Oblig 1 MatMek4270

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Task 1.2.3

First lets solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

for

$$u = e^{i(k_x x + k_y y - wt)}$$

We get

$$(-wi)^2 e^{i(k_x x + k_y y - wt)} = c^2 e^{i(k_x x + k_y y - wt)}((k_x i)^2 + (k_y i)^2)$$
$$w = \sqrt{c^2(k_x^2 + k_y^2)}$$

This is exactly the dispersion coefficient found in task 1.2.1 when using equation 1.4 in the task as an exact solution to the wave equation.

Task 1.2.4

First find exact dispersion coefficient using that $k_x = k_y = k$

$$w = \sqrt{c^2(k_x^2 + k_y^2)} = \sqrt{2}ck$$

Inserting equation 1.7 into 1.3 gives

$$\frac{be^{-\hat{i}\overline{w}\Delta t} - 2b + be^{\hat{i}\overline{w}\Delta t}}{(\Delta t)^2} = c^2\frac{be^{-\hat{i}kh} - 2b + be^{\hat{i}kh} + be^{-\hat{i}kh} - 2b + be^{\hat{i}kh}}{h^2}$$
$$e^{-\hat{i}\overline{w}\Delta t} + e^{\hat{i}\overline{w}\Delta t} - 2 = 2CFL^2(e^{-\hat{i}kh} + e^{\hat{i}kh} - 2)$$
$$CFL^2 = \frac{1}{2}\frac{\cos\overline{w}\Delta t - 4}{\cos kh - 4}$$

where $b = e^{\hat{i}kh(i+j)+\overline{w}n\Delta t}$ and $CFL = c\Delta t/h$

Now use $k = \frac{w}{\sqrt{2}c}$ and $h = \frac{c\Delta t}{CFL}$ and insert $CFL = 1/\sqrt{2}$, we get

$$\begin{aligned} cos(\overline{w}\Delta t) - 4 &= cos(kh) - 4 \\ \overline{w}\Delta t &= kh \\ \overline{w} &= \frac{\frac{w}{\sqrt{2}c}\frac{c\Delta t}{CFL}}{\Delta t} \\ \overline{w} &= w \end{aligned}$$