P R O J E C T

Development of low-order shock-capturing scheme for discontinuous Galerkin method

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MIDTERM PUBLIC PRESENTATION

21rst March 2019

von Karman Institute for Fluid Dynamics

Aeronautics & Aerospace department

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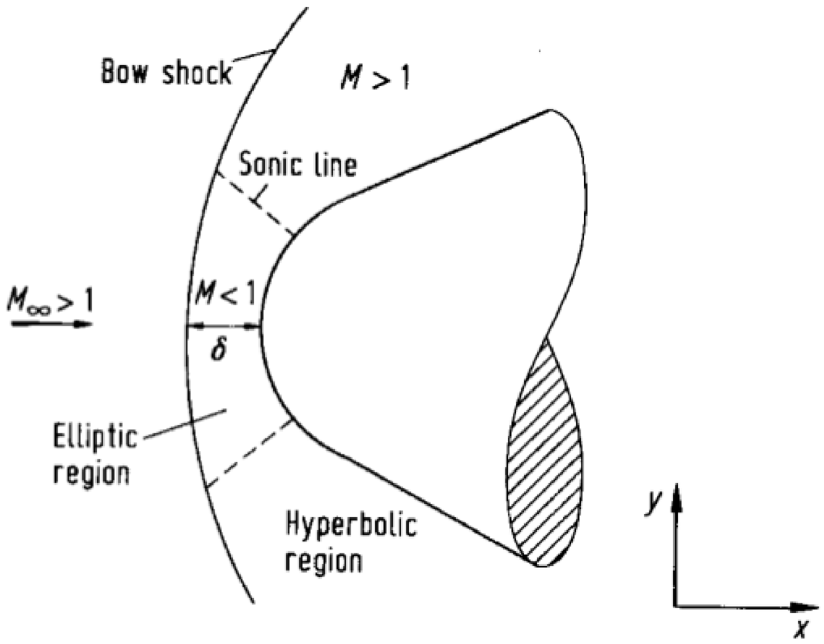
OUTLINE

1. Motivation
2. State of the art
3. Aim of the project
4. Progress up to today
5. Conclusion & future work

CHALLENGES

- **Measure the aerodynamic forces and heat flux on a body:**
  → Experiments in wind tunnels
  → Simulation by means of CFD
- **Hypersonic flow:** → Multi-physics phenomena
  → High temperature and compressible effects

"***The role of CFD in engineering predictions has become so strong that today it can be viewed as a new 'third dimension' in fluid dynamics, the other two dimensions being the classical cases of pure experiment and pure theory.***"
[J.D. Anderson]



J.D. Anderson et al. Computational Fluid Dynamics: An Introduction (Von Karman Institute Book). Edited by John F.Wendt. Springer; 3rd ed., 2008.

CHALLENGES

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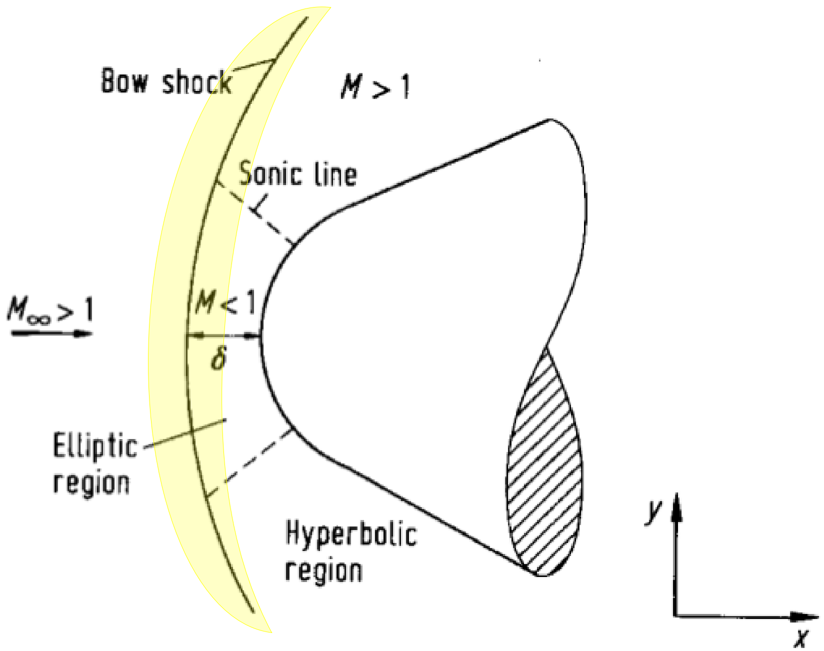
The shock Capturing method consist in letting appear naturally within the computational space the shock region without any specific treatment of the shocks themselves.

The shock fitting method treats the shock region independently from the flowfield by treating the shock as an internal boundary condition.



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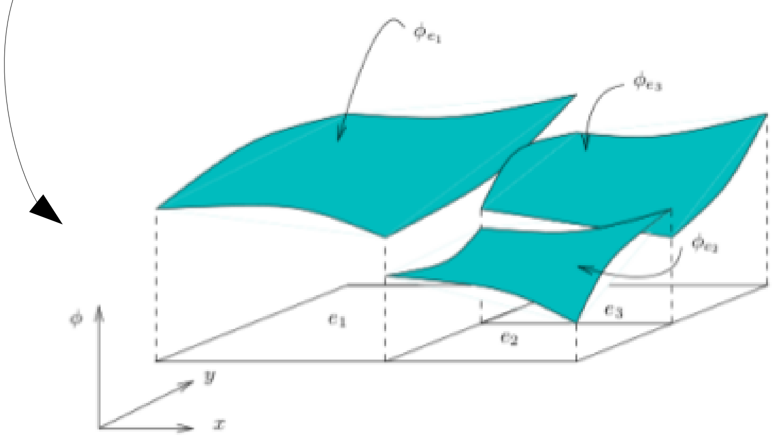
Criteria for high order scheme

	Compact stencil	hp- adaptability	Unstructured grid approach	Parallel implementation	Shock capturing at high Mach number
Finite Difference (FD)	🔴	🟠	🔴	🟠	🔴
Finite Volume (FV)	🔴	🟠	🟢	🟠	🟢
Finite Element (FE)	🟢	🟢	🟢	🟠	🟠
Discontinuous Galerkin (DG)	🟢	🟢	🟢	🟢	?



- 🟢 Well adapted
- 🟠 Can be adapted with some constraints
- 🔴 Not adapted

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Shock detector and artificial viscosity

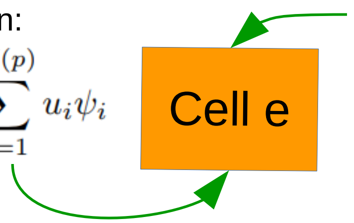
Shock detector: identify the location of discontinuities in the computational domain
Artificial viscosity: makes shocks broader

Approximation of the solution:

$$\hat{u} = \sum_{i=1}^{N(p-1)} u_i \psi_i$$

Solution:

$$u = \sum_{i=1}^{N(p)} u_i \psi_i$$



IF(the polynomials shapes are similar**)**
Then the solution on the cell e is smooth
Else the solution contains discontinuities

Sensor of the discontinuity on the cell e:

$$S_e = \frac{(u-\hat{u}, u-\hat{u})_e}{(u,u)_e}$$

$(\cdot,\cdot)_e$ inner product in $L_2(\Omega_e)$

Artificial viscosity for each cell e:

$$\varepsilon_e = \begin{cases} 0 & \text{if} \quad s_e < s_0 - \kappa \\ \frac{\varepsilon_0}{2}\left(1 + \sin\frac{\pi(s_e - s_0)}{2\kappa}\right) & \text{if} \quad s_0 - \kappa \le s_e \le s_0 + \kappa \\ \varepsilon_0 & \text{if} \quad s_e > s_0 + \kappa \end{cases}$$

with $s_e = \log_{10} S_e$

Input defined by the user:

- The threshold $s_0 \sim 1/p^4$
- The interval κ
- The amount of artificial viscosity $\varepsilon_0 \sim h/p$

Conservative form of the NS equations in 2D for a calorically and thermally perfect gas

Let $\Omega_{adm} = \left\{ \boldsymbol{w} \in \mathbb{R}^4 : \rho > 0, \boldsymbol{u} \in \mathbb{R}^2, e > 0 \right\}$

Find $\boldsymbol{w} = \begin{pmatrix} \rho \\ \rho \boldsymbol{u} \\ \rho E \end{pmatrix} \in \Omega_{adm}$ **such that:**

$$\partial_t \boldsymbol{w} + \nabla.\mathbb{F}_c(\boldsymbol{w}) - \nabla.\mathbb{F}_v(\boldsymbol{w}, \nabla \boldsymbol{w}) = 0, \ \forall (\boldsymbol{x}, t) \in \Omega \times]0; +\infty[$$

$$\boldsymbol{w}(\boldsymbol{x}, 0) = \boldsymbol{w}_{t=0}(\boldsymbol{x}), \ \forall \boldsymbol{x} \in \Omega$$

$$p = \rho R T$$

+ Boundary conditions on $\Gamma = \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_{wall}$

$$\mathbb{F}_c(\boldsymbol{w}) = \begin{pmatrix} \rho \boldsymbol{u}^T \\ \rho \boldsymbol{u} \otimes \boldsymbol{u} + p\mathbb{I} \\ (\rho E + p)\boldsymbol{u}^T \end{pmatrix}_{4 \times 2} \qquad \mathbb{F}_v(\boldsymbol{w}) = \begin{pmatrix} \boldsymbol{0}^T_{\mathbb{R}^2} \\ \boldsymbol{\tau} \\ \boldsymbol{u}^T.\boldsymbol{\tau} - \boldsymbol{q}^T \end{pmatrix}_{4 \times 2}$$

Generic Form of the Variational Formulation

$$\sum_{K \in \Omega_h} \int_K \phi_i^K \partial_t \boldsymbol{w}_h dV + \mathcal{L}_c(\boldsymbol{w}_h, \phi^K) + \mathcal{L}_v(\boldsymbol{w}_h, \nabla \boldsymbol{w}_h, \phi^K) = 0$$

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Lax Friedrich Riemann Solver

$$\hat{\mathbb{F}}_c(\boldsymbol{w}_h^+, \boldsymbol{w}_h^-, \boldsymbol{n}) = \{\mathbb{F}_c(\boldsymbol{w}_h)\}\, \boldsymbol{n} + \frac{a}{2} [\![\boldsymbol{w}_h]\!]$$

$$a = max \left\{ \left| \frac{\partial \mathbb{F}_c \boldsymbol{n}}{\partial \boldsymbol{w}} \right| : \boldsymbol{w} = \boldsymbol{w}_h^\pm \right\}$$

$$z^\pm(x_e) = \lim_{y \to x_e,\ y \in K^\pm} z(y) \qquad [\![z]\!] = \begin{cases} z^+ - z^- & if\ e \in \epsilon_i \\ z^+ - z_{wall} & if\ e \in \epsilon_{wall} \end{cases} \qquad \{z\} = \begin{cases} \frac{z_+ + z_-}{2} & if\ e \in \epsilon_i \\ z_{wall} & if\ e \in \epsilon_{wall} \end{cases}$$

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Internal penalty method

$$z^{\pm}(x_e) = \lim_{y \to x_e,\ y \in K^{\pm}} z(y)$$

$$[\![z]\!] = \begin{cases} z^+ - z^- & if\ e \in \epsilon_i \\ z^+ - z_{wall} & if\ e \in \epsilon_{wall} \end{cases}$$

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$$\sum_{K \in \Omega_h} \int_K \phi_i^K \partial_t \boldsymbol{w}_h dV + \mathcal{L}_c(\boldsymbol{w}_h, \phi^K) + \mathcal{L}_v(\boldsymbol{w}_h, \nabla \boldsymbol{w}_h, \phi^K) = 0$$

~~**Internal penalty method**~~

Inconsistent at p=0

$$z^{\pm}(x_e) = \lim_{y \to x_e,\ y \in K^{\pm}} z(y) \qquad [\![z]\!] = \begin{cases} z^+ - z^- & if\ e \in \epsilon_i \\ z^+ - z_{wall} & if\ e \in \epsilon_{wall} \end{cases} \qquad \{z\} = \begin{cases} \frac{z_+ + z_-}{2} & if\ e \in \epsilon_i \\ z_{wall} & if\ e \in \epsilon_{wall} \end{cases}$$

Generic Form of the Variational Formulation

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The second scheme of Bassi Rebay

Introduce a new equation in the initial problem To solve

$$\begin{cases} \Upsilon_h = \nabla \boldsymbol{w}_h \\ \partial_t \boldsymbol{w} + \nabla . \mathbb{F}_c(\boldsymbol{w}) - \nabla . \mathbb{F}_v(\boldsymbol{w}, \nabla \boldsymbol{w}) = 0 \end{cases}$$

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New Variational Formulation

$$z^{\pm}(x_e) = \lim_{y \to x_e,\ y \in K^{\pm}} z(y) \qquad [\![z]\!] = \begin{cases} z^+ - z^- & if\ e \in \epsilon_i \\ z^+ - z_{wall} & if\ e \in \epsilon_{wall} \end{cases} \qquad \{z\} = \begin{cases} \frac{z_+ + z_-}{2} & if\ e \in \epsilon_i \\ z_{wall} & if\ e \in \epsilon_{wall} \end{cases}$$

Generic Form of the Variational Formulation

$$\begin{cases} \mathcal{L}_{aux}(\boldsymbol{w}_h, \Upsilon_h, \alpha^K) = 0 \\ \sum_{K\in\Omega_h} \int_K \phi_i^K \partial_t \boldsymbol{w}_h dV + \mathcal{L}_c(\boldsymbol{w}_h, \phi^K) + \mathcal{L}_v(\boldsymbol{w}_h, \Upsilon_h, \phi^K) = 0 \\ \end{cases}$$

The second scheme of Bassi Rebay is consistent in p0

$$z^{\pm}(x_e) = \lim_{y\to x_e,\ y\in K^{\pm}} z(y)$$

$$[\![z]\!] = \begin{cases} z^+ - z^- & if\ e \in \epsilon_i \\ z^+ - z_{wall} & if\ e \in \epsilon_{wall} \end{cases}$$

$$\{z\} = \begin{cases} \frac{z_+ + z_-}{2} & if\ e \in \epsilon_i \\ z_{wall} & if\ e \in \epsilon_{wall} \end{cases}$$

Shock capturing methods

Low order approach

1. Compute in P0 in all the domain
2. Localize the shock with a sensor
3. Restrain the shock region to P0 and run in a higher order everywhere else on the domain

High order approach

Run in high order in all the domain & use the AV to reduce the spurious oscillations in the shock region

OUTLINE

Aim of the project

Develop a robust shock capturing scheme for the DG method

Main tasks

1. Review the hypersonic flow challenges for **high order numerical simulation** ✓

2. Identify the limit of the **artificial viscosity** into the Argo platform ✓

3. Optimize and adapt a hybrid solver for hypersonic applications which uses a **degraded order scheme** (DG(P0)) with LF-BR2) in the shock region and a high order method (DG(P>1) with LF-BR2) everywhere else ⌛

4. Validation on a specific test case ✗

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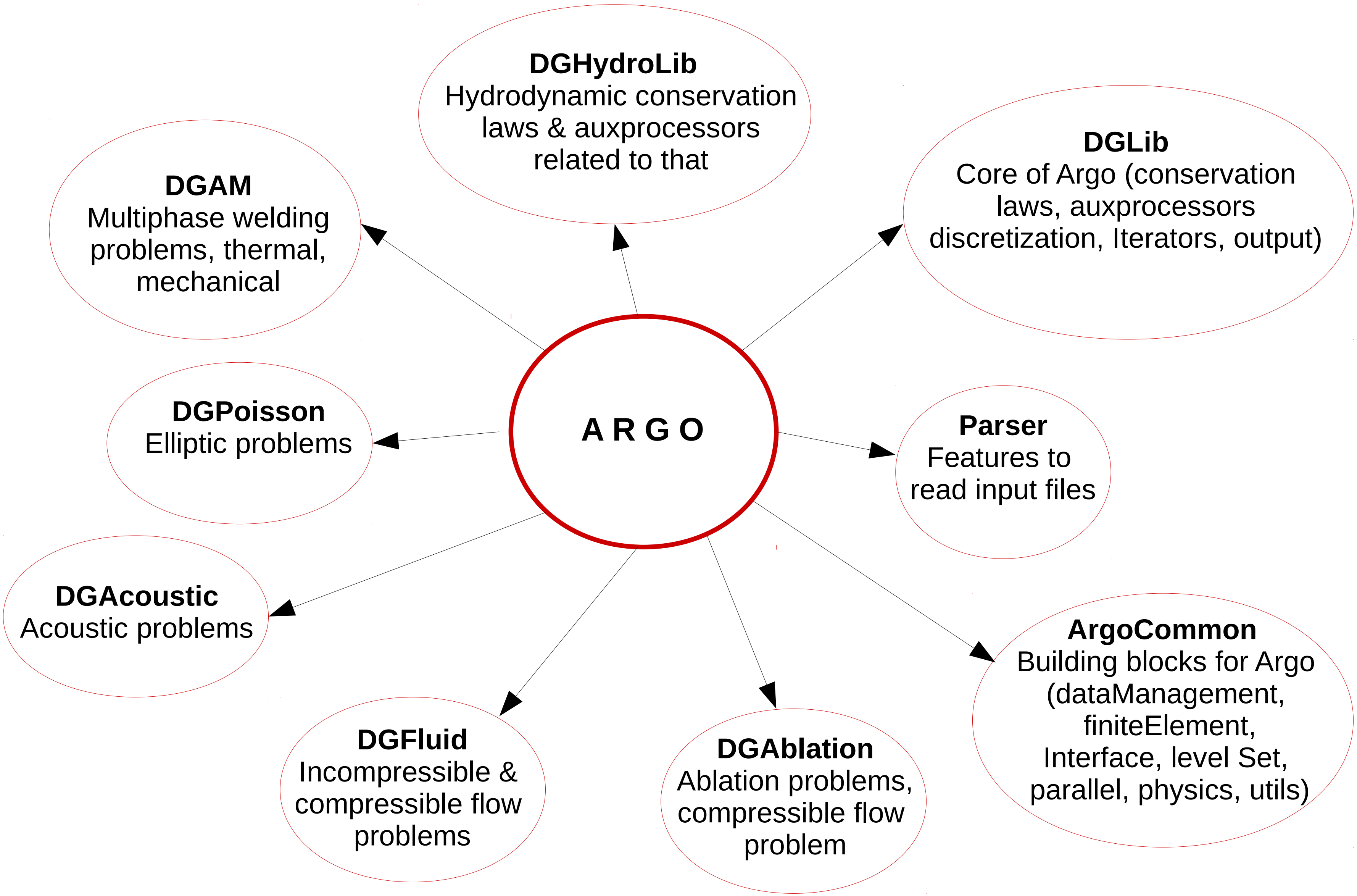
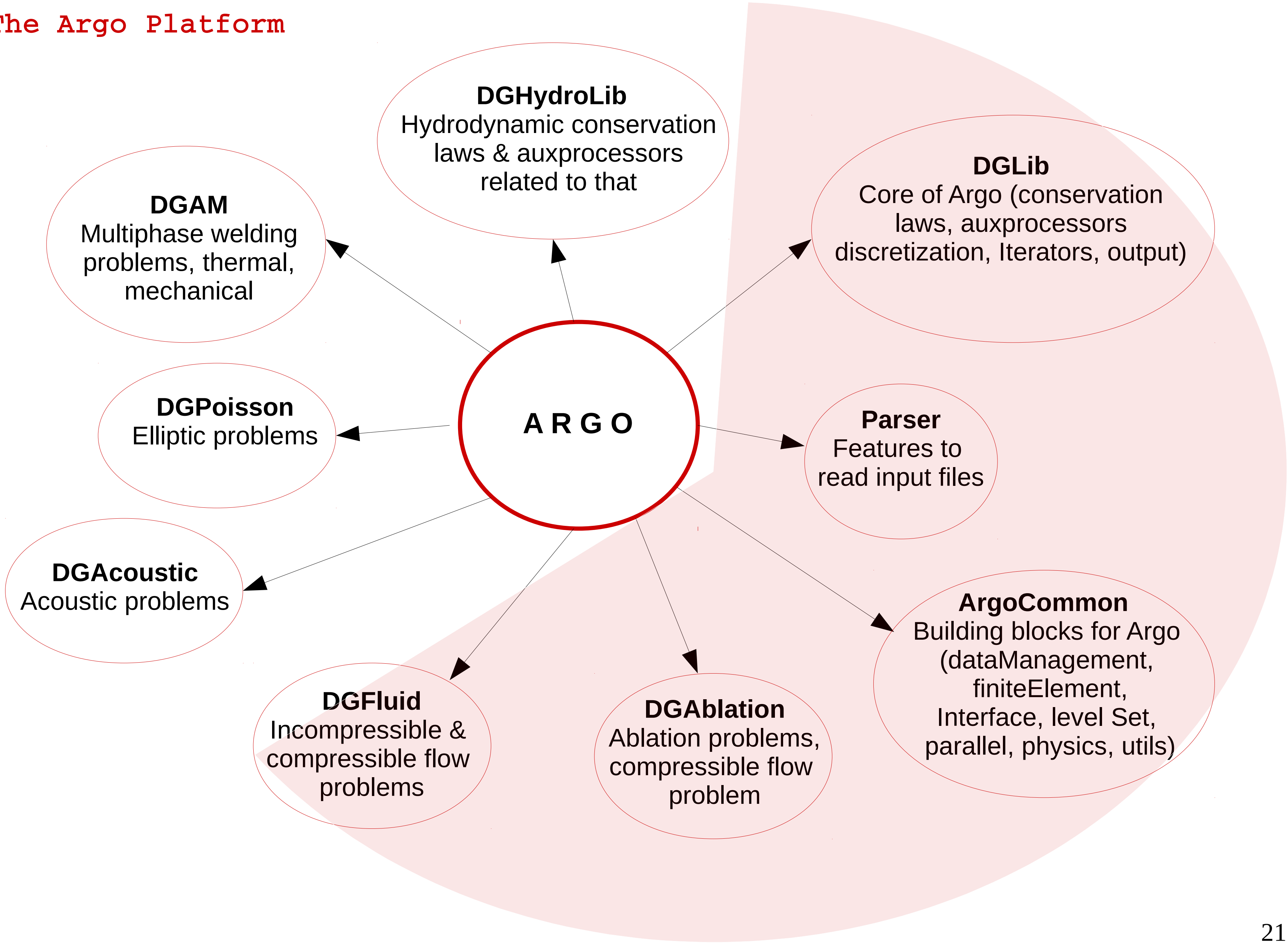
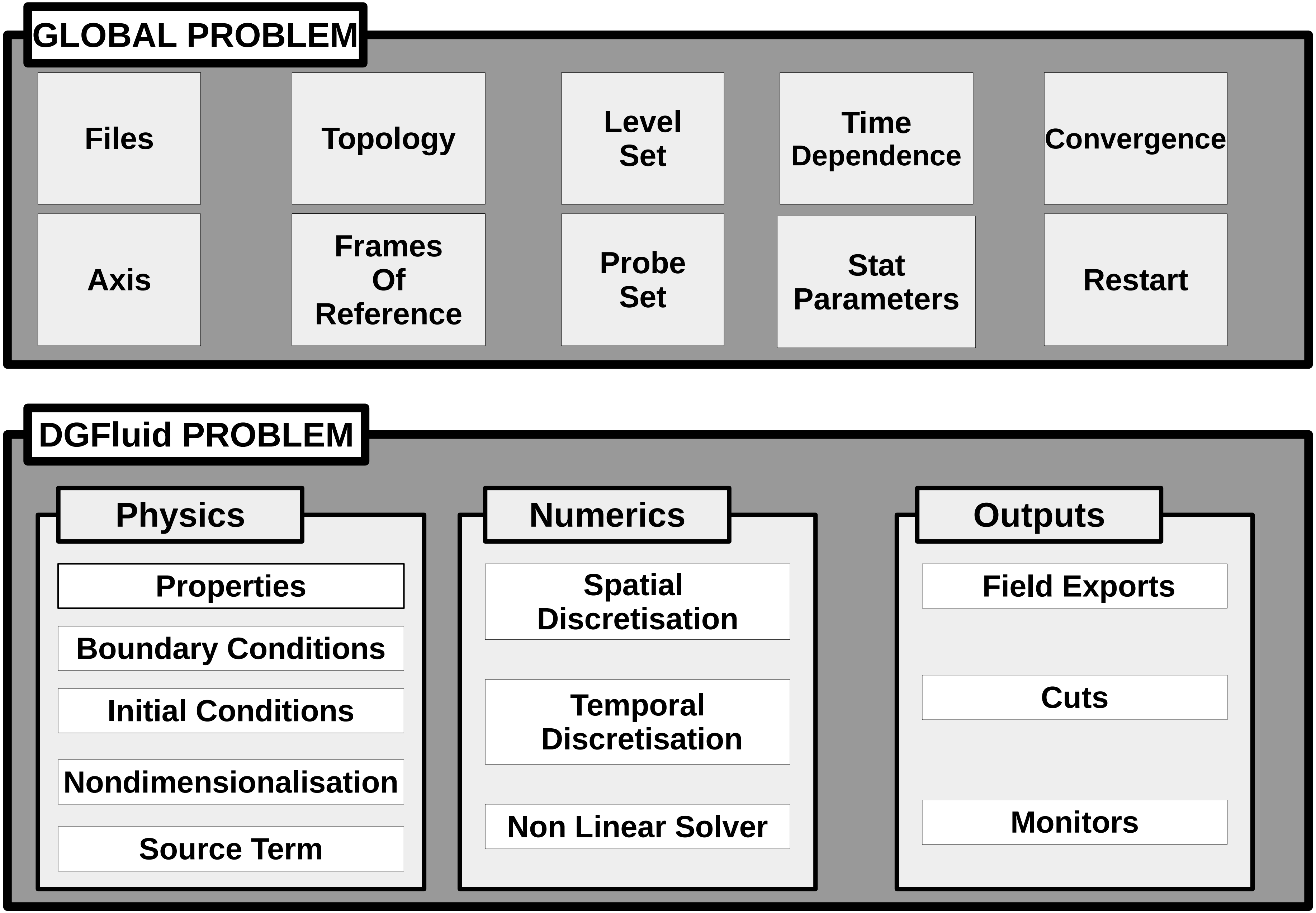
The Argo Platform



The Argo Platform

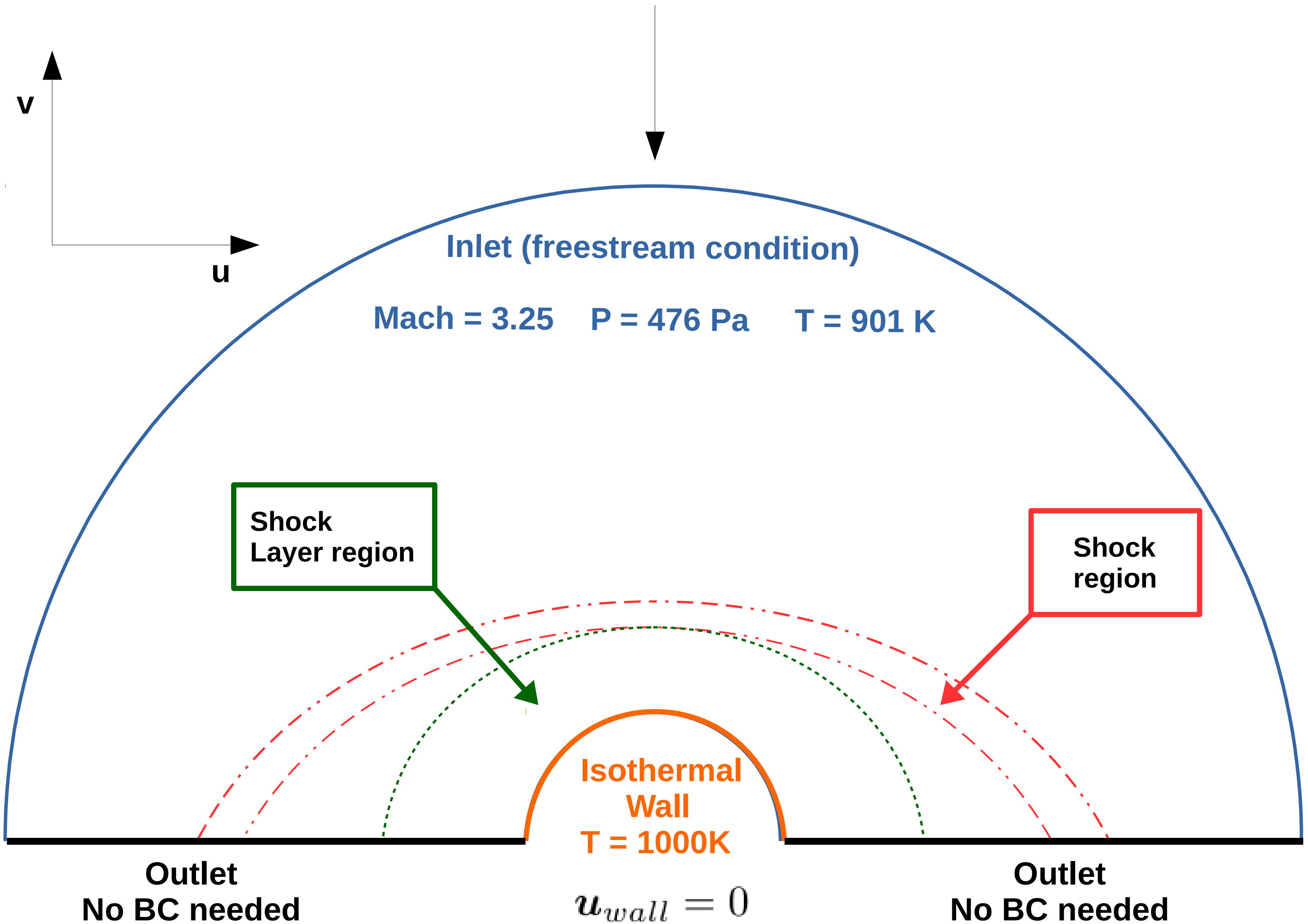


Structure of the input file



Test case configuration

2D Perfect Gas



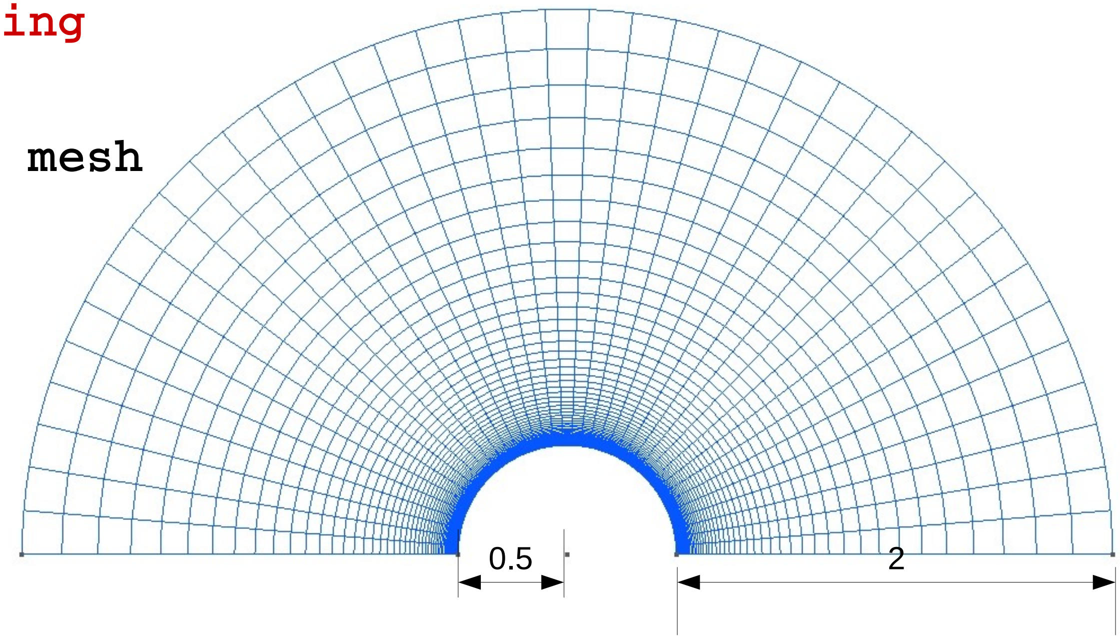
$$u_{wall} = 0$$

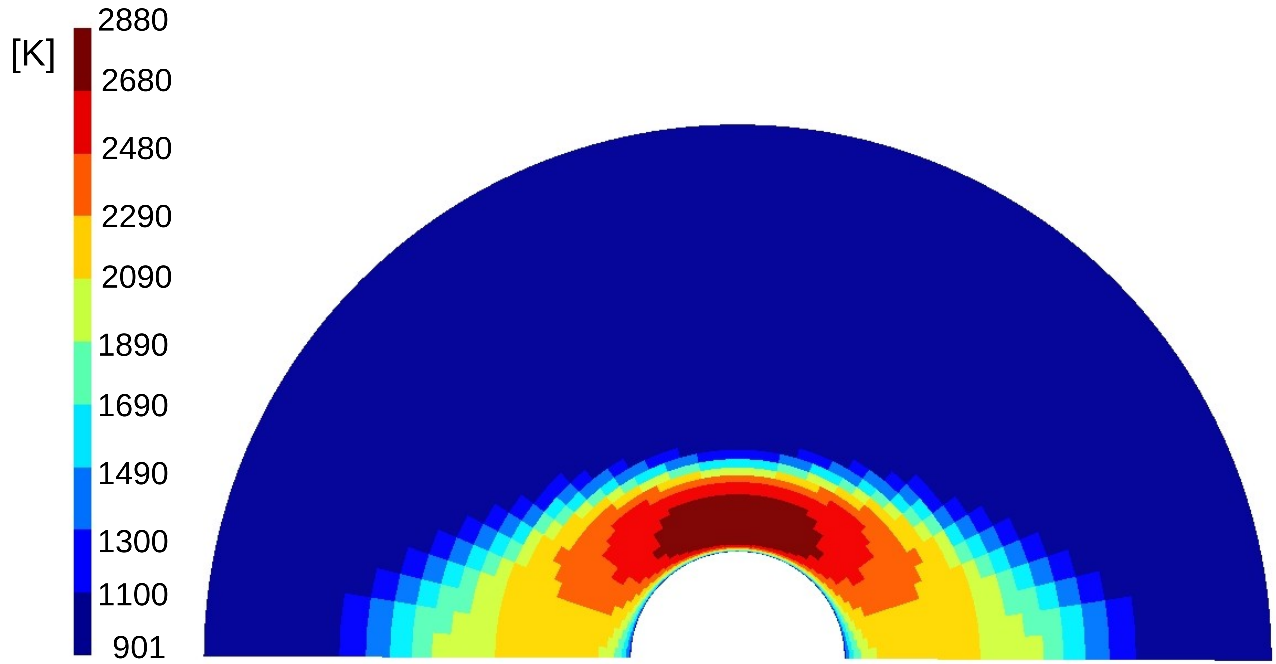
$$\nabla T_{wall}.n = 0$$

First Test P0 (shock capturing method)

Coarse structured mesh

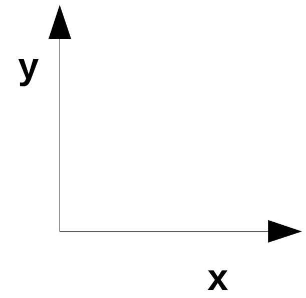
Method	DG(P=0)
Diffusive flux	BR2
Mesh	3323 elements

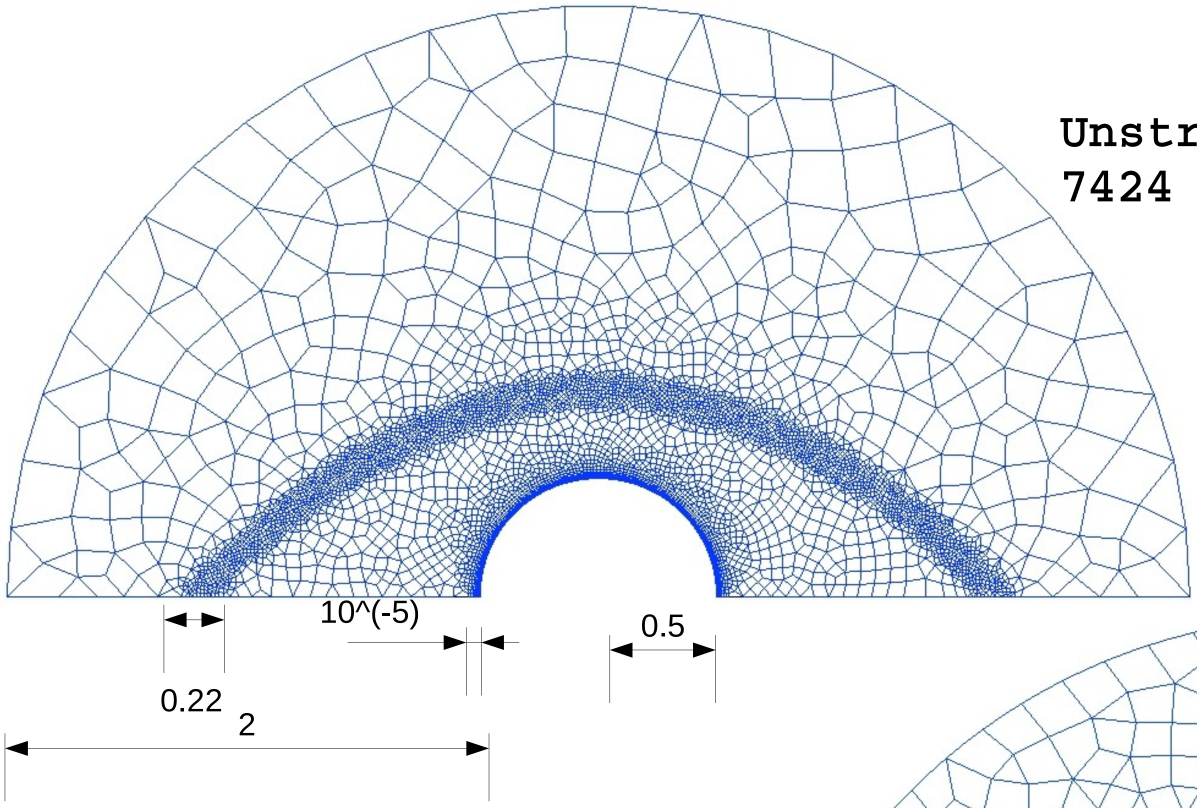




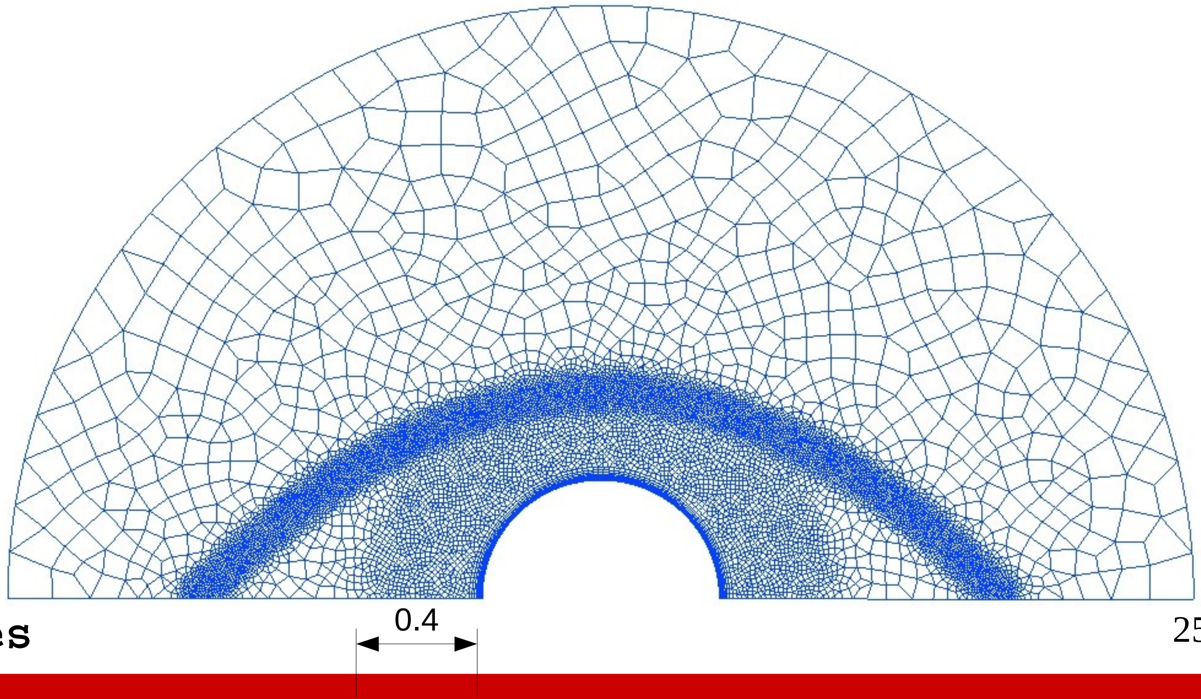
Temperature [K]

Refined Meshes (Gmsh)



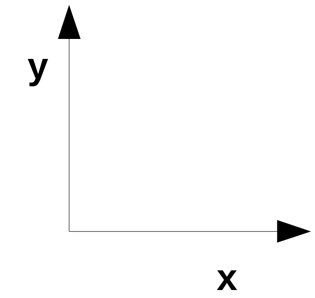


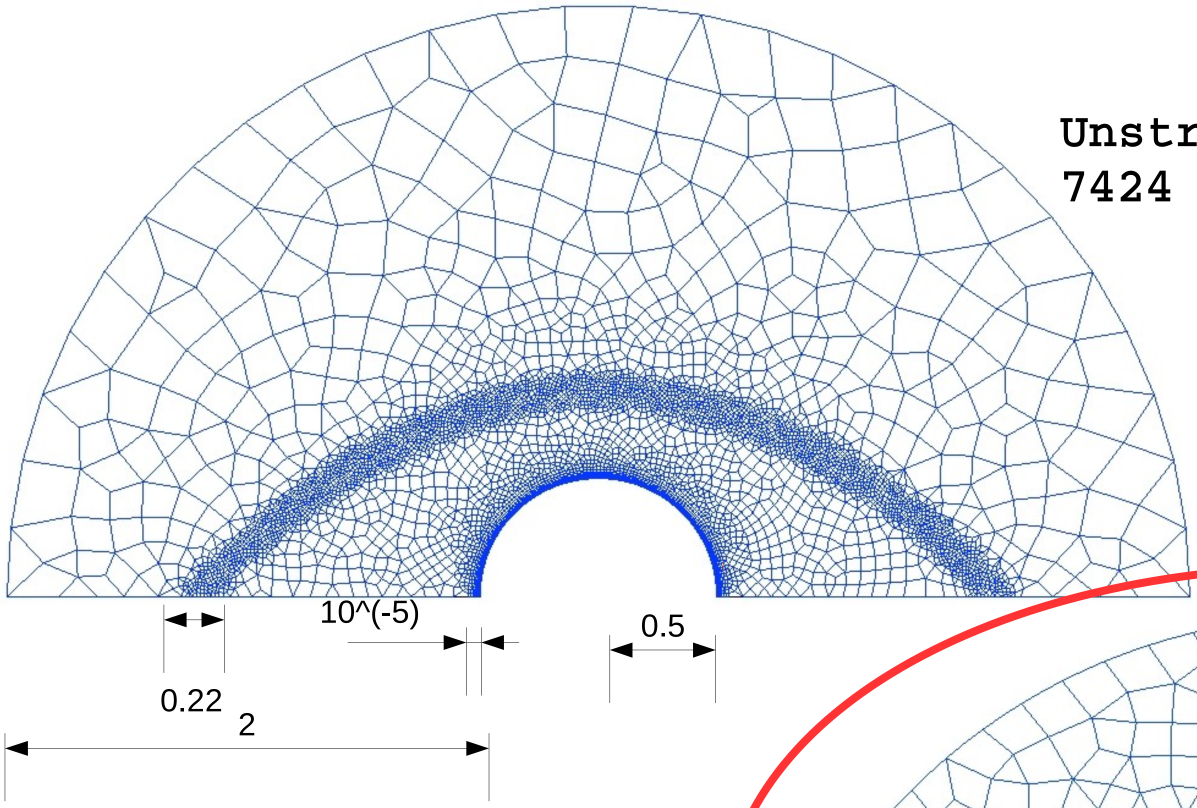
Unstructured mesh
7424 elements
- **1154 triangles**
- **5764 quadrangles**



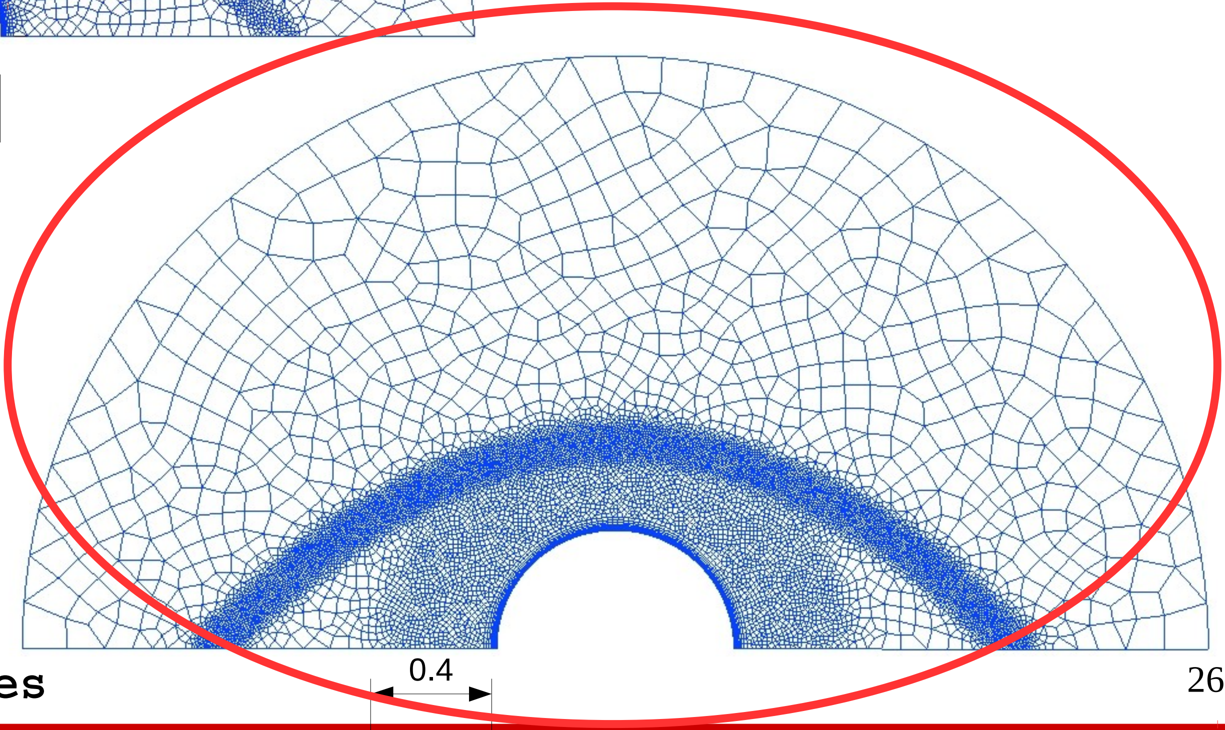
Unstructured mesh
15 606 elements
- **2865 triangles**
- **11652 quadrangles**

Refined Meshes (Gmsh)





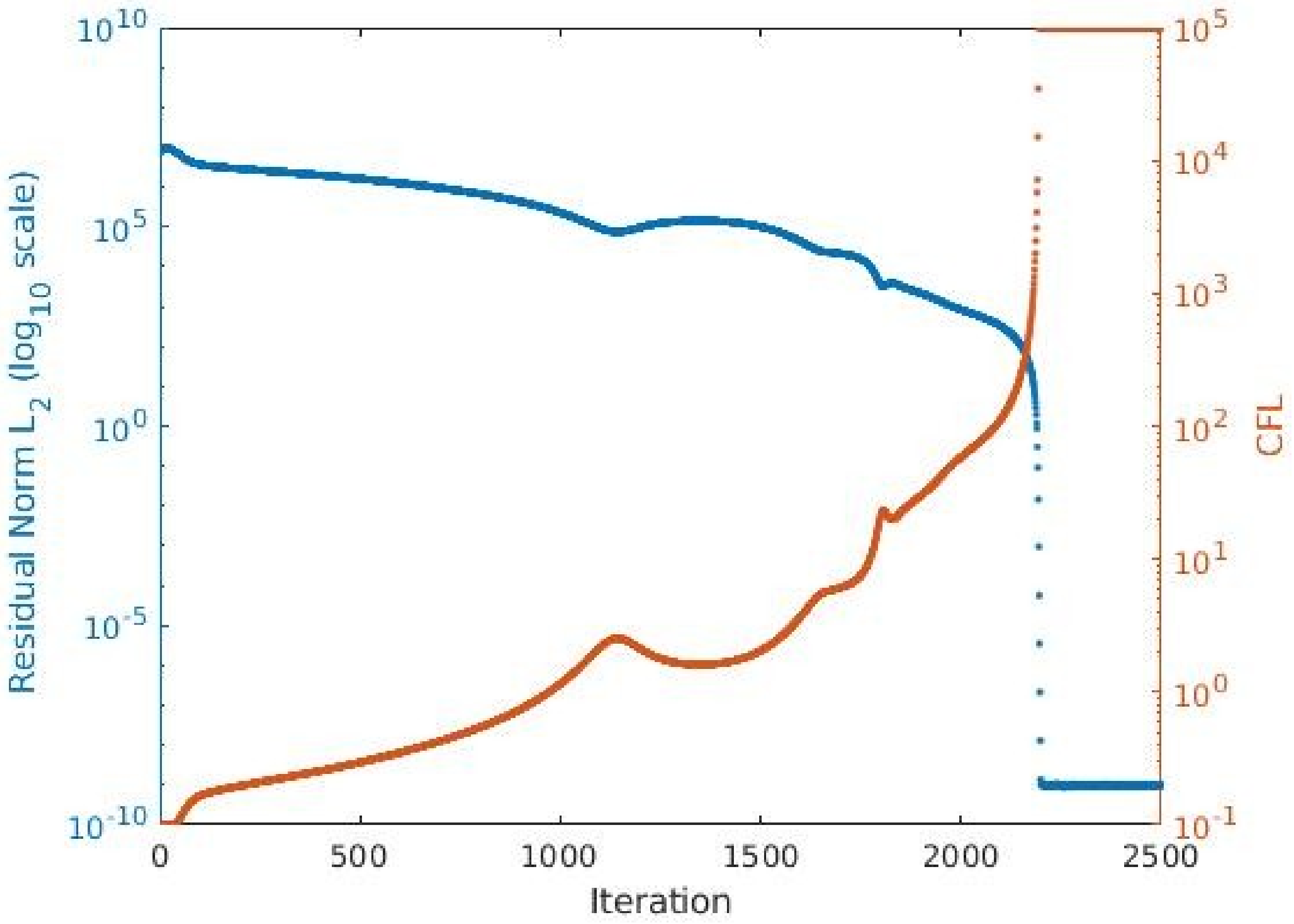
Unstructured mesh
7424 elements
- 1154 triangles
- 5764 quadrangles



Unstructured mesh
15 606 elements
- 2865 triangles
- 11652 quadrangles

Solution on refined 15606 elements mesh P0

Method	DG(P=0)
Diffusive flux	BR2
Mesh	Refined 15606 elements



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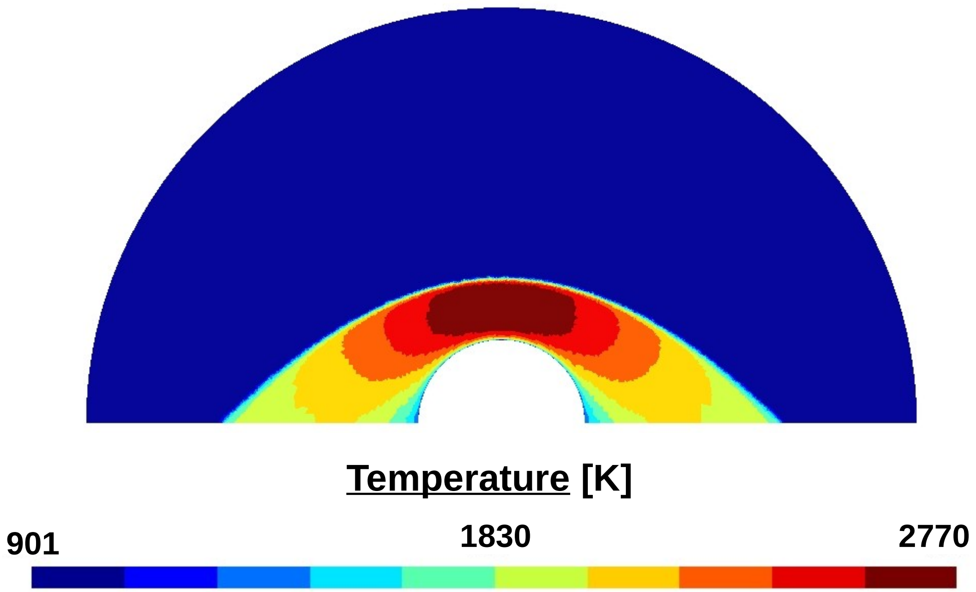
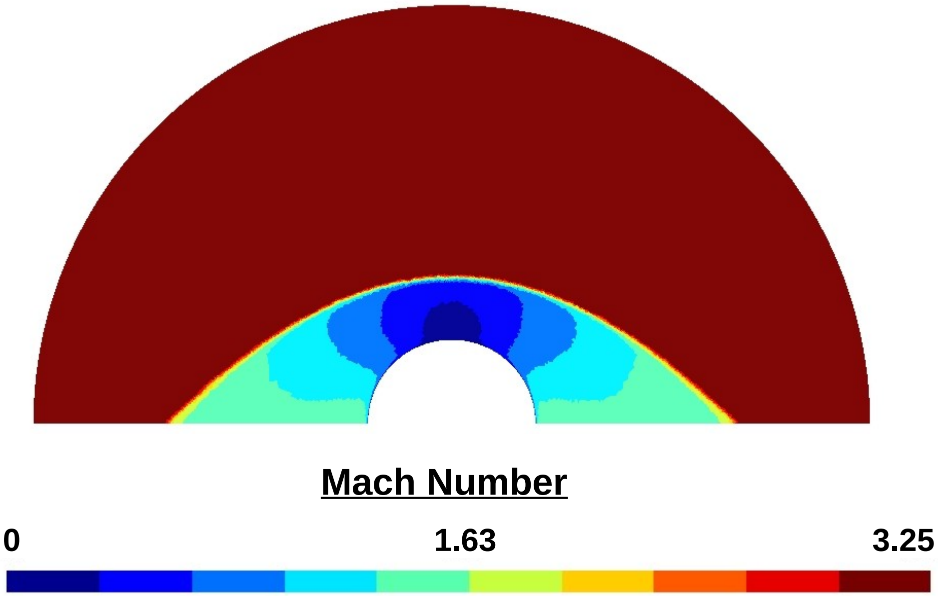
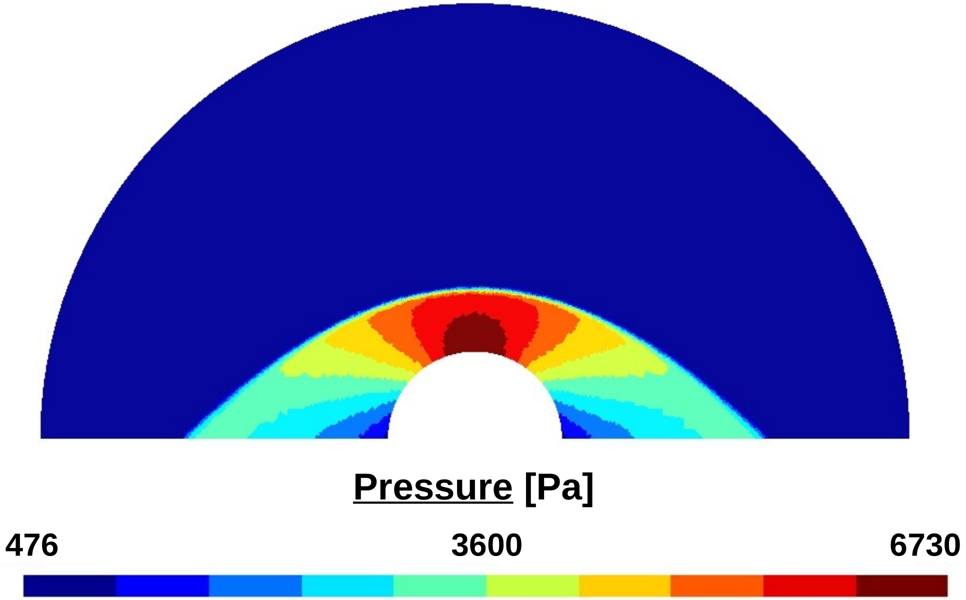
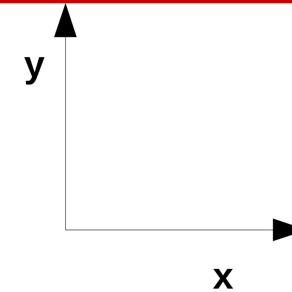
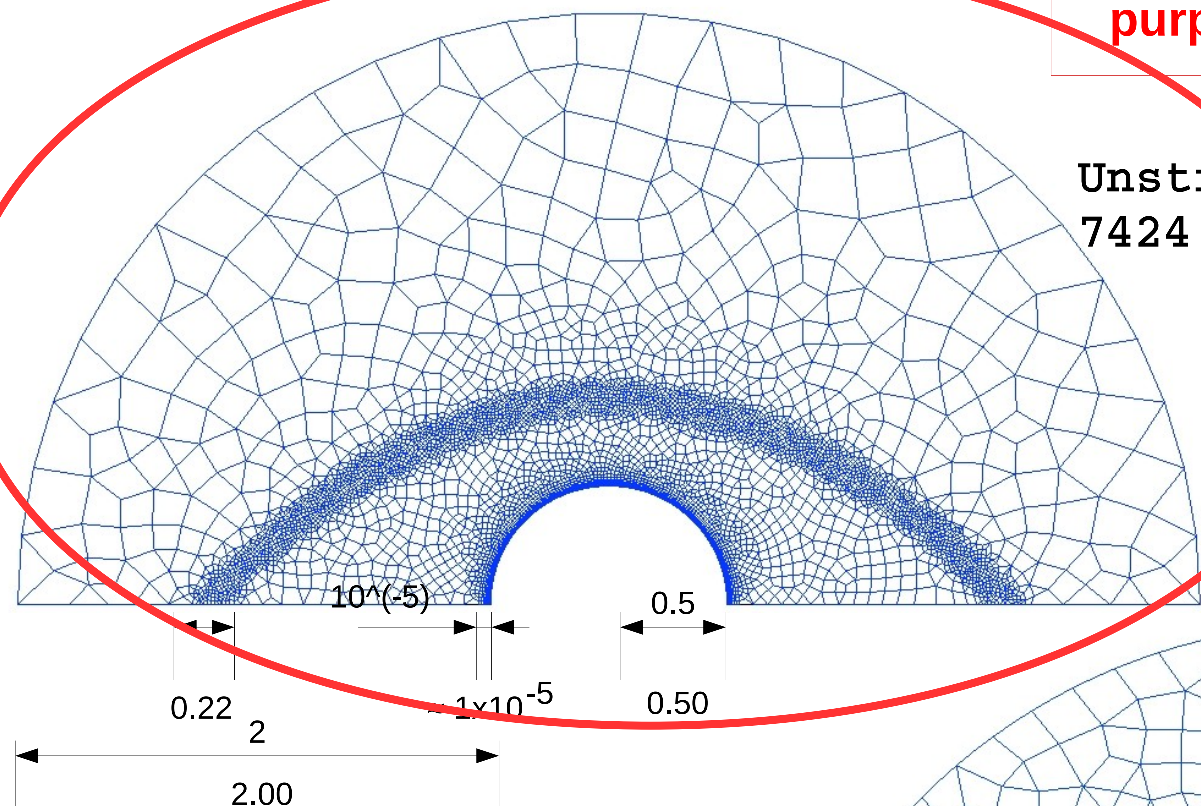






For testing purposes





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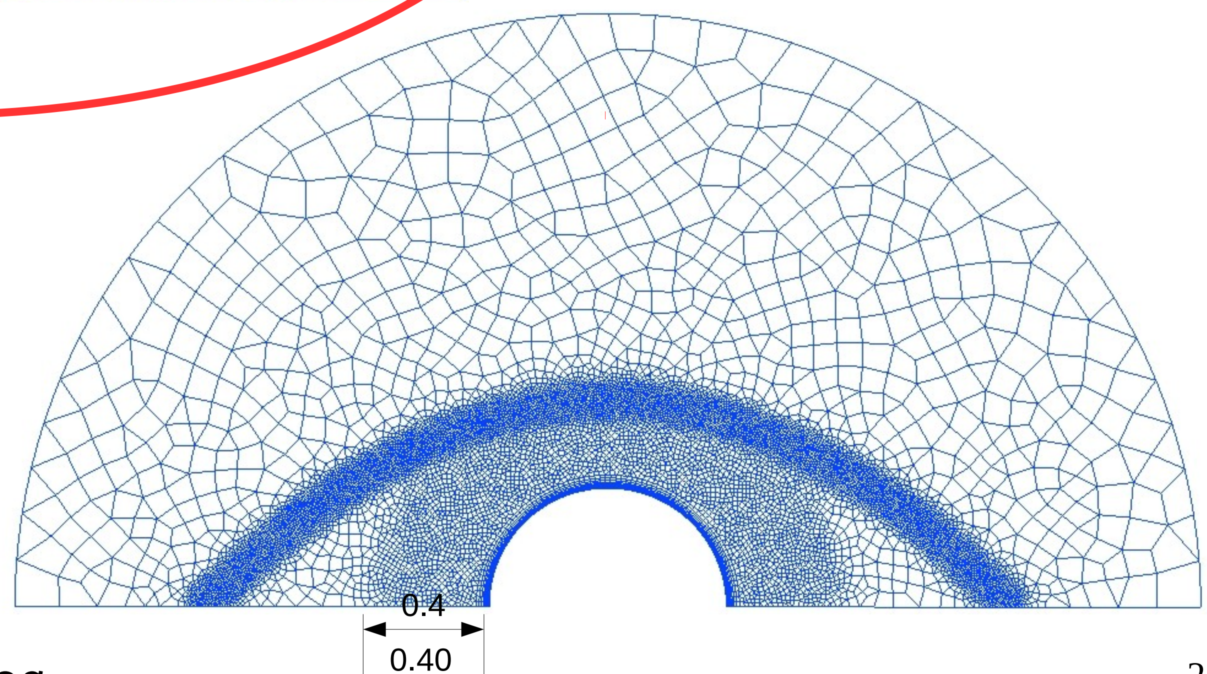

Unstructured mesh
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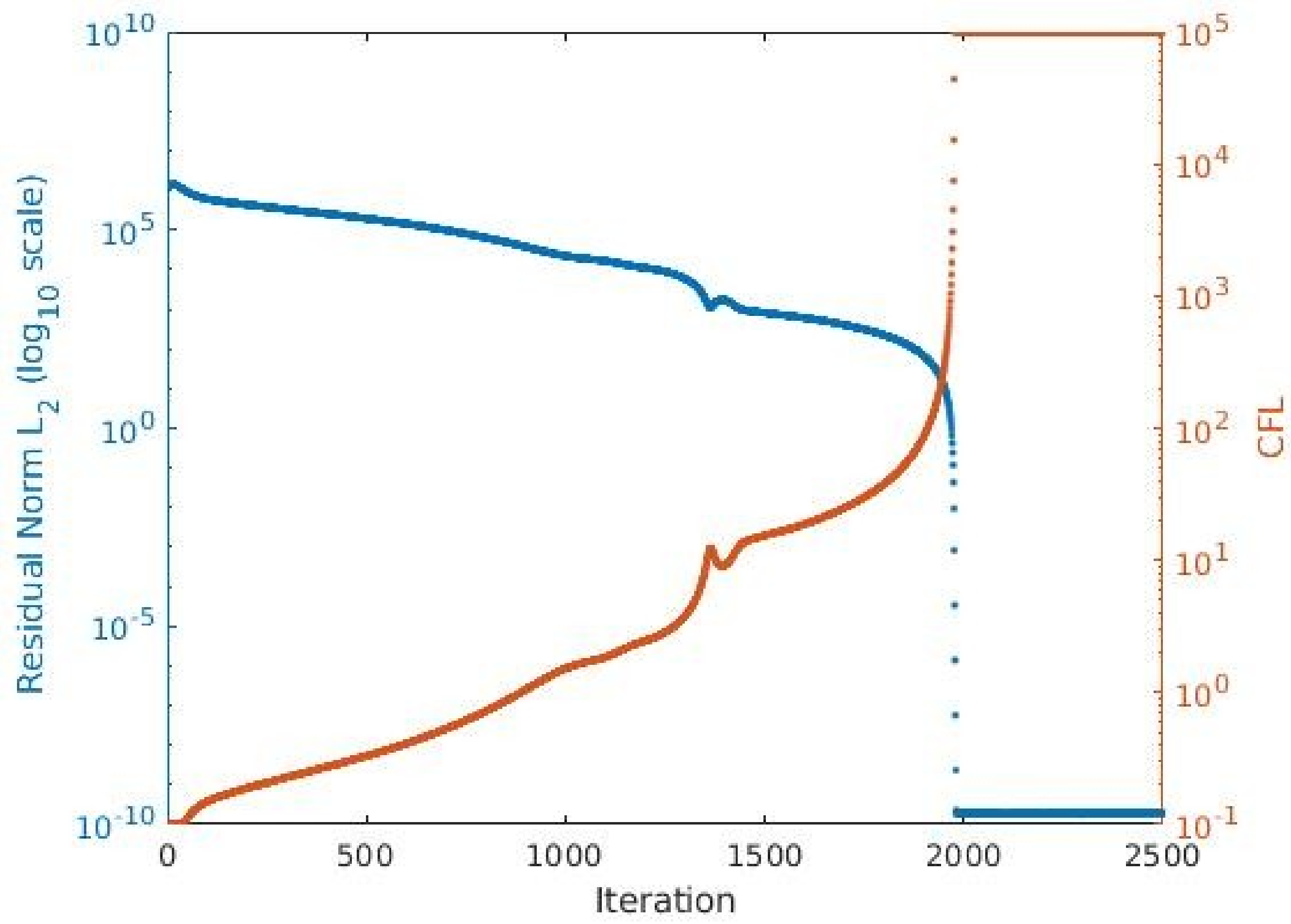
Residual on refined 7424 elements mesh P0

Method	DG(P=0)
Diffusive flux	BR2
Mesh	Refined 7424 elements



Comparison of the results obtained for the temperature And the Mach number with those obtained by Marc Cruellas Bordes (using IP+AV in DG(P=1))

Test case configuration

Parameter	Freestream	Wall	Outlet
Velocity [m/s]	1956	No-slip	NA
Static pressure [Pa]	476	NA	NA
Static temperature [K]	901	1000	NA

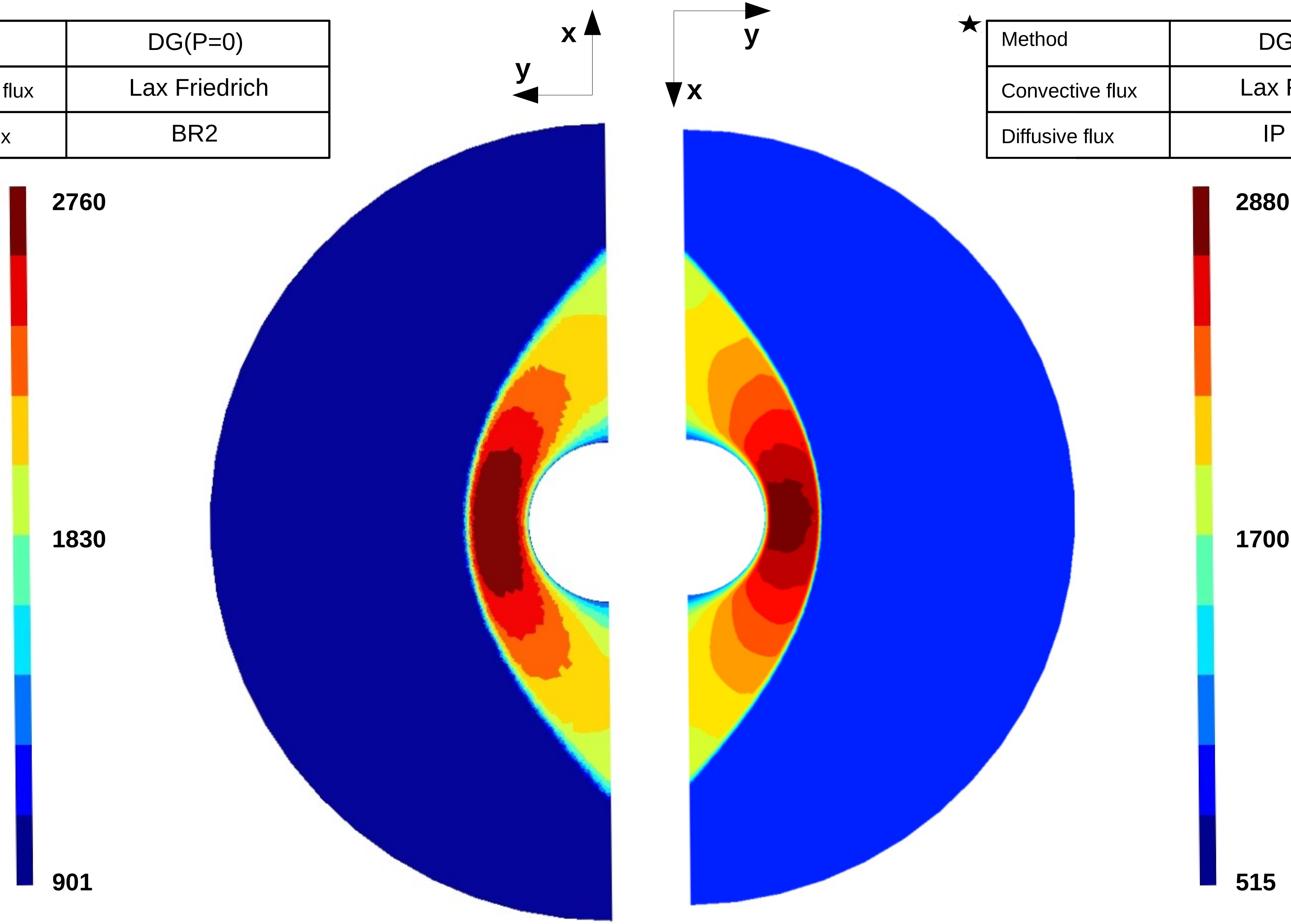
Method	DG(**P=0**)
Convective flux	Lax Friedrich
Diffusive flux	**BR2**
Mesh	Refined 7424 elements

Method	DG(**P=1**)
Convective flux	Lax Friedrich
Diffusive flux	**IP + AV**
Mesh	Refined 7424 elements

Method	DG(P=0)
Convective flux	Lax Friedrich
Diffusive flux	BR2

★

Method	DG(P=1)
Convective flux	Lax Friedrich
Diffusive flux	IP + AV

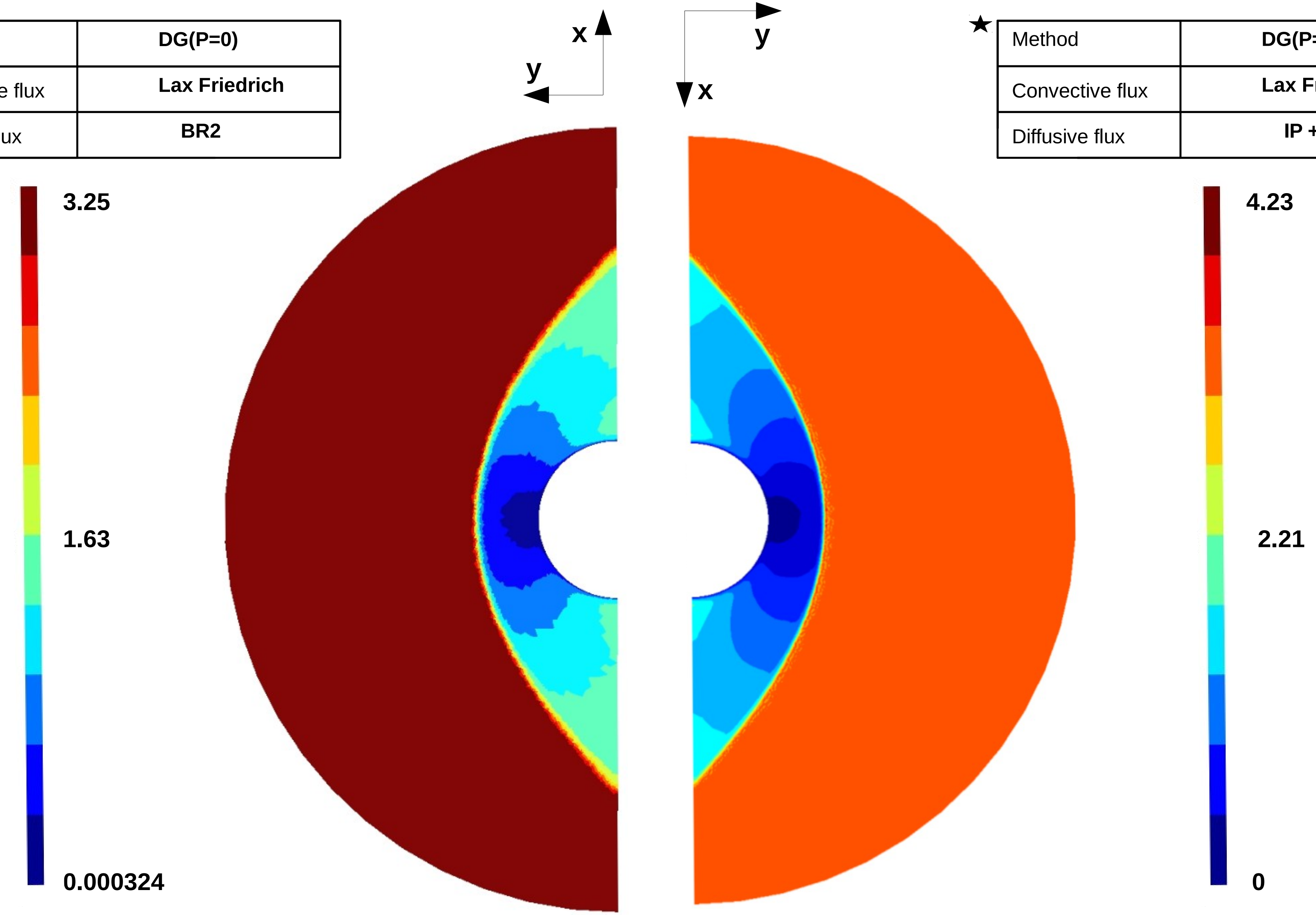


Temperature [K] on refined 7424 elements mesh P0

Method	DG(P=0)
Convective flux	Lax Friedrich
Diffusive flux	BR2

★

Method	DG(P=1)
Convective flux	Lax Friedrich
Diffusive flux	IP + AV



Mach number on refined 7424 elements mesh P0

★ Simulation made by Marc Cruellas Bordes

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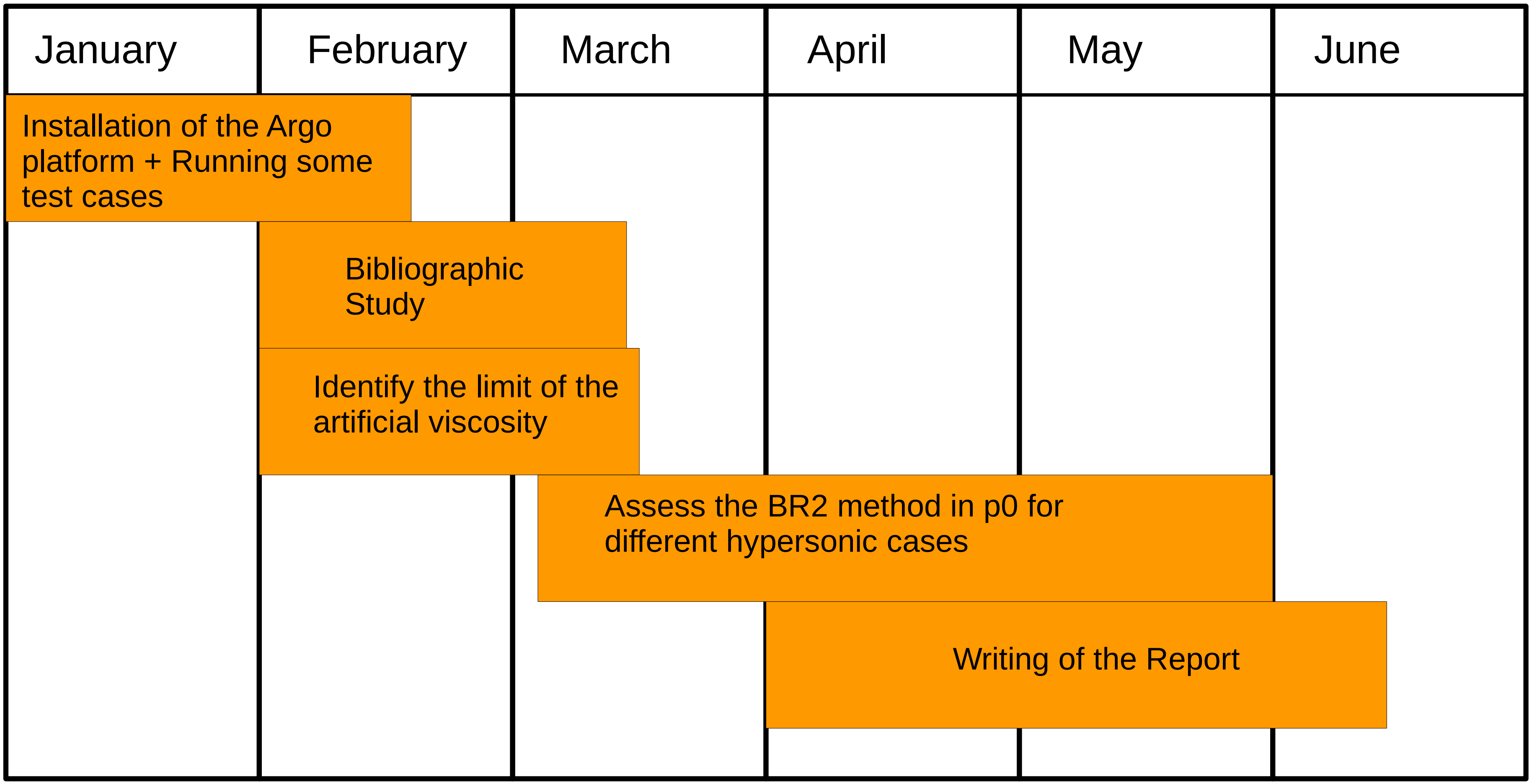
Conclusion & Futur Work

- The BR2 method showed promising results compared to the internal penalty method
- The artificial viscosity requires the tunning of three parameters

- Suggestion of future work :
  - Assess the BR2 method on coarse mesh with different Mach number

Planning



A repository on Github has been created. It contains all the current test cases and results.
SanaAmri/Argo_TestCases

BACKUP SLIDES

Variational Formulation

Each cell K is approximate by a polynomial that results from a linear combination of N_p basis functions ϕ_i^K that are polynomial in the cell K and equal to zero in $\Omega_h \backslash \mathrm{K}$

Find $\boldsymbol{w}_h \in (\upsilon_h^p)^4$ such that $\forall \phi_i^K,\ i \in [\![1, N_p]\!],\ \boldsymbol{w}_h$ is solution of

$$\sum_{K\in\Omega_h} \int_K \phi_i^K \partial_t \boldsymbol{w}_h dV - \sum_{K\in\Omega_h} \int_K \left(\mathbb{F}_c(\boldsymbol{w}_h) - \mathbb{F}_v(\boldsymbol{w}_h, \Upsilon_h)\right) \nabla \phi_i^K dv + \sum_{K\in\Omega_h} \oint_{\partial K} \phi_i^K \left(\hat{\mathbb{F}}_c - \hat{\mathbb{F}}_v\right) dS = 0$$

Υ_h is an auxiliary variable that represents the gradient $\nabla \boldsymbol{w}_h$ in the space of discretization

$$\upsilon_h^p = \left\{ f \in L^2(\Omega_h) : \phi_{|K} \in \mathcal{P}^p(K),\ \forall K \in \Omega_h \right\}$$

$$dim(\upsilon_h^p) := N \times N_p$$

$\mathcal{P}^p(K)$: Polynomial of degree inferior or equal to p on K

$$dim(\mathcal{P}^p) := \prod_{i=1}^{d} \frac{(p+i)}{i} = N_p$$

$$\mathbb{F}_v(\boldsymbol{w}_h, \nabla \boldsymbol{w}_h) := (G(\boldsymbol{w}_h)\Upsilon_h)$$

$$\mathcal{L}_c(\boldsymbol{w}_h, \phi^K) = -\sum_{K \in \Omega_h} \int_K \nabla(\phi^K)\mathbb{F}_c(\boldsymbol{w}_h) + \sum_{e \in \epsilon_i} \int_e [\![\phi^K]\!]\, \hat{\mathbb{F}}_c(\boldsymbol{w}_h^+, \boldsymbol{w}_h^-, \boldsymbol{n}) + \sum_{e \in \epsilon_{wall}} \int_e \phi^+ \mathbb{F}_c(\boldsymbol{w}_{wall})$$

Global lifting operator:

$$\Upsilon_h := \nabla \boldsymbol{w}_h + R_h\left([\![\boldsymbol{w}_h]\!]\right)$$

Local lifting operator:

$$\{G\Upsilon_h\} = \{G\nabla \boldsymbol{w}_h\} + \{G r_h^e\}$$

η_r : Stabilization parameter

$$\mathcal{L}_c(\boldsymbol{w}_h, \phi^K) = -\sum_{K \in \Omega_h} \int_K \nabla(\phi^K)\mathbb{F}_c(\boldsymbol{w}_h) + \sum_{e \in \epsilon_i} \int_e [\![\phi^K]\!]\, \hat{\mathbb{F}}_c(\boldsymbol{w}_h^+, \boldsymbol{w}_h^-, \boldsymbol{n}) + \sum_{e \in \epsilon_{wall}} \int_e \phi^+ \mathbb{F}_c(\boldsymbol{w}_{wall})$$

Local lifting

$$\int_{K^L \cup K^R} \phi^K r_h^e \left([\![\boldsymbol{w}_h]\!] \right) dV = \begin{cases} -\int_e \left\{ \phi^K \right\} [\![\boldsymbol{w}_h]\!] \otimes \boldsymbol{n} dS & if \ e \in \epsilon_i \\ -\int_e \phi^{K,+} [\![\boldsymbol{w}_h]\!] \otimes \boldsymbol{n} dS & if \ e \in \epsilon_{wall} \end{cases}$$

Global Lifting

$$\sum_{K \in \Omega_h} \int_K \phi^K R_h \left([\![\boldsymbol{w}_h]\!] \right) dV = - \sum_{e \in \epsilon_i} \int_e \left\{ \phi^K \right\} [\![\boldsymbol{w}_K]\!] \otimes \boldsymbol{n} ds + \sum_{e \in \epsilon_{wall}} \int_e \phi^{K,+} [\![\boldsymbol{w}_K]\!] \otimes \boldsymbol{n} ds$$

Discretisation of the **diffusive term** : The Interior penalty method limitation

Gouverning PDE’s expressed in conservative and hypervectorial form:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = \nabla \cdot \mathbf{F}_v(\mathbf{U}, \nabla \mathbf{U}) + \mathbf{S}(\mathbf{U}, \nabla \mathbf{U})$$

Weak form of the equation (DG method):

$$\sum_{\Omega_e} \int_{\Omega_e} v \frac{\partial u}{\partial t} d\Omega_e + \sum_{\Omega_e} \int_{\Omega_e} v \frac{\partial F(u)}{\partial x} d\Omega_e = \sum_{\Omega_e} \int_{\Omega_e} v \frac{\partial}{\partial x} \left(D \frac{\partial u}{\partial x} \right) d\Omega_e$$

$$[v] = v^- n^- + v^+ n^+$$

$$\langle \bullet \rangle = \frac{1}{2} (\bullet^- + \bullet^+)$$

Rewritting the diffusive term as:

$$\boxed{\begin{aligned} \sum_{\Omega_e} \int_{\Omega_e} v \frac{\partial}{\partial x} \left(D \frac{\partial u}{\partial x} \right) d\Omega_e &= \sum_{\Omega_e} \int_{\Omega_e} \frac{\partial v}{\partial x} \left(D \frac{\partial u}{\partial x} \right) d\Omega_e \\ &- \sum_{\partial\Omega_e} \int_{\partial\Omega_e} \left\langle D(u) \frac{\partial u}{\partial x} \right\rangle [v] dS - \theta \sum_{\partial\Omega_e} \int_{\partial\Omega_e} \left\langle D(u) \frac{\partial v}{\partial x} \right\rangle [u] dS + \alpha \sum_{\partial\Omega_e} \int_{\partial\Omega_e} [v][u] dS \end{aligned}}$$

The **interior penalty coefficient θ** can take the value -1, 0 or 1.