

21 含参变量的积分、向量场的微分运算

21.1 知识结构

第12章重积分

12.7 含参变量的积分

12.7.1 引言

12.7.2 含参变量的定积分

12.7.3 含参变量的广义积分

第13章向量场的微积分

13.1 向量场的微分运算

(a) 数量场的梯度算子

(b) 向量场的散度算子

(c) 向量场的旋度算子

21.2 习题12.7解答

1. 求下列含参变量积分的导数:

$$(1) f(x) = \int_0^\pi \sin(xy) dy; \quad (2) f(x) = \int_0^x \sin(xy) dy;$$

$$(3) f(x) = \int_0^1 \frac{xdy}{\sqrt{1-x^2y^2}}; \quad (4) f(x) = \int_0^x f(y+x, y-x) dy.$$

解: (1) $f'(x) = \int_0^\pi \frac{\partial}{\partial x} \sin(xy) dy = \int_0^\pi y \cos(xy) dy = \frac{1}{x} \int_0^\pi y \cos(xy) d(xy) = \frac{1}{x} \int_0^\pi y d \sin(xy)$
 $= \frac{y}{x} \sin(xy) \Big|_0^\pi - \frac{1}{x} \int_0^\pi \sin(xy) dy = \frac{\pi \sin(\pi x)}{x} + \frac{1}{x^2} \cos(xy) \Big|_0^\pi = \frac{\pi \sin(\pi x)}{x} + \frac{\cos(\pi x)}{x^2} - \frac{1}{x^2}.$

(2) $f'(x) = \int_0^x \frac{\partial}{\partial x} \sin(xy) dy + \sin(x \cdot x) \cdot \frac{dx}{dx} - \sin(x \cdot 0) \cdot \frac{d0}{dx} = \int_0^x y \cos(xy) dy - \sin(x^2)$
 $= \frac{1}{x} \int_0^x y d \sin(xy) + \sin(x^2) = \frac{y}{x} \sin(xy) \Big|_0^x - \frac{1}{x} \int_0^x \sin(xy) dy + \sin(x^2)$
 $= \sin(x^2) - \frac{1}{x^2} \int_0^x \sin(xy) d(xy) + \sin(x^2) = 2 \sin(x^2) + \frac{1}{x^2} \cos(xy) \Big|_0^x$
 $= 2 \sin(x^2) + \frac{\cos(x^2)}{x^2} - \frac{1}{x^2}.$

(3) 方法1: $f'(x) = \int_0^1 \frac{\partial}{\partial x} \left[\frac{x}{\sqrt{1-x^2y^2}} \right] dy = \int_0^1 \frac{\sqrt{1-x^2y^2} - x \frac{-2xy^2}{2\sqrt{1-x^2y^2}}}{1-x^2y^2} dy = \int_0^1 \frac{1}{(1-x^2y^2)^{\frac{3}{2}}} dy$
 $= \frac{1}{x} \int_0^1 \frac{1}{(1-x^2y^2)^{\frac{3}{2}}} d(xy) \xrightarrow{xy=\sin \theta} \frac{1}{x} \int_0^{\arcsin x} \frac{1}{\cos^3 \theta} d \sin \theta = \frac{1}{x} \int_0^{\arcsin x} \frac{\cos \theta}{\cos^3 \theta} d\theta = \frac{1}{x} \int_0^{\arcsin x} \sec^2 \theta d\theta$
 $= \frac{1}{x} \tan \theta \Big|_0^{\arcsin x} = \frac{1}{x} \frac{x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}.$

方法2: $\because f(x) = \int_0^1 \frac{xdy}{\sqrt{1-x^2y^2}} = \int_0^1 \frac{d(xy)}{\sqrt{1-x^2y^2}} = \arcsin(xy) \Big|_0^1 = \arcsin x,$

$$\therefore f'(x) = \frac{1}{\sqrt{1-x^2}}.$$

$$(4) f'(x) = \int_0^x \frac{\partial}{\partial x} f(y+x, y-x) dy + f(x+x, x-x) \frac{dx}{dx} - f(0+x, 0-x) \frac{d0}{dx} \\ = \int_0^x [f'_1(y+x, y-x) - f'_2(y+x, y-x)] dy + f(2x, 0).$$

2. 设 $f(x)$ 是 $[0, 1]$ 上的连续函数, 对 $x \in [0, 1]$, 令 $F(x) = \int_0^x f(t)(x-t)^{n-1} dt$, 求 $F^{(n)}(x)$.

$$\text{解: } F'(x) = \int_0^x \frac{\partial}{\partial x} [f(t)(x-t)^{n-1}] dt + f(x)(x-x)^{n-1} \frac{dx}{dx} - f(0)(x-0) \frac{d0}{dx} \\ = (n-1) \int_0^x f(t)(x-t)^{n-2} dt,$$

$$F''(x) = (n-1) \int_0^x \frac{\partial}{\partial x} [f(t)(x-t)^{n-2}] dt + (n-1)f(x)(x-x)^{n-2} \frac{dx}{dx} - (n-1)f(0)(x-0)^{n-2} \frac{d0}{dx} \\ = (n-1)(n-2) \int_0^x f(t)(x-t)^{n-3} dt,$$

...

$$F^{(n-1)}(x) = (n-1)(n-2) \cdots [n-(n-1)] \int_0^x f(t)(x-t)^{n-(n-1)-1} dt = (n-1)! \int_0^x f(t) dt,$$

$$F^{(n)}(x) = (n-1)! f(x).$$

3. 设 $f(y) = \int_0^1 (x-1)x^y \ln^{-1} x dx$, 求 $f'(y)$ 和 $\lim_{y \rightarrow +\infty} f(y)$, 并证明 $f(y) = \ln \frac{2+y}{1+y} (y > -1)$.

$$\text{证明: } f'(y) = \int_0^1 \frac{\partial}{\partial y} [(x-1)x^y \ln^{-1} x] dx = \int_0^1 (x-1)x^y \ln x \ln^{-1} x dx = \int_0^1 (x-1)x^y dx \\ = \int_0^1 (x^{y+1} - x^y) dx = \left(\frac{1}{y+2} x^{y+2} - \frac{1}{y+1} x^{y+1} \right) \Big|_0^1 = \frac{1}{y+2} - \frac{1}{y+1},$$

$$\lim_{y \rightarrow +\infty} f(y) = \int_0^1 \lim_{y \rightarrow +\infty} (x-1)x^y \ln^{-1} x dx = \int_0^1 0 dx = 0,$$

$$\therefore f(y) = \int_{+\infty}^y f'(t) dt = \int_{+\infty}^y \left(\frac{1}{t+2} - \frac{1}{t+1} \right) dt = \ln \frac{t+2}{t+1} \Big|_{+\infty}^y = \ln \frac{y+2}{y+1} - \lim_{y \rightarrow +\infty} \ln(1 + \frac{1}{t+1}) \\ = \ln \frac{y+2}{y+1}.$$

21.3 第12章补充题解答

1. 设 $f(u)$ 是连续函数, 求证:

$$\int_a^b dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} f(x_n) dx_n = \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f(x) dx.$$

证明: 当 $n=1$ 时 $\int_a^b f(x_1) dx_1 = \frac{1}{(1-1)!} \int_a^b (b-x)^{1-1} f(x) dx$, 命题成立,

当 $n=2$ 时 $\int_a^b dx_1 \int_a^{x_1} f(x_2) dx_2 = \int_a^b dx_2 \int_{x_2}^b f(x_2) dx_1 = \int_a^b (b-x_2) f(x_2) dx_2 \\ = \int_a^b (b-x) f(x) dx$, 命题成立,

假设当 $n=k$ 时命题成立, 即

$$\int_a^b dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{k-1}} f(x_k) dx_k = \frac{1}{(k-1)!} \int_a^b (b-x)^{k-1} f(x) dx.$$

则当 $n = k + 1$ 时, 令 $F(x) = \int_a^x f(x_{k+1})dx_{k+1}$

$$\begin{aligned}
 & \int_a^b dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{k-1}} dx_k \int_a^{x_k} f(x_{k+1})dx_{k+1} \\
 &= \int_a^b dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{k-1}} F(x_k)dx_k \\
 &= \frac{1}{(k-1)!} \int_a^b (b-x)^{k-1} F(x)dx \\
 &= \frac{1}{(k-1)!} \int_a^b (b-x)^{k-1} \left[\int_a^x f(x_{k+1})dx_{k+1} \right] dx \\
 &= \frac{1}{(k-1)!} \int_a^b f(x_{k+1}) \left[\int_{x_{k+1}}^b (b-x)^{k-1} dx \right] dx_{k+1} \\
 &= \frac{1}{(k-1)!} \int_a^b f(x_{k+1}) \left[-\frac{1}{k-1+1} (b-x)^{k-1+1} \Big|_{x_{k+1}}^b \right] dx_{k+1} \\
 &= \frac{1}{(k-1)!} \int_a^b f(x_{k+1}) \frac{1}{k} (b-x_{k+1})^k dx_{k+1} \\
 &= \frac{1}{k!} \int_a^b f(x) (b-x)^k dx,
 \end{aligned}$$

故当 $n = k + 1$ 时命题也成立. 证毕.

2. 求椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ (质量均匀) 绕直线 $y = kx$ 的转动惯量, 并说明 k 为何值时转动惯量最大.

解: 设 $D = \{(x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$, $\forall (x, y) \in D$ 到直线 $y = kx$ 的距离为 $d = \frac{|kx-y|}{\sqrt{1+k^2}}$, 设椭圆的密度为 μ ,

则已知椭圆绕直线 $y = kx$ 的转动惯量

$$\begin{aligned}
 J &= \iint_D d^2 \mu dx dy = \iint_D \frac{(kx-y)^2}{1+k^2} \mu dx dy \\
 &= \frac{\mu}{1+k^2} \iint_D (kar \cos \theta - br \sin \theta)^2 \cdot ab r dr d\theta = \frac{ab\mu}{1+k^2} \int_0^1 r^3 dr \int_0^{2\pi} (ka \cos \theta - b \sin \theta)^2 d\theta \\
 &= \frac{ab\mu}{4(1+k^2)} \int_0^{2\pi} (k^2 a^2 \cos^2 \theta + b^2 \sin^2 \theta - 2kab \sin \theta \cos \theta) d\theta \\
 &= \frac{ab\mu}{4(1+k^2)} (k^2 a^2 \int_0^{2\pi} \cos^2 \theta d\theta + b^2 \int_0^{2\pi} \sin^2 \theta d\theta - 2ab \int_0^{2\pi} \sin \theta \cos \theta d\theta) \\
 &= \frac{ab\mu}{4(1+k^2)} (k^2 a^2 4 \cdot \frac{\pi}{4} + b^2 4 \cdot \frac{\pi}{4} - 0) = \frac{\pi ab\mu}{4(1+k^2)} (k^2 a^2 + b^2) = \frac{\pi}{4} ab\mu \frac{k^2 a^2 + b^2}{1+k^2} = \frac{\pi}{4} ab\mu (a^2 + \frac{b^2-a^2}{1+k^2}),
 \end{aligned}$$

i) 若 $a > b > 0$, 则 $k = \infty$ 时转动惯量 J 取得最大值;

ii) 若 $b > a > 0$, 则 $k = 0$ 时转动惯量 J 取得最大值;

iii) 若 $a = b > 0$, 则转动惯量 J 恒为常数.

3. 设有半径为 R , 高为 H 的正圆锥体(质量均匀), 试求:

(1) 该圆锥体对位于其顶点处质量为 m 的质点的引力;

(2) 该圆锥体关于它的中心轴的转动惯量.

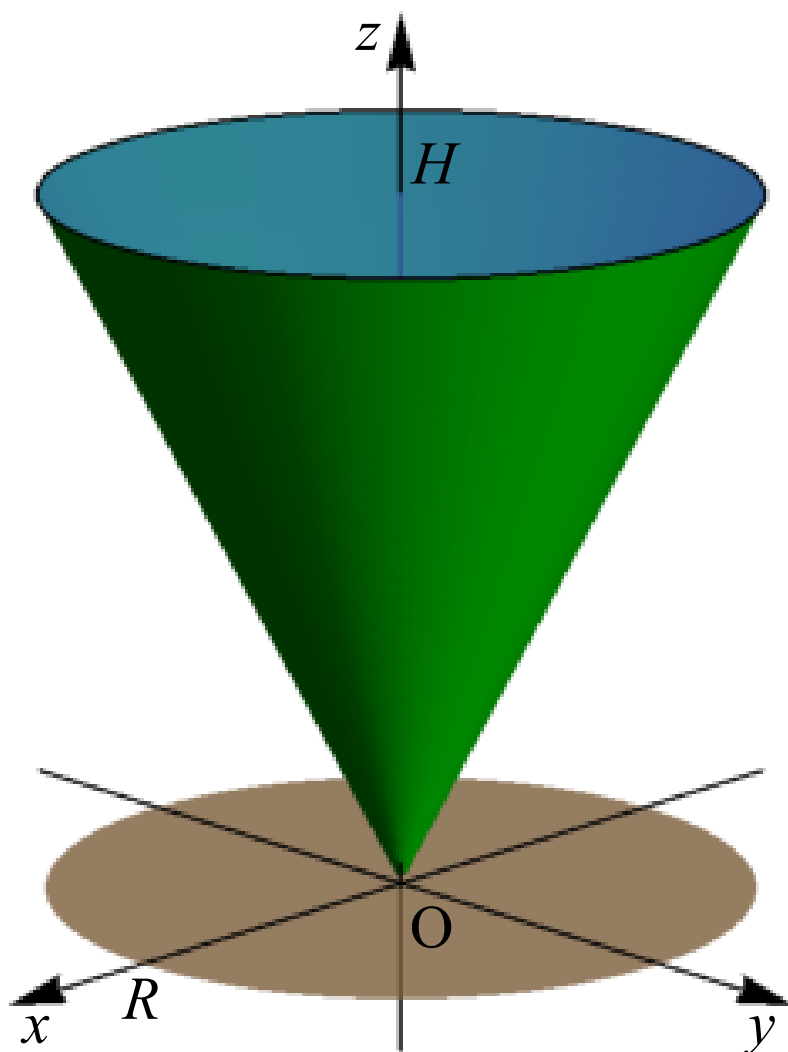


图 1: 第12章补充题 3.题图示

解：以该圆锥体的顶点为坐标原点，该圆锥体的旋转轴为 z 轴，建立空间直角坐标系，使该圆锥体在 xOy 平面上方，该圆锥体占据的空间区域可表示为 $\Omega = \{(x, y, z) \mid \frac{H}{R}\sqrt{x^2 + y^2} \leq z \leq H, x^2 + y^2 \leq R^2\} = \{(\theta, r, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq R, \frac{H}{R}r \leq z \leq H\}$ 根据对称性可知引力的 x 和 y 分量 $F_x = F_y = 0$ ，设该圆锥体的密度为 ρ ，则引力的 z 分量

$$\begin{aligned} F_z &= \iiint_{\Omega} \frac{G\rho m}{x^2 + y^2 + z^2} \frac{z}{\sqrt{x^2 + y^2 + z^2}} dx dy dz = \int_0^{2\pi} d\theta \int_0^R dr \int_{\frac{H}{R}r}^H \frac{G\rho m}{r^2 + z^2} \frac{z}{\sqrt{r^2 + z^2}} r dz \\ &= 2\pi G\rho m \int_0^R dr \int_{\frac{H}{R}r}^H \frac{rz}{(r^2 + z^2)^{\frac{3}{2}}} dz = 2\pi G\rho m \int_0^R \frac{1}{2} r^{\frac{1}{1-\frac{3}{2}}} (r^2 + z^2)^{-\frac{3}{2}+1} \Big|_{\frac{H}{R}r}^H dr \end{aligned}$$

$$\begin{aligned}
&= 2\pi G\rho m \int_0^R r \left(\frac{1}{\sqrt{r^2 + \frac{H^2}{R^2} r^2}} - \frac{1}{\sqrt{r^2 + H^2}} \right) dr = 2\pi G\rho m \int_0^R \left(\frac{1}{\sqrt{1 + \frac{H^2}{R^2}}} - \frac{r}{\sqrt{r^2 + H^2}} \right) dr \\
&= 2\pi G\rho m \left(\frac{1}{\sqrt{1 + \frac{H^2}{R^2}}} r - \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} \sqrt{r^2 + H^2} \right) \Big|_0^R = 2\pi G\rho m \left(\frac{R}{\sqrt{1 + \frac{H^2}{R^2}}} - \sqrt{R^2 + H^2} + H \right) \\
&= 2\pi G\rho m \left(H + \frac{R^2 - R^2 - H^2}{\sqrt{R^2 + H^2}} \right) = 2\pi G\rho m \left(H - \frac{H^2}{\sqrt{R^2 + H^2}} \right),
\end{aligned}$$

故该圆锥体对位于其顶点处质量为 m 的质点的引力为 $(0, 0, 2\pi G\rho m(H - \frac{H^2}{\sqrt{R^2 + H^2}}))$.

(2) 该圆锥体关于它的中心轴的转动惯量

$$\begin{aligned}
J_z &= \iiint_{\Omega} (x^2 + y^2) \rho dx dy dz = \int_0^{2\pi} d\theta \int_0^R dr \int_{\frac{H}{R}r}^H r^2 \rho \cdot r dz = 2\pi \rho \int_0^R r^3 (H - \frac{H}{R}r) dr \\
&= 2\pi \rho \left(\frac{1}{4} H r^4 - \frac{1}{5} \frac{H}{R} r^5 \right) \Big|_0^R = 2\pi \rho \left(\frac{1}{4} H R^4 - \frac{1}{5} H R^4 \right) = \frac{1}{10} \pi \rho H R^4.
\end{aligned}$$

4. 计算积分 $\iint_D (x^2 + y^2)^{\frac{1}{2}} dx dy$, 其中 $D = \{(x, y) \mid 0 \leq x, y \leq a\}$.

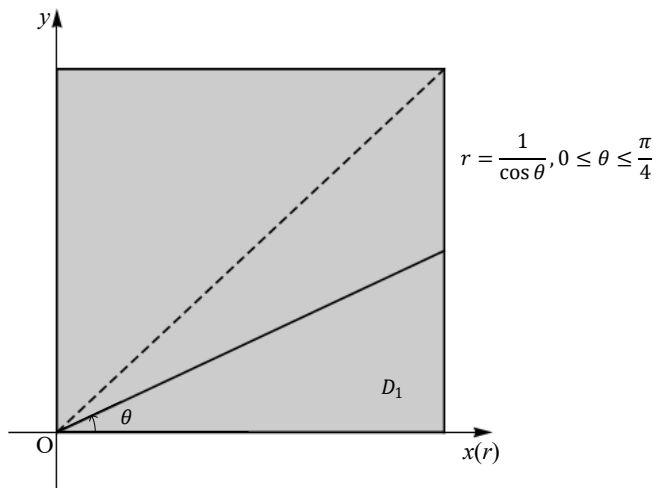


图 2: 第12章补充题 4.题图示

解: 设 $D_1 = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\} = \{(r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq r \leq \frac{a}{\cos \theta}\}$, 根据对称性可知

$$\begin{aligned}
\iint_D (x^2 + y^2)^{\frac{1}{2}} dx dy &= 2 \iint_{D_1} (x^2 + y^2)^{\frac{1}{2}} dx dy = 2 \int_0^{\frac{\pi}{4}} d\theta \int_0^{\frac{a}{\cos \theta}} r \cdot r dr = 2 \int_0^{\frac{\pi}{4}} \frac{1}{3} r^3 \Big|_0^{\frac{a}{\cos \theta}} d\theta \\
&= 2 \int_0^{\frac{\pi}{4}} \frac{1}{3} \frac{a^3}{\cos^3 \theta} d\theta = \frac{2}{3} a^3 \int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta = \frac{2}{3} a^3 \int_0^{\frac{\pi}{4}} \sec \theta d \tan \theta = \frac{2}{3} a^3 (\sec \theta \tan \theta \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan \theta d \sec \theta) \\
&= \frac{2}{3} a^3 (\sqrt{2} - \int_0^{\frac{\pi}{4}} \tan^2 \theta \sec \theta d\theta) = \frac{2}{3} a^3 [\sqrt{2} - \int_0^{\frac{\pi}{4}} (\sec^2 \theta - 1) \sec \theta d\theta] \\
&= \frac{2}{3} a^3 [\sqrt{2} - \int_0^{\frac{\pi}{4}} (\sec^3 \theta - \sec \theta) d\theta] = \frac{2}{3} a^3 (\sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta + \int_0^{\frac{\pi}{4}} \sec \theta d\theta) \\
&= \frac{\frac{2}{3} a^3}{\frac{2}{3} a^3 + \frac{2}{3} a^3} \left(\frac{2}{3} \sqrt{2} a^3 + \frac{2}{3} a^3 \int_0^{\frac{\pi}{4}} \sec \theta d\theta \right) = \frac{1}{2} \left(\frac{2}{3} \sqrt{2} a^3 + \frac{2}{3} a^3 \ln |\tan \theta + \sec \theta| \Big|_0^{\frac{\pi}{4}} \right) \\
&= \frac{1}{3} a^3 [\sqrt{2} + \ln(1 + \sqrt{2})].
\end{aligned}$$

5. 设 $t > 0$, $f(x)$ 在 $[0, 1]$ 上连续, 求证 $\int_0^t dx \int_0^x dy \int_0^y f(x)f(y)f(z) dz = \frac{1}{6} (\int_0^t f(s) ds)^3$.

证明：方法1：设 $F(x) = \int_0^x f(s)ds$ ，则

$$\begin{aligned} \int_0^t dx \int_0^x dy \int_0^y f(x)f(y)f(z)dz &= \int_0^t f(x)dx \int_0^x f(y)dy \int_0^y f(z)dz \\ &= \int_0^t f(x)dx \int_0^x f(y)dy \int_0^y dF(z) = \int_0^t f(x)dx \int_0^x f(y)[F(y) - F(0)]dy \\ &= \int_0^t f(x)dx \int_0^x f(y)F(y)dy = \int_0^t f(x)dx \int_0^x F(y)dF(y) \\ &= \int_0^t f(x)\left\{\frac{1}{2}[F(x)]^2 - [F(0)]^2\right\}dx = \frac{1}{2} \int_0^t [F(x)]^2 dF(x) = \frac{1}{6}[F(x)]^3 \Big|_0^t = \frac{1}{6}[F(t)]^3 \\ &= \frac{1}{6}\left(\int_0^t f(s)ds\right)^3. \end{aligned}$$

方法2：记 $\varphi(y) = (\int_0^y f(z)dz)^2$ ，则 $\varphi'(y) = 2f(y) \int_0^y f(z)dz$. 于是

$$\begin{aligned} \int_0^x dy \int_0^y f(y)f(z)dz &= \int_0^x dy \left[f(y) \int_0^y f(z)dz \right] = \frac{1}{2} \int_0^x \varphi'(y)dy = \frac{1}{2} \int_0^x d\varphi(y) = \frac{1}{2}\varphi(x) \\ &= \frac{1}{2}\left(\int_0^x f(s)ds\right)^2, \end{aligned}$$

$$\begin{aligned} \int_0^t dx \int_0^x dy \int_0^y f(x)f(y)f(z)dz &= \frac{1}{2} \int_0^t dx \left[f(x) \left(\int_0^x f(s)ds\right)^2 \right] = \frac{1}{6} \int_0^t d\left(\int_0^x f(s)ds\right)^3 \\ &= \frac{1}{6}\left(\int_0^t f(s)ds\right)^3. \end{aligned}$$

6. 计算 $\int_L (x^{\frac{4}{3}} + y^{\frac{4}{3}})dl$ ，其中 L 为星形线 $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} (a > 0)$.

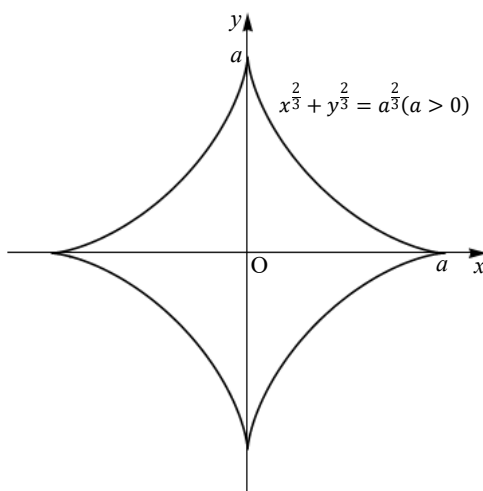


图 3: 第12章补充题 6.题图示

解： L 的参数方程可表示为
$$\begin{cases} x = a \cos^3 \theta, \\ y = a \sin^3 \theta, \end{cases} \quad 0 \leq \theta \leq 2\pi,$$

$$\begin{aligned}
& \therefore \int_L (x^{\frac{4}{3}} + y^{\frac{4}{3}}) dl = \int_0^{2\pi} (a^{\frac{4}{3}} \cos^4 \theta + a^{\frac{4}{3}} \sin^4 \theta) \sqrt{[(a \cos^3 \theta)']^2 + [(a \sin^3 \theta)']^2} d\theta \\
& = \int_0^{2\pi} (a^{\frac{4}{3}} \cos^4 \theta + a^{\frac{4}{3}} \sin^4 \theta) \sqrt{[-3a \cos^2 \theta \sin \theta]^2 + [3a \sin^2 \theta \cos \theta]^2} d\theta \\
& = \int_0^{2\pi} (a^{\frac{4}{3}} \cos^4 \theta + a^{\frac{4}{3}} \sin^4 \theta) \sqrt{9a^2 \cos^2 \theta \sin^2 \theta} d\theta = 4 \int_0^{\frac{\pi}{2}} (a^{\frac{4}{3}} \cos^4 \theta + a^{\frac{4}{3}} \sin^4 \theta) \sqrt{9a^2 \cos^2 \theta \sin^2 \theta} d\theta \\
& = 4 \int_0^{\frac{\pi}{2}} (a^{\frac{4}{3}} \cos^4 \theta + a^{\frac{4}{3}} \sin^4 \theta) 3a \cos \theta \sin \theta d\theta = 12a^{\frac{7}{3}} \int_0^{\frac{\pi}{2}} (\cos^5 \theta \sin \theta + \sin^5 \theta \cos \theta) d\theta \\
& = 12a^{\frac{7}{3}} \left(\int_0^{\frac{\pi}{2}} \cos^5 \theta \sin \theta d\theta + \int_0^{\frac{\pi}{2}} \sin^5 \theta \cos \theta d\theta \right) = 12a^{\frac{7}{3}} \left(-\frac{1}{6} \cos^6 \theta \Big|_0^{\frac{\pi}{2}} + \frac{1}{6} \sin^6 \theta \Big|_0^{\frac{\pi}{2}} \right) \\
& = 4a^{\frac{7}{3}}.
\end{aligned}$$

7. 设 $f(u)$ 是连续函数, $D = \{(x, y) \mid x^2 + y^2 \leq a^2\}$. 试求

$$\iint_D \frac{af(x) + bf(y)}{f(x) + f(y)} dx dy.$$

解: \because 积分域 D 关于 $y = x$ 对称, 且在关于 $y = x$ 的对称点 (x, y) 和 $(x', y') = (y, x)$ 处

$$\frac{f(x')}{f(x') + f(y')} = \frac{f(y)}{f(y) + f(x)},$$

$$\therefore \iint_D \frac{f(x)}{f(x) + f(y)} dx dy = \iint_D \frac{f(y)}{f(x) + f(y)} dx dy = \frac{1}{2} \iint_D \frac{f(x) + f(y)}{f(x) + f(y)} dx dy = \frac{1}{2} \iint_D dx dy = \frac{1}{2} \pi a^2,$$

$$\therefore \iint_D \frac{af(x) + bf(y)}{f(x) + f(y)} dx dy = a \iint_D \frac{f(x)}{f(x) + f(y)} dx dy + b \iint_D \frac{f(y)}{f(x) + f(y)} dx dy = \frac{a+b}{2} \pi a^2.$$

8. 设 $D = \{(x, y) \mid |x| + |y| \leq 1\}$, 将 $\iint_D f(x+y) dx dy$ 化为定积分.

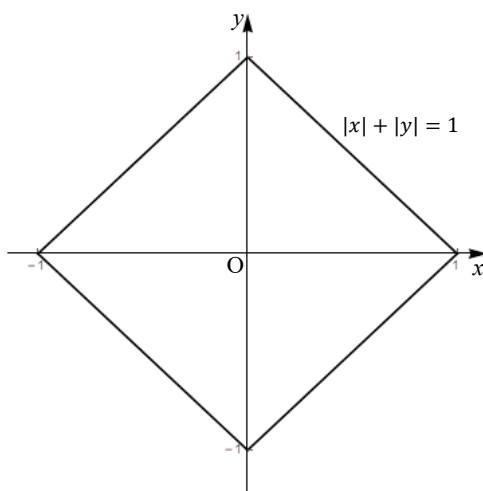


图 4: 第12章补充题 8.题图示

解: 方法1: 令 $\begin{cases} u = x + y, \\ v = x - y, \end{cases}$ 则 $D = \{(u, v) \mid -1 \leq u \leq 1, -1 \leq v \leq 1\}$,

$$\frac{D(u,v)}{D(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2,$$

$$\begin{aligned} \therefore \iint_D f(x+y) dx dy &= \iint_D f(u) \frac{1}{|\frac{D(u,v)}{D(x,y)}|} du dv = \iint_D \frac{1}{2} f(u) du dv = \frac{1}{2} \int_{-1}^1 dv \int_{-1}^1 f(u) du \\ &= \int_{-1}^1 f(u) du. \end{aligned}$$

方法2: 将重积分化为累次积分, 有

$$\iint_{|x|+|y|\leq 1} f(x+y) dx dy = \int_{-1}^0 dx \int_{-1-x}^{1+x} f(x+y) dy + \int_0^1 dx \int_{-1+x}^{1-x} f(x+y) dy,$$

令 $x+y=u$,

$$\begin{aligned} \text{上式} &= \int_{-1}^0 dx \int_{-1}^{1+2x} f(u) du + \int_0^1 x d \int_{2x-1}^1 f(u) du \\ &= \int_{-1}^1 f(u) du \int_{\frac{u-1}{2}}^{\frac{u+1}{2}} dx = \int_{-1}^1 f(u) du. \end{aligned}$$

9. 设一元函数 f 连续, $\Omega_t = \{(x, y, z) \mid x^2 + y^2 \leq t^2, 0 \leq z \leq h\}$. 令 $F(t) = \iiint_{\Omega_t} [z^2 + f(x^2 + y^2)] dV$, 试求 $\frac{dF}{dt}$ 和 $\lim_{t \rightarrow 0} \frac{F(t)}{t^2}$.

$$\begin{aligned} \text{解: } F(t) &= \iiint_{\Omega_t} [z^2 + f(x^2 + y^2)] dx dy dz = \int_0^{2\pi} d\theta \int_0^t dr \int_0^h [z^2 + f(r^2)] r dz \\ &= 2\pi \int_0^t \left[\frac{1}{3} r z^3 + r f(r^2) z \right] \Big|_0^h dr = 2\pi \int_0^t \left[\frac{1}{3} h^3 r + h r f(r^2) \right] dr, \end{aligned}$$

$$\therefore \frac{dF}{dt} = \frac{2\pi}{3} h^3 t + 2\pi h t f(t^2),$$

$$\therefore \lim_{t \rightarrow 0} \frac{1}{t^2} F(t) = \lim_{t \rightarrow 0} \frac{\frac{dF}{dt}}{2t} = \lim_{t \rightarrow 0} \frac{2\pi [\frac{1}{3} h^3 t + h t f(t^2)]}{2t} = \lim_{t \rightarrow 0} \pi [\frac{1}{3} h^3 + h f(t^2)] = \frac{\pi}{3} h^3 + \pi h f(0).$$

10. 计算下列积分:

(1) $\iiint_{\Omega} (ax + by + cz)^2 dV$, 其中 $\Omega = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq R^2\}$;

(2) $\iiint_{\Omega} (\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}) dV$, 其中 $\Omega = \{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\}$.

解: (1) 方法1: 由对称性可知

$$\begin{aligned} \iiint_{\Omega} (ax + by + cz)^2 dV &= \iiint_{\Omega} (a^2 x^2 + b^2 y^2 + c^2 z^2 + 2abxy + 2acxz + 2bcyz) dV \\ &= \iiint_{\Omega} (a^2 x^2 + b^2 y^2 + c^2 z^2) dV = (a^2 + b^2 + c^2) \iiint_{\Omega} z^2 dV \\ &= (a^2 + b^2 + c^2) \iiint_{\Omega} r^2 \cos^2 \varphi \cdot r^2 \sin \varphi d\theta d\varphi dr \\ &= (a^2 + b^2 + c^2) \int_0^{2\pi} d\theta \int_0^R r^4 dr \int_0^{\pi} \cos^2 \varphi \sin \varphi d\varphi \\ &= \frac{2}{5} \pi R^4 (a^2 + b^2 + c^2) \left(-\frac{1}{3} \cos^3 \varphi \right) \Big|_0^{\pi} = \frac{4}{15} \pi R^4 (a^2 + b^2 + c^2). \end{aligned}$$

$$\begin{aligned} \text{方法2: } \iiint_{\Omega} (ax + by + cz)^2 dV &= \iiint_{\Omega} (a^2 x^2 + b^2 y^2 + c^2 z^2 + 2abxy + 2acxz + 2bcyz) dV \\ &= \iiint_{\Omega} (a^2 x^2 + b^2 y^2 + c^2 z^2) dV = (a^2 + b^2 + c^2) \iiint_{\Omega} z^2 dV = \frac{1}{3} (a^2 + b^2 + c^2) \iiint_{\Omega} (x^2 + y^2 + z^2) dV \\ &= \frac{1}{3} (a^2 + b^2 + c^2) \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^R r^2 \cdot r^2 \sin \varphi dr = \frac{2}{3} \pi (a^2 + b^2 + c^2) \int_0^{\pi} \sin \varphi d\varphi \int_0^R r^4 dr \\ &= \frac{2}{3} \pi (a^2 + b^2 + c^2) \left(-\cos \varphi \right) \Big|_0^{\pi} \frac{1}{5} r^5 \Big|_0^R = \frac{4}{15} \pi R^4 (a^2 + b^2 + c^2). \end{aligned}$$

$$(2) \text{ 令 } \begin{cases} x = au, \\ y = bv, \\ z = cw, \end{cases} \quad \text{则 } \Omega = \{(u, v, w) \mid u^2 + v^2 + w^2 \leq 1\},$$

$$\frac{D(x, y, z)}{D(u, v, w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc,$$

$$\begin{aligned} \therefore \iiint_{\Omega} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dV &= \iiint_{\Omega} (u^2 + v^2 + w^2) \left| \frac{D(u, v, w)}{D(x, y, z)} \right| du dv dw = abc \iiint_{\Omega} (u^2 + v^2 + w^2) du dv dw \\ &= abc \frac{4}{15} \pi 1^4 (1^2 + 1^2 + 1^2) = \frac{4}{5} \pi abc. \end{aligned}$$

21.4 习题13.1解答

1. 验证梯度算子 ∇ 的下列性质，其中 α, β 为任意常数， f, g 为任意可微函数：

$$(1) \nabla(\alpha f + \beta g) = \alpha \nabla f + \beta \nabla g;$$

$$(2) \nabla(fg) = g \nabla f + f \nabla g;$$

$$(3) \nabla\left(\frac{f}{g}\right) = \frac{g \nabla f - f \nabla g}{g^2} \text{ (在 } g \text{ 不等于零处成立)}.$$

证明：(1)

$$\begin{aligned} \nabla(\alpha f + \beta g) &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) (\alpha f + \beta g) \\ &= \frac{\partial(\alpha f + \beta g)}{\partial x} \mathbf{i} + \frac{\partial(\alpha f + \beta g)}{\partial y} \mathbf{j} + \frac{\partial(\alpha f + \beta g)}{\partial z} \mathbf{k} \\ &= \left(\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial g}{\partial x} \right) \mathbf{i} + \left(\alpha \frac{\partial f}{\partial y} + \beta \frac{\partial g}{\partial y} \right) \mathbf{j} + \left(\alpha \frac{\partial f}{\partial z} + \beta \frac{\partial g}{\partial z} \right) \mathbf{k} \\ &= \alpha \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) + \beta \left(\frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial g}{\partial z} \mathbf{k} \right) \\ &= \alpha \nabla f + \beta \nabla g. \end{aligned}$$

(2)

$$\begin{aligned} \nabla(fg) &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) (fg) \\ &= \frac{\partial(fg)}{\partial x} \mathbf{i} + \frac{\partial(fg)}{\partial y} \mathbf{j} + \frac{\partial(fg)}{\partial z} \mathbf{k} \\ &= \left(g \frac{\partial f}{\partial x} + f \frac{\partial g}{\partial x} \right) \mathbf{i} + \left(g \frac{\partial f}{\partial y} + f \frac{\partial g}{\partial y} \right) \mathbf{j} + \left(g \frac{\partial f}{\partial z} + f \frac{\partial g}{\partial z} \right) \mathbf{k} \\ &= g \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) + f \left(\frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial g}{\partial z} \mathbf{k} \right) \\ &= g \nabla f + f \nabla g. \end{aligned}$$

(3)

$$\begin{aligned}
\nabla\left(\frac{f}{g}\right) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\left(\frac{f}{g}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{f}{g}\right)\mathbf{i} + \frac{\partial}{\partial y}\left(\frac{f}{g}\right)\mathbf{j} + \frac{\partial}{\partial z}\left(\frac{f}{g}\right)\mathbf{k} \\
&= \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\mathbf{i} + \frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\mathbf{j} + \frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\mathbf{k} \\
&= \frac{g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) - f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right)}{g^2} \\
&= \frac{g\nabla f - f\nabla g}{g^2}.
\end{aligned}$$

2. 验证散度算子的下列性质(其中 f 为函数, \mathbf{u}, \mathbf{v} 是向量场):

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = -\mathbf{u} \cdot \nabla \times \mathbf{v} + \mathbf{v} \cdot \nabla \times \mathbf{u}.$$

证明:

$$\begin{aligned}
 \nabla \cdot (\mathbf{u} \times \mathbf{v}) &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1(x, y, z) & u_2(x, y, z) & u_3(x, y, z) \\ v_1(x, y, z) & v_2(x, y, z) & v_3(x, y, z) \end{vmatrix} \\
 &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \cdot [(u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}] \\
 &= \frac{\partial(u_2 v_3 - u_3 v_2)}{\partial x} + \frac{\partial(u_3 v_1 - u_1 v_3)}{\partial y} + \frac{\partial(u_1 v_2 - u_2 v_1)}{\partial z} \\
 &= \frac{\partial u_2}{\partial x} v_3 + u_2 \frac{\partial v_3}{\partial x} - \frac{\partial u_3}{\partial x} v_2 - u_3 \frac{\partial v_2}{\partial x} \\
 &\quad + \frac{\partial u_3}{\partial y} v_1 + u_3 \frac{\partial v_1}{\partial y} - \frac{\partial u_1}{\partial y} v_3 - u_1 \frac{\partial v_3}{\partial y} \\
 &\quad + \frac{\partial u_1}{\partial z} v_2 + u_1 \frac{\partial v_2}{\partial z} - \frac{\partial u_2}{\partial z} v_1 - u_2 \frac{\partial v_1}{\partial z} \\
 &= u_1 \left(\frac{\partial v_2}{\partial z} - \frac{\partial v_3}{\partial y} \right) + u_2 \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right) + u_3 \left(\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x} \right) \\
 &\quad + v_1 \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \right) + v_2 \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \right) + v_3 \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) \\
 &= -u_1 \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) - u_2 \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) - u_3 \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \\
 &\quad + v_1 \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \right) + v_2 \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \right) + v_3 \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) \\
 &= -(u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}) \cdot \left[\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \mathbf{k} \right] \\
 &\quad + (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \cdot \left[\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) \mathbf{k} \right] \\
 &= -(u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} + (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix} \\
 &= -\mathbf{u} \cdot (\nabla \times \mathbf{v}) + \mathbf{v} \cdot (\nabla \times \mathbf{u})
 \end{aligned}$$

3. 设 $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = \sqrt{x^2 + y^2 + z^2}$:

(1) 设 $f(u)$ 为可微函数, 求 $\nabla f(r)$;

(2) 设 $\mathbf{F} = f(r)\mathbf{r}$, 求证 $\nabla \times \mathbf{F} \equiv \mathbf{0}$. 又问当 f 满足什么条件时, $\nabla \cdot \mathbf{F} = 0$?

解: (1)

$$\begin{aligned}
 \nabla f(r) &= \frac{\partial f(r)}{\partial x} \mathbf{i} + \frac{\partial f(r)}{\partial y} \mathbf{j} + \frac{\partial f(r)}{\partial z} \mathbf{k} \\
 &= f'(r) \frac{\partial r}{\partial x} \mathbf{i} + f'(r) \frac{\partial r}{\partial y} \mathbf{j} + f'(r) \frac{\partial r}{\partial z} \mathbf{k} = f'(r) \left(\frac{\partial r}{\partial x} \mathbf{i} + \frac{\partial r}{\partial y} \mathbf{j} + \frac{\partial r}{\partial z} \mathbf{k} \right) \\
 &= f'(r) \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \mathbf{k} \right) \\
 &= \frac{f'(r)}{\sqrt{x^2 + y^2 + z^2}} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{f'(r)}{r} \mathbf{r}.
 \end{aligned}$$

(2)

i) 证明:

$$\begin{aligned}
 \nabla \times \mathbf{F} &= \nabla \times [f(r)\mathbf{r}] = \nabla f(r) \times \mathbf{r} + f(r) \nabla \times \mathbf{r} \\
 &= \frac{f'(r)}{r} \mathbf{r} \times \mathbf{r} + f(r) \nabla \times \mathbf{r} \\
 &= 0 + f(r) \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
 &= f(r) \left[\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \mathbf{k} \right] \\
 &= f(r) (0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) = 0.
 \end{aligned}$$

ii)

\therefore

$$\begin{aligned}
 \nabla \cdot \mathbf{F} &= \nabla \cdot [f(r)\mathbf{r}] = \nabla f(r) \cdot \mathbf{r} + f(r) \nabla \cdot \mathbf{r} = \frac{f'(r)}{r} \mathbf{r} \cdot \mathbf{r} + f(r) \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) \\
 &= \frac{df(r)}{dr} r + 3f(r) = 0,
 \end{aligned}$$

\therefore 当 $f(r) \neq 0$ 时, $r \neq 0$,

$$\frac{df(r)}{f(r)} = -3 \frac{dr}{r},$$

\therefore

$$\ln |f(r)| = -3 \ln |r| + C_1,$$

\therefore

$$f(r) = \frac{C}{r^3}, C \text{ 为任意常数.}$$

4. 验证旋度算子的下列基本公式:

$$(1) \nabla \times (\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha \nabla \times \mathbf{u} + \beta \nabla \times \mathbf{v};$$

$$(2) \nabla \times (\nabla f) = \mathbf{0};$$

$$(3) \nabla \cdot (\nabla \times \mathbf{v}) = 0.$$

证明: (1)

$$\begin{aligned} \nabla \times (\alpha \mathbf{u} + \beta \mathbf{v}) &= \nabla \times (\alpha u_1 \mathbf{i} + \alpha u_2 \mathbf{j} + \alpha u_3 \mathbf{k} + \beta v_1 \mathbf{i} + \beta v_2 \mathbf{j} + \beta v_3 \mathbf{k}) \\ &= \nabla \times [(\alpha u_1 + \beta v_1) \mathbf{i} + (\alpha u_2 + \beta v_2) \mathbf{j} + (\alpha u_3 + \beta v_3) \mathbf{k}] \\ &= \frac{\partial(\alpha u_1 + \beta v_1)}{\partial x} + \frac{\partial(\alpha u_2 + \beta v_2)}{\partial y} + \frac{\partial(\alpha u_3 + \beta v_3)}{\partial z} \\ &= \alpha \frac{\partial u_1}{\partial x} + \beta \frac{\partial v_1}{\partial x} + \alpha \frac{\partial u_2}{\partial y} + \beta \frac{\partial v_2}{\partial y} + \alpha \frac{\partial u_3}{\partial z} + \beta \frac{\partial v_3}{\partial z} \\ &= \alpha \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \right) + \beta \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) \\ &= \alpha \nabla \cdot \mathbf{u} + \beta \nabla \cdot \mathbf{v}. \end{aligned}$$

(2) 当 $f \in C^2$ 时

$$\begin{aligned} \nabla \times (\nabla f) &= \nabla \times \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k} \\ &= 0 \mathbf{i} + 0 \mathbf{j} + 0 \mathbf{k} = \mathbf{0}. \end{aligned}$$

(3) 当 $\mathbf{v} \in C^2$ 时

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{v}) &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1(x, y, z) & v_2(x, y, z) & v_3(x, y, z) \end{vmatrix} \\ &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \cdot \left[\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \mathbf{k} \right] \\ &= \left(\frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} \right) + \left(\frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} \right) + \left(\frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y} \right) \\ &= \left(\frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_3}{\partial y \partial x} \right) + \left(\frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_1}{\partial z \partial y} \right) + \left(\frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_2}{\partial x \partial z} \right) \\ &= 0. \end{aligned}$$

5. 求下列向量场的散度:

$$(1) \mathbf{v} = xyz(\mathbf{i} + \mathbf{j} + \mathbf{k});$$

$$(2) \mathbf{v} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times \mathbf{c};$$

$$(3) \mathbf{v} = [(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{c}](x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \text{ (其中 } \mathbf{c} \text{ 为常值向量)}.$$

解: (1)

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \nabla(xyz) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) + xyz \nabla \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= (yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) + xyz\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) \\ &= xyz + xyz + xyz + 3xyz = 6xyz.\end{aligned}$$

(2)

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \mathbf{c} \cdot [\nabla \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})] - (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (\nabla \times \mathbf{c}) \\ &= \mathbf{c} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} - (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{0} \\ &= \mathbf{c} \cdot \left[\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\mathbf{k}\right] - 0 \\ &= \mathbf{c} \cdot \mathbf{0} - 0 = 0.\end{aligned}$$

(3) 记 $\mathbf{c} = (c_1, c_2, c_3)$

$$\begin{aligned}\nabla \cdot \mathbf{v} &= [(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{c}] \nabla \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) + (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \nabla [(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{c}] \\ &= [(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{c}] \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) + (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \nabla (c_1x + c_2y + c_3z) \\ &= 3[(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{c}] + (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}) \\ &= 3[(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{c}] + (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{c} \\ &= 4[(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{c}].\end{aligned}$$

6. 求下列向量场的旋度:

(1) $\mathbf{v} = y^2z\mathbf{i} + z^2x\mathbf{j} + x^2y\mathbf{k}$;

(2) $\mathbf{v} = f(\sqrt{x^2 + y^2 + z^2})\mathbf{c}$ (其中 \mathbf{c} 为常值向量).

解: (1)

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z & z^2x & x^2y \end{vmatrix} = (x^2 - 2xz)\mathbf{i} + (y^2 - 2xy)\mathbf{j} + (z^2 - 2yz)\mathbf{k}.$$

(2)

$$\begin{aligned}\nabla \times \mathbf{v} &= \nabla \times [f(\sqrt{x^2 + y^2 + z^2})\mathbf{c}] \\ &= \nabla f(\sqrt{x^2 + y^2 + z^2}) \times \mathbf{c} + f(\sqrt{x^2 + y^2 + z^2}) \nabla \times \mathbf{c} \\ &= f'(\sqrt{x^2 + y^2 + z^2}) \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k} \right) \times \mathbf{c} + \mathbf{0} \\ &= \frac{f'(\sqrt{x^2 + y^2 + z^2})}{\sqrt{x^2 + y^2 + z^2}} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times \mathbf{c}.\end{aligned}$$