16 泰勒公式、几何应用

16.1 知识结构

第10章多元函数微分学

10.6 二元函数的泰勒公式

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10.6.2 二元函数的泰勒公式

第11章多元函数微分学的应用

11.1 向量值函数的导数和积分

11.1.1 向量值函数

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16.2 习题10.6解答

1. 写出$f(x, y) = x^y$在点$(1, 1)$带佩亚诺余项的三阶泰勒公式，由此计算$1.1^{1.02}$.

   解：$f(1, 1) = 1$,

   $\frac{\partial f}{\partial x} = yx^{y-1}, \frac{\partial f}{\partial y} = x^y \ln x$,

   $\frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2}, \frac{\partial^2 f}{\partial x \partial y} = x^{y-1} + yx^{y-1}\ln x, \frac{\partial^2 f}{\partial y^2} = x^y(\ln x)^2$,

   $\frac{\partial^3 f}{\partial x^3} = y(y-1)(y-2)x^{y-3}, \frac{\partial^3 f}{\partial x \partial y^2} = yx^{y-1}(\ln x)^2$,

   $\frac{\partial^3}{\partial y \partial x^2} = (2y-1)x^{y-2} + y(y-1)x^{y-2}\ln x = [(2y-1) + y(y-1)\ln x]x^{y-2}$,

   $\frac{\partial^3 f}{\partial y^3} = x^y(\ln x)^3$,

$\therefore$

$$
\begin{aligned}
&f(x,y)\\
=&f(1,1)+[\frac{\partial f(1,1)}{\partial x}(x-1)+\frac{\partial f(1,1)}{\partial y}(y-1)]\\
&+\frac{1}{2}[\frac{\partial^2 f(1,1)}{\partial x^2}(x-1)^2+2\frac{\partial^2 f(1,1)}{\partial x\partial y}(x-1)(y-1)+\frac{\partial^2 f(1,1)}{\partial y^2}(y-1)^2]\\
&+\frac{1}{3}[\frac{\partial^3 f(1,1)}{\partial x^3}(x-1)^3+3\frac{\partial^3 f(1,1)}{\partial x\partial y^2}(x-1)(y-1)^2+3\frac{\partial^3 f(1,1)}{\partial y\partial x^2}(x-1)^2(y-1)\\
&\qquad+\frac{\partial^3 f(1,1)}{\partial y^3}(y-1)^3]+o[(\sqrt{(x-1)^2+(y-1)^2})^3]\\
=&1+(x-1)+\frac{1}{2}[2(x-1)(y-1)]+\frac{1}{3}[3(x-1)^2(y-1)]+o[(\sqrt{(x-1)^2+(y-1)^2})^3]\\
=&x+(x-1)(y-1)+(x-1)^2(y-1)+o[(\sqrt{(x-1)^2+(y-1)^2})^3]
\end{aligned}
$$

$\therefore 1.1^{1.02}=f(1.1,1.02)\approx 1+0.1+0.1\times 0.02+0.1^2\times 0.02=1.1022.$

2. 证明当$|x|,|y|$充分小时，有$\frac{\cos x}{\cos y}\approx 1-\frac{1}{2}(x^2-y^2)$.

证明：记$f(x,y)=\frac{\cos x}{\cos y}$,

$f(0,0)=1,$

$$
\begin{aligned}
&\frac{\partial f}{\partial x}=-\frac{\sin x}{\cos y},\frac{\partial f}{\partial y}=\frac{\cos x\sin y}{\cos^2 y},\\
&\frac{\partial^2 f}{\partial x^2}=-\frac{\cos x}{\cos y},\frac{\partial^2 f}{\partial x\partial y}=-\frac{\sin x\sin y}{\cos^2 y},\\
&\frac{\partial^2 f}{\partial y^2}=\frac{\cos x\cos y\cos^2 y+\cos x\sin y 2\cos y\sin y}{\cos^4 y}=\frac{\cos x\cos^2 y+2\cos x\sin^2 y}{\cos^4 y},
\end{aligned}
$$

$\therefore$

$$
\begin{aligned}
f(x,y)=&f(0,0)+[\frac{\partial f(0,0)}{\partial x}x+\frac{\partial f(0,0)}{\partial y}y]+\frac{1}{2}[\frac{\partial^2 f(0,0)}{\partial x^2}x^2+2\frac{\partial^2 f(0,0)}{\partial x\partial y}xy+\frac{\partial^2 f(0,0)}{\partial y^2}y^2]\\
&+o[(\sqrt{x^2+y^2})^2]\\
=&1+(0+0)+\frac{1}{2}(-x^2+0+y^2)+o[(\sqrt{x^2+y^2})^2]\\
=&1-\frac{1}{2}(x^2-y^2)+o[(\sqrt{x^2+y^2})^2],
\end{aligned}
$$

$\therefore$当$|x|,|y|$充分小时，有$\frac{\cos x}{\cos y}\approx 1-\frac{1}{2}(x^2-y^2)$.

3. 写出$f(x,y)=\sqrt{1+y^2}\cos x$在点$(0,1)$的一阶泰勒多项式及拉格朗日余项.

解：$f(0,1)=\sqrt{2}$,

$$\frac{\partial f}{\partial x}=-\sqrt{1+y^2}\sin x,\frac{\partial f}{\partial y}=\frac{2y\cos x}{2\sqrt{1+y^2}}=\frac{y\cos x}{\sqrt{1+y^2}},$$

$$\frac{\partial^2 f}{\partial x^2} = -\sqrt{1+y^2}\cos x, \frac{\partial^2 f}{\partial x \partial y} = \frac{-2y\sin x}{2\sqrt{1+y^2}} = \frac{-y\sin x}{\sqrt{1+y^2}},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\cos x\sqrt{1+y^2} - y\cos x\frac{2y}{2\sqrt{1+y^2}}}{1+y^2} = \frac{\cos x}{(1+y^2)^{\frac{3}{2}}},$$

$\therefore$

$$\begin{aligned}
f(x,y) =& f(0,1) + [\frac{\partial f(0,1)}{\partial x}x + \frac{\partial f}{\partial y}(y-1)] \\
&+ \frac{1}{2}[\frac{\partial^2 f(\theta x, 1+\theta(y-1))}{\partial x^2}x^2 + 2\frac{\partial^2 f(\theta x, 1+\theta(y-1))}{\partial x \partial y}x(y-1) \\
&\qquad + \frac{\partial^2 f(\theta x, 1+\theta(y-1))}{\partial y^2}(y-1)^2] \\
=& \sqrt{2} + \frac{\sqrt{2}}{2}(y-1) \\
&+ \frac{1}{2}\Big(-\sqrt{1+[1+\theta(y-1)]^2}\cos(\theta x)x^2 - \frac{2[1+\theta(y-1)]\sin\theta x}{\sqrt{1+[1+\theta(y-1)]^2}}x(y-1) \\
&\qquad + \frac{\cos\theta x}{\{1+[1+\theta(y-1)]^2\}^{\frac{3}{2}}}(y-1)^2\Big) \\
=& \sqrt{2} + \frac{\sqrt{2}}{2}(y-1) \\
&+ \frac{1}{2}\Big\{ -x^2\sqrt{1+(1+\theta(y-1))^2}\cos\theta x - 2x(y-1)\frac{1+\theta(y-1)}{\sqrt{1+(1+\theta(y-1))^2}}\sin\theta x \\
&\qquad + (y-1)^2\frac{\cos\theta x}{\{1+(1+\theta(y-1))^2\}^{\frac{3}{2}}}\Big\}, 0<\theta<1.
\end{aligned}$$

16.3 第10章补充题

1. 设$f(x,y)$是定义在整个平面上的连续函数，当$x^2+y^2\to+\infty$时，$f(x,y)\to+\infty$. 求证存在(x_0,y_0)，使

$$f(x_0,y_0) = \min\left\{f(x,y) \mid (x,y)\in\mathbb{R}^2\right\}.$$

证明：$\because$当$x^2+y^2\to+\infty$时，$f(x,y)\to+\infty$,

$\therefore$对于$f(0,0), \exists N>0, s.t. f(x,y)>f(0,0), x^2+y^2>N^2$,

$\because$在有界闭区域$D=\{(x,y)\mid x^2+y^2\le N^2\}$内部$f(x,y)$连续，

$\therefore \exists(x_0,y_0)\in D, s.t. f(x_0,y_0)\le f(x,y), (x,y)\in D$，此时$f(x_0,y_0)\le f(0,0)$，

$\therefore f(x_0,y_0)\le f(0,0)<f(x,y), x^2+y^2>N^2$,

$\therefore f(x_0,y_0)\le f(x,y), (x,y)\in\mathbb{R}^2$,

$\therefore$存在(x_0,y_0)，使$f(x_0,y_0)=\min\{f(x,y)\mid(x,y)\in\mathbb{R}^2\}$.

2. 设$f(x,y)$是定义在整个平面上的连续函数，$f(0,0)=0$，且当$(x,y)\neq(0,0)$时，$f(x,y)>0$，又设对于任意的(x,y)和任意实数c，都有

$$f(cx,cy)=c^2f(x,y).$$

求证存在正数a,b，使得对于任意的(x,y)，都有

$$a(x^2+y^2)\leq f(x,y)\leq b(x^2+y^2).$$

证明：$\because f(x,y)$是定义在整个平面上的连续函数，

$\therefore f(x,y)$在有界闭区域$D=\{(x,y)\mid 0.5<x^2+y^2\leq 1.5\}$上连续，

$\because$当$(x,y)\neq(0,0)$时，$f(x,y)>0$，

$\therefore \exists b\geq a>0, s.t. a\leq f(x,y)\leq b, (x,y)\in D,$

$\therefore$当$(x,y)\in D^*=\{(x,y)\mid x^2+y^2=1\}\subset D$时，$a\leq f(x,y)\leq b,$

$\because f(x,y)=f(\sqrt{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}},\sqrt{x^2+y^2}\frac{y}{x^2+y^2})=(x^2+y^2)f(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}),$

又$\because a\leq f(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}})\leq b,$

$\therefore a(x^2+y^2)\leq f(x,y)\leq b(x^2+y^2).$

3. 若对于任意实数t，函数$f(x,y,z)$满足$f(tx,ty,tz)=t^kf(x,y,z)$，则称$f(x,y,z)$为k次齐次函数. 试证k次齐次函数$f(x,y,z)$满足方程

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=kf(x,y,z).$$

证明：方法1：等式$f(tx,ty,tz)=t^kf(x,y,z)$两边分别对t求偏导[1]得

$$x\frac{\partial f(tx,ty,tz)}{\partial x}+y\frac{\partial f(tx,ty,tz)}{\partial y}+z\frac{\partial f(tx,ty,tz)}{\partial z}=kt^{k-1}f(x,y,z),$$

令$t=1$得

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=kf(x,y,z).$$

方法2：$\because f(tx,ty,tz)=t^kf(x,y,z),$

$\therefore$

$$\frac{\partial f(x,y,z)}{\partial x}=\frac{\partial}{\partial x}\frac{f(tx,ty,tz)}{t^k}=\frac{1}{t^k}\frac{\partial f(tx,ty,tz)}{\partial x}t=\frac{1}{t^{k-1}}\frac{\partial f(tx,ty,tz)}{\partial x}, \tag{1a}$$

$$\frac{\partial f(x,y,z)}{\partial y}=\frac{\partial}{\partial y}\frac{f(tx,ty,tz)}{t^k}=\frac{1}{t^k}\frac{\partial f(tx,ty,tz)}{\partial y}t=\frac{1}{t^{k-1}}\frac{\partial f(tx,ty,tz)}{\partial y}, \tag{1b}$$

[1]这里应是偏导。

$$\frac{\partial f(x,y,z)}{\partial z}=\frac{\partial}{\partial z}\frac{f(tx,ty,tz)}{t^k}=\frac{1}{t^k}\frac{\partial f(tx,ty,tz)}{\partial z}t=\frac{1}{t^{k-1}}\frac{\partial f(tx,ty,tz)}{\partial z},\tag{1c}$$

方程$f(tx,ty,tz)=t^kf(x,y,z)$两边分别对t求导

$$x\frac{\partial f(tx,ty,tz)}{\partial x}+y\frac{\partial f(tx,ty,tz)}{\partial y}+z\frac{\partial f(tx,ty,tz)}{\partial z}=kt^{k-1}f(x,y,z),$$

将式 (1a)-(1c)代入上式

$$xt^{k-1}\frac{\partial f(x,y,z)}{\partial x}+yt^{k-1}\frac{\partial f(x,y,z)}{\partial y}+zt^{k-1}\frac{\partial f(x,y,z)}{\partial z}=kt^{k-1}f(x,y,z),$$

即

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=kf(x,y,z).$$

4. 设F为三元可微函数，$u=u(x,y,z)$是由方程$F(u^2-x^2,u^2-y^2,u^2-z^2)=0$确定的隐函数. 求证

$$\frac{u'_x}{x}+\frac{u'_y}{y}+\frac{u'_z}{z}=\frac{1}{u}.$$

证明：方程$F(u^2-x^2,u^2-y^2,u^2-z^2)=0$两边分别对$x$求偏导

$$F'_1(2u\frac{\partial u}{\partial x}-2x)+F'_22u\frac{\partial u}{\partial x}+F'_32u\frac{\partial u}{\partial x}=0$$

得

$$\frac{\partial u}{\partial x}=\frac{xF'_1}{u(F'_1+F'_2+F'_3)}$$

同理

$$\frac{\partial u}{\partial y}=\frac{yF'_2}{u(F'_1+F'_2+F'_3)},$$
$$\frac{\partial u}{\partial z}=\frac{zF'_3}{u(F'_1+F'_2+F'_3)},$$

$\therefore$

$$\frac{u'_x}{x}+\frac{u'_y}{y}+\frac{u'_z}{z}=\frac{F'_1+F'_2+F'_3}{u(F'_1+F'_2+F'_3)}=\frac{1}{u}.$$

5. 求方程$\frac{\partial^2 z}{\partial x\partial y}=x+y$满足条件$z(x,0)=x,z(0,y)=y^2$的解$z(z,y)$.

解：方法1：$\because \frac{\partial^2 z}{\partial x\partial y}=x+y$,

$\therefore \frac{\partial z}{\partial y}=\int_0^x(x+y)\mathrm{d}x+\varphi_0(y)=\frac{x^2}{2}+xy+\varphi_0(y)$,

$\because z(0,y)=y^2$,

$\therefore \frac{z(0,y)}{\partial y}=2y=\varphi_0(y)$,

$\therefore z(x,y)=\int_0^y[\frac{x^2}{2}+xy+2y]\mathrm{d}y+\psi(x)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+\psi(x)$,

$\because z(x,0)=x=\psi(x)$,

$\therefore z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x.$

方法2：$\because \frac{\partial^2 z}{\partial x\partial y}=x+y$,

$\therefore \int \frac{\partial^2 z}{\partial x\partial y}\mathrm{d}x=\int(x+y)\mathrm{d}x=\frac{1}{2}x^2+xy+C_1(y)=\frac{\partial z}{\partial y}+C_1(y)$,

$\therefore$可设$\frac{\partial z}{\partial y}=\frac{1}{2}x^2+xy+C(y)$，其中$C(y)$是与$x$无关的$y$的函数，

$\because z(0,y)=y^2$,

$\therefore \frac{\partial z(0,y)}{\partial y}=2y=C(y)$,

$\therefore \frac{\partial z}{\partial y}=\frac{1}{2}x^2+xy+2y$,

$\therefore \int \frac{\partial z}{\partial y}\mathrm{d}y=\int(\frac{1}{2}x^2+xy+2y)\mathrm{d}y=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+C_3(x)=z(x,y)+C_4(x)$,

$\therefore$可设$z(x,y)=\int(\frac{1}{2}x^2+xy+2y)\mathrm{d}y+C^*(x)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+C^*(x)$，其中$C^*(x)$是与$y$无关的$x$的函数，

$\because z(x,0)=x=C^*(x)$,

$\therefore z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x.$

方法3：$\because \frac{\partial^2 z}{\partial x\partial y}=x+y$,

$\therefore \int \frac{\partial^2 z}{\partial x\partial y}\mathrm{d}x=\int(x+y)\mathrm{d}x=\frac{1}{2}x^2+xy+C_1(y)=\frac{\partial z}{\partial y}+C_2(y)$,

$\therefore$可设$\frac{\partial z}{\partial y}=\frac{1}{2}x^2+xy+C(y)$，其中$C(y)$是与$x$无关的$y$的函数，

$\therefore \int \frac{\partial z}{\partial y}\mathrm{d}y=\frac{1}{2}x^2y+\frac{1}{2}xy^2+\int C(y)\mathrm{d}y=z(x,y)+C_3(x)$,

$\therefore$可设$z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+F(y)+C^*(x)$，其中$F(y)$是$C(y)$的一个与$x$无关的原函数，$C^*(x)$是与$y$无关的$x$的函数，

$\because z(0,y)=y^2, z(x,0)=x$,

$\therefore F(y)+C^*(0)=y^2, F(0)+C^*(x)=x$,

$\therefore F(y)=y^2-C^*(0), C^*(x)=x-F(0)$，且$F(0)+C^*(0)=0$,

$\therefore z(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x-[F(0)+C^*(0)]=\frac{1}{2}x^2y+\frac{1}{2}xy^2+y^2+x.$

6. 设$z=f(x,y)$处处可微，a,b不全等于零. 求证满足方程$bz'_x=az'_y$的充分条件是存在一元函数$g(u)$，使得$z=f(x,y)=g(ax+by)$.

证明：$\because$存在一元函数$g(u)$，使得$z=f(x,y)=g(ax+by)$,

$\therefore$对于任意常数C，z在直线$ax+by=C$上恒等于常数，

$\therefore z=f(x,y)$在直线$ax+by=C$上任意一点处沿该直线方向的方向导数均等于零，[2]

[2]这里之前的版本中是"$\therefore z$在该直线方向的方向导数恒等于零，"，不准确。

由a,b不全为零知直线$ax+by=C$的方向向量可表示为$(-b,a)$，[3]

又$\because z=f(x,y)$处处可微，[4]

$\therefore z=f(x,y)$在直线$ax+by=C$上的每一点处沿$(-b,a)$方向的方向导数[5]

$$\frac{\partial z}{\partial \boldsymbol{l}}=\mathrm{grad}z\cdot\frac{1}{a^2+b^2}(-b,a)=(z_x',z_y')\cdot\frac{1}{a^2+b^2}(-b,a)=\frac{1}{\sqrt{a^2+b^2}}(-bz_x'+az_y')=0,$$

$\therefore$在直线$ax+by=C$上的每一点处$bz_x'=az_y'$，由C的任意性知$bz_x'=az_y'$处处成立.[6]

7. 设D为包含原点$O(0,0)$的一个圆域. $f(x,y)$在D中处处有连续偏导数，并且满足$xf_x'+yf_y'=0$. 求证$f(x,y)$在D中恒等于某个常数.

   证明：$\because f(x,y)$在D中处处有连续偏导数，

   $\therefore f(x,y)$在D中处处可微，

   $\therefore f(x,y)$在点$(x,y)\in D((x,y)\neq(0,0))$处由原点$(0,0)$指向点$(x,y)$方向的方向导数[7]

$$\frac{\partial f(x,y)}{\partial \boldsymbol{v}}=\mathrm{grad}f(x,y)\cdot\frac{1}{\sqrt{x^2+y^2}}(x,y)=\frac{1}{\sqrt{x^2+y^2}}(f_x',f_y')\cdot(x,y)=\frac{xf_x'+yf_y'}{\sqrt{x^2+y^2}}=0,$$

   $\therefore f(x,y)$在D中由原点出发且不含原点的每一条射线上任一点处沿该射线方向的方向导数均为0,[8]

   $\therefore f(x,y)$在D中由原点出发且不含原点的每一条射线上均为常数，[9]

   $\because f(x,y)$在D中处处可微，故处处连续，故在原点$(0,0)$处连续，[10]

   $\therefore f(x,y)$在在D中由原点出发的每一条射线上均等于$f(0,0)$，[11]

   $\therefore f(x,y)=f(0,0),(x,y)\in D.$

16.4 习题11.1解答

1. 设$\boldsymbol{u}(t),\boldsymbol{v}(t)$是可导的向量值函数，$\lambda(t)$为可导数值函数，求证：
   (1)$\frac{\mathrm{d}}{\mathrm{d}t}(\lambda(t)\boldsymbol{u}(t))=\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}\boldsymbol{u}(t)+\lambda(t)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t)$;
   (2)$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}(t)\cdot\boldsymbol{v}(t))=(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t))\cdot\boldsymbol{v}(t)+\boldsymbol{u}(t)\cdot(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(t))$.

[3]这里之前的版本中是“该直线的方向向量为$(-b,a)$。

[4]这里加上了这句。

[5]这里之前的版本中是“$\therefore z$在该直线方向的方向导数,”，不准确。

[6]这里之前的版本中是“$\therefore bz_x'=az_y'$.”，不准确。

[7]这里之前的版本中是“$\therefore f(x,y)$在点$(x,y)\in D((x,y)\neq(0,0))$处的方向导数”，表述不很清楚。

[8]这里在第一版的基础上增加了这句。

[9]这里之前的版本中是“$\therefore f(x,y)$在经过原点且不包括原点的直线上恒为常数,”，不准确。

[10]这里加上了“故在原点$(0,0)$处连续,”。

[11]这里新加上了这句。

证明：(1)设$\boldsymbol{u}(t)=(u_1(t),u_2(t),u_3(t))$，则$\lambda(t)\boldsymbol{u}(t)=(\lambda(t)u_1(t),\lambda(t)u_2(t),\lambda(t)u_3(t))$，

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\lambda(t)\boldsymbol{u}(t))&=(\mathrm{d}t[\lambda(t)u_1(t)]',[\lambda(t)u_2(t)]',[\lambda(t)u_3(t)]')\\&=(\lambda'(t)u_1(t)+\lambda(t)u_1'(t),\lambda'(t)u_2(t)+\lambda(t)u_2'(t),\lambda'(t)u_3(t)+\lambda(t)u_3'(t))\\&=(\lambda'(t)u_1(t),\lambda'(t)u_2(t),\lambda'(t)u_3(t))+(\lambda(t)u_1'(t),\lambda(t)u_2'(t),\lambda(t)u_3'(t))\\&=\lambda'(t)(u_1(t),u_2(t),u_3(t))+\lambda(t)(u_1'(t),u_2'(t),u_3'(t))\\&=\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}\boldsymbol{u}(t)+\lambda(t)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t).\end{aligned}$$

(2)设$\boldsymbol{u}(t)=(u_1(t),u_2(t),u_3(t)),\boldsymbol{v}(t)=(v_1(t),v_2(t),v_3(t))$，

则$\boldsymbol{u}(t)\cdot\boldsymbol{v}(t)=u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t)$，

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}(t)\cdot\boldsymbol{v}(t))&=\frac{\mathrm{d}}{\mathrm{d}t}[u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t)]\\&=u_1'(t)v_1(t)+u_1(t)v_1'(t)+u_2'(t)v_2(t)+u_2(t)v_2'(t)+u_3'(t)v_3(t)+u_3(t)v_3'(t)\\&=u_1'(t)v_1(t)+u_2'(t)v_2(t)+u_3'(t)v_3(t)+u_1(t)v_1'(t)+u_2(t)v_2'(t)+u_3(t)v_3'(t)\\&=(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t))\cdot\boldsymbol{v}(t)+\boldsymbol{u}(t)\cdot(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(t)).\end{aligned}$$

2. 求下列曲线在指定点的单位切向量：

(1)$\boldsymbol{r}(t)=(\mathrm{e}^{2t},\mathrm{e}^{-2t},t\mathrm{e}^{2t}),t=0$；

(2)$\boldsymbol{r}(t)=t\boldsymbol{i}+2\sin t\boldsymbol{j}+3\cos t\boldsymbol{k},t=\frac{\pi}{6}$.

解：(1)单位切向量$\boldsymbol{t}=\frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}=\frac{(2\mathrm{e}^{2t},-2\mathrm{e}^{-2t},(1+2t)\mathrm{e}^{2t})}{\|(2\mathrm{e}^{2t},-2\mathrm{e}^{-2t},(1+2t)\mathrm{e}^{2t})\|}\Big|_{t=0}=\frac{(2,-2,1)}{\sqrt{4+4+1}}=(\frac{2}{3},-\frac{2}{3},\frac{1}{3})$.

(2)单位切向量$\boldsymbol{t}=\frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}=\frac{\boldsymbol{i}+2\cos t\boldsymbol{j}-3\sin t\boldsymbol{k}}{\|\boldsymbol{i}+2\cos t\boldsymbol{j}-3\sin t\boldsymbol{k}\|}\Big|_{t=\frac{\pi}{6}}=\frac{\boldsymbol{i}+\sqrt{3}\boldsymbol{j}-\frac{3}{2}\boldsymbol{k}}{\sqrt{1+3+\frac{9}{4}}}=\frac{2}{5}\boldsymbol{i}+\frac{2\sqrt{3}}{5}\boldsymbol{j}-\frac{3}{5}\boldsymbol{k}$.

3. 求下列曲线在指定点的切线方程：

(1)$\boldsymbol{r}(t)=(1+2t,1+t-t^2,1-t+t^2-t^3),M(1,1,1)$；

(2)$\boldsymbol{r}(t)=\sin(\pi t)\boldsymbol{i}+\sqrt{t}\boldsymbol{j}+\cos(\pi t)\boldsymbol{k},M(0,1,-1)$.

解：(1)在$M(1,1,1)$点处$t=0$，切向量$\boldsymbol{t}=\boldsymbol{r}'(0)=(2,1-2t,-1+2t-3t^2)_{t=0}=(2,1,-1)$，则切线方程为

$$\frac{x-1}{2}=y-1=-(z-1).$$

(2)在$M(0,1,-1)$点处$t=1$，切向量$\boldsymbol{t}=\boldsymbol{r}'(1)=\pi\cos(\pi t)\boldsymbol{i}+\frac{1}{2\sqrt{t}}\boldsymbol{j}-\pi\sin(\pi t)\boldsymbol{k}\big|_{t=1}=-\pi\boldsymbol{i}+\frac{1}{2}\boldsymbol{j}$，则切线方程为$\begin{cases}\frac{x}{-\pi}=2(y-1),\\z=-1,\end{cases}$ 即$\begin{cases}x+2\pi y=2\pi,\\z=-1.\end{cases}$

4. 求下列向量值函数的积分：

(1)$\int_0^{\frac{\pi}{4}}[\cos(2t)\boldsymbol{i}+\sin(2t)\boldsymbol{j}+t\sin t\boldsymbol{k}]\mathrm{d}t$；

(2)$\int_1^4(\sqrt{t}\boldsymbol{i}+t\mathrm{e}^{-t}\boldsymbol{j}+\frac{1}{t^2}\boldsymbol{k})\mathrm{d}t$.

解：(1)$\int_0^{\frac{\pi}{4}}[\cos(2t)\boldsymbol{i}+\sin(2t)\boldsymbol{j}+t\sin t\boldsymbol{k}]\mathrm{d}t=\int_0^{\frac{\pi}{4}}\cos(2t)\mathrm{d}t\boldsymbol{i}+\int_0^{\frac{\pi}{4}}\sin(2t)\mathrm{d}t\boldsymbol{j}+\int_0^{\frac{\pi}{4}}t\sin t\mathrm{d}t\boldsymbol{k}$,

$\because\int_0^{\frac{\pi}{4}}\cos(2t)\mathrm{d}t=\frac{1}{2}\sin(2t)\Big|_0^{\frac{\pi}{4}}=\frac{1}{2},\ \int_0^{\frac{\pi}{4}}\sin(2t)\mathrm{d}t=-\frac{1}{2}\cos(2t)\Big|_0^{\frac{\pi}{4}}=\frac{1}{2}$,

$\int_0^{\frac{\pi}{4}}t\sin t\mathrm{d}t=-\int_0^{\frac{\pi}{4}}t\mathrm{d}\cos t=-t\cos t\Big|_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}\cos t\mathrm{d}t=-\frac{\sqrt{2}\pi}{8}+\sin t\Big|_0^{\frac{\pi}{4}}=-\frac{\sqrt{2}\pi}{8}+\frac{\sqrt{2}}{2}$,

$\therefore\int_0^{\frac{\pi}{4}}[\cos(2t)\boldsymbol{i}+\sin(2t)\boldsymbol{j}+t\sin t\boldsymbol{k}]\mathrm{d}t=\frac{1}{2}\boldsymbol{i}+\frac{1}{2}\boldsymbol{j}+(-\frac{\sqrt{2}\pi}{8}+\frac{\sqrt{2}}{2})\boldsymbol{k}$.

(2)$\int_1^4(\sqrt{t}\boldsymbol{i}+t\mathrm{e}^{-t}\boldsymbol{j}+\frac{1}{t^2}\boldsymbol{k})\mathrm{d}t=\int_1^4\sqrt{t}\mathrm{d}t\boldsymbol{i}+\int_1^4 t\mathrm{e}^{-t}\mathrm{d}t\boldsymbol{j}+\int_1^4\frac{1}{t^2}\mathrm{d}t\boldsymbol{k}$,

$\because\int_1^4\sqrt{t}\mathrm{d}t=\frac{1}{1+\frac{1}{2}}t^{\frac{1}{2}+1}\Big|_1^4=\frac{14}{3},\int_1^4 t\mathrm{e}^{-t}\mathrm{d}t=-\int_1^4 t\mathrm{d}\mathrm{e}^{-t}=-t\mathrm{e}^{-t}\Big|_1^4+\int_1^4\mathrm{e}^{-t}\mathrm{d}t$

$=-4\mathrm{e}^{-4}+\mathrm{e}^{-1}-\mathrm{e}^{-t}\Big|_1^4=-4\mathrm{e}^{-4}+\mathrm{e}^{-1}-\mathrm{e}^{-4}+\mathrm{e}^{-1}=-5\mathrm{e}^{-4}+2\mathrm{e}^{-1}$,

$\int_1^4\frac{1}{t^2}\mathrm{d}t=\frac{1}{-2+1}t^{-2+1}\Big|_1^4=-\frac{1}{4}+1=\frac{3}{4}$,

$\therefore\int_1^4(\sqrt{t}\boldsymbol{i}+t\mathrm{e}^{-t}\boldsymbol{j}+\frac{1}{t^2}\boldsymbol{k})\mathrm{d}t=\frac{14}{3}\boldsymbol{i}+(-5\mathrm{e}^{-4}+2\mathrm{e}^{-1})\boldsymbol{j}+\frac{3}{4}\boldsymbol{k}$.

5. 已知$\boldsymbol{r}'(t),\boldsymbol{r}(0)$，求$\boldsymbol{r}(t)$：

(1)$\boldsymbol{r}'(t)=(t^2,4t^3,-t^2),\boldsymbol{r}(0)=(0,1,0)$；

(2)$\boldsymbol{r}'(t)=\sin t\boldsymbol{i}-\cos t\boldsymbol{j}+2t\boldsymbol{k},\boldsymbol{r}(0)=\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$.

解：(1)方法1：

$$\begin{aligned}\boldsymbol{r}(t)&=\int_0^t\boldsymbol{r}'(t)\mathrm{d}t+\boldsymbol{C}=(\int_0^t t^2\mathrm{d}t+C_1,\int_0^t 4t^3\mathrm{d}t+C_2,\int_0^t(-t^2)\mathrm{d}t+C_3)\\&=(\frac{1}{3}t^3+C_1,t^4+C_2,-\frac{1}{3}t^3+C_3),\end{aligned}$$

$\therefore\boldsymbol{r}(0)=(C_1,C_2,C_3)=(0,1,0)$,

$\therefore\boldsymbol{r}(t)=(\frac{1}{3}t^3,t^4+1,-\frac{1}{3}t^3)$.

方法2：$\because\int t^2\mathrm{d}t=\frac{1}{3}t^3+C,\ \int 4t^3\mathrm{d}t=t^4+C,\ \int(-t^2)\mathrm{d}t=-\frac{1}{3}t^3+C$,

$\therefore r(t)=\int\boldsymbol{r}'(t)\mathrm{d}t=(\frac{1}{3}t^3+C_1,t^4+C_2,-\frac{1}{3}t^3+C_3)$,

$\because\boldsymbol{r}(0)=(0,1,0)$,

$\therefore C_1=0,\ C_2=1,\ C_3=0$,

$\therefore\boldsymbol{r}(t)=(\frac{1}{3}t^3,t^4+1,-\frac{1}{3}t^3)$.

(2)方法1：

$$\begin{aligned}\boldsymbol{r}(t)=\int_0^t\boldsymbol{r}'(t)\mathrm{d}t+\boldsymbol{C}&=(\int_0^t\sin t\mathrm{d}t+C_1)\boldsymbol{i}+(-\int_0^t\cos t\mathrm{d}t+C_2)\boldsymbol{j}+(\int_0^t 2t\mathrm{d}t+C_3)\boldsymbol{k}\\&=(-\cos t+C_1)\boldsymbol{i}+(-\sin t+C_2)\boldsymbol{j}+(t^2+C_3)\boldsymbol{k},\end{aligned}$$

$\because\boldsymbol{r}(0)=(-1+C_1)\boldsymbol{i}+C_2\boldsymbol{j}+C_3\boldsymbol{k}=\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$,

$\therefore C_1=2,C_2=1,C_3=2$,

$\therefore\boldsymbol{r}(t)=(-\cos t+2)\boldsymbol{i}+(-\sin t+1)\boldsymbol{j}+(t^3+2)\boldsymbol{k}$.

方法2：$\because \int \sin t\mathrm{d}t = -\cos t + C,\ \int(-\cos t)\mathrm{d}t = -\sin t + C,\ \int 2t\mathrm{d}t = t^2 + C,$

$\therefore \boldsymbol{r}(t) = \int \boldsymbol{r}'(t)\mathrm{d}t = (-\cos t + C_1)\boldsymbol{i} + (-\sin t + C_2)\boldsymbol{j} + (t^2 + C_3)\boldsymbol{k},$

$\because \boldsymbol{r}(0) = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k},$

$\therefore C_1 = 2,\ C_2 = 1,\ C_3 = 2,$

$\therefore \boldsymbol{r}(t) = (-\cos t + 2)\boldsymbol{i} + (-\sin t + 1)\boldsymbol{j} + (t^3 + 2)\boldsymbol{k}.$

6. 证明下列等式：

(1)$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}(t) \times \boldsymbol{r}'(t)) = \boldsymbol{r}(t) \times \boldsymbol{r}''(t)$；

(2)$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{r}(t)\| = \frac{\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)}{\|\boldsymbol{r}(t)\|}(\boldsymbol{r}(t) \neq \boldsymbol{0})$；

(3)$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{r}(t) \cdot (\boldsymbol{r}'(t) \times \boldsymbol{r}''(t))] = \boldsymbol{r}(t) \cdot [\boldsymbol{r}'(t) \times \boldsymbol{r}'''(t)]$.

证明：(1)$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}(t) \times \boldsymbol{r}'(t)) = \boldsymbol{r}'(t) \times \boldsymbol{r}'(t) + \boldsymbol{r}(t) \times \boldsymbol{r}''(t) = \boldsymbol{r}(t) \times \boldsymbol{r}''(t)$.

(2)$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{r}(t)\| = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)} = \frac{1}{2\sqrt{\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)}}\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)]$

$= \frac{1}{2\sqrt{\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)}}[\boldsymbol{r}'(t)\cdot\boldsymbol{r}(t) + \boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)] = \frac{2\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)}{2\sqrt{\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)}} = \frac{\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)}{\|\boldsymbol{r}(t)\|}(\boldsymbol{r}(t) \neq \boldsymbol{0})$.

(3)$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{r}(t) \cdot (\boldsymbol{r}'(t) \times \boldsymbol{r}''(t))] = \boldsymbol{r}'(t) \cdot (\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)) + \boldsymbol{r}(t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}'(t) \times \boldsymbol{r}''(t))$

$= \boldsymbol{r}(t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)) = \boldsymbol{r}(t) \cdot (\boldsymbol{r}''(t) \times \boldsymbol{r}''(t) + \boldsymbol{r}'(t) \times \boldsymbol{r}'''(t)) = \boldsymbol{r}(t) \cdot [\boldsymbol{r}'(t) \times \boldsymbol{r}'''(t)]$.

7. 求等速圆周运动$\boldsymbol{r} = R\cos(\omega t)\boldsymbol{i} + R\sin(\omega t)\boldsymbol{j}$在$t$时刻的速度与加速度.

解：t时刻的速度$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = -R\omega\sin(\omega t)\boldsymbol{i} + R\omega\cos(\omega t)\boldsymbol{j}$,

t时刻的加速度$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = -R\omega^2\cos(\omega t)\boldsymbol{i} - R\omega^2\sin(\omega t)\boldsymbol{j}$.

8. 已知螺旋线的向量方程为$\boldsymbol{r} = a\cos\theta\boldsymbol{i} + a\sin\theta\boldsymbol{j} + b\theta\boldsymbol{k}(a > 0, b > 0)$，求在$\theta_0$处的切线方程.

解：在θ_0处的切向量$\boldsymbol{r}'(\theta_0) = -a\sin\theta_0\boldsymbol{i} + a\cos\theta_0\boldsymbol{j} + b\boldsymbol{k}$，切线方程

$$\frac{x - a\cos\theta_0}{-a\sin\theta_0} = \frac{y - a\sin\theta_0}{a\cos\theta_0} = \frac{z - b\theta_0}{b}.$$

9. 设$\boldsymbol{r} = -a\sin\theta\boldsymbol{i} + a\cos\theta\boldsymbol{j} + b\theta\boldsymbol{k}$，求$\frac{1}{2}\int_0^{2\pi}(\boldsymbol{r} \times \boldsymbol{r}')\mathrm{d}\theta$.

解：$\boldsymbol{r}'(\theta) = -a\cos\theta\boldsymbol{i} - a\sin\theta\boldsymbol{j} + b\boldsymbol{k}$,

$$\boldsymbol{r}(\theta)\times\boldsymbol{r}'(\theta) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a\sin\theta & a\cos\theta & b\theta \\ -a\cos\theta & -a\sin\theta & b \end{vmatrix} = \begin{vmatrix} a\cos\theta & b\theta \\ -a\sin\theta & b \end{vmatrix}\boldsymbol{i} + \begin{vmatrix} b\theta & -a\sin\theta \\ b & -a\cos\theta \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} -a\sin\theta & a\cos\theta \\ -a\cos\theta & -a\sin\theta \end{vmatrix}\boldsymbol{k}$$

$= (ab\cos\theta + ab\theta\sin\theta)\boldsymbol{i} - (-ab\sin\theta + ab\theta\cos\theta)\boldsymbol{j} + a^2\boldsymbol{k}$,

$\because \int_0^{2\pi}(ab\cos\theta + ab\theta\sin\theta)\mathrm{d}\theta = ab\sin\theta\Big|_0^{2\pi} - \int_0^{2\pi} ab\theta\mathrm{d}\cos\theta$

$= -ab\theta\cos\theta\Big|_0^{2\pi} + \int_0^{2\pi} ab\cos\theta\mathrm{d}\theta = -2\pi ab + ab\sin\theta\Big|_0^{2\pi} = -2\pi ab$,

$\int_0^{2\pi}-(-ab\sin\theta+ab\theta\cos\theta)\mathrm{d}\theta=\int_0^{2\pi}(ab\sin\theta-ab\theta\cos\theta)\mathrm{d}\theta=-ab\cos\theta\Big|_0^{2\pi}-ab\int_0^{2\pi}\theta\mathrm{d}\sin\theta$
$=-ab\theta\sin\theta\Big|_0^{2\pi}+ab\int_0^{2\pi}\sin\theta\mathrm{d}\theta=-ab\cos\theta\Big|_0^{2\pi}=0,$
$\int_0^{2\pi}a^2\mathrm{d}\theta=2\pi a^2,$
$\therefore\frac{1}{2}\int_0^{2\pi}(\boldsymbol{r}\times\boldsymbol{r}')\mathrm{d}\theta=-\pi ab\boldsymbol{i}+\pi a^2\boldsymbol{k}.$

16.5 习题11.2解答

1. 求下列曲面在指定点的法线方程与切平面的方程：
(1)$x^2+y^2+z^2=14$，在点$(1,2,3)$；
(2)$z=\frac{1}{2}x^2-y^2$，在点$(2,-1,1)$；
(3)$(2a^2-z^2)x^2-a^2y^2=0$，在点(a,a,a)；
(4)$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$，在点$(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}})$；
(5)$\begin{cases}x=u\cos v,\\ y=u\sin v,\\ z=av,\end{cases}$ 在$(u,v)=(u_0,v_0)$处.

解：(1)法向量$\boldsymbol{n}=(2x,2y,2z)\Big|_{(1,2,3)}=2(1,2,3)$,

法线方程$x-1=\frac{y-2}{2}=\frac{z-3}{3}$,

切平面方程$(x-1)+2(y-2)+3(z-3)=0$，即$x+2y+3z=14$.

(2)法向量$\boldsymbol{n}=(x,-2y,-1)\Big|_{(2,-1,1)}=(2,2,-1)$,

法线方程$\frac{x-2}{2}=\frac{y+1}{2}=-(z-1)$,

切平面方程$2(x-2)+2(y+1)-(z-1)=0$，即$2x+2y-z=1$.

(3)法向量$\boldsymbol{n}=(2(2a^2-z^2)x,-2a^2y,-2x^2z)\Big|_{(a,a,a)}=2a^3(1,-1,-1)$,

法线方程$x-a=-(y-a)=-(z-a)$,

切平面方程$(x-a)-(y-a)-(z-a)=0$，即$x-y-z=-a$.

(4)法向量$\boldsymbol{n}=(\frac{2x}{a^2},\frac{2y}{b^2},\frac{2z}{c^2})\Big|_{(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}})}=\frac{2}{\sqrt{3}}(\frac{1}{a},\frac{1}{b},\frac{1}{c})$,

法线方程$a(x-\frac{a}{\sqrt{3}})=b(y-\frac{b}{\sqrt{3}})=c(z-\frac{c}{\sqrt{3}})$,

切平面方程$\frac{1}{a}(x-\frac{a}{\sqrt{3}})+\frac{1}{b}(y-\frac{b}{\sqrt{3}})+\frac{1}{c}(z-\frac{c}{\sqrt{3}})=0$，即$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\sqrt{3}$.

(5)法向量$\boldsymbol{n}=(\cos v,\sin v,0)\times(-u\sin v,u\cos v,a)\Big|_{(u_0,v_0)}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ \cos v_0&\sin v_0&0\\ -u_0\sin v_0&u_0\cos v_0&a\end{vmatrix}$

$=(\begin{vmatrix}\sin v_0&0\\ u_0\cos v_0&a\end{vmatrix},\begin{vmatrix}0&a\\ a&u_0\cos v_0\end{vmatrix},\begin{vmatrix}\cos v_0&\sin v_0\\ -u_0\sin v_0&u_0\cos v_0\end{vmatrix})=(a\sin v_0,-a\cos v_0,u_0),$

法线方程$\frac{x-u_0\cos v_0}{a\sin v_0}=\frac{y-u_0\sin v_0}{-a\cos v_0}=\frac{z-av_0}{u_0}$,

切平面方程$a\sin v_0(x-u_0\cos v_0)-a\cos v_0(y-u_0\sin v_0)+u_0(z-av_0)=0$，

即$ax\sin v_0-ay\cos v_0+zu_0=au_0v_0$.

2. 按要求求下列曲面的切平面方程：

(1)曲面$x^2+2y^2+3z^2=21$的与平面$x+4y+6z=0$平行的切平面；

(2)曲面$z=x^2+y^2$的与直线$\begin{cases}x+2z=1,\\ y+2z=2\end{cases}$垂直的切平面；

(3)双曲抛物面$\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}(a>0)$在任意点处的切平面在各坐标轴上的截距之和为$a$.

解：(1)曲面的法向量$\boldsymbol{n}=(2x,4y,6z)$，平面的法向量$\boldsymbol{n}_1=(1,4,6)$，

则由$\begin{cases}(2x,4y,6z)=a(1,4,6),\\ x^2+2y^2+3z^2=21\end{cases}$得曲面上与该平面相切的切平面的切点为$\pm(1,2,2)$，

切平面方程$x-1+4(y-2)+6(z-2)=0$或$x+1+4(y+2)+6(z+2)=0$，即$x+4y+6z=\pm21$.

(2)直线的切向量$\boldsymbol{t}=(1,0,2)\times(0,1,2)=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&0&2\\0&1&2\end{vmatrix}=(\begin{vmatrix}0&2\\1&2\end{vmatrix},\begin{vmatrix}2&1\\2&0\end{vmatrix},\begin{vmatrix}1&0\\0&1\end{vmatrix})=(-2,-2,1)$，

曲面的法向量$\boldsymbol{n}=(2x,2y,-1)$，曲面上与直线垂直的切平面的法向量$\boldsymbol{n}_0=a\boldsymbol{t}$，

由$\begin{cases}(2x,2y,-1)=a(-2,-2,1),\\ z=x^2+y^2\end{cases}$可得切点为$(1,1,2)$,

切平面方程$-2(x-1)-2(y-1)+z-2=0$，即$2x+2y-z=2$.

(3)法向量$\boldsymbol{n}=(1,1,v)\times(1,-1,u)\big|_{(1,-1)}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&1&-1\\1&-1&1\end{vmatrix}=(\begin{vmatrix}1&-1\\-1&1\end{vmatrix},\begin{vmatrix}-1&1\\1&1\end{vmatrix},\begin{vmatrix}1&1\\1&-1\end{vmatrix})$

$=(0,-2,-2)$，切点为$(0,2,-1)$，

切平面的方程$-2(y-2)-2(z+1)=0$，即$y+z=1$.

3. 求证：曲面$\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a},a>0$在任意点处的切平面在各坐标轴上的截距之和为$a$.

证明：曲面在任意点(x_0,y_0,z_0)处的法向量$\boldsymbol{n}=\frac{1}{2}(\frac{1}{\sqrt{x_0}},\frac{1}{\sqrt{y_0}},\frac{1}{\sqrt{z_0}})$,

切平面方程$\frac{1}{\sqrt{x_0}}(x-x_0)+\frac{1}{\sqrt{y_0}}(y-y_0)+\frac{1}{\sqrt{z_0}}(z-z_0)=0$，即$\frac{x}{ax_0}+\frac{y}{ay_0}+\frac{z}{az_0}=1$,

切平面在x,y,z轴上的截距之和

$$\sqrt{ax_0}+\sqrt{ay_0}+\sqrt{az_0}=\sqrt{a}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})=a.$$

4. 证明二次曲面$ax^2+by^2+cz^2=1$在点$M_0(x_0,y_0,z_0)$处的切平面方程为

$$ax_0x+by_0y+cz_0z=1.$$

证明：曲面在点$M_0(x_0,y_0,z_0)$处的法向量$\boldsymbol{n}=2(ax_0,by_0,cz_0)$,

切平面方程

$$\begin{aligned}&ax_0(x-x_0)+by_0(y-y_0)+cz_0(z-z_0)\\=&ax_0x-ax_0^2+by_0y-by_0^2+cz_0z-cz_0^2\\=&ax_0x-by_0y-cz_0z-1=0,\end{aligned}$$

即

$$ax_0x+by_0y+cz_0z=1.$$

5. 设函数f可微，试证曲面$z=yf(\frac{x}{y})$的所有切平面相交于一个公共点.

证明：曲面在点(x_0,y_0,z_0)处的法向量$\boldsymbol{n}=(yf'(\frac{x}{y})\frac{1}{y},f(\frac{x}{y})+yf'(\frac{x}{y})(-\frac{x}{y^2}),-1)\big|_{(x_0,y_0,z_0)}$
$=(f'(\frac{x_0}{y_0}),f(\frac{x_0}{y_0})-\frac{x_0}{y_0}f'(\frac{x_0}{y_0}),-1)$,

切平面的方程

$$\begin{aligned}&f'(\frac{x_0}{y_0})(x-x_0)+[f(\frac{x_0}{y_0})-\frac{x_0}{y_0}f'(\frac{x_0}{y_0})](y-y_0)-(z-z_0)\\=&f'(\frac{x_0}{y_0})x+[f(\frac{x_0}{y_0})-\frac{x_0}{y_0}f'(\frac{x_0}{y_0})]y-z-x_0f'(\frac{x_0}{y_0})-y_0f(\frac{x_0}{y_0})+x_0f'(\frac{x_0}{y_0})+z_0\\=&f'(\frac{x_0}{y_0})x+[f(\frac{x_0}{y_0})-\frac{x_0}{y_0}f'(\frac{x_0}{y_0})]y-z\\=&0,\end{aligned}$$

$\because$无论切点(x_0,y_0,z_0)取在何处，点$(x,y,z)=(0,0,0)$始终满足以上方程，

$\therefore$曲面$z=yf(\frac{x}{y})$的所有切平面相交于一个公共点$(0,0,0)$.

6. 已知函数f可微，证明曲面$f(\frac{x-a}{z-c},\frac{y-b}{z-c})=0$上任意一点处的切平面通过一定点，并求出此点的位置.

证明：曲面上任一点(x_0,y_0,z_0)处的法向量

$$\boldsymbol{n}=(\frac{1}{z_0-c}f_1',\frac{1}{z_0-c}f_2',-\frac{x_0-a}{(z_0-c)^2}f_1'-\frac{y_0-b}{(z_0-c)^2}f_2'),$$

切平面方程

$$\begin{aligned}&\frac{x-x_0}{z_0-c}f_1'+\frac{y-y_0}{z_0-c}f_2'+[-\frac{x_0-a}{(z_0-c)^2}f_1'-\frac{y_0-b}{(z_0-c)^2}f_2'](z-z_0)\\=&[\frac{x-x_0}{z_0-c}-\frac{x_0-a}{(z_0-c)^2}(z-z_0)]f_1'+[\frac{y-y_0}{z_0-c}-\frac{y_0-b}{(z_0-c)^2}(z-z_0)]f_2'\\=&0,\end{aligned}$$

其中偏导数均在$(\frac{x_0-a}{z_0-c},\frac{y_0-b}{z_0-c})$处取值，

$\because$无论切点(x_0,y_0,z_0)取在何处，$(x,y,z)=(a,b,c)$始终满足以上方程，

$\therefore$曲面$f(\frac{x-a}{z-c},\frac{y-b}{z-c})=0$上任意一点处的切平面通过定点$(a,b,c)$.

7. 设曲面S_1和S_2的方程分别为$F_1(x,y,z)=0,F_2(x,y,z)=0$，其中$F_1$和$F_2$是可微函数，试证$S_1$与$S_2$垂直的充分必要条件是对交线上的任意一点$(x,y,z)$，均有

$$\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial x}+\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial y}+\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial z}=0.$$

证明：必要性：$\because S_1$与S_2垂直，

$\therefore$交线上的任意一点(x,y,z)处两曲面的法向量互相垂直，即

$$(\frac{\partial F_1}{\partial x},\frac{\partial F_1}{\partial y},\frac{\partial F_1}{\partial z})\cdot(\frac{\partial F_2}{\partial x},\frac{\partial F_2}{\partial y},\frac{\partial F_2}{\partial z})=\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial x}+\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial y}+\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial z}=0;$$

充分性：$\because$交线上的任意一点(x,y,z)处

$$\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial x}+\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial y}+\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial z}=(\frac{\partial F_1}{\partial x},\frac{\partial F_1}{\partial y},\frac{\partial F_1}{\partial z})\cdot(\frac{\partial F_2}{\partial x},\frac{\partial F_2}{\partial y},\frac{\partial F_2}{\partial z})=0,$$

$\therefore$交线上的任意一点(x,y,z)处曲面S_1与S_2的法向量互相垂直，

$\therefore S_1$与S_2垂直.

8. 已知函数F可微，若T为曲面S： $F(x,y,z)=0$在点$M_0(x_0,y_0,z_0)$处的切平面，l为T上任意一条过M_0的直线，求证：在S上存在一条曲线，该曲线在M_0处的切线恰好为l.

证明：方法1：设直线l的方程为$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$，因$l$在$T$上，其方向向量$(a,b,c)$应满足

$$(a,b,c)\cdot\mathrm{grad}F(x_0,y_0,z_0)=aF'_x+bF'_y+cF'_z=0,$$

过直线l且与切平面T垂直的平面A的法向量

$$\boldsymbol{n}=(a,b,c)\times\mathrm{grad}F(x_0,y_0,z_0)=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\a&b&c\\F'_x&F'_y&F'_z\end{vmatrix}=(\begin{vmatrix}b&c\\F'_y&F'_z\end{vmatrix},\begin{vmatrix}c&a\\F'_z&F'_x\end{vmatrix},\begin{vmatrix}a&b\\F'_x&F'_y\end{vmatrix})$$

$$=(bF'_z-cF'_y,cF'_x-aF'_z,aF'_y-bF'_x),$$

曲面S与平面A的交线在点M_0处的切向量

$$
\begin{aligned}
\boldsymbol{t} =&\operatorname{grad}F(x_0,y_0,z_0)\times\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ F'_x & F'_y & F'_z\\ bF'_z-cF'_y & cF'_x-aF'_z & aF'_y-bF'_x\end{vmatrix}\\
=&(\begin{vmatrix}F'_y & F'_z\\ cF'_x-aF'_z & aF'_y-bF'_x\end{vmatrix},\begin{vmatrix}F'_z & F'_x\\ aF'_y-bF'_x & bF'_z-cF'_y\end{vmatrix},\begin{vmatrix}F'_x & F'_y\\ bF'_z-cF'_y & cF'_x-aF'_z\end{vmatrix})\\
=&[a(F'_y)^2-bF'_xF'_y-cF'_xF'_z+a(F'_z)^2]\boldsymbol{i}-[aF'_xF'_y-b(F'_x)^2-b(F'_z)^2+cF'_yF'_z]\boldsymbol{j}\\
&+[c(F'_x)^2-aF'_xF'_z-bF'_yF'_z+c(F'_y)^2]\boldsymbol{k}\\
=&[a(F'_y)^2-F'_x(bF'_y+cF'_z)+a(F'_z)^2]\boldsymbol{i}-[F'_y(aF'_x+cF'_z)-b(F'_x)^2-b(F'_z)^2]\boldsymbol{j}\\
&+[c(F'_x)^2-F'_z(aF'_x+bF'_y)+c(F'_y)^2]\boldsymbol{k}\\
=&[a(F'_y)^2+a(F'_x)^2+a(F'_z)^2]\boldsymbol{i}-[-(F'_y)^2-b(F'_x)^2-b(F'_z)^2]\boldsymbol{j}+[c(F'_x)^2+c(F'_z)^2+c(F'_y)^2]\boldsymbol{k}\\
=&[(F'_x)^2+(F'_z)^2+(F'_y)^2](a,b,c),
\end{aligned}
$$

$\therefore \boldsymbol{t}\parallel(a,b,c)$,

$\therefore$曲面S与平面A的交线在点M_0处的切线为l，即在S上存在一条曲线，该曲线在M_0处的切线恰好为l.

方法2：设直线l的方程为$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$，因$l$在$T$上，其方向向量$(a,b,c)$应满足

$$(a,b,c)\perp\operatorname{grad}F(x_0,y_0,z_0),$$

过直线l且与切平面T垂直的平面A的法向量

$$\boldsymbol{n}=(a,b,c)\times\operatorname{grad}F(x_0,y_0,z_0),$$

曲面S与平面A的交线在点M_0处的切向量

$$\boldsymbol{t}=[\operatorname{grad}F(x_0,y_0,z_0)\times\boldsymbol{n}]\parallel(a,b,c),$$

$\therefore$曲面S与平面A的交线在点M_0处的切线为l，即在S上存在一条曲线，该曲线在M_0处的切线恰好为l.