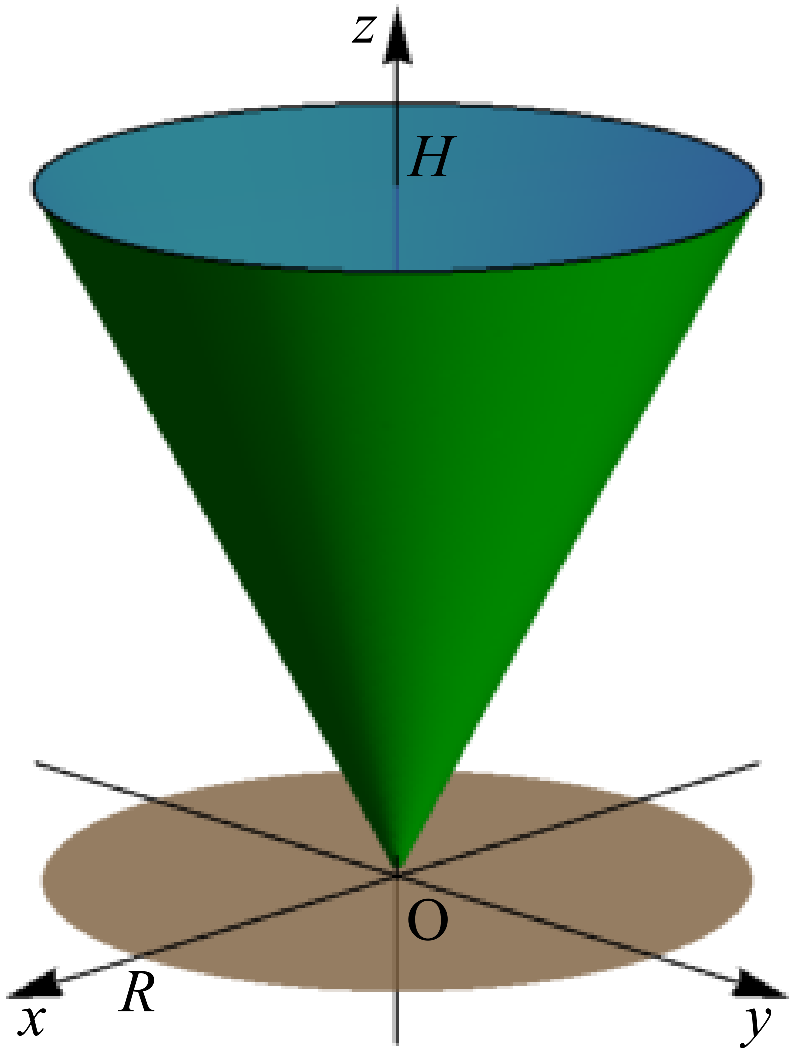

图 1: 第12章补充题 3.题图示

解：以该圆锥体的顶点为坐标原点，该圆锥体的旋转轴为z轴，建立空间直角坐标系，使该圆锥体在xOy平面上方，该圆锥体占据的空间区域可表示为$\Omega = \{(x,y,z) \mid \frac{H}{R}\sqrt{x^2+y^2} \leqslant z \leqslant H, x^2+y^2 \leqslant R^2\} = \{(\theta,r,z) \mid 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant r \leqslant R, \frac{H}{R}r \leqslant z \leqslant H\}$根据对称性可知引力的$x$和$y$分量$F_x = F_y = 0$，设该圆锥体的密度为$\rho$，则引力的$z$分量

$F_z = \iiint\limits_{\Omega} \frac{G\rho m}{x^2+y^2+z^2}\frac{z}{\sqrt{x^2+y^2+z^2}}\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta\int_0^R \mathrm{d}r\int_{\frac{H}{R}r}^{H}\frac{G\rho m}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}}r\mathrm{d}z$

$= 2\pi G\rho m\int_0^R \mathrm{d}r\int_{\frac{H}{R}r}^{H}\frac{rz}{(r^2+z^2)^{\frac{3}{2}}}\mathrm{d}z = 2\pi G\rho m\int_0^R \frac{1}{2}r\frac{1}{1-\frac{3}{2}}(r^2+z^2)^{-\frac{3}{2}+1}\Big|_{\frac{H}{R}r}^{H}\mathrm{d}r$

$$= 2\pi G\rho m\int_0^R r(\frac{1}{\sqrt{r^2+\frac{H^2}{R^2}r^2}}-\frac{1}{\sqrt{r^2+H^2}})\mathrm{d}r = 2\pi G\rho m\int_0^R(\frac{1}{\sqrt{1+\frac{H^2}{R^2}}}-\frac{r}{\sqrt{r^2+H^2}})\mathrm{d}r$$
$$= 2\pi G\rho m(\frac{1}{\sqrt{1+\frac{H^2}{R^2}}}r-\frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}}\sqrt{r^2+H^2})\Big|_0^R = 2\pi G\rho m(\frac{R}{\sqrt{1+\frac{H^2}{R^2}}}-\sqrt{R^2+H^2}+H)$$
$$= 2\pi G\rho m(H+\frac{R^2-R^2-H^2}{\sqrt{R^2+H^2}}) = 2\pi G\rho m(H-\frac{H^2}{\sqrt{R^2+H^2}}),$$

故该圆锥体对位于其顶点处质量为m的质点的引力为$(0,0,2\pi G\rho m(H-\frac{H^2}{\sqrt{R^2+H^2}}))$.

(2)该圆锥体关于它的中心轴的转动惯量

$$J_z = \iiint\limits_{\Omega}(x^2+y^2)\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta\int_0^R\mathrm{d}r\int_{\frac{H}{R}r}^{H}r^2\rho\cdot r\mathrm{d}z = 2\pi\rho\int_0^R r^3(H-\frac{H}{R}r)\mathrm{d}r$$
$$= 2\pi\rho(\frac{1}{4}Hr^4-\frac{1}{5}\frac{H}{R}r^5)\Big|_0^R = 2\pi\rho(\frac{1}{4}HR^4-\frac{1}{5}HR^4) = \frac{1}{10}\pi\rho HR^4.$$

4. 计算积分$\iint\limits_D(x^2+y^2)^{\frac{1}{2}}\mathrm{d}x\mathrm{d}y$，其中$D=\{(x,y)\mid 0\leqslant x,y\leqslant a\}$.

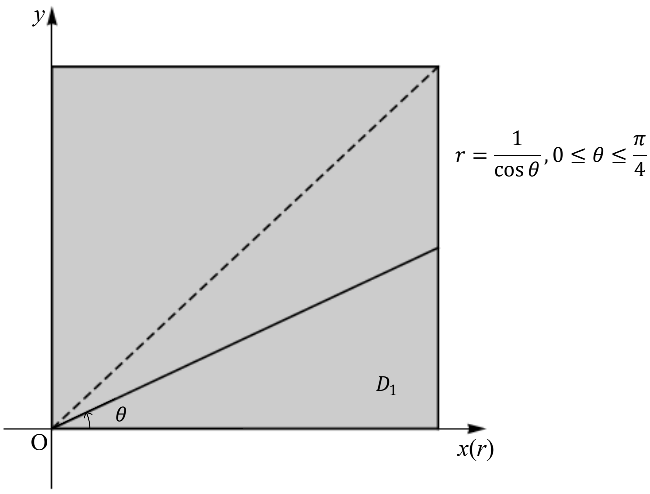

图 2: 第12章补充题 4.题图示

解：设$D_1=\{(x,y)\mid 0\leqslant x\leqslant 1,0\leqslant y\leqslant x\}=\{(r,\theta)\mid 0\leqslant\theta\leqslant\frac{\pi}{4},0\leqslant r\leqslant\frac{a}{\cos\theta}\}$，根据对称性可知

$$\iint\limits_D(x^2+y^2)^{\frac{1}{2}}\mathrm{d}x\mathrm{d}y = 2\iint\limits_{D_1}(x^2+y^2)^{\frac{1}{2}}\mathrm{d}x\mathrm{d}y = 2\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^{\frac{a}{\cos\theta}}r\cdot r\mathrm{d}r = 2\int_0^{\frac{\pi}{4}}\frac{1}{3}r^3\Big|_0^{\frac{a}{\cos\theta}}\mathrm{d}\theta$$
$$= 2\int_0^{2\pi}\frac{1}{3}\frac{a^3}{\cos^3\theta}\mathrm{d}\theta = \frac{2}{3}a^3\int_0^{\frac{\pi}{4}}\sec^3\theta\mathrm{d}\theta = \frac{2}{3}a^3\int_0^{\frac{\pi}{4}}\sec\theta\mathrm{d}\tan\theta = \frac{2}{3}a^3(\sec\theta\tan\theta\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan\theta\mathrm{d}\sec\theta)$$
$$= \frac{2}{3}a^3(\sqrt{2}-\int_0^{\frac{\pi}{4}}\tan^2\theta\sec\theta\mathrm{d}\theta) = \frac{2}{3}a^3[\sqrt{2}-\int_0^{\frac{\pi}{4}}(\sec^2\theta-1)\sec\theta\mathrm{d}\theta]$$
$$= \frac{2}{3}a^3[\sqrt{2}-\int_0^{\frac{\pi}{4}}(\sec^3\theta-\sec\theta)\mathrm{d}\theta] = \frac{2}{3}a^3(\sqrt{2}-\int_0^{\frac{\pi}{4}}\sec^3\theta\mathrm{d}\theta+\int_0^{\frac{\pi}{4}}\sec\theta\mathrm{d}\theta)$$
$$= \frac{\frac{2}{3}a^3}{\frac{2}{3}a^3+\frac{2}{3}a^3}(\frac{2}{3}\sqrt{2}a^3+\frac{2}{3}a^3\int_0^{\frac{\pi}{4}}\sec\theta\mathrm{d}\theta) = \frac{1}{2}(\frac{2}{3}\sqrt{2}a^3+\frac{2}{3}a^3\ln|\tan\theta+\sec\theta|\Big|_0^{\frac{\pi}{4}})$$
$$= \frac{1}{3}a^3[\sqrt{2}+\ln(1+\sqrt{2})].$$

5. 设$t>0,f(x)$在$[0,1]$上连续，求证$\int_0^t\mathrm{d}x\int_0^x\mathrm{d}y\int_0^y f(x)f(y)f(z)\mathrm{d}z=\frac{1}{6}(\int_0^t f(s)\mathrm{d}s)^3$.

证明：方法1：设$F(x)=\int_0^x f(s)\mathrm{d}s$，则

$$
\begin{aligned}
&\int_0^t \mathrm{d}x\int_0^x \mathrm{d}y\int_0^y f(x)f(y)f(z)\mathrm{d}z=\int_0^t f(x)\mathrm{d}x\int_0^x f(y)\mathrm{d}y\int_0^y f(z)\mathrm{d}z\\
=&\int_0^t f(x)\mathrm{d}x\int_0^x f(y)\mathrm{d}y\int_0^y \mathrm{d}F(z)=\int_0^t f(x)\mathrm{d}x\int_0^x f(y)[F(y)-F(0)]\mathrm{d}y\\
=&\int_0^t f(x)\mathrm{d}x\int_0^x f(y)F(y)\mathrm{d}y=\int_0^t f(x)\mathrm{d}x\int_0^x F(y)\mathrm{d}F(y)\\
=&\int_0^t f(x)\{\frac{1}{2}[F(x)]^2-[F(0)]^2\}\mathrm{d}x=\frac{1}{2}\int_0^t[F(x)]^2\mathrm{d}F(x)=\frac{1}{6}[F(x)]^3\big|_0^t=\frac{1}{6}[F(t)]^3\\
=&\frac{1}{6}(\int_0^t f(s)\mathrm{d}s)^3.
\end{aligned}
$$

方法2：记$\varphi(y)=(\int_0^y f(z)\mathrm{d}z)^2$，则$\varphi'(y)=2f(y)\int_0^y f(z)\mathrm{d}z$. 于是

$$
\begin{aligned}
\int_0^x \mathrm{d}y\int_0^y f(y)f(z)\mathrm{d}z=\int_0^x \mathrm{d}y[f(y)\int_0^y f(z)\mathrm{d}z]=\frac{1}{2}\int_0^x \varphi' y\mathrm{d}y=\frac{1}{2}\int_0^x \mathrm{d}\varphi(y)=\frac{1}{2}\varphi(x)\\
=\frac{1}{2}(\int_0^x f(s)\mathrm{d}s)^2,
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t \mathrm{d}x\int_0^x \mathrm{d}y\int_0^y f(x)f(y)f(z)\mathrm{d}z=\frac{1}{2}\int_0^t \mathrm{d}x[f(x)(\int_0^x f(s)\mathrm{d}s)^2]=\frac{1}{6}\int_0^t \mathrm{d}(\int_0^x f(s)\mathrm{d}s)^3\\
=\frac{1}{6}(\int_0^t f(s)\mathrm{d}s)^3.
\end{aligned}
$$

6. 计算$\int_L(x^{\frac{4}{3}}+y^{\frac{4}{3}})\mathrm{d}l$，其中$L$为星形线$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}(a>0)$.

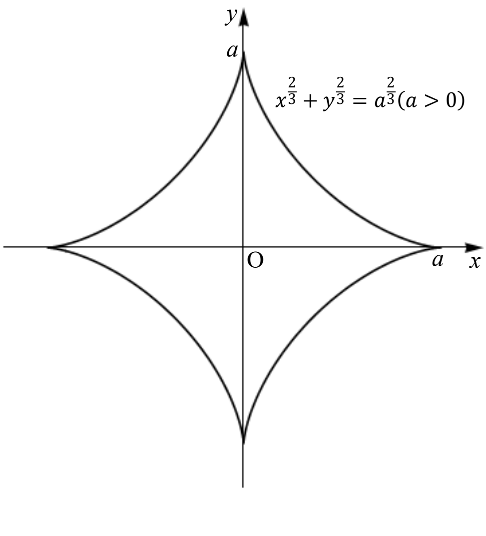

图 3: 第12章补充题 6.题图示

解：L的参数方程可表示为$\begin{cases}x=a\cos^3\theta,\\ y=a\sin^3\theta,\end{cases}\quad 0\leqslant\theta\leqslant 2\pi,$

$\therefore \int_L (x^{\frac{4}{3}}+y^{\frac{4}{3}})\mathrm{d}l = \int_0^{2\pi}(a^{\frac{4}{3}}\cos^4\theta + a^{\frac{4}{3}}\sin^4\theta)\sqrt{[(a\cos^3\theta)']^2+[(a\sin^3\theta)']^2}\mathrm{d}\theta$
$= \int_0^{2\pi}(a^{\frac{4}{3}}\cos^4\theta + a^{\frac{4}{3}}\sin^4\theta)\sqrt{[-3a\cos^2\theta\sin\theta]^2+[3a\sin^2\theta\cos\theta]^2}\mathrm{d}\theta$
$= \int_0^{2\pi}(a^{\frac{4}{3}}\cos^4\theta + a^{\frac{4}{3}}\sin^4\theta)\sqrt{9a^2\cos^2\theta\sin^2\theta}\mathrm{d}\theta = 4\int_0^{\frac{\pi}{2}}(a^{\frac{4}{3}}\cos^4\theta + a^{\frac{4}{3}}\sin^4\theta)\sqrt{9a^2\cos^2\theta\sin^2\theta}\mathrm{d}\theta$
$= 4\int_0^{\frac{\pi}{2}}(a^{\frac{4}{3}}\cos^4\theta + a^{\frac{4}{3}}\sin^4\theta)3a\cos\theta\sin\theta\mathrm{d}\theta = 12a^{\frac{7}{3}}\int_0^{\frac{\pi}{2}}(\cos^5\theta\sin\theta+\sin^5\theta\cos\theta)\mathrm{d}\theta$
$= 12a^{\frac{7}{3}}(\int_0^{\frac{\pi}{2}}\cos^5\theta\sin\theta\mathrm{d}\theta + \int_0^{\frac{\pi}{2}}\sin^5\theta\cos\theta\mathrm{d}\theta) = 12a^{\frac{7}{3}}(-\frac{1}{6}\cos^6\theta\Big|_0^{\frac{\pi}{2}} + \frac{1}{6}\sin^6\theta\Big|_0^{\frac{\pi}{2}})$
$= 4a^{\frac{7}{3}}.$

7. 设$f(u)$是连续函数，$D=\{(x,y)\mid x^2+y^2\leqslant a^2\}$. 试求

$$\iint\limits_D \frac{af(x)+bf(y)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y.$$

解：$\because$积分域D关于$y=x$对称，且在关于$y=x$的对称点(x,y)和$(x',y')=(y,x)$处
$\frac{f(x')}{f(x')+f(y')}=\frac{f(y)}{f(y)+f(x)}$,
$\therefore \iint\limits_D \frac{f(x)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y = \iint\limits_D \frac{f(y)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y = \frac{1}{2}\iint\limits_D \frac{f(x)+f(y)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y = \frac{1}{2}\iint\limits_D \mathrm{d}x\mathrm{d}y = \frac{1}{2}\pi a^2$,
$\therefore \iint\limits_D \frac{af(x)+bf(y)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y = a\iint\limits_D \frac{f(x)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y + b\iint\limits_D \frac{f(y)}{f(x)+f(y)}\mathrm{d}x\mathrm{d}y = \frac{a+b}{2}\pi a^2$.

8. 设$D=\{(x,y)\mid |x|+|y|\leqslant 1\}$，将$\iint\limits_D f(x+y)\mathrm{d}x\mathrm{d}y$化为定积分.

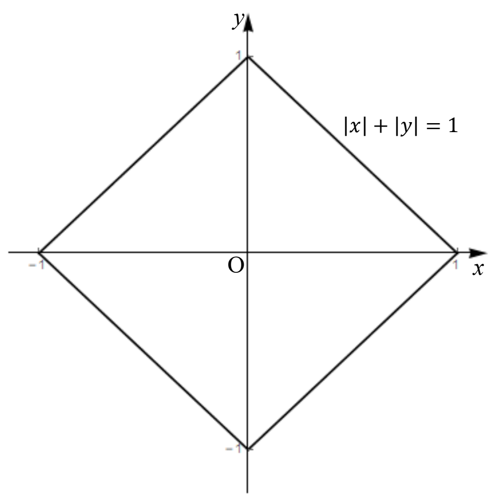

图 4: 第12章补充题 8.题图示

解：方法1：令$\begin{cases} u=x+y, \\ v=x-y, \end{cases}$ 则$D=\{(u,v)\mid -1\leqslant u\leqslant 1, -1\leqslant v\leqslant 1\}$,

$\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2,$

$\therefore \iint\limits_{D} f(x+y)\mathrm{d}x\mathrm{d}y = \iint\limits_{D} f(u)\frac{1}{|\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}|}\mathrm{d}u\mathrm{d}v = \iint\limits_{D} \frac{1}{2}f(u)\mathrm{d}u\mathrm{d}v = \frac{1}{2}\int_{-1}^{1}\mathrm{d}v\int_{-1}^{1}f(u)\mathrm{d}u$
$= \int_{-1}^{1} f(u)\mathrm{d}u.$

方法2：将重积分化为累次积分，有

$$\iint\limits_{|x|+|y|\leqslant 1} f(x+y)\mathrm{d}x\mathrm{d}y = \int_{-1}^{0}\mathrm{d}x\int_{-1-x}^{1+x} f(x+y)\mathrm{d}y + \int_{0}^{1}\mathrm{d}x\int_{-1+x}^{1-x} f(x+y)\mathrm{d}y,$$

令$x+y=u$,

$$\begin{aligned}\text{上式} &= \int_{-1}^{0}\mathrm{d}x\int_{-1}^{1+2x} f(u)\mathrm{d}u + \int_{0}^{1} x\mathrm{d}\int_{2x-1}^{1} f(u)\mathrm{d}u \\ &= \int_{-1}^{1} f(u)\mathrm{d}u\int_{\frac{u-1}{2}}^{\frac{u+1}{2}}\mathrm{d}x = \int_{-1}^{1} f(u)\mathrm{d}u.\end{aligned}$$

9. 设一元函数f连续，$\Omega_t = \{(x,y,z) \mid x^2+y^2 \leqslant t^2, 0 \leqslant z \leqslant h\}$. 令$F(t) = \iiint\limits_{\Omega_t}[z^2 + f(x^2+y^2)]\mathrm{d}V$，试求$\frac{\mathrm{d}F}{\mathrm{d}t}$和$\lim\limits_{t\to 0}\frac{F(t)}{t^2}$.

解：$F(t) = \iiint\limits_{\Omega_t}[z^2+f(x^2+y^2)]\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta\int_0^t\mathrm{d}r\int_0^h[z^2+f(r^2)]r\mathrm{d}z$
$= 2\pi\int_0^t[\frac{1}{3}rz^3 + rf(r^2)z]\big|_0^h\mathrm{d}r = 2\pi\int_0^t[\frac{1}{3}h^3r + hrf(r^2)]\mathrm{d}r,$
$\therefore \frac{\mathrm{d}F}{\mathrm{d}t} = \frac{2\pi}{3}h^3t + 2\pi htf(t^2),$
$\therefore \lim\limits_{t\to 0}\frac{1}{t^2}F(t) = \lim\limits_{t\to 0}\frac{\frac{\mathrm{d}F}{\mathrm{d}t}}{2t} = \lim\limits_{t\to 0}\frac{2\pi[\frac{1}{3}h^3t+htf(t^2)]}{2t} = \lim\limits_{t\to 0}\pi[\frac{1}{3}h^3+hf(t^2)] = \frac{\pi}{3}h^3 + \pi hf(0).$

10. 计算下列积分：
(1)$\iiint\limits_{\Omega}(ax+by+cz)^2\mathrm{d}V$，其中$\Omega = \{(x,y,z) \mid x^2+y^2+z^2 \leqslant R^2\}$；
(2)$\iiint\limits_{\Omega}(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2})\mathrm{d}V$，其中$\Omega = \{(x,y,z) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1\}$.

解：(1)方法1：由对称性可知

$\iiint\limits_{\Omega}(ax+by+cz)^2\mathrm{d}V = \iiint\limits_{\Omega}(a^2x^2+b^2y^2+c^2z^2+2abxy+2acxz+2bcyz)\mathrm{d}V$
$= \iiint\limits_{\Omega}(a^2x^2+b^2y^2+c^2z^2)\mathrm{d}V = (a^2+b^2+c^2)\iiint\limits_{\Omega} z^2\mathrm{d}V$
$= (a^2+b^2+c^2)\iiint\limits_{\Omega} r^2\cos^2\varphi \cdot r^2\sin\varphi\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r$
$= (a^2+b^2+c^2)\int_0^{2\pi}\mathrm{d}\theta\int_0^R r^4\mathrm{d}r\int_0^\pi\cos^2\varphi\sin\varphi\mathrm{d}\varphi$
$= \frac{2}{5}\pi R^4(a^2+b^2+c^2)(-\frac{1}{3}\cos^3\varphi)\big|_0^\pi = \frac{4}{15}\pi R^4(a^2+b^2+c^2).$

方法2：$\iiint\limits_{\Omega}(ax+by+cz)^2\mathrm{d}V = \iiint\limits_{\Omega}(a^2x^2+b^2y^2+c^2z^2+2abxy+2acxz+2bcyz)\mathrm{d}V$
$= \iiint\limits_{\Omega}(a^2x^2+b^2y^2+c^2z^2)\mathrm{d}V = (a^2+b^2+c^2)\iiint\limits_{\Omega} z^2\mathrm{d}V = \frac{1}{3}(a^2+b^2+c^2)\iiint\limits_{\Omega}(x^2+y^2+z^2)\mathrm{d}V$
$= \frac{1}{3}(a^2+b^2+c^2)\int_0^{2\pi}\mathrm{d}\theta\int_0^\pi\mathrm{d}\varphi\int_0^R r^2\cdot r^2\sin\varphi\mathrm{d}r = \frac{2}{3}\pi(a^2+b^2+c^2)\int_0^\pi\sin\varphi\mathrm{d}\varphi\int_0^R r^4\mathrm{d}r$
$= \frac{2}{3}\pi(a^2+b^2+c^2)(-\cos\varphi)\big|_0^\pi\frac{1}{5}r^5\big|_0^R = \frac{4}{15}\pi R^4(a^2+b^2+c^2).$

(2)令$\begin{cases} x = au, \\ y = bv, \\ z = cw, \end{cases}$ 则$\Omega = \{(u,v,w) \mid u^2+v^2+w^2 \leqslant 1\}$,

$\frac{\mathrm{D}(x,y,z)}{\mathrm{D}(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$,

$\therefore \iiint\limits_{\Omega}(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2})\mathrm{d}V = \iiint\limits_{\Omega}(u^2+v^2+w^2)|\frac{\mathrm{D}(u,v,w)}{\mathrm{D}(x,y,z)}|\mathrm{d}u\mathrm{d}v\mathrm{d}w = abc\iiint\limits_{\Omega}(u^2+v^2+w^2)\mathrm{d}u\mathrm{d}v\mathrm{d}w$
$= abc\frac{4}{15}\pi 1^4(1^2+1^2+1^2) = \frac{4}{5}\pi abc.$

21.4 习题13.1解答

1. 验证梯度算子$\boldsymbol{\nabla}$的下列性质，其中α, β为任意常数，f, g为任意可微函数：
   (1)$\boldsymbol{\nabla}(\alpha f+\beta g) = \alpha\boldsymbol{\nabla} + \beta\boldsymbol{\nabla} g$；
   (2)$\boldsymbol{\nabla}(fg) = g\boldsymbol{\nabla} f + f\boldsymbol{\nabla} g$；
   (3)$\boldsymbol{\nabla}(\frac{f}{g}) = \frac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2}$(在$g$不等于零处成立).

   证明：(1)

$$\begin{aligned}
\boldsymbol{\nabla}(\alpha f+\beta g) &= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k})(\alpha f+\beta g) \\
&= \frac{\partial(\alpha f+\beta g)}{\partial x}\boldsymbol{i} + \frac{\partial(\alpha f+\beta g)}{\partial y}\boldsymbol{j} + \frac{\partial(\alpha f+\beta g)}{\partial z}\boldsymbol{k} \\
&= (\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial g}{\partial x})\boldsymbol{i} + (\alpha\frac{\partial f}{\partial y} + \beta\frac{\partial g}{\partial y})\boldsymbol{j} + (\alpha\frac{\partial f}{\partial z} + \beta\frac{\partial g}{\partial z})\boldsymbol{k} \\
&= \alpha(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}) + \beta(\frac{\partial g}{\partial x}\boldsymbol{i} + \frac{\partial g}{\partial y}\boldsymbol{j} + \frac{\partial g}{\partial z}\boldsymbol{k}) \\
&= \alpha\boldsymbol{\nabla} f + \beta\boldsymbol{\nabla} g.
\end{aligned}$$

   (2)

$$\begin{aligned}
\boldsymbol{\nabla}(fg) &= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k})(fg) \\
&= \frac{\partial(fg)}{\partial x}\boldsymbol{i} + \frac{\partial(fg)}{\partial y}\boldsymbol{j} + \frac{\partial(fg)}{\partial z}\boldsymbol{k} \\
&= (g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x})\boldsymbol{i} + (g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y})\boldsymbol{j} + (g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z})\boldsymbol{k} \\
&= g(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}) + f(\frac{\partial g}{\partial x}\boldsymbol{i} + \frac{\partial g}{\partial y}\boldsymbol{j} + \frac{\partial g}{\partial z}\boldsymbol{k}) \\
&= g\boldsymbol{\nabla} f + f\boldsymbol{\nabla} g.
\end{aligned}$$

(3)

$$\begin{aligned}
\boldsymbol{\nabla}(\frac{f}{g}) &= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k})(\frac{f}{g}) \\
&= \frac{\partial}{\partial x}(\frac{f}{g})\boldsymbol{i} + \frac{\partial}{\partial y}(\frac{f}{g})\boldsymbol{j} + \frac{\partial}{\partial z}(\frac{f}{g})\boldsymbol{k} \\
&= \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\boldsymbol{i} + \frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\boldsymbol{j} + \frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\boldsymbol{k} \\
&= \frac{g(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}) - f(\frac{\partial g}{\partial x}\boldsymbol{i} + \frac{\partial g}{\partial y}\boldsymbol{j} + \frac{\partial g}{\partial z}\boldsymbol{k})}{g^2} \\
&= \frac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2}.
\end{aligned}$$

2. 验证散度算子的下列性质(其中f为函数，$\boldsymbol{u}, \boldsymbol{v}$是向量场)：

$$\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = -\boldsymbol{u} \cdot \boldsymbol{\nabla} \times \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \times \boldsymbol{u}.$$

证明：

$$
\begin{aligned}
\nabla\cdot(\boldsymbol{u}\times\boldsymbol{v}) &= (\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})\cdot\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ u_1(x,y,z) & u_2(x,y,z) & u_3(x,y,z)\\ v_1(x,y,z) & v_2(x,y,z) & v_3(x,y,z)\end{vmatrix}\\
&= (\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})\cdot[(u_2v_3-u_3v_2)\boldsymbol{i}+(u_3v_1-u_1v_3)\boldsymbol{j}+(u_1v_2-u_2v_2)\boldsymbol{k}]\\
&= \frac{\partial(u_2v_3-u_3v_2)}{\partial x}+\frac{\partial(u_3v_1-u_1v_3)}{\partial y}+\frac{\partial(u_1v_2-u_2v_1)}{\partial z}\\
&= \frac{\partial u_2}{\partial x}v_3+u_2\frac{\partial v_3}{\partial x}-\frac{\partial u_3}{\partial x}v_2-u_3\frac{\partial v_2}{\partial x}\\
&+ \frac{\partial u_3}{\partial y}v_1+u_3\frac{\partial v_1}{\partial y}-\frac{\partial u_1}{\partial y}v_3-u_1\frac{\partial v_3}{\partial y}\\
&+ \frac{\partial u_1}{\partial z}v_2+u_1\frac{\partial v_2}{\partial z}-\frac{\partial u_2}{\partial z}v_1-u_2\frac{\partial v_1}{\partial z}\\
&= u_1(\frac{\partial v_2}{\partial z}-\frac{\partial v_3}{\partial y})+u_2(\frac{\partial v_3}{\partial x}-\frac{\partial v_1}{\partial z})+u_3(\frac{\partial v_1}{\partial y}-\frac{\partial v_2}{\partial x})\\
&+ v_1(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z})+v_2(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x})+v_3(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y})\\
&= -u_1(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z})-u_2(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x})-u_3(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y})\\
&\quad + v_1(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z})+v_2(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x})+v_3(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y})\\
&= -(u_1\boldsymbol{i}+u_2\boldsymbol{j}+u_3\boldsymbol{k})\cdot[(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z})\boldsymbol{i}+(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x})\boldsymbol{j}+(\frac{\partial v_3}{\partial x}-\frac{\partial v_1}{\partial y})\boldsymbol{k}]\\
&\quad + (v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k})\cdot[(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z})\boldsymbol{i}+(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x})\boldsymbol{j}+(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y})\boldsymbol{k}]\\
&= -(u_1\boldsymbol{i}+u_2\boldsymbol{j}+u_3\boldsymbol{k})\cdot\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}+(v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k})\cdot\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u_1 & u_2 & u_3\end{vmatrix}\\
&= -\boldsymbol{u}\cdot(\nabla\times\boldsymbol{v})+\boldsymbol{v}\cdot(\nabla\times\boldsymbol{u})
\end{aligned}
$$

3. 设$\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}, r=\sqrt{x^2+y^2+z^2}$：
(1)设$f(u)$为可微函数，求$\nabla f(r)$；
(2)设$\boldsymbol{F}=f(r)\boldsymbol{r}$，求证$\nabla\times\boldsymbol{F}\equiv\boldsymbol{0}$. 又问当$f$满足什么条件时，$\nabla\cdot\boldsymbol{F}=0$?

解：(1)

$$\begin{aligned}
\boldsymbol{\nabla} f(r) &= \frac{\partial f(r)}{\partial x}\boldsymbol{i} + \frac{\partial f(r)}{\partial y}\boldsymbol{j} + \frac{\partial f(r)}{\partial z}\boldsymbol{k} \\
&= f'(r)\frac{\partial r}{\partial x}\boldsymbol{i} + f'(r)\frac{\partial r}{\partial y}\boldsymbol{j} + f'(r)\frac{\partial r}{\partial z}\boldsymbol{k} = f'(r)(\frac{\partial r}{\partial x}\boldsymbol{i} + \frac{\partial r}{\partial y}\boldsymbol{j} + \frac{\partial r}{\partial z}\boldsymbol{k}) \\
&= f'(r)(\frac{x}{\sqrt{x^2+y^2+z^2}}\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\boldsymbol{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\boldsymbol{k}) \\
&= \frac{f'(r)}{\sqrt{x^2+y^2+z^2}}(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}) = \frac{f'(r)}{r}\boldsymbol{r}.
\end{aligned}$$

(2)

i)证明：

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{\nabla} \times [f(r)\boldsymbol{r}] &= \boldsymbol{\nabla} f(r) \times \boldsymbol{r} + f(r)\boldsymbol{\nabla} \times \boldsymbol{r} \\
&= \frac{f'(r)}{r}\boldsymbol{r} \times \boldsymbol{r} + f(r)\boldsymbol{\nabla} \times \boldsymbol{r} \\
&= 0 + f(r)\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
&= f(r)[(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z})\boldsymbol{i} + (\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x})\boldsymbol{j} + (\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y})\boldsymbol{k}] \\
&= f(r)(0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}) = 0.
\end{aligned}$$

ii)

$\because$

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{F} &= \boldsymbol{\nabla} \cdot [f(r)\boldsymbol{r}] = \boldsymbol{\nabla} f(r) \cdot \boldsymbol{r} + f(r)\boldsymbol{\nabla} \cdot \boldsymbol{r} = \frac{f'(r)}{r}\boldsymbol{r} \cdot \boldsymbol{r} + f(r)(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}) \\
&= \frac{\mathrm{d}f(r)}{\mathrm{d}r}r + 3f(r) = 0,
\end{aligned}$$

$\therefore$当$f(r) \neq 0$时，$r \neq 0$,

$$\frac{\mathrm{d}f(r)}{f(r)} = -3\frac{\mathrm{d}r}{r},$$

$\therefore$

$$\ln|f(r)| = -3\ln|r| + C_1,$$

$\therefore$

$$f(r) = \frac{C}{r^3}, C\text{为任意常数}.$$

4. 验证旋度算子的下列基本公式：

(1)$\boldsymbol{\nabla} \times (\alpha\boldsymbol{u} + \beta\boldsymbol{v}) = \alpha\boldsymbol{\nabla} \times \boldsymbol{u} + \beta\boldsymbol{\nabla} \times \boldsymbol{v}$;

(2)$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f)=\mathbf{0}$;

(3)$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})=0$.

证明：(1)

$$\begin{aligned}\boldsymbol{\nabla}\times(\alpha\boldsymbol{u}+\beta\boldsymbol{v})&=\boldsymbol{\nabla}\times(\alpha u_1\boldsymbol{i}+\alpha u_2\boldsymbol{j}+\alpha u_3\boldsymbol{k}+\beta v_1\boldsymbol{i}+\beta v_2\boldsymbol{j}+\beta v_3\boldsymbol{k})\\&=\boldsymbol{\nabla}\times[(\alpha u_1+\beta v_1)\boldsymbol{i}+(\alpha u_2+\beta v_2)\boldsymbol{j}+(\alpha u_3+\beta v_3)\boldsymbol{k}]\\&=\frac{\partial(\alpha u_1+\beta v_1)}{\partial x}+\frac{\partial(\alpha u_2+\beta v_2)}{\partial y}+\frac{\partial(\alpha u_3+\beta v_3)}{\partial z}\\&=\alpha\frac{\partial u_1}{\partial x}+\beta\frac{\partial v_1}{\partial x}+\alpha\frac{\partial u_2}{\partial y}+\beta\frac{\partial v_2}{\partial y}+\alpha\frac{\partial u_3}{\partial z}+\beta\frac{\partial v_3}{\partial z}\\&=\alpha(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z})+\beta(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z})\\&=\alpha\boldsymbol{\nabla}\cdot\boldsymbol{u}+\beta\boldsymbol{\nabla}\cdot\boldsymbol{v}.\end{aligned}$$

(2)当$f\in C^2$时

$$\begin{aligned}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f)&=\boldsymbol{\nabla}\times(\frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}+\frac{\partial f}{\partial z}\boldsymbol{k})=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\\frac{\partial f}{\partial x}&\frac{\partial f}{\partial y}&\frac{\partial f}{\partial z}\end{vmatrix}\\&=(\frac{\partial^2 f}{\partial x\partial z}-\frac{\partial^2 f}{\partial z\partial x})\boldsymbol{i}+(\frac{\partial^2 f}{\partial z\partial x}-\frac{\partial^2 f}{\partial x\partial z})\boldsymbol{j}+(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x})\boldsymbol{k}\\&=0\boldsymbol{i}+0\boldsymbol{j}+0\boldsymbol{k}=\mathbf{0}.\end{aligned}$$

(3)当$\boldsymbol{v}\in C^2$时

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})&=(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})\cdot\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\v_1(x,y,z)&v_2(x,y,z)&v_3(x,y,z)\end{vmatrix}\\&=(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})\cdot[(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z})\boldsymbol{i}+(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x})\boldsymbol{j}+(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y})\boldsymbol{k}]\\&=(\frac{\partial^2 v_3}{\partial x\partial y}-\frac{\partial^2 v_2}{\partial x\partial z})+(\frac{\partial^2 v_1}{\partial y\partial z}-\frac{\partial^2 v_3}{\partial y\partial x})+(\frac{\partial^2 v_2}{\partial z\partial x}-\frac{\partial^2 v_1}{\partial z\partial y})\\&=(\frac{\partial^2 v_3}{\partial x\partial y}-\frac{\partial^2 v_3}{\partial y\partial x})+(\frac{\partial^2 v_1}{\partial y\partial z}-\frac{\partial^2 v_1}{\partial z\partial y})+(\frac{\partial^2 v_2}{\partial z\partial x}-\frac{\partial^2 v_2}{\partial x\partial z})\\&=0.\end{aligned}$$

5. 求下列向量场的散度：

(1)$\boldsymbol{v}=xyz(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})$;

(2)$\boldsymbol{v}=(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\times\boldsymbol{c}$;

(3)$\boldsymbol{v}=[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}](x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})$(其中$\boldsymbol{c}$为常值向量).

解：(1)

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{v}&=\boldsymbol{\nabla}(xyz)\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})+xyz\boldsymbol{\nabla}\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\\&=(yz\boldsymbol{i}+xz\boldsymbol{j}+xy\boldsymbol{k})\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})+xyz(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z})\\&=xyz+xyz+xyz+3xyz=6xyz.\end{aligned}$$

(2)

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{v}&=\boldsymbol{c}\cdot[\boldsymbol{\nabla}\times(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})]-(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot(\boldsymbol{\nabla}\times\boldsymbol{c})\\&=\boldsymbol{c}\cdot\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\x&y&z\end{vmatrix}-(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{0}\\&=\boldsymbol{c}\cdot[(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z})\boldsymbol{i}+(\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x})\boldsymbol{j}+(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y})\boldsymbol{k}]-0\\&=\boldsymbol{c}\cdot\boldsymbol{0}-0=0.\end{aligned}$$

(3)记$\boldsymbol{c}=(c_1,c_2,c_3)$

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{v}&=[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}]\boldsymbol{\nabla}\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})-(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{\nabla}[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}]\\&=[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}](\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z})-(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{\nabla}(c_1x+c_2y+c_3z)\\&=3[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}]-(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot(c_1\boldsymbol{i}+c_2\boldsymbol{j}+c_3\boldsymbol{k})\\&=3[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}]-(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}\\&=2[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot\boldsymbol{c}.\end{aligned}$$

6. 求下列向量场的旋度：

(1)$\boldsymbol{v}=y^2z\boldsymbol{i}+z^2x\boldsymbol{j}+x^2y\boldsymbol{k}$;

(2)$\boldsymbol{v}=f(\sqrt{x^2+y^2+z^2})\boldsymbol{c}$(其中$\boldsymbol{c}$)为常值向量.

解：(1)

$$\boldsymbol{\nabla}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\y^2z&z^2x&x^2y\end{vmatrix}=(x^2-2xz)\boldsymbol{i}+(y^2-2xy)\boldsymbol{j}+(z^2-2yz)\boldsymbol{k}.$$

(2)

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{v}&=\boldsymbol{\nabla}\times[f(\sqrt{x^2+y^2+z^2})\boldsymbol{c}]\\&=\boldsymbol{\nabla}f(\sqrt{x^2+y^2+z^2})\times\boldsymbol{c}+f(\sqrt{x^2+y^2+z^2})\boldsymbol{\nabla}\times\boldsymbol{c}\\&=f'(\sqrt{x^2+y^2+z^2})(\frac{x}{\sqrt{x^2+y^2+z^2}}\boldsymbol{i}+\frac{y}{\sqrt{x^2+y^2+z^2}}\boldsymbol{j}+\frac{z}{\sqrt{x^2+y^2+z^2}}\boldsymbol{k})\times\boldsymbol{c}+\boldsymbol{0}\\&=\frac{f'(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\times\boldsymbol{c}.\end{aligned}$$