MA10230 Methods and Applications 1A, 2017/18

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Lecture notes and audio recording

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Overview and Aims

Semester 1: MA10230 Methods and Applications 1A: Calculus

Semester 2: MA10236 Methods and Applications 1B: Vectors and Mechanics

- Revise integration, and use integration techniques to integrate a variety of functions.
- Introduce calculus in several variables: partial differentiation, double and triple integrals.
- Introduce various types of ordinary differential equations (ODEs).
- Show how these methods help us solve many problems arising in applications.
- Provide foundations for Mechanics in Semester 2.
- Links with Analysis – which explains the maths behind the methods.

Introduction

The following example is intended to illustrate how calculus is applied in other fields. You don't have to follow every step at this stage (although some of you will). By the end of the unit, you should be able to tackle more complex examples.

Example. Application: predicting global population.

Population is a function $u(t)$ of time t.

Express this as:

A. Table

Year	Population (millions)
1000	310
1800	978
1900	1650
1990	5263
2000	6070
2008	6707
2008	6707
2010	6873

B. Graph (figure (1))

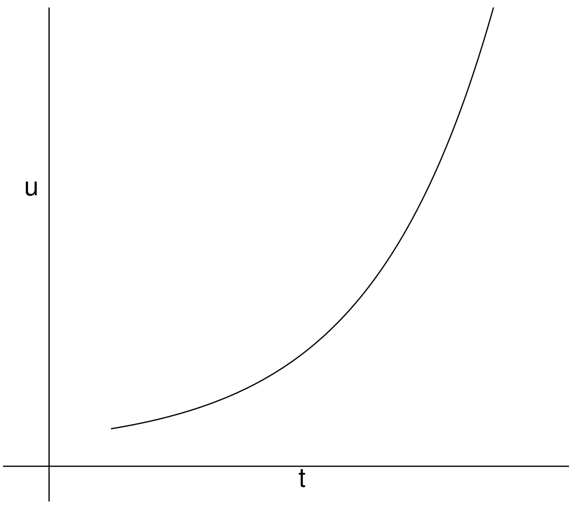

Figure 1: Exponential growth

C. Differential equation

Population is $u(t)$. What is the **rate of change** of the population?

In a time interval δt:

- A proportion $a\delta t$ of population produce children.
- A proportion $b\delta t$ of population die.
- A proportion $c\delta t$ of population emigrate to Mars.

So,

$$\begin{aligned} u(t+\delta t) &= u(t) + (a-b-c)u(t)\delta t \\ \Longrightarrow \frac{u(t+\delta t) - u(t)}{\delta t} &= \lambda u(t), \end{aligned}$$

where $\lambda = a - b - c$.

Let $\delta t \to 0$ (see analysis course).

Then

$$\begin{aligned} \frac{du(t)}{dt} &= \lambda u(t) \\ u(t_0) &= u_0 \end{aligned}$$

A differential equation like this, describing population growth, was used by Thomas Malthus, early 19th century.

Parameter estimation

The UN estimates that the population grows by 1.14% per year.

Let t be time in years.

We separate the variables in our differential equation, then integrate (details later, when we study differential equations).

$$\begin{aligned} \frac{du}{dt} &= \lambda u \\ \Longrightarrow \int \frac{1}{u}\, du &= \int \lambda \, dt \\ \Longrightarrow u(t) &= u_0 e^{\lambda(t-t_0)} \end{aligned}$$

We are interested in the change over one year, so let $t_0 = t$, so $u_0 = u(t)$, and consider

$$\begin{aligned} u(t+1) &= u(t)e^{\lambda(t+1-t)} \\ &= u(t)e^{\lambda} \\ \Longrightarrow \quad \frac{u(t+1) - u(t)}{u(t)} &= \frac{u(t)e^{\lambda} - u(t)}{u(t)} \\ &= e^{\lambda} - 1. \end{aligned}$$

This is the proportional increase in population over one year, so the UN estimate tells us that

$$\begin{aligned} e^{\lambda} - 1 &= 0.0114 \qquad \text{(1.14\% growth)} \\ \implies \lambda &= \ln(1 + 0.0114) \approx 0.01133551 \end{aligned}$$

Hence

$$u(t) = u_0 e^{0.01133551(t - t_0)}$$

Application

How long would it take for the population to double?

Pros:

- Model easy to solve
- Gives testable predictions

Cons:

- Unrealistic – does not include effects of resource depletion, over-crowding, medical advances, agricultural advances, etc.

To construct more realistic models, we will need to be able to

- work with more than one independent variable;
- construct and solve more complex differential equations;
- evaluate more complex integrals.

1 Integration

Recall: There are two different types of integral. Suppose we have a function f.

1. The *indefinite integral* of f is a function F such that
$$\frac{dF}{dx} = f(x).$$
We write
$$F = \int f\, dx.$$
This defines integration as the inverse of differentiation (antidifferentiation). F is called the *antiderivative* of f.

2. The *definite integral* is defined as the *net signed area* between f and the horizontal axis between a and b. It is written
$$\int_a^b f(x)\, dx.$$

You are probably used to using indefinite integrals to evaluate definite integrals, e.g.

$$\begin{aligned}\int_0^{\pi/2} \sin(x) dx &= [-\cos(x)]_0^{\pi/2} \\ &= 0 - (-1) \\ &= 1.\end{aligned}$$

But why are we allowed to combine definite and indefinite integrals in this way?

1.1 Fundamental Theorem of Calculus

This theorem establishes the astonishing connection between indefinite and definite integrals.

Theorem 1.1. *Let f be a continuous function on $u \in [a, b]$.*

Part A.

The function f has an antiderivative g given by
$$g(x) = \int_a^x f(u) du$$
(signed area between a and x, definite integral).

Part B.

Given any antiderivative h of f,
$$\begin{aligned}\int_a^b f(u)\, du &= h(b) - h(a) \\ &= g(b) - g(a) \\ &= \int f(x) dx\Big|_{x=b} - \int f(x) dx\Big|_{x=a}\end{aligned}$$

We don't have the formal mathematics required to prove this rigorously, and won't have until Semester 2 of Analysis 1. Isaac Newton didn't have that formal mathematics yet either, so we (roughly) followed the method of justification he used in 1669.

Justification of Part A. We wish to differentiate

$$g(x) = \int_a^x f(u)\,du.$$

Suppose x changes by a small increment Δx. The corresponding change in g is $g(x + \Delta x) - g(x)$, so the average rate of change from x to $x + \Delta x$ is

$$\frac{\Delta g}{\Delta x} = \frac{g(x + \Delta x) - g(x)}{\Delta x}.$$

As $\Delta x \to 0$, this approaches the derivative $\dfrac{dg}{dx}$.

Now, consider the diagram in figure (2).

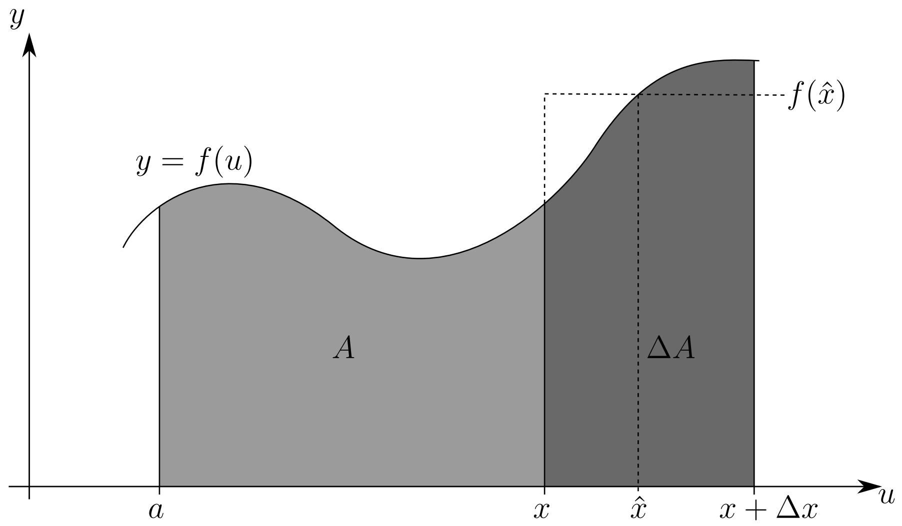

Figure 2: The area $g(x + \Delta x) = \text{area } A + \text{area } \Delta A$

By definition of the definite integral defining g,

$$g(x + \Delta x) = \text{area } A + \text{area } \Delta A = g(x) + \text{area } \Delta A.$$

There is a point $\hat{x}$ with $x \le \hat{x} \le x + \Delta x$ such that $\Delta A = f(\hat{x})\Delta x$, as shown in figure (2). So

$$\begin{aligned} g(x + \Delta x) - g(x) &= \text{area } \Delta A \\ &= f(\hat{x})\Delta x \end{aligned}$$

Hence

$$\frac{g(x+\Delta x)-g(x)}{\Delta x} = f(\hat{x})$$

As $\Delta x \to 0$, $\hat{x} \to x$ (since it gets sandwiched between x and $x+\Delta x$), so $f(\hat{x}) \to f(x)$ and

$$\frac{dg}{dx} = f(x)$$

Hence g is an antiderivative for f.

(see Analysis 1 Semester 2 for a rigorous proof).

□

Note that this result allows us to differentiate certain functions defined in terms of integrals. Explicitly, the statement "g is an antiderivative for f" means that

$$\frac{dg}{dx} = \frac{d}{dx}\left(\int_a^x f(u)\,du\right) = f(x).$$

Notice also that the derivative of g does not depend on the lower limit of the integral, a. If we were to evaluate the integral and write an explicit expression for g (assuming that this is possible), this constant a would only contribute a constant term to that expression; the derivative of that constant term would therefore be 0.

Justification of Part B. Let h be an antiderivative of f.

So

$$\begin{aligned} & \frac{dh}{dx} = f = \frac{dg}{dx} \\ \Longrightarrow\quad & \frac{dh}{dx} - \frac{dg}{dx} = 0 \\ \Longrightarrow\quad & \frac{d}{dx}(h-g) = 0 \\ & \text{by linearity of derivative} \\ \Longrightarrow\quad & h-g = c, \quad \text{constant.} \end{aligned}$$

Hence

$$h = g + c.$$

Now

$$\begin{aligned} \int_a^b f(x)dx &= \left.\int_a^x f(u)du\right|_{x=b} \\ &= \left.g(x)\right|_{x=b} = g(b) \end{aligned}$$

$$\int_a^a f(x)dx = g(a) = 0,$$

so

$$\begin{aligned}\int_a^b f(x)dx &= g(b) - g(a) \\ &= (g(b) + c) - (g(a) + c) \\ &= h(b) - h(a)\end{aligned}$$

□

Corollary 1.2.

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(u)du = f(b)\frac{db}{dx} - f(a)\frac{da}{dx}$$

Proof. Let $g(x)$ be an antiderivative of $f(x)$, so

$$\frac{dg}{dx} = f(x)$$

Then by Fundamental Theorem of Calculus,

$$\int_{a(x)}^{b(x)} f(u)du = g(b(x)) - g(a(x))$$

$$\begin{aligned}\implies \quad \frac{d}{dx}\int_{a(x)}^{b(x)} f(u)du &= \frac{dg}{db}\frac{db}{dx} - \frac{dg}{da}\frac{da}{dx} \\ &\quad \text{(chain rule)} \\ &= f(b)\frac{db}{dx} - f(a)\frac{da}{dx}\end{aligned}$$

□

1.2 Hyperbolic functions

Hyperbolic functions are a type of function related to trigonometric functions. We will use them to help us integrate various types of functions, using a technique called *integration by substitution*, which we'll meet in the next section.

$$\begin{aligned}\cosh(x) &= \frac{1}{2}(e^x + e^{-x}) \\ \sinh(x) &= \frac{1}{2}(e^x - e^{-x}) \\ \tanh(x) &= \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}\end{aligned}$$

Sketches

What do cosh and sinh look like?

$\cosh(0) = 1$

As $x \to \infty$, $\cosh(x) \to \infty$.

As $x \to -\infty$, $\cosh(x) \to \infty$.

$\cosh(x) > 0 \ \forall x$.

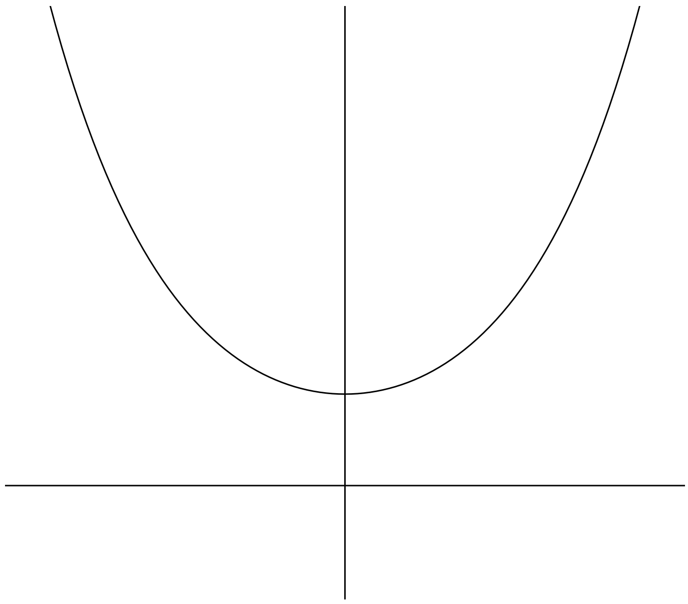

Figure 3: $\cosh(x)$

Hanging chains (e.g. those in suspension bridges) have cosh shape, called a "catenary".

Soap bubbles between wands have a surface derived from the cosh shape.

$\sinh(0) = 0$

As $x \to \infty$, $\sinh(x) \to \infty$.

As $x \to -\infty$, $\sinh(x) \to -\infty$.

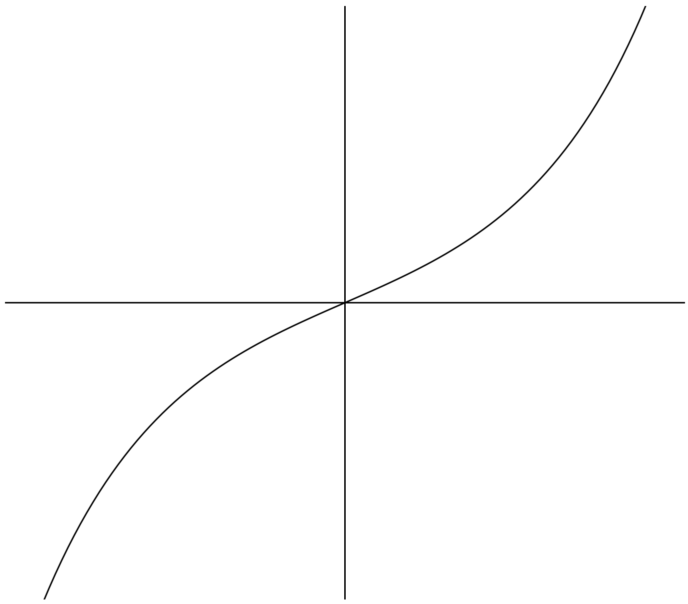

Figure 4: $\sinh(x)$

$\tanh(0) = 0$

As $x \to \infty$, $\tanh(x) \to 1$.

As $x \to -\infty$, $\tanh(x) \to -1$.

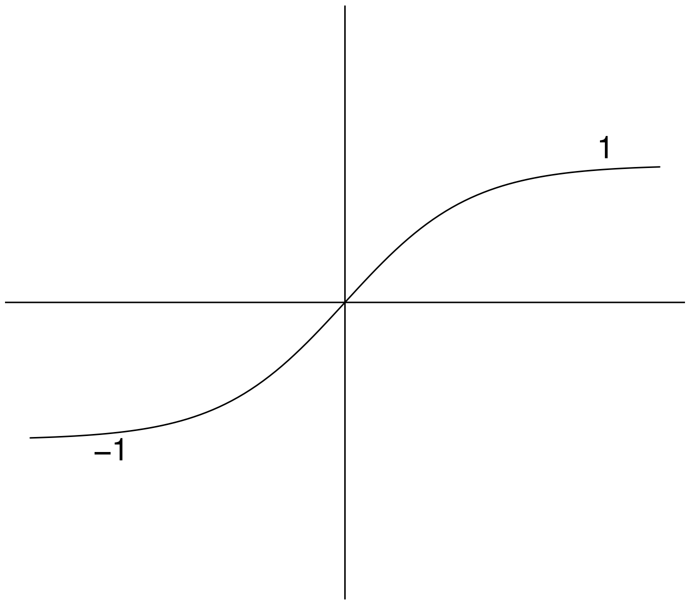

Figure 5: $\tanh(x)$

Hyperbolic identities

Hyperbolic functions satisfy identities similar to trigonometric identities.
Examples.

1. $\cosh^2(x) - \sinh^2(x) = 1$ because

$$\begin{aligned}\cosh^2(x) - \sinh^2(x) &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) \\ &= \frac{1}{2} + \frac{1}{2} = 1.\end{aligned}$$

   Compare this with the trigonometric identity

$$\cos^2(x) + \sin^2(x) = 1.$$

2. $1 - \tanh^2(x) = \operatorname{sech}^2(x)$. Using $\cosh^2(x) - \sinh^2(x) = 1$ and dividing through by $\cosh^2(x)$ gives

$$\frac{\cosh^2(x)}{\cosh^2(x)} - \frac{\sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)},$$

   i.e. $1 - \tanh^2(x) = \operatorname{sech}^2(x)$.

   Compare this with the trigonometric identity

$$1 + \tan^2(x) = \sec^2(x).$$

3. The hyperbolic addition formula for sinh is

$$\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\cosh(y).$$

   The corresponding trigonometric identity is

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\cos(y).$$

   (See Exercise Sheet 1.)

4. The hyperbolic addition formula for cosh is

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y).$$

   The corresponding trigonometric identity is

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

In fact, *Osborn's Rule* states that any trigonometric identity can be converted into a hyperbolic identity by

- replacing trigonometric functions with their hyperbolic counterparts,
- swapping the sign of any product of two sinhs.

Note: be careful with the second step, e.g.

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)},$$

so $\tan^2(x)$ becomes $-\tanh^2(x)$ (see Example 2).

This arises because hyperbolic and trigonometric functions are related as follows:

$$\begin{aligned}
\cosh(ix) &= \cos(x), \\
\cosh(x) &= \cos(ix), \\
\sinh(ix) &= i\sin(x), \\
\sinh(x) &= -i\sin(ix),
\end{aligned}$$

where $i^2 = -1$.

This comes from Euler's relation:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Geometric interpretation

In trigonometric expressions such as $\sin(\theta)$, $\cos(\theta)$, etc., θ can be interpreted as an angle. Similarly, in $\sinh(t)$ and $\cosh(t)$, t can be interpreted as an area.

Since $\cosh^2(t) - \sinh^2(t) = 1$, for any t, the point $(x, y) = (\cosh(t), \sinh(t))$ lies on the curve $x^2 - y^2 = 1$. Then t corresponds to twice the area bounded by this curve, the x-axis, and the line from the origin to the point $(\cosh(t), \sinh(t))$, as shown in figure (6).

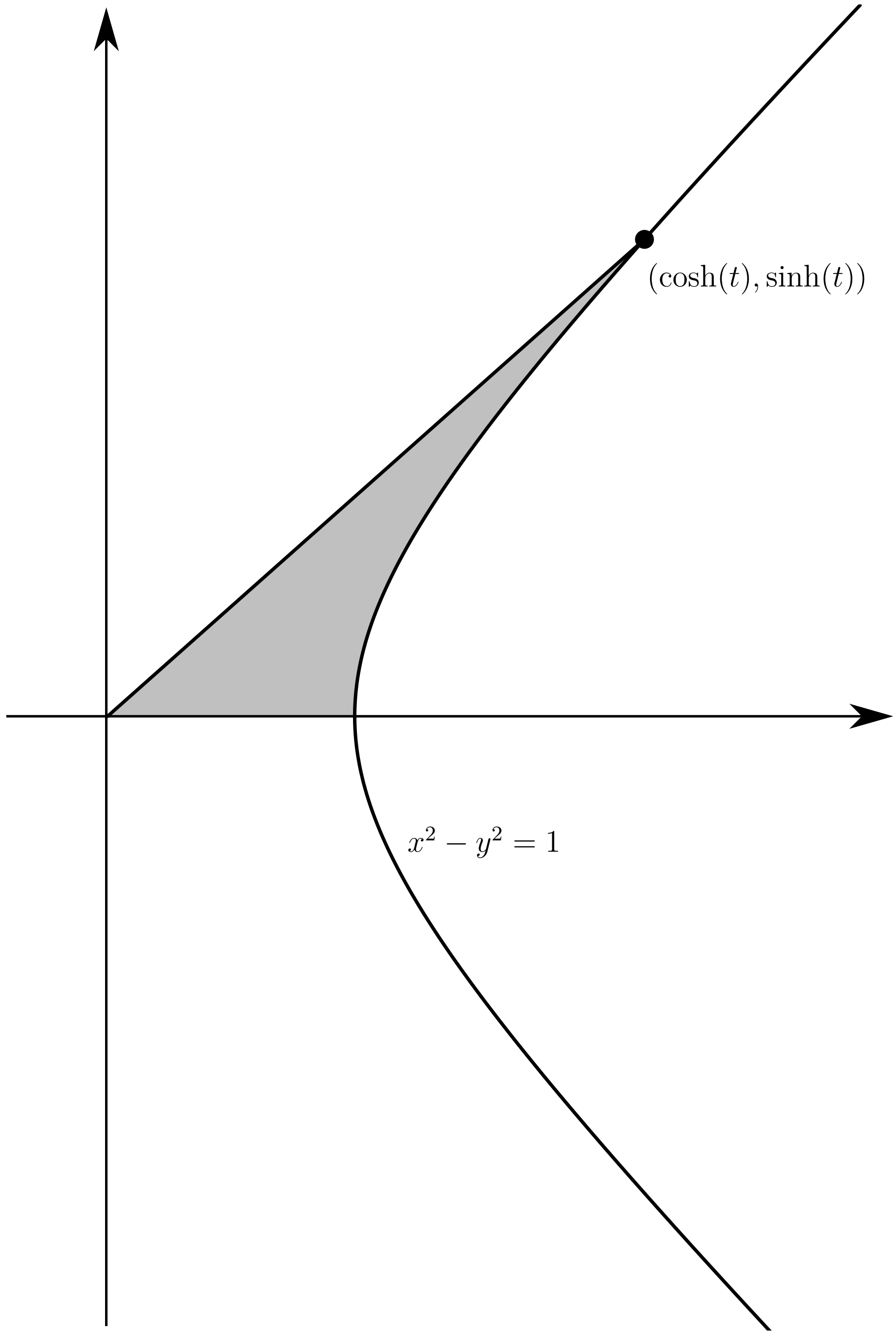

Figure 6: t corresponds to twice the shaded area

Because of this, the inverse hyperbolic functions are denoted arsinh, arcosh, etc. – "ar" is an abbreviation of "area".

Differentiating hyperbolic functions

Since hyperbolic functions are defined in terms of the exponential function, they are straightforward to differentiate.

$$\begin{aligned}\frac{d}{dx}\sinh(x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)\\ &= \frac{e^x + e^{-x}}{2}\\ &= \cosh(x)\end{aligned}$$

Similarly,

$$\frac{d}{dx}\cosh(x) = \sinh(x),$$

$$\frac{d}{dx}\tanh x = \frac{1}{\cosh^2(x)} = \operatorname{sech}^2(x).$$

1.3 Integrating functions involving square roots

Integrals involving functions with square roots often arise in mechanics. In this section, we will investigate methods for integrating them using trigonometric and hyperbolic substitutions.

Recall that integration by substitution is a technique for evaluating integrals involving composite functions, using the formula

$$\int f(g(x))\frac{dg}{dx}dx = F(g(x)) + c,$$

derived from the chain rule for differentiation. (For a more detailed reminder of integration by substitution, see the additional notes on the course Moodle page.)

Example. Evaluate the integral

$$\int \sqrt{a^2 - x^2}\, dx,$$

where $|x| < a$.

The integrand is a semicircle with radius a. To inform which choice of substitution to use, consider figure (7). The coordinates of any point $(x, \sqrt{a^2 - x^2})$ on the curve can be expressed in terms of θ, the angle between the y-axis and the line segment from the origin to that point, using trigonometric functions.

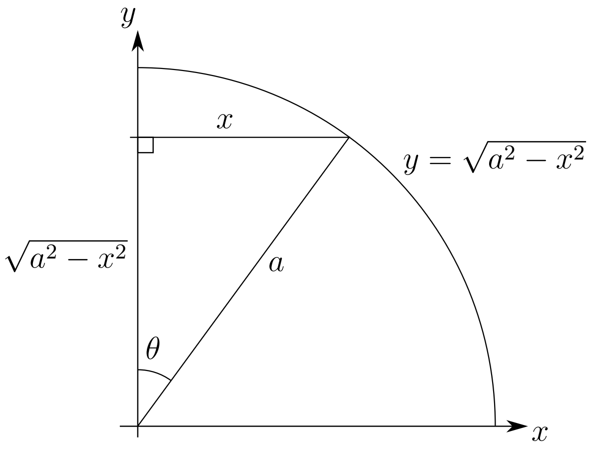

Figure 7: Finding a substitution to integrate the curve $\sqrt{a^2 - x^2}$