MArefSetup

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Test

This section should be in the contents page for the PDF but not the web or word.

1 Quadratic equations

A quadratic equation is an equation with the form $ax^2 + bx + c = 0$ where x represents an unknown and a , b and c are known numbers with $a \neq 0$.

1.1 Solutions to a quadratic equation

A solution to a quadratic equation is a value of x such that the equation balances. The solutions to quadratic equations can be found by using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

Example.
For instance, the solutions to $x^2 + 2x - 3 = 0$ are:

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times -3}}{2 \times 1} \\ &= \frac{-2 \pm \sqrt{4 + 12}}{2} \\ &= \frac{-2 \pm \sqrt{16}}{2} \\ &= \frac{-2 \pm 4}{2} \end{aligned}$$

Hence, $x = 1$ or $x = -3$.

1.2 The discriminant

Definition 1.1 (Discriminant)**.**
The **discriminant** of a quadratic equation with coefficients $a, b, c \in \mathbb{R}$ is:

$$\Delta = b^2 - 4ac.$$

Remark 1.2.
Note that this is the expression beneath the square root symbol in the quadratic formula (eq. (1)).

We can use the discriminant to determine the number of real roots of a quadratic equation. The number depends on the value of Δ as in table table 1.

Value of Δ	Real roots
$\Delta > 0$	Two, distinct
$\Delta = 0$	One, repeated
$\Delta < 0$	Zero

Table 1: Number of real roots of a quadratic equation, given the discriminant

Figure fig. 1 shows an example of each possibility[1].

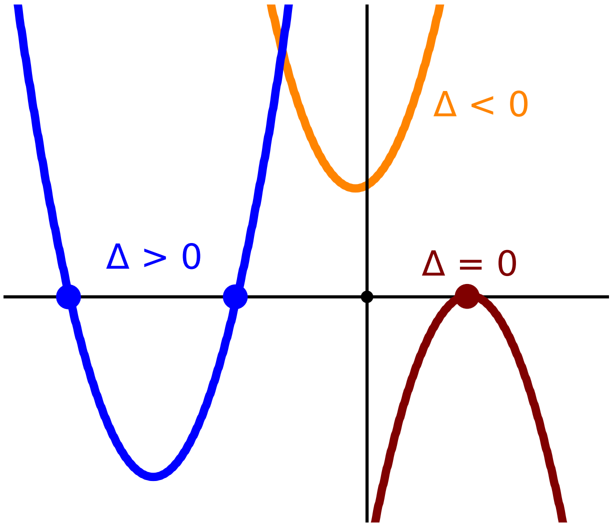

Figure 1: Examples of quadratic functions with zero, one and two real roots.

[1]The image is due to Olin, CC-BY-AS 3.0 downloaded from Wikimedia Commons

2 The scalar product

Consider two vectors $\mathbf{a}$ and $\mathbf{b}$ drawn so their tails are at the same point.

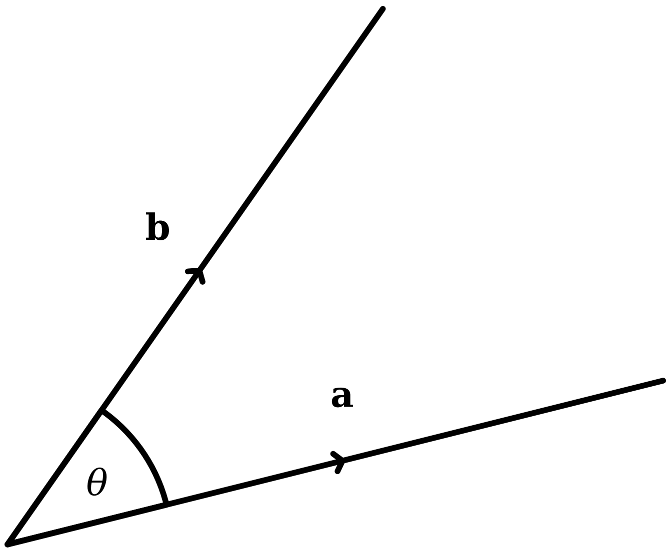

Figure 2: Two vectors with angle between them.

We define the scalar product of $\mathbf{a}$ and $\mathbf{b}$ as follows.

Definition 2.1 (Scalar product)**.**
The **scalar product** of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

where

- $|\mathbf{a}|$ is the modulus of $\mathbf{a}$,
- $|\mathbf{b}|$ is the modulus of $\mathbf{b}$, and
- θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Remark 2.2.
It is important to use the dot symbol for the scalar product (also called the dot product). You must not use a $\times$ symbol as this denotes the vector product which is defined differently.

Example 2.3.
Let

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

The angle between these vectors is $\theta = 45°$. Then $|\mathbf{a}| = \sqrt{8}$ and $|\mathbf{b}| = 4$. So,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \end{pmatrix} &= |\mathbf{a}||\mathbf{b}| \cos\theta \\ &= \sqrt{8} \times 4 \times \cos 45° \\ &= 4\sqrt{8} \times \frac{1}{\sqrt{2}} = 4\frac{\sqrt{8}}{\sqrt{2}} = 4\sqrt{4} = 8. \end{aligned}$$

Note that the result is a scalar, not a vector.

2.1 Vectors in cartesian form

When vectors are given in cartesian form there is an alternative formula for calculating the scalar product.

Proposition 2.4.
If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

Proof. Consider the vector $\mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$. The modulus of this is

$$|\mathbf{b} - \mathbf{a}| = \sqrt{(b_1 - a_2)^2 + (b_2 - a_2)^2}.$$

Note from figure fig. 3 that the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{b} - \mathbf{a}$ form a triangle:

Let θ denote the angle between $\mathbf{a}$ and $\mathbf{b}$. Then, the cosine rule yields:

$$|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos\theta. \tag{2}$$

Substituting the definition of the scalar product of $\mathbf{a}$ and $\mathbf{b}$ into equation eq. (2) gives:

$$|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,(\mathbf{a} \cdot \mathbf{b}).$$

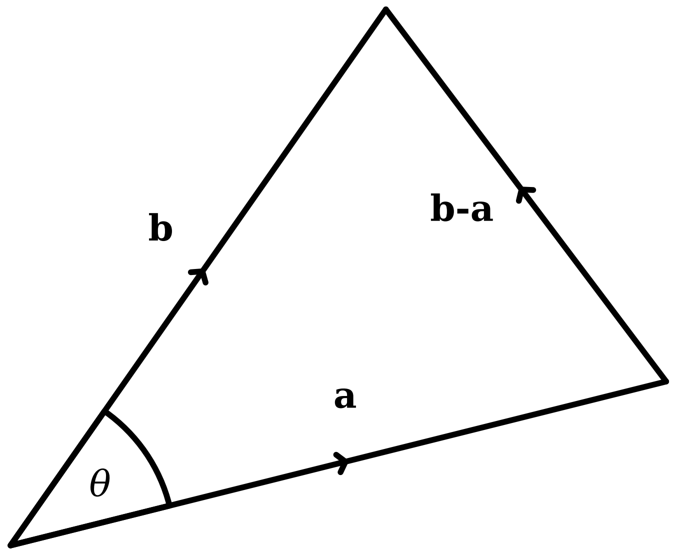

Figure 3: A triangle is formed by two vectors and their difference.

Rearranging:

$$(\mathbf{a}\cdot\mathbf{b}) = \frac{1}{2}\left(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{b}-\mathbf{a}|^2\right).$$

Writing this in terms of components produces:

$$\begin{aligned}(\mathbf{a}\cdot\mathbf{b}) &= \frac{1}{2}\left(a_1^2 + a_2^2 + b_1^2 + b_2^2 - (b_1-a_1)^2 - (b_2-a_2)^2\right)\\ &= \frac{1}{2}\left(a_1^2 + a_2^2 + b_1^2 + b_2^2 - b_1^2 + 2b_1a_1 - a_1^2 - b_2^2 + 2b_2a_2 - a_2^2\right)\\ &= \frac{1}{2}\left(2b_1a_1 + 2b_2a_2\right)\\ &= a_1b_1 + a_2b_2\end{aligned}$$

as required. □

Example 2.5.
Consider again the vectors

$$\mathbf{a} = \begin{pmatrix}2\\2\end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix}4\\0\end{pmatrix}.$$

Calculating the scalar product using the components:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 = 2\times 4 + 2\times 0 = 8.$$

Note that if we are given vectors in this form, the scalar product may be used to calculate the angle between them. Since $\mathbf{a} \cdot \mathbf{b} = 8$ and we have:

$$|\mathbf{a}| = \sqrt{8}$$

$$|\mathbf{b}| = 4.$$

Hence,

$$\begin{aligned} 8 = \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos\theta \\ &= 4\sqrt{8}\cos\theta. \end{aligned}$$

Rearranging:

$$\theta = \cos^{-1}\left(\frac{8}{4\sqrt{8}}\right) = 45°.$$

3 Using Matlab

Two calculate the scalar product in Matlab the `dot` function is used.

Create two vectors:

```
> A = [4 -1 2];
> B = [2 -2 -1];
```

Calculate the scalar product:

```
> C = dot(A,B)

    C = 8
```

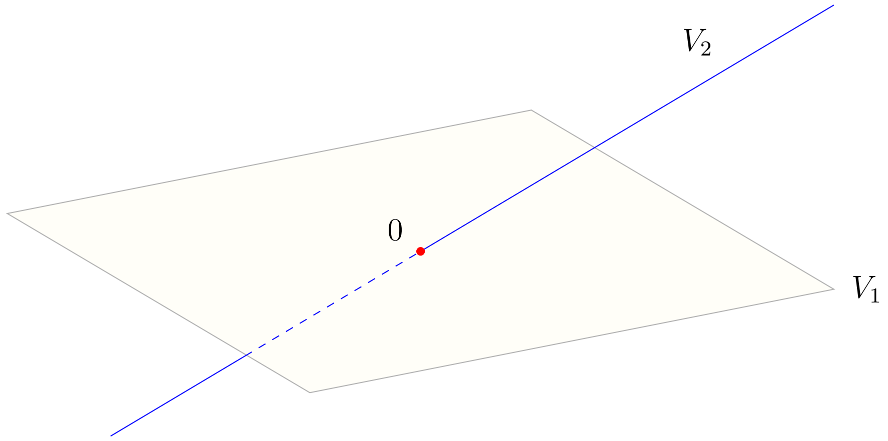

Figure 4: $\mathbb{R}^3$ as a direct sum of a line and a plane

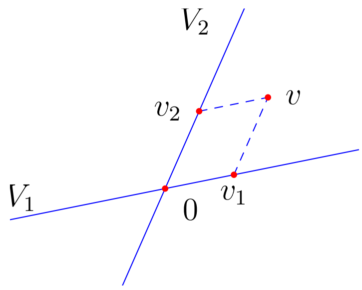

Figure 5: $\mathbb{R}^2 = V_1 \oplus V_2$

4 Using Tikz

5 Testing pdftex graphics

6 Package tests

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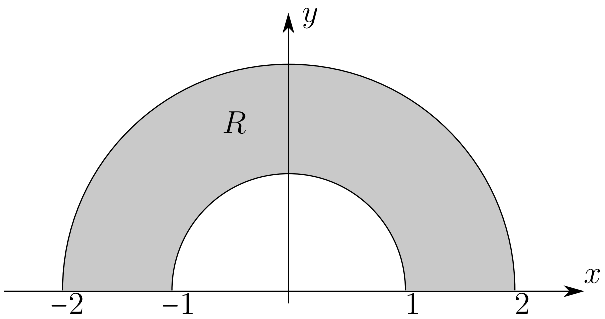

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x
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