P R O J E C T

Development of low-order shock-capturing scheme for discontinuous Galerkin method

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MIDTERM EVALUATION TEAM MEETING

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von Karman Institute for Fluid Dynamics

Aeronautics & Aerospace department

Supervisors & Advisors:

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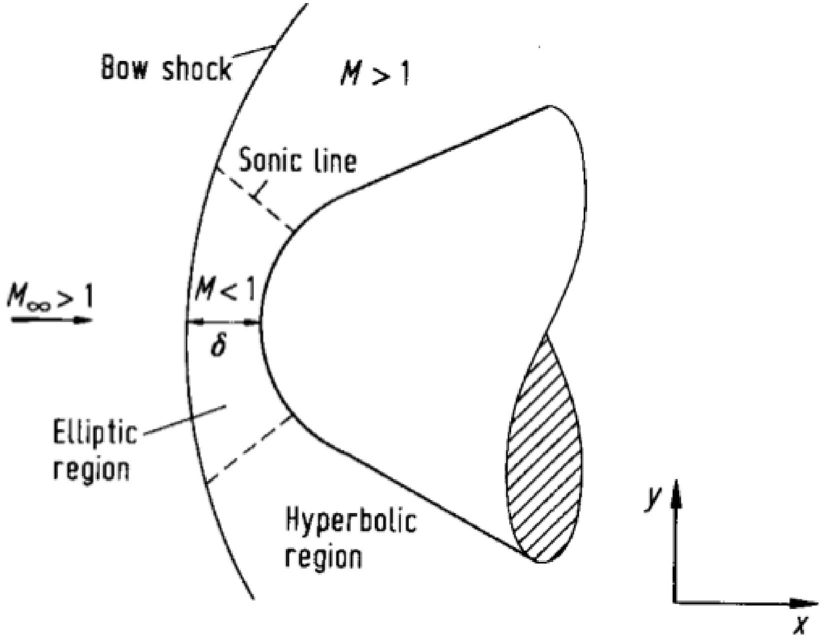
OUTLINE

1. Motivation
2. Bibliographic research approach
3. Aims of the project
4. Progress up to today
5. Conclusion & future work

CHALLENGES

- **Measure the aerodynamic forces and heat flux on a body:**
  → Experiments in wind tunnels
  → Simulation by means of CFD
- **Hypersonic flow:** → Multi-physics phenomena
  → High temperature and compressible effects

"***The role of CFD in engineering predictions has become so strong that today it can be viewed as a new 'third dimension' in fluid dynamics, the other two dimensions being the classical cases of pure experiment and pure theory.***"



J.D. Anderson et al. Computational Fluid Dynamics: An Introduction (Von Karman Institute Book). Edited by John F.Wendt. Springer; 3rd ed., 2008.

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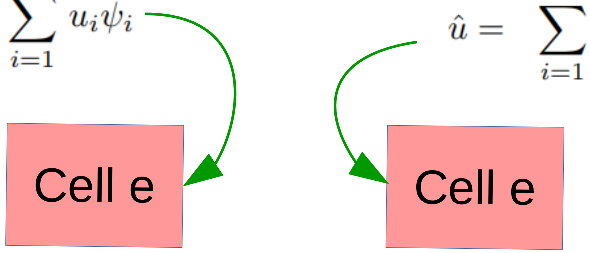
Shock detector and artificial viscosity

Shock detectors are part of most shock capturing techniques and have for purpose to identify the location of discontinuities in the computational domain
Artificial viscosity makes shocks broader, so that they can be resolved over several grid cells.

Method:

$$u = \sum_{i=1}^{N(p)} u_i \psi_i \qquad \hat{u} = \sum_{i=1}^{N(p-1)} u_i \psi_i$$



IF(the polynomials shapes are similar**)**
Then the solution on the cell e is smooth
Else the solution contains discontinuities

Sensor of the discontinuity on the cell e:

$$S_e = \frac{(u-\hat{u}, u-\hat{u})_e}{(u,u)_e}$$

$(\cdot,\cdot)_e$ inner product in $L_2(\Omega_e)$

Inputs defined by the user:

- The threshold $s_0 \sim 1/p^4$
- The interval κ
- The amount of artificial viscosity $\varepsilon_0 \sim h/p$

Artificial viscosity for each cell e:

$$\varepsilon_e = \begin{cases} 0 & \text{if} \quad s_e < s_0 - \kappa \\ \frac{\varepsilon_0}{2}\left(1 + \sin \frac{\pi(s_e - s_0)}{2\kappa}\right) & \text{if} \quad s_0 - \kappa \le s_e \le s_0 + \kappa \\ \varepsilon_0 & \text{if} \quad s_e > s_0 + \kappa \end{cases}$$

with $s_e = \log_{10} S_e$

Main weakness of sensors

- The quantity needs to be compared to a threshold defined arbitrarily
- The method can not be used for p=0

The Navier-Stokes equations

$$\boldsymbol{w} = \begin{pmatrix} \rho \\ \rho \boldsymbol{u} \\ \rho E \end{pmatrix} \in \Omega_{adm}$$

Let $\Omega_{adm} = \left\{ \boldsymbol{w} \in \mathbb{R}^4 : \rho > 0, \boldsymbol{u} \in \mathbb{R}^2, e > 0 \right\}$

The conservative form of the NS equations in 2D for a perfect calorically and thermally perfect gas are defined as

$$\partial_t \boldsymbol{w} + \nabla . \mathbb{F}_c(\boldsymbol{w}) - \nabla . \mathbb{F}_v(\boldsymbol{w}, \nabla \boldsymbol{w}) = 0, \ \forall (\boldsymbol{x}, t) \in \Omega \times]0; +\infty[$$

$$\boldsymbol{w}(\boldsymbol{x}, 0) = \boldsymbol{w}_{t=0}(\boldsymbol{x}), \forall \boldsymbol{x} \in \Omega$$

$$p = \rho R T$$

$$\mathbb{F}_c(\boldsymbol{w}) = \begin{pmatrix} \rho \boldsymbol{u}^T \\ \rho \boldsymbol{u} \otimes \boldsymbol{u} + p \mathbb{I} \\ (\rho E + p) \boldsymbol{u}^T \end{pmatrix}_{4 \times 2} \qquad \mathbb{F}_v(\boldsymbol{w}) = \begin{pmatrix} \boldsymbol{0}_{\mathbb{R}^2}^T \\ \boldsymbol{\tau} \\ \boldsymbol{u}^T . \boldsymbol{\tau} - \boldsymbol{q}^T \end{pmatrix}_{4 \times 2}$$

Physical boundary conditions (defined on the following slides): $\Gamma = \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_{wall}$

- **Two important vector spaces**: $H^1(\Omega_h) = \{f \in L^2(\Omega_h) : \phi_{|K} \in H^1(K),\ \forall K \in \Omega_h\}$

  $v_h^p = \{f \in L^2(\Omega_h) : \phi_{|K} \in \mathcal{P}^p(K),\ \forall K \in \Omega_h\}$

- $\mathcal{P}^p(K)$: **Polynomial of degree inferior or equal to p on K**

- **Each cell K is approximate by a polynomial that results from a linear combination of N_p basis functions ϕ_i^K that are polynomial in the cell K and equal to zero in $\Omega_h \backslash$K**

Variational Formulation: **Find $\boldsymbol{w}_h \in (v_h^p)^4$ such that $\forall \phi_i^K,\ i \in [\![1, N_p]\!]$, $\boldsymbol{w}_h$ is solution of**

$$\sum_{K \in \Omega_h} \int_K \phi_i^K \partial_t \boldsymbol{w}_h dV - \sum_{K \in \Omega_h} \int_K \left(\mathbb{F}_c(\boldsymbol{w}_h) - \mathbb{F}_v(\boldsymbol{w}_h, \Upsilon_h)\right) \nabla \phi_i^K dv$$
$$+ \sum_{K \in \Omega_h} \oint_{\partial K} \phi_i^K \left(\hat{\mathbb{F}}_c - \hat{\mathbb{F}}_v\right) dS = 0$$

- Υ_h **is an auxiliary variable that represents the gradient** ∇w_h **in the space of discretisation**

$dim(\mathcal{P}^p) := \prod_{i=1}^{d} \frac{(p+i)}{i} = N_p \quad dim(v_h^p) := N \times N_p$

Generic Form of the Variational Formulation

$$\sum_{K \in \Omega_h} \int_K \phi_i^K \partial_t \boldsymbol{w}_h dV + \mathcal{L}_c(\boldsymbol{w}_h, \phi^K) + \mathcal{L}_v(\boldsymbol{w}_h, \nabla \boldsymbol{w}_h, \phi^K) = 0$$

$$z^{\pm}(x_e) = \lim_{y \to x_e,\ y \in K^{\pm}} z(y)$$

$$[\![z]\!] = \begin{cases} z^+ - z^- & if\ e \in \epsilon_i \\ z^+ - z_{wall} & if\ e \in \epsilon_{wall} \end{cases}$$

$$\{z\} = \begin{cases} \frac{z_+ + z_-}{2} & if\ e \in \epsilon_i \\ z_{wall} & if\ e \in \epsilon_{wall} \end{cases}$$

$$\sum_{K\in\Omega_h}\int_K \phi_i^K \partial_t \boldsymbol{w}_h dV + \mathcal{L}_c(\boldsymbol{w}_h,\phi^K) + \mathcal{L}_v(\boldsymbol{w}_h,\nabla\boldsymbol{w}_h,\phi^K) = 0$$

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Lax Friedrich Riemann Solver

$$\hat{\mathbb{F}}_c(\boldsymbol{w}_h^+,\boldsymbol{w}_h^-,\boldsymbol{n}) = \{\mathbb{F}_c(\boldsymbol{w}_h)\}\,\boldsymbol{n} + \frac{a}{2}[\![\boldsymbol{w}_h]\!]$$

$$a = max\left\{\left|\frac{\partial\mathbb{F}_c\boldsymbol{n}}{\partial\boldsymbol{w}}\right| : \boldsymbol{w} = \boldsymbol{w}_h^{\pm}\right\}$$

$$\begin{aligned}\mathcal{L}_c(\boldsymbol{w}_h,\phi^K) = &-\sum_{K\in\Omega_h}\int_K \nabla(\phi^K)\mathbb{F}_c(\boldsymbol{w}_h)\\ &+\sum_{e\in\epsilon_i}\int_e [\![\phi^K]\!]\,\hat{\mathbb{F}}_c(\boldsymbol{w}_h^+,\boldsymbol{w}_h^-,\boldsymbol{n})\\ &+\sum_{e\in\epsilon_{wall}}\int_e \phi^+\mathbb{F}_c(\boldsymbol{w}_{wall})\end{aligned}$$

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$$\sum_{K \in \Omega_h} \int_K \phi_i^K \partial_t \boldsymbol{w}_h dV + \mathcal{L}_c(\boldsymbol{w}_h, \phi^K) + \mathcal{L}_v(\boldsymbol{w}_h, \nabla \boldsymbol{w}_h, \phi^K) = 0$$

Internal penalty inconsistent at p=0

$$\sum_{K\in\Omega_h}\int_K \phi_i^K \partial_t \boldsymbol{w}_h dV + \mathcal{L}_c(\boldsymbol{w}_h, \phi^K) + \mathcal{L}_v(\boldsymbol{w}_h, \nabla \boldsymbol{w}_h, \phi^K) = 0$$

The second scheme of Bassi Rebay

$$z^{\pm}(x_e) = \lim_{y\to x_e,\ y\in K^{\pm}} z(y)$$

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$$\begin{cases} \Upsilon_h = \nabla \boldsymbol{w}_h \\ \sum_{K\in\Omega_h} \int_K \phi_i^K \partial_t \boldsymbol{w}_h dV + \mathcal{L}_c(\boldsymbol{w}_h, \phi^K) + \mathcal{L}_v(\boldsymbol{w}_h, \Upsilon_h, \phi^K) = 0 \end{cases}$$

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$$\mathbb{F}_v(\boldsymbol{w}_h, \nabla \boldsymbol{w}_h) := (G(\boldsymbol{w}_h)\Upsilon_h)$$

Find $(\Upsilon_h, \boldsymbol{w}_h)$ **in** $(v_h^p)^{4\times 2} \times (v_h^p)^4$ **such that**

for all (α^K, ϕ^K) **in** $(v_h^p)^2 \times v_h^p$**,**

$$\begin{cases} \sum_{K\in\Omega_h} \int_K \Upsilon_h \alpha^K dV + \sum_{K\in\Omega_h} \int_K (\nabla.\alpha^K) \boldsymbol{w}_h dV \\ - \sum_{e\in\epsilon_i} \int_e (\{\boldsymbol{w}_h\} \otimes \boldsymbol{n}) [\![\alpha^K]\!] dS - \sum_{e\in\epsilon_{wall}} \int_e (\boldsymbol{w}_{wall} \otimes \boldsymbol{n}) \alpha^{K,+} dS = 0 \\ \sum_{K\in\Omega_h} \int_K \phi_i^K \partial_t \boldsymbol{w}_h dV + \mathcal{L}_c(\boldsymbol{w}_h, \phi^K) + \sum_{K\in\Omega_h} \int_K \nabla\phi^K (G\Upsilon_h) dV \\ - \sum_{e\in\epsilon_i} \int_e [\![\phi^K]\!] \{G\Upsilon_h\} \boldsymbol{n} dS - \sum_{e\in\epsilon_{wall}} \int_e \phi^{K,+} \mathbb{F}_v(\boldsymbol{w}_{wall}, \nabla \boldsymbol{w}_{wall}) \boldsymbol{n} dS = 0 \end{cases}$$

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$$\mathbb{F}_v(\boldsymbol{w}_h, \nabla \boldsymbol{w}_h) := (G(\boldsymbol{w}_h)\Upsilon_h)$$

Global lifting operator:

$$\Upsilon_h := \nabla \boldsymbol{w}_h + R_h([\![\boldsymbol{w}_h]\!])$$

Local lifting operator:

$$\{G\Upsilon_h\} = \{G\nabla \boldsymbol{w}_h\} + \{G r_h^e\}$$

Find $(\Upsilon_h, \boldsymbol{w}_h)$ **in** $(\upsilon_h^p)^{4\times 2} \times (\upsilon_h^p)^4$ **such that**

for all (α^K, ϕ^K) **in** $(\upsilon_h^p)^2 \times \upsilon_h^p$**,**

$$\begin{cases} \sum_{K\in\Omega_h} \int_K \Upsilon_h \alpha^K dV + \sum_{K\in\Omega_h} \int_K (\nabla.\alpha^K) \boldsymbol{w}_h dV \\ - \sum_{e\in\epsilon_i} \int_e (\{\boldsymbol{w}_h\} \otimes \boldsymbol{n}) [\![\alpha^K]\!] dS - \sum_{e\in\epsilon_{wall}} \int_e (\boldsymbol{w}_{wall} \otimes \boldsymbol{n}) \alpha^{K,+} dS = 0 \\ \sum_{K\in\Omega_h} \int_K \phi_i^K \partial_t \boldsymbol{w}_h dV + \mathcal{L}_c(\boldsymbol{w}_h, \phi^K) + \sum_{K\in\Omega_h} \int_K \nabla\phi^K (G\Upsilon_h) dV \\ - \sum_{e\in\epsilon_i} \int_e [\![\phi^K]\!] \{G\Upsilon_h\} \boldsymbol{n} dS - \sum_{e\in\epsilon_{wall}} \int_e \phi^{K,+} \mathbb{F}_v(\boldsymbol{w}_{wall}, \nabla \boldsymbol{w}_{wall}) \boldsymbol{n} dS = 0 \end{cases}$$

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$$

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$$
\begin{aligned}
\mathcal{L}_v(\boldsymbol{w}_h, \Upsilon_h, \phi^K) = &- \sum_{K\in\Omega_h} \int_K \nabla \phi^K G(\boldsymbol{w}_h) \left(\nabla \boldsymbol{w}_h + R_h([\![\boldsymbol{w}_h]\!])\right) dV \\
&+ \sum_{e\in\epsilon_i} \int_e \eta_r [\![\phi^K]\!] \left\{G(\boldsymbol{w}_h)\left(\nabla \boldsymbol{w}_h + \eta_r r_h^e([\![\boldsymbol{w}_h]\!])\right)\right\} \boldsymbol{n} dS \\
&+ \sum_{e\in\epsilon_{wall}} \int_e \phi^{K,+} \mathbb{F}_v(\boldsymbol{w}_{wall}, \nabla \boldsymbol{w}_{wall} + \eta_r r_h^e([\![\boldsymbol{w}_h]\!])) \boldsymbol{n} dS
\end{aligned}
$$

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Aim of the project

Develop a robust shock capturing scheme for the DG method

Main tasks

1. Review the hypersonic flow challenges for **high order numerical simulation** ✓

2. Identify the limit of the **artificial viscosity** into the Argo platform ✓

3. Optimize and adapt a hybrid solver for hypersonic applications which uses a **degraded order scheme** (BR2) in the shock region and a high order method (DG) everywhere else ⌛

4. Validation on a specific test case ✗

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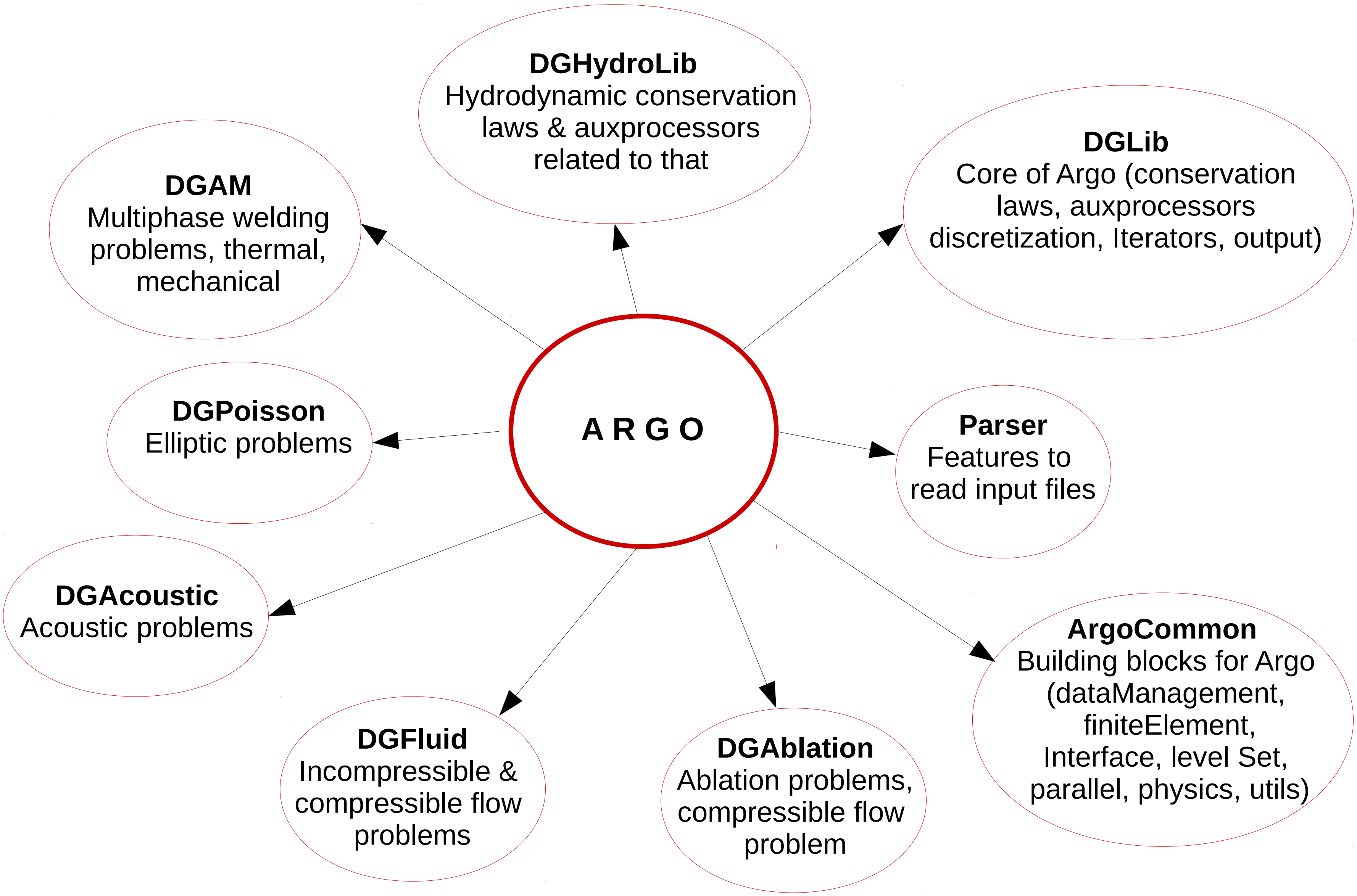
DGHydroLib
Hydrodynamic conservation laws & auxprocessors related to that
DGAM
Multiphase welding problems, thermal, mechanical
DGLib
Core of Argo (conservation laws, auxprocessors discretization, Iterators, output)
DGPoisson
Elliptic problems
A R G O
Parser
Features to read input files
DGAcoustic
Acoustic problems
ArgoCommon
Building blocks for Argo (dataManagement, finiteElement, Interface, level Set, parallel, physics, utils)
DGFluid
Incompressible & compressible flow problems
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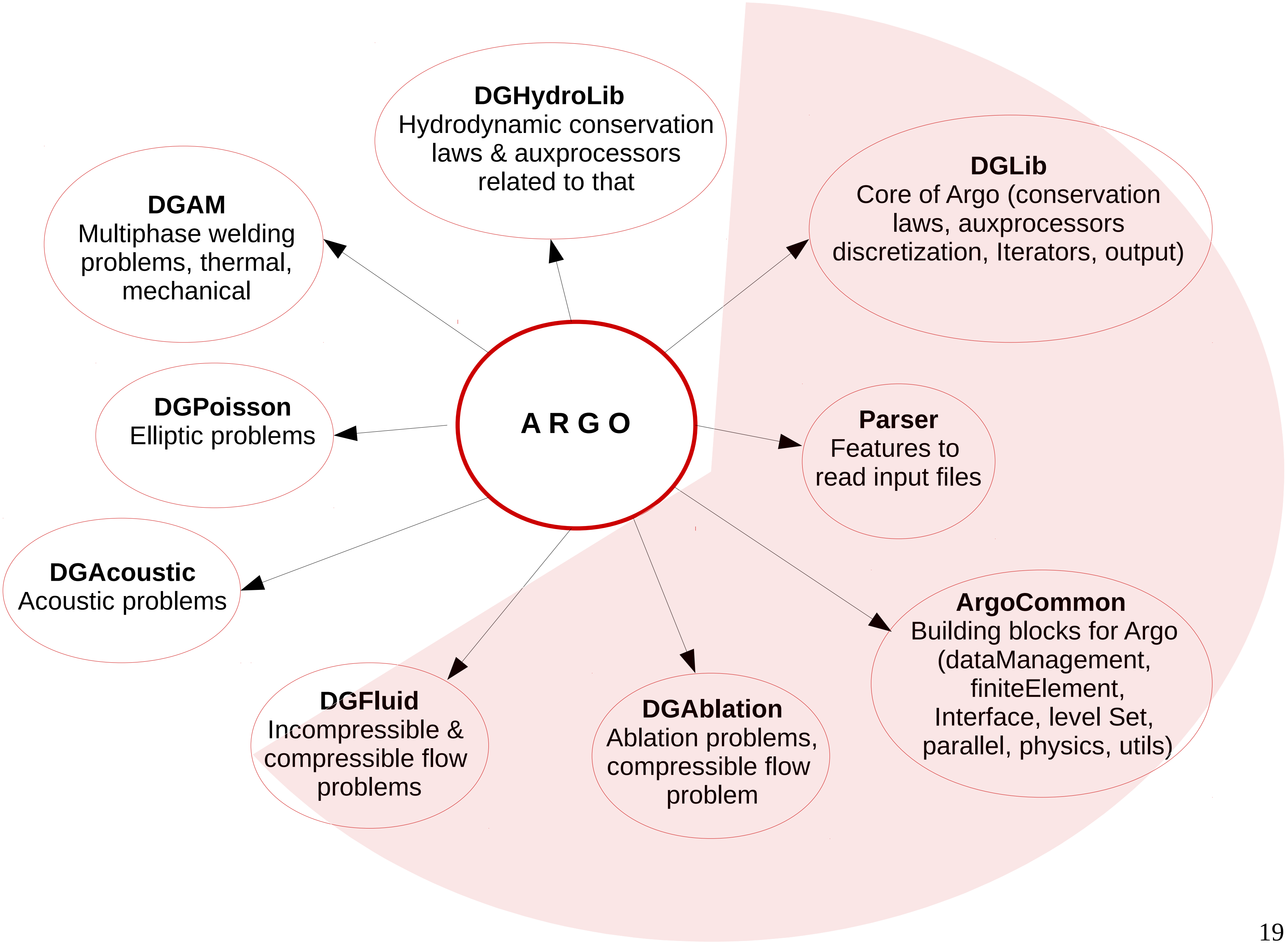
Ablation problems, compressible flow problem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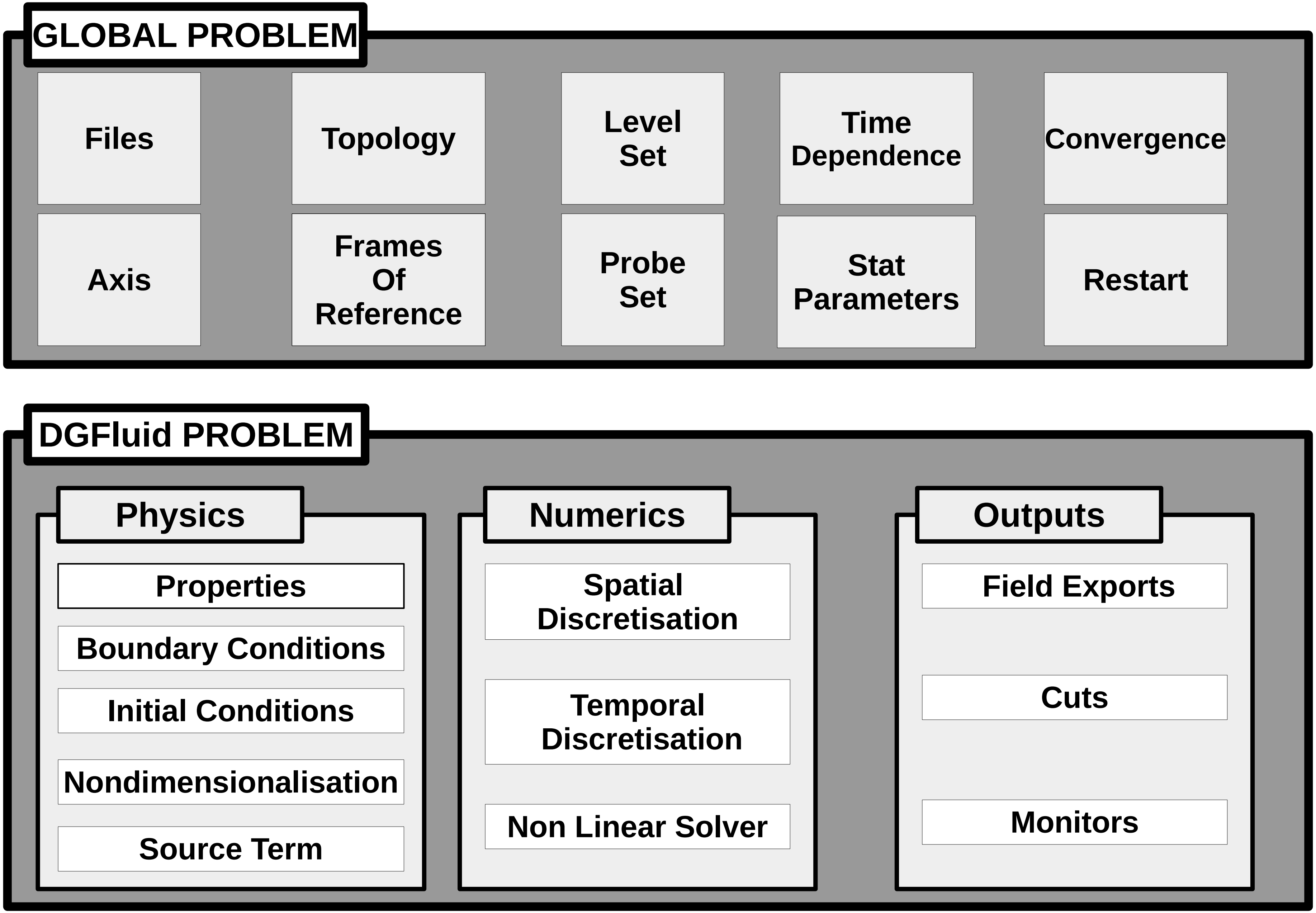
Structure of the input file



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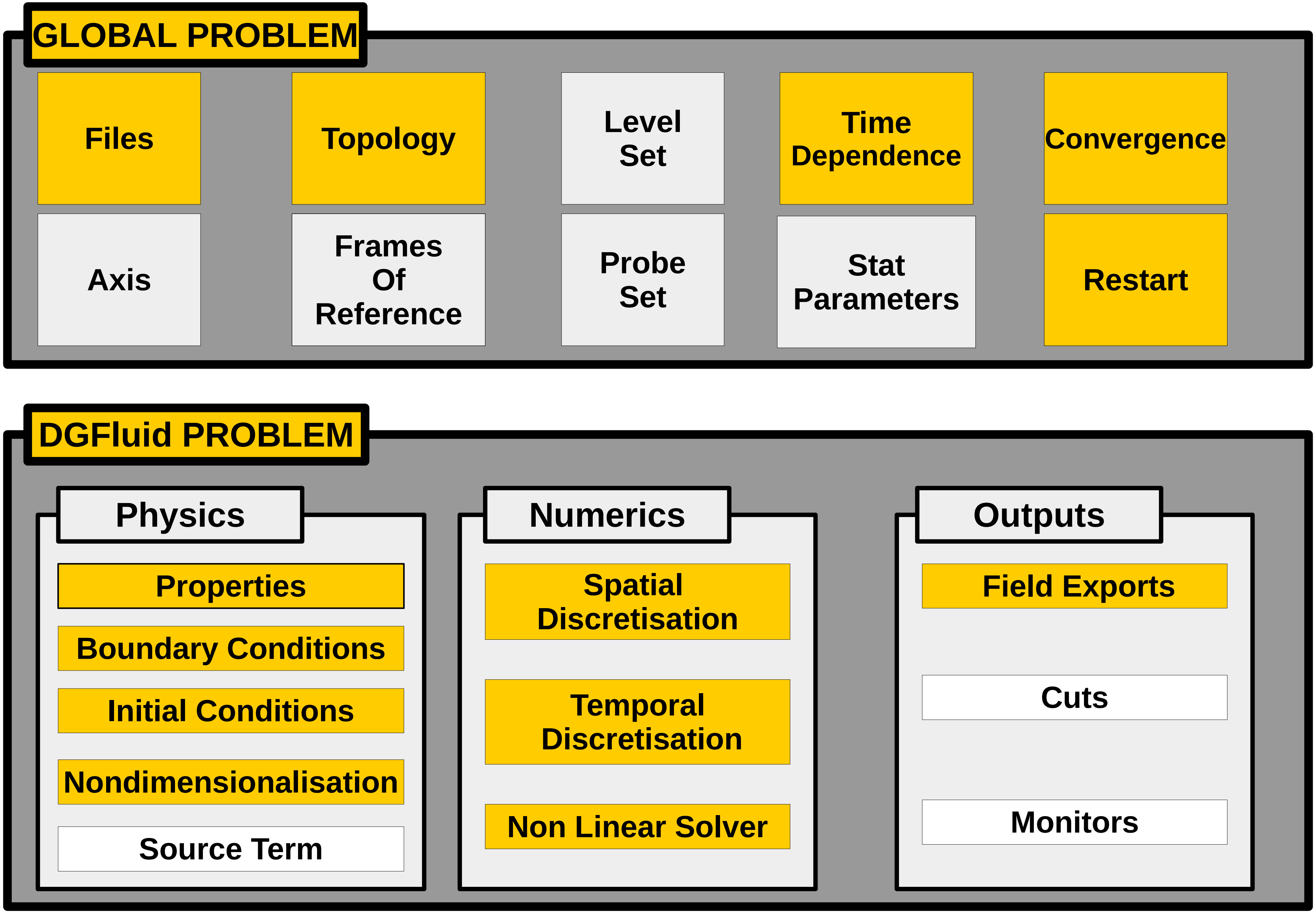
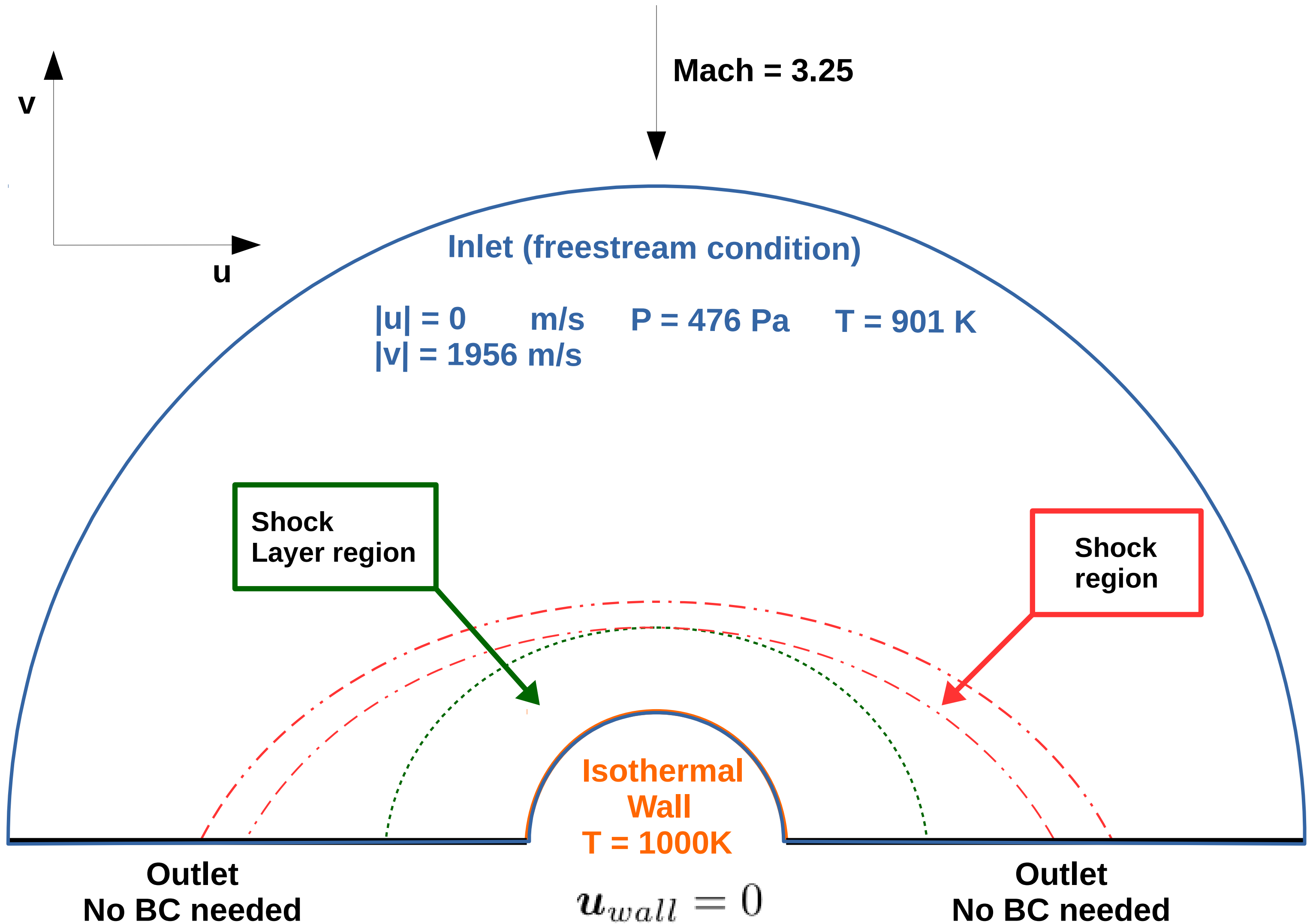


Scheme of the Physical model

2D Perfect Gas

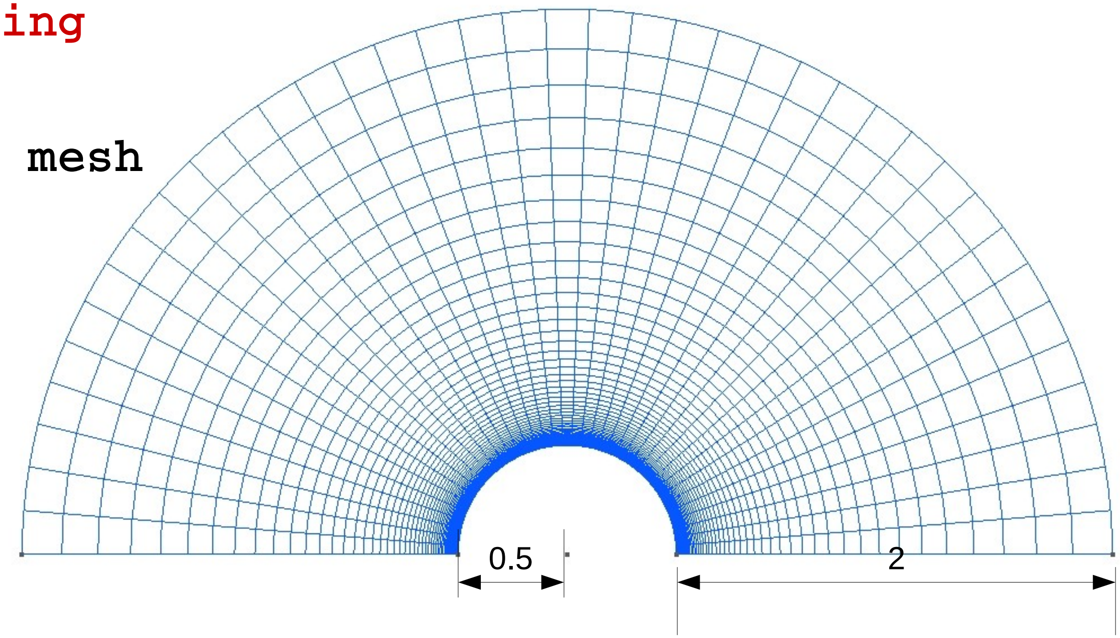


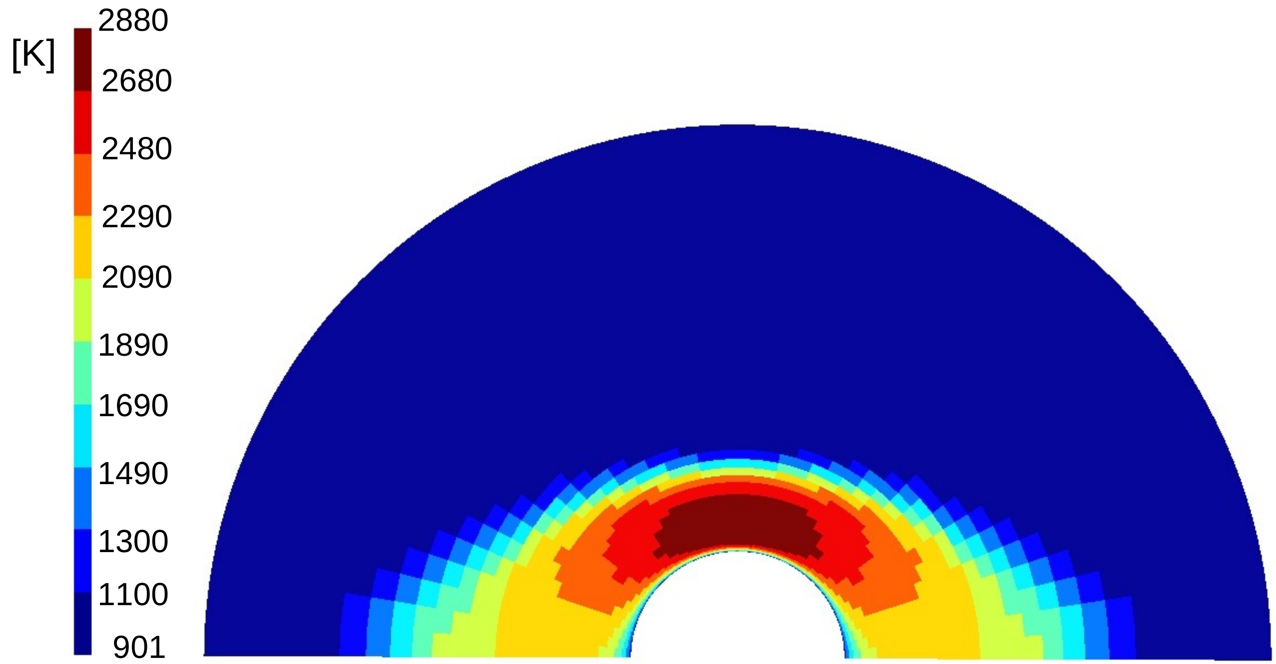
$$\boldsymbol{u}_{wall} = 0$$

$$\nabla T_{wall} . \boldsymbol{n} = 0$$

First Test P0 (shock capturing method)

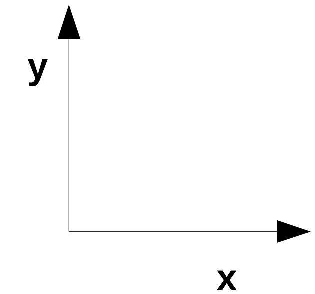
Coarse structured mesh
3323 elements

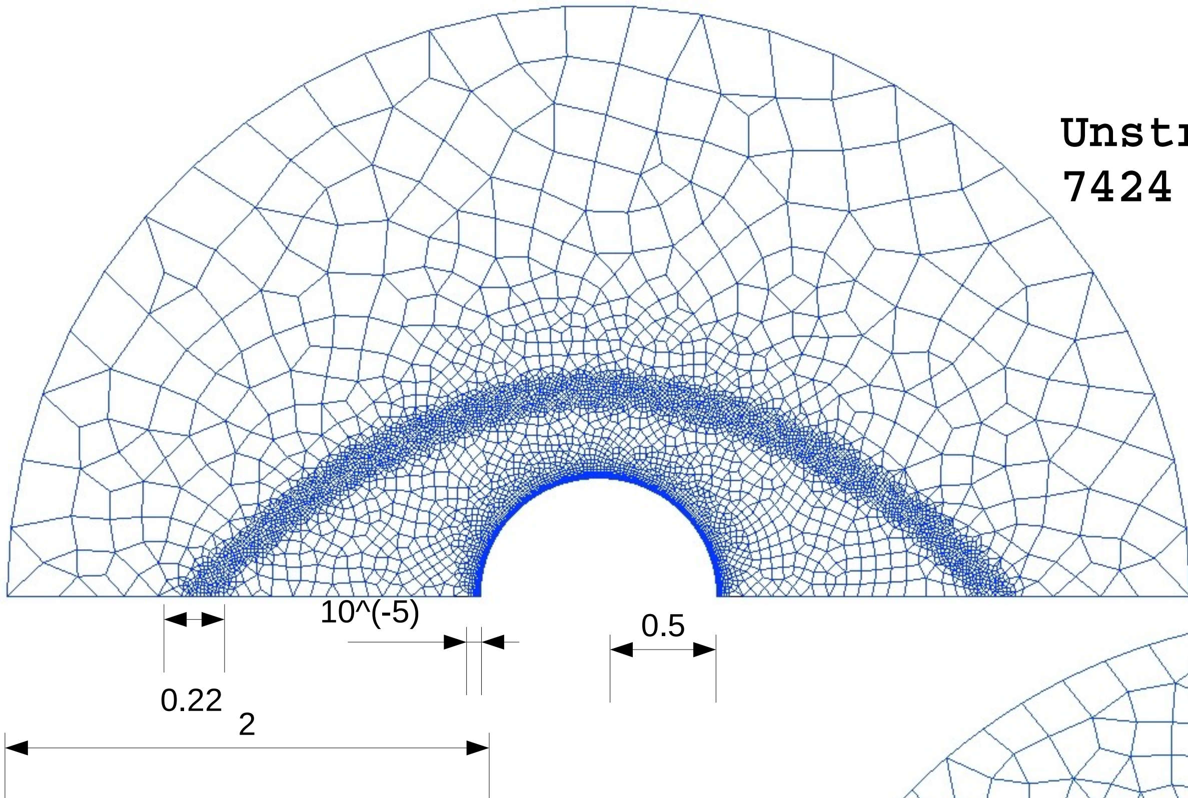




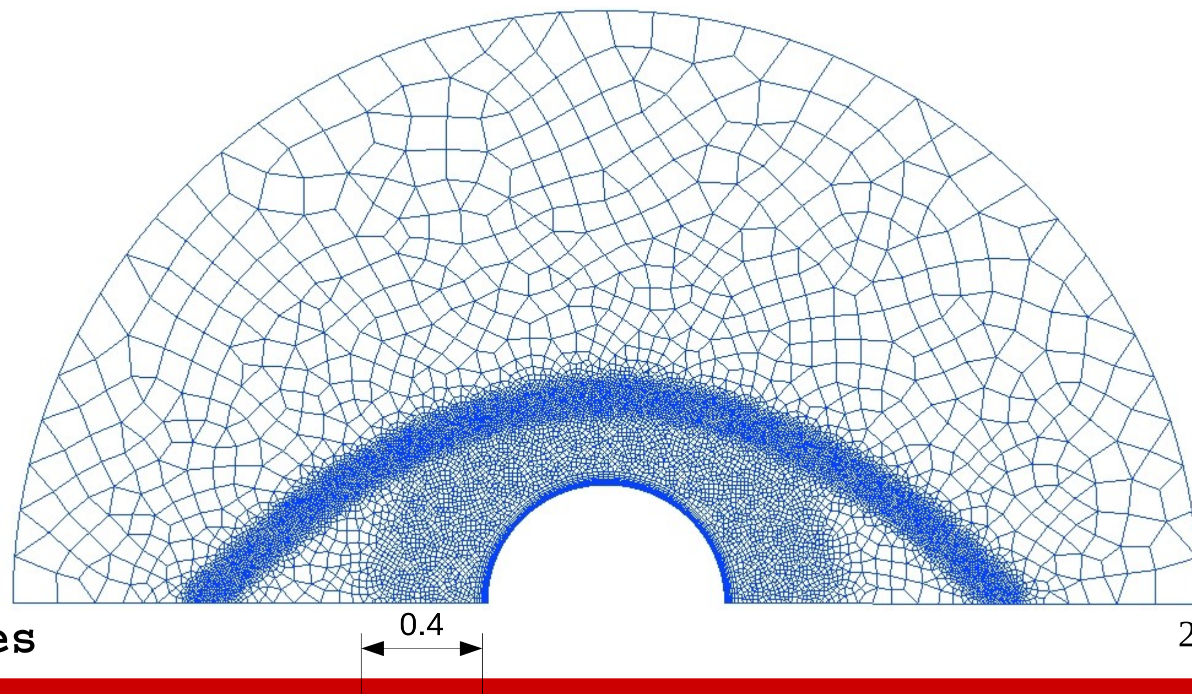
Temperature [K]

Refined Meshes (Gmsh)



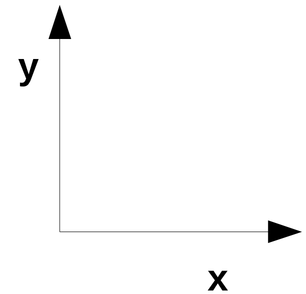


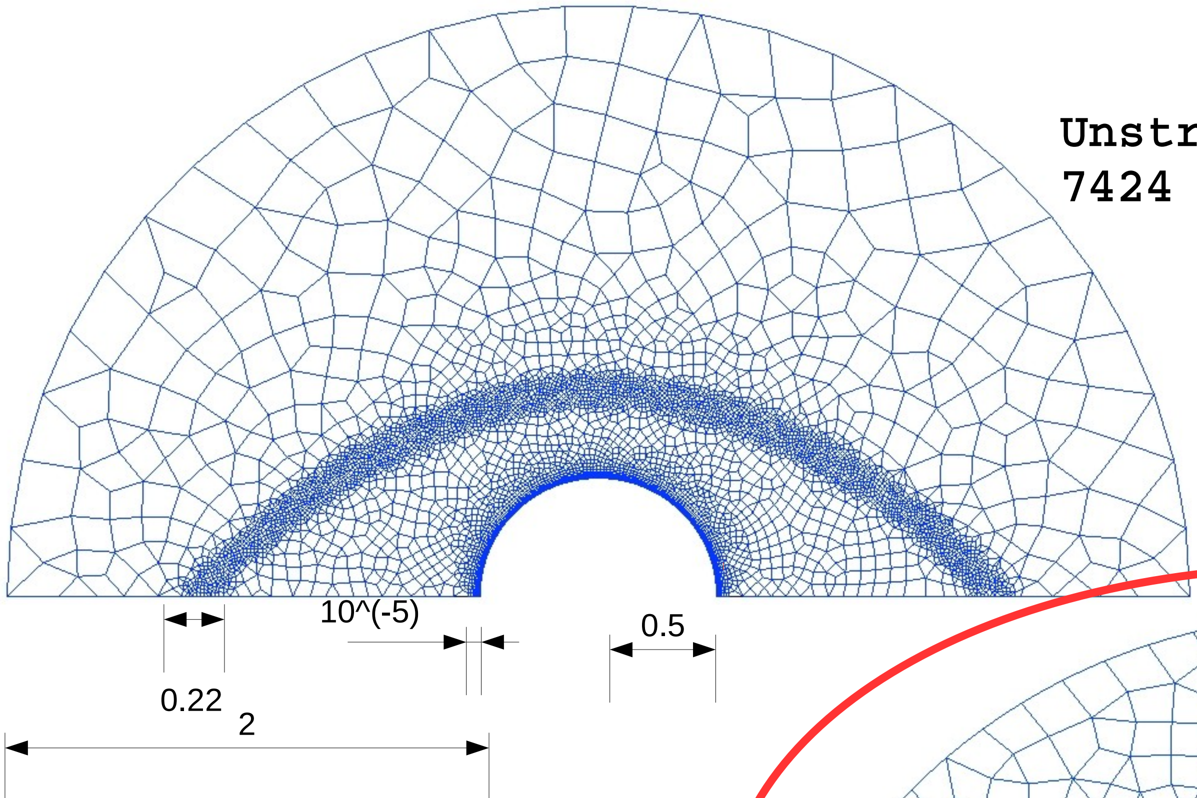
Unstructured mesh
7424 elements
- 1154 triangles
- 5764 quadrangles



Unstructured mesh
15 606 elements
- 2865 triangles
- 11652 quadrangles

Refined Meshes (Gmsh)





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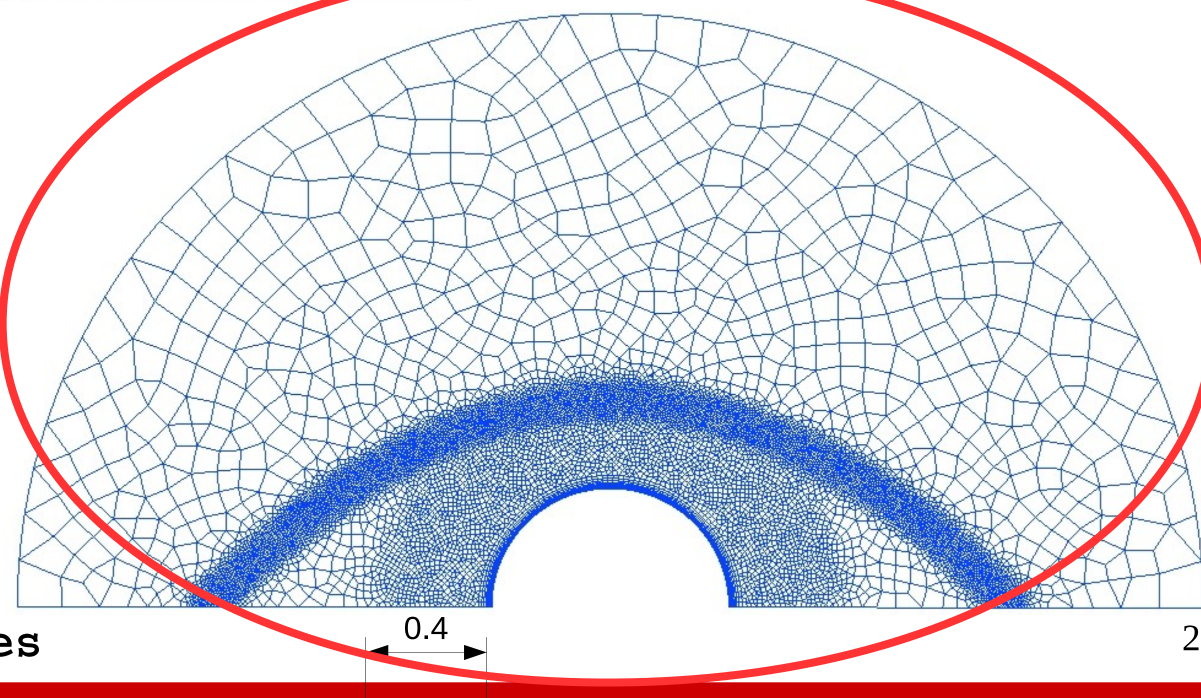

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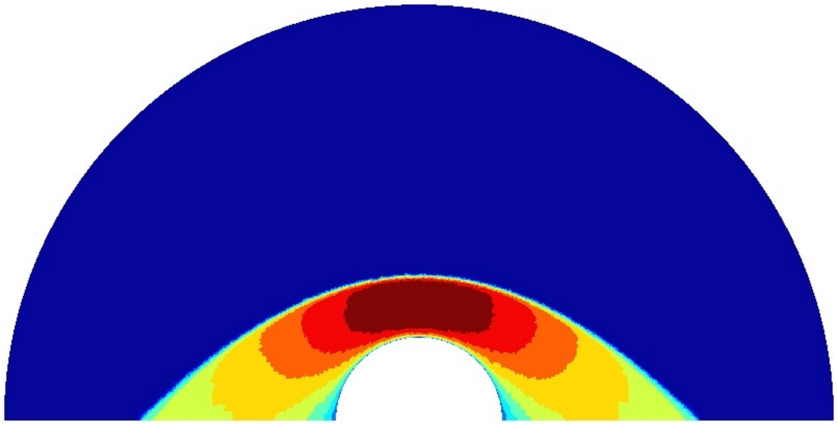
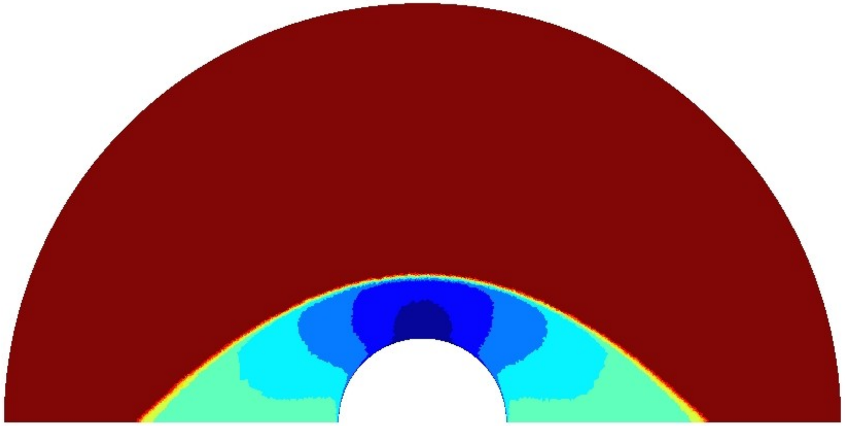
Solution on Refined 15606 elements mesh P0



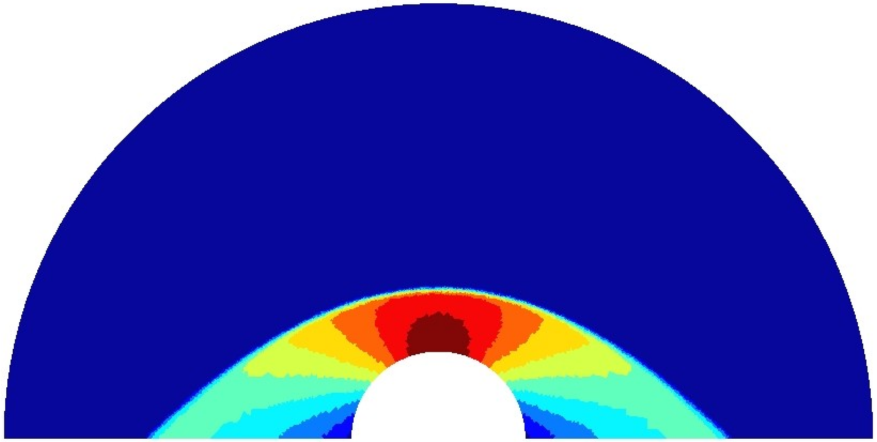
Temperature [K]

901 1830 2770



Mach Number

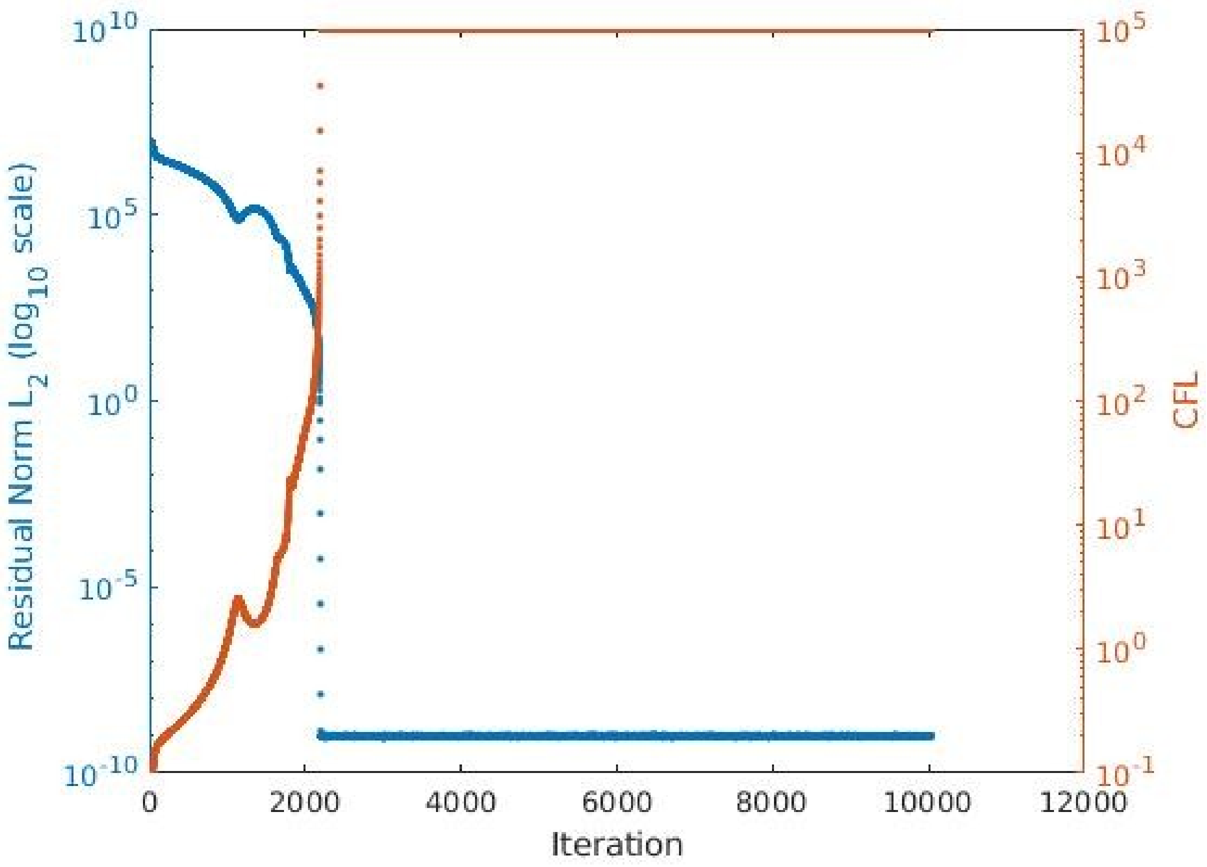
0 1.63 3.25



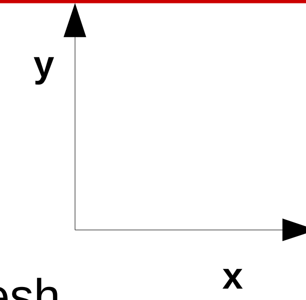
Pressure [Pa]

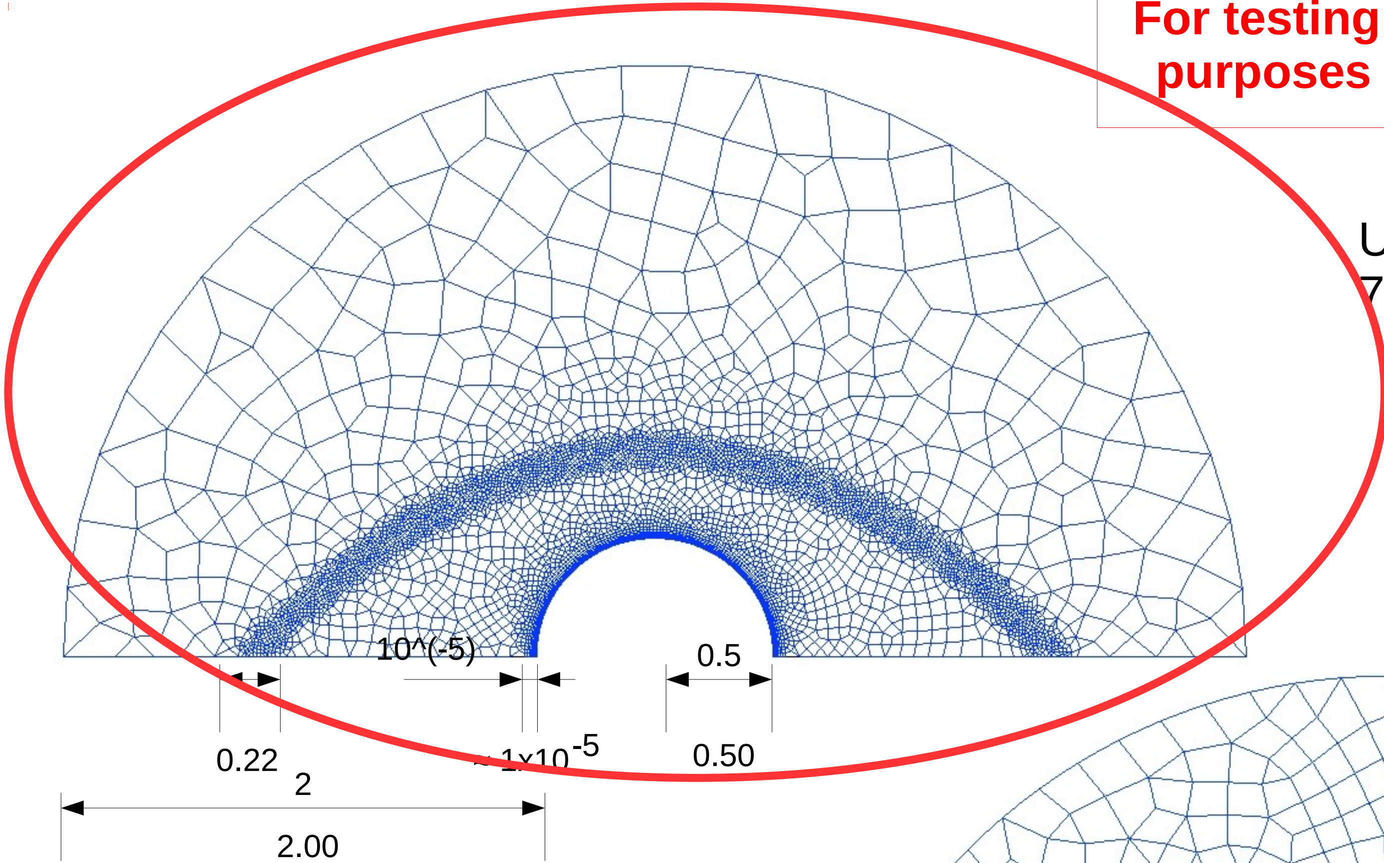
476 3600 6730

Solution on Refined 15606 elements mesh P0



For testing purposes





Unstructured mesh
7424 elements
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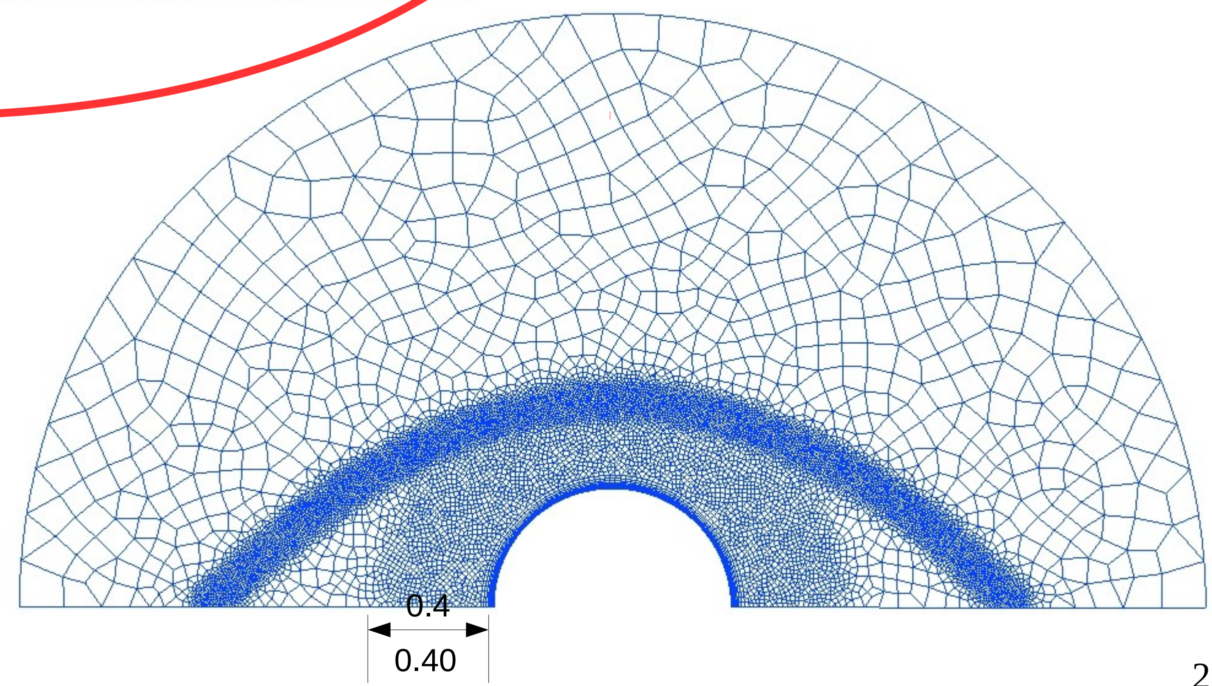
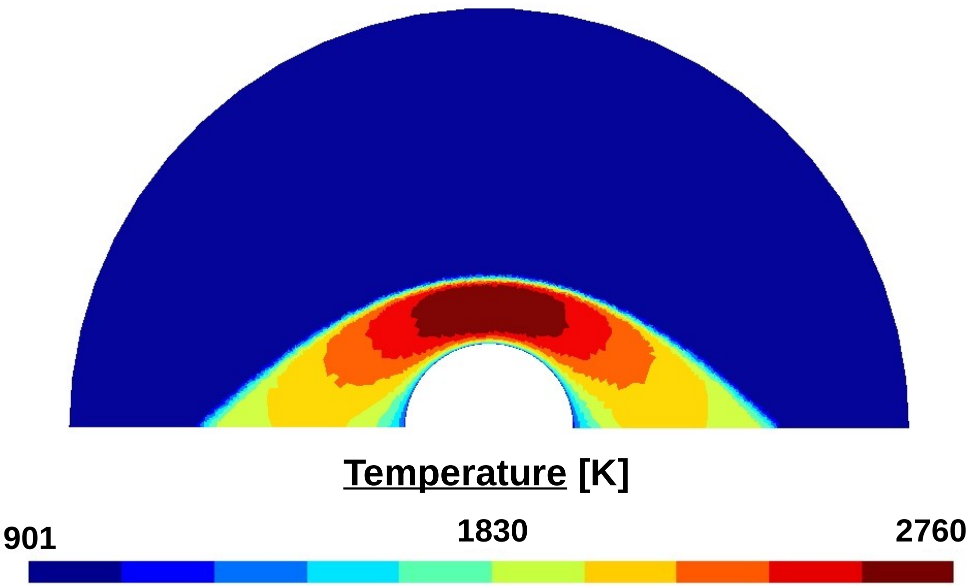
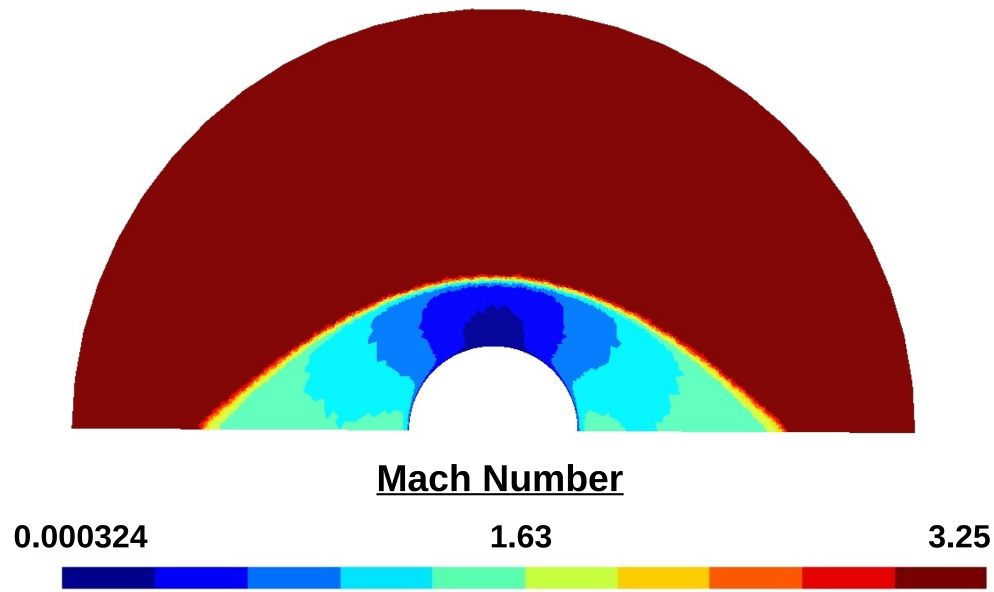
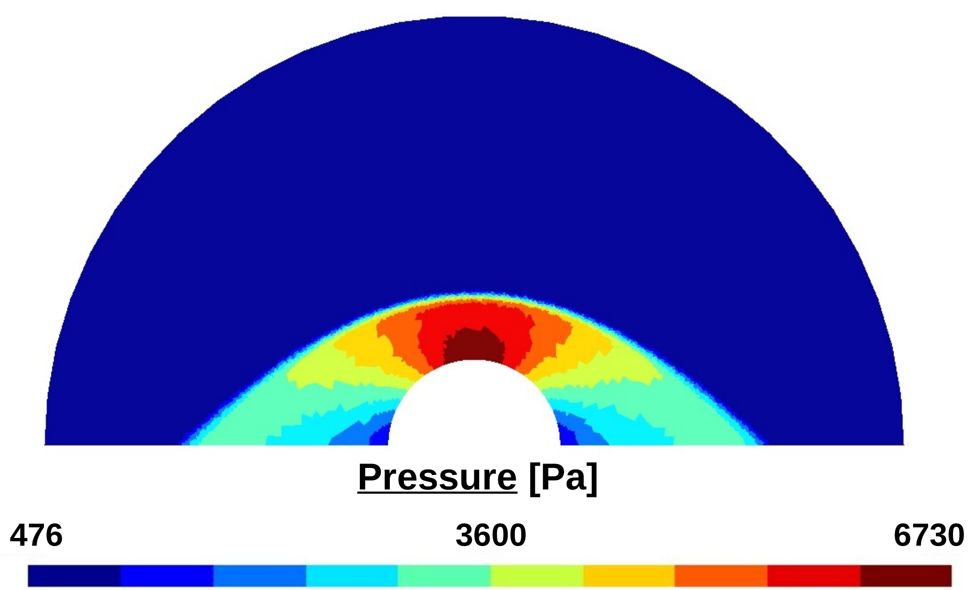
- - 2865 triangles
- - 11652 quadrangles



Solution on Refined 7424 element mesh P0

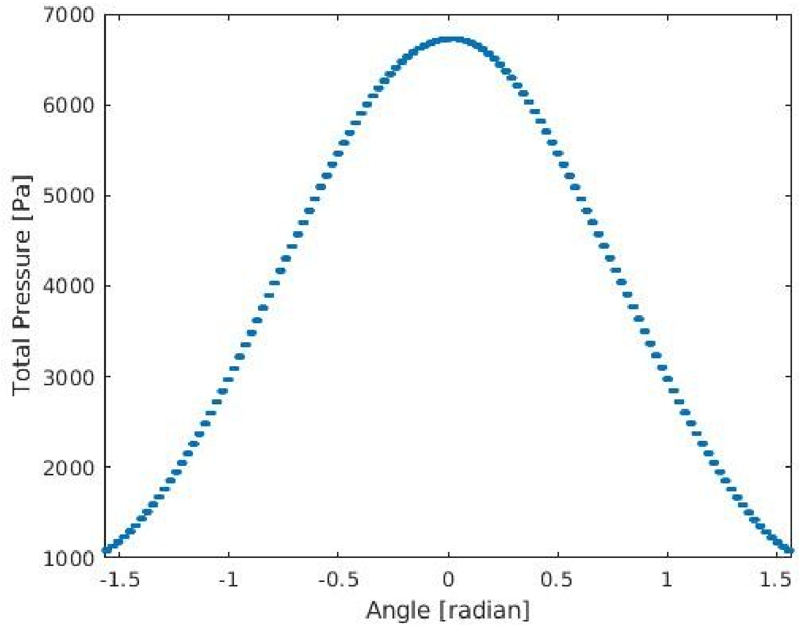


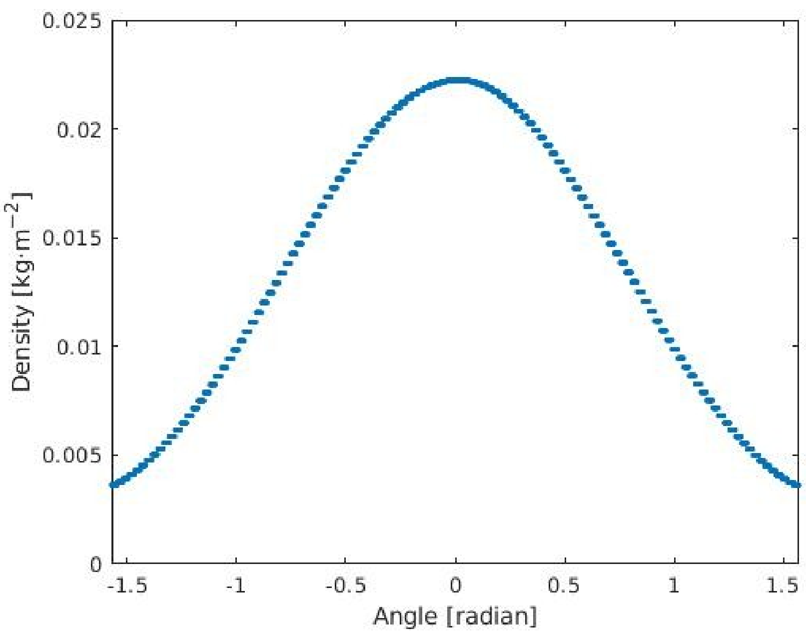


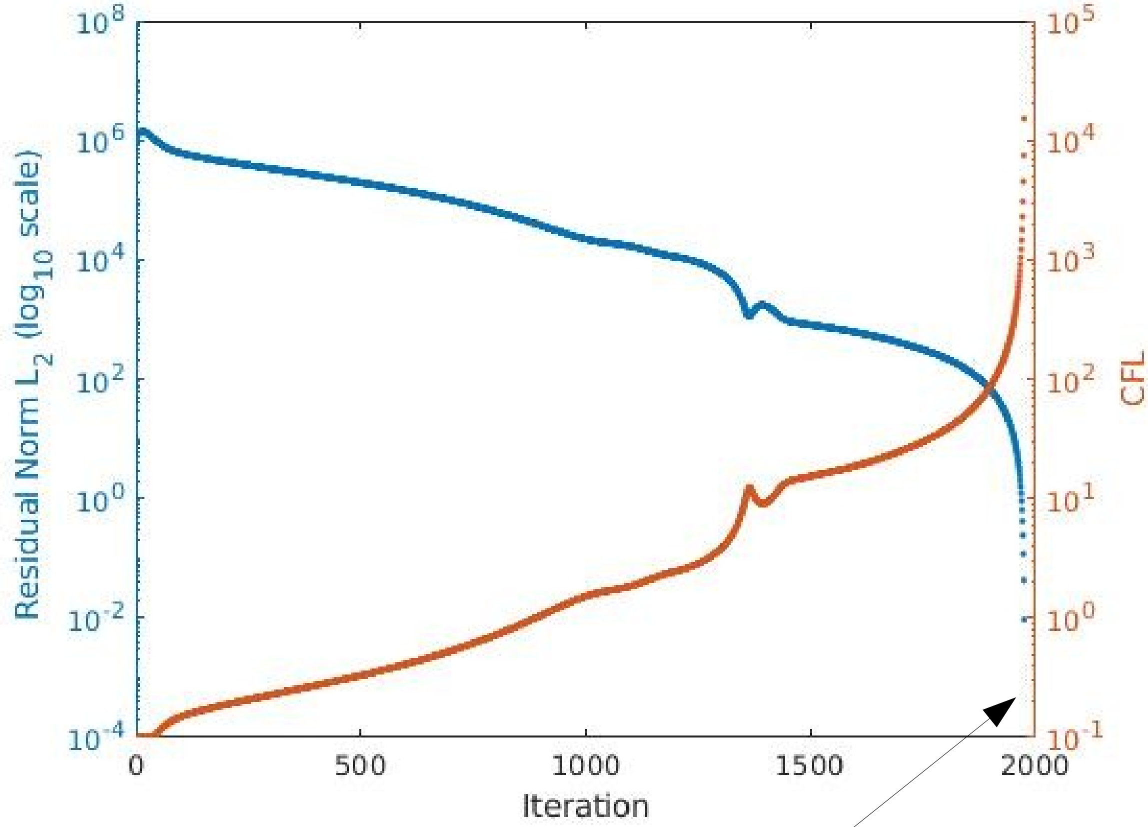


Solution along the wall on Refined 7424 element mesh P0

CPU time: 1306.6 s File Here

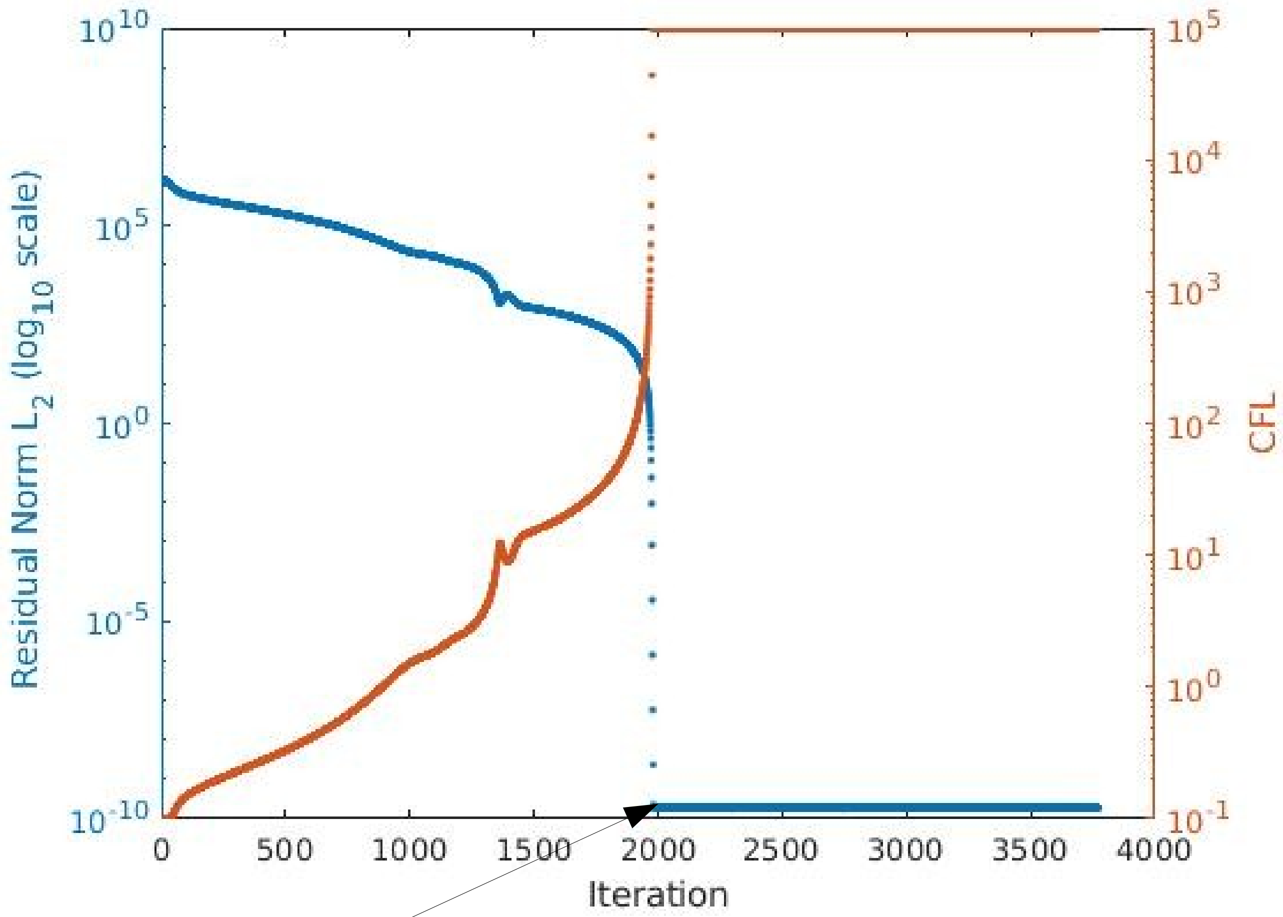






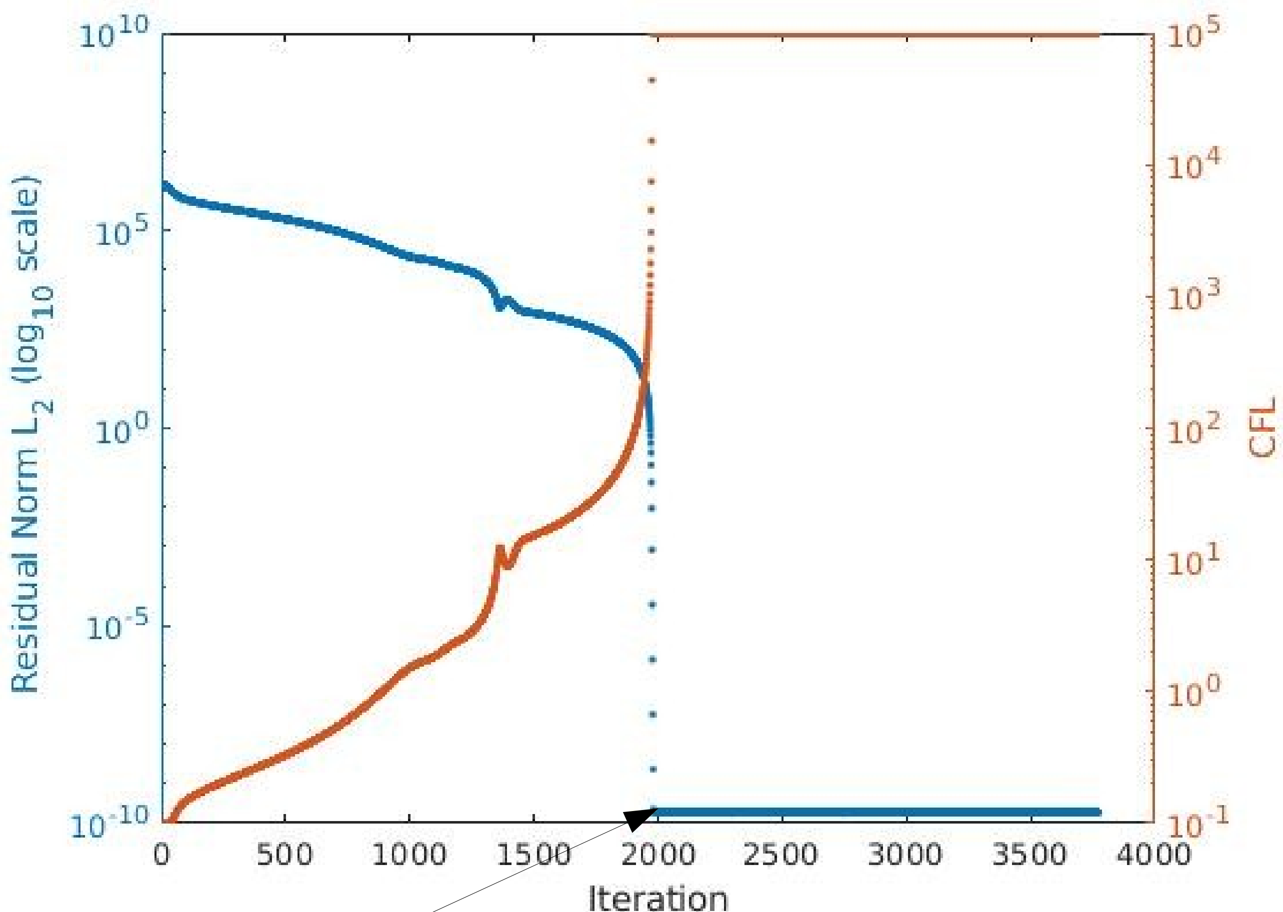
The solution seems to be more precise than 10^(-4), that is why I reduced the tolerance **for the next case** to see if we can obtain a better precision

Residual on refined 7424 elements mesh P0 (tol=10^(-16))



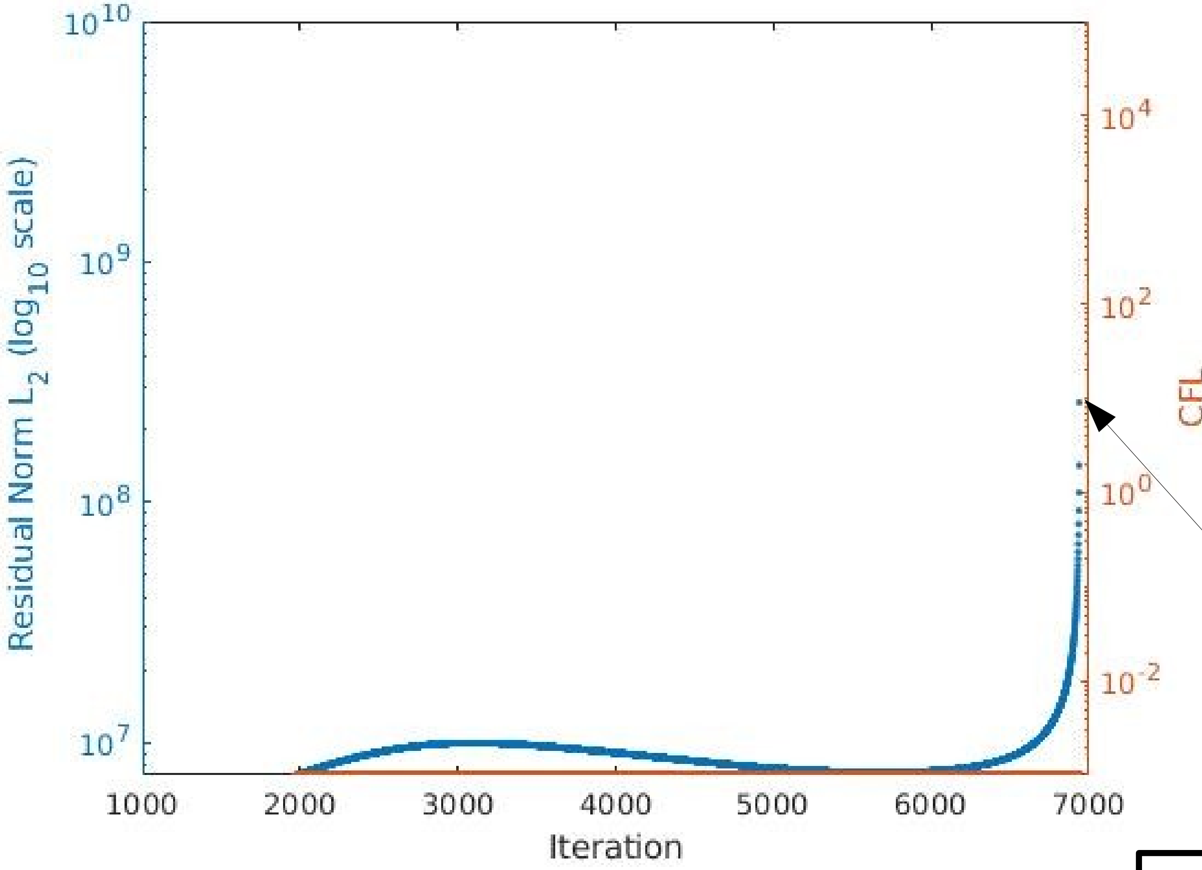
We reached a precision of 10^(-10) that seems to be the best result (I had to stop the code manually at one point)

Residual on refined 7424 elements mesh P0 (tol=10^(-16))



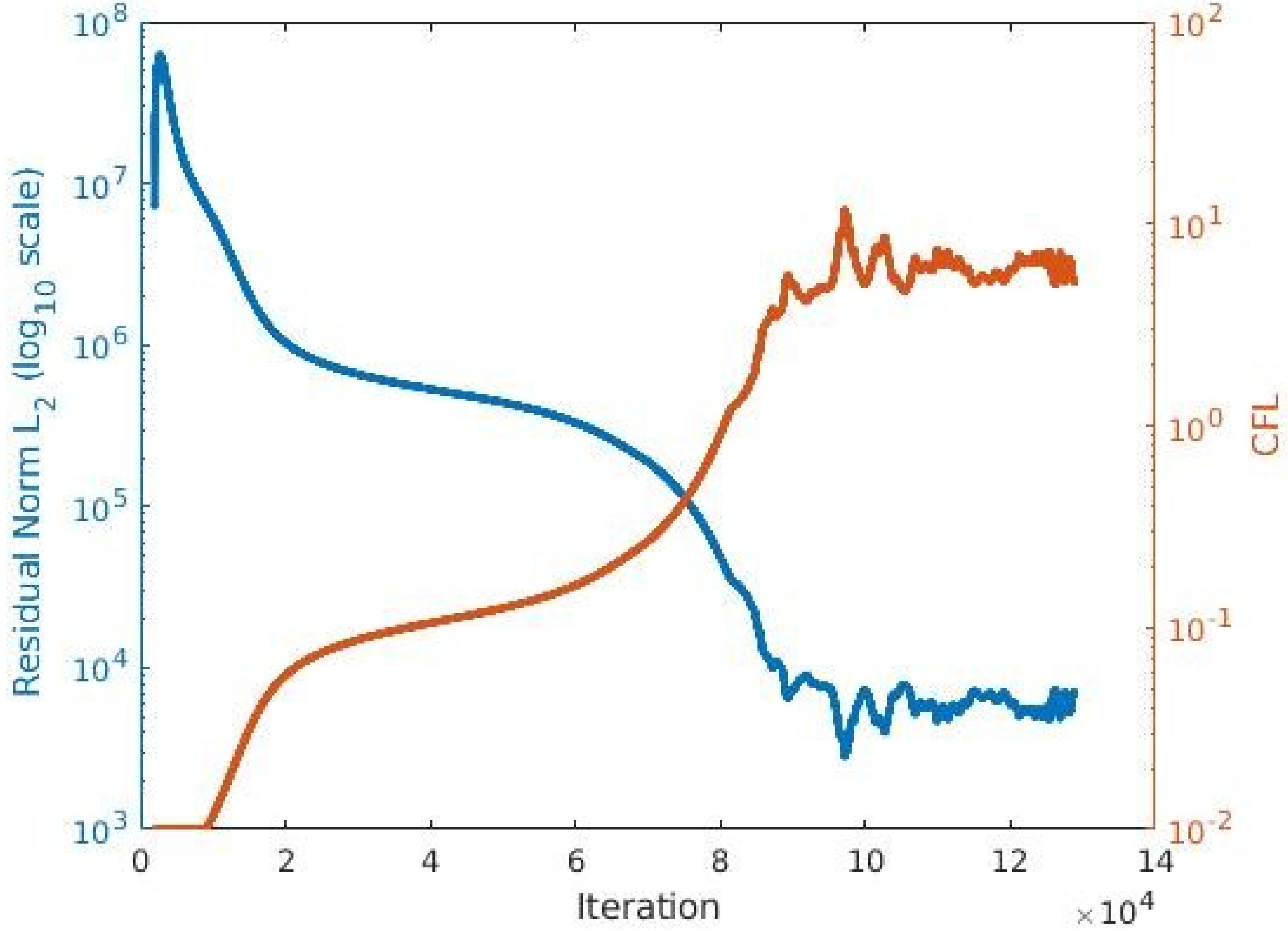
How can we improve the convergence ?
By increasing the order of the DGM !

Solution along the wall on Refined 7424 element mesh P1 without Artificial Viscosity



Without any surprise the code crashed… Next step Adding the AV!

Solution along the wall on Refined 7424 element mesh P1 WITH Artificial Viscosity



OUTLINE

1. Motivation
2. Bibliographic research approach
3. Aims of the project
4. Progress up to today
5. Conclusion & future work

Conclusion & Futur Work

- The artificial viscosity and the CFL are the two main components that allow the convergence for the Galerkin Method in high order in the Argo platform.

- Suggestion of future work :
  - Adapt the the modified Localized Laplacian Artificial Viscosity (LLAV) applied in the flux reconstruction method in COOLFluiD from the following article.

    Ray Vandenhoeck, *“Massively Parallel and Robust High-Order Methods for Transitional Hypersonic Flow Modelling on Unstructured Grids”*

BACKUP SLIDES

Local lifting

$$\int_{K^L \cup K^R} \phi^K r_h^e \left([\![\boldsymbol{w}_h]\!] \right) dV = \begin{cases} -\int_e \left\{ \phi^K \right\} [\![\boldsymbol{w}_h]\!] \otimes \boldsymbol{n} dS & if \;\; e \in \epsilon_i \\ -\int_e \phi^{K,+} [\![\boldsymbol{w}_h]\!] \otimes \boldsymbol{n} dS & if \;\; e \in \epsilon_{wall} \end{cases}$$

Global Lifting

$$\sum_{K \in \Omega_h} \int_K \phi^K R_h \left([\![\boldsymbol{w}_h]\!] \right) dV = -\sum_{e \in \epsilon_i} \int_e \left\{ \phi^K \right\} [\![\boldsymbol{w}_K]\!] \otimes \boldsymbol{n} ds + \sum_{e \in \epsilon_{wall}} \int_e \phi^{K,+} [\![\boldsymbol{w}_K]\!] \otimes \boldsymbol{n} ds$$

Discretisation of the **diffusive term** : The Interior penalty method limitation

Gouverning PDE's expressed in conservative and hypervectorial form:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = \nabla \cdot \mathbf{F}_v(\mathbf{U}, \nabla \mathbf{U}) + \mathbf{S}(\mathbf{U}, \nabla \mathbf{U})$$

Weak form of the equation (DG method):

$$\sum_{\Omega_e} \int_{\Omega_e} v \frac{\partial u}{\partial t} d\Omega_e + \sum_{\Omega_e} \int_{\Omega_e} v \frac{\partial F(u)}{\partial x} d\Omega_e = \sum_{\Omega_e} \int_{\Omega_e} v \frac{\partial}{\partial x} \left(D \frac{\partial u}{\partial x} \right) d\Omega_e$$

$$[v] = v^- n^- + v^+ n^+$$

$$\langle \bullet \rangle = \frac{1}{2} (\bullet^- + \bullet^+)$$

Rewritting the diffusive term as:

$$\boxed{\begin{aligned} \sum_{\Omega_e} \int_{\Omega_e} v \frac{\partial}{\partial x} \left(D \frac{\partial u}{\partial x} \right) d\Omega_e &= \sum_{\Omega_e} \int_{\Omega_e} \frac{\partial v}{\partial x} \left(D \frac{\partial u}{\partial x} \right) d\Omega_e \\ &- \sum_{\partial\Omega_e} \int_{\partial\Omega_e} \left\langle D(u) \frac{\partial u}{\partial x} \right\rangle [v] dS - \theta \sum_{\partial\Omega_e} \int_{\partial\Omega_e} \left\langle D(u) \frac{\partial v}{\partial x} \right\rangle [u] dS + \alpha \sum_{\partial\Omega_e} \int_{\partial\Omega_e} [v][u] dS \end{aligned}}$$

The **interior penalty coefficient θ** can take the value -1, 0 or 1.