MArefSetup

Emma Cliffe

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Contents

Test

This section should be in the contents page for the PDF but not the web or word.

1 Quadratic equations

A quadratic equation is an equation with the form $ax^2 + bx + c = 0$ where x represents an unknown and a, b and c are known numbers with $a \neq 0$.

1.1 Solutions to a quadratic equation

A solution to a quadratic equation is a value of x such that the equation balances. The solutions to quadratic equations can be found by using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

Example. *For instance, the solutions to* $x^2 + 2x - 3 = 0$ *are:*

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times -3}}{2 \times 1} \\ &= \frac{-2 \pm \sqrt{4 + 12}}{2} \\ &= \frac{-2 \pm \sqrt{16}}{2} \\ &= \frac{-2 \pm 4}{2} \end{aligned}$$

Hence, $x = 1$ *or* $x = -3$*.*

1.2 The discriminant

Definition 1.1 (Discriminant)**.** The *discriminant* of a quadratic equation with coefficients $a, b, c \in \mathbb{R}$ is:

$$\Delta = b^2 - 4ac.$$

Remark 1.2. **Note that** this is the expression beneath the square root symbol in the quadratic formula (1).

We can use the discriminant to determine the number of real roots of a quadratic equation. The number depends on the value of Δ as in table 1.

Value of Δ	Real roots
$\Delta > 0$	Two, distinct
$\Delta = 0$	One, repeated
$\Delta < 0$	Zero

Table 1: Number of real roots of a quadratic equation, given the discriminant

Figure 1 shows an example of each possibility[1].

[1] The image is due to Olin, CC-BY-AS 3.0 downloaded from Wikimedia Commons

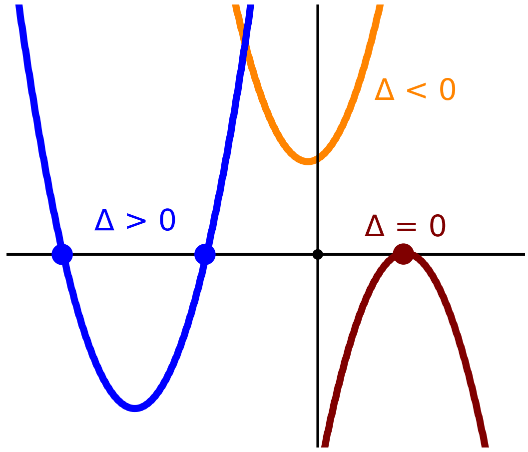

Figure 1: Examples of quadratic functions with zero, one and two real roots.

2 The scalar product

Consider two vectors **a** and **b** drawn so their tails are at the same point.

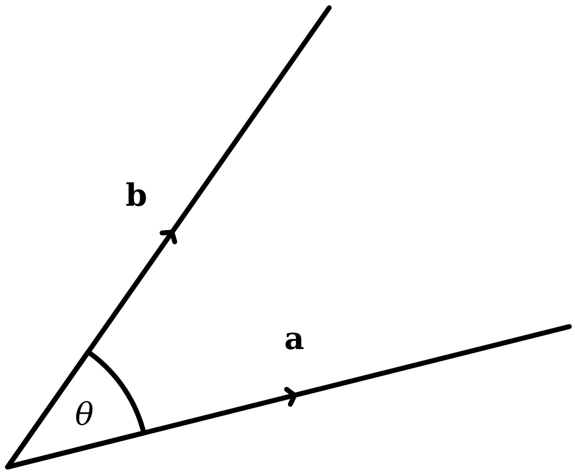

Figure 2: Two vectors with angle between them.

We define the scalar product of **a** and **b** as follows.

Definition 2.1 (Scalar product)**.** The *scalar product* of **a** and **b** is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

where

- $|\mathbf{a}|$ is the modulus of $\mathbf{a}$,
- $|\mathbf{b}|$ is the modulus of $\mathbf{b}$, and
- θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Remark 2.2. It is important to use the dot symbol for the scalar product (also called the dot product). You must not use a $\times$ symbol as this denotes the vector product which is defined differently.

Example 2.3. Let

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

The angle between these vectors is $\theta = 45°$. Then $|\mathbf{a}| = \sqrt{8}$ and $|\mathbf{b}| = 4$. So,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \end{pmatrix} &= |\mathbf{a}||\mathbf{b}| \cos\theta \\ &= \sqrt{8} \times 4 \times \cos 45° \\ &= 4\sqrt{8} \times \frac{1}{\sqrt{2}} = 4\frac{\sqrt{8}}{\sqrt{2}} = 4\sqrt{4} = 8. \end{aligned}$$

Note that the result is a scalar, not a vector.

2.1 Vectors in cartesian form

When vectors are given in cartesian form there is an alternative formula for calculating the scalar product.

Proposition 2.4. *If* $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ *and* $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ *then*

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

Proof. Consider the vector $\mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$. The modulus of this is

$$|\mathbf{b} - \mathbf{a}| = \sqrt{(b_1 - a_2)^2 + (b_2 - a_2)^2}.$$

Note from figure 3 that the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{b} - \mathbf{a}$ form a triangle:

Let θ denote the angle between $\mathbf{a}$ and $\mathbf{b}$. Then, the cosine rule yields:

$$|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos\theta. \tag{2}$$

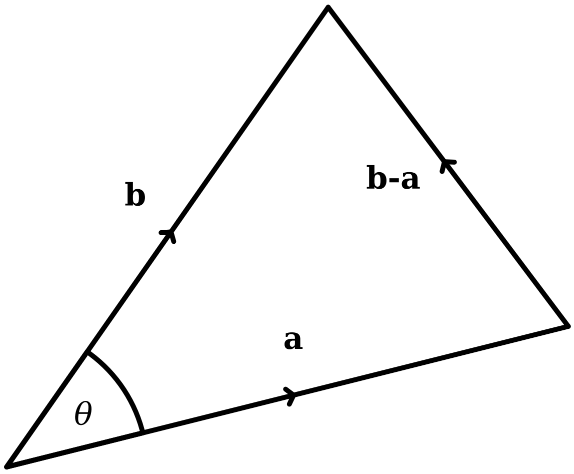

Figure 3: A triangle is formed by two vectors and their difference.

Substituting the definition of the scalar product of **a** and **b** into equation 2 gives:

$$|\mathbf{b}-\mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\left(\mathbf{a}\cdot\mathbf{b}\right).$$

Rearranging:

$$(\mathbf{a}\cdot\mathbf{b}) = \frac{1}{2}\left(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{b}-\mathbf{a}|^2\right).$$

Writing this in terms of components produces:

$$\begin{aligned}
(\mathbf{a}\cdot\mathbf{b}) &= \frac{1}{2}\left(a_1^2 + a_2^2 + b_1^2 + b_2^2 - (b_1-a_1)^2 - (b_2-a_2)^2\right)\\
&= \frac{1}{2}\left(a_1^2 + a_2^2 + b_1^2 + b_2^2 - b_1^2 + 2b_1a_1 - a_1^2 - b_2^2 + 2b_2a_2 - a_2^2\right)\\
&= \frac{1}{2}\left(2b_1a_1 + 2b_2a_2\right)\\
&= a_1b_1 + a_2b_2
\end{aligned}$$

as required. □

Example 2.5. Consider again the vectors

$$\mathbf{a} = \begin{pmatrix}2\\2\end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix}4\\0\end{pmatrix}.$$

Calculating the scalar product using the components:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 = 2\times 4 + 2\times 0 = 8.$$

Note that if we are given vectors in this form, the scalar product may be used to calculate the angle between them. Since $\mathbf{a} \cdot \mathbf{b} = 8$ and we have:

$$
\begin{aligned}
|\mathbf{a}| &= \sqrt{8} \\
|\mathbf{b}| &= 4.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
8 = \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos\theta \\
&= 4\sqrt{8}\cos\theta.
\end{aligned}
$$

Rearranging:

$$\theta = \cos^{-1}\left(\frac{8}{4\sqrt{8}}\right) = 45^\circ.$$

3 Using Matlab

Two calculate the scalar product in Matlab the `dot` function is used.

Create two vectors:

```
> A = [4 -1 2];
> B = [2 -2 -1];
```

Calculate the scalar product:

```
> C = dot(A,B)

    C = 8
```