

# MArefSetup

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# Test

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## 1 Quadratic equations

A quadratic equation is an equation with the form  $ax^2 + bx + c = 0$  where  $x$  represents an unknown and  $a$ ,  $b$  and  $c$  are known numbers with  $a \neq 0$ .

### 1.1 Solutions to a quadratic equation

A solution to a quadratic equation is a value of  $x$  such that the equation balances. The solutions to quadratic equations can be found by using the

quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (1)$$

**Example.**

For instance, the solutions to  $x^2 + 2x - 3 = 0$  are:

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times -3}}{2 \times 1} \\ &= \frac{-2 \pm \sqrt{4 + 12}}{2} \\ &= \frac{-2 \pm \sqrt{16}}{2} \\ &= \frac{-2 \pm 4}{2} \end{aligned}$$

Hence,  $x = 1$  or  $x = -3$ .

## 1.2 The discriminant

**Definition 1.1** (Discriminant).

The **discriminant** of a quadratic equation with coefficients  $a, b, c \in \mathbb{R}$  is:

$$\Delta = b^2 - 4ac.$$

**Remark 1.2.**

**Note that** this is the expression beneath the square root symbol in the quadratic formula (1).

We can use the discriminant to determine the number of real roots of a quadratic equation. The number depends on the value of  $\Delta$  as in table 2.

Figure 1 shows an example of each possibility<sup>1</sup>.

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<sup>1</sup>The image is due to Olin, CC-BY-AS 3.0 downloaded from Wikimedia Commons

Value of $\Delta$	Real roots
$\Delta > 0$	Two, distinct
$\Delta = 0$	One, repeated
$\Delta < 0$	Zero

Table 2: Number of real roots of a quadratic equation, given the discriminant

## 2 The scalar product

Consider two vectors  $\mathbf{a}$  and  $\mathbf{b}$  drawn so their tails are at the same point.

We define the scalar product of  $\mathbf{a}$  and  $\mathbf{b}$  as follows.

**Definition 2.1** (Scalar product).

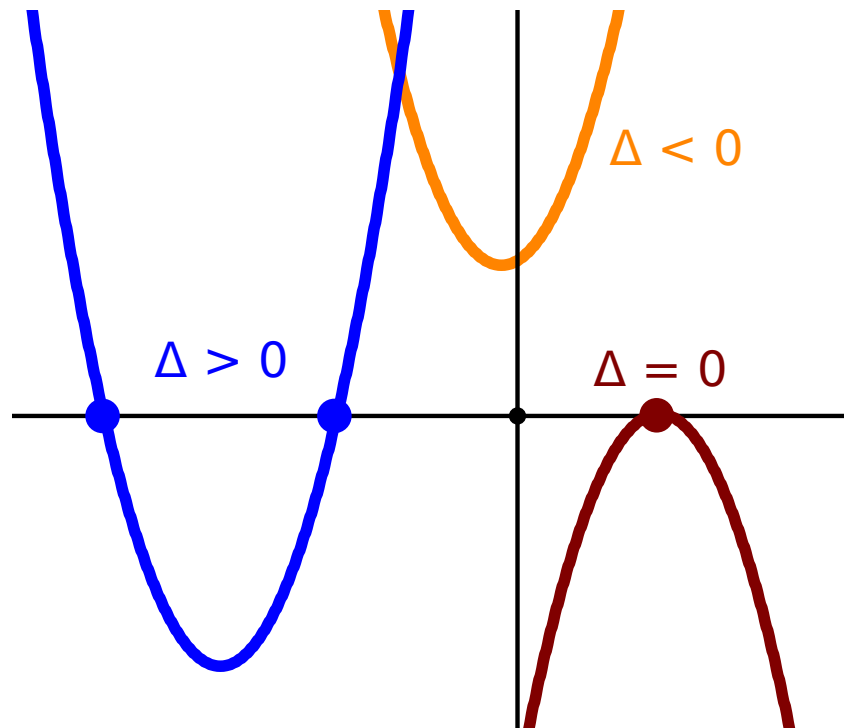


Figure 1: Examples of quadratic functions with zero, one and two real roots.

The **scalar product** of  $\mathbf{a}$  and  $\mathbf{b}$  is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

where

- $|\mathbf{a}|$  is the modulus of  $\mathbf{a}$ ,
- $|\mathbf{b}|$  is the modulus of  $\mathbf{b}$ , and



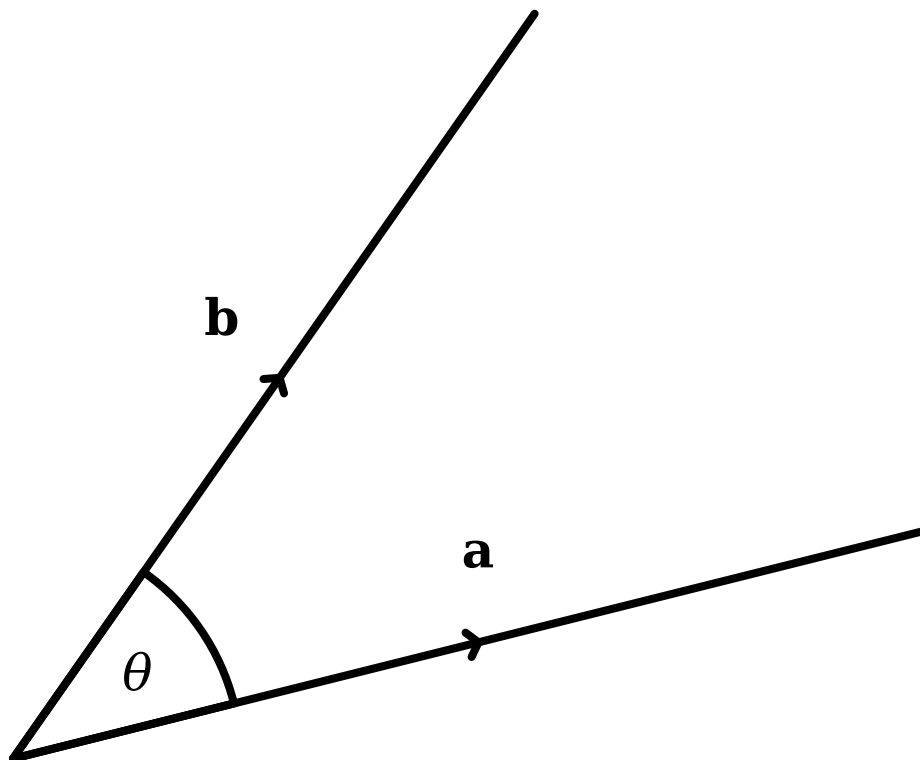


Figure 2: Two vectors with angle between them.

- $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ .

**Remark 2.2.**

It is important to use the dot symbol for the scalar product (also called the dot product). You must not use a  $\times$  symbol as this denotes the vector product which is defined differently.

**Example 2.3.**

Let

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

The angle between these vectors is  $\theta = 45^\circ$ . Then  $|\mathbf{a}| = \sqrt{8}$  and  $|\mathbf{b}| = 4$ . So,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \end{pmatrix} = |\mathbf{a}| |\mathbf{b}| \cos \theta \\ &= \sqrt{8} \times 4 \times \cos 45^\circ \\ &= 4\sqrt{8} \times \frac{1}{\sqrt{2}} = 4 \frac{\sqrt{8}}{\sqrt{2}} = 4\sqrt{4} = 8. \end{aligned}$$

Note that the result is a scalar, not a vector.

## 2.1 Vectors in cartesian form

When vectors are given in cartesian form there is an alternative formula for calculating the scalar product.

### Proposition 2.4.

If  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$  and  $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$  then

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2.$$

**Proof.** Consider the vector  $\mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$ .

The modulus of this is

$$|\mathbf{b} - \mathbf{a}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}.$$

Note from figure 3 that the vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{b} - \mathbf{a}$  form a triangle:

Let  $\theta$  denote the angle between  $\mathbf{a}$  and  $\mathbf{b}$ . Then,

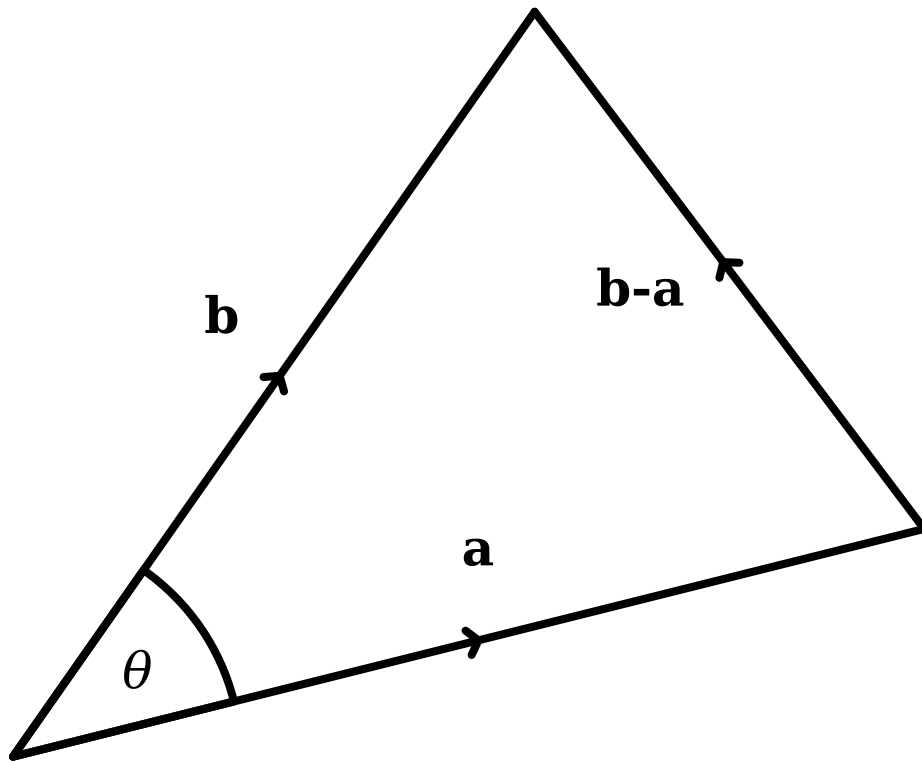


Figure 3: A triangle is formed by two vectors and their difference.

the cosine rule yields:

$$|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta. \quad (2)$$

Substituting the definition of the scalar product of **a** and **b** into equation 2 gives:

$$|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2(\mathbf{a} \cdot \mathbf{b}).$$

Rearranging:

$$(\mathbf{a} \cdot \mathbf{b}) = \frac{1}{2} (|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{b} - \mathbf{a}|^2).$$

Writing this in terms of components produces:

$$\begin{aligned}(\mathbf{a} \cdot \mathbf{b}) &= \frac{1}{2} (a_1^2 + a_2^2 + b_1^2 + b_2^2 - (b_1 - a_1)^2 - (b_2 - a_2)^2) \\&= \frac{1}{2} (a_1^2 + a_2^2 + b_1^2 + b_2^2 - b_1^2 + 2b_1a_1 - a_1^2 - b_2^2 + 2b_2a_2 \\&\quad - a_2^2) \\&= \frac{1}{2} (2b_1a_1 + 2b_2a_2) \\&= a_1b_1 + a_2b_2\end{aligned}$$

as required. □

**Example 2.5.**

Consider again the vectors

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

Calculating the scalar product using the components:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 = 2 \times 4 + 2 \times 0 = 8.$$

Note that if we are given vectors in this form, the scalar product may be used to calculate the angle between them. Since  $\mathbf{a} \cdot \mathbf{b} = 8$  and we have:

$$|\mathbf{a}| = \sqrt{8}$$

$$|\mathbf{b}| = 4.$$

Hence,

$$8 = \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

$$= 4\sqrt{8} \cos \theta.$$

Rearranging:

$$\theta = \cos^{-1} \left( \frac{8}{4\sqrt{8}} \right) = 45^\circ.$$

### 3 Using Matlab

To calculate the scalar product in Matlab the dot function is used.

Create two vectors:

```
> A = [4 -1 2];
```

```
> B = [2 -2 -1];
```

Calculate the scalar product:

```
> C = dot(A,B)
```

```
C = 8
```