15 多元函数微分学(2)

15.1 知识结构

第10章多元函数微分学

10.4 复合函数微分法

- 10.4.1 复合函数求导法则
- 10.4.2 函数的方向导数和梯度
  - i. 方向导数
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- 10.4.3 雅可比矩阵

10.5 隐函数微分法

- 10.5.1 隐函数的背景和概念
- 10.5.2 一个方程确定的隐函数
- 10.5.3 方程组确定的隐函数

15.2 隐函数求导两种方法等价性的证明

设二元函数$z = z(x, y)$由方程$F(x, y, z) = 0$确定，$F(x, y, z)$有连续的偏导数，求$z = z(x, y)$关于x, y的偏导数有以下两种方法：

(1) 将方程$F(x, y, z) = 0$两边分别对x, y求偏导数：

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0,\ \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0,$$

解得

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z},\ \frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}.$$

(2) 将方程$F(x, y, z) = 0$两边求全微分：

$$\mathrm{d}F(x, y, z) = F'_x\mathrm{d}x + F'_y\mathrm{d}y + F'_z\mathrm{d}z = 0,$$

整理得

$$\mathrm{d}z = -\frac{F'_x}{F'_z}\mathrm{d}x - \frac{F'_y}{F'_z}\mathrm{d}y,$$

则

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z},\ \frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}.$$

以上这两种方法是等价的，可做如下证明。因为$z=z(x,y)$是方程$F(x,y,z)=0$确定的隐函数，所以将$z=z(x,y)$代入函数$F=F(x,y,z)$得到一个关于x,y的常数函数$f(x,y)=F(x,y,z(x,y))=0$（比如方程$F(x,y,z)=x^2+y^2+z^2-R^2=0$确定隐函数$z(x,y)=\sqrt{R^2-x^2-y^2}$，将$z(x,y)$代入$F(x,y,z)=x^2+y^2+z^2-R^2$得到$f(x,y)=F(x,y,z(x,y))=x^2+y^2+(\sqrt{R^2-x^2-y^2})^2-R^2\equiv 0$，为一恒等于0的常数函数），该常数函数关于$x,y$的两个偏导数都等于0，即

$$\frac{\partial f}{\partial x}=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial z}=0,\ \frac{\partial f}{\partial y}=\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}=0,$$

可据此求出$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$，即方法(1).

因为$f(x,y)=F(x,y,z(x,y))=0$是一个常数函数，所以对该函关于x,y求全微分等于0，即

$$\mathrm{d}f(x,y)=(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x})\mathrm{d}x+(\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y})\mathrm{d}y=0\mathrm{d}x+0\mathrm{d}y=0$$

将该式做如下整理：

$$\begin{aligned}0=\mathrm{d}f(x,y)&=\frac{\partial F}{\partial x}\mathrm{d}x+\frac{\partial F}{\partial y}\mathrm{d}y+(\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\mathrm{d}y)\\&=\frac{\partial F}{\partial x}\mathrm{d}x+\frac{\partial F}{\partial y}\mathrm{d}y+\frac{\partial F}{\partial z}(\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y)\\&=\frac{\partial F}{\partial x}\mathrm{d}x+\frac{\partial F}{\partial y}\mathrm{d}y+\frac{\partial F}{\partial z}\mathrm{d}z\\&=\mathrm{d}F(x,y,z)\end{aligned}\tag{1}$$

可得到

$$\mathrm{d}z=-\frac{F'_x}{F'_z}\mathrm{d}x-\frac{F'_y}{F'_z}\mathrm{d}y=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y,$$

可据此求出$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$，即方法(2). 式 (1)的推导过程即是全微分形式不变性的证明过程.

15.3 习题10.4解答

1. 求下列复合函数的偏导数：

(1)$z=xy+xf(u),u=\frac{y}{x}$，其中f为C^1类函数，求$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$；

(2)$z=f(u,v),u=x,v=\frac{x}{y}$，其中$f$为$C^2$类函数，求$\frac{\partial^2 z}{\partial y^2}$；

(3)$z=xf(\frac{y}{x})+yg(\frac{x}{y})$，其中$f,g$为$C^2$类函数，求$\frac{\partial^2 z}{\partial x\partial y}$；

(4)$z=\frac{y}{f(x^2-y^2)}$，其中f为可微函数，求$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}$；

(5)$u=f(x,xy,xyz)$，其中f为可微函数，求$\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$；

(6)$z=\mathrm{e}^{x-2y},x=\sin t,y=t^3$，求$\frac{\mathrm{d}z}{\mathrm{d}t}$.

解：(1)$\because z=xy+xf(u)=xy+xf(\frac{y}{x})$,

$\therefore \frac{\partial z}{\partial x}=y+f(\frac{y}{x})+xf'(\frac{y}{x})(-\frac{y}{x^2})=y+f(\frac{y}{x})-\frac{y}{x}f'(\frac{y}{x}),\frac{\partial z}{\partial y}=x+xf'(\frac{y}{x})\frac{1}{x}=x+f'(\frac{y}{x})$,

$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x[y+f(\frac{y}{x})-\frac{y}{x}f'(\frac{y}{x})]+y[x+f'(\frac{y}{x})]=2xy+yf(\frac{y}{x}).$

(2)$\frac{\partial z}{\partial y}=\frac{\partial f(u,v)}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f(u,v)}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial f(u,v)}{\partial v}(-\frac{x}{y^2}),$

$\frac{\partial^2 z}{\partial y^2}=-\frac{x}{y^2}[\frac{\partial^2 f(u,v)}{\partial u\partial v}\frac{\partial u}{\partial y}+\frac{\partial^2 f(u,v)}{\partial v^2}\frac{\partial v}{\partial y}]+\frac{\partial f(u,v)}{\partial v}\frac{2x}{y^3}=-\frac{x}{y^2}[\frac{\partial^2 f(u,v)}{\partial v^2}(-\frac{x}{y^2})]+\frac{\partial f(u,v)}{\partial v}\frac{2x}{y^3}$
$=\frac{x^2}{y^4}\frac{\partial^2 f(u,v)}{\partial v^2}+\frac{2x}{y^3}\frac{\partial f(u,v)}{\partial v}.$

(3)$\frac{\partial z}{\partial y}=xf'(\frac{y}{x})\frac{1}{x}+g(\frac{x}{y})+yg'(\frac{x}{y})(-\frac{x}{y^2})=f'(\frac{y}{x})+g(\frac{x}{y})-\frac{x}{y}g'(\frac{x}{y}),$

$\frac{\partial^2 z}{\partial x\partial y}=f''(\frac{y}{x})(-\frac{y}{x^2})+g'(\frac{x}{y})\frac{1}{y}-\frac{1}{y}g'(\frac{x}{y})-\frac{x}{y}g''(\frac{x}{y})\frac{1}{y}=-\frac{y}{x^2}f''(\frac{y}{x})-\frac{x}{y^2}g''(\frac{x}{y}).$

(4)$\frac{\partial z}{\partial x}=\frac{-yf'(x^2-y^2)2x}{[f(x^2-y^2)]^2}=\frac{-2xyf'(x^2-y^2)}{[f(x^2-y^2)]^2},\frac{\partial z}{\partial y}=\frac{f(x^2-y^2)-yf'(x^2-y^2)(-2y)}{[f(x^2-y^2)]^2}=\frac{f(x^2-y^2)+2y^2f'(x^2-y^2)}{[f(x^2-y^2)]^2},$

$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{1}{x}\frac{-2xyf'(x^2-y^2)}{[f(x^2-y^2)]^2}+\frac{1}{y}\frac{f(x^2-y^2)+2y^2f'(x^2-y^2)}{[f(x^2-y^2)]^2}=\frac{f(x^2-y^2)}{y[f(x^2-y^2)]^2}=\frac{1}{yf(x^2-y^2)}.$

(5)$\frac{\partial u}{\partial x}=f_1'+f_2'y+f_3'yz=f_1'+yf_2'+yzf_3',\frac{\partial u}{\partial y}=xf_2'+xzf_3',\frac{\partial u}{\partial z}=xyf_3'.$

(6)$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^{x-2y}\cos t+\mathrm{e}^{x-2y}(-2)3t^2=\mathrm{e}^{x-2y}(\cos t-6t^2)$
$=\mathrm{e}^{\sin t-2t^3}(\cos t-6t^2).$

2. 已知$z=f(x+y^2)$，其中函数f二阶可导，试求$\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial y^2},\frac{\partial^2 z}{\partial x\partial y}$.

解：$\frac{\partial z}{\partial x}=f'(x^2+y^2)2x,\frac{\partial^2 z}{\partial x^2}=2f'(x^2+y^2)+2xf''(x^2+y^2)2x$
$=2f'(x^2+y^2)+4x^2f''(x^2+y^2),$

$\frac{\partial z}{\partial y}=f'(x^2+y^2)2y,\frac{\partial^2 z}{\partial y^2}=2f'(x^2+y^2)+2yf''(x^2+y^2)2y=2f'(x^2+y^2)+4y^2f''(x^2+y^2),$

$\frac{\partial^2 z}{\partial x\partial y}=2yf''(x^2+y^2)2x=4xyf''(x^2+y^2).$

3. 设$z=yf(x^2y,\frac{y}{x})$，其中f具有连续的二阶偏导数，求z''_{xx},z''_{xy}.

解：$z'_x=y[f_1'(x^2y,\frac{y}{x})2xy+f_2'(x^2y,\frac{y}{x})(-\frac{y}{x^2})]=2xy^2f_1'(x^2y,\frac{y}{x})-\frac{y^2}{x^2}f_2'(x^2y,\frac{y}{x}),$
$z''_{xx}=2y^2f_1'+2xy^2[f''_{11}2xy+f''_{12}(-\frac{y}{x^2})]-(-\frac{2y^2}{x^3})f_2'-\frac{y^2}{x^2}[f''_{21}2xy+f''_{22}(-\frac{y}{x^2})]$
$=2y^2f_1'+\frac{2y^2}{x^3}f_2'+4x^2y^3f''_{11}-\frac{4y^3}{x}f''_{12}+\frac{y^3}{x^4}f''_{22}$
$=2y^2f_1'+\frac{2y^2}{x^3}f_2'+4x^2y^3f''_{11}-\frac{4y^3}{x}f''_{12}+\frac{y^3}{x^4}f''_{22},$

$z''_{xy}=4xyf_1'+2xy^2[f''_{11}x^2+f''_{12}\frac{1}{x}]-\frac{2y}{x^2}f_2'-\frac{y^2}{x^2}[f''_{21}x^2+f''_{22}\frac{1}{x}]$
$=4xyf_1'-\frac{2y}{x^2}f_2'+2x^3y^2f''_{11}+y^2f''_{12}-\frac{y^2}{x^3}f''_{22}$
$=4xyf_1'-\frac{2y}{x^2}f_2'+2x^3y^2f''_{11}+y^2f''_{12}-\frac{y^2}{x^3}f''_{22}.$

4. 设函数f,g有连续导数，令$u=yf(\frac{x}{y})+xg(\frac{y}{x})$，求$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}$.

解：**【该做法应加上f,g有二阶连续导数的条件：】**

$\frac{\partial u}{\partial x}=yf'(\frac{x}{y})\frac{1}{y}+g(\frac{y}{x})+xg'(\frac{y}{x})(-\frac{y}{x^2})=f'(\frac{x}{y})+g(\frac{y}{x})-\frac{y}{x}g'(\frac{y}{x})$,
$\frac{\partial^2 u}{\partial x^2}=f''(\frac{x}{y})\frac{1}{y}+g'(\frac{y}{x})(-\frac{y}{x^2})-(-\frac{y}{x^2})g'(\frac{y}{x})-\frac{y}{x}g''(\frac{y}{x})(-\frac{y}{x^2})=\frac{1}{y}f''(\frac{x}{y})+\frac{y^2}{x^3}g''(\frac{y}{x})$,

$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}=f''(\frac{x}{y})(-\frac{x}{y^2})+g'(\frac{y}{x})\frac{1}{x}-\frac{1}{x}g'(\frac{y}{x})-\frac{y}{x}g''(\frac{y}{x})\frac{1}{x}=-\frac{x}{y^2}f''(\frac{x}{y})-\frac{y}{x^2}g''(\frac{y}{x})$,

$\therefore x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=x[\frac{1}{y}f''(\frac{x}{y})+\frac{y^2}{x^3}g''(\frac{y}{x})]+y[-\frac{x}{y^2}f''(\frac{x}{y})-\frac{y}{x^2}g''(\frac{y}{x})]=0.$

【正确做法：】$\because f,g$有连续导数，

$\therefore u=yf(\frac{x}{y})+xg(\frac{y}{x}))\in C^1(\mathbb{R}^2\backslash\{(x,y)|x=0\text{或}y=0\})$,

$\therefore$

$$\frac{\partial u}{\partial x}=yf'(\frac{x}{y})\frac{1}{y}+g(\frac{y}{x})+xg'(\frac{y}{x})(-\frac{y}{x^2})=g(\frac{y}{x})+f'(\frac{x}{y})-\frac{y}{x}g'(\frac{y}{x}),$$
$$\frac{\partial u}{\partial y}=f(\frac{x}{y})+yf'(\frac{x}{y})(-\frac{x}{y^2})+xg'(\frac{y}{x})\frac{1}{x}=f(\frac{x}{y})-\frac{x}{y}f'(\frac{x}{y})+g'(\frac{y}{x}),$$

$\therefore$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=xg(\frac{y}{x})+yf(\frac{x}{y})=u\in C^1,$$

$\therefore$

$$\frac{\partial}{\partial x}(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y})-\frac{\partial u}{\partial x}=x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=0.$$

5. 求$z=\ln(\mathrm{e}^{-x}+\frac{x^2}{y})$在点$(1,1)$处沿$\boldsymbol{v}=(a,b)^T(a\neq 0)$的方向导数.

解：$\because \frac{\partial z}{\partial x}=\frac{-\mathrm{e}^{-x}+\frac{2x}{y}}{\mathrm{e}^{-x}+\frac{x^2}{y}},\frac{\partial z}{\partial y}=\frac{-\frac{x^2}{y^2}}{\mathrm{e}^{-x}+\frac{x^2}{y}}$,

$\therefore \mathrm{grad}z(1,1)=(\frac{\partial z(1,1)}{\partial x},\frac{\partial z(1,1)}{\partial y})=(\frac{-\mathrm{e}^{-1}+2}{\mathrm{e}^{-1}+1},\frac{-1}{\mathrm{e}^{-1}+1})=(\frac{2\mathrm{e}-1}{\mathrm{e}+1},-\frac{\mathrm{e}}{\mathrm{e}+1})$,

$\because \frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$在点$(1,1)$及其附近存在且在点$(1,1)$处连续,

$\therefore f(x,y)$在点$(1,1)$处可微,

$\therefore \frac{\partial z(1,1)}{\partial \boldsymbol{v}}=\mathrm{grad}z(1,1)\cdot\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}=(\frac{2\mathrm{e}-1}{\mathrm{e}+1},-\frac{\mathrm{e}}{\mathrm{e}+1})\cdot\frac{1}{\sqrt{a^2+b^2}}(a,b)^T=\frac{1}{\sqrt{a^2+b^2}}(\frac{2a\mathrm{e}-a-b\mathrm{e}}{\mathrm{e}+1})$.

6. 已知$f(x,y)=x^2-xy+y^2$.
(1)当$\boldsymbol{v}$分别为何向量时，方向导数$\frac{\partial f(1,1)}{\partial \boldsymbol{v}}$会取到最大值、最小值和零值？并求出其最大值和最小值.
(2)试求$\mathrm{grad}f(1,1)$，并说明其方向与大小的意义.

解：(1)$\because \frac{\partial f(x,y)}{\partial x}=2x-y,\frac{\partial f(x,y)}{\partial y}=2y-x$,

$\therefore \mathrm{grad}f(1,1)=(\frac{\partial f(1,1)}{\partial x},\frac{\partial f(1,1)}{\partial y})=(1,1)$,

$\because \frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$在点$(1,1)$及其附近存在且在点$(1,1)$处连续,

$\therefore f(x,y)$在点$(1,1)$处可微,

$\therefore \frac{\partial f(1,1)}{\partial \boldsymbol{v}} = \mathrm{grad} f(1,1)\cdot \boldsymbol{v} = \|\mathrm{grad} f(1,1)\|\|\boldsymbol{v}\|\cos\theta = \|\mathrm{grad} f(1,1)\|\cos\theta$，

当$\boldsymbol{v}$与梯度向量的夹角$\theta=0$即$\boldsymbol{v}=\frac{1}{\sqrt{2}}(1,1)$时，方向导数$\frac{\partial f(1,1)}{\partial \boldsymbol{v}}$取得最大值$\|\mathrm{grad} f(1,1)\| = \sqrt{2}$；

当$\boldsymbol{v}$与梯度向量的夹角$\theta=\pi$即$\boldsymbol{v}=-\frac{1}{\sqrt{2}}(1,1)$时，方向导数$\frac{\partial f(1,1)}{\partial \boldsymbol{v}}$取得最小值$-\|\mathrm{grad} f(1,1)\| = -\sqrt{2}$；

当$\boldsymbol{v}$与梯度向量的夹角$\theta=\frac{\pi}{2}$即$\boldsymbol{v}=\frac{1}{\sqrt{2}}(-1,1)$或$\frac{1}{\sqrt{2}}(1,-1)$时，方向导数$\frac{\partial f(1,1)}{\partial \boldsymbol{v}}=0$.

(2)$\mathrm{grad} f(1,1) = (1,1)$，其方向表示方向导数最大的方向，其大小为方向导数的最大值.

15.4 习题10.5解答

1. 设$y=y(x), z=z(x)$是由方程$z=xf(x+y)$和$F(x,y,z)=0$所确定的函数，其中f和F分别具有连续导数和偏导数，求$\frac{\mathrm{d}z}{\mathrm{d}x}$.

解：方法1：将$z=xf(x+y), F(x,y,z)=0$两边分别对x求导：

$$\begin{aligned}&\frac{\mathrm{d}z}{\mathrm{d}x} = f(x+y)+xf'(x+y)[1+\frac{\mathrm{d}y}{\mathrm{d}x}],\\ &F'_x + F'_y\frac{\mathrm{d}y}{\mathrm{d}x} + F'_z\frac{\mathrm{d}z}{\mathrm{d}x} = 0\end{aligned}$$

由该方程组解得

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{[f(x+y)+xf'(x+y)]F'_y - xf'(x+y)F'_x}{F'_y + xf'(x+y)F'_z}.$$

方法2：将$xf(x+y)-z=0, F(x,y,z)=0$两边求全微分：

$$\begin{aligned}&[f(x+y)+xf'(x+y)]\mathrm{d}x + xf'(x+y)\mathrm{d}y - \mathrm{d}z = 0,\\ &F'_x\mathrm{d}x + F'_y\mathrm{d}y + F'_z\mathrm{d}z = 0,\end{aligned}$$

因为$y=y(x), z=z(x)$，将以上方程两边分别除以$\mathrm{d}x$得

$$\begin{aligned}&[f(x+y)+xf'(x+y)] + xf'(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\mathrm{d}z}{\mathrm{d}x} = 0,\\ &F'_x + F'_y\frac{\mathrm{d}y}{\mathrm{d}x} + F'_z\frac{\mathrm{d}z}{\mathrm{d}x} = 0,\end{aligned}$$

由该方程组解得

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{(f+xf')F'_y - xf'F'_x}{F'_y + xf'F'_z}.$$

2. 设由方程$F(\frac{x}{z},\frac{z}{y})=0$可以确定隐函数$z=z(x,y)$，求$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

解：方法1：将$F(\frac{x}{z},\frac{z}{y})=0$两边分别对$x,y$求偏导：

$$F_1'(\frac{1}{z}+\frac{-x}{z^2}\frac{\partial z}{\partial x})+F_2'\frac{1}{y}\frac{\partial z}{\partial x}=0,$$
$$F_1'\frac{-x}{z^2}\frac{\partial z}{\partial y}+F_2'(\frac{1}{y}\frac{\partial z}{\partial y}+\frac{-z}{y^2})=0,$$

∴

$$\frac{\partial z}{\partial x}=\frac{\frac{1}{z}F_1'}{\frac{x}{z^2}F_1'-\frac{1}{y}F_2'}=\frac{F_1'}{\frac{x}{z}F_1'-\frac{z}{y}F_2'},$$
$$\frac{\partial z}{\partial y}=\frac{\frac{z}{y^2}F_2'}{\frac{1}{y}F_2'-\frac{x}{z^2}F_1'}=\frac{\frac{z^2}{y^2}F_2'}{\frac{z}{y}F_2'-\frac{x}{z}F_1'}.$$

方法2：将方程$F(\frac{x}{z},\frac{z}{y})=0$两边求全微分：

$$F_1'\frac{1}{z}\mathrm{d}x+F_2'(-\frac{z}{y^2})\mathrm{d}y+[F_1'(-\frac{x}{z^2})+F_2'\frac{1}{y}]\mathrm{d}z=0,$$

即

$$\mathrm{d}z=-\frac{F_1'\frac{1}{z}}{F_1'(-\frac{x}{z^2})+F_2'\frac{1}{y}}\mathrm{d}x-\frac{F_2'(-\frac{z}{y^2})}{F_1'(-\frac{x}{z^2})+F_2'\frac{1}{y}}\mathrm{d}y,$$

∴

$$\frac{\partial z}{\partial x}=\frac{-F_1'\frac{1}{z}}{F_1'(-\frac{x}{z^2})+F_2'\frac{1}{y}}=\frac{F_1'}{\frac{x}{z}F_1'-\frac{z}{y}F_2'},$$
$$\frac{\partial z}{\partial y}=\frac{-F_2'(-\frac{z}{y^2})}{F_1'(-\frac{x}{z^2})+F_2'\frac{1}{y}}=\frac{\frac{z^2}{y^2}F_2'}{\frac{z}{y}F_2'-\frac{x}{z}F_1'}.$$

3. 证明：方程$F(x+\frac{z}{y},y+\frac{z}{x})=0$所确定的隐函数$z=z(x,y)$满足方程

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy.$$

证明：方法1：

∵

$$\mathrm{d}F(x+\frac{z}{y},y+\frac{z}{x})=[F_1'+F_2'(-\frac{z}{x^2})]\mathrm{d}x+[F_1'(\frac{-z}{y^2})+F_2']\mathrm{d}y+(F_1'\frac{1}{y}+F_2'\frac{1}{x})\mathrm{d}z=0,$$

∴

$$\mathrm{d}z=\frac{F_1'+F_2'(-\frac{z}{x^2})}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}\mathrm{d}x+\frac{F_1'(\frac{-z}{y^2})+F_2'}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}\mathrm{d}y,$$

$\therefore$

$$\frac{\partial z}{\partial x}=\frac{F_1'+F_2'(-\frac{z}{x^2})}{F_1'\frac{1}{y}+F_2'\frac{1}{x}},\frac{\partial z}{\partial y}=\frac{F_1'(\frac{-z}{y^2})+F_2'}{F_1'\frac{1}{y}+F_2'\frac{1}{x}},$$

$\therefore$

$$\begin{aligned}x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}&=x\frac{F_1'+F_2'(-\frac{z}{x^2})}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}+y\frac{F_1'(\frac{-z}{y^2})+F_2'}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}=\frac{xF_1'+F_2'(-\frac{z}{x})+F_1'(\frac{-z}{y})+yF_2'}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}\\&=\frac{(x-\frac{z}{y})F_1'+(y-\frac{z}{x})F_2'}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}=(xy-z)\frac{\frac{1}{y}F_1'+\frac{1}{x}F_2'}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}\\&=xy-z.\end{aligned}$$

方法2：将方程$F(x+\frac{z}{y},y+\frac{z}{x})=0$两边分别对$x,y$求偏导：

$$\begin{cases}F_1'(1+\frac{1}{y}\frac{\partial z}{\partial x})+F_2'(\frac{-z}{x^2}+\frac{1}{x}\frac{\partial z}{\partial x})=0,\\F_1'(\frac{-z}{y^2}+\frac{1}{y}\frac{\partial z}{\partial y})+F_2'(1+\frac{1}{x}\frac{\partial z}{\partial y})=0,\end{cases}$$

这是一个关于F_1',F_2'的齐次方程组，要使该方程组有非零解，则必须

$$\begin{vmatrix}1+\frac{1}{y}\frac{\partial z}{\partial x} & \frac{-z}{x^2}+\frac{1}{x}\frac{\partial z}{\partial x}\\ \frac{-z}{y^2}+\frac{1}{y}\frac{\partial z}{\partial y} & 1+\frac{1}{x}\frac{\partial z}{\partial y}\end{vmatrix}=0,$$

即

$$(1+\frac{1}{y}\frac{\partial z}{\partial x})(1+\frac{1}{x}\frac{\partial z}{\partial y})-(\frac{-z}{x^2}+\frac{1}{x}\frac{\partial z}{\partial x})(\frac{-z}{y^2}+\frac{1}{y}\frac{\partial z}{\partial y})=0,$$

整理得

$$\begin{aligned}&1+\frac{1}{y}\frac{\partial z}{\partial x}+\frac{1}{x}\frac{\partial z}{\partial y}-\frac{z^2}{x^2y^2}+\frac{z}{x^2y}\frac{\partial z}{\partial y}+\frac{z}{xy^2}\frac{\partial z}{\partial x}\\=&1-\frac{z^2}{x^2y^2}+\frac{xy+z}{xy^2}\frac{\partial z}{\partial x}+\frac{xy+z}{x^2y}\frac{\partial z}{\partial y}\\=&\frac{(xy+z)(xy-z)}{x^2y^2}+\frac{xy+z}{xy^2}\frac{\partial z}{\partial x}+\frac{xy+z}{x^2y}\frac{\partial z}{\partial y}\\=&\frac{xy+z}{x^2y^2}[(xy-z)+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}]\\=&0,\end{aligned}$$

$\therefore xy+z=0$或$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy,$[1]

$\because xy+z=0$也满足$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy,$

故$z=z(x,y)$满足方程$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy.$

[1]在修订版中，这里去掉了“且$xy\neq 0$”，因为由$F(x+\frac{z}{y},y+\frac{z}{x})=0$可直接得到$xy\neq 0$.

4. 设$z=f(u)$，且$u=u(x,y)$满足$u=\varphi(u)+\int_y^x p(t)\mathrm{d}t$(其中$f$可导，$\varphi\in C^1$，且$\varphi'(u)\neq 1, p\in C$). 求证：$p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}=0$.

证明：
$$\frac{\partial z}{\partial x}=f'(u)\frac{\partial u}{\partial x},\frac{\partial z}{\partial y}=f'(u)\frac{\partial u}{\partial y},$$
$\because$
$$u=\varphi(u)+\int_y^x p(t)\mathrm{d}t=\varphi(u)+\int_0^x p(t)\mathrm{d}t+\int_y^0 p(t)\mathrm{d}t,$$
$\therefore$
$$\frac{\partial u}{\partial x}=\varphi'(u)\frac{\partial u}{\partial x}+p(x),\frac{\partial u}{\partial y}=\varphi'(u)\frac{\partial u}{\partial y}-p(y),(*)$$
$\because \varphi'(u)\neq 1,$

$\therefore$
$$\frac{\partial u}{\partial x}=\frac{p(x)}{1-\varphi'(u)},\frac{\partial u}{\partial y}=\frac{-p(y)}{1-\varphi'(u)},$$
$\therefore$
$$\begin{aligned}\frac{\partial z}{\partial x}&=f'(u)\frac{\partial u}{\partial x}=f'(u)\frac{p(x)}{1-\varphi'(u)},\\ \frac{\partial z}{\partial y}&=f'(u)\frac{\partial u}{\partial y}=f'(u)\frac{-p(y)}{1-\varphi'(u)},\end{aligned}$$
$\therefore$
$$\begin{aligned}&p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}\\ =&p(y)f'(u)\frac{p(x)}{1-\varphi'(u)}+p(x)f'(u)\frac{-p(y)}{1-\varphi'(u)}\\ =&0.\end{aligned}$$

5. 已知方程$F(x+y,y+z)=1$确定了隐函数$z=z(x,y)$，其中F具有连续的二阶偏导数，求$\frac{\partial^2 z}{\partial y\partial x}$.

解：方法1：将方程$F(x+y,y+z)=1$两边对x求偏导：
$$F_1'(x+y,y+z)+F_2'(x+y,y+z)\frac{\partial z}{\partial x}=0,$$
得
$$\frac{\partial z}{\partial x}=-\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)},$$
$\therefore$
$$\frac{\partial^2 z}{\partial y\partial x}=-\frac{[F_{11}''+F_{12}''(1+\frac{\partial z}{\partial y})]F_2'-F_1'[F_{21}''+F_{22}''(1+\frac{\partial z}{\partial y})]}{(F_2')^2},$$

方程$F(x+y,y+z)=1$两边对y求偏导：

$$F_1'(x+y,y+z)+F_2'(x+y,y+z)(1+\frac{\partial z}{\partial y})=0,$$

得

$$1+\frac{\partial z}{\partial y}=-\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)},$$

$\therefore$

$$\begin{aligned}\frac{\partial^2 z}{\partial y\partial x}&=-\frac{[F_{11}''+F_{12}''(-\frac{F_1'}{F_2'})]F_2'-F_1'[F_{21}''+F_{22}''(-\frac{F_1'}{F_2'})]}{(F_2')^2}\\&=-\frac{(F_2')^2F_{11}''-F_1'F_2'F_{12}''-F_1'F_2'F_{21}''+(F_1')^2F_{22}''}{(F_2')^3}\\&=\frac{-(F_2')^2F_{11}''+2F_1'F_2'F_{12}''-(F_1')^2F_{22}''}{(F_2')^3}.\end{aligned}$$

方法2：方程$F(x+y,y+z)=1$两边求全微分：

$$\mathrm{d}F(x+y,y+z)=F_1'(x+y,y+z)\mathrm{d}x+[F_1'(x+y,y+z)+F_2'(x+y,y+z)]\mathrm{d}y+F_2'(x+y,y+z)\mathrm{d}z=0,$$

即

$$\begin{aligned}\mathrm{d}z&=-\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)}\mathrm{d}x-\frac{F_1'(x+y,y+z)+F_2'(x+y,y+z)}{F_2'(x+y,y+z)}\mathrm{d}z\\&=-\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)}\mathrm{d}x-[1+\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)}]\mathrm{d}z,\end{aligned}$$

$\therefore$

$$\frac{\partial z}{\partial x}=-\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)},$$

$\therefore$

$$\frac{\partial^2 z}{\partial y\partial x}=-\frac{[F_{11}''+F_{12}''(1+\frac{\partial z}{\partial y})]F_2'-F_1'[F_{21}''+F_{22}''(1+\frac{\partial z}{\partial y})]}{(F_2')^2},$$

$\because$

$$\frac{\partial z}{\partial y}=-1-\frac{F_1'(x+y,y+z)}{F_2'(x+y,y+z)},$$

$\therefore$

$$\begin{aligned}\frac{\partial^2 z}{\partial y\partial x}&=-\frac{[F_{11}''+F_{12}''(-\frac{F_1'}{F_2'})]F_2'-F_1'[F_{21}''+F_{22}''(-\frac{F_1'}{F_2'})]}{(F_2')^2}\\&=-\frac{(F_2')^2F_{11}''-F_1'F_2'F_{12}''-F_1'F_2'F_{21}''+(F_1')^2F_{22}''}{(F_2')^3}\\&=\frac{-(F_2')^2F_{11}''+2F_1'F_2'F_{12}''-(F_1')^2F_{22}''}{(F_2')^3}.\end{aligned}$$

6. 设方程组$\begin{cases} x^2+y^2+z^2=3x, \\ 2x-3y+5z=4, \end{cases}$ 确定y与z是x的函数，求$\frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}z}{\mathrm{d}x}$.

解：方法1：将方程组的两个方程两边分别对x求导：

$$\begin{cases} 2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}+2z\frac{\mathrm{d}z}{\mathrm{d}x}=3, \\ 2-3\frac{\mathrm{d}y}{\mathrm{d}x}+5\frac{\mathrm{d}z}{\mathrm{d}x}=0, \end{cases}$$

可解得 $\frac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\begin{vmatrix} 3-2x & 2z \\ -2 & 5 \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ -3 & 5 \end{vmatrix}}=\frac{15-10x+4z}{10y+6z}, \frac{\mathrm{d}z}{\mathrm{d}y}=\dfrac{\begin{vmatrix} 2y & 3-2x \\ -3 & -2 \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ -3 & 5 \end{vmatrix}}=\frac{-4y+9-6x}{10y+6z}$.

方法2：将方程组的两个方程两边分别求全微分：

$$\begin{cases} 2\mathrm{d}x+2y\mathrm{d}y+2z\mathrm{d}z=3, \\ 2\mathrm{d}x-3\mathrm{d}y+5\mathrm{d}z=0, \end{cases}$$

$\because y$与z是x的函数

$$\therefore \begin{cases} 2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}+2z\frac{\mathrm{d}z}{\mathrm{d}x}=3, \\ 2-3\frac{\mathrm{d}y}{\mathrm{d}x}+5\frac{\mathrm{d}z}{\mathrm{d}x}=0, \end{cases}$$

可解得 $\frac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\begin{vmatrix} 3-2x & 2z \\ -2 & 5 \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ -3 & 5 \end{vmatrix}}=\frac{15-10x+4z}{10y+6z}, \frac{\mathrm{d}z}{\mathrm{d}y}=\dfrac{\begin{vmatrix} 2y & 3-2x \\ -3 & -2 \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ -3 & 5 \end{vmatrix}}=\frac{-4y+9-6x}{10y+6z}$.