4. (15) Consider the model, described by the following state and observation equation

$$x_t = x_{t-1} + u_{t-1} + \epsilon_{t-1}, \quad z_t = x_t + \eta_t \tag{6}$$

with $P(\epsilon_t) = \mathcal{N}(0, \sigma)$, $P(\eta_t) = \mathcal{N}(0, \rho)$. As was shown on the lecture, the max-likelihood estimate of the hidden state conditioned on all information observed by the moment t reads:

$$\hat{x}_t = K_t z_t + (1 - K_t)(\hat{x}_{t-1} + u_{t-1}). \tag{7}$$

Compute $\lim_{t\to\infty} K_t$ and $\lim_{t\to\infty} \left\langle (\hat{x}_t - x_t)^2 \right\rangle$.

В лекции было выведено, что

$$K_t = \frac{err_{t-1}^2 + \sigma^2}{err_{t-1}^2 + \sigma^2 + \rho^2} = \frac{err_t^2}{\rho^2}$$

$$err_t^2 = \rho^2 \frac{err_{t-1}^2 + \sigma^2}{err_{t-1}^2 + \rho^2 + \sigma^2}$$

Предположим, что ответ на задачу существует, т.е. при $t \to \infty$ err и K выходят на константу. Тогда $err_t = err_{t-1}$

$$err_\infty^2 = \rho^2 \frac{err_\infty^2 + \sigma^2}{err_\infty^2 + \rho^2 + \sigma^2}$$

$$err_\infty^2 = \frac{\sigma^2}{2}\left(\sqrt{1+\frac{4p^2}{\sigma^2}} - 1\right) =: \langle (\hat{X}_t - X_t)^2 \rangle$$

$$K_\infty = \frac{err_\infty^2}{p^2}$$