24 高斯公式与斯托克斯公式

24.1 知识结构

第13章向量场的微积分

13.5 高斯公式与斯托克斯公式

13.5.1 高斯公式

13.5.2 斯托克斯公式

24.2 高斯公式与斯托克斯公式

1. 高斯公式

$\Omega \subset \mathbb{R}^3, \partial\Omega$逐片光滑，
向量场$\boldsymbol{F}(x,y,z) = X(x,y,z)\boldsymbol{i} + Y(x,y,z)\boldsymbol{j} + Z(x,y,z)\boldsymbol{k} \in C^1(\Omega)$,

则

$$\begin{aligned}\oiint_{\partial\Omega} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{S} = \oiint_{\partial\Omega} \boldsymbol{F}\cdot \boldsymbol{n}\mathrm{d}S &= \oiint_{\partial\Omega} X\mathrm{d}y\wedge\mathrm{d}z + Y\mathrm{d}z\wedge\mathrm{d}x + Z\mathrm{d}x\wedge\mathrm{d}y \\ &= \iiint_{\Omega} (\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \iiint_{\Omega} \boldsymbol{\nabla}\cdot\boldsymbol{F}\mathrm{d}x\mathrm{d}y\mathrm{d}z^1 \\ &= \iiint_{\Omega} \mathrm{div}\boldsymbol{F}\mathrm{d}x\mathrm{d}y\mathrm{d}z.\end{aligned}$$

【直接代入公式计算的题目：1.(1)/(2)/(3)/(4)/(6). (第一类题目)】

注意：

(a) 在非封闭曲面上的积分，可添加简单曲面构造封闭区域，利用高斯公式计算.

【利用该点计算的题目：1.(5). (第二类题目)】

(b) $\boldsymbol{F}(x,y,z) = X(x,y,z)\boldsymbol{i} + Y(x,y,z)\boldsymbol{j} + Z(x,y,z)\boldsymbol{k} \in C^1(\Omega)$，即$X,Y,Z$有一阶连续偏导数，也即$X,Y,Z$连续可微.

【考察该点的题目：1.(7). (第三类题目)】

(c) $\partial\Omega$外侧为正，即在$\partial\Omega$的外侧积分.

【考察该点的题目：1.(2).】

[1]这里的符号$\boldsymbol{\nabla}$可读作nabla.

(d) $\mathrm{d}y\wedge\mathrm{d}z, \mathrm{d}z\wedge\mathrm{d}x, \mathrm{d}x\wedge\mathrm{d}y$与$X, Y, Z$的对应关系.

【考察该点的题目：1.(4)/(6).】

(e) 利用奇偶性和对称性可简化计算.

【考察该点的题目：1.(2)(奇偶性),(3)/(4)(轮换对称性).】

2. 斯托克斯公式

$\Omega\subset\mathbb{R}^3, \Sigma\subset\Omega$为逐片光滑的有向曲面，$\partial\Sigma$逐段光滑，

向量场$\boldsymbol{F}(x,y,z)=X(x,y,z)\boldsymbol{i}+Y(x,y,z)\boldsymbol{j}+Z(x,y,z)\boldsymbol{k}\in C^1(\Omega)$,

则

$$\begin{aligned}
\oint_{\partial\Sigma}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{l}=\oint_{\partial\Sigma}\boldsymbol{F}\cdot\boldsymbol{\tau}\mathrm{d}l=\oint_{\partial\Sigma}X\mathrm{d}x+Y\mathrm{d}y+Z\mathrm{d}z&\\
&=\iint\limits_{\Sigma}\begin{vmatrix}\mathrm{d}y\wedge\mathrm{d}z & \mathrm{d}z\wedge\mathrm{d}x & \mathrm{d}x\wedge\mathrm{d}y\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ X & Y & Z\end{vmatrix}\\
&=\iint\limits_{\Sigma}(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z})\mathrm{d}y\wedge\mathrm{d}z+(\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x})\mathrm{d}z\wedge\mathrm{d}x+(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y})\mathrm{d}x\wedge\mathrm{d}y\\
&=\iint\limits_{\Sigma}\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ X & Y & Z\end{vmatrix}\cdot\mathrm{d}\boldsymbol{S}=\iint\limits_{\Sigma}\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ X & Y & Z\end{vmatrix}\cdot\boldsymbol{n}\mathrm{d}S\\
&=\iint\limits_{\Sigma}(\boldsymbol{\nabla}\times\boldsymbol{F})\cdot\mathrm{d}\boldsymbol{S}=\iint\limits_{\Sigma}(\mathrm{rot}\boldsymbol{F})\cdot\mathrm{d}\boldsymbol{S}.
\end{aligned}$$

注意：

(a) Σ与$\partial\Sigma$的方向符合右手法则. 在用斯托克斯公式进行计算时，选取的曲面方向应与已知曲线方向满足右手法则.

【如本节习题中2.(1)/(2),3.(1)/(2)的图形所示(图 8, 9, 10, 11).】

(b) 曲面Σ应尽量简单，便于求面积或便于计算曲面积分. 如本节习题中的2.(1)，已知曲线L是一个大圆，可直接求出其围成的圆形平面的面积，故Σ取为L围成的圆形平面，根据右手法则，上侧为正. 本节习题中的2.(2)中，已知曲线是一个平面曲线，Σ可取为该平面曲线围成的平面，根据右手法则，上侧为正.

【第一类习题：2.(1)/(2).】

(c) $\boldsymbol{F}(x,y,z)=X(x,y,z)\boldsymbol{i}+Y(x,y,z)\boldsymbol{j}+Z(x,y,z)\boldsymbol{k}\in C^1(\Sigma)$，即$X, Y, Z$有一阶连续偏导数，也即$X, Y, Z$连续可微.

【考察该点的题目：3.(1)/(2). (第二类习题)】

【其他类型的题目：4. (第三类习题)】

24.3 习题13.5解答

1. 用高斯公式计算下列曲面积分：

(1)$\iint\limits_S x^3\mathrm{d}y\wedge\mathrm{d}z+y^3\mathrm{d}z\wedge\mathrm{d}x+z^3\mathrm{d}x\wedge\mathrm{d}y$，其中$S$为区域$x^2+y^2\leqslant z,0\leqslant z\leqslant 4$的边界外侧；

(2)$\iint\limits_S x^2\mathrm{d}y\wedge\mathrm{d}z+y^2\mathrm{d}z\wedge\mathrm{d}x+z^2\mathrm{d}x\wedge\mathrm{d}y$，其中$S$为$x^2+y^2+z^2=R^2$的内侧；

(3)$\iint\limits_S x^2\mathrm{d}y\wedge\mathrm{d}z+y^2\mathrm{d}z\wedge\mathrm{d}x+z^2\mathrm{d}x\wedge\mathrm{d}y$，其中$S$为正方体$\Omega$：$0\leqslant x\leqslant 1,0\leqslant y\leqslant 1,0\leqslant z\leqslant 1$的外表面；

(4)$\iint\limits_S xz\mathrm{d}x\wedge\mathrm{d}y+xy\mathrm{d}y\wedge\mathrm{d}z+yz\mathrm{d}z\wedge\mathrm{d}x$，其中$S$为平面$x+y+z=1$与三个坐标面围成的区域边界外侧；

(5)$\iint\limits_S x^2\mathrm{d}y\wedge\mathrm{d}z+(z^2-2z)\mathrm{d}x\wedge\mathrm{d}y$，其中$S$为$z=\sqrt{x^2+y^2}$被平面$z=0$和$z=1$所截部分的外侧；

(6)$\oiint\limits_S(x-y)\mathrm{d}x\wedge\mathrm{d}y+(y-z)x\mathrm{d}y\wedge\mathrm{d}z$，其中$S$为柱面$x^2+y^2=1$与平面$z=1,z=3$所围柱体的边界外侧；

(7)$\oiint\limits_S\frac{x\mathrm{d}y\wedge\mathrm{d}z+y\mathrm{d}z\wedge\mathrm{d}x+z\mathrm{d}x\wedge\mathrm{d}y}{(x^2+y^2+z^2)^{\frac{3}{2}}}$，其中$S$为椭球面$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

解：(1)记S围成的区域为Ω，则$x^3,y^3,z^3\in C^1(\Omega)$,

$\therefore\iint\limits_S x^3\mathrm{d}y\wedge\mathrm{d}z+y^3\mathrm{d}z\wedge\mathrm{d}x+z^3\mathrm{d}x\wedge\mathrm{d}y=\iiint\limits_\Omega(\frac{\partial x^3}{\partial x}+\frac{\partial y^3}{\partial y}+\frac{\partial z^3}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z$

$=\iiint\limits_\Omega(3x^2+3y^2+3z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z=3\int_0^{2\pi}\mathrm{d}\theta\int_0^2\mathrm{d}r\int_{r^2}^4(r^2+z^2)\cdot r\mathrm{d}z=6\pi\int_0^2(r^3z+\frac{1}{3}rz^3)\Big|_{r^2}^4\mathrm{d}r$

$=6\pi\int_0^2(4r^3+\frac{64}{3}r-r^5-\frac{1}{3}r^7)\mathrm{d}r=6\pi(r^4+\frac{32}{3}r^2-\frac{1}{6}r^6-\frac{1}{3}\frac{1}{8}r^8)\Big|_0^2$

$=6\pi(16+\frac{32}{3}\cdot 4-\frac{1}{6}\cdot 64-\frac{1}{3}\cdot\frac{1}{8}\cdot 2^8)=6\pi(16+\frac{64}{3})=(96+128)\pi=224\pi.$

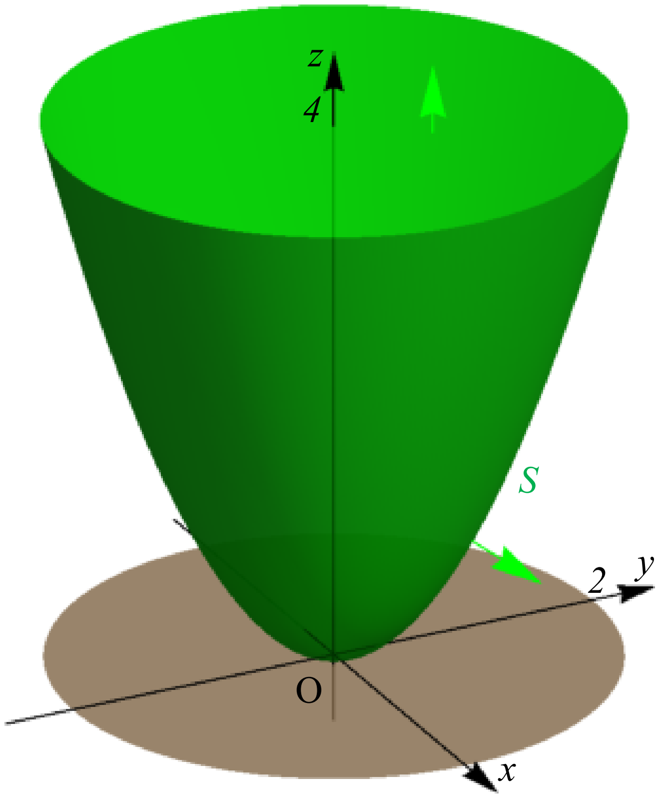

图 1: 习题13.5 1.(1)题图示

(2)记Ω为S^-围成的区域，

$\because x^2, y^2, z^2 \in C^1(\Omega)$,

$\therefore \iint\limits_S x^2 \mathrm{d}y \wedge \mathrm{d}z + y^2 \mathrm{d}z \wedge \mathrm{d}x + z^2 \mathrm{d}x \wedge \mathrm{d}y = -\iint\limits_{S^-} x^2 \mathrm{d}y \wedge \mathrm{d}z + y^2 \mathrm{d}z \wedge \mathrm{d}x + z^2 \mathrm{d}x \wedge \mathrm{d}y$

$= -\iiint\limits_{\Omega} (\frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z}) \mathrm{d}x\mathrm{d}y\mathrm{d}z = -\iiint\limits_{\Omega} (2x + 2y + 2z) \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0.$

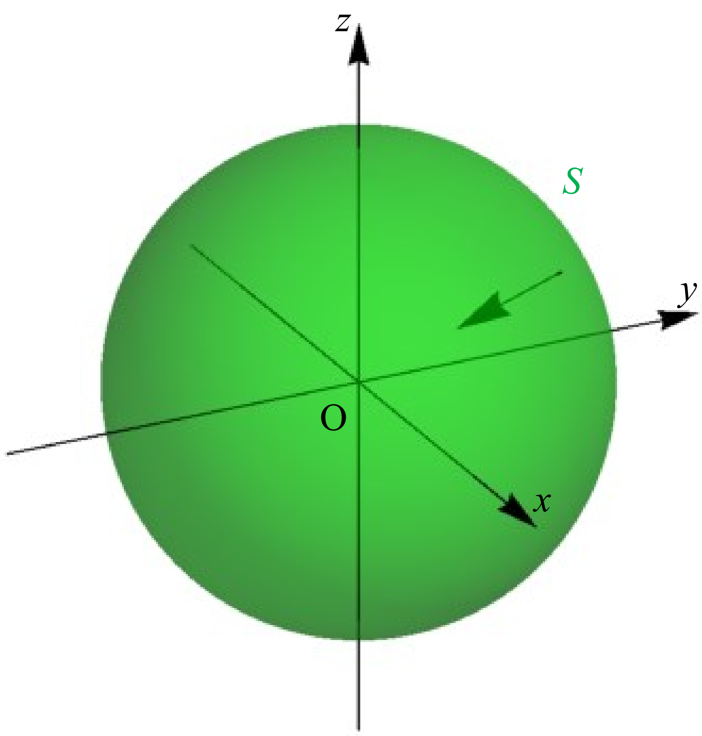

图 2: 习题13.5 1.(2)题图示

(3)$\because x^2, y^2, z^2 \in C^1(\Omega)$,

$\therefore \iint\limits_S x^2 \mathrm{d}y \wedge \mathrm{d}z + y^2 \mathrm{d}z \wedge \mathrm{d}x + z^2 \mathrm{d}x \wedge \mathrm{d}y = \iiint\limits_\Omega (\frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z}) \mathrm{d}x\mathrm{d}y\mathrm{d}z$

$= 2 \iiint\limits_\Omega (x + y + z) \mathrm{d}x\mathrm{d}y\mathrm{d}z = 6 \iiint\limits_\Omega x \mathrm{d}x\mathrm{d}y\mathrm{d}z = 6 \int_0^1 x \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z = 6 \int_0^1 x \mathrm{d}x \int_0^1 \mathrm{d}y$

$= 6 \int_0^1 x \mathrm{d}x = 3x^2\Big|_0^1 = 3.$

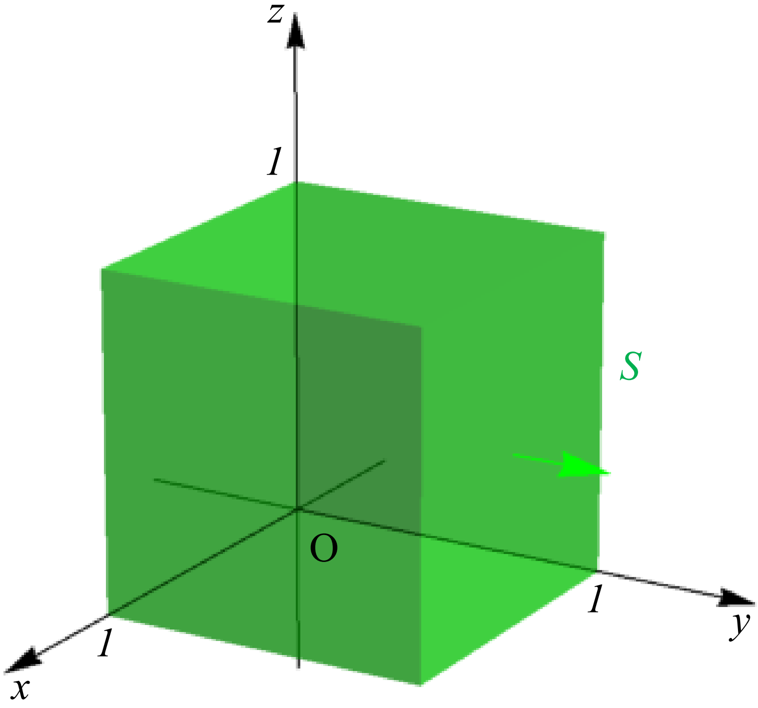

图 3: 习题13.5 1.(3)题图示

(4)记S围成的区域为Ω，则$xz, xy, yz \in C^1(\Omega)$,

$$\therefore \iint\limits_{S} xz\mathrm{d}x\wedge\mathrm{d}z+xy\mathrm{d}y\wedge\mathrm{d}z+yz\mathrm{d}z\wedge\mathrm{d}x = \iiint\limits_{\Omega}(\frac{\partial xy}{\partial x}+\frac{\partial yz}{\partial y}+\frac{\partial xz}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{\Omega}(y+z+x)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$= 3\iiint\limits_{\Omega} x\mathrm{d}x\mathrm{d}y\mathrm{d}z = 3\int_0^1 x\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int_0^{1-x-y}\mathrm{d}z = 3\int_0^1 x\mathrm{d}x\int_0^{1-x}(1-x-y)\mathrm{d}y$$
$$= 3\int_0^1 x(y-xy-\frac{1}{2}y^2)\Big|_0^{1-x}\mathrm{d}x = 3\int_0^1 x(1-x-\frac{1}{2}y)y\Big|_0^{1-x}\mathrm{d}x$$
$$= 3\int_0^1 x[1-x-\frac{1}{2}(1-x)](1-x)\mathrm{d}x = 3\int_0^1 x\frac{1}{2}(1-x)^2\mathrm{d}x = \frac{3}{2}\int_0^1 x(1-2x+x^2)\mathrm{d}x$$
$$= \frac{3}{2}\int_0^1(x-2x^2+x^3)\mathrm{d}x = \frac{3}{2}(\frac{1}{2}x^2-\frac{2}{3}x^3+\frac{1}{4}x^4)\Big|_0^1 = \frac{3}{2}(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}) = \frac{3}{2}(\frac{3}{4}-\frac{2}{3}) = \frac{3}{2}\cdot\frac{1}{12} = \frac{1}{8}.$$

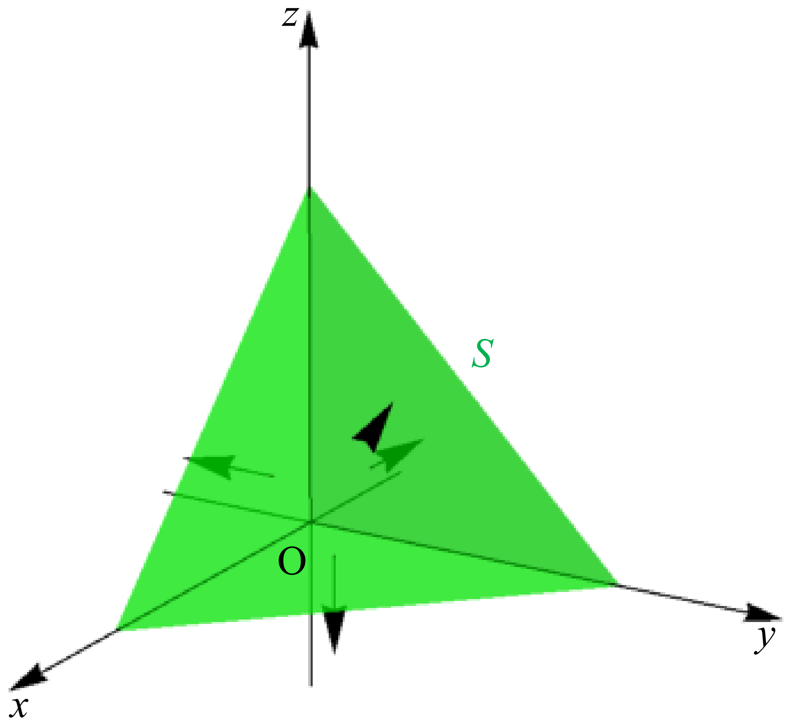

图 4: 习题13.5 1.(4)题图示

(5)取平面$S_1: z=1, x^2+y^2\leqslant 1$，上侧为正，记$S$与$S_1$围成的区域为$\Omega$,

$\because x^2, z^2-2z\in C^1(\Omega)$,

$\therefore \iint\limits_{S+S_1} x^2\mathrm{d}y\wedge\mathrm{d}z+(z^2-2z)\mathrm{d}x\wedge\mathrm{d}y=\iiint\limits_{\Omega}(\frac{\partial x^2}{\partial x}+\frac{\partial 0}{\partial y}+\frac{\partial(z^2-2z)}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z$

$=\iiint\limits_{\Omega}(2x+2z-2)\mathrm{d}x\mathrm{d}y\mathrm{d}z$,

由对称性可知$\iiint\limits_{\Omega}2x\mathrm{d}x\mathrm{d}y\mathrm{d}z=0$,

$\therefore$上式$=\iiint\limits_{\Omega}(2z-2)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_0^1(2z-2)\mathrm{d}z\iint\limits_{x^2+y^2\leqslant z^2}\mathrm{d}x\mathrm{d}y=\int_0^1(2z-2)\pi z^2\mathrm{d}z$

$=2\pi\int_0^1(z^3-z^2)\mathrm{d}z=2\pi(\frac{1}{4}z^4-\frac{1}{3}z^3)\Big|_0^1=-\frac{1}{6}\pi$,

$\because$在S_1上$\mathrm{d}\boldsymbol{S}=(-\frac{\partial z}{\partial x},-\frac{\partial z}{\partial y},1)\mathrm{d}x\mathrm{d}y=(0,0,1)\mathrm{d}x\mathrm{d}y=(\mathrm{d}y\wedge\mathrm{d}z,\mathrm{d}z\wedge\mathrm{d}x,\mathrm{d}x\wedge\mathrm{d}y)$,

$\therefore \iint\limits_{S_1}x^2\mathrm{d}y\wedge\mathrm{d}z+(z^2-2z)\mathrm{d}x\wedge\mathrm{d}y=\iint\limits_{x^2+y^2\leqslant 1}[x^2\cdot 0+(1^2-2\cdot 1)]\mathrm{d}x\mathrm{d}y=-\iint\limits_{x^2+y^2\leqslant 1}\mathrm{d}x\mathrm{d}y=-\pi$,

$\therefore \iint\limits_{S}x^2\mathrm{d}y\wedge\mathrm{d}z+(z^2-2z)\mathrm{d}x\wedge\mathrm{d}y=-\frac{\pi}{6}-\iint\limits_{S_1}x^2\mathrm{d}y\wedge\mathrm{d}z+(z^2-2z)\mathrm{d}x\wedge\mathrm{d}y=-\frac{\pi}{6}+\pi=\frac{5}{6}\pi$.

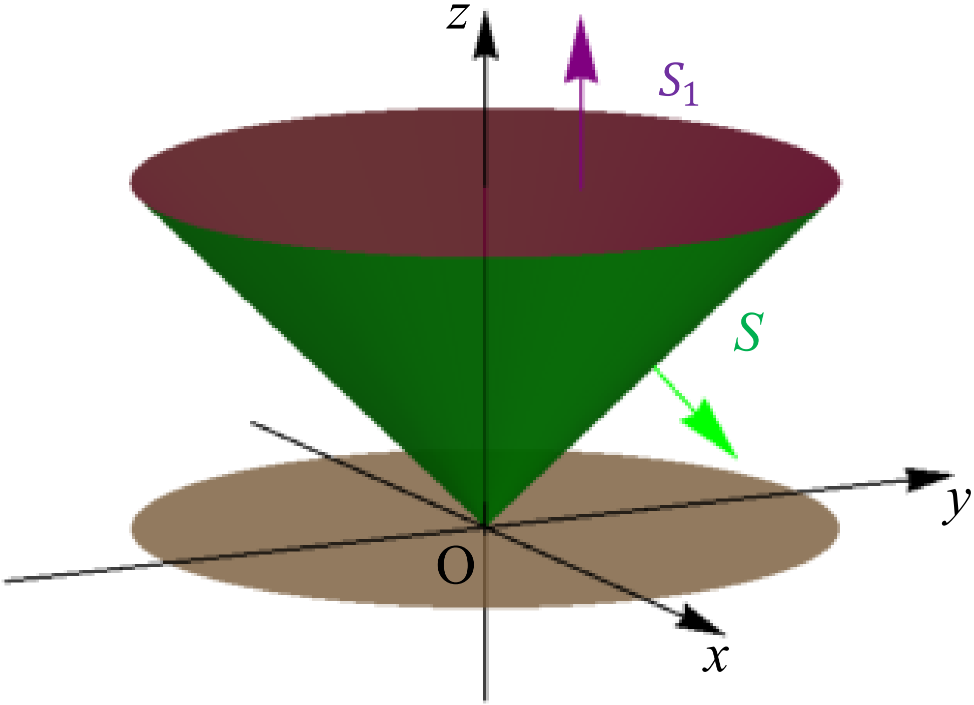

图 5: 习题13.5 1.(5)题图示

(6)记S围成的圆柱体区域为Ω，则$x-y,(y-z)x\in C^1(\Omega)$,

$\therefore \oiint\limits_S (x-y)\mathrm{d}x\wedge\mathrm{d}y+(y-z)x\mathrm{d}y\wedge\mathrm{d}z=\iiint\limits_\Omega[\frac{\partial(y-z)x}{\partial x}+\frac{\partial 0}{\partial y}+\frac{\partial(x-y)}{\partial z}]\mathrm{d}x\mathrm{d}y\mathrm{d}z$
$=\iiint\limits_\Omega(y-z)\mathrm{d}x\mathrm{d}y\mathrm{d}z,$

由对称性可知$\iiint\limits_\Omega y\mathrm{d}x\mathrm{d}y\mathrm{d}z=0,$

$\therefore$上式$=-\iiint\limits_\Omega z\mathrm{d}x\mathrm{d}y\mathrm{d}z=-\int_1^3 z\mathrm{d}z\iint\limits_{x^2+y^2\leqslant 1}\mathrm{d}x\mathrm{d}y=-\pi\int_1^3 z\mathrm{d}z=-\frac{1}{2}\pi z^2\Big|_1^3=-4\pi.$

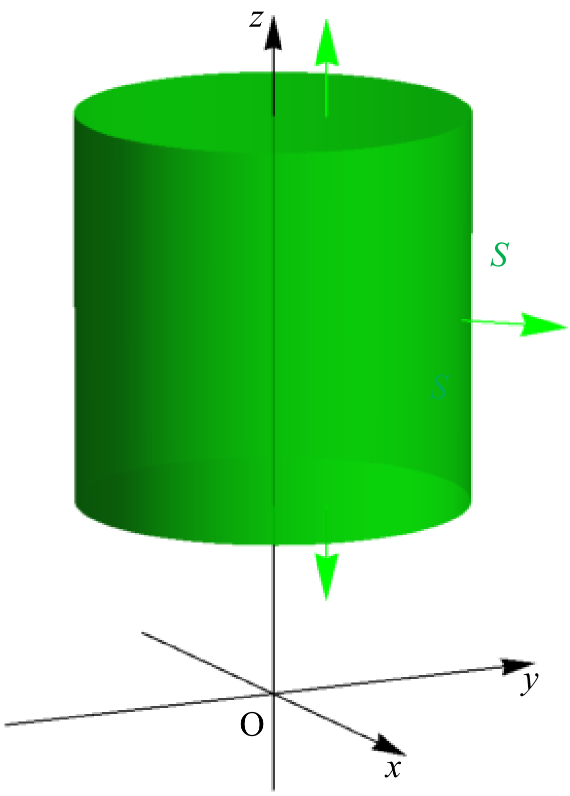

图 6: 习题13.5 1.(6)题图示

(7)取球面S_1：$x^2+y^2+z^2=r^2, r<\min\{a,b,c\}$，外侧为正，设$S$与$S_1^-$围成的区域为$\Omega$，$S_1$围成的区域为$\Omega_1$，

则$X=\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$，$Y=\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}$，$Z=\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\in C^1(\Omega)$，

$$\frac{\partial X}{\partial x}=\frac{(x^2+y^2+z^2)^{\frac{3}{2}}-x\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2x}{(x^2+y^2+z^2)^3}=\frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}},$$
$$\frac{\partial Y}{\partial y}=\frac{(x^2+y^2+z^2)^{\frac{3}{2}}-y\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2y}{(x^2+y^2+z^2)^3}=\frac{z^2+x^2-2y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}},$$
$$\frac{\partial Z}{\partial z}=\frac{(x^2+y^2+z^2)^{\frac{3}{2}}-z\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2z}{(x^2+y^2+z^2)^3}=\frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}},$$

$\therefore$

$$\oiint\limits_{S+S_1^-}\frac{\partial X}{\partial x}\mathrm{d}y\wedge\mathrm{d}z+\frac{\partial Y}{\partial y}\mathrm{d}z\wedge\mathrm{d}x+\frac{\partial Z}{\partial z}\mathrm{d}x\wedge\mathrm{d}y=\iiint\limits_{\Omega}(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$=\iiint\limits_{\Omega}\frac{y^2+z^2-2x^2+z^2+x^2-2y^2+x^2+y^2-2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint\limits_{\Omega}0\mathrm{d}x\mathrm{d}y\mathrm{d}z=0,$$

$\therefore$

$$\begin{aligned}
&\iint\limits_{S}\!\!\!\!\!\!\bigcirc \frac{x\mathrm{d}y\wedge\mathrm{d}z+y\mathrm{d}z\wedge\mathrm{d}x+z\mathrm{d}x\wedge\mathrm{d}y}{(x^2+y^2+z^2)^{\frac{3}{2}}}=0-\iint\limits_{S_1^-}\!\!\!\!\!\!\bigcirc \frac{x\mathrm{d}y\wedge\mathrm{d}z+y\mathrm{d}z\wedge\mathrm{d}x+z\mathrm{d}x\wedge\mathrm{d}y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\\
=&\iint\limits_{S_1}\!\!\!\!\!\!\bigcirc \frac{x\mathrm{d}y\wedge\mathrm{d}z+y\mathrm{d}z\wedge\mathrm{d}x+z\mathrm{d}x\wedge\mathrm{d}y}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\iint\limits_{S_1}\!\!\!\!\!\!\bigcirc \frac{x\mathrm{d}y\wedge\mathrm{d}z+y\mathrm{d}z\wedge\mathrm{d}x+z\mathrm{d}x\wedge\mathrm{d}y}{(r^2)^{\frac{3}{2}}}\\
=&\frac{1}{r^3}\iint\limits_{S_1}\!\!\!\!\!\!\bigcirc x\mathrm{d}y\wedge\mathrm{d}z+y\mathrm{d}z\wedge\mathrm{d}x+z\mathrm{d}x\wedge\mathrm{d}y=\frac{1}{r^3}\iiint\limits_{\Omega}(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
=&\frac{3}{r^3}\iiint\limits_{\Omega}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\frac{3}{r^3}\frac{4}{3}\pi r^3=4\pi.
\end{aligned}$$

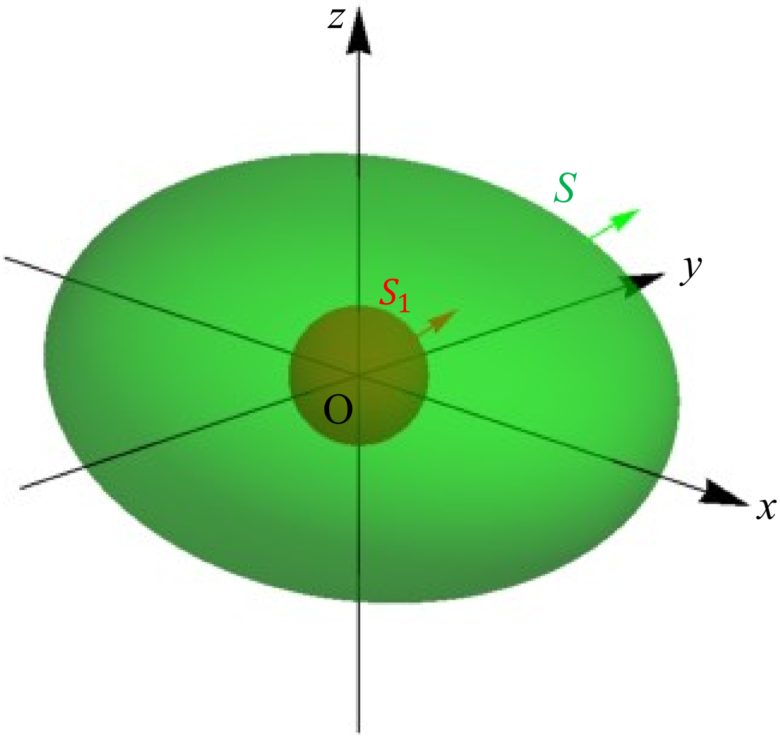

图 7: 习题13.5 1.(7)题图示

2. 用斯托克斯公式计算下列曲线积分：

(1)$\oint_L y\mathrm{d}x+z\mathrm{d}y+x\mathrm{d}z$，其中$L$是圆周$\begin{cases}x^2+y^2+z^2=R^2,\\ x+y+z=0,\end{cases}$ 从x轴正向看去为逆时针方向；

(2)$\oint_L (y-x)\mathrm{d}x+(z-y)\mathrm{d}y+(x-z)\mathrm{d}z$，其中$L$是柱面$x^2+y^2=a^2$与平面$x+z=a(a>0)$的交线，从$x$轴正向看去为逆时针方向.

解：(1)记Σ是平面$x+y+z=0$上L围成的部分，Σ与L的方向符合右手法则，记Σ的上侧为正，

$\because y,z,x\in C^1$,

$$\therefore \oint_L y\mathrm{d}x+z\mathrm{d}y+x\mathrm{d}z=\iint\limits_{\Sigma}\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & z & x\end{vmatrix}\cdot\boldsymbol{n}\mathrm{d}S=\iint\limits_{\Sigma}(\frac{\partial x}{\partial y}-\frac{\partial z}{\partial z},\frac{\partial z}{\partial y}-\frac{\partial x}{\partial x},\frac{\partial z}{\partial x}-\frac{\partial y}{\partial y})\cdot\boldsymbol{n}\mathrm{d}S$$

$$=\iint\limits_{\Sigma}(0-1,0-1,0-1)\cdot\boldsymbol{n}\mathrm{d}S=\iint\limits_{\Sigma}(-1,-1,-1)\cdot\boldsymbol{n}\mathrm{d}S,$$

$\because \Sigma$的单位法向量为$\boldsymbol{n}=(1,1,1)\frac{1}{\sqrt{3}}$,

$$\therefore \oint_L y\mathrm{d}x+z\mathrm{d}y+x\mathrm{d}z=\iint\limits_{\Sigma}(-1,-1,-1)\cdot(1,1,1)\frac{1}{\sqrt{3}}\mathrm{d}S=\frac{1}{\sqrt{3}}\iint\limits_{\Sigma}-3\mathrm{d}S=-\sqrt{3}\pi R^2.$$

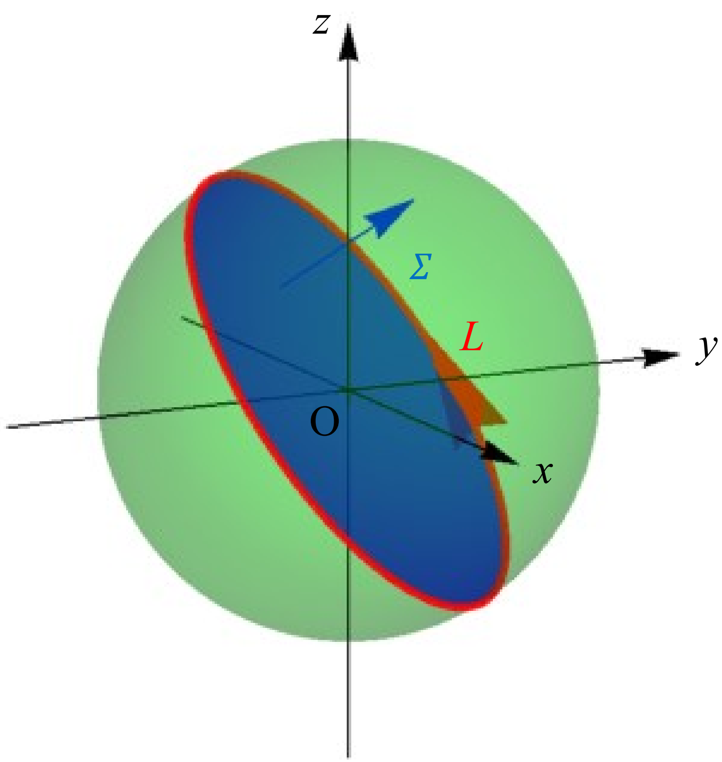

图 8: 习题13.5 2.(1)题图示

(2)记Σ是平面$x+z=a$上L围成的部分，Σ的方向与L的方向符合右手法则，即Σ的上

侧为正，

$\because y-x, z-y, x-z \in C^1$,

$\therefore \oint_L (y-x)\mathrm{d}x+(z-y)\mathrm{d}y+(x-z)\mathrm{d}z = \iint\limits_{\Sigma} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-x & z-y & x-z \end{vmatrix} \cdot \mathrm{d}\boldsymbol{S}$

$= \iint\limits_{\Sigma} (\frac{\partial(x-z)}{\partial y} - \frac{\partial(z-y)}{\partial z}, \frac{\partial(y-x)}{\partial z} - \frac{\partial(x-z)}{\partial x}, \frac{\partial(z-y)}{\partial x} - \frac{\partial(y-x)}{\partial y}) \cdot \mathrm{d}\boldsymbol{S} = \iint\limits_{\Sigma} (0-1, 0-1, 0-1) \cdot \mathrm{d}\boldsymbol{S}$,

$\because$在$\Sigma: z = a-x, x^2+y^2 \leqslant a^2$的上侧

$\mathrm{d}\boldsymbol{S} = (-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1)\mathrm{d}x\mathrm{d}y = (1,0,1)\mathrm{d}x\mathrm{d}y = (\mathrm{d}y\wedge\mathrm{d}z, \mathrm{d}z\wedge\mathrm{d}x, \mathrm{d}x\wedge\mathrm{d}y)$,

$\therefore$上式$= \iint\limits_{x^2+y^2\leqslant a^2} (0-1,0-1,0-1)\cdot(1,0,1)\mathrm{d}x\mathrm{d}y = -2\iint\limits_{x^2+y^2\leqslant a^2} \mathrm{d}x\mathrm{d}y = -2\pi a^2$.

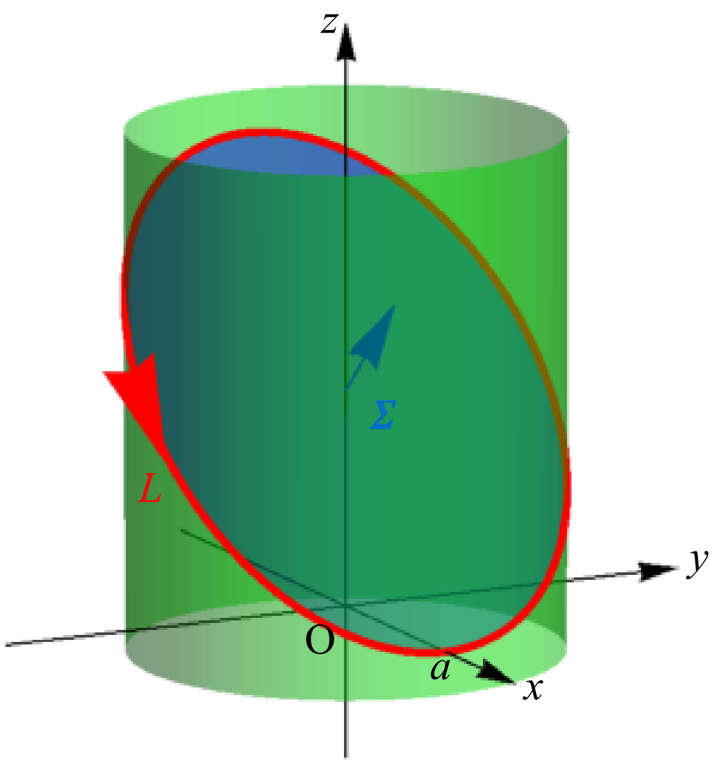

图 9: 习题13.5 2.(2)题图示

3. 计算$I = \oint_L \frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2} + z\mathrm{d}z$，其中$L$是：
(1)任意一条既不环绕z轴，也不与z轴相交的简单闭曲线；
(2)任意一条环绕z轴一圈且不与z轴相交的简单闭曲线，从z轴正向看去为逆时针方向.

解：(1)取曲面Σ为曲线L围成的逐片光滑有向曲面，Σ和L的方向符合右手法则，z轴不穿过Σ,

则$X=\frac{-y}{x^2+y^2}$，$Y=\frac{x}{x^2+y^2}$，$Z=z\in C^1$,

$\therefore$

$$
\begin{aligned}
I=\oint_L X\mathrm{d}x+Y\mathrm{d}y+Z\mathrm{d}z &= \iint\limits_{\Sigma}\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ X & Y & Z\end{vmatrix}\cdot\mathrm{d}\boldsymbol{S}=\iint\limits_{\Sigma}\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & z\end{vmatrix}\cdot\mathrm{d}\boldsymbol{S}\\
&=\iint\limits_{\Sigma}(\frac{\partial z}{\partial y}-\frac{\partial}{\partial z}\frac{x}{x^2+y^2},\frac{\partial}{\partial z}\frac{-y}{x^2+y^2}-\frac{\partial z}{\partial x},\frac{\partial}{\partial x}\frac{x}{x^2+y^2}-\frac{\partial}{\partial y}\frac{-y}{x^2+y^2})\cdot\mathrm{d}\boldsymbol{S}\\
&=\iint\limits_{\Sigma}(0-0,0-0,\frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}-\frac{-(x^2+y^2)+y\cdot 2y}{(x^2+y^2)^2})\cdot\mathrm{d}\boldsymbol{S}\\
&=\iint\limits_{\Sigma}(0,0,0)\cdot\mathrm{d}\boldsymbol{S}=0.
\end{aligned}
$$

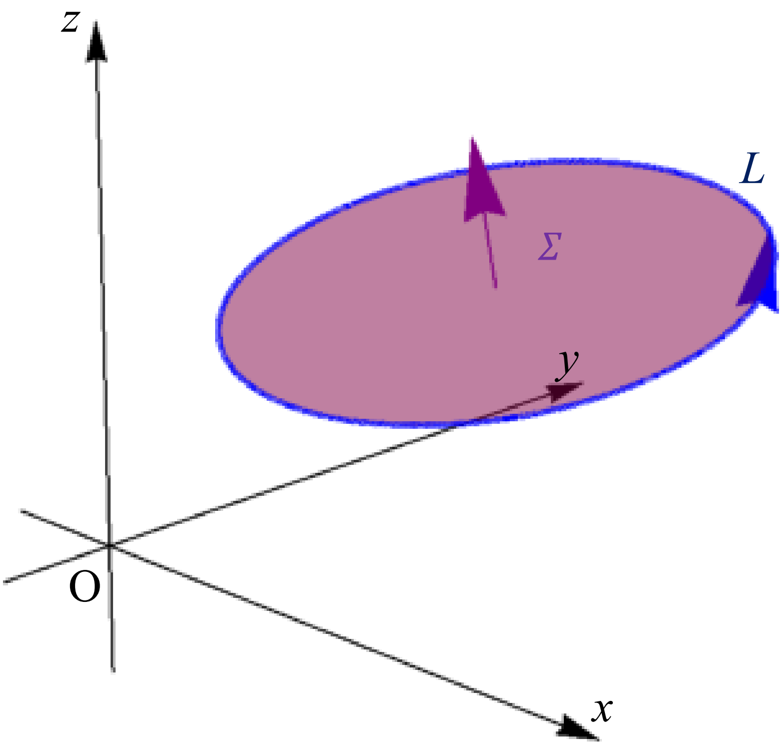

图 10: 习题13.5 3.(1)题图示

(2)设曲面Σ为曲线L围成的逐片光滑有向曲面，Σ与L的方向符合右手法则，即Σ的上侧为正，此时z轴穿过Σ. 取柱面$x^2+y^2=r^2$与Σ交于曲线L_1，r应足够小使得闭合交线L_1全部位于Σ上，设从z轴正向看去L_1为逆时针方向. 记Σ_1为L_1围成的逐片光滑正向曲面，Σ_2为L与L_1^-围成的逐片光滑正向曲面, $\Omega_2=\mathbb{R}^3\backslash\{(x,y)|x^2+y^2<r^2\}$.

则$X=\frac{-y}{x^2+y^2}$, $Y=\frac{x}{x^2+y^2}$, $Z=z\in C^1(\Omega_2)$,

与(1)同理可知$\oint_{L+L_1^-}X\mathrm{d}x+Y\mathrm{d}y+Z\mathrm{d}z=0$,

$\therefore$

$$\begin{aligned}
I&=0-\oint_{L_1^-}X\mathrm{d}x+Y\mathrm{d}y+Z\mathrm{d}z=\oint_{L_1}X\mathrm{d}x+Y\mathrm{d}y+Z\mathrm{d}z\\
&=\oint_{L_1}\frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}+z\mathrm{d}z=\oint_{L_1}\frac{-y\mathrm{d}x+x\mathrm{d}y}{r^2}+z\mathrm{d}z\\
&=\iint\limits_{\Sigma_1}\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ -\frac{y}{r^2}&\frac{x}{r^2}&z\end{vmatrix}\cdot\mathrm{d}\boldsymbol{S}=\iint\limits_{\Sigma_1}(\frac{\partial z}{\partial y}-\frac{\partial}{\partial z}\frac{x}{r^2},\frac{\partial}{\partial z}(-\frac{y}{r^2})-\frac{\partial z}{\partial x},\frac{\partial}{\partial x}\frac{x}{r^2}-\frac{\partial}{\partial y}(-\frac{y}{r^2}))\cdot\mathrm{d}\boldsymbol{S}\\
&=\iint\limits_{\Sigma_1}(0-0,0-0,\frac{1}{r^2}-(-\frac{1}{r^2}))\cdot\mathrm{d}\boldsymbol{S}=\iint\limits_{\Sigma_1}(0,0,\frac{2}{r^2})\cdot(\mathrm{d}y\wedge\mathrm{d}z,\mathrm{d}z\wedge\mathrm{d}x,\mathrm{d}x\wedge\mathrm{d}y)\\
&=\iint\limits_{\Sigma_1}\frac{2}{r^2}\mathrm{d}x\wedge\mathrm{d}y=\frac{2}{r^2}\iint\limits_{\Sigma_1}\mathrm{d}x\wedge\mathrm{d}y,
\end{aligned}$$

$\because\Sigma_1$方程可表示为$z=f(x,y),x^2+y^2\leqslant r^2$，且$\Sigma_1$上侧为正，

$\therefore I=\frac{2}{r^2}\iint\limits_{\Sigma_1}\mathrm{d}x\wedge\mathrm{d}y=\frac{2}{r^2}\iint\limits_{x^2+y^2\leqslant r^2}\mathrm{d}x\mathrm{d}y=\frac{2}{r^2}\pi r^2=2\pi$.

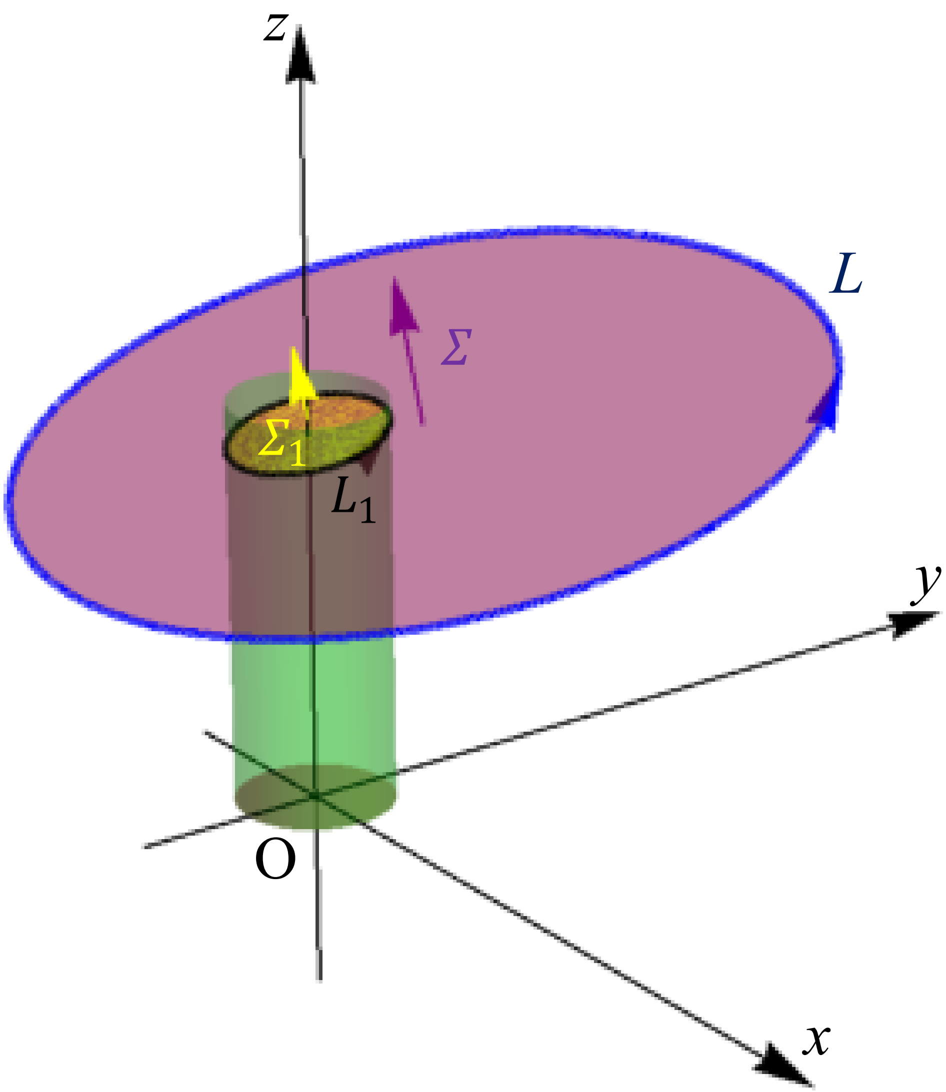

图 11: 习题13.5 3.(2)题图示

4. 证明：

$$\oint_L \begin{vmatrix} \mathrm{d}x & \mathrm{d}y & \mathrm{d}z \\ \cos\alpha & \cos\beta & \cos\gamma \\ x & y & z \end{vmatrix} = 2S,$$

其中L是$\mathbb{R}^3$中某个平面上的一条简单逐段光滑闭曲线，$\boldsymbol{n} = (\cos\alpha, \cos\beta, \cos\gamma)$是该平面的单位法向量，$L$的方向与$\boldsymbol{n}$的方向服从右手法则，$S$是$L$所围的面积.

证明：

$$\oint_L \begin{vmatrix} \mathrm{d}x & \mathrm{d}y & \mathrm{d}z \\ \cos\alpha & \cos\beta & \cos\gamma \\ x & y & z \end{vmatrix} = \oint_L (z\cos\beta - y\cos\gamma)\mathrm{d}x + (x\cos\gamma - z\cos\alpha)\mathrm{d}y + (y\cos\alpha - x\cos\beta)\mathrm{d}z,$$

记Σ是L在该平面上围成的部分，Σ的方向与$\boldsymbol{n}$的方向一致，则Σ的面积为S，$z\cos\beta - y\cos\gamma, x\cos\gamma - z\cos\alpha, y\cos\alpha - x\cos\beta \in C^1(\Sigma)$,

$\therefore$

$$\begin{aligned}
\text{上式} &= \iint\limits_{\Sigma} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\cos\beta - y\cos\gamma & x\cos\gamma - z\cos\alpha & y\cos\alpha - x\cos\beta \end{vmatrix} \\
&= \iint\limits_{\Sigma} \{[\frac{\partial(y\cos\alpha - x\cos\beta)}{\partial x} - \frac{\partial(x\cos\gamma - z\cos\alpha)}{\partial z}]\boldsymbol{i} \\
&\qquad + [\frac{\partial(z\cos\beta - y\cos\gamma)}{\partial z} - \frac{\partial(y\cos\alpha - x\cos\beta)}{\partial x}]\boldsymbol{j} \\
&\qquad + [\frac{\partial(x\cos\beta - z\cos\alpha)}{\partial x} - \frac{\partial(z\cos\beta - y\cos\gamma)}{\partial y}]\boldsymbol{k}\} \cdot \boldsymbol{n}\mathrm{d}S \\
&= \iint\limits_{\Sigma} \{[\cos\alpha - (-\cos\alpha)]\boldsymbol{i} + [\cos\beta - (-\cos\beta)]\boldsymbol{j} + [\cos\gamma - (-\cos\gamma)]\boldsymbol{k}\} \cdot \boldsymbol{n}\mathrm{d}S \\
&= \iint\limits_{\Sigma} (2\cos\alpha\boldsymbol{i} + 2\cos\beta\boldsymbol{j} + 2\cos\gamma\boldsymbol{k}) \cdot \boldsymbol{n}\mathrm{d}S \\
&= \iint\limits_{\Sigma} (2\cos\alpha, 2\cos\beta, 2\cos\gamma) \cdot (\cos\alpha, \cos\beta, \cos\gamma)\mathrm{d}S \\
&= \iint\limits_{\Sigma} (2\cos^2\alpha + 2\cos^2\beta + 2\cos^2\gamma)\mathrm{d}S \\
&= 2\iint\limits_{\Sigma} \mathrm{d}S = 2S.
\end{aligned}$$