3. (35) Consider the model, described on the lecture:

$$x_t = x_{t-1} + u_{t-1} + \epsilon_{t-1}, \quad z_t = x_t + \eta_t \tag{4}$$

where z_t is observed state, x_t is hidden state and, for simplicity, $u_t \equiv u$. Assume that the measurement and the control noise have the following distributions:

$$P(\epsilon_t) = \mathcal{N}(0, \sigma), \quad P(\eta_t) = \mathcal{N}(0, \rho = 2\sigma). \tag{5}$$

Compute, analytically, the following distributions:

(a) (5) $P(x_0|z_0)$: distribution of x_0 after the first 1 observation

$$P(z_0|x_0) = \frac{1}{\rho\sqrt{2\pi}} \exp\left(-\frac{(x_0 - z_0)^2}{2\rho^2}\right)$$

$$P(x_0|z_0) = \frac{P(z_0|x_0)\overbrace{P(x_0)}^{\text{Пусть } 1}}{\underbrace{P(z_0)}_{\text{тогда это тоже } 1}} = P(z_0|x_0)$$

$$= \frac{1}{\rho\sqrt{2\pi}} \exp\left(-\frac{(x_0 - z_0)^2}{2\rho^2}\right)$$

(b) (10) $P(x_0|z_0, z_1)$: distribution of x_0 after the first 2 observations

$$z_1 = x_0 + u_0 + \epsilon_0 + \eta_1 \sim \mathcal{N}(x_0 + u_0, \sigma^2 + p^2)$$

$$P(x_0|z_0 z_1) = \frac{p(z_1|x_0)\, p(z_0|x_0)\, P(x_0)}{p(z_1 z_0)}$$

Если забыть о константах и априорной вероятности $P(x_0)$, то:

$$P(x_0|z_0 z_1) \sim \exp\left(-\frac{(z_1 - (x_0+u_0))^2}{2\underbrace{(\sigma^2+p^2)}_{\sigma_1^2}}\right) \exp\left(-\frac{(z_0-x_0)^2}{2\underbrace{p^2}_{\sigma_0^2}}\right)$$

$$= \exp\left[-\frac{1}{2}\underbrace{\left(\frac{1}{\sigma_0^2}+\frac{1}{\sigma_1^2}\right)}_{1/\sigma_*^2}\left(x_0^2 - 2x_0\left(\frac{\sigma_*^2}{\sigma_1^2}(z_1-u_0) + \frac{\sigma_*^2}{\sigma_0^2} z_0\right) + \dots\right]\right.$$

уйдет в константу

$$x_0|z_0, z_1 \sim \mathcal{N}\left(\frac{\sigma_*^2}{\sigma_1^2}(z_1-u_0) + \frac{\sigma_*^2}{\sigma_0^2} z_0\,;\ \sigma_*^2\right),$$

где $\sigma_*^2 = (\sigma_0^{-2} + \sigma_1^{-2})^{-1}$, $\sigma_0^2 = p^2$, $\sigma_1^2 = p^2 + \sigma^2$

(c) (20) $P(x_1|z_0, z_2)$: distribution of x_1 after the first 3 observations, when the second observation is missing.

Note that each of these distributions is Gaussian, so its enough to specify mean and variance in each case. Feel free to use Eq. (3) (`Mathematica` or `sympy` may also be useful for algebraic manipulations.)

$$z_0 = x_0 + \eta_0 = (x_1 - u_0 - \epsilon_0) + \eta_0 \sim \mathcal{N}(x_1 - u_0, \sigma^2 + \rho^2)$$

$$z_2 = x_2 + \eta_2 = (x_1 + u_1 + \epsilon_1) + \eta_2 \sim \mathcal{N}(x_1 - u_1, \sigma^2 + \rho^2)$$

Аналогично предыдущему пункту,

Если ввести

$$\sigma_0^2 = \sigma_1^2 = \sigma^2 + \rho^2$$

$$\sigma_*^2 = (\sigma_0^{-2} + \sigma_2^{-2})^{-1} = \frac{1}{2}(\sigma^2 + \rho^2)$$

то ответом будет

$$x_1 | z_2 z_0 \sim \mathcal{N}\left(\frac{\sigma_*^2}{\sigma_0^2}(z_0 + u) + \frac{\sigma_*^2}{\sigma_2^2}(z_2 - u), \ \sigma_*^2\right)$$

$$\mathcal{N}\left(\frac{z_0 + z_1}{2}, \ \sigma_*^2\right)$$