14 保守场、向量场的微分运算

14.1 复习计划

14.2　知识结构

向量场的微积分

- 向量场在曲线上的积分、格林公式
- 向量场在曲面上的积分
- 高斯公式与斯托克斯公式

保守场

- 平面保守场
  - 保守场的几个概念
    - 保守场 $\boldsymbol{F}(x,y)$　曲线积分 $\int_{L(A)}^{(B)}\boldsymbol{F}(x,y)\cdot\mathrm{d}\boldsymbol{l}$ 与路径无关
    - 有势场 $\boldsymbol{F}(x,y)$
      - $\exists\varphi(x,y),s.t.\operatorname{grad}\varphi(x,y)=\boldsymbol{F}(x,y),(x,y)\in D$
      - $\varphi(x,y)$为势函数
      - 可微的有势场是无旋场
      - 即 $\operatorname{rot}\boldsymbol{F}=(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y})\boldsymbol{k}=\boldsymbol{0}$
    - 全微分式 $X(x,y)\mathrm{d}x+Y(x,y)\mathrm{d}y$
      - $\exists\varphi(x,y),s.t.\mathrm{d}\varphi(x,y)=X(x,y)\mathrm{d}x+Y(x,y)\mathrm{d}y$
      - $\varphi(x,y)$为原函数
      - $\boldsymbol{F}(x,y)=(X,Y)$有势与$X\mathrm{d}x+Y\mathrm{d}y$是全微分式等价
        - 势函数也是原函数
        - 求原函数的方法
  - 一般区域上保守场的充要条件
    - $\int_{L(A)}^{(B)}\boldsymbol{F}(x,y)\cdot\mathrm{d}\boldsymbol{l}$与路径无关
    - $\oint_L\boldsymbol{F}(x,y)\cdot\mathrm{d}\boldsymbol{l}=0$
    - $\boldsymbol{F}(x,y)=(X,Y)$有势
    - $X\mathrm{d}x+Y\mathrm{d}y$是全微分式
    - 四个条件等价
  - 单连通区域D上保守场的充要条件
    - $\boldsymbol{F}(x,y)=(X,Y)\in C^1(D)$且$\operatorname{rot}\boldsymbol{F}=(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y})\boldsymbol{k}=\boldsymbol{0}$
    - 即$\boldsymbol{F}(x,y)$是D上的无旋场
    - 注意该充要条件只能应用于单连通域

- 空间保守场
  - 保守场的几个概念
    - 保守场 $\boldsymbol{F}(x,y,z)$　曲线积分 $\int_{L(A)}^{(B)}\boldsymbol{F}(x,y,z)\cdot\mathrm{d}\boldsymbol{l}$ 与路径无关
    - 有势场 $\boldsymbol{F}(x,y,z)$
      - $\exists\varphi(x,y,z),s.t.\operatorname{grad}\varphi(x,y,z)=\boldsymbol{F}(x,y,z),(x,y,z)\in\Omega$
      - $\varphi(x,y,z)$为势函数
      - 可微的有势场是无旋场
      - 即$\operatorname{rot}\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\X&Y&z\end{vmatrix}=\boldsymbol{0}$
    - 全微分式 $X(x,y,z)\mathrm{d}x+Y(x,y,z)\mathrm{d}y+Z(x,y,z)\mathrm{d}z$
      - $\exists\varphi(x,y,z),s.t.\mathrm{d}\varphi(x,y,z)=X(x,y,z)\mathrm{d}x+Y(x,y,z)\mathrm{d}y+Z(x,y,z)\mathrm{d}z$
      - $\varphi(x,y,z)$为原函数
      - $\boldsymbol{F}(x,y,z)=(X,Y)$有势与$X\mathrm{d}x+Y\mathrm{d}y+Z\mathrm{d}z$是全微分式等价　势函数也是原函数
      - 求原函数的方法
        - 曲线积分法
        - 不定积分法
  - 一般区域上保守场的充要条件
    - $\int_{L(A)}^{(B)}\boldsymbol{F}(x,y,z)\cdot\mathrm{d}\boldsymbol{l}$与路径无关
    - $\oint_L\boldsymbol{F}(x,y,z)\cdot\mathrm{d}\boldsymbol{l}=0$
    - $\boldsymbol{F}(x,y,z)=(X,Y,Z)$有势
    - $X\mathrm{d}x+Y\mathrm{d}y+Z\mathrm{d}z$是全微分式
    - 四个条件等价
  - 曲面单连通区域上保守场的充要条件
    - $\boldsymbol{F}(x,y,z)=(X,Y,Z)\in C^1(\Omega)$且$\operatorname{rot}\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\X&Y&Z\end{vmatrix}=\boldsymbol{0}$
    - 即$\boldsymbol{F}(x,y)$是Ω上的无旋场
    - 注意该充要条件只能应用与曲面单连通区域

- 调和场
  - 调和场的概念
    - 既是有势场又是无源场
    - $\boldsymbol{F}(x,y,z)=\boldsymbol{\nabla}\varphi(x,y,z),\boldsymbol{\nabla}\cdot\boldsymbol{F}(x,y,z)=0$
  - 调合场势函数的性质
    - $\varphi(x,y,z)\in C^2\Rightarrow\Delta\varphi(x,y,z)=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0$
    - $\Delta\varphi(x,y,z)=0$调和方程，其解为调和函数
    - $\varphi(x,y,z)\in C^2(\Omega)$是调和函数，且在$\partial\Omega$上的函数值等于$0\Rightarrow\varphi(x,y,z)=0,(x,y,z)\in\Omega$

向量场的微分运算

- 数量场f梯度
  - 二维 $\operatorname{grad}f(x,y)=\boldsymbol{\nabla}f(x,y)=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})$
  - 三维 $\operatorname{grad}f(x,y,z)=\boldsymbol{\nabla}f(x,y,z)=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})$
- 向量场$\boldsymbol{F}$的散度
  - 二维 $\operatorname{div}\boldsymbol{F}=\boldsymbol{\nabla}\cdot\boldsymbol{F}=\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}$
  - 三维 $\operatorname{div}\boldsymbol{F}=\boldsymbol{\nabla}\cdot\boldsymbol{F}=\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}$
  - $\operatorname{div}\boldsymbol{F}=0$的场为无源场
- 向量场$\boldsymbol{F}$的旋度
  - 二维 $\operatorname{rot}\boldsymbol{F}=\operatorname{curl}\boldsymbol{F}=\boldsymbol{\nabla}\times\boldsymbol{F}=(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y})\boldsymbol{k}$
  - 三维 $\operatorname{rot}\boldsymbol{F}=\operatorname{curl}\boldsymbol{F}=\boldsymbol{\nabla}\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\X&Y&Z\end{vmatrix}$
  - $\operatorname{rot}\boldsymbol{F}=\boldsymbol{0}$的场为无旋场
  - 散度和旋度的积分定义可看扈老师的讲课提纲

14.3 保守场

平面保守场的性质如下所示。

平面保守场

$F(x, y) = X(x, y)i + Y(x, y)j \in C(D)$

保守场
(积分)

① $\int_{L_1(A)}^B F \cdot dl = \int_{L_2(A)}^B F \cdot dl$ （定义、选取简单路径计算复杂路径上的积分）

② $\oint_L F \cdot dl = 0$ （环量为零）

等价

$$\int_{L(A)}^B F \cdot dl = \int_{L(A)}^B X dx + Y dy$$
$$= \varphi(B) - \varphi(A)$$
曲线积分的
牛顿-莱布尼茨公式

$$\varphi(x, y) = \int_{(x_0, y_0)}^{(x,y)} F \cdot dl$$
$$= \int_{(x_0, y_0)}^{(x,y)} X dx + Y dy$$
求解势（原）函数的方法
（平面变上限积分）

有势场
(微分)

③ $\exists \varphi(x, y) \in C^1, s.t. \mathbf{grad}\varphi(x, y) = F$ （有势）

④ $\exists \varphi(x, y) \in C^1, s.t. d\varphi(x, y) = X(x, y)dx + Y(x, y)dy$ （有原函数）

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$F(x, y) \in C^1(D),$
rot$F$
$= \nabla \times F(x, y)$
$= \nabla \times (\nabla \varphi)$
$= \mathbf{0}$

无旋场

说明：

1. 性质①中的简单路径可选为与坐标轴平行的直线或折线路径.
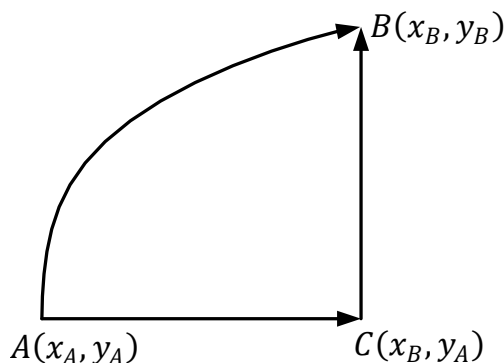
图 1: 与坐标轴平行的折线路径.

在与x轴平行的直线路径AC上，积分$\int_{L(A)}^{C} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l} = \int_{L(A)}^{C} X(x,y)\mathrm{d}x + Y(x,y)\mathrm{d}y$的被积表达式$X(x,y)\mathrm{d}x + Y(x,y)\mathrm{d}y$中$\mathrm{d}y = 0, y = y_A$，故

$$\int_{L(A)}^{C} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l} = \int_{(x_A, y_A)}^{(x_B, y_A)} X(x,y)\mathrm{d}x + Y(x,y)\mathrm{d}y = \int_{x_A}^{x_B} X(x, y_A)\mathrm{d}x.$$

在与y轴平行的直线路径CB上，积分$\int_{L(C)}^{B} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l} = \int_{L(C)}^{B} X(x,y)\mathrm{d}x + Y(x,y)\mathrm{d}y$的被积表达式$X(x,y)\mathrm{d}x + Y(x,y)\mathrm{d}y$中$\mathrm{d}x = 0, x = x_B$，故

$$\int_{L(C)}^{B} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l} = \int_{(x_B, y_A)}^{(x_B, y_B)} X(x,y)\mathrm{d}x + Y(x,y)\mathrm{d}y = \int_{y_A}^{y_B} Y(x_B, y)\mathrm{d}y.$$

【考察这一点的习题：1.(1)/(2)/(3)/(4),3.(1)/(2). (第一类题目)】

2. ①,②,③,④相互等价，既是性质定理也是判定定理，知道其中一个即可判定该向量场是保守场，也就可以得到另外三个性质.

3. 保守场是有势场，有势场也是保守场，二者是完全等价的概念. 保守场的两个性质从积分角度描述了向量场，有势场的两个性质从微分角度描述了向量场.

4. 可通过对势函数这样一个数量场的分析来对向量场进行分析，比如可以用引力势能分析引力场，用重力势能分析重力场，用电势能分析电场力的场，用电势分析电场。

5. 势函数与原函数相同.

6. 可微的保守场是无旋场（但无旋场不一定是保守场）.

【考察这一点的习题：2. (第二类题目)】

7. 平面单连通域上的无旋场是保守场. 常利用平面单连通区域上无旋来判定一个向量场是保守场.

【考察这一点的习题：1.(1)/(2)/(3)/(4),3.(1)/(2). (第一类题目)】

8. 可微场才可用旋度算子计算旋度. 如一个向量场不可微，则不可用旋度算子计算旋度，难以用旋度为零来判断该向量场是保守场.

　　【考察这一点的习题：4. (第三类题目)】

9. 求解全微分式的原函数有两种方法，变上限积分法和不定积分法.

　　【考察这一点的习题：3.(1)/(2). (第四类题目)】

10. 空间保守场的性质与平面保守场类似，主要的不同是三维空间中的曲面单连通域（比如球壳是曲面单连通域，圆环体则不是曲面单连通域）上的无旋场是保守场。

14.4　无源场、无旋场

1. 无源场$\mathrm{div}\boldsymbol{F} = \boldsymbol{\nabla} \cdot \boldsymbol{F} = 0$.

　　【与无源场有关的题目：5,7,8.(1)/(2),9. (第五类题目)】

2. 无旋场$\mathrm{rot}\boldsymbol{F} = \boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$.

　　【与无旋场有关的题目：6,10.(1)/(2). (第六类题目)】

　　【综合题目：9. (主要考察通量的概念、高斯公式、无源场)】

14.5　向量场的微分运算习题分类

1. 验证三个算子的性质.

　　【习题13.1中的1., 2., 4.】

2. 求梯度、散度、旋度.

　　【习题13.1中的3., 5., 6.】

14.6　习题13.6解答

1. 利用积分域与路线无关的性质计算下列积分：

(1)$\int_L (x^3 + xy^2)\mathrm{d}x + (y^3 + x^2 y)\mathrm{d}y$，其中$L$为从$O(0,0)$经$A(1,1)$到$B(2,0)$的折线；

(2)$\int_L (y+1)\tan x\mathrm{d}x - \ln\cos x\mathrm{d}y$，其中$L$为曲线$x = \cos t, y = 2\sin t(0 \leqslant t \leqslant \pi)$，顺时针方向；

(3)$\int_L (\ln\frac{y}{x} - 1)\mathrm{d}x + \frac{x}{y}\mathrm{d}y$，其中$L$为由点$A(1,1)$出发到$B(\mathrm{e}, 3\mathrm{e})$的任何一条不与$x$轴以及$y$轴相交的曲线；

(4)$\int_L \frac{1+y^2 f(xy)}{y}\mathrm{d}x + \frac{x}{y^2}[y^2 f(xy) - 1]\mathrm{d}y$，其中$L$为由点$A(0,1)$出发到$B(1,2)$的任何一条不与$x$轴相交的曲线，$f$是连续可微的函数.

解：(1)
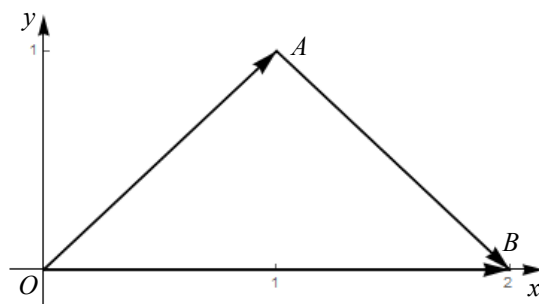
图 2: 习题13.6 1.(1)题图示

令 $\begin{cases} X(x,y) = x^3 + xy^2, \\ Y(x,y) = y^3 + x^2y, \end{cases}$ 则 $\boldsymbol{F}(x,y) = (X,Y) \in C^1(\mathbb{R})$,

$\because \mathbb{R}$为单连通域，

且$\mathrm{rot}\boldsymbol{F} = (\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y})\boldsymbol{k} = (2xy - 2xy)\boldsymbol{k} = \boldsymbol{0}$,

$\therefore \int_L X\mathrm{d}x + Y\mathrm{d}y = \int_{(0,0)}^{(2,0)} X\mathrm{d}x + Y\mathrm{d}y = \int_0^2 x^3\mathrm{d}x = \frac{1}{4}x^4\big|_0^2 = 4$.

(2)
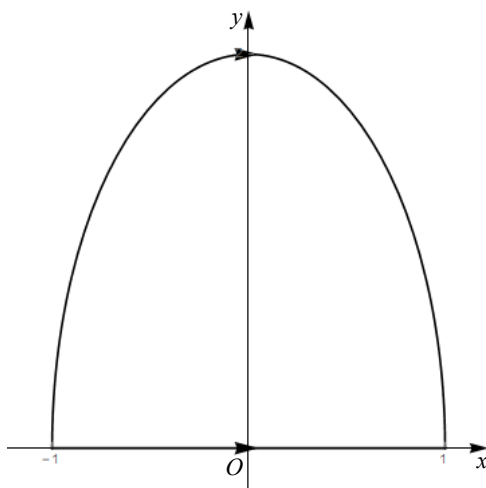
图 3: 习题13.6 1.(2)题图示

曲线L：$\begin{cases} x = \cos t, \\ y = 2\sin t, \end{cases}$ $(0 \leqslant t \leqslant \pi)$为一椭圆弧$x^2 + \frac{y^2}{4} = 1, y \geqslant 0$，顺时针的起点

为$(-1,0)$终点为$(1,0)$,

设$D = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\infty, \infty)$，则$L \in D$,

令$\begin{cases} X(x,y) = (y+1)\tan x, \\ Y(x,y) = -\ln\cos x, \end{cases}$ 则$\boldsymbol{F}(x,y) = (X,Y) \in C^1(D)$,

$\because D$是单连通区域，

且$\mathrm{rot}\boldsymbol{F} = (\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y})\boldsymbol{k} = (-\frac{-\sin x}{\cos x} - \tan x)\boldsymbol{k} = \boldsymbol{0}$,

$\therefore \int_L X\mathrm{d}x + Y\mathrm{d}y = \int_{(-1,0)}^{(1,0)} X\mathrm{d}x + Y\mathrm{d}y = \int_{-1}^{1} \tan x\mathrm{d}x = 0$.

(3)
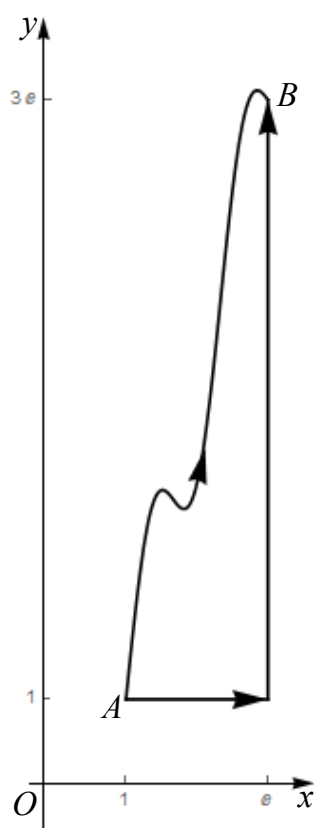
图 4: 习题13.6 1.(3)题图示

设$D = \{(x,y) \mid x > 0, y > 0\}$，则$L \in D$,

令$\begin{cases} X(x,y) = \ln\frac{y}{x} - 1, \\ Y(x,y) = \frac{x}{y}, \end{cases}$ 则$\boldsymbol{F}(x,y) = (X,Y) \in C^1(D)$,

$\because D$是单连通域，

且$\mathrm{rot}\boldsymbol{F} = (\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y})\boldsymbol{k} = (\frac{1}{y} - \frac{1}{\frac{y}{x}}\frac{1}{x})\boldsymbol{k} = \boldsymbol{0}$,

$\therefore \int_L X\mathrm{d}x + Y\mathrm{d}y = \int_{(1,1)}^{(\mathrm{e},3\mathrm{e})} X\mathrm{d}x + Y\mathrm{d}y = \int_{(1,1)}^{(\mathrm{e},1)} X\mathrm{d}x + Y\mathrm{d}y + \int_{(\mathrm{e},1)}^{(\mathrm{e},3\mathrm{e})} X\mathrm{d}x + Y\mathrm{d}y$

$= \int_1^{\mathrm{e}}(\ln\frac{1}{x} - 1)\mathrm{d}x + \int_1^{3\mathrm{e}}\frac{\mathrm{e}}{y}\mathrm{d}y = x(-\ln x - 1)\big|_1^{\mathrm{e}} - \int_1^{\mathrm{e}} x\mathrm{d}(-\ln x - 1) + \mathrm{e}\ln y\big|_1^{3\mathrm{e}}$

$= \mathrm{e}(-1 - 1) - 1 \cdot (0 - 1) + \int_1^{\mathrm{e}} x\frac{1}{x}\mathrm{d}x + \mathrm{e}\ln 3\mathrm{e} - 0 = -2\mathrm{e} + 1 + (\mathrm{e} - 1) + \mathrm{e}\ln 3 + \mathrm{e} = \mathrm{e}\ln 3$.

(4)
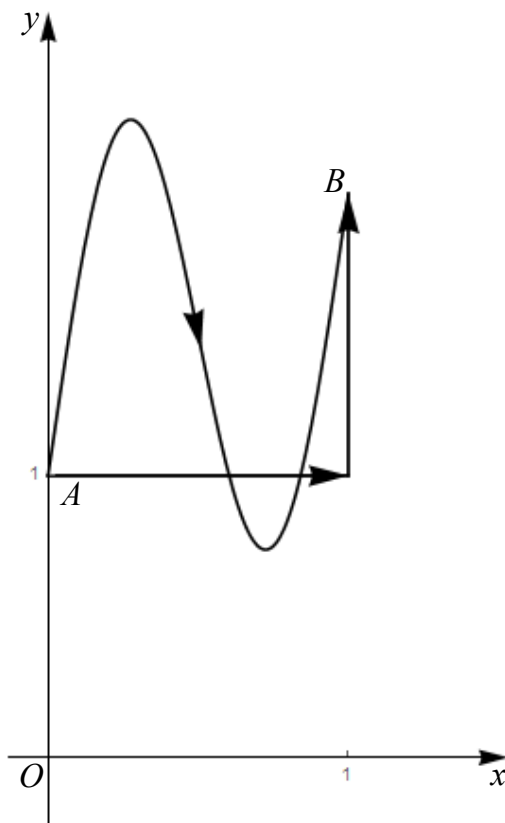
图 5: 习题13.6 1.(4)题图示

设$D = \{(x,y) \mid y > 0\}$,

令$\begin{cases} X(x,y) = \frac{1+y^2 f(xy)}{y} = \frac{1}{y} + yf(xy), \\ Y(x,y) = \frac{x}{y^2}[y^2 f(xy) - 1] = xf(xy) - \frac{x}{y^2}, \end{cases}$ 则$\boldsymbol{F}(x,y) = (X,Y) \in C^1(D)$,

$\because D$单连通，

且$\mathrm{rot}\boldsymbol{F} = (\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y})\boldsymbol{k} = \{f(xy) + xf'(xy)y - \frac{1}{y^2} - [-\frac{1}{y^2} + f(xy) + yf'(xy)x]\}\boldsymbol{k} = \boldsymbol{0}$,

$\therefore \int_L X\mathrm{d}x + Y\mathrm{d}y = \int_{(0,1)}^{(1,2)} X\mathrm{d}x + Y\mathrm{d}y = \int_{(0,1)}^{(1,1)} X\mathrm{d}x + Y\mathrm{d}y + \int_{(1,1)}^{(1,2)} X\mathrm{d}x + Y\mathrm{d}y$
$= \int_0^1 [1 + f(x)]\mathrm{d}x + \int_1^2 [f(y) - \frac{1}{y^2}]\mathrm{d}y = \int_0^1 \mathrm{d}x + \int_0^1 f(x)\mathrm{d}x + \int_1^2 f(y)\mathrm{d}y - \int_1^2 \frac{1}{y^2}\mathrm{d}y$
$= 1 + \int_0^1 f(x)\mathrm{d}x + \frac{1}{y}\big|_1^2 = \frac{1}{2} + \int_0^1 f(x)\mathrm{d}x.$

2. 确定p的值，使积分$\int_A^B (x^4 + 4xy^p)\mathrm{d}x + (6x^{p-1}y^2 - 5y^4)\mathrm{d}y$与路线无关. 当$A = (0,0), B = (1,2)$时，计算积分的值.
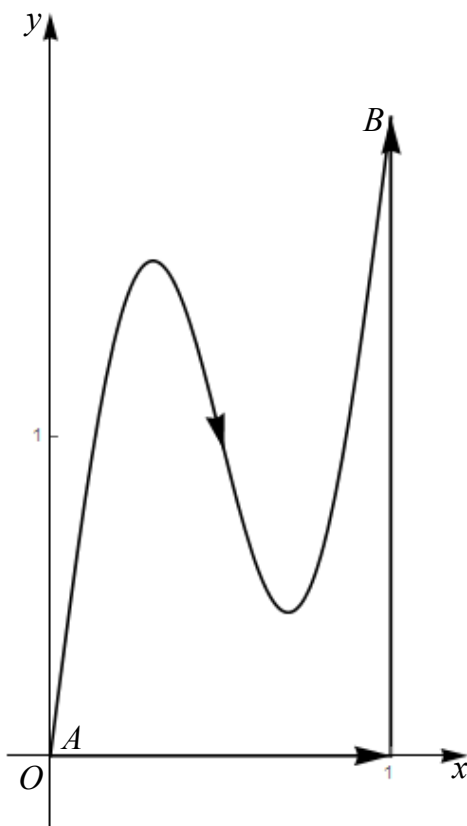
图 6: 习题13.6 2.题图示

解：$\because$积分$\int_A^B (x^4 + 4xy^p)\mathrm{d}x + (6x^{p-1}y^2 - 5y^4)\mathrm{d}y$与路线无关,

令 $\begin{cases} X(x,y) = x^4 + 4xy^p, \\ Y(x,y) = 6x^{p-1}y^2 - 5y^4, \end{cases}$ 则$\boldsymbol{F}(x,y) = (X, Y)$是保守场，

$\because \boldsymbol{F}(x,y) \in C^1$,

$\therefore \mathrm{rot}\boldsymbol{F} = (\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y})\boldsymbol{k} = [6(p-1)x^{p-2}y^2 - 4pxy^{p-1}]\boldsymbol{k} = \boldsymbol{0}$,

$\therefore p = 3$.

$\because A = (0,0), B = (1,2)$,

$\therefore \int_A^B X\mathrm{d}x + Y\mathrm{d}y = \int_{(0,0)}^{(1,2)} X\mathrm{d}x + Y\mathrm{d}y = \int_{(0,0)}^{(1,0)} X\mathrm{d}x + Y\mathrm{d}y + \int_{(1,0)}^{(1,2)} X\mathrm{d}x + Y\mathrm{d}y$

$= \int_0^1 x^4\mathrm{d}x + \int_0^2 (6y^2 - 5y^4)\mathrm{d}y = \frac{1}{5}x^5\big|_0^1 + (2y^3 - y^5)\big|_0^2 = \frac{1}{5} + (2\cdot 8 - 32) = -\frac{79}{5}.$

3. 判定下列微分形式是否为全微分，若是，求出其原函数：

(1)$(2x\cos y - y^2\sin x)\mathrm{d}x + (2y\cos x - x^2\sin y)\mathrm{d}y$;

(2)$(\mathrm{e}^x\cos y + 2xy^2)\mathrm{d}x + (2x^2 y - \mathrm{e}^x\sin y)\mathrm{d}y$.

解：(1)令 $\begin{cases} X(x,y) = 2x\cos y - y^2\sin x, \\ Y(x,y) = 2y\cos x - x^2\sin y, \end{cases}$ 则 $\boldsymbol{F}(x,y) = (X,Y) \in C^1(\mathbb{R})$,

$\because \mathbb{R}$是单连通区域，

且 $\mathrm{rot}\boldsymbol{F} = [\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y})\boldsymbol{k} = (-2y\sin x - 2x\sin y - (-2x\sin y - 2y\sin x)]\boldsymbol{k} = \boldsymbol{0}$,

$\therefore X\mathrm{d}x + Y\mathrm{d}y$是全微分式，

方法1：原函数$\varphi(x,y) = \int_{(0,0)}^{(x,y)} X(s,t)\mathrm{d}s + Y(s,t)\mathrm{d}t + C$

$= \int_{(0,0)}^{(x,0)} X(s,t)\mathrm{d}s + Y(s,t)\mathrm{d}t + \int_{(x,0)}^{(x,y)} X(s,t)\mathrm{d}s + Y(s,t)\mathrm{d}t + C$

$= \int_0^x 2s\mathrm{d}s + \int_0^y (2t\cos x - x^2\sin t)\mathrm{d}t + C = s^2\big|_0^x + (t^2\cos x + x^2\cos t)\big|_0^y$

$= x^2 + (y^2\cos x - x^2\sin y - x^2) + C = y^2\cos x - x^2\sin y + C.$

方法2：设原函数为$\varphi(x,y)$,

则$\frac{\partial\varphi(x,y)}{\partial x} = 2x\cos y - y^2\sin x$,

$\therefore \varphi(x,y) = x^2\cos y + y^2\cos x + C(y)$,

$\therefore \frac{\partial\varphi(x,y)}{\partial y} = -x^2\sin y + 2y\cos x + C'(y) = 2y\cos x - x^2\sin y$,

$\therefore C'(y) = 0,\ C(y) = C$,

$\therefore \varphi(x,y) = x^2\cos y + y^2\cos x + C.$

(2)令 $\begin{cases} X(x,y) = \mathrm{e}^x\cos y + 2xy^2, \\ Y(x,y) = 2x^2 y - \mathrm{e}^x\sin y, \end{cases}$ 则 $\boldsymbol{F}(x,y) = (X,Y) \in C^1(\mathbb{R})$,

$\because \mathbb{R}$是单连通区域，

且 $\mathrm{rot}\boldsymbol{F} = (\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y})\boldsymbol{k} = [4xy - \mathrm{e}^x\sin y - (-\mathrm{e}^x\sin y + 4xy)]\boldsymbol{k} = \boldsymbol{0}$,

$\therefore X\mathrm{d}x + Y\mathrm{d}y$是全微分式.

方法1：原函数$\varphi(x,y) = \int_{(0,0)}^{(x,y)} X(s,t)\mathrm{d}s + Y(s,t)\mathrm{d}t + C_1$

$= \int_{(0,0)}^{(x,0)} X(s,t)\mathrm{d}s + Y(s,t)\mathrm{d}t + \int_{(x,0)}^{(x,y)} X(s,t)\mathrm{d}s + Y(s,t)\mathrm{d}t + C_1$

$= \int_0^x \mathrm{e}^s\mathrm{d}s + \int_0^y (2x^2 t - \mathrm{e}^x\sin t)\mathrm{d}t + C_1 = \mathrm{e}^s\big|_0^x + (x^2 t^2 + \mathrm{e}^x\cos t)\big|_0^y + C_1$

$= \mathrm{e}^x - 1 + (x^2 y^2 + \mathrm{e}^x\cos y - \mathrm{e}^x) + C_1 = x^2 y^2 + \mathrm{e}^x\cos y + C.$

方法：设原函数为$\varphi(x,y)$,

则 $\frac{\partial \varphi(x,y)}{\partial x} = \mathrm{e}^x \cos y + 2xy^2$,

$\therefore \varphi(x,y) = \mathrm{e}^x \cos y + x^2 y^2 + C(y)$,

$\therefore \frac{\partial \varphi(x,y)}{\partial y} = -\mathrm{e}^x \sin y + 2x^2 y + C'(y) = 2x^2 y - \mathrm{e}^x \sin y$,

$\therefore C'(y) = 0, C(y) = C$,

$\therefore \varphi(x,y) = \mathrm{e}^x \cos y + x^2 y^2 + C.$

4. 设 $f(u)$ 连续，L 为逐段光滑简单闭曲线，求证：

$$\oint_L f(x^2 + y^2)(x\mathrm{d}x + y\mathrm{d}y) = 0.$$

证明：令 $\varphi(x,y) = \frac{1}{2} \int_0^{x^2+y^2} f(u)\mathrm{d}u$,

$\frac{\partial \varphi}{\partial x} = f(x^2+y^2)x, \frac{\partial \varphi}{\partial y} = f(x^2+y^2)y$,

$\therefore \mathrm{d}\varphi(x,y) = \frac{\partial \varphi}{\partial x}\mathrm{d}x + \frac{\partial \varphi}{\partial y}\mathrm{d}y = f(x^2+y^2)(x\mathrm{d}x + y\mathrm{d}y)$,

$\therefore \oint_L f(x^2+y^2)(x\mathrm{d}x + y\mathrm{d}y) = 0.$

5.设一元函数 f 有连续的导数，计算 $\boldsymbol{\nabla} \cdot (f(r)\boldsymbol{r})$，其中

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, \; r = \sqrt{x^2 + y^2 + z^2},$$

并说明 f 满足什么条件时，$f(r)\boldsymbol{r}$ 为无源场.

解：$\boldsymbol{\nabla} \cdot (f(r)\boldsymbol{r}) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (xf(r), yf(r), zf(r)) = \frac{\partial [xf(r)]}{\partial x} + \frac{\partial [yf(r)]}{\partial y} + \frac{\partial [zf(r)]}{\partial z}$
$= f(r) + xf'(r)\frac{x}{r} + f(r) + yf'(r)\frac{y}{r} + f(r) + zf'(r)\frac{z}{r}$
$= 3f(r) + f'(r)\frac{x^2+y^2+z^2}{r} = 3f(r) + rf'(r).$

$\because f(r)\boldsymbol{r}$ 无源，

$\therefore 3f(r) + rf'(r) = 0$,

当 $f(r) \not\equiv 0$ 时 $\frac{\mathrm{d}f(r)}{f(r)} = -\frac{3}{r}\mathrm{d}r$,

$\therefore \ln|f(r)| = -3\ln|r| + C$,

$\therefore f(r)r^3 = \pm\mathrm{e}^C$,

$\therefore f(r)r^3 = C_0, \; C_0, C$ 为任意常数，$f(r) \equiv 0$ 也满足该式.

5. 设 $\boldsymbol{F} = f(r)\boldsymbol{r}$($r$ 与 $\boldsymbol{r}$ 的意义与上题同)，证明 $\mathrm{rot}\boldsymbol{F} = \boldsymbol{0}$.

证明：$\mathrm{rot}\boldsymbol{F} = \boldsymbol{\nabla} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$
$= (\frac{\partial zf(r)}{\partial y} - \frac{\partial yf(r)}{\partial z}, \frac{\partial xf(r)}{\partial z} - \frac{\partial zf(r)}{\partial x}, \frac{\partial yf(r)}{\partial x} - \frac{\partial xf(r)}{\partial y})$
$= (zf'(r)\frac{y}{r} - yf'(r)\frac{z}{r}, xf'(r)\frac{z}{r} - zf'(r)\frac{x}{r}, yf'(r)\frac{y}{r} - xf'(r)\frac{y}{r})$
$= (0,0,0) = \boldsymbol{0}.$

6. 设f有连续的二阶导数，计算$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f(r))$，其中$r, \boldsymbol{r}$同题5，并说明$f$满足什么条件时，$\boldsymbol{\nabla} f$为无源场.

解：$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f(r)) = \boldsymbol{\nabla} \cdot (f'(r)\frac{x}{r}, f'(r)\frac{y}{r}, f'(r)\frac{z}{r}) = \frac{\partial}{\partial x}[f'(r)\frac{x}{r}] + \frac{\partial}{\partial y}[f'(r)\frac{y}{r}] + \frac{\partial}{\partial z}[f'(r)\frac{z}{r}]$

$= f''(r)\frac{x}{r} \cdot \frac{x}{r} + f'(r)\frac{r - x\frac{x}{r}}{r^2} + f''(r)\frac{y}{r} \cdot \frac{y}{r} + f'(r)\frac{r - y\frac{y}{r}}{r^2} + f''(r)\frac{z}{r} \cdot \frac{z}{r} + f'(r)\frac{r - z\frac{z}{r}}{r^2}$

$= f''(r)\frac{x^2}{r^2} + f'(r)\frac{y^2+z^2}{r^3} + f''(r)\frac{y^2}{r^2} + f'(r)\frac{z^2+x^2}{r^3} + f''(r)\frac{z^2}{r^2} + f'(r)\frac{x^2+y^2}{r^3}$

$= f''(r)\frac{x^2+y^2+z^2}{r^2} + f'(r)\frac{2(x^2+y^2+z^2)}{r^3} = f''(r) + \frac{2}{r}f'(r)$.

$\because \boldsymbol{\nabla} f$为无源场，

$\therefore f''(r) + \frac{2}{r}f'(r) = 0$,

当$f'(r) \not\equiv 0$时$\frac{\mathrm{d}f'(r)}{f'(r)} = -\frac{2}{r}\mathrm{d}r$,

$\therefore \ln|f'(r)| = -2\ln|r| + C$,

$\therefore f'(r)r^2 = \pm \mathrm{e}^C$,

$\therefore f'(r) = \frac{C_0}{r^2}$，$f'(r) \equiv 0$也满足该式.

$\therefore f(r) = -\frac{C_0}{r} + C_2 = \frac{C_1}{r} + C_2$.

7. 证明下列向量场为无源场：

(1)$\boldsymbol{v} = \boldsymbol{u}_1 \times \boldsymbol{u}_2$，其中$\boldsymbol{u}_1, \boldsymbol{u}_2$是无旋场；

(2)$\boldsymbol{v} = \frac{\boldsymbol{r}}{r^3}$，其中$r, \boldsymbol{r}$同题5.

证明：(1)$\mathrm{div}\boldsymbol{v} = \boldsymbol{\nabla} \cdot \boldsymbol{v} = \boldsymbol{\nabla} \cdot (\boldsymbol{u}_1 \times \boldsymbol{u}_2) = \boldsymbol{u}_2 \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}_1) - \boldsymbol{u}_1 \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}_2)$

$= \boldsymbol{u}_2 \cdot \boldsymbol{0} - \boldsymbol{u}_1 \cdot \boldsymbol{0} = 0 - 0 = 0$.

注：该公式的证明见习题13.1中的2题.

(2)$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \boldsymbol{\nabla} \cdot (\frac{\boldsymbol{r}}{r^3}) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}) = \frac{\partial}{\partial x}(\frac{x}{r^3}) + \frac{\partial}{\partial y}(\frac{y}{r^3}) + \frac{\partial}{\partial z}(\frac{z}{r^3})$

$= \frac{r^3 - x3r^2\frac{x}{r}}{r^6} + \frac{r^3 - y3r^2\frac{y}{r}}{r^6} + \frac{r^3 - z3r^2\frac{z}{r}}{r^6} = \frac{r^2-3x^2}{r^5} + \frac{r^2-3y^2}{r^5} + \frac{r^2-3z^2}{r^5}$

$= \frac{y^2+z^2-2x^2}{r^5} + \frac{z^2+x^2-2y^2}{r^5} + \frac{x^2+y^2-2z^2}{r^5} = 0$.

9. 求电场$\boldsymbol{v} = \frac{\boldsymbol{r}}{r^3}$穿过包围原点的任意简单光滑闭曲面的电通量，其中$r, \boldsymbol{r}$同题5.

解：设S是包围原点的任意简单光滑闭曲面，S_1是S围成区域中的包围原点的任意简单光滑闭曲面，S_1, S外侧为正，记S, S_1^-围成的区域为Ω，

则$\oiint\limits_{S} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} - \oiint\limits_{S_1} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \oiint\limits_{S} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} + \oiint\limits_{S_1^-} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \oiint\limits_{S+S_1^-} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}$,

$\because \Omega$不包含原点，

$\therefore \boldsymbol{v} \in C^1(\Omega)$且由上述题8(2)可知$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$,

$\therefore \oiint\limits_{S+S_1^-} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \iiint\limits_{\Omega} \boldsymbol{\nabla} \cdot \boldsymbol{v}\mathrm{d}V = \iiint\limits_{\Omega} 0\mathrm{d}V = 0$,

$\therefore \oiint\limits_{S} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \oiint\limits_{S_1} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}$,

$\therefore \boldsymbol{v} = \frac{\boldsymbol{r}}{r^3}$穿过包围原点的任意简单光滑闭曲面的电通量都相等，故可取一个特殊的曲面计算电通量的值.

不妨取$S_1 : r = a, a > 0$，记Ω_1是S_1围成的区域，

$$\therefore \oiint\limits_{S_1} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \oiint\limits_{S_1} \frac{\boldsymbol{r}}{r^3} \cdot \mathrm{d}\boldsymbol{S} = \oiint\limits_{S_1} \frac{\boldsymbol{r}}{a^3} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{a^3} \oiint\limits_{S_1} \boldsymbol{r} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{a^3} \iiint\limits_{\Omega_1} \boldsymbol{\nabla} \cdot \boldsymbol{r} \mathrm{d}V$$
$$= \frac{1}{a^3} \iiint\limits_{\Omega_1} (\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}) \mathrm{d}V = \frac{1}{a^3} \iiint\limits_{\Omega_1} 3 \mathrm{d}V = \frac{3}{a^3} \iiint\limits_{\Omega_1} \mathrm{d}V = \frac{3}{a^3} \frac{4}{3} \pi a^3 = 4\pi.\text{[1]}$$

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[1]该题给出了真空中点电荷电场的高斯定理的证明. 真空中位于原点的点电荷q产生的电场

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0} \frac{\boldsymbol{r}}{r^3} = \frac{q}{4\pi\varepsilon_0} \boldsymbol{v},$$

故该电场穿过包围该电荷的任意简单光滑闭曲面的电通量

$$\oiint\limits_{S} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{q}{4\pi\varepsilon_0} \oiint\limits_{S} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \frac{q}{\varepsilon_0}.$$

即真空中的电荷的电场穿过包围该点电荷的任意简单光滑闭曲面的电通量与该电荷的电量成正比.

$$Q = \iint\limits_S \mathrm{d}Q = \iint\limits_S \vec{v} \cdot \vec{n}\mathrm{d}S = \iint\limits_S \vec{v} \cdot \mathrm{d}\vec{S}$$

(a)

$$\vec{\nabla} \cdot \vec{v} = \vec{\nabla} \cdot \frac{\vec{r}}{r} = 0$$

$$\oiint_{S+S_1{}^-} \vec{v} \cdot \mathrm{d}\vec{S} = \iiint_{\Omega} \vec{\nabla} \cdot \vec{v}\, \mathrm{d}V$$

(b)
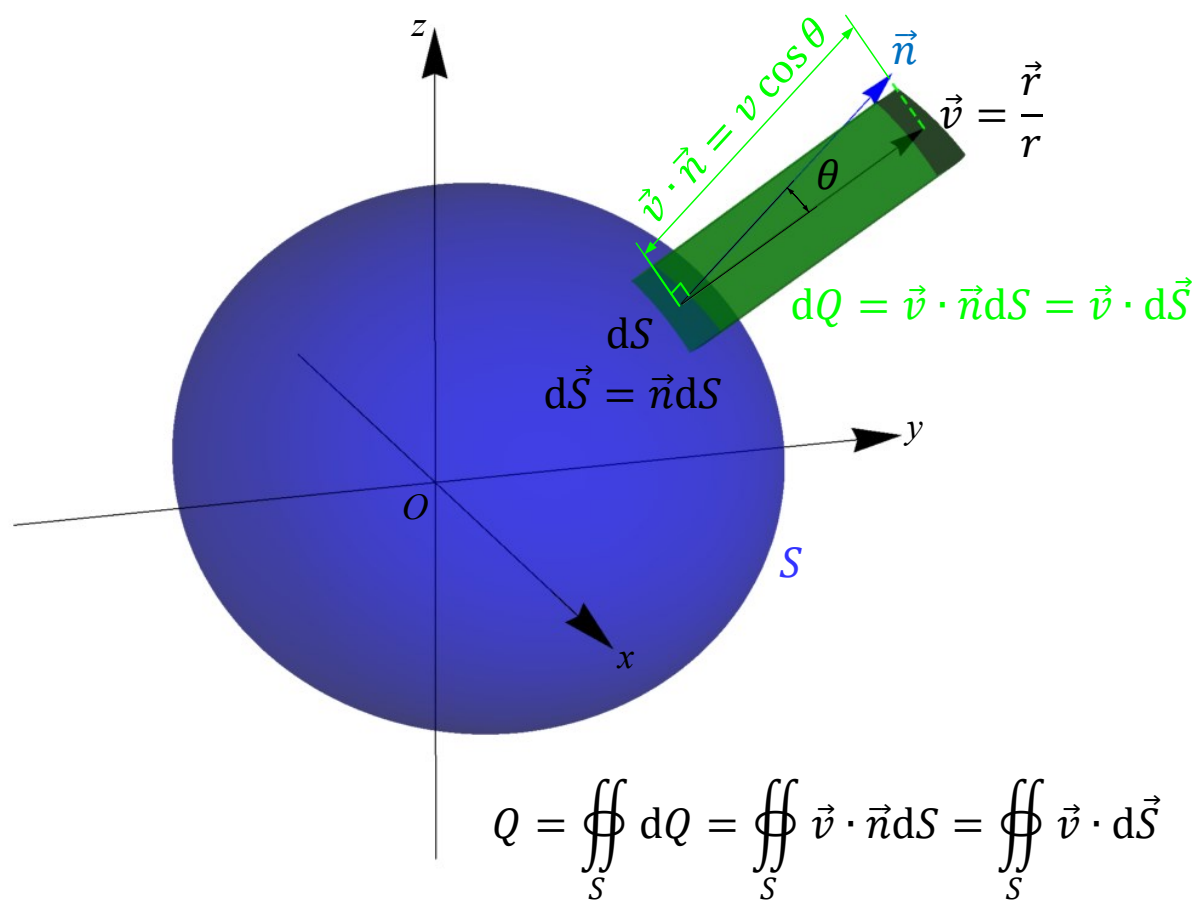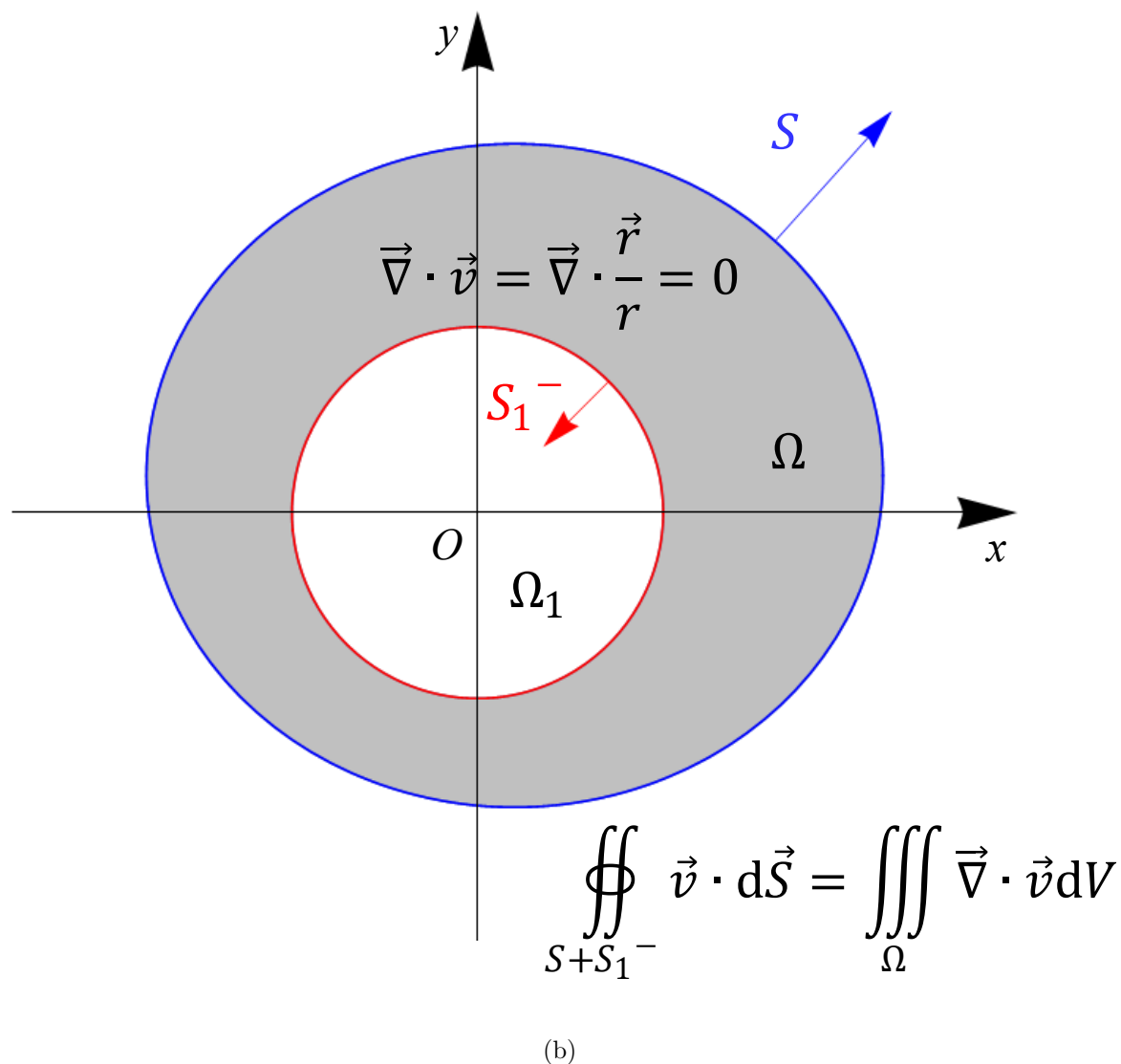
图 7: 习题13.6 9.题图示

8. 证明下列向量场为无旋场：

(1)$\boldsymbol{v} = (x - x_0)\boldsymbol{i} + (y - y_0)\boldsymbol{j} + (z - z_0)\boldsymbol{k}$;

(2)$\boldsymbol{v} = yz(2x + y + z)\boldsymbol{i} + zx(x + 2y + z)\boldsymbol{j} + xy(x + y + 2z)\boldsymbol{k}$.

证明：(1)$\mathrm{rot}\boldsymbol{v} = \boldsymbol{\nabla} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - x_0 & y - y_0 & z - z_0 \end{vmatrix}$

$= \left(\frac{\partial(z-z_0)}{\partial y} - \frac{\partial(y-y_0)}{\partial z}, \frac{\partial(x-x_0)}{\partial z} - \frac{\partial(z-z_0)}{\partial x}, \frac{\partial(y-y_0)}{\partial x} - \frac{\partial(x-x_0)}{\partial y} \right) = (0, 0, 0) = \boldsymbol{0}$.

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$(2)\boldsymbol{v} = (2xyz + y^2z + yz^2)\boldsymbol{i} + (zx^2 + 2xyz + xz^2)\boldsymbol{j} + (x^2y + xy^2 + 2xyz)\boldsymbol{k},$

$$\mathrm{rot}\boldsymbol{v} = \boldsymbol{\nabla} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz + y^2z + yz^2 & zx^2 + 2xyz + xz^2 & x^2y + xy^2 + 2xyz \end{vmatrix}$$

$$= (\frac{\partial(x^2y + xy^2 + 2xyz)}{\partial y} - \frac{\partial(zx^2 + 2xyz + xz^2)}{\partial z},$$

$$\frac{\partial(2xyz + y^2z + yz^2)}{\partial z} - \frac{\partial(x^2y + xy^2 + 2xyz)}{\partial x},$$

$$\frac{\partial(zx^2 + 2xyz + xz^2)}{\partial x} - \frac{\partial(2xyz + y^2z + yz^2)}{\partial y})$$

$$= (x^2 + 2xy + 2xz - (x^2 + 2xy + 2zx),$$

$$2xy + y^2 + 2yz - (2xy + y^2 + 2yz),$$

$$2zx + 2yz + z^2 - (2zx + 2yz + z^2))$$

$$= \boldsymbol{0}.$$

14.7 习题13.1解答

1. 验证梯度算子$\boldsymbol{\nabla}$的下列性质，其中α, β为任意常数，f, g为任意可微函数：

   $(1)\boldsymbol{\nabla}(\alpha f + \beta g) = \alpha\boldsymbol{\nabla} + \beta\boldsymbol{\nabla}g$；

   $(2)\boldsymbol{\nabla}(fg) = g\boldsymbol{\nabla}f + f\boldsymbol{\nabla}g$；

   $(3)\boldsymbol{\nabla}(\frac{f}{g}) = \frac{g\boldsymbol{\nabla}f - f\boldsymbol{\nabla}g}{g^2}$(在$g$不等于零处成立).

   证明：(1)

   $$\boldsymbol{\nabla}(\alpha f + \beta g) = (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k})(\alpha f + \beta g)$$

   $$= \frac{\partial(\alpha f + \beta g)}{\partial x}\boldsymbol{i} + \frac{\partial(\alpha f + \beta g)}{\partial y}\boldsymbol{j} + \frac{\partial(\alpha f + \beta g)}{\partial z}\boldsymbol{k}$$

   $$= (\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial g}{\partial x})\boldsymbol{i} + (\alpha\frac{\partial f}{\partial y} + \beta\frac{\partial g}{\partial y})\boldsymbol{j} + (\alpha\frac{\partial f}{\partial z} + \beta\frac{\partial g}{\partial z})\boldsymbol{k}$$

   $$= \alpha(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}) + \beta(\frac{\partial g}{\partial x}\boldsymbol{i} + \frac{\partial g}{\partial y}\boldsymbol{j} + \frac{\partial g}{\partial z}\boldsymbol{k})$$

   $$= \alpha\boldsymbol{\nabla}f + \beta\boldsymbol{\nabla}g.$$

(2)

$$\begin{aligned}
\boldsymbol{\nabla}(fg) &= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k})(fg) \\
&= \frac{\partial(fg)}{\partial x}\boldsymbol{i} + \frac{\partial(fg)}{\partial y}\boldsymbol{j} + \frac{\partial(fg)}{\partial z}\boldsymbol{k} \\
&= (g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x})\boldsymbol{i} + (g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y})\boldsymbol{j} + (g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z})\boldsymbol{k} \\
&= g(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}) + f(\frac{\partial g}{\partial x}\boldsymbol{i} + \frac{\partial g}{\partial y}\boldsymbol{j} + \frac{\partial g}{\partial z}\boldsymbol{k}) \\
&= g\boldsymbol{\nabla} f + f\boldsymbol{\nabla} g.
\end{aligned}$$

(3)

$$\begin{aligned}
\boldsymbol{\nabla}(\frac{f}{g}) &= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k})(\frac{f}{g}) \\
&= \frac{\partial}{\partial x}(\frac{f}{g})\boldsymbol{i} + \frac{\partial}{\partial y}(\frac{f}{g})\boldsymbol{j} + \frac{\partial}{\partial z}(\frac{f}{g})\boldsymbol{k} \\
&= \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\boldsymbol{i} + \frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\boldsymbol{j} + \frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\boldsymbol{k} \\
&= \frac{g(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}) - f(\frac{\partial g}{\partial x}\boldsymbol{i} + \frac{\partial g}{\partial y}\boldsymbol{j} + \frac{\partial g}{\partial z}\boldsymbol{k})}{g^2} \\
&= \frac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2}.
\end{aligned}$$

2. 验证散度算子的下列性质(其中f为函数，$\boldsymbol{u}, \boldsymbol{v}$是向量场)：

$$\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = -\boldsymbol{u} \cdot \boldsymbol{\nabla} \times \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \times \boldsymbol{u}.$$

证明：

$$\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}) \cdot \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1(x,y,z) & u_2(x,y,z) & u_3(x,y,z) \\ v_1(x,y,z) & v_2(x,y,z) & v_3(x,y,z) \end{vmatrix}$$

$$= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}) \cdot [(u_2v_3 - u_3v_2)\boldsymbol{i} + (u_3v_1 - u_1v_3)\boldsymbol{j} + (u_1v_2 - u_2v_2)\boldsymbol{k}]$$

$$= \frac{\partial(u_2v_3 - u_3v_2)}{\partial x} + \frac{\partial(u_3v_1 - u_1v_3)}{\partial y} + \frac{\partial(u_1v_2 - u_2v_1)}{\partial z}$$

$$= \frac{\partial u_2}{\partial x}v_3 + u_2\frac{\partial v_3}{\partial x} - \frac{\partial u_3}{\partial x}v_2 - u_3\frac{\partial v_2}{\partial x}$$

$$+ \frac{\partial u_3}{\partial y}v_1 + u_3\frac{\partial v_1}{\partial y} - \frac{\partial u_1}{\partial y}v_3 - u_1\frac{\partial v_3}{\partial y}$$

$$+ \frac{\partial u_1}{\partial z}v_2 + u_1\frac{\partial v_2}{\partial z} - \frac{\partial u_2}{\partial z}v_1 - u_2\frac{\partial v_1}{\partial z}$$

$$= u_1(\frac{\partial v_2}{\partial z} - \frac{\partial v_3}{\partial y}) + u_2(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}) + u_3(\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x})$$

$$+ v_1(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}) + v_2(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}) + v_3(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y})$$

$$= -u_1(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}) - u_2(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}) - u_3(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y})$$

$$+ v_1(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}) + v_2(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}) + v_3(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y})$$

$$= -(u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}) \cdot [(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z})\boldsymbol{i} + (\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x})\boldsymbol{j} + (\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial y})\boldsymbol{k}]$$

$$+ (v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}) \cdot [(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z})\boldsymbol{i} + (\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x})\boldsymbol{j} + (\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y})\boldsymbol{k}]$$

$$= -(u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}) \cdot \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} + (v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}) \cdot \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$= -\boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) + \boldsymbol{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u})$$

3. 设 $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, r = \sqrt{x^2 + y^2 + z^2}$：

(1)设 $f(u)$ 为可微函数，求 $\boldsymbol{\nabla} f(r)$；

(2)设 $\boldsymbol{F} = f(r)\boldsymbol{r}$，求证 $\boldsymbol{\nabla} \times \boldsymbol{F} \equiv \boldsymbol{0}$. 又问当 f 满足什么条件时，$\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0$？

解：(1)

$$\boldsymbol{\nabla} f(r) = \frac{\partial f(r)}{\partial x}\boldsymbol{i} + \frac{\partial f(r)}{\partial y}\boldsymbol{j} + \frac{\partial f(r)}{\partial z}\boldsymbol{k}$$

$$= f'(r)\frac{\partial r}{\partial x}\boldsymbol{i} + f'(r)\frac{\partial r}{\partial y}\boldsymbol{j} + f'(r)\frac{\partial r}{\partial z}\boldsymbol{k} = f'(r)(\frac{\partial r}{\partial x}\boldsymbol{i} + \frac{\partial r}{\partial y}\boldsymbol{j} + \frac{\partial r}{\partial z}\boldsymbol{k})$$

$$= f'(r)(\frac{x}{\sqrt{x^2+y^2+z^2}}\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\boldsymbol{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\boldsymbol{k})$$

$$= \frac{f'(r)}{\sqrt{x^2+y^2+z^2}}(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) = \frac{f'(r)}{r}\boldsymbol{r}.$$

(2)

i)证明：

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{\nabla} \times [f(r)\boldsymbol{r}] = \boldsymbol{\nabla} f(r) \times \boldsymbol{r} + f(r)\boldsymbol{\nabla} \times \boldsymbol{r}$$

$$= \frac{f'(r)}{r}\boldsymbol{r} \times \boldsymbol{r} + f(r)\boldsymbol{\nabla} \times \boldsymbol{r}$$

$$= 0 + f(r)\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= f(r)[(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z})\boldsymbol{i} + (\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x})\boldsymbol{j} + (\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y})\boldsymbol{k}]$$

$$= f(r)(0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}) = 0.$$

ii)

∵

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \boldsymbol{\nabla} \cdot [f(r)\boldsymbol{r}] = \boldsymbol{\nabla} f(r) \cdot \boldsymbol{r} + f(r)\boldsymbol{\nabla} \cdot \boldsymbol{r} = \frac{f'(r)}{r}\boldsymbol{r} \cdot \boldsymbol{r} + f(r)(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z})$$

$$= \frac{\mathrm{d}f(r)}{\mathrm{d}r}r + 3f(r) = 0,$$

∴当$f(r) \neq 0$时，$r \neq 0,$

$$\frac{\mathrm{d}f(r)}{f(r)} = -3\frac{\mathrm{d}r}{r},$$

∴

$$\ln|f(r)| = -3\ln|r| + C_1,$$

∴

$$f(r) = \frac{C}{r^3}, C为任意常数.$$

4. 验证旋度算子的下列基本公式：

(1)$\boldsymbol{\nabla} \times (\alpha\boldsymbol{u} + \beta\boldsymbol{v}) = \alpha\boldsymbol{\nabla} \times \boldsymbol{u} + \beta\boldsymbol{\nabla} \times \boldsymbol{v}$;

(2)$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \boldsymbol{0}$;

(3)$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) = 0$.

证明：(1)

$$\begin{aligned}
\boldsymbol{\nabla} \times (\alpha\boldsymbol{u} + \beta\boldsymbol{v}) &= \boldsymbol{\nabla} \times (\alpha u_1\boldsymbol{i} + \alpha u_2\boldsymbol{j} + \alpha u_3\boldsymbol{k} + \beta v_1\boldsymbol{i} + \beta v_2\boldsymbol{j} + \beta v_3\boldsymbol{k}) \\
&= \boldsymbol{\nabla} \times [(\alpha u_1 + \beta v_1)\boldsymbol{i} + (\alpha u_2 + \beta v_2)\boldsymbol{j} + (\alpha u_3 + \beta v_3)\boldsymbol{k}] \\
&= \frac{\partial(\alpha u_1 + \beta v_1)}{\partial x} + \frac{\partial(\alpha u_2 + \beta v_2)}{\partial y} + \frac{\partial(\alpha u_3 + \beta v_3)}{\partial z} \\
&= \alpha\frac{\partial u_1}{\partial x} + \beta\frac{\partial v_1}{\partial x} + \alpha\frac{\partial u_2}{\partial y} + \beta\frac{\partial v_2}{\partial y} + \alpha\frac{\partial u_3}{\partial z} + \beta\frac{\partial v_3}{\partial z} \\
&= \alpha(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}) + \beta(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}) \\
&= \alpha\boldsymbol{\nabla} \cdot \boldsymbol{u} + \beta\boldsymbol{\nabla} \cdot \boldsymbol{v}.
\end{aligned}$$

(2)当$f \in C^2$时

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) &= \boldsymbol{\nabla} \times (\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= (\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x})\boldsymbol{i} + (\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z})\boldsymbol{j} + (\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x})\boldsymbol{k} \\
&= 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k} = \boldsymbol{0}.
\end{aligned}$$

(3)当$\boldsymbol{v} \in C^2$时

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) &= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}) \cdot \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1(x,y,z) & v_2(x,y,z) & v_3(x,y,z) \end{vmatrix} \\
&= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}) \cdot [(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z})\boldsymbol{i} + (\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x})\boldsymbol{j} + (\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y})\boldsymbol{k}] \\
&= (\frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z}) + (\frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_3}{\partial y \partial x}) + (\frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}) \\
&= (\frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_3}{\partial y \partial x}) + (\frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_1}{\partial z \partial y}) + (\frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_2}{\partial x \partial z}) \\
&= 0.
\end{aligned}$$

5. 求下列向量场的散度：

(1)$\boldsymbol{v} = xyz(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$;

$(2)\boldsymbol{v} = (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \times \boldsymbol{c};$

$(3)\boldsymbol{v} = [(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{c}](x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$(其中$\boldsymbol{c}$为常值向量).

解：(1)

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{v} &= \boldsymbol{\nabla}(xyz) \cdot (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) + xyz\boldsymbol{\nabla} \cdot (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \\
&= (yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}) \cdot (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) + xyz(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}) \\
&= xyz + xyz + xyz + 3xyz = 6xyz.
\end{aligned}$$

(2)

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{v} &= \boldsymbol{c} \cdot [\boldsymbol{\nabla} \times (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})] - (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{c}) \\
&= \boldsymbol{c} \cdot \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} - (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{0} \\
&= \boldsymbol{c} \cdot [(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z})\boldsymbol{i} + (\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x})\boldsymbol{j} + (\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y})\boldsymbol{k}] - 0 \\
&= \boldsymbol{c} \cdot \boldsymbol{0} - 0 = 0.
\end{aligned}$$

(3)记$\boldsymbol{c} = (c_1, c_2, c_3)$

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{v} &= [(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{c}]\boldsymbol{\nabla} \cdot (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) - (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{\nabla}[(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{c}] \\
&= [(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{c}](\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}) - (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{\nabla}(c_1 x + c_2 y + c_3 z) \\
&= 3[(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{c}] - (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot (c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}) \\
&= 3[(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{c}] - (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{c} \\
&= 2[(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{c}].
\end{aligned}$$

6. 求下列向量场的旋度：

$(1)\boldsymbol{v} = y^2 z\boldsymbol{i} + z^2 x\boldsymbol{j} + x^2 y\boldsymbol{k};$

$(2)\boldsymbol{v} = f(\sqrt{x^2 + y^2 + z^2})\boldsymbol{c}$(其中$\boldsymbol{c}$)为常值向量.

解：(1)

$$\boldsymbol{\nabla} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix} = (x^2 - 2xz)\boldsymbol{i} + (y^2 - 2xy)\boldsymbol{j} + (z^2 - 2yz)\boldsymbol{k}.$$

(2)

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{v} &= \boldsymbol{\nabla} \times [f(\sqrt{x^2 + y^2 + z^2})\boldsymbol{c}] \\
&= \boldsymbol{\nabla} f(\sqrt{x^2 + y^2 + z^2}) \times \boldsymbol{c} + f(\sqrt{x^2 + y^2 + z^2})\boldsymbol{\nabla} \times \boldsymbol{c} \\
&= f'(\sqrt{x^2 + y^2 + z^2})(\frac{x}{\sqrt{x^2 + y^2 + z^2}}\boldsymbol{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\boldsymbol{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\boldsymbol{k}) \times \boldsymbol{c} + \boldsymbol{0} \\
&= \frac{f'(\sqrt{x^2 + y^2 + z^2})}{\sqrt{x^2 + y^2 + z^2}}(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \times \boldsymbol{c}.
\end{aligned}$$