16　泰勒公式、几何应用

16.1　知识结构

第10章多元函数微分学

10.6　二元函数的泰勒公式

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　　10.6.2　二元函数的泰勒公式

第11章多元函数微分学的应用

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　　11.1.1　向量值函数

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11.2　空间曲面的切平面与法向量

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16.2　习题10.6解答

1. 写出$f(x, y) = x^y$在点$(1, 1)$带佩亚诺余项的三阶泰勒公式，由此计算$1.1^{1.02}$.

　　解：$f(1, 1) = 1$,

　　$\frac{\partial f}{\partial x} = yx^{y-1}, \frac{\partial f}{\partial y} = x^y \ln x$,

　　$\frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2}, \frac{\partial^2 f}{\partial x \partial y} = x^{y-1} + yx^{y-1} \ln x, \frac{\partial^2 f}{\partial y^2} = x^y(\ln x)^2$,

　　$\frac{\partial^3 f}{\partial x^3} = y(y-1)(y-2)x^{y-3}, \frac{\partial^3 f}{\partial x \partial y^2} = yx^{y-1}(\ln x)^2$,

　　$\frac{\partial^3}{\partial y \partial x^2} = (2y-1)x^{y-2} + y(y-1)x^{y-2} \ln x = [(2y-1) + y(y-1) \ln x]x^{y-2}$,

　　$\frac{\partial^3 f}{\partial y^3} = x^y(\ln x)^3$,

$\therefore$

$f(x,y)$

$$=f(1,1)+[\frac{\partial f(1,1)}{\partial x}(x-1)+\frac{\partial f(1,1)}{\partial y}(y-1)]$$

$$+\frac{1}{2}[\frac{\partial^2 f(1,1)}{\partial x^2}(x-1)^2+2\frac{\partial^2 f(1,1)}{\partial x\partial y}(x-1)(y-1)+\frac{\partial^2 f(1,1)}{\partial y^2}(y-1)^2]$$

$$+\frac{1}{6}[\frac{\partial^3 f(1,1)}{\partial x^3}(x-1)^3+3\frac{\partial^3 f(1,1)}{\partial x\partial y^2}(x-1)(y-1)^2+3\frac{\partial^3 f(1,1)}{\partial y\partial x^2}(x-1)^2(y-1)$$

$$+\frac{\partial^3 f(1,1)}{\partial y^3}(y-1)^3]+o[(\sqrt{(x-1)^2+(y-1)^2})^3]$$

$$=1+(x-1)+\frac{1}{2}[2(x-1)(y-1)]+\frac{1}{6}[3(x-1)^2(y-1)]+o[(\sqrt{(x-1)^2+(y-1)^2})^3]$$

$$=x+(x-1)(y-1)+\frac{1}{2}(x-1)^2(y-1)+o[(\sqrt{(x-1)^2+(y-1)^2})^3]$$

$\therefore 1.1^{1.02}=f(1.1,1.02)\approx 1+0.1+0.1\times 0.02+\frac{1}{2}\times 0.1^2\times 0.02=1.1021.$

2. 证明当$|x|,|y|$充分小时，有$\frac{\cos x}{\cos y}\approx 1-\frac{1}{2}(x^2-y^2)$.

证明：记$f(x,y)=\frac{\cos x}{\cos y}$,

$f(0,0)=1$,

$\frac{\partial f}{\partial x}=-\frac{\sin x}{\cos y},\frac{\partial f}{\partial y}=\frac{\cos x\sin y}{\cos^2 y}$,

$\frac{\partial^2 f}{\partial x^2}=-\frac{\cos x}{\cos y},\frac{\partial^2 f}{\partial x\partial y}=-\frac{\sin x\sin y}{\cos^2 y}$,

$\frac{\partial^2 f}{\partial y^2}=\frac{\cos x\cos y\cos^2 y+\cos x\sin y 2\cos y\sin y}{\cos^4 y}=\frac{\cos x\cos^2 y+2\cos x\sin^2 y}{\cos^3 y}$,

$\therefore$

$$f(x,y)=f(0,0)+[\frac{\partial f(0,0)}{\partial x}x+\frac{\partial f(0,0)}{\partial y}y]+\frac{1}{2}[\frac{\partial^2 f(0,0)}{\partial x^2}x^2+2\frac{\partial^2 f(0,0)}{\partial x\partial y}xy+\frac{\partial^2 f(0,0)}{\partial y^2}y^2]$$

$$+o[(\sqrt{x^2+y^2})^2]$$

$$=1+(0+0)+\frac{1}{2}(-x^2+0+y^2)+o[(\sqrt{x^2+y^2})^2]$$

$$=1-\frac{1}{2}(x^2-y^2)+o[(\sqrt{x^2+y^2})^2],$$

$\therefore$当$|x|,|y|$充分小时，有$\frac{\cos x}{\cos y}\approx 1-\frac{1}{2}(x^2-y^2)$.

3. 写出$f(x,y)=\sqrt{1+y^2}\cos x$在点$(0,1)$的一阶泰勒多项式及拉格朗日余项.

解：$f(0,1)=\sqrt{2}$,

$\frac{\partial f}{\partial x}=-\sqrt{1+y^2}\sin x,\frac{\partial f}{\partial y}=\frac{2y\cos x}{2\sqrt{1+y^2}}=\frac{y\cos x}{\sqrt{1+y^2}}$,

$$\frac{\partial^2 f}{\partial x^2} = -\sqrt{1+y^2}\cos x, \frac{\partial^2 f}{\partial x \partial y} = \frac{-2y\sin x}{2\sqrt{1+y^2}} = \frac{-y\sin x}{\sqrt{1+y^2}},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\cos x\sqrt{1+y^2} - y\cos x\frac{2y}{2\sqrt{1+y^2}}}{1+y^2} = \frac{\cos x}{(1+y^2)^{\frac{3}{2}}},$$

$$\therefore$$

$$
\begin{aligned}
f(x,y) =& f(0,1) + [\frac{\partial f(0,1)}{\partial x}x + \frac{\partial f}{\partial y}(y-1)] \\
&+ \frac{1}{2}[\frac{\partial^2 f(\theta x, 1+\theta(y-1))}{\partial x^2}x^2 + 2\frac{\partial^2 f(\theta x, 1+\theta(y-1))}{\partial x \partial y}x(y-1) \\
&\qquad\qquad\qquad\qquad + \frac{\partial^2 f(\theta x, 1+\theta(y-1))}{\partial y^2}(y-1)^2] \\
=& \sqrt{2} + \frac{\sqrt{2}}{2}(y-1) \\
&+ \frac{1}{2}\Big(-\sqrt{1+[1+\theta(y-1)]^2}\cos(\theta x)x^2 - \frac{2[1+\theta(y-1)]\sin\theta x}{\sqrt{1+[1+\theta(y-1)]^2}}x(y-1) \\
&\qquad\qquad\qquad\qquad + \frac{\cos\theta x}{\{1+[1+\theta(y-1)]^2\}^{\frac{3}{2}}}(y-1)^2\Big) \\
=& \sqrt{2} + \frac{\sqrt{2}}{2}(y-1) \\
&+ \frac{1}{2}\Big\{ -x^2\sqrt{1+(1+\theta(y-1))^2}\cos\theta x - 2x(y-1)\frac{1+\theta(y-1)}{\sqrt{1+(1+\theta(y-1))^2}}\sin\theta x \\
&\qquad\qquad\qquad\qquad + (y-1)^2\frac{\cos\theta x}{\{1+(1+\theta(y-1))^2\}^{\frac{3}{2}}}\Big\}, 0 < \theta < 1.
\end{aligned}
$$

16.3 第10章补充题

1. 设$f(x,y)$是定义在整个平面上的连续函数，当$x^2+y^2 \to +\infty$时，$f(x,y) \to +\infty$. 求证存在(x_0,y_0)，使

$$f(x_0,y_0) = \min\left\{ f(x,y) \mid (x,y) \in \mathbb{R}^2 \right\}.$$

证明：$\because$当$x^2+y^2 \to +\infty$时，$f(x,y) \to +\infty$,

$\therefore$对于$f(0,0), \exists N > 0, s.t. f(x,y) > f(0,0), x^2+y^2 > N^2$,

$\because$在有界闭区域$D = \{(x,y) \mid x^2+y^2 \le N^2\}$内部$f(x,y)$连续，

$\therefore \exists (x_0,y_0) \in D, s.t. f(x_0,y_0) \le f(x,y), (x,y) \in D$，此时$f(x_0,y_0) \le f(0,0)$,

$\therefore f(x_0,y_0) \le f(0,0) < f(x,y), x^2+y^2 > N^2$,

$\therefore f(x_0,y_0) \le f(x,y), (x,y) \in \mathbb{R}^2$,

$\therefore$存在(x_0,y_0)，使$f(x_0,y_0) = \min\{f(x,y) \mid (x,y) \in \mathbb{R}^2\}$.

2. 设$f(x,y)$是定义在整个平面上的连续函数，$f(0,0) = 0$，且当$(x,y) \neq (0,0)$时，$f(x,y) > 0$，又设对于任意的(x,y)和任意实数c，都有

$$f(cx, cy) = c^2 f(x,y).$$

求证存在正数a,b，使得对于任意的(x,y)，都有

$$a(x^2 + y^2) \leq f(x,y) \leq b(x^2 + y^2).$$

证明：$\because f(x,y)$是定义在整个平面上的连续函数，

$\therefore f(x,y)$在有界闭区域$D = \{(x,y) \mid 0.5 < x^2 + y^2 \leq 1.5\}$上连续，

$\because$当$(x,y) \neq (0,0)$时，$f(x,y) > 0$，

$\therefore \exists b \geq a > 0, s.t. a \leq f(x,y) \leq b, (x,y) \in D,$

$\therefore$当$(x,y) \in D^* = \{(x,y) \mid x^2 + y^2 = 1\} \subset D$时，$a \leq f(x,y) \leq b,$

$\because f(x,y) = f(\sqrt{x^2+y^2} \frac{x}{\sqrt{x^2+y^2}}, \sqrt{x^2+y^2} \frac{y}{x^2+y^2}) = (x^2+y^2) f(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}),$

又$\because a \leq f(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}) \leq b,$

$\therefore a(x^2+y^2) \leq f(x,y) \leq b(x^2+y^2).$

3. 若对于任意实数t，函数$f(x,y,z)$满足$f(tx,ty,tz) = t^k f(x,y,z)$，则称$f(x,y,z)$为$k$次齐次函数. 试证$k$次齐次函数$f(x,y,z)$满足方程

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = kf(x,y,z).$$

证明：方法1：等式$f(tx,ty,tz) = t^k f(x,y,z)$两边分别对$t$求偏导[1]得

$$x\frac{\partial f(tx,ty,tz)}{\partial x} + y\frac{\partial f(tx,ty,tz)}{\partial y} + z\frac{\partial f(tx,ty,tz)}{\partial z} = kt^{k-1} f(x,y,z),$$

令$t = 1$得

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = kf(x,y,z).$$

方法2：$\because f(tx,ty,tz) = t^k f(x,y,z),$

$\therefore$

$$\frac{\partial f(x,y,z)}{\partial x} = \frac{\partial}{\partial x} \frac{f(tx,ty,tz)}{t^k} = \frac{1}{t^k} \frac{\partial f(tx,ty,tz)}{\partial x} t = \frac{1}{t^{k-1}} \frac{\partial f(tx,ty,tz)}{\partial x}, \tag{1a}$$

$$\frac{\partial f(x,y,z)}{\partial y} = \frac{\partial}{\partial y} \frac{f(tx,ty,tz)}{t^k} = \frac{1}{t^k} \frac{\partial f(tx,ty,tz)}{\partial y} t = \frac{1}{t^{k-1}} \frac{\partial f(tx,ty,tz)}{\partial y}, \tag{1b}$$

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[1]这里应是偏导。

$$\frac{\partial f(x,y,z)}{\partial z} = \frac{\partial}{\partial z}\frac{f(tx,ty,tz)}{t^k} = \frac{1}{t^k}\frac{\partial f(tx,ty,tz)}{\partial z}t = \frac{1}{t^{k-1}}\frac{\partial f(tx,ty,tz)}{\partial z}, \qquad (1c)$$

方程$f(tx,ty,tz) = t^k f(x,y,z)$两边分别对$t$求导

$$x\frac{\partial f(tx,ty,tz)}{\partial x} + y\frac{\partial f(tx,ty,tz)}{\partial y} + z\frac{\partial f(tx,ty,tz)}{\partial z} = kt^{k-1}f(x,y,z),$$

将式 (1a)-(1c)代入上式

$$xt^{k-1}\frac{\partial f(x,y,z)}{\partial x} + yt^{k-1}\frac{\partial f(x,y,z)}{\partial y} + zt^{k-1}\frac{\partial f(x,y,z)}{\partial z} = kt^{k-1}f(x,y,z),$$

即

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = kf(x,y,z).$$

4. 设F为三元可微函数，$u = u(x,y,z)$是由方程$F(u^2 - x^2, u^2 - y^2, u^2 - z^2) = 0$确定的隐函数. 求证

$$\frac{u'_x}{x} + \frac{u'_y}{y} + \frac{u'_z}{z} = \frac{1}{u}.$$

证明：方程$F(u^2 - x^2, u^2 - y^2, u^2 - z^2) = 0$两边分别对$x$求偏导

$$F'_1(2u\frac{\partial u}{\partial x} - 2x) + F'_2 2u\frac{\partial u}{\partial x} + F'_3 2u\frac{\partial u}{\partial x} = 0$$

得

$$\frac{\partial u}{\partial x} = \frac{xF'_1}{u(F'_1 + F'_2 + F'_3)}$$

同理

$$\frac{\partial u}{\partial y} = \frac{yF'_2}{u(F'_1 + F'_2 + F'_3)},$$
$$\frac{\partial u}{\partial z} = \frac{zF'_3}{u(F'_1 + F'_2 + F'_3)},$$

$\therefore$

$$\frac{u'_x}{x} + \frac{u'_y}{y} + \frac{u'_z}{z} = \frac{F'_1 + F'_2 + F'_3}{u(F'_1 + F'_2 + F'_3)} = \frac{1}{u}.$$

5. 求方程$\frac{\partial^2 z}{\partial x \partial y} = x + y$满足条件$z(x,0) = x, z(0,y) = y^2$的解$z(z,y)$.

解：方法1：$\because \frac{\partial^2 z}{\partial x \partial y} = x + y$,

$\therefore \frac{\partial z}{\partial y} = \int_0^x (x+y)\mathrm{d}x + \varphi_0(y) = \frac{x^2}{2} + xy + \varphi_0(y)$,

$\because z(0,y) = y^2$,

$\therefore \frac{z(0,y)}{\partial y} = 2y = \varphi_0(y)$,

$\therefore z(x,y) = \int_0^y [\frac{x^2}{2} + xy + 2y]\mathrm{d}y + \psi(x) = \frac{1}{2}x^2 y + \frac{1}{2}xy^2 + y^2 + \psi(x)$,

$\because z(x,0) = x = \psi(x)$,

$\therefore z(x,y) = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + y^2 + x$.

方法2：$\because \frac{\partial^2 z}{\partial x \partial y} = x + y$,

$\therefore \int \frac{\partial^2 z}{\partial x \partial y}\mathrm{d}x = \int (x+y)\mathrm{d}x = \frac{1}{2}x^2 + xy + C_1(y) = \frac{\partial z}{\partial y} + C_1(y)$,

$\therefore$可设$\frac{\partial z}{\partial y} = \frac{1}{2}x^2 + xy + C(y)$，其中$C(y)$是与$x$无关的$y$的函数，

$\because z(0,y) = y^2$,

$\therefore \frac{\partial z(0,y)}{\partial y} = 2y = C(y)$,

$\therefore \frac{\partial z}{\partial y} = \frac{1}{2}x^2 + xy + 2y$,

$\therefore \int \frac{\partial z}{\partial y}\mathrm{d}y = \int (\frac{1}{2}x^2 + xy + 2y)\mathrm{d}y = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + y^2 + C_3(x) = z(x,y) + C_4(x)$,

$\therefore$可设$z(x,y) = \int (\frac{1}{2}x^2 + xy + 2y)\mathrm{d}y + C^*(x) = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + y^2 + C^*(x)$，其中$C^*(x)$是与$y$无关的$x$的函数，

$\because z(x,0) = x = C^*(x)$,

$\therefore z(x,y) = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + y^2 + x$.

方法3：$\because \frac{\partial^2 z}{\partial x \partial y} = x + y$,

$\therefore \int \frac{\partial^2 z}{\partial x \partial y}\mathrm{d}x = \int (x+y)\mathrm{d}x = \frac{1}{2}x^2 + xy + C_1(y) = \frac{\partial z}{\partial y} + C_2(y)$,

$\therefore$可设$\frac{\partial z}{\partial y} = \frac{1}{2}x^2 + xy + C(y)$，其中$C(y)$是与$x$无关的$y$的函数，

$\therefore \int \frac{\partial z}{\partial y}\mathrm{d}y = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + \int C(y)\mathrm{d}y = z(x,y) + C_3(x)$,

$\therefore$可设$z(x,y) = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + F(y) + C^*(x)$，其中$F(y)$是$C(y)$的一个与$x$无关的原函数，$C^*(x)$是与$y$无关的$x$的函数，

$\because z(0,y) = y^2, z(x,0) = x$,

$\therefore F(y) + C^*(0) = y^2, F(0) + C^*(x) = x$,

$\therefore F(y) = y^2 - C^*(0), C^*(x) = x - F(0)$，且$F(0) + C^*(0) = 0$,

$\therefore z(x,y) = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + y^2 + x - [F(0) + C^*(0)] = \frac{1}{2}x^2y + \frac{1}{2}xy^2 + y^2 + x$.

6. 设$z = f(x,y)$处处可微，a,b不全等于零. 求证满足方程$bz'_x = az'_y$的充分条件是存在一元函数$g(u)$，使得$z = f(x,y) = g(ax + by)$.

   证明：$\because$存在一元函数$g(u)$，使得$z = f(x,y) = g(ax + by)$,

   $\therefore$对于任意常数C，z在直线$ax + by = C$上恒等于常数，

   $\therefore z = f(x,y)$在直线$ax + by = C$上任意一点处沿该直线方向的方向导数均等于零，[2]

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[2]这里之前的版本中是"$\therefore z$在该直线方向的方向导数恒等于零，"，不准确。

由a,b不全为零知直线$ax+by=C$的方向向量可表示为$(-b,a)$，[3]

又$\because z=f(x,y)$处处可微，[4]

$\therefore z=f(x,y)$在直线$ax+by=C$上的每一点处沿$(-b,a)$方向的方向导数[5]

$$\frac{\partial z}{\partial \boldsymbol{l}}=\mathrm{grad}\,z\cdot\frac{1}{a^2+b^2}(-b,a)=(z'_x,z'_y)\cdot\frac{1}{a^2+b^2}(-b,a)=\frac{1}{\sqrt{a^2+b^2}}(-bz'_x+az'_y)=0,$$

$\therefore$在直线$ax+by=C$上的每一点处$bz'_x=az'_y$，由C的任意性知$bz'_x=az'_y$处处成立.[6]

7. 设D为包含原点$O(0,0)$的一个圆域. $f(x,y)$在D中处处有连续偏导数，并且满足$xf'_x+yf'_y=0$. 求证$f(x,y)$在D中恒等于某个常数.

证明：$\because f(x,y)$在D中处处有连续偏导数，

$\therefore f(x,y)$在D中处处可微，

$\therefore f(x,y)$在点$(x,y)\in D((x,y)\neq(0,0))$处由原点$(0,0)$指向点$(x,y)$方向的方向导数[7]

$$\frac{\partial f(x,y)}{\partial \boldsymbol{v}}=\mathrm{grad}\,f(x,y)\cdot\frac{1}{\sqrt{x^2+y^2}}(x,y)=\frac{1}{\sqrt{x^2+y^2}}(f'_x,f'_y)\cdot(x,y)=\frac{xf'_x+yf'_y}{\sqrt{x^2+y^2}}=0,$$

$\therefore f(x,y)$在D中由原点出发且不含原点的每一条射线上任一点处沿该射线方向的方向导数均为0,[8]

$\therefore f(x,y)$在D中由原点出发且不含原点的每一条射线上均为常数，[9]

$\because f(x,y)$在D中处处可微，故处处连续，故在原点$(0,0)$处连续，[10]

$\therefore f(x,y)$在D中由原点出发的每一条射线上均等于$f(0,0)$，[11]

$\therefore f(x,y)=f(0,0),(x,y)\in D.$

【注意:】如果区域D不包含原点，但仍有$f(x,y)$在D中处处有连续偏导数，并且满足$xf'_x+yf'_y=0$，则$f(x,y)$在D中不一定恒等于常数，比如函数

$$f(x,y)=\begin{cases}\cos(\arccos\frac{x}{\sqrt{x^2+y^2}}), & y\geqslant 0,\\\cos(2\pi-\arccos\frac{x}{\sqrt{x^2+y^2}}), & y<0.\end{cases}$$

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[3]这里之前的版本中是"该直线的方向向量为$(-b,a)$。

[4]这里加上了这句。

[5]这里之前的版本中是"$\therefore z$在该直线方向的方向导数，"，不准确。

[6]这里之前的版本中是"$\therefore bz'_x=az'_y.$"，不准确。

[7]这里之前的版本中是"$\therefore f(x,y)$在点$(x,y)\in D((x,y)\neq(0,0))$处的方向导数"，表述不很清楚。

[8]这里在第一版的基础上增加了这句。

[9]这里之前的版本中是"$\therefore f(x,y)$在经过原点且不包括原点的直线上恒为常数，"，不准确。

[10]这里加上了"故在原点$(0,0)$处连续，"。

[11]这里新加上了这句。

该函数在极坐标下的方程是$f(r\cos\theta, r\sin\theta) = \cos\theta$. 函数$f(x,y)$在$D$中处处有连续偏导数，且在由原点出发的每一条射线$\theta = C$上均为常数$\cos C$，故满足$xf'_x + yf'_y = 0$，但$f(x,y) \not\equiv const$.

函数$f(x,y), (x,y) \in \{(x,y) \mid 0 < x^2 + y^2 \leqslant 1\}$的图形如图1所示.
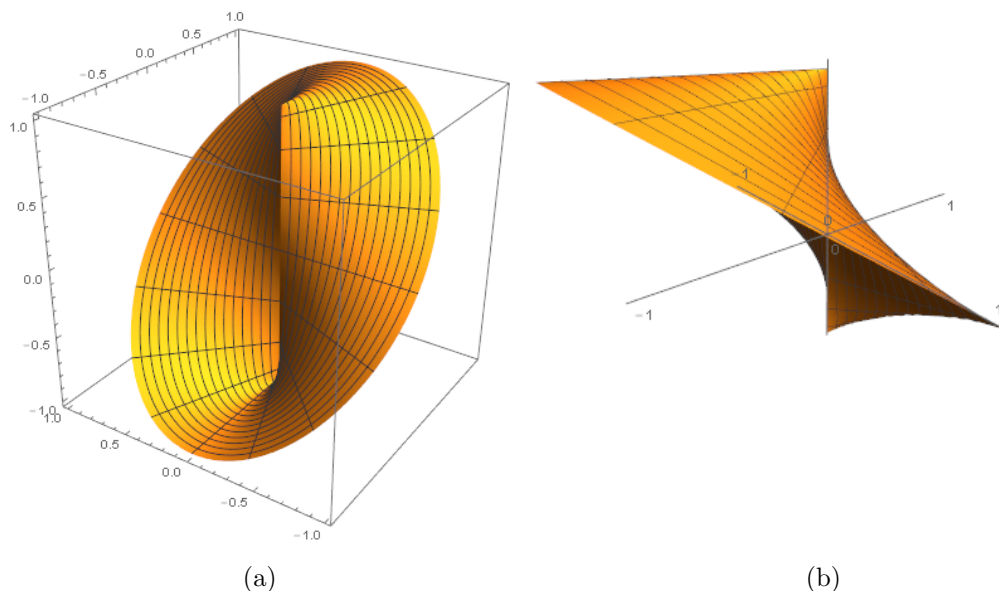

(a)　　　　　　　　　　　　　　(b)

图 1: 函数$f(r\cos\theta, r\sin\theta) = \cos\theta, (r,\theta) \in \{(r,\theta) \mid 0 < r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$的图形

16.4　习题11.1解答

1. 设$\boldsymbol{u}(t), \boldsymbol{v}(t)$是可导的向量值函数，$\lambda(t)$为可导数值函数，求证：

    (1)$\dfrac{\mathrm{d}}{\mathrm{d}t}(\lambda(t)\boldsymbol{u}(t)) = \dfrac{\mathrm{d}\lambda(t)}{\mathrm{d}t}\boldsymbol{u}(t) + \lambda(t)\dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t)$；

    (2)$\dfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}(t) \cdot \boldsymbol{v}(t)) = (\dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t)) \cdot \boldsymbol{v}(t) + \boldsymbol{u}(t) \cdot (\dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(t))$.

    证明：(1)设$\boldsymbol{u}(t) = (u_1(t), u_2(t), u_3(t))$，则$\lambda(t)\boldsymbol{u}(t) = (\lambda(t)u_1(t), \lambda(t)u_2(t), \lambda(t)u_3(t))$，

    $$\begin{aligned}
    \frac{\mathrm{d}}{\mathrm{d}t}(\lambda(t)\boldsymbol{u}(t)) &= (\mathrm{d}t[\lambda(t)u_1(t)]', [\lambda(t)u_2(t)]', [\lambda(t)u_3(t)]') \\
    &= (\lambda'(t)u_1(t) + \lambda(t)u'_1(t), \lambda'(t)u_2(t) + \lambda(t)u'_2(t), \lambda'(t)u_3(t) + \lambda(t)u'_3(t)) \\
    &= (\lambda'(t)u_1(t), \lambda'(t)u_2(t), \lambda'(t)u_3(t)) + (\lambda(t)u'_1(t), \lambda(t)u'_2(t), \lambda(t)u'_3(t)) \\
    &= \lambda'(t)(u_1(t), u_2(t), u_3(t)) + \lambda(t)(u'_1(t), u'_2(t), u'_3(t)) \\
    &= \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}\boldsymbol{u}(t) + \lambda(t)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t).
    \end{aligned}$$

    (2)设$\boldsymbol{u}(t) = (u_1(t), u_2(t), u_3(t)), \boldsymbol{v}(t) = (v_1(t), v_2(t), v_3(t))$，

则$\boldsymbol{u}(t) \cdot \boldsymbol{v}(t) = u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)$，

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}(t) \cdot \boldsymbol{v}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}[u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)]\\ &= u_1'(t)v_1(t) + u_1(t)v_1'(t) + u_2'(t)v_2(t) + u_2(t)v_2'(t) + u_3'(t)v_3(t) + u_3(t)v_3'(t)\\ &= u_1'(t)v_1(t) + u_2'(t)v_2(t) + u_3'(t)v_3(t) + u_1(t)v_1'(t) + u_2(t)v_2'(t) + u_3(t)v_3'(t)\\ &= (\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t)) \cdot \boldsymbol{v}(t) + \boldsymbol{u}(t) \cdot (\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(t)).\end{aligned}$$

2. 求下列曲线在指定点的单位切向量：

   (1)$\boldsymbol{r}(t) = (\mathrm{e}^{2t}, \mathrm{e}^{-2t}, t\mathrm{e}^{2t}), t = 0$；

   (2)$\boldsymbol{r}(t) = t\boldsymbol{i} + 2\sin t\boldsymbol{j} + 3\cos t\boldsymbol{k}, t = \frac{\pi}{6}$.

   解：(1)单位切向量$\boldsymbol{t} = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \frac{(2\mathrm{e}^{2t}, -2\mathrm{e}^{-2t}, (1+2t)\mathrm{e}^{2t})}{\|(2\mathrm{e}^{2t}, -2\mathrm{e}^{-2t}, (1+2t)\mathrm{e}^{2t})\|}\Big|_{t=0} = \frac{(2, -2, 1)}{\sqrt{4+4+1}} = (\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})$.

   (2)单位切向量$\boldsymbol{t} = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \frac{\boldsymbol{i} + 2\cos t\boldsymbol{j} - 3\sin t\boldsymbol{k}}{\|\boldsymbol{i} + 2\cos t\boldsymbol{j} - 3\sin t\boldsymbol{k}\|}\Big|_{t=\frac{\pi}{6}} = \frac{\boldsymbol{i} + \sqrt{3}\boldsymbol{j} - \frac{3}{2}\boldsymbol{k}}{\sqrt{1+3+\frac{9}{4}}} = \frac{2}{5}\boldsymbol{i} + \frac{2\sqrt{3}}{5}\boldsymbol{j} - \frac{3}{5}\boldsymbol{k}$.

3. 求下列曲线在指定点的切线方程：

   (1)$\boldsymbol{r}(t) = (1 + 2t, 1 + t - t^2, 1 - t + t^2 - t^3), M(1, 1, 1)$；

   (2)$\boldsymbol{r}(t) = \sin(\pi t)\boldsymbol{i} + \sqrt{t}\boldsymbol{j} + \cos(\pi t)\boldsymbol{k}, M(0, 1, -1)$.

   解：(1)在$M(1, 1, 1)$点处$t = 0$，切向量$\boldsymbol{t} = \boldsymbol{r}'(0) = (2, 1 - 2t, -1 + 2t - 3t^2)_{t=0} = (2, 1, -1)$，则切线方程为

   $$\frac{x-1}{2} = y - 1 = -(z-1).$$

   (2)在$M(0, 1, -1)$点处$t = 1$，切向量$\boldsymbol{t} = \boldsymbol{r}'(1) = \pi\cos(\pi t)\boldsymbol{i} + \frac{1}{2\sqrt{t}}\boldsymbol{j} - \pi\sin(\pi t)\boldsymbol{k}\big|_{t=1} = -\pi\boldsymbol{i} + \frac{1}{2}\boldsymbol{j}$，则切线方程为$\begin{cases}\frac{x}{-\pi} = 2(y-1), \\ z = -1,\end{cases}$ 即 $\begin{cases}x + 2\pi y = 2\pi, \\ z = -1.\end{cases}$

4. 求下列向量值函数的积分：

   (1)$\int_0^{\frac{\pi}{4}}[\cos(2t)\boldsymbol{i} + \sin(2t)\boldsymbol{j} + t\sin t\boldsymbol{k}]\mathrm{d}t$；

   (2)$\int_1^4(\sqrt{t}\boldsymbol{i} + t\mathrm{e}^{-t}\boldsymbol{j} + \frac{1}{t^2}\boldsymbol{k})\mathrm{d}t$.

   解：(1)$\int_0^{\frac{\pi}{4}}[\cos(2t)\boldsymbol{i} + \sin(2t)\boldsymbol{j} + t\sin t\boldsymbol{k}]\mathrm{d}t = \int_0^{\frac{\pi}{4}}\cos(2t)\mathrm{d}t\boldsymbol{i} + \int_0^{\frac{\pi}{4}}\sin(2t)\mathrm{d}t\boldsymbol{j} + \int_0^{\frac{\pi}{4}}t\sin t\mathrm{d}t\boldsymbol{k}$，

   $\because \int_0^{\frac{\pi}{4}}\cos(2t)\mathrm{d}t = \frac{1}{2}\sin(2t)\big|_0^{\frac{\pi}{4}} = \frac{1}{2}$, $\int_0^{\frac{\pi}{4}}\sin(2t)\mathrm{d}t = -\frac{1}{2}\cos(2t)\big|_0^{\frac{\pi}{4}} = \frac{1}{2}$,

   $\int_0^{\frac{\pi}{4}}t\sin t\mathrm{d}t = -\int_0^{\frac{\pi}{4}}t\mathrm{d}\cos t = -t\cos t\big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}}\cos t\mathrm{d}t = -\frac{\sqrt{2}\pi}{8} + \sin t\big|_0^{\frac{\pi}{4}} = -\frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2}$,

   $\therefore \int_0^{\frac{\pi}{4}}[\cos(2t)\boldsymbol{i} + \sin(2t)\boldsymbol{j} + t\sin t\boldsymbol{k}]\mathrm{d}t = \frac{1}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j} + (-\frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2})\boldsymbol{k}$.

   (2)$\int_1^4(\sqrt{t}\boldsymbol{i} + t\mathrm{e}^{-t}\boldsymbol{j} + \frac{1}{t^2}\boldsymbol{k})\mathrm{d}t = \int_1^4\sqrt{t}\mathrm{d}t\boldsymbol{i} + \int_1^4 t\mathrm{e}^{-t}\mathrm{d}t\boldsymbol{j} + \int_1^4\frac{1}{t^2}\mathrm{d}t\boldsymbol{k}$，

   $\because \int_1^4\sqrt{t}\mathrm{d}t = \frac{1}{1+\frac{1}{2}}t^{\frac{1}{2}+1}\big|_1^4 = \frac{14}{3}, \int_1^4 t\mathrm{e}^{-t}\mathrm{d}t = -\int_1^4 t\mathrm{d}\mathrm{e}^{-t} = -t\mathrm{e}^{-t}\big|_1^4 + \int_1^4\mathrm{e}^{-t}\mathrm{d}t$

   $= -4\mathrm{e}^{-4} + \mathrm{e}^{-1} - \mathrm{e}^{-t}\big|_1^4 = -4\mathrm{e}^{-4} + \mathrm{e}^{-1} - \mathrm{e}^{-4} + \mathrm{e}^{-1} = -5\mathrm{e}^{-4} + 2\mathrm{e}^{-1}$,

   $\int_1^4\frac{1}{t^2}\mathrm{d}t = \frac{1}{-2+1}t^{-2+1}\big|_1^4 = -\frac{1}{4} + 1 = \frac{3}{4}$,

$\therefore \int_1^4(\sqrt{t}\boldsymbol{i} + t\mathrm{e}^{-t}\boldsymbol{j} + \frac{1}{t^2}\boldsymbol{k})\mathrm{d}t = \frac{14}{3}\boldsymbol{i} + (-5\mathrm{e}^{-4} + 2\mathrm{e}^{-1})\boldsymbol{j} + \frac{3}{4}\boldsymbol{k}.$

5. 已知$\boldsymbol{r}'(t), \boldsymbol{r}(0)$，求$\boldsymbol{r}(t)$：

   (1)$\boldsymbol{r}'(t) = (t^2, 4t^3, -t^2), \boldsymbol{r}(0) = (0, 1, 0)$；

   (2)$\boldsymbol{r}'(t) = \sin t\boldsymbol{i} - \cos t\boldsymbol{j} + 2t\boldsymbol{k}, \boldsymbol{r}(0) = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}.$

   解：(1)方法1：

$$\boldsymbol{r}(t) = \int_0^t \boldsymbol{r}'(t)\mathrm{d}t + \boldsymbol{C} = (\int_0^t t^2\mathrm{d}t + C_1, \int_0^t 4t^3\mathrm{d}t + C_2, \int_0^t(-t^2)\mathrm{d}t + C_3)$$
$$= (\frac{1}{3}t^3 + C_1, t^4 + C_2, -\frac{1}{3}t^3 + C_3),$$

$\therefore \boldsymbol{r}(0) = (C_1, C_2, C_3) = (0, 1, 0),$

$\therefore \boldsymbol{r}(t) = (\frac{1}{3}t^3, t^4 + 1, -\frac{1}{3}t^3).$

方法2：$\because \int t^2\mathrm{d}t = \frac{1}{3}t^3 + C, \ \int 4t^3\mathrm{d}t = t^4 + C, \ \int(-t^2)\mathrm{d}t = -\frac{1}{3}t^3 + C,$

$\therefore r(t) = \int \boldsymbol{r}'(t)\mathrm{d}t = (\frac{1}{3}t^3 + C_1, t^4 + C_2, -\frac{1}{3}t^3 + C_3),$

$\because \boldsymbol{r}(0) = (0, 1, 0),$

$\therefore C_1 = 0, \ C_2 = 1, \ C_3 = 0,$

$\therefore \boldsymbol{r}(t) = (\frac{1}{3}t^3, t^4 + 1, -\frac{1}{3}t^3).$

(2)方法1：

$$\boldsymbol{r}(t) = \int_0^t \boldsymbol{r}'(t)\mathrm{d}t + \boldsymbol{C} = (\int_0^t \sin t\mathrm{d}t + C_1)\boldsymbol{i} + (-\int_0^t \cos t\mathrm{d}t + C_2)\boldsymbol{j} + (\int_0^t 2t\mathrm{d}t + C_3)\boldsymbol{k}$$
$$= (-\cos t + C_1)\boldsymbol{i} + (-\sin t + C_2)\boldsymbol{j} + (t^2 + C_3)\boldsymbol{k},$$

$\because \boldsymbol{r}(0) = (-1 + C_1)\boldsymbol{i} + C_2\boldsymbol{j} + C_3\boldsymbol{k} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k},$

$\therefore C_1 = 2, C_2 = 1, C_3 = 2,$

$\therefore \boldsymbol{r}(t) = (-\cos t + 2)\boldsymbol{i} + (-\sin t + 1)\boldsymbol{j} + (t^3 + 2)\boldsymbol{k}.$

方法2：$\because \int \sin t\mathrm{d}t = -\cos t + C, \ \int(-\cos t)\mathrm{d}t = -\sin t + C, \ \int 2t\mathrm{d}t = t^2 + C,$

$\therefore \boldsymbol{r}(t) = \int \boldsymbol{r}'(t)\mathrm{d}t = (-\cos t + C_1)\boldsymbol{i} + (-\sin t + C_2)\boldsymbol{j} + (t^2 + C_3)\boldsymbol{k},$

$\because \boldsymbol{r}(0) = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k},$

$\therefore C_1 = 2, \ C_2 = 1, \ C_3 = 2,$

$\therefore \boldsymbol{r}(t) = (-\cos t + 2)\boldsymbol{i} + (-\sin t + 1)\boldsymbol{j} + (t^3 + 2)\boldsymbol{k}.$

6. 证明下列等式：

   (1)$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}(t) \times \boldsymbol{r}'(t)) = \boldsymbol{r}(t) \times \boldsymbol{r}''(t)$；

(2)$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{r}(t)\| = \frac{\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)}{\|\boldsymbol{r}(t)\|}(\boldsymbol{r}(t)\neq\boldsymbol{0})$;

(3)$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{r}(t)\cdot(\boldsymbol{r}'(t)\times\boldsymbol{r}''(t))] = \boldsymbol{r}(t)\cdot[\boldsymbol{r}'(t)\times\boldsymbol{r}'''(t)]$.

证明：(1)$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}(t)\times\boldsymbol{r}'(t)) = \boldsymbol{r}'(t)\times\boldsymbol{r}'(t) + \boldsymbol{r}(t)\times\boldsymbol{r}''(t) = \boldsymbol{r}(t)\times\boldsymbol{r}''(t)$.

(2)$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{r}(t)\| = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)} = \frac{1}{2\sqrt{\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)}}\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)]$

$= \frac{1}{2\sqrt{\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)}}[\boldsymbol{r}'(t)\cdot\boldsymbol{r}(t)+\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)] = \frac{2\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)}{2\sqrt{\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)}} = \frac{\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)}{\|\boldsymbol{r}(t)\|}(\boldsymbol{r}(t)\neq\boldsymbol{0})$.

(3)$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{r}(t)\cdot(\boldsymbol{r}'(t)\times\boldsymbol{r}''(t))] = \boldsymbol{r}'(t)\cdot(\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)) + \boldsymbol{r}(t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}'(t)\times\boldsymbol{r}''(t))$

$= \boldsymbol{r}(t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)) = \boldsymbol{r}(t)\cdot(\boldsymbol{r}''(t)\times\boldsymbol{r}''(t)+\boldsymbol{r}'(t)\times\boldsymbol{r}'''(t)) = \boldsymbol{r}(t)\cdot[\boldsymbol{r}'(t)\times\boldsymbol{r}'''(t)]$.

7. 求等速圆周运动$\boldsymbol{r} = R\cos(\omega t)\boldsymbol{i} + R\sin(\omega t)\boldsymbol{j}$在$t$时刻的速度与加速度.

   解：t时刻的速度$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = -R\omega\sin(\omega t)\boldsymbol{i} + R\omega\cos(\omega t)\boldsymbol{j}$,

   t时刻的加速度$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = -R\omega^2\cos(\omega t)\boldsymbol{i} - R\omega^2\sin(\omega t)\boldsymbol{j}$.

8. 已知螺旋线的向量方程为$\boldsymbol{r} = a\cos\theta\boldsymbol{i} + a\sin\theta\boldsymbol{j} + b\theta\boldsymbol{k}(a>0, b>0)$，求在$\theta_0$处的切线方程.

   解：在θ_0处的切向量$\boldsymbol{r}'(\theta_0) = -a\sin\theta_0\boldsymbol{i} + a\cos\theta_0\boldsymbol{j} + b\boldsymbol{k}$，切线方程

   $$\frac{x - a\cos\theta_0}{-a\sin\theta_0} = \frac{y - a\sin\theta_0}{a\cos\theta_0} = \frac{z - b\theta_0}{b}.$$

9. 设$\boldsymbol{r} = -a\sin\theta\boldsymbol{i} + a\cos\theta\boldsymbol{j} + b\theta\boldsymbol{k}$，求$\frac{1}{2}\int_0^{2\pi}(\boldsymbol{r}\times\boldsymbol{r}')\mathrm{d}\theta$.

   解：$\boldsymbol{r}'(\theta) = -a\cos\theta\boldsymbol{i} - a\sin\theta\boldsymbol{j} + b\boldsymbol{k}$,

   $\boldsymbol{r}(\theta)\times\boldsymbol{r}'(\theta) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a\sin\theta & a\cos\theta & b\theta \\ -a\cos\theta & -a\sin\theta & b \end{vmatrix} = \begin{vmatrix} a\cos\theta & b\theta \\ -a\sin\theta & b \end{vmatrix}\boldsymbol{i} + \begin{vmatrix} b\theta & -a\sin\theta \\ b & -a\cos\theta \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} -a\sin\theta & a\cos\theta \\ -a\cos\theta & -a\sin\theta \end{vmatrix}\boldsymbol{k}$

   $= (ab\cos\theta + ab\theta\sin\theta)\boldsymbol{i} - (-ab\sin\theta + ab\theta\cos\theta)\boldsymbol{j} + a^2\boldsymbol{k}$,

   $\because \int_0^{2\pi}(ab\cos\theta + ab\theta\sin\theta)\mathrm{d}\theta = ab\sin\theta\big|_0^{2\pi} - \int_0^{2\pi}ab\theta\mathrm{d}\cos\theta$

   $= -ab\theta\cos\theta\big|_0^{2\pi} + \int_0^{2\pi}ab\cos\theta\mathrm{d}\theta = -2\pi ab + ab\sin\theta\big|_0^{2\pi} = -2\pi ab$,

   $\int_0^{2\pi}-(-ab\sin\theta + ab\theta\cos\theta)\mathrm{d}\theta = \int_0^{2\pi}(ab\sin\theta - ab\theta\cos\theta)\mathrm{d}\theta = -ab\cos\theta\big|_0^{2\pi} - ab\int_0^{2\pi}\theta\mathrm{d}\sin\theta$

   $= -ab\theta\sin\theta\big|_0^{2\pi} + ab\int_0^{2\pi}\sin\theta\mathrm{d}\theta = -ab\cos\theta\big|_0^{2\pi} = 0$,

   $\int_0^{2\pi}a^2\mathrm{d}\theta = 2\pi a^2$,

   $\therefore \frac{1}{2}\int_0^{2\pi}(\boldsymbol{r}\times\boldsymbol{r}')\mathrm{d}\theta = -\pi ab\boldsymbol{i} + \pi a^2\boldsymbol{k}$.

16.5 习题11.2解答

1. 求下列曲面在指定点的法线方程与切平面的方程：

   (1)$x^2 + y^2 + z^2 = 14$，在点$(1, 2, 3)$；

(2)$z = \frac{1}{2}x^2 - y^2$，在点$(2, -1, 1)$；

(3)$(2a^2 - z^2)x^2 - a^2y^2 = 0$，在点$(a, a, a)$；

(4)$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$，在点$(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}})$；

(5)$\begin{cases} x = u\cos v, \\ y = u\sin v, \\ z = av, \end{cases}$ 在$(u, v) = (u_0, v_0)$处.

解：(1)法向量$\boldsymbol{n} = (2x, 2y, 2z)\big|_{(1,2,3)} = 2(1, 2, 3)$,

法线方程$x - 1 = \frac{y-2}{2} = \frac{z-3}{3}$,

切平面方程$(x - 1) + 2(y - 2) + 3(z - 3) = 0$，即$x + 2y + 3z = 14$.

(2)法向量$\boldsymbol{n} = (x, -2y, -1)\big|_{(2,-1,1)} = (2, 2, -1)$,

法线方程$\frac{x-2}{2} = \frac{y+1}{2} = -(z - 1)$,

切平面方程$2(x - 2) + 2(y + 1) - (z - 1) = 0$，即$2x + 2y - z = 1$.

(3)法向量$\boldsymbol{n} = (2(2a^2 - z^2)x, -2a^2y, -2x^2z)\big|_{(a,a,a)} = 2a^3(1, -1, -1)$,

法线方程$x - a = -(y - a) = -(z - a)$,

切平面方程$(x - a) - (y - a) - (z - a) = 0$，即$x - y - z = -a$.

(4)法向量$\boldsymbol{n} = (\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2})\big|_{(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}})} = \frac{2}{\sqrt{3}}(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$,

法线方程$a(x - \frac{a}{\sqrt{3}}) = b(y - \frac{b}{\sqrt{3}}) = c(z - \frac{c}{\sqrt{3}})$,

切平面方程$\frac{1}{a}(x - \frac{a}{\sqrt{3}}) + \frac{1}{b}(y - \frac{b}{\sqrt{3}}) + \frac{1}{c}(z - \frac{c}{\sqrt{3}}) = 0$，即$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \sqrt{3}$.

(5)法向量$\boldsymbol{n} = (\cos v, \sin v, 0) \times (-u\sin v, u\cos v, a)\big|_{(u_0, v_0)} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos v_0 & \sin v_0 & 0 \\ -u_0\sin v_0 & u_0\cos v_0 & a \end{vmatrix}$

$= (\begin{vmatrix} \sin v_0 & 0 \\ u_0\cos v_0 & a \end{vmatrix}, \begin{vmatrix} 0 & a \\ a & u_0\cos v_0 \end{vmatrix}, \begin{vmatrix} \cos v_0 & \sin v_0 \\ -u_0\sin v_0 & u_0\cos v_0 \end{vmatrix}) = (a\sin v_0, -a\cos v_0, u_0)$,

法线方程$\frac{x - u_0\cos v_0}{a\sin v_0} = \frac{y - u_0\sin v_0}{-a\cos v_0} = \frac{z - av_0}{u_0}$,

切平面方程$a\sin v_0(x - u_0\cos v_0) - a\cos v_0(y - u_0\sin v_0) + u_0(z - av_0) = 0$，

即$ax\sin v_0 - ay\cos v_0 + zu_0 = au_0v_0$.

2. 按要求求下列曲面的切平面方程：

(1)曲面$x^2 + 2y^2 + 3z^2 = 21$的与平面$x + 4y + 6z = 0$平行的切平面；

(2)曲面$z = x^2 + y^2$的与直线$\begin{cases} x + 2z = 1, \\ y + 2z = 2 \end{cases}$ 垂直的切平面；

(3)双曲抛物面$\boldsymbol{r} = (u + v, u - v, uv)$在$u = 1, v = -1$处的切平面.

解：(1)曲面的法向量 $\boldsymbol{n} = (2x, 4y, 6z)$，平面的法向量 $\boldsymbol{n}_1 = (1, 4, 6)$，

则由 $\begin{cases} (2x, 4y, 6z) = a(1, 4, 6), \\ x^2 + 2y^2 + 3z^2 = 21 \end{cases}$ 得曲面上与该平面相切的切平面的切点为 $\pm(1, 2, 2)$，

切平面方程 $x - 1 + 4(y - 2) + 6(z - 2) = 0$ 或 $x + 1 + 4(y + 2) + 6(z + 2) = 0$，即 $x + 4y + 6z = \pm 21$.

(2)直线的切向量 $\boldsymbol{t} = (1, 0, 2) \times (0, 1, 2) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{vmatrix} = \left(\begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right) = (-2, -2, 1)$，

曲面的法向量 $\boldsymbol{n} = (2x, 2y, -1)$，曲面上与直线垂直的切平面的法向量 $\boldsymbol{n}_0 = a\boldsymbol{t}$，

由 $\begin{cases} (2x, 2y, -1) = a(-2, -2, 1), \\ z = x^2 + y^2 \end{cases}$ 可得切点为 $(1, 1, 2)$，

切平面方程 $-2(x - 1) - 2(y - 1) + z - 2 = 0$，即 $2x + 2y - z = 2$.

(3)法向量 $\boldsymbol{n} = (1, 1, v) \times (1, -1, u) \big|_{(1, -1)} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \left(\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \right)$

$= (0, -2, -2)$，切点为 $(0, 2, -1)$，

切平面的方程 $-2(y - 2) - 2(z + 1) = 0$，即 $y + z = 1$.

3. 求证：曲面 $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}, a > 0$ 在任意点处的切平面在各坐标轴上的截距之和为 a.

   证明：曲面在任意点 (x_0, y_0, z_0) 处的法向量 $\boldsymbol{n} = \frac{1}{2}\left(\frac{1}{\sqrt{x_0}}, \frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{z_0}} \right)$，

   切平面方程 $\frac{1}{\sqrt{x_0}}(x - x_0) + \frac{1}{\sqrt{y_0}}(y - y_0) + \frac{1}{\sqrt{z_0}}(z - z_0) = 0$，即 $\frac{x}{ax_0} + \frac{y}{ay_0} + \frac{z}{az_0} = 1$，

   切平面在 x, y, z 轴上的截距之和

   $$\sqrt{ax_0} + \sqrt{ay_0} + \sqrt{az_0} = \sqrt{a}(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}) = a.$$

4. 证明二次曲面 $ax^2 + by^2 + cz^2 = 1$ 在点 $M_0(x_0, y_0, z_0)$ 处的切平面方程为

   $$ax_0 x + by_0 y + cz_0 z = 1.$$

   证明：曲面在点 $M_0(x_0, y_0, z_0)$ 处的法向量 $\boldsymbol{n} = 2(ax_0, by_0, cz_0)$，

   切平面方程

   $$ax_0(x - x_0) + by_0(y - y_0) + cz_0(z - z_0)$$
   $$= ax_0 x - ax_0^2 + by_0 y - by_0^2 + cz_0 z - cz_0^2$$
   $$= ax_0 x - by_0 y - cz_0 z - 1 = 0,$$

即

$$ax_0x + by_0y + cz_0z = 1.$$

5. 设函数f可微，试证曲面$z = yf(\frac{x}{y})$的所有切平面相交于一个公共点.

证明：曲面在点(x_0, y_0, z_0)处的法向量$\boldsymbol{n} = (yf'(\frac{x}{y})\frac{1}{y}, f(\frac{x}{y}) + yf'(\frac{x}{y})(-\frac{x}{y^2}), -1)\big|_{(x_0,y_0,z_0)}$

$= (f'(\frac{x_0}{y_0}), f(\frac{x_0}{y_0}) - \frac{x_0}{y_0}f'(\frac{x_0}{y_0}), -1)$,

切平面的方程

$$f'(\frac{x_0}{y_0})(x - x_0) + [f(\frac{x_0}{y_0}) - \frac{x_0}{y_0}f'(\frac{x_0}{y_0})](y - y_0) - (z - z_0)$$
$$= f'(\frac{x_0}{y_0})x + [f(\frac{x_0}{y_0}) - \frac{x_0}{y_0}f'(\frac{x_0}{y_0})]y - z - x_0f'(\frac{x_0}{y_0}) - y_0f(\frac{x_0}{y_0}) + x_0f'(\frac{x_0}{y_0}) + z_0$$
$$= f'(\frac{x_0}{y_0})x + [f(\frac{x_0}{y_0}) - \frac{x_0}{y_0}f'(\frac{x_0}{y_0})]y - z$$
$$= 0,$$

$\because$无论切点(x_0, y_0, z_0)取在何处，点$(x, y, z) = (0, 0, 0)$始终满足以上方程，

$\therefore$曲面$z = yf(\frac{x}{y})$的所有切平面相交于一个公共点$(0, 0, 0)$.

6. 已知函数f可微，证明曲面$f(\frac{x-a}{z-c}, \frac{y-b}{z-c}) = 0$上任意一点处的切平面通过一定点，并求出此点的位置.

证明：曲面上任一点(x_0, y_0, z_0)处的法向量

$$\boldsymbol{n} = (\frac{1}{z_0 - c}f_1', \frac{1}{z_0 - c}f_2', -\frac{x_0 - a}{(z_0 - c)^2}f_1' - \frac{y_0 - b}{(z_0 - c)^2}f_2'),$$

切平面方程

$$\frac{x - x_0}{z_0 - c}f_1' + \frac{y - y_0}{z_0 - c}f_2' + [-\frac{x_0 - a}{(z_0 - c)^2}f_1' - \frac{y_0 - b}{(z_0 - c)^2}f_2'](z - z_0)$$
$$= [\frac{x - x_0}{z_0 - c} - \frac{x_0 - a}{(z_0 - c)^2}(z - z_0)]f_1' + [\frac{y - y_0}{z_0 - c} - \frac{y_0 - b}{(z_0 - c)^2}(z - z_0)]f_2'$$
$$= 0,$$

其中偏导数均在$(\frac{x_0-a}{z_0-c}, \frac{y_0-b}{z_0-c})$处取值，

$\because$无论切点(x_0, y_0, z_0)取在何处，$(x, y, z) = (a, b, c)$始终满足以上方程，

$\therefore$曲面$f(\frac{x-a}{z-c}, \frac{y-b}{z-c}) = 0$上任意一点处的切平面通过定点$(a, b, c)$.

7. 设曲面S_1和S_2的方程分别为$F_1(x, y, z) = 0, F_2(x, y, z) = 0$，其中$F_1$和$F_2$是可微函数，试证$S_1$与$S_2$垂直的充分必要条件是对交线上的任意一点$(x, y, z)$，均有

$$\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial z} = 0.$$

证明：必要性：$\because S_1$与S_2垂直，

$\therefore$交线上的任意一点(x,y,z)处两曲面的法向量互相垂直，即

$$(\frac{\partial F_1}{\partial x},\frac{\partial F_1}{\partial y},\frac{\partial F_1}{\partial z})\cdot(\frac{\partial F_2}{\partial x},\frac{\partial F_2}{\partial y},\frac{\partial F_2}{\partial z})=\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial x}+\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial y}+\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial z}=0;$$

充分性：$\because$交线上的任意一点(x,y,z)处

$$\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial x}+\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial y}+\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial z}=(\frac{\partial F_1}{\partial x},\frac{\partial F_1}{\partial y},\frac{\partial F_1}{\partial z})\cdot(\frac{\partial F_2}{\partial x},\frac{\partial F_2}{\partial y},\frac{\partial F_2}{\partial z})=0,$$

$\therefore$交线上的任意一点(x,y,z)处曲面S_1与S_2的法向量互相垂直，

$\therefore S_1$与S_2垂直.

8. 已知函数F可微，若T为曲面S：$F(x,y,z)=0$在点$M_0(x_0,y_0,z_0)$处的切平面，l为T上任意一条过M_0的直线，求证：在S上存在一条曲线，该曲线在M_0处的切线恰好为l.

证明：方法1：设直线l的方程为$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$，因$l$在$T$上，其方向向量$(a,b,c)$应满足

$$(a,b,c)\cdot\mathrm{grad}F(x_0,y_0,z_0)=aF_x'+bF_y'+cF_z'=0,$$

过直线l且与切平面T垂直的平面A的法向量

$$\boldsymbol{n}=(a,b,c)\times\mathrm{grad}F(x_0,y_0,z_0)=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\a&b&c\\F_x'&F_y'&F_z'\end{vmatrix}=(\begin{vmatrix}b&c\\F_y'&F_z'\end{vmatrix},\begin{vmatrix}c&a\\F_z'&F_x'\end{vmatrix},\begin{vmatrix}a&b\\F_x'&F_y'\end{vmatrix})$$

$$=(bF_z'-cF_y',cF_x'-aF_z',aF_y'-bF_x'),$$

曲面S与平面A的交线在点M_0处的切向量

$$\boldsymbol{t}=\mathrm{grad}F(x_0,y_0,z_0)\times\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\F_x'&F_y'&F_z'\\bF_z'-cF_y'&cF_x'-aF_z'&aF_y'-bF_x'\end{vmatrix}$$

$$=(\begin{vmatrix}F_y'&F_z'\\cF_x'-aF_z'&aF_y'-bF_x'\end{vmatrix},\begin{vmatrix}F_z'&F_x'\\aF_y'-bF_x'&bF_z'-cF_y'\end{vmatrix},\begin{vmatrix}F_x'&F_y'\\bF_z'-cF_y'&cF_x'-aF_z'\end{vmatrix})$$

$$=[a(F_y')^2-bF_x'F_y'-cF_x'F_z'+a(F_z')^2]\boldsymbol{i}-[aF_x'F_y'-b(F_x')^2-b(F_z')^2+cF_y'F_z']\boldsymbol{j}$$

$$\quad+[c(F_x')^2-aF_x'F_z'-bF_y'F_z'+c(F_y')^2]\boldsymbol{k}$$

$$=[a(F_y')^2-F_x'(bF_y'+cF_z')+a(F_z')^2]\boldsymbol{i}-[F_y'(aF_x'+cF_z')-b(F_x')^2-b(F_z')^2]\boldsymbol{j}$$

$$\quad+[c(F_x')^2-F_z'(aF_x'+bF_y')+c(F_y')^2]\boldsymbol{k}$$

$$=[a(F_y')^2+a(F_x')^2+a(F_z')^2]\boldsymbol{i}-[-(F_y')^2-b(F_x')^2-b(F_z')^2]\boldsymbol{j}+[c(F_x')^2+c(F_z')^2+c(F_y')^2]\boldsymbol{k}$$

$$=[(F_x')^2+(F_z')^2+(F_y')^2](a,b,c),$$

$\therefore \boldsymbol{t} \parallel (a,b,c)$,

$\therefore$曲面S与平面A的交线在点M_0处的切线为l，即在S上存在一条曲线，该曲线在M_0处的切线恰好为l.

方法2：设直线l的方程为$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$，因$l$在$T$上，其方向向量$(a,b,c)$应满足

$$(a,b,c) \perp \operatorname{grad}F(x_0,y_0,z_0),$$

过直线l且与切平面T垂直的平面A的法向量

$$\boldsymbol{n} = (a,b,c) \times \operatorname{grad}F(x_0,y_0,z_0),$$

曲面S与平面A的交线在点M_0处的切向量

$$\boldsymbol{t} =[\operatorname{grad}F(x_0,y_0,z_0) \times \boldsymbol{n}] \parallel (a,b,c),$$

$\therefore$曲面S与平面A的交线在点M_0处的切线为l，即在S上存在一条曲线，该曲线在M_0处的切线恰好为l.