Implementing Finite Element Methods Using Firedrake

November 2024

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- Firedrake allows high level specification of finite element problems through code that looks like maths.
- These automatically translated by the system into high performance compiled code.
- Advanced and customizable solvers enabled by seamless integration with PETSc.

- Consider the following problem

$$\begin{aligned} -\Delta u + u &= f, \quad &&\textbf{in} \;\; \Omega \subset (0,1)^2, \\ \nabla u \cdot \vec{n} &= 0, \quad &&\textbf{on} \;\; \partial\Omega, \end{aligned}$$

where $\vec{n}$ is the outward unit normal to the boundary $\partial\Omega$

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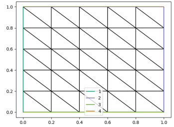
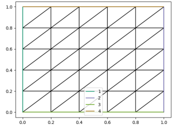
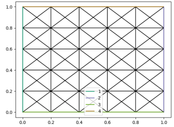
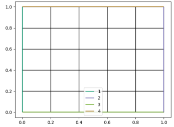
- We find $u \in H^1(\Omega)$ such that

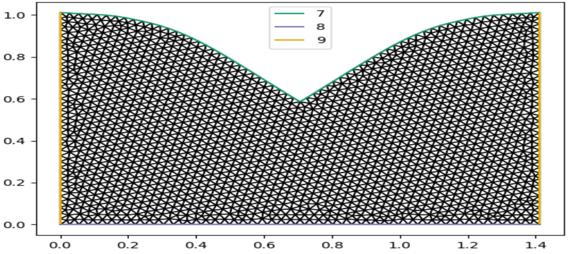
$$\int_{\Omega} \nabla u \cdot \nabla v - \underbrace{\int_{\partial\Omega} v \nabla u \cdot \vec{n}}_{=0, \text{ by BCs}} + \int_{\Omega} uv \quad = \quad \int_{\Omega} fv, \quad \forall v \in H^1(\Omega),$$

Discretising the domain Ω

- Firedrake can build many standard meshes, including UnitSquareMesh.

- For complicated geometries, one can use mesh generators such as Gmsh.



The discrete level

Given $V_h \subset H_0^1(\Omega)$ a finite-dimensional space. At the discrete level, we solve for

$$\int_\Omega \nabla u_h \cdot \nabla v_h + \int_\Omega u_h v_h \quad = \quad \int_\Omega f v_h, \quad \forall v_h \in V_h \subset H^1(\Omega),$$

Nodal basis functions (Lagrange elements)

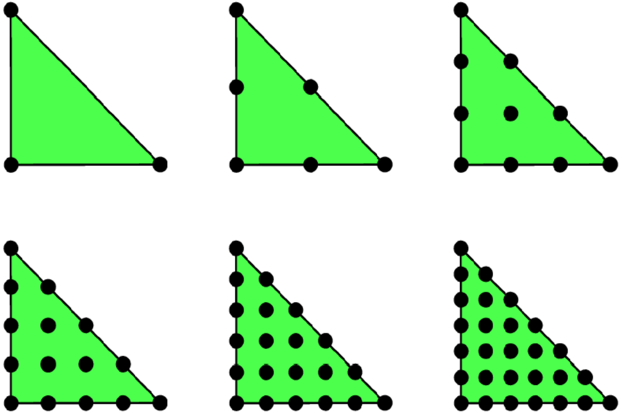


Figure: The Lagrange triangle for $q = \{1, 2, 3, 4, 5, 6\}$

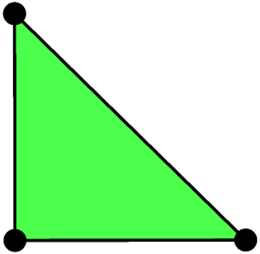
Nodal Basis

Given the location of N degrees of freedoms x_i, $i = 0,\ 1,\ 2,\ \ldots,\ N-1.$ the associated nodal basis function ϕ satisfies

$$\phi_i(x_j) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta.

Linear Lagrange basis functions



At vertices $p_1 = (0,0)$, $p_2 = (0,1)$, $p_3 = (1,0)$, we have

$$\phi_1 = 1 - x - y, \quad \phi_2 = y, \quad \phi_3 = x.$$

Linear algebraic formulation

- Let $V_h = \mathrm{Span}\{\phi_1,\ \phi_2, \cdots, \phi_n\}$,
- u_h can be expanded as

$$u_h = \sum_{j=1}^{N} U_j \phi_j, \quad v_h = \phi_i.$$

Thus, we solve for U_i such that

$$\sum_j a(\phi_j, \phi_i) U_j = F(\phi_i),$$

where

$$\begin{aligned} a(\phi_j, \phi_i) &= \int_\Omega \nabla \phi_j \cdot \nabla \phi_i + \phi_j \phi_i \\ F(\phi_j) &= \int_\Omega f \phi_i. \end{aligned}$$

Firedrake implementation

```python
from firedrake import *
# Constuct 50 by 50 square mesh
mesh = UnitSquareMesh(50, 50)

# Piecewise continuous linear polynomials.
V = FunctionSpace(mesh, "CG", 1)

# The trial and Test functions.
u = TrialFunction(V)
v = TestFunction(V)
```

Firedrake implementation continued

```
#The right hand side
f = Function(V)
x, y = SpatialCoordinate(mesh)
f.interpolate((1+8*pi*pi)*cos(x*pi*2)*cos(y*pi*2))

# The bilinear and linear forms
a = (inner(grad(u), grad(v)) + inner(u, v)) * dx
L = inner(f, v) * dx

#Define a function that holds the solution.
u = Function(V)
```

Firedrake implementation continued

```
# How to solver the linear system. LU factorisation!
sp={'ksp_type': 'preonly', 'pc_type': 'lu'}

# Solve
solve(a == L, u, solver_parameters=sp)
```

From Strong PDE to Solution

Strong PDE

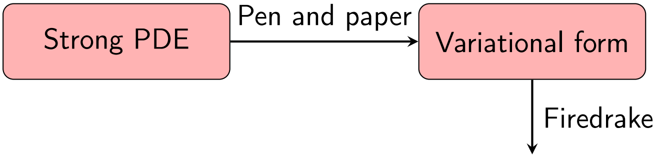
From Strong PDE to Solution

Strong PDE $\xrightarrow{\text{Pen and paper}}$

From Strong PDE to Solution

Strong PDE $\xrightarrow{\text{Pen and paper}}$ Variational form

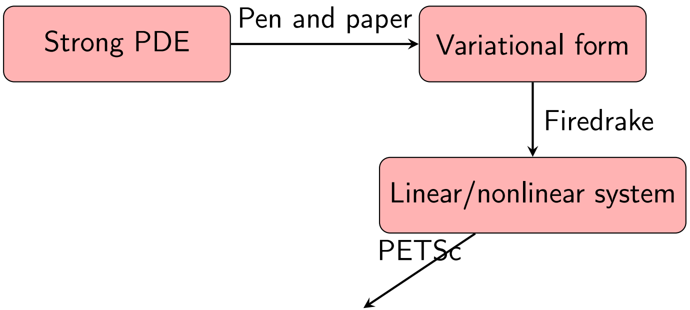
From Strong PDE to Solution



From Strong PDE to Solution

$$\begin{tikzcd}
\text{Strong PDE} \arrow[r, "\text{Pen and paper}"] & \text{Variational form} \arrow[d, "\text{Firedrake}"] \\
& \text{Linear/nonlinear system}
\end{tikzcd}$$

From Strong PDE to Solution



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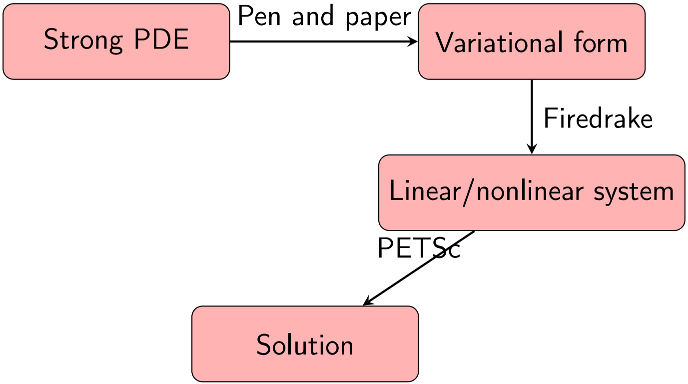


Advection equation

Governing PDE

Consider the problem

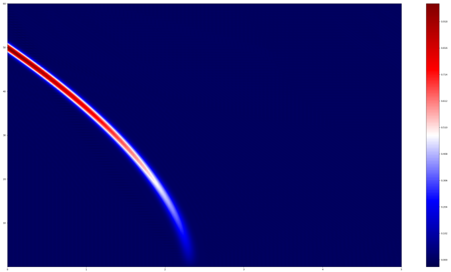
$$\begin{aligned} \nabla \cdot (\boldsymbol{b} u) &= 0, \qquad \textbf{in} \;\; \Omega \subset [0,5] \times [10^{-3}, 60], \\ +\text{BCs} & \qquad\qquad\qquad \text{on } \partial\Omega \end{aligned}$$

Here $\boldsymbol{b}$ determines the advection direction, taken to be $\boldsymbol{b} = \left[1, \; \frac{-1}{c\sqrt[4]{y^3}}\right]$ and c is a constant.

- This problem can be written in the variational form as

$$-\int_\Omega (\boldsymbol{b} u) \cdot \nabla v + \int_{\partial\Omega} v(\boldsymbol{b} u) \cdot \vec{n}$$

Advection solution



Double Slit Experiment

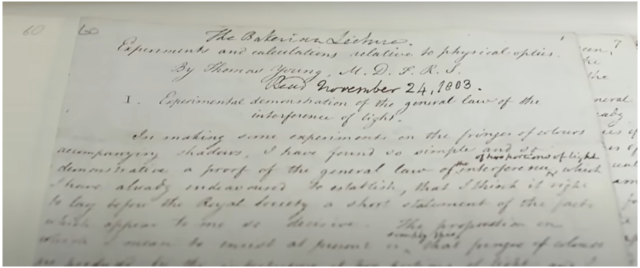
What is light?

A YouTuber asked people on the street: "What is light?" Here's how they responded:

- Light is brightness, I guess.
- We have auras, which are light!
- Lights up the room, it makes it not dark!
- It goes in your eyes and then you see stuff.

This is not an easy question!

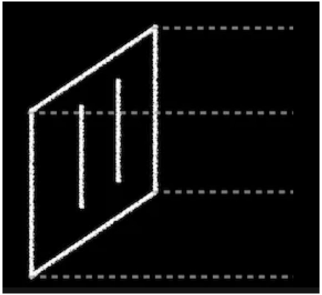
- In the late 1600s, Newton proposed light was a stream of particles.
- Huygens proposed that light was a wave.
- This debate was settled by the Thomas Young's double slit experiment.



The Bakerian Lecture.
Experiments and calculations relative to physical optics.
By Thomas Young, M.D. F.R.S.
Read November 24, 1803.
I. Experimental demonstration of the general law of the interference of light.
In making some experiments on the fringes of colours accompanying shadows, I have found so simple and so demonstrative a proof of the general law of the interference of two portions of light, which I have already endeavoured to establish, that I think it right to lay before the Royal Society a short statement of the facts which appear to me so decisive. The proposition on which I mean to insist at present is simply this, that fringes of colours

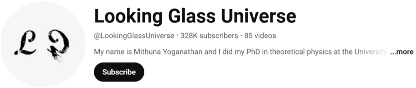
Figure: Thomas Young's handwritten notes from 1803.

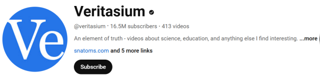
Double slit experiments

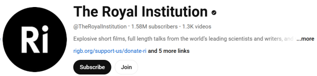


- Particles go through each slit and produce two spots underneath.
- A wave from one slit interacts with waves from others slits.
- If the peak of one wave meets up with the bottom from the other, we get destructive interference (no waves).

Youtube channels citations







Governing PDEs

- Consider the **Wave equation** of the form

$$u_{tt} - \Delta u = 0, \quad \text{in } \Omega$$
$$\text{suitable BCs on } \partial\Omega$$

- Finite element methods are used to discretise in space, and explicit-Euler to discretise the time coordinate.

Flow Past a Cylinder

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- These three components form what is called "Reynold number".
- At a large Reynold number, the fluid becomes unstable and vortex shedding occur.
- This pattern called the Von Carman Vortex street.

Vortex shedding modelling

- Consider the following Navier-Stokes problem,

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \ \text{ in } \ \Omega, \\
\nabla \cdot \boldsymbol{u} = 0 \ \text{ in } \ \Omega, \\
+ \text{ BCs}
\end{aligned}
$$

where f is a source function, $\boldsymbol{u}$ is the fluid velocity, p is its pressure, and ν is the viscosity constant.

Thank You!