28C 第14章补充题

28C.1 第14章补充题解答

1. 设$f(x)$在$[0,+\infty)$连续，且$\lim\limits_{x\to+\infty}f(x)=b$. 求证：

   (1)若$a>0$, 则方程$y'+ay=f(x)$的每个解$y(x)$都满足$\lim\limits_{x\to+\infty}y(x)=\frac{b}{a}$;

   (2)若$a<0$, 则方程$y'+ay=f(x)$只有一个解$y_0(x)$满足$\lim\limits_{x\to+\infty}y_0(x)=\frac{b}{a}$.

   证明：$y'+ay=f(x)$的解为

   $$y=\mathrm{e}^{-\int_0^x a\mathrm{d}t}[\int_0^x f(t)\mathrm{e}^{\int_0^t a\mathrm{d}u}\mathrm{d}t+C]=\mathrm{e}^{-ax}[\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C]=\frac{\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C}{\mathrm{e}^{ax}},$$

   (1)当$a>0$时

   $$\lim_{x\to+\infty}y(x)=\lim_{x\to+\infty}\frac{\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C}{\mathrm{e}^{ax}}=\lim_{x\to+\infty}\frac{f(x)\mathrm{e}^{ax}}{a\mathrm{e}^{ax}}=\lim_{x\to+\infty}\frac{f(x)}{a}=\frac{b}{a}.$$

   (2)当$a<0$时，$\lim\limits_{x\to+\infty}\mathrm{e}^{ax}=0$, 要使极限$\lim\limits_{x\to+\infty}y(x)=\lim\limits_{x\to+\infty}\frac{\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C}{\mathrm{e}^{ax}}$存在，则

   $$\lim_{x\to+\infty}[\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C]=0,$$

   即$C=-\int_0^{+\infty}f(t)\mathrm{e}^{at}\mathrm{d}x$,

   此时

   $$\lim_{x\to+\infty}y(x)=\lim_{x\to+\infty}\frac{\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C}{\mathrm{e}^{ax}}=\lim_{x\to+\infty}\frac{f(x)\mathrm{e}^{ax}}{a\mathrm{e}^{ax}}=\lim_{x\to+\infty}\frac{f(x)}{a}=\frac{b}{a}.$$

   故只有一个解$y_0(x)=\mathrm{e}^{-ax}[\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x-\int_0^{+\infty}f(t)\mathrm{e}^{at}\mathrm{d}x]$满足$\lim\limits_{x\to+\infty}y_0(x)=\frac{b}{a}$.

2. 设$f(x)$连续.

   (1)求方程$y'+ay=f(x)$满足$y\big|_{x=0}=0$的解$y(x)(a>0)$;

   (2)若$|f(x)|\leqslant k$, 求证当$x\geqslant0$时，有$|y(x)|\leqslant\frac{k}{a}(1-\mathrm{e}^{-ax})$.

   解：(1)一阶线性非齐次微分方程$y'+ay=f(x)$的解为

   $$y=\mathrm{e}^{-\int_0^x a\mathrm{d}t}[\int_0^x f(t)\mathrm{e}^{\int_0^t a\mathrm{d}u}\mathrm{d}t+C]=\mathrm{e}^{-ax}[\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C]=\frac{\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x+C}{\mathrm{e}^{ax}},$$

   $\because y\big|_{x=0}=\frac{C}{1}=C=0$,

   $\therefore$满足$y\big|_{x=0}=0$的解$y(x)=\frac{\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x}{\mathrm{e}^{ax}}$.

   (2)$\because |f(x)|\leqslant k$,

   $\therefore$当$x\geqslant0$时

   $|y(x)|=\frac{|\int_0^x f(t)\mathrm{e}^{at}\mathrm{d}x|}{\mathrm{e}^{ax}}\leqslant\frac{\int_0^x |f(t)|\mathrm{e}^{at}\mathrm{d}t}{\mathrm{e}^{ax}}\leqslant\frac{\int_0^x k\mathrm{e}^{at}\mathrm{d}t}{\mathrm{e}^{ax}}=\frac{k\int_0^x \mathrm{e}^{at}\mathrm{d}t}{\mathrm{e}^{ax}}=\frac{\frac{k}{a}\mathrm{e}^{ax}\big|_0^x}{\mathrm{e}^{ax}}=\frac{\frac{k}{a}(\mathrm{e}^{ax}-1)}{\mathrm{e}^{ax}}=\frac{k}{a}(1-\mathrm{e}^{-ax}).$

3. 设$y_1(x),y_2(x)$是方程$y''+p(x)y'+q(x)y=0$的两个解，并且函数$f(x)=\frac{y_2(x)}{y_1(x)}$在某个点$x_0$处取得极值. 问$y_1(x)$和$y_2(x)$能否构成该方程的一个基本解组？

解：$\because f(x)=\frac{y_2(x)}{y_1(x)}$在点$x_0$处取得极值,

$\therefore f'(x_0)=\frac{y_2'(x_0)y_1(x_0)-y_1'(x_0)y_2(x_0)}{[y_1(x_0)]^2}=0$, 即$y_2'(x_0)y_1(x_0)-y_1'(x_0)y_2(x_0)=0$,

$\therefore W[y_1,y_2](x_0)=\begin{vmatrix}y_1(x_0)&y_2(x_0)\\y_1'(x_0)&y_2'(x_0)\end{vmatrix}=y_2'(x_0)y_1(x_0)-y_1'(x_0)y_2(x_0)=0$,

$\therefore y_1(x),y_2(x)$线性相关，故$y_1(x)$和$y_2(x)$不能构成该方程的一个基本解组.

4. 已知$f(x)$二阶连续可导，并且对于xOy平面上每一条逐段光滑的有向曲线L都有

$$\oint_L [f'(x)+6f(x)+4\mathrm{e}^{-x}]y\mathrm{d}x+f'(x)\mathrm{d}y=0.$$

试求$f(x)$.

解：$\because f(x)\in C^2(\mathbb{R})$,

$\therefore [f'(x)+6f(x)+4\mathrm{e}^{-x}]y,f'(x)\in C^2(\mathbb{R}^2)$,

$\because$对于xOy平面上每一条逐段光滑的有向曲线L都有

$\oint_L[f'(x)+6f(x)+4\mathrm{e}^{-x}]y\mathrm{d}x+f'(x)\mathrm{d}y=0$,

$\therefore \frac{\partial}{\partial y}[f'(x)+6f(x)+4\mathrm{e}^{-x}]y=f'(x)+6f(x)+4\mathrm{e}^{-x}=\frac{\partial}{\partial x}f'(x)=f''(x)$,

即$f''(x)-f'(x)-6f(x)=4\mathrm{e}^{-x}$,

该二阶常系数线性非齐次微分方程的齐次方程的特征方程为$\lambda^2-\lambda-6=(\lambda-3)(\lambda+2)=0$，特征根$\lambda_1=3,\lambda_2=-2$, 故齐次方程的通解为$y=C_1\mathrm{e}^{3x}+C_2\mathrm{e}^{-2x}$.

非齐次方程的自由项$4\mathrm{e}^{-x}$中的$\lambda=-1$, 不是特征方程的解，故可设非齐次方程的特解为$y=a\mathrm{e}^{-x}$, 代入原非齐次方程得$a\mathrm{e}^{-x}+a\mathrm{e}^{-x}-6a\mathrm{e}^{-x}=-4a\mathrm{e}^{-x}=4\mathrm{e}^{-x}$,

$\therefore a=-1$,

$\therefore f(x)=C_1\mathrm{e}^{3x}+C_2\mathrm{e}^{-2x}-\mathrm{e}^{-x},C_1,C_2\in\mathbb{R}$.

5. 假定对于半空间$x>0$的任意光滑封闭曲面S，有

$$\iint\limits_S xf(x)\mathrm{d}y\wedge\mathrm{d}z-xyf(x)\mathrm{d}z\wedge\mathrm{d}x+\mathrm{e}^{2x}z\mathrm{d}x\wedge\mathrm{d}y=0,$$

其中$f(x)$在$(0,+\infty)$有连续导数，且满足$\lim\limits_{x\to0_+}f(x)=1$. 求$f(x)$.

解：$\because f(x)\in C^1(0,+\infty)$,

$\therefore xf(x),-xyf(x),\mathrm{e}^{2x}z\in C^1((0,+\infty)\times(-\infty,+\infty)\times(-\infty,+\infty))$,

$\forall (x,y,z) \in (0,+\infty) \times (-\infty,+\infty) \times (-\infty,+\infty)$, 设$S$为包围该点的任意光滑封闭曲面，记$S$围成的区域为$\Omega$, 则

$$\oiint\limits_{S} xf(x)\mathrm{d}y \wedge \mathrm{d}z - xyf(x)\mathrm{d}z \wedge \mathrm{d}x + \mathrm{e}^{2x}z\mathrm{d}x \wedge \mathrm{d}y$$

$$= \iiint\limits_{\Omega} \{\frac{\partial}{\partial x}[xf(x)] + \frac{\partial}{\partial y}[-xyf(x)] + \frac{\partial}{\partial z}(\mathrm{e}^{2x}z)\}\mathrm{d}V$$

$$= \iiint\limits_{\Omega} [f(x) + xf'(x) - xf(x) + \mathrm{e}^{2x}]\mathrm{d}V = 0,$$

$\because f(x) + xf'(x) - xf(x) + \mathrm{e}^{2x}$连续，

$\therefore$根据积分中值定理$\exists(\xi,\eta,\zeta) \in \Omega$, 使得

$$\iiint\limits_{\Omega} [f(x) + xf'(x) - xf(x) + \mathrm{e}^{2x}]\mathrm{d}V = [f(\xi) + \xi f'(\xi) - \xi f(\xi) + \mathrm{e}^{2\xi}]V(\Omega) = 0,$$

其中$V(\Omega)$表示区域Ω的体积，

$\therefore$

$$f(x) + xf'(x) - xf(x) + \mathrm{e}^{2x} = \lim_{(\xi,\eta,\zeta)\to(x,y,z)}[f(\xi) + \xi f'(\xi) - \xi f(\xi) + \mathrm{e}^{2\xi}]$$

$$= \lim_{\Omega\to(x,y,z)} \frac{\iiint\limits_{\Omega}[f(x) + xf'(x) - xf(x) + \mathrm{e}^{2x}]\mathrm{d}V}{V(\Omega)}$$

$$= \lim_{\Omega\to(x,y,z)} \frac{0}{V(\Omega)}$$

$$= 0, (x,y,z) \in (0,+\infty) \times (-\infty,+\infty) \times (-\infty,+\infty),$$

$\therefore f'(x) + \frac{1-x}{x}f(x) = -\frac{1}{x}\mathrm{e}^{2x}, x > 0$,

$\therefore f(x) = \mathrm{e}^{-\int \frac{1-x}{x}\mathrm{d}x}[\int(-\frac{1}{x}\mathrm{e}^{2x})\mathrm{e}^{\int \frac{1-x}{x}\mathrm{d}x}\mathrm{d}x + C] = \mathrm{e}^{x-\ln x}[\int(-\frac{1}{x}\mathrm{e}^{2x})\mathrm{e}^{\ln x - x}\mathrm{d}x + C]$

$= \frac{1}{x}\mathrm{e}^x[\int(-\frac{1}{x}\mathrm{e}^{2x})\frac{x}{\mathrm{e}^x}\mathrm{d}x + C] = \frac{1}{x}\mathrm{e}^x[\int(-\mathrm{e}^x)\mathrm{d}x + C] = \frac{1}{x}(-\mathrm{e}^{-2x} + C\mathrm{e}^x) = \frac{1}{x}(C\mathrm{e}^x - \mathrm{e}^{2x})$,

$\because \lim\limits_{x\to 0_+} f(x) = \lim\limits_{x\to 0_+} \frac{C\mathrm{e}^x - \mathrm{e}^{2x}}{x} = \lim\limits_{x\to 0_+} \frac{C\mathrm{e}^x - 2\mathrm{e}^{2x}}{1} = C - 2 = 1$,

$\therefore C = 3$,

$\therefore f(x) = \frac{\mathrm{e}^x}{x}(3 - \mathrm{e}^x)$.

6. 设$f(x)$有二阶连续导数，并满足方程$f(x) = \int_0^x f(1-t)\mathrm{d}t + 1$, 求$f(x)$.

解：$\because f(x) = \int_0^x f(1-t)\mathrm{d}t + 1$,

$\therefore f'(x) = f(1-x)(*)$,

$\therefore f'(1-x) = f(x)$,

$\therefore(*)$式两边求导得$f''(x) = -f'(1-x) = -f(x)$,

即$f''(x) + f(x) = 0$, 该齐次线性微分方程的特征方程$\lambda^2 + 1 = 0$的根为$\lambda_{1,2} = \pm i$, 通解为$f(x) = C_1 \cos x + C_2 \sin x$,

$\because f(0) = 1$, 由(*)式得$f'(0) = f(1)$,

$\therefore C_1 = 1, C_2 = C_1 \cos 1 + C_2 \sin 1$,

$\therefore C_2 = \frac{\cos 1}{1 - \sin 1}$,

$\therefore f(x) = \cos x + \frac{\cos 1}{1 - \sin 1} \sin x$.

7. 求级数$x + \frac{1}{1 \times 3} x^3 + \frac{1}{1 \times 3 \times 5} x^5 + \cdots + \frac{1}{(2n+1)!!} x^{2n+1} + \cdots$的收敛域以及和函数.

解：原级数可以记为$\sum_{n=1}^{\infty} \frac{1}{(2n+1)!!} x^{2n+1} = \sum_{n=1}^{\infty} a_n$,

$\because \forall x \in \mathbb{R}, \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{\frac{1}{(2n+3)!!} |x|^{2n+3}}{\frac{1}{(2n+1)!!} |x|^{2n+1}} = \lim_{n \to \infty} \frac{1}{2n+3} |x|^2 = 0$,

$\therefore$由比值判别法可知，$\sum_{n=1}^{\infty} a_n$对所有x都绝对收敛. 故收敛域为$(-\infty, +\infty)$.

记和函数为$S(x) = \sum_{n=1}^{\infty} \frac{1}{(2n+1)!!} x^{2n+1}$, 则

$$S'(x) = [\sum_{n=1}^{\infty} \frac{1}{(2n+1)!!} x^{2n+1}]' = \sum_{n=1}^{\infty} [\frac{1}{(2n+1)!!} x^{2n+1}]'$$

$$= 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n-1)!!} = 1 + x \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!!} = 1 + xS(x),$$

即$S(x)$应满足一阶线性非齐次微分方程

$$S'(x) - xS(x) = 1,$$

其通解为$S(x) = e^{-\int_0^x (-x) dx} [\int_0^x 1 e^{\int_0^t (-u) du} dt + C] = e^{\frac{1}{2} x^2} [\int_0^x e^{-\frac{1}{2} t^2} dt + C]$,

$\because S(0) = C = 0$,

$\therefore S(x) = e^{\frac{1}{2} x^2} \int_0^x e^{-\frac{1}{2} t^2} dt$.

8. 求方程$y'' \cos x - 2y' \sin x + 3y \cos x = e^x$的通解.

解：令$u(x) = y \cos x$, 则$u' = y' \cos x - y \sin x$, $u'' = y'' \cos x - y' \sin x - y' \sin x - y \cos x = y'' \cos x - 2y' \sin x - y \cos x$,

$\therefore u'' + 4u = e^x$(*), 该非齐次线性微分方程对应的齐次方程的特征方程为$\lambda^2 + 4 = 0$, 特征根为$\lambda_{1,2} = \pm 2i$, 齐次方程的通解为$u = C_1 \cos 2x + C_2 \sin 2x$,

非齐次方程(*)的自由项e^x中的$\lambda = 1$不是特征方程的根，故可设非齐次方程的特解为$u^* = ae^x$，代入该非齐次方程得$ae^x + 4ae^x = 5ae^x = e^x$, $a = \frac{1}{5}$,

$\therefore$非齐次方程(*)的通解为$u = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{5} e^x$,

$\therefore$原方程的通解为$y = u \cos x = \cos x (C_1 \cos 2x + C_2 \sin 2x + \frac{1}{5} e^x)$.

9. 设 $f(x)$ 是定解问题

$$\begin{cases} y' = x^2 + y^2, \\ y(0) = 0, \end{cases}$$

的解. 试研究函数 $f(x)$ 的增减性和凸凹性, 并求 $\lim\limits_{x \to 0} \dfrac{f(x)}{x^3}$.

解：$\because f(x)$ 是定解问题 $\begin{cases} y' = x^2 + y^2, \\ y(0) = 0, \end{cases}$ 的解,

$\therefore f'(x) = x^2 + [f(x)]^2 \geqslant 0$, 且当 $x \neq 0$ 时 $f'(x) > 0$,

$\therefore f(x)$ 在 $(-\infty, +\infty)$ 上是单调增加的函数.

$\because f(0) = 0$,

$\therefore$ 当 $x > 0$ 时 $f(x) > 0$，当 $x < 0$ 时 $f(x) < 0$,

$\because f''(x) = 2x + 2f(x)f'(x)$,

又 $\because f'(x) > 0, x \neq 0$,

$\therefore$ 当 $x > 0$ 时 $f''(x) > 0$，当 $x < 0$ 时 $f''(x) < 0$,

$\therefore f(x)$ 在 $(0, +\infty)$ 上是下凸函数，在 $(-\infty, 0)$ 上是上凸函数.

$$\begin{aligned}
\lim_{x \to 0} \frac{f(x)}{x^3} &= \lim_{x \to 0} \frac{f'(x)}{3x^2} = \lim_{x \to 0} \frac{f''(x)}{6x} = \lim_{x \to 0} \frac{f'''(x)}{6} \\
&= \lim_{x \to 0} \frac{2 + 2[f'(x)]^2 + 2f(x)f''(x)}{6} \\
&= \lim_{x \to 0} \frac{2 + 2\{x^2 + [f(x)]^2\}^2 + 2f(x)f''(x)}{6} = \frac{2}{6} = \frac{1}{3}.
\end{aligned}$$