Artificial variable

LPP in which constraints may also have $\geq$ and $=$ signs after insuring that all $b_i \geq 0$ are considered in this concept. In such cases basis matrix can't be obtained as an identity matrix in the starting simplex table, therefore we introduce a new type of variable called the artificial variable (no physical meaning)

To solve such LPP there are two methods

1. The Big M method (or) method of penalties
2. The two-phase simplex method

I The Big M method

1. Express the given problem in the std form.
2. Add non -ve artificial variables to the left side of each of the equations corresponding to the constraints of the type $\geq$ or $=$.

   $A_1, A_2, \ldots$

   However ~~addn~~ of addition of these artificial variables causes ~~void~~ voilation of the corresponding constraints. Therefore we would like to get rid of these variables and would not allow them to appear in the final solution. This is achieved by assigning a very large penalty ($-M$ for maximization and M for minimisation) in the objective function.

3. Solve the modified LPP by simplex method, until any one of ~~these~~ the 3 cases may arise

   (i) If no artificial variable appear in the basis and the optimality conditions are satisfied, then the current solution is an optimal basic feasible solution.

(ii) If atleast one artificial variable in the basis at zero level (RHS = 0) and the optimality condition is satisfied then the current solution is an optimal basic feasible solution.

(iii) If atleast one artificial variable appears in the basis at +ve level (RHS ≠ 0, ≥0) and the optimality condition satisfied, then the original solution has no feasible solution. The solution satisfies the constraints but does not optimise the objective function. Since it contains a very large penalty M and is called pseudo optimal solution.

Note: while applying simplex method whenever an artificial variable happens to leave the basis, we drop that artificial variable and omit all the entries corresponding to its column from the simplex table.

Ex ① Use penalty method (Big M method) to maximize $Z = 3x_1 + 2x_2$

$2x_1 + x_2 \le 2$, $3x_1 + 4x_2 \ge 12$, $x_1, x_2 \ge 0$.

Soln: By introducing slack variable S_1 to $\le$, surplus variable S_2 to ≥ 0 and artificial variable A_1 to ≥ 0, the given LPP can be formulated as

$$\text{Max } Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA_1 \Rightarrow Z - 3x_1 - 2x_2 + 0S_1 + 0S_2 + MA_1 = 0$$

$$\left.\begin{aligned} 2x_1 + x_2 + S_1 &= 2 \\ 3x_1 + 4x_2 - S_2 + A_1 &= 12 \end{aligned}\right\}$$

x_1, x_2, S_1, S_2, A_1 there are 5 variables

$m = 2$ (Equations)

$n - m = 5 - 2 = 3$

$\hookrightarrow$ $A_1 = 12 - 3x_1 - 4x_2 + S_2$ Substitute in the objective function

BV Basis	x_1	x_2	S_1	S_2	A_1	RHS	ratio
S_1	2	1	1	0	0	2	
A_1	3	4	0	-1	1	12	
z	-3	-2	0	0	M	0	
S_1	2	$\boxed{1}$	1	0	0	2	$\leftarrow$ $2/1 = 2$
A_1	3	4	0	-1	1	12	$12/4 = 3$
z	$-(3+3M)$	$-(2+4M)$ $\uparrow$	0	$-M$	0	$-12M$	$R_2' \to R_2 - 4R_1$
x_2	2	1	1	0	0	2	$R_3' \to R_3 + (2+4M)R_1$
A_1	-5	0	-4	-1	1	4	
z	$5M+1$	0	$4M+2$	M	0	$4-4M$	

In the bottom row all elements are ≥ 0 and an artificial variable appear in the basis [basic variables] at +ve level, the given LPP does not possess any feasible solution. But the LPP possess a pseudo optimal solution.

* Solve the LPP by Big-M method

Min $z = 2x_1 + x_2$, $3x_1 + x_2 = 3$, $4x_1 + 3x_2 \geq 6$, $x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0.$

Soln. Converting the objective function into maximization type

~~Max ...~~ $-\text{Min}(z) = \boxed{\text{Max } z^* = -2x_1 - x_2}$

Consider

$$\boxed{\begin{aligned} &3x_1 + x_2 + A_1 = 3 \\ &4x_1 + 3x_2 - S_1 + A_2 = 6 \\ &x_1 + 2x_2 + S_2 = 4 \end{aligned}}$$

Surplus variable

Slack variable

Max z

(or)

Max $z^* = -2x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

$z^* + 2x_1 + x_2 + MA_1 + MA_2 = 0$

~~$z^* + 2x_1 + x_2 + M$~~

Basis	x_1	x_2	S_1	S_2	A_1	A_2	RHS	ratio
S_2	1	2	0	1	0	0	4	
A_1	3	1	0	0	1	0	3	
A_2	4	3	-1	0	0	1	6	
z^*	2	1	0	0	M	M	0	

Since $z^* + 2x_1 + x_2 + MA_1 + MA_2 = 0$ Eliminating A_1 & A_2

$z^* + 2x_1 + x_2 + M[3 - 3x_1 - x_2] + M[6 + S_1 - 4x_1 - 3x_2] = 0$

$z^* + 2x_1 + x_2 + 3M - 3Mx_1 - Mx_2 + 6M + S_1M - 4Mx_1 - 3Mx_2 = 0$

$z^* + (2 - 7M)x_1 + (1 - 4M)x_2 + MS_1 = -9M$

Basis	x_1	x_2	S_1	S_2	A_1	A_2	RHS	ratio
S_2	1	2	0	1	0	0	4	$4/1 = 4$
A_1	$\boxed{3}$	1	0	0	1	0	3	$3/3 = 1$
A_2	4	3	-1	0	0	1	6	$6/4 = 1.5$
z^*	$2 - 7M$	$1 - 4M$	M	0	0	0	$-9M$	

$R_2 \to \frac{1}{3} R_2$

Basis	x_1	x_2	S_1	S_2	A_1	A_2	RHS	ratio
S_2	1	2	0	1	0	0	4	
x_1	$\boxed{1}$	$1/3$	0	0	$1/3$	0	1	
A_2	4	3	-1	0	0	1	6	
z^*	$2-7M$	$1-4M$	M	0	0	0	$-9M$	
S_2	0	$5/3$	0	1	$-1/3$	0	3	$\frac{3}{5/3}=9/5$
x_1	1	$1/3$	0	0	$1/3$	0	1	3
A_2	0	$\boxed{5/3}$	-1	0	$-4/3$	1	2 ←	$\frac{2}{5/3}=\frac{6}{5}$
z^*	0	$\frac{1}{3}-\frac{5M}{3}$ ↑	M	0	$-\frac{2}{3}+\frac{7}{3}M$	0	$-2M-2$	
S_2	0	$5/3$	0	1	$-1/3$	0	3	
x_1	1	$1/3$	0	0	$1/3$	0	1	
x_2	0	$\boxed{1}$	$-3/5$	0	$-4/5$	$3/5$	$6/5$	
z^*	0	$\frac{1}{3}-\frac{5M}{3}$	M	0	$-\frac{2}{3}+\frac{7}{3}M$	0	$-2M-2$	
S_2	0	0	1	1	1	-1	1	
x_1	1	0	$1/5$	0	$3/5$	$-1/5$	$3/5$	
x_2	0	1	$-3/5$	0	$-4/5$	$3/5$	$6/5$	
z^*	0	0	$\frac{1}{5}$	0	$-\frac{2}{5}+M$	$-\frac{1}{5}-M$	$-\frac{12}{5}$	

$R_1' \to R_1 - R_2$

$R_3' \to R_3 - 4R_2$

$R_4' \to R_4 - (2-7M)R_2$

$R_3' \to \frac{3}{5}R_3$

$R_1' \to R_1 - \frac{5}{3}R_3$

$R_2' \to R_2 - \frac{1}{3}R_3$

$R_4' \to R_4 - \left(\frac{1}{3}-\frac{5M}{3}\right)R_3$

$\therefore \; x_1 = 3/5 \qquad x_2 = 6/5$

$\text{Max } z^* = -12/5$

$\boxed{\text{Min } z = \frac{12}{5}}$

Verification: $\text{max } z^* = -2x_1 - x_2$

$= -2(3/5) - 6/5$

$= \frac{-12}{5}$

Two phase Simplex method (or) Two-phase method

Steps involved: This is Another method to deal with the
phase-I artificial variables wherein the LPP is solved in two
phases

phase-I

① Express the given problem in the standard form by introducing slack, surplus and artificial variables

② Formulate an artificial objective function by assigning (−1) cost to each of the artificial variables A_i and zero cost to all other variables

$$z^* = -A_1 - A_2 - \cdots - A_m$$

③ Maximize z^* subject to the constraints of the original problem using simplex method. Then 3-cases arise:

ⓐ Max $z^* < 0$ and atleast one artificial variable appears in the optimal basis at a +ve level (RHS).
In this case, the original problem does't possess any feasible solution and the procedure comes to an end.

ⓑ Max $z^* = 0$ and no artificial variable appears in the optimal basis.
In this case, a basic feasible solution is obtained and we proceed to phase-II for finding the optimal basic feasible solution to the original problem.

ⓒ Max $z^* = 0$ and at least one artificial variable appears in the optimal basis at zero level (RHS = 0).
Here a feasible solution to the auxillary LPP is also a feasible solution to the original problem with all the artificial variables set $= 0$

To obtain a basic feasible solution, we prolong phase I for pushing all the artificial variables out of the basis (without proceeding on to phase II).

Phase II. The basic feasible solution found at the end of phase I is used as the starting solution for the original problem in this phase *i.e.* the final simplex table of phase I is taken as the initial simplex table of phase II and the artificial objective function is replaced by the original objective function. Then we find the optimal solution.

Example 12.21. *Use two-phase method to*

Minimize $Z = 7.5x_1 - 3x_2$

subject to the constraints $3x_1 - x_2 - x_3 \geq 3,\ x_1 - x_2 + x_3 \geq 2,\ x_1, x_2, x_3 \geq 0.$

(*Andhra, B.E., 1999*)

Sol. Phase I. *Step 1. Express the problem in standard form.*

Introducing surplus variables s_1, s_2 and artificial variables A_1, A_2, the phase I problem in standard form becomes

$$\text{Max. } Z^* = 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 - A_1 - A_2$$

subject to
$$3x_1 - x_2 - x_3 - s_1 + 0s_2 + A_1 + 0A_2 = 3$$
$$x_1 - x_2 + x_3 + 0s_1 - s_2 + 0A_1 + A_2 = 2$$
$$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0.$$

Step 2. Find an initial basic feasible solution.

Setting $x_1 = x_2 = x_3 = s_1 = s_2 = 0$,

we have $A_1 = 3, A_2 = 2$ and $Z^* = -5$

$\therefore$ Initial simplex table is

c_B	c_j / Basis	0	0	0	0	0	− 1	− 1		
		x_1	x_2	x_3	s_1	s_2	A_1	A_2	b	θ
− 1	A_1	(3)	− 1	− 1	− 1	0	1	0	3	1 ←
− 1	A_2	1	− 1	1	0	− 1	0	1	2	2
	$Z_j^* = \Sigma c_B a_{ij}$	− 4	2	0	1	1	− 1	− 1	− 5	
	$C_j = c_j - Z_j^*$	4 ↑	− 2	0	− 1	− 1	0	0		

As C_j is positive under x_1 column, this solution is not optimal.

Step 3. Iterate towards an optimal solution.

Making key element (3) unity and replacing A_1 by x_1, we have the new simplex table :

c_B	c_j / Basis	0	0	0	0	0	− 1	− 1		
		x_1	x_2	x_3	s_1	s_2	A_1	A_2	b	θ
0	x_1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	1	− 3
− 1	A_2	0	$-\frac{2}{3}$	$\left(\frac{4}{3}\right)$	$\frac{1}{3}$	− 1	$-\frac{1}{3}$	1	1	$\frac{3}{4}$ ←
	Z_j^*	0	$\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{1}{3}$	1	$\frac{1}{3}$	− 1	− 1	
	C_j	0	$-\frac{2}{3}$	$\frac{4}{3}$ ↑	$\frac{1}{3}$	− 1	$-\frac{1}{3}$	0		

Since C_j is positive under x_3 and s_1 columns, this solution is not optimal.

Making key element (4/3) unity and replacing A_2 by x_3, we obtain the revised simplex table :

	c_j	0	0	0	0	0	– 1	– 1	
c_B	Basis	x_1	x_2	x_3	s_1	s_2	A_1	A_2	b
0	x_1	1	$-\frac{1}{2}$	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{4}$
0	x_3	0	$-\frac{1}{2}$	1	$\frac{1}{4}$	$-\frac{3}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
Z_j^*		0	0	0	0	0	0	0	0
C_j		0	0	0	0	0	– 1	– 1	

Since all $C_j \leq 0$, this table gives the optimal solution. Also $Z^*_{max} = 0$ and no artificial variable appears in the basis. Thus an optimal basic feasible solution to the auxiliary problem and therefore to the original problem, has been attained.

Phase II. Considering the actual costs associated with the original variables, the objective function is

$$\text{Max. } Z' = -15/2x_1 + 3x_2 + 0x_3 + 0s_1 + 0s_2 - 0A_1 - 0A_2$$

subject to $\quad 3x_1 - x_2 - x_3 - s_1 + 0s_2 + A_1 + 0A_2 = 3,$

$$x_1 - x_2 + x_3 + 0s_1 - s_2 + 0A_1 + A_2 = 2,$$

$$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$$

The optimal initial feasible solution thus obtained, will be an optimal basic feasible solution to the original *L.P.P.*

Using final table of phase I, the initial simplex table of phase II is as follows :

	c_j	– 15/2	3	0	0	0	
c_B	Basis	x_1	x_2	x_3	s_1	s_2	b
– 15/2	x_1	1	– 1/2	0	– 1/4	– 1/4	5/4
0	x_3	0	– 1/2	1	1/4	– 3/4	3/4
Z_j		– 15/2	15/4	0	15/8	15/8	– 75/8
C_j		0	– 3/4	0	– 15/8	– 15/8	

Since all $C_j \leq 0$, this solution is optimal.

Hence an optimal basic feasible solution to the given problem is

$$x_1 = 5/4,\ x_2 = 0,\ x_3 = 3/4 \text{ and min. } Z = 75/8.$$

12.10. EXCEPTIONAL CASES

(1) Tie for the incoming variable. When more than one variable has the same largest positive value in C_j row (in maximization problem), a tie for the choice of incoming variable occurs. As there is no method to break this tie, we choose any one of the prospective incoming variables arbitrarily. Such an arbitrary choice does'nt in any way affect the optimal solution.

(2) Tie for the outgoing variable. When more than one variable has the same least positive ratio under the θ-column, a tie for the choice of outgoing variable occurs. If the equal values of said ratio are > 1, choose any one of the prospective leaving variables arbitrarily. Such an arbitrary choice does'nt affect the optimal solution.