9/5/24

UNIT-II : Continuous Probability Distribution Function and Joint Probability Function

Exponential Distribution

Although binomial and poisson distribution can be used to solve many problems in engg. and science there are still numerous situations that require different type of density function. One such funcn is exponential distribution.

~~It is a~~ It has large number of application in the field of queuing theory and reliability process. Some of the important examples of exponential distribution are the waiting and arrival time, a person spend a time in a queue for service, and waiting time.

The continuous pdf $f(x)$ is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

is called exponential distribution where $\alpha > 0$

Q) Obtain mean, variance and S.D in case of exponential distribution function.

WKT pdf of exponential distribution is

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

I Mean, $\mu = E(X)$

$$= \int_{-\infty}^{\infty} x \, f(x) \, dx$$

$$= \int_{0}^{\infty} x \, \cancel{f(x)} \, \alpha e^{-\alpha x} \, dx$$

$$= \alpha \left[\frac{x e^{-\alpha x}}{-\alpha} - \frac{e^{-\alpha x}}{\alpha^2} \right]_0^\infty$$

$$= \alpha \left[0 - 0 - 0 + \frac{1}{\alpha^2} \right]$$

$$\boxed{\mu = \frac{1}{\alpha}}$$

II Variance, $V = E(X^2) - \mu^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^\infty x^2 f(x)\,dx$$

$$= \int_0^\infty x^2 \alpha e^{-\alpha x}\,dx$$

$$= \alpha \left[\frac{x^2 e^{-\alpha x}}{-\alpha} - 2x\frac{e^{-\alpha x}}{\alpha^2} + 2\frac{e^{-\alpha x}}{-\alpha^3} \right]_0^\infty$$

$$= \alpha \left[0 - 0 + 0 + \frac{2}{\alpha^3} \right]$$

$$\cancel{\boxed{V = \frac{1}{\alpha^2}}} \qquad E(X^2) = \frac{2}{\alpha^2}$$

III. $\therefore$ $V = E(X^2) - \mu^2$

$$= \frac{2}{\alpha^2} - \frac{1}{\alpha^2}$$

$$\boxed{V = \frac{1}{\alpha^2}}$$

IV. Standard deviation, $\sigma = \sqrt{V} = \sqrt{1/\alpha^2}$

$$\therefore \boxed{\sigma = \frac{1}{\alpha}}$$

① If X is exponential random variable with mean 5, Evaluate
i) $P[0 < X < 1]$
ii) $P[-\infty < X < 10]$
iii) $P[X \leq 0 \text{ or } X \geq 1]$

Given mean, $\mu = 5$

$\frac{1}{\alpha} = 5 \quad \therefore \boxed{\alpha = \frac{1}{5}}$

$\therefore$ The corresponding $f(x)$ is,

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{5} e^{-x/5} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a) $P[0 < X < 1] = \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{5} e^{-\frac{x}{5}}\,dx.$

$$= \frac{1}{\cancel{5}}\left[-\cancel{5}\, e^{-x/5}\right]_0^1$$

$$= -e^{-1/5} + 1$$

$$= \underline{0.1812}$$

b) $P[-\infty < X < 10] = \int_{-\infty}^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$

$$= \int_0^{10} \frac{1}{5} e^{-x/5}\,dx$$

$$= \left[-e^{-x/5}\right]_0^{10}$$

$$= -e^{-2} + 1$$

$$= \underline{0.8646}$$

c) $P[X \leq 0 \text{ or } X \geq 1]$

note: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P[X \leq 0 \text{ or } X \geq 1] = P[X \leq 0] + P[X \geq 1] - \cancel{P[X \leq 0 \text{ and } X \geq 1]}^{\,0}$

$$= \cancel{\int_{-\infty}^{0} f(x)\,dx}^{\,0} + \int_1^{\infty} f(x)\,dx$$

$$= \int_1^{\infty} \frac{1}{5} e^{-x/5}\,dx = \left[-e^{-x/5}\right]_1^{\infty}$$

$$= 0 + e^{-1/5}$$

$$= \underline{0.8187}$$

(2) The length of telephone conversation has been exponential distribution and found always to be 5 min. Find the probability that a random call made will

i) end less than 5 min

ii) b/w 5 and 10 min

Given mean, $\mu = 5$

$$\frac{1}{\alpha} = 5 \Rightarrow \boxed{\alpha = \frac{1}{5}}$$

$\therefore$ The corresponding pdf is

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases} = \begin{cases} \frac{1}{5} e^{-x/5}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

i) $$P[X < 5] = \int_{-\infty}^{5} f(x)\,dx = \int_{0}^{5} f(x)\,dx$$

$$= \int_{0}^{5} \frac{1}{5} e^{-x/5}\,dx$$

$$= \left[-e^{-x/5}\right]_0^5$$

$$= -e^{-1} + 1$$

$$= \underline{\underline{0.632}}$$

ii) $$P[\cancel{0}5 < X < 10] = \int_{5}^{10} f(x)\,dx = \int_{5}^{10} \frac{1}{5} e^{-x/5}\,dx$$

$$= \left[-e^{-x/5}\right]_5^{10}$$

$$= -e^{-2} + e^{-1}$$

$$= \underline{\underline{0.2325}}$$

(3) The increase in sales per day in a shop is exponentially distributed with mean Rs 600. The sales tax is to be levied at the rate of 9%. What is the probability that the sale tax will exceed Rs 81 on a particular day.

Mean $\mu = 600$

$$\frac{1}{\alpha} = 600 \Rightarrow \alpha = \frac{1}{600}$$

Let X be the amount of sales

$$\therefore f(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases} = \begin{cases} \frac{1}{600} e^{-x/600} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Let A be the amount of sales for which the sales tax is 81

Sales	Tax
100	9
A	81

$$A = \frac{81 \times 100}{9}$$

$$A = 900$$

$$\therefore P[X > 900] = \int_{900}^{\infty} f(x)\,dx = \int_{900}^{\infty} \frac{1}{600} e^{-x/600}\,dx$$

$$= \left[-e^{-x/600}\right]_{900}^{\infty}$$

$$= 0 + e^{-3/2}$$

$$= 0.2231$$

4. The sales per day in a shop is exponentially distributed with the mean sale amounting to be Rs 100 and the net profit is 8%. Find prob. for net profit exceeds Rs 30 on two consecutive day

Gamma Distribution

The continuous probability distribution having density function $f(x)$ defined as

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha} \cdot \Gamma(\alpha)} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

is called gamma distribution.

① $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$

② $\Gamma(n+1) = n\Gamma(n)$

③ $\Gamma(n+1) = n!$ if n is integer

Obtain mean, variance and S.D of gamma distribution

Let $f(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha} \Gamma(\alpha)} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$

I. Mean, $\mu = E(X)$

$$= \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{0}^{\infty} x \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)} dx.$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_{0}^{\infty} x^{\alpha} e^{-x/\beta} dx$$

put $\frac{x}{\beta} = t \Rightarrow x = \beta t \Rightarrow dx = \beta dt$

$$= \frac{1}{\cancel{\beta^{\alpha}}\, \Gamma(\alpha)} \int_{0}^{\infty} e^{-t} (\cancel{\beta} t)^{\alpha}\, \cancel{dt}\, \beta dt$$

$$= \frac{\beta}{\Gamma(\alpha)} \int_{0}^{\infty} e^{-t} t^{\alpha} dt$$

$$= \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha+1)$$

$$= \frac{\beta\, \alpha\, \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}$$

$$\boxed{\mu = \alpha\beta}$$

II. Variance, $V = E(X^2) - (E(X))^2$

But $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$$= \int_{0}^{\infty} x^2 \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)} dx.$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_{0}^{\infty} e^{-x/\beta} x^{\alpha+1} dx$$

put $\frac{x}{\beta} = t \Rightarrow dx = \beta dt$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^\infty \cancel{x^{\alpha+1}} \cancel{e^{-}} e^{-t} (\beta t)^{\alpha+1} \cdot \beta\, dt$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha+1} dt$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha+2)$$

$$= \frac{\beta^2 (\alpha+1) \Gamma(\alpha+1)}{\Gamma(\alpha)}$$

$$= \frac{\beta^2 (\alpha+1) \alpha \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}$$

$$= \alpha\beta^2(\alpha+1)$$

$$E(X^2) = \alpha^2\beta^2 + \alpha\beta^2$$

$$V = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2$$

$$\boxed{V = \alpha\beta^2}$$

IV. S.D, $\sigma = \sqrt{V}$

$$\boxed{\sigma = \beta\sqrt{\alpha}}$$

① The number of accidents per day on a certain highway follows the gamma distribution with an average 6 and variance 18. Find the probability that
i) there will be more than 8 accidents
ii) b/w ~~8~~ 5 and 8 accidents.

given $\mu = 6$ $\quad V = 18$

$\alpha\beta = 6$ $\quad \alpha\beta^2 = 18$

$\alpha\beta\beta = 18$

$6\beta = 18$

$\boxed{\beta = 3}$ $\quad \boxed{\alpha = 2}$

i) The corresponding pdf is

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

$$= \begin{cases} \dfrac{x\, e^{-x/3}}{9 \cdot 2!} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

$$f(x) = \begin{cases} \dfrac{x\, e^{-x/3}}{\cancel{18}\,9} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

i) $P[X > 8] = \int_8^\infty f(x)\,dx$

$$= \int_8^\infty \frac{x e^{-x/3}}{\cancel{18}\,9}\,dx$$

$$= \frac{1}{\cancel{18}\,9}\left[x e^{-x/3}(-3) - e^{-x/3}\cdot 9\right]_8^\infty$$

$$= \frac{1}{9\,\cancel{18}}\left[+24 e^{-8/3} + 9 e^{-8/3}\right]$$

$$= e^{-8/3} + \frac{8}{3} e^{-8/3}$$

$$= \underline{\underline{0.2547}}$$

ii) $P[5 < X < 8] = \int_5^8 f(x)\,dx$

$$= \int_5^8 \frac{x e^{-x/3}}{9}\,dx$$

$$= \frac{1}{9}\left[\cancel{0} - 3x e^{-x/3} - 9 e^{-x/3}\right]_5^8$$

$$= \frac{1}{9}\left[-24 e^{-8/3} - 9 e^{-8/3} + 15 e^{-5/3} + 9 e^{-5/3}\right]$$

$$= \underline{\underline{0.2489}}$$

1) The daily sales of certain brand of bicycles in a city in excess of 1000 piece is distributed as the gamma distribution with parameter $\alpha = 2$ and $\beta = 5$. The city has a daily stock 1500 pieces of the brand. Find the probability that the stock is insufficient on a particular day.

Given,

$\alpha = 2$, $\beta = 500$,

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

$$= \begin{cases} \dfrac{x e^{-x/5}}{500^2 \Gamma(2)} & x \geq 0 \\ 0 & x < 0 \end{cases} = \begin{cases} \dfrac{x e^{-x/5}}{250000} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Let X be sales of bicycles.

P[~~&~~ The stock is insufficient]

$= P[X > 1500]$

$$= \int_{1500}^{\infty} f(x)\,dx = \frac{1}{500^2}\int_{1500}^{\infty} x e^{-x/5}\,dx.$$

$$= \frac{1}{500^2}\left[-5x e^{-x/5} \cancel{+} -25 e^{-x/5}\right]_{1500}^{\infty}$$

$$= \frac{1}{500^2}\left[7500 e^{-300} \cancel{+} + 25 e^{-300}\right]$$

$=$

15/5/24 Uniform distribution (OR) Rectangular distribution

A continuous probability distribution having density function $f(x)$ given by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

is called uniform distribution.

Q) Obtain mean, variance and S.D of uniform distribution.

Let $f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$

I. Mean, $\mu = E(X)$

$$= \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{a}^{b} x \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)}$$

$$\boxed{\mu = \frac{b+a}{2}}$$

II. Variance, $V = E(X^2) - \mu^2$

But $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$

$$= \int_a^b x^2 \frac{1}{b-a} dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{1}{3\cancel{(b-a)}} \cancel{(b-a)}(b^2+ab+a^2)$$

$$E(X^2) = \frac{b^2+ab+a^2}{3}$$

$$V = \frac{b^2}{3} + \frac{ab}{3} + \frac{a^2}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{b^2}{3} + \frac{ab}{3} + \frac{a^2}{3} - \frac{b^2}{4} - \frac{2ab}{4} - \frac{a^2}{4}$$

$$= \cancel{b^3} \; \cancel{b^3} \; \frac{a^2 \cancel{+} - 2ab + b^2}{12}$$

$$\boxed{V = \frac{(a-b)^2}{12}}$$

III. S.D, $\sigma = \sqrt{V}$

$$\boxed{\sigma = \frac{a-b}{2\sqrt{3}}}$$

A random variable X is uniformly distributed over $(-3, 3)$. Find the value of k such that $P[X > k] = \frac{1}{3}$.

Given $a = -3$, $b = 3$

~~Let X~~ $f(x) = \begin{cases} \frac{1}{b-a} & \cancel{x \geq} \; a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$

$$= \begin{cases} \frac{1}{6} & -3 \leq x \leq 3 \\ 0 & \cancel{\text{otherwise}} \end{cases}$$

$$P[X > k] = \frac{1}{3}$$

$$\int_k^3 f(x)\,dx = \frac{1}{3} \quad \Rightarrow \cancel{\left[\frac{1}{6}x\right]} \; \frac{1}{6}$$

$$\frac{1}{2}\int_k^3 1\,dx = 1$$

$$[x]_k^3 = 2$$

$$3 - k = 2$$

$$k = 3 - 2$$

$$\boxed{k = 1}$$

(2) On a certain city transport route, buses arrive uniformly between zero and 30 min. If a person reaches the bus stop on this route a random time during this period, what is the probability he will have to wait for 20 min?

Since the buses arrive uniformly between 0 to 30, Let $a = 0$, $b = 30$.

$$f(x) = \begin{cases} \frac{1}{b-a} & 0 \leq x \leq 30 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{30} & 0 < x < 30 \\ 0 & \text{otherwise} \end{cases}$$

where X represents waiting time in bus stop

$$P[X \geq 20] = \int_{20}^{\infty} f(x)\,dx$$

$$= \int_{20}^{30} \frac{1}{30}\,dx$$

$$= \frac{1}{30}[x]_{20}^{30}$$

$$= \frac{30-20}{30}$$

$$= \frac{1}{3} = \underline{0.333}$$

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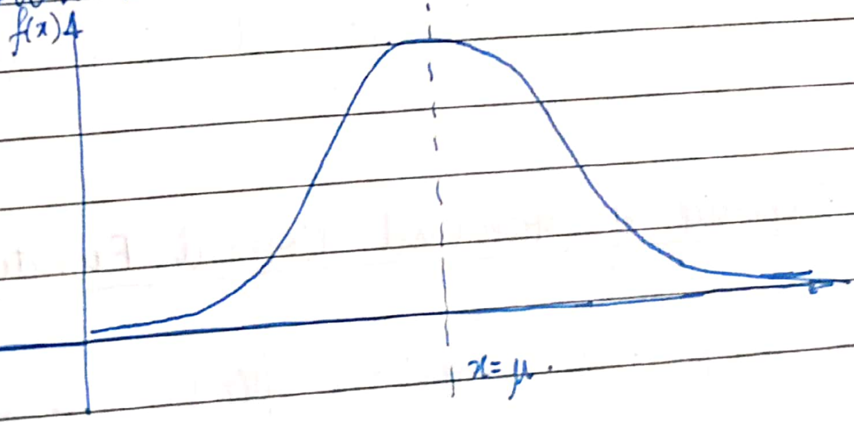
Normal Distribution (Gaussian distribution)

In probability theory, and statistics, the normal distribution also called the gaussian distribution is the most significant continuous probability distribution. Sometimes it is also called a bell shaped curve. A large number of random variables are either nearly or exactly represented by the normal distribution, in every physical science and economics. Further more, it can be used to approximate other probability distributions, Therefore supporting the usage of the word normal as about the one mostly used.

Let μ and σ be two arbitrary real constants such that $-\infty < \mu < +\infty$ and $\sigma > 0$. When the continuous probability distribution having probability density function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is called normal distribution. or gaussian distribution. named after a German Mathematician Gauss.

Properties of Normal Distribution curve:

Clearly it is found that a normal distribution curve is a bell shaped curve symmetrical about $x = \mu$ as shown below



② Since $f(x)$ is a valid probability distribution function, therefore the area under $\int_{-\infty}^{\infty} f(x)\,dx = 1$, this curve is 1.

Evaluation of normal probabilities

According to the definition,

$$P[a<X<b] = \int_a^b f(x)\,dx$$

$$= \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

put $\frac{x-\mu}{\sigma} = z$ then $x = \mu + \sigma z$

$dx = \sigma dz$

When $x = a$, $z = \frac{a-\mu}{\sigma} = z_1$

$x = b$, $z = \frac{b-\mu}{\sigma} = z_2$

$$P[a<x<b] = \int_{z_1}^{z_2} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot \cancel{\sigma} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{z^2}{2}} dz$$

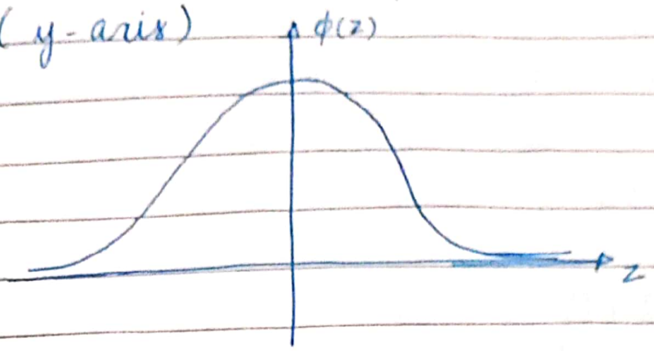
$$P[a<x<b] = \int_{z_1}^{z_2} \phi(z)\,dz \quad \text{where } \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

and $\phi(z)$ is referred to as standard normal (distribution) function.. The area under the standard normal curve is obtained using standard normal table.

Properties of Standard Normal Function $\phi(z)$

① Clearly the function $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is ~~symmetric~~

~~abo~~ also a bell shaped curve symmetric about $z = 0$. (y-axis)



clearly $\int_{-\infty}^{\infty} \phi(z)\, dz = 1$ i.e. the total area under the curve is 1. The area bounded by the curve in 1^{st} quadrant is 0.5. similarly in 2^{nd} quadrant the area bound is 0.5.

The area under the standard normal curve is obtained using standard normal distribution table.

$$z = \frac{x - \mu}{\sigma}$$

For the standard normal distribution of random variable z, evaluate the followings:

i) ~~Bso~~ $P[0 \leq Z \leq 1.45]$

$$P[0 \leq Z \leq 1.45] = \int_{0}^{1.45} \phi(z)\, dz$$

$$= \int_{0}^{1.45} \frac{1}{\sqrt{2}\sqrt{\pi}} e^{-z^2/2} dz$$

$$= \underline{\underline{0.4265}}$$

ii)

ii) $P[-2.6 \leq x \leq 0]$

$$= \int_{-2.6}^{0} \phi(z)\,dz = \int_{0}^{2.60} \phi(z)\,dz$$

$$= 0.4953$$

iii) $P[1.25 \leq z \leq 2.1]$

$$= \int_{1.25}^{2.1} \phi(z)\,dz = A(2.1) - A(1.25)$$

$$= 0.4821 - 0.3944$$

$$= 0.0877$$

iv) $P[z \geq 1.5] = \int_{1.5}^{\infty} \phi(z)\,dz$

$$= 0.5 - A(1.5)$$

$$= 0.5 - 0.4332$$

$$= 0.0668$$

(2) For the normal distribution with mean 2 and S.D 4 evaluate the following:

i) $P[X \geq 5]$ ii) $P[|X| \leq 4]$ iii) $P[|X| > 3)$

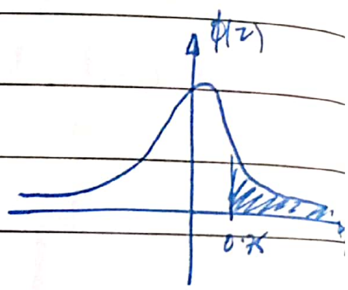
$\mu = 2 \quad \sigma = 4$

$$Z = \frac{x-\mu}{\sigma} = \frac{x-2}{4}$$

i) $P[X \geq 5]$

~~Z6~~ At $x = 5$, $Z = \frac{5-2}{4} = 0.75$

$$P(X \geq 5) = P(Z \geq 0.75)$$



$$= \int_{0.75}^{\infty} \phi(z)\,dz$$

$$= 0.\cancel{7}5 - A(0.75)$$

$$= 0.5 - 0.2734$$

$$= 0.2266$$

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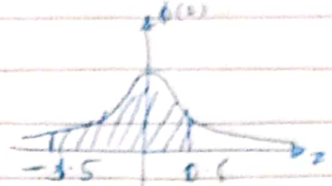
ii) $P[|X|<4] = P[-4<X<4]$

At $X=4$, $z = \frac{4-\cancel{8}2}{4} = 0.5$

$X=-4$, $z = \frac{-4-2}{4} = \cancel{-3}-1.5$



$\therefore P[-4<X<4] = P[\cancel{0}-1.5 < z < 0.5] = \int_{-1.5}^{0.5} \phi(z)dz$

$= A(1.5) + A(0.5)$

$= 0.4332 + 0.1915$

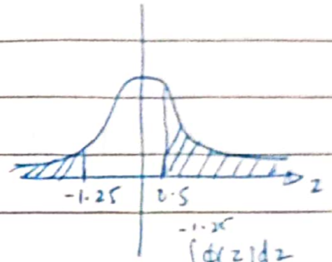
$= \underline{\underline{0.6247}}$

iii) $P[|X|>3] = P[X \leq -3$ and $X \geq 3]$

$= P[X \leq -3] + P[X \geq 3]$

when $x = -3$, $z = \frac{-3-2}{4} = -1.25$

$x = 3$, $z = \frac{3-2}{4} = 0.25$



$\int_{-\infty}^{-1.25} \phi(z)dz + \int_{0.25}^{\infty} \phi(z)dz$

$\therefore P[X \leq -3] + P[X \geq 3] = P[z \leq -1.25] + P[z \geq 0.25]$

$= [0.5 - A(1.25)] + [0.5 - A(0.25)]$

$= (0.5 - 0.3944) + (0.5 - 0.0987)$

$= \underline{\underline{0.5069}}$

(or)

$P[|X| \cancel{<} > 3] = 1 - P[|X| \leq 3] = 1 - (A(+1.25) + A(0.25))$

$= \cancel{0.5069}$

$= 1 - (0.3944 + 0.0987)$

$= \underline{\underline{0.5069}}$

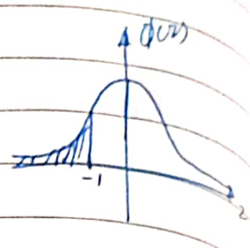
③ The marks of thousand students in an exam follow a normal distribution with mean 70 and S.D 5. Find the no. of students whose marks will be a) less than 65 b) more than 75 c) b/w 65 and 75

Here $\mu = 70$, $\sigma = 5$

$\therefore Z = \dfrac{X-\mu}{\sigma} = \dfrac{x-70}{5}$ Let X be the marks of the

i) P[less than 65] = P[X < 65]

When $x = 65$, $Z = \dfrac{65-70}{5} = -1$



$\therefore P[X<65] = P[Z < -1]$

$= \int_{-\infty}^{-1} \phi(z)\,dz$

$= 0.5 - A(1)$

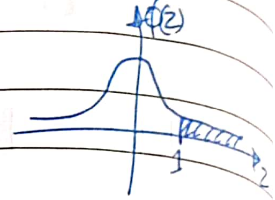
$= 0.5 - 0.3413$

$= \underline{0.1587}$

$\therefore$ Approx No. of students $= 0.1587 \times 1000 = 158.7 \approx \underline{159}$

ii) P[more than 75] = P[X > 75]

When $x = 75$, $Z = \dfrac{75-70}{5} = 1$.



$\therefore P[X>75] = P[Z>1]$

$= \int_{1}^{\infty} \phi(z)\,dz$

$= 0.5 - A(1)$

$= 0.1587$.

$\therefore$ Approx no. of students $= 0.1587 \times 1000$

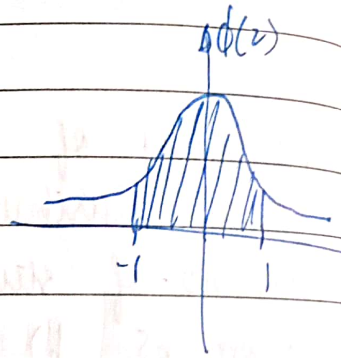
$= 158.7$

$\approx \underline{159}$

iii) P[b/w 65 and 75] = P[65 < X < 75]

When $x = 65$; $Z = -1$

$x = 75 \Rightarrow Z = 1$



$\therefore P[65<X<75] = P[-1<Z<1]$

$= \int_{-1}^{1} \phi(z)\,dz$

$= A(1) + A(1)$

$= \cancel{0}$

$= \cancel{0.1587 + 0.1587}$ $= 0.3413 + 0.3413$

$\cancel{= 0.3174}$ $= 0.6826$

$\therefore$ Approx No. of students $= 0.6826 \times 1000$

$= \cancel{317.4}\ 682.6$

$\cancel{\approx 318}$ ≈ 682

④ In a test of 2000 electric bulbs, it was found that the life of particular bulb is normally distributed with the mean life of ~~2000~~ 2040 hrs and S.D of 60 hrs. Estimate the number of bulbs likely to burn for
i) More than 2150 hrs
ii) Less than 1950 hrs
iii) more than 1920 hrs but less than 2160 hrs.

$\mu = 2040$ $\sigma = 60$

$z = \frac{x-\mu}{\sigma} = \frac{x-2040}{60}$

Let x be life of particular bulb

i) $P[x > 2150]$.

When $x = 2150$, $z = \frac{2150-2040}{60} = 1.833$

$$P[x > 2150] = P[z > 1.833] = \int_{1.83}^{\infty} \phi(z)\,dz$$

$= 0.5 - A(1.83)$

$= 0.5 - 0.4664$

$=$

ii) $P[x < 1950]$

When $x = 1950$, $z = \frac{1950-2040}{60} =$

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(5) In a normal distribution, 7% are under 35 and 89% are under 63. Find the mean and S.D.

(OR)

In an exam, 7% of the students scored less than 35 marks and 89% of student scored less than 63. Find the mean and S.D of their marks. Assume that marks are normally distributed.

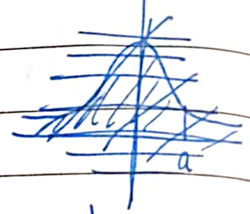
Let x be the marks of student.

Given 7% of students under 35 marks.

$$P[X<35] = 7\% = 0.07$$

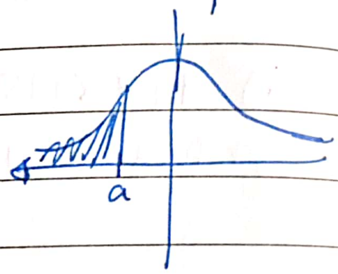
When $x = 35$, $z = \frac{x-\mu}{\sigma} = \frac{35-\mu}{\sigma} = a$ (say)

$$P[x<35] = P[z<a] = 0.07$$



$$\int_{-\infty}^{a} \phi(z)\,dz = 0.07.$$

$$\cancel{0.5 - A(a) + 0.5 = 0.07}$$



$$0.5 - A(-a) = 0.07$$

$$A(-a) = 0.43$$

From table,

$\boxed{\cancel{-a = 0.11}}$ $\boxed{\cancel{a = -0.11}}$

$\boxed{a = -1.48}$ $\Rightarrow \frac{35-\mu}{\sigma} = -1.48$.

$\Rightarrow \boxed{\mu - 1.48\sigma = 35}$ — (1)

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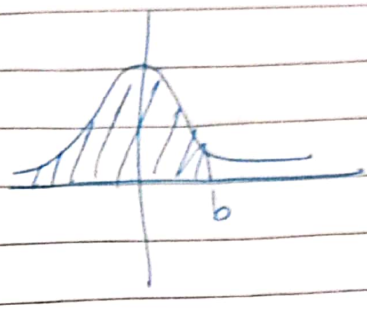
given 89% of students under 63 marks

$P[X<63] = 89\% = 0.89$

when $x = 63$, $z = \frac{63-\mu}{\sigma} = b$.



$P[X<63] = P[Z<b] = 0.89$.

$\int_{-\infty}^{b} \phi(z)\, dz = 0.89$.

$0.5 + A(b) = 0.89$

$A(b) = 0.39$

$\therefore$ ~~b = 0.8~~ $\boxed{b = 1.23}$

$\frac{63-\mu}{\sigma} = 1.23$.

$\mu + 1.23\sigma = 63$ — (2)

$\cancel{\mu = 42.86}$

$\cancel{\sigma = 16.37}$

$\boxed{\begin{aligned}\mu &= 50.29 \\ \sigma &= 10.33\end{aligned}}$

In a normal distribution, 16% of the items are under 39 and 72% are over 80. Find the mean and S.D.

Joint Probability Distribution

→ Two Dimension Random Variable

Let S be the sample space ~~set~~ associated with the random experiment. Let $X = X(s)$ and $Y = Y(s)$ be the two random variables on the sample space S. The two real value functions $X(s)$ and $Y(s)$ ~~is~~ each assign a real number to each outcome of the experiment.

For example, ① X - Age and Y - BP of a person

② X - Crop yield and Y - amount of rainfall

③ X - IQ and Y - nutrition of individual

→ **Joint Probability Function (Discrete case)**

If X and Y are two discrete random variables, the the joint probability function is given by

$$f(x_i, y_j) = P[x = x_i, y = y_j]$$

which gives the probability of simultaneous occurance of x_i and y_j.

Further $f(x_i, y_i)$ satisfies the following conditions:

i) $f(x_i, y_j) \geq 0$

ii) $\sum_i \sum_j f(x_i, y_j) = 1$

Mathematically a joint probability function is represented as below:

X \ Y	y_1	y_2	y_3
x_1	J_{11}	J_{12}	J_{13}
x_2	J_{21}	J_{22}	J_{23}
y_3	J_{31}	J_{32}	J_{33}

where $J_{ij} = P[x = x_i, y = y_i]$

Marginal Distribution

The marginal distribution wrt the random variable is the probability ~~of x~~ distribution of x alone obtained by summing over the values of y ie.

x_i	x_1	x_2	x_3
$f(x_i)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

Similarly the marginal distribution $g(y)$ wrt y is the probability distribution of y alone obtained by summing over the value of x.

y_i	y_1	y_2	y_3
$f(y_i)$	$f(y_1)$	$f(y_2)$	$f(y_3)$

Explanation:

$$E(X) = \sum_i x_i f(x_i)$$

$$E(Y) = \sum_j y_j f(x_j)$$

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$$E(XY) = \sum_i \sum_j x_i y_j J_{ij}$$

Covariance

Covariance is a measure of how much two var, variables changes together. If X and Y are two random variables, then the covariance of x & y is defined as

$$Cov(x,y) = E(XY) - E(X) \cdot E(Y)$$

Correlation coefficient

If change in one variable corresponds to change another variable, y, then x and y are correlated and it is defined as

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

where $\sigma_x^2 = E(X^2) - E(X)^2$

$\sigma_y^2 = E(Y^2) - E(Y)^2$

① An unbiased coin is tossed three times. Let X denotes 0 and 1 according as the head or tail occurs on first toss. Let y denotes number of heads which occur. i) Find the joint distribution of x and y

ii) Find the marginal distribution of x and y

iii) Covariance of x & y i.e $Cov(x,y)$

iv) $\rho(x,y)$

Let the sample space of expt is

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

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S	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
X	0	0	0	0	1	1	1	1
Y	3	2	2	1	2	1	1	0

$\therefore$ Range of $X = \{0, 1\}$
Range of $Y = \{0, 1, 2, 3\}$

$P[X=0, Y=0] = 0$ $\quad$ $P[X=1, Y=0] = 1/8$
$P[X=0, Y=1] = 1/8$ $\quad$ $P[X=1, Y=1] = 1/4$
$P[X=0, Y=2] = 1/4$ $\quad$ $P[X=1, Y=2] = 1/8$
$P[X=0, Y=3] = 1/8$ $\quad$ $P[X=1, Y=3] = 0$
~~P[X=2~~

i) $\therefore$ The joint probability function is

X \ Y	0	1	2	3
0	0	1/8	1/4	1/8
1	1/8	1/4	1/8	0

) Marginal distribution wrt X:

X	0	1
f(x)	1/2	1/2

Marginal distribution wrt Y:

Y	0	1	2	3
f(y)	1/8	3/8	3/8	1/8

(c) $E(XY) = \sum_i \sum_j x_i y_j J_{ij}$

$= \cancel{0\times 0\times 0} + \cancel{0\times 1\times \frac{1}{8}} + \cancel{0\times 2\times \frac{1}{4}} + \cancel{0\times 3\times \frac{1}{8}} +$
$\cancel{1\times 0\times \frac{1}{8}} + 1\times 1\times \frac{1}{4} + 1\times 2\times \frac{1}{8} + \cancel{1\times 3\times 0}$

$= \frac{1}{\ldots}$

$E(X) = \sum_i x_i f(x_i) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$

$E(Y) = \sum_j y_j f(y_j) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$

$= \frac{3}{2}$

$\therefore Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$

$= \frac{1}{2} - \frac{1}{2} \times \frac{3}{2}$

$= -\frac{1}{4} = \underline{\underline{-0.25}}$

iv) $\sigma_x^2 = E(x^2) - E(X)^2$

$= \sum x^2 f(x) - \left(\frac{1}{2}\right)^2$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\boxed{\sigma_x = \sqrt{\frac{1}{4}} = \frac{1}{2}}$

$\sigma_y^2 = E(Y^2) - E(Y)^2$

$= \sum y^2 f(y) - \left(\frac{3}{2}\right)^2$

$= \frac{3}{8} + \frac{12}{8} + \frac{9}{8} - \frac{9}{4}$

$= \frac{24 - 18}{8}$

$= \frac{3}{4}$

$\boxed{\sigma_y = \frac{\sqrt{3}}{2}}$

$\therefore \rho(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$

$= \frac{-1/4}{\frac{1}{2} \times \frac{\sqrt{3}}{2}}$

$= \frac{-1}{\sqrt{3}} = \underline{\underline{-0.5773}}$

23/5/24

(2) Two cards are selected at a random from a box which contains five cards numbered 1, 1, 2, 2, 3. Find the joint distribution of X and Y where X denotes the sum and Y denotes the maximum of two numbers. Determine Cov(X, Y) and correlation coefficient.

~~Let the sample space S = {(1,1), (1,2), (1,2), (1,3),~~
~~(1,1), (1,2), (1,2), (1,3), (2,1), (2,1), (2,2), (2,3),~~
~~(2,1), (2,1), (2,2), (2,3), (3,1),~~

Let the sample space S

$S = \{(1,1), (1,2), (1,2), (1,3), (1,1), (1,2), (1,2), (1,3)$
$(2,1), (2,1), (2,2), (2,3), (2,1), (2,1), (2,2), (2,3),$
$(3,1), (3,1), (3,2), (3,2), \cancel{(3,2)}\}$

X - Sum of two numbers.

Range of X = {2, 3, 4, 5}

Y - max of two numbers

Range of Y = {1, 2, 3}.

∴ The corresponding joint distribution is

X \ Y	1	2	3
2	1/10	0	0
3	0	2/5	0
4	0	1/10	~~2/5~~ 1/5
5	0	0	1/5

$$E(XY) = \sum_i \sum_j x_i y_j J_{ij}$$

$$= 2 \times 1 \times \frac{1}{10} + 3 \times 2 \times \frac{4}{10} + 4 \times 2 \times \frac{1}{10} + 4 \times 3 \times \frac{2}{10} + 5 \times 3 \times \frac{2}{10}$$

$$= \underline{\underline{8.8}}$$

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Marginal distribution wrt X:

x_i	2	3	4	5
$f(x_i)$	$1/10$	$2/5$	$3/10$	$1/5$

Marginal distribution wrt Y:

y	1	2	3
$f(y)$	$1/10$	$1/2$	$2/5$

$E(X) = \sum x f(x) = \frac{2}{10} + \frac{12}{10} + \frac{12}{10} + \frac{10}{10} = 3.6.$

$E(Y) = \sum y f(y) = \frac{1}{10} + \frac{10}{10} + \frac{12}{10} = 2.3$

$Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$

$= 8.8 - 3.6 \times 2.3$

$= \underline{\underline{0.52}}.$

$\sigma_x^2 = E(X^2) - E(X)^2$

$= \sum x^2 f(x) - (3.6)^2$

$= \frac{4}{10} + \frac{36}{10} + \frac{48}{10} + \frac{50}{10} - (3.6)^2$

$= 0.84$

$\sigma_x = \frac{\sqrt{21}}{5}$

$\sigma_y^2 = E(Y^2) - E(Y)^2$

$= \sum y^2 f(y) - (2.3)^2$

$= \frac{1}{10} + \frac{20}{10} + \frac{36}{10} - (2.3)^2$

$= 0.41$

$\sigma_y = \frac{\sqrt{41}}{10}$

$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \frac{\frac{52}{100}}{\frac{\sqrt{21}\,\sqrt{41}}{50}}.$

$\underline{\underline{\rho(X,Y) = 0.8860}}$

A bag contains 2 red, 3 blue and 3 green bulbs. If X denotes no. of blue bulbs and Y denotes no. of red bulbs. Suppose two bulbs are selected at random. Find the joint distribution of X and Y. Also find the marginal distribution of X and Y. And also find ~~probability of X+Y <~~ $P[X+Y \leq 1]$.

The no. of ways of selecting two bulbs out of 8 is $^8C_2 = 28$.

Range of $X = \{0, 1, 2\}$

Range of $Y = \{0, 1, 2\}$

The corresponding joint distribution is

X \ Y	0	1	2
0	3/28	6/28	1/28
1	9/28	6/28	0
2	3/28	0	0

$P[X=0, Y=0] = \frac{^3C_2}{^8C_2} = \frac{3}{28}$

$P[X=1, Y=0] = \frac{^3C_1 \times ^3C_1}{^8C_2} = \frac{9}{28}$

$P[X=0\ Y=1] = \frac{^2C_1 \times ^3C_1}{^8C_2} = \frac{6}{28}$

$P[X=1, Y=1] = \frac{^3C_1 \times ^2C_1}{^8C_2} = \frac{6}{28}$

$P[X=0, Y=2] = \frac{^2C_2}{^8C_2} = \frac{1}{28}$

$P[X=1, Y=2] = 0$

$P[X=2, Y=0] = \frac{^3C_2}{^8C_2} = \frac{3}{28}$

$P[X=2, Y=2] = 0$

$P[X=2, Y=1] = 0$

Marginal distribution wrt X

x_i	0	1	2
$f(x_i)$	10/28	15/28	3/28

marginal distribution wrt Y:

y_i	0	1	2
$f(y_i)$	15/28	12/28	1/28

$P[X+Y \leq 1] = P[X=0, Y=0] + P[X=0, Y=1] + P[X=1, Y=0]$

$= \frac{3}{28} + \frac{6}{28} + \frac{9}{28}$

$= \frac{18}{28}$

$= 0.6428$

Joint Distribution (Continuous Random Variable)

Let X and Y be two continuous random variable. If $f(x,y)$ is a real value function satisfying the following conditions:

i) $f(x,y) \geq 0$

ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Then the function $f(x,y)$ is called joint probability function / density function.

Consequently the probability that X lies ~~bet~~ in the interval (a, b) and Y lies in the interval (c, d) is given by $P[a < X < b, c < Y < d] = \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dx\,dy$

Marginal probability distribution

WRT random variable X is defined as

$$h(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

WRT random variable Y is defined as

$$g(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

27/5/24 **Expectation:**

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$E(Y) = \int_{-\infty}^{\infty} y\, g(y)\,dy$$

note: If X and ~~of~~ Y are independent random variables, then
$E(XY) = E(X)\cdot E(Y)$

Covariance:

$Cov(X, Y) = E(XY) - E(X)\cdot E(Y)$

Again, if X and Y are independent then
$Cov(X, Y) = 0$

① The joint ~~prot~~ pdf of two random variables X and Y is given by $f(x,y) = \begin{cases} K(x+y), & 0 < x < 2 \quad 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$

i) Find the value of k.
ii) Find the marginal pdf of X and Y
iii) Are X and Y are independent?

given $f(x,y) = \begin{cases} K(x+y) & 0 < x < 2 \quad 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$

i) Since $f(x,y)$ is a valid pdf,

WKT $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) = 1$

$$\int_0^2\int_0^2 k(x+y)\,dx\,dy = 1 \Rightarrow k\int_0^2\int_0^2 k(x+y)\,dy\,dx$$

$$k\int_0^2 \left| xy + \frac{y^2}{2} \right|_0^2 dx = 1$$

$$k\int_0^2 2x + 2 \, dx = 1$$

$$k\left[x^2 + 2x\right]_0^2 = 1$$

$$k\left[4 + 4\right] = 1$$

$$\boxed{k = \frac{1}{8}}$$

ii) Marginal distribution wrt ~~X~~ Y

$$g(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^2 k(x+y)\,dx.$$

$$= \frac{1}{8}\left[\frac{x^2}{2} + xy\right]_0^2$$

$$g(y) = \frac{1}{8}\left[2 + 2y\right] = \frac{y+1}{4}$$

Marginal distribution wrt X

$$h(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$$

$$= \int_0^2 \frac{1}{8}(x+y)\,dy$$

$$= \frac{1}{8}\left[xy + \frac{y^2}{2}\right]_0^2$$

$$h(x) = \frac{1}{8}\left[2x + 2\right] = \frac{x+1}{4}$$

iii) $$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, k(x+y)\,dx\,dy$$

$$= \frac{1}{8}\int_0^2\int_0^2 x^2y + xy^2\,dx\,dy$$

$$= \frac{1}{8}\int_0^2 \left[\frac{x^3 y}{3} + \frac{x^2 y^2}{2}\right]_0^2 dy$$

$$= \frac{1}{8}\int_0^2 \left[\frac{8}{3}y + 2y^2\right]dy$$

$$= \frac{1}{8}\left[\frac{\cancel{8}4y^2}{\cancel{6}3} + \frac{2}{3}y^3\right]_0^2$$

$$= \frac{1}{8}\left[\frac{8\times 2}{3} + \frac{8\times 2}{3}\right] = \frac{4}{3}.$$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x, y)\, dx$$

$$= \int_0^2 x\, \frac{1}{8}[2x+2]\, dx = \frac{1}{8}\int_0^2 2x^2 + 2x\, dx$$

$$= \frac{1}{8}\left[\frac{2}{3}x^3 + x^2\right]_0^2$$

$$= \frac{1}{2}\left[\frac{4}{3} + 1\right] = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}.$$

$$E(Y) = \int_{-\infty}^{\infty} y\, g(y)\, dy = \int_0^2 \frac{1}{8} y\, (2+2y)\, dy$$

$$= \frac{1}{8}\int_0^2 2y + 2y^2\, dy = \frac{1}{8}\left[y^2 + \frac{2y^3}{3}\right]_0^2$$

$$= \frac{1}{8}\left[4 + \frac{16}{3}\right]$$

$$= \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$$

$$\mathrm{COV}(X,Y) = E(XY) - E(X).E(Y)$$

$$= \frac{4}{3} - \frac{49}{36} \neq 0.$$

$\therefore$ ~~X are~~ $\therefore$ X and Y are not independent.

The joint pdf of a two dimensional random variable X and Y is given by

$$f(x,y) = xy^2 + \frac{x^2}{8} \quad 0 < x < 2 \quad 0 \le y \le 1$$

i) $P[X>1]$ ii) $P[Y<1/2]$

Let $f(x,y) = xy^2 + \frac{x^2}{8}$ $0 \le x \le 2$, $0 \le y \le 1$

i) $P[X>1] = \int_{y=0}^{1}\int_{x=1}^{2} f(x,y)\,dx\,dy$

$$= \int_0^1\int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx\,dy$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_1^2 dy$$

$$= \int_0^1 \left[2y^2 + \frac{1}{3} - \frac{y^2}{2} - \frac{1}{24}\right] dy$$

$$= \int_0^1 \left[\frac{3}{2}y^2 + \frac{7}{24}\right] dy$$

$$= \left[\frac{y^3}{2} + \frac{7y}{24}\right]_0^1$$

$$= \frac{1}{2} + \frac{7}{24}$$

$$= \frac{19}{24}$$

ii) $P[Y<1/2] = \int_{x=0}^{2}\int_{y=0}^{1/2} f(x,y)\,dy\,dx$

$$= \int_0^2\int_0^{0.5} xy^2 + \frac{x^2}{8}\,dy\,dx$$

$$= \int_0^2 \left[\frac{xy^3}{3} + \frac{x^2}{8}y\right]_0^{1/2} dx$$

$= \int_0^2 \frac{x}{24} + \frac{x^2}{64/16}\, dx$

$= \left[\frac{x^2}{48} + \frac{x^3}{16 \times 3} \right]_0^2$

$= \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$

(3) Let X and Y be continuous random variables having joint density function given by

$$f(x,y) = \begin{cases} 4xy & 0<x<1 \quad 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

i) Verify that $E(XY) = E(X) \cdot E(Y)$

ii) $E(X+Y) = E(X) + E(Y)$

Let $f(x,y) = \begin{cases} 4xy & 0<x<1 \quad 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

i) $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$

$= \int_0^1\int_0^1 4xy \cdot xy\, dx\, dy$

$= \int_0^1\int_0^1 4x^2y^2\, dx\, dy = \int_0^1 \left[\frac{4}{3}x^3y^2\right]_0^1 dy$

$= \int_0^1 \frac{4}{3}y^2\, dy = \left[\frac{4}{9}y^3\right]_0^1$

$= \frac{4}{9}$

$h(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^1 4xy\, dy = 4\left[\frac{xy^2}{2}\right]_0^1 = 2x$

$g(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^1 4xy\, dx = 4\left[\frac{x^2y}{2}\right]_0^1 = 2y$

$E(X) = \int_{-\infty}^{\infty} x\, h(x)\, dx = \int_0^1 x\, 2x\, dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$

$E(Y) = \int_{-\infty}^{\infty} y\, g(y)\, dy = \int_0^1 y\, 2y\, dx = 2\left[\frac{y^3}{3}\right]_0^1 = \frac{2}{3}$

$E(X)\cdot E(Y) = \frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9} = E(XY)$

$\therefore E(XY) = E(X)\cdot E(Y)$ [~~They~~ They are independent

)

$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f(x,y)\, dx\, dy$

$= \int_0^1\int_0^1 (x+y)\, 4xy\, dx\, dy = \int_0^1\int_0^1 4x^2y + 4xy^2\, dx\, dy$

$= \int_0^1 \left[\frac{4}{3}x^3y + 2x^2y^2\right]_0^1 dy$

$= \int_0^1 \left[\frac{4}{3}y + 2y^2\right] dy$

$= \left[\frac{2y^2}{3} + \frac{2}{3}y^3\right]_0^1$

$= \frac{2}{3} + \frac{2}{3}$

$= \frac{4}{3}$

$E(X) + E(Y) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} = E(X+Y)$

$\therefore E(X+Y) = \underline{E(X)} + E(Y)$