Path tracking for an automated vehicle

Case study report

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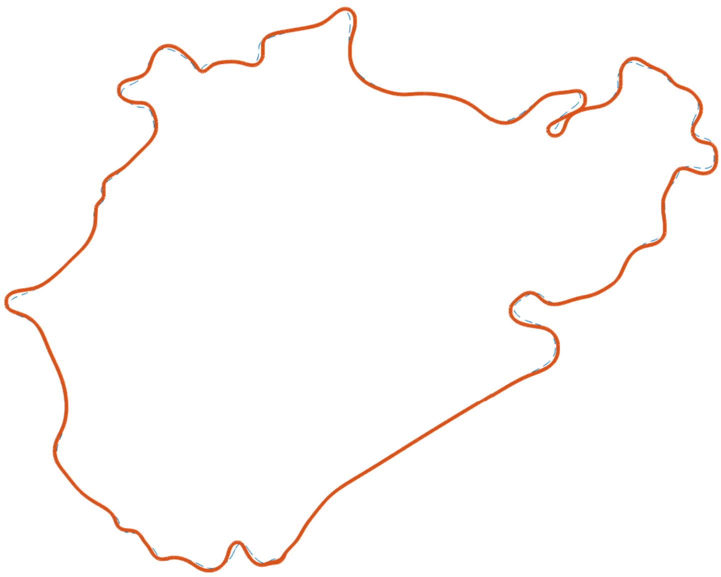


Multivariable control and coordination systems (EE-477)
Prof. Gillet Denis
EPFL
Fall 2022

Contents

1 Modeling, linearization, and discretization

1.1 Problem definition

System's variable

- Reference path π, $\pi : \mathbb{R} \longrightarrow \mathbb{R}^3$, $\quad \pi(s_\pi) = (x_\pi, y_\pi, \theta_\pi)$
- Distance s : $s \in \mathbb{R}$, distance along the path corresponding to the vehicle's position projections.
- Lateral error d : $d \in \mathbb{R}$, $\quad d = ||P_r - P_{r,\pi}||$, distance between the path and the vehicle.
- heading error s : $\theta_e \in \mathbb{R}$, $\quad \theta_e = (\theta - \theta_\pi)$, difference between vehicle yaw angle and path's orientation.
- Longitudinal speed v : $v \in \mathbb{R}^+$
- Longitudinal speed reference v_{ref} : $v_{ref} \in \mathbb{R}^+$
- Steering wheel position ϕ : $\phi \in \mathbb{R}$
- Steering wheel reference ϕ_{ref} : $\phi_{ref} \in [-4\pi, 4\pi]$
- Path's curvature κ

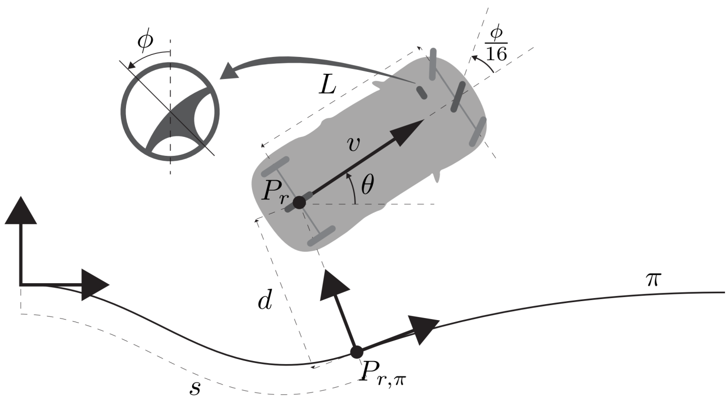


Figure 1: Coordinate system

System's model

With the variable defined above, one may write the dynamic of the system with the following equations :

$$\dot{s} = \frac{v\cos(\theta_e)}{1 - d\kappa(s)} \tag{1}$$

$$\dot{d} = v\sin(\theta_e) \tag{2}$$

$$\dot{\theta}_e = \frac{v}{L}\tan(\frac{\phi}{16}) - \kappa(s)\dot{s} \tag{3}$$

$$\dot{v} = \sigma_v(v_{ref} - v) \tag{4}$$

$$\dot{\phi} = \sigma_{ref}(\phi_{ref} - \phi) \tag{5}$$

$$\tag{6}$$

In the following, state measurement is assumed unless stated otherwise and the states and control input of the system are :

$$\mathbf{x} = [x_1, x_2, x_3, x_4, x_5] = [s, d, \theta_e, v, \phi] \tag{7}$$
$$\mathbf{u} = [u_1, u_2] = [v_{ref}, \phi_{ref}] \tag{8}$$
$$\mathbf{y} = \mathbf{x} \tag{9}$$

1.2 Questions

Question 0.1 : Rewrite the non-linear model in the standart form $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \end{bmatrix} = \begin{bmatrix} \frac{x_4 \cos(x_3)}{1 - x_2 \kappa(x_1)} \\ x_4 \sin(x_3) \\ \frac{x_4}{L} \tan(\frac{x_5}{16}) - \frac{\kappa(x_1) x_4 \cos(x_3)}{1 - x_2 \kappa(x_1)} \\ \sigma_v (u_1 - x_4) \\ \sigma_{ref} (u_2 - x_5) \end{bmatrix} \tag{10}$$

Question 0.2 : Given the chosen state space, why it wouldn't be appropriate linearizing the system around a nominal point.

For the given state space, there is no valid point on the whole path, so approximation errors would be inevitable. This can be easily represented by considering the variable x_1, which is nothing but the distance travelled on the path. The only way for a point to represent it correctly is for the vehicle to be stationary, and for the reference trajectory to be limited to a point. Otherwise, as soon as the vehicle is in motion, x_1 has a time dependency and cannot be correctly represented by a point. It is therefore necessary to linearise around a trajectory in order to limit the error of the model.

Question 0.3 : Calculate the nominal trajectory representing the constant-speed perfect tracking situation (that is, a situation where the vehicle drives perfectly on the path and at a constant speed $\mathbf{v_{ref}}$) for a path of constant curvature $\kappa_{\mathbf{ref}}$.

Several assumption are made that implies the following :

- Constant path curvature : $\kappa(s) \longrightarrow \kappa_{ref}$
- Constant speed : $v(s) \longrightarrow v_{ref}$
- Vehicle drives perfectly on the the path $\longrightarrow$ no error and no error dynamics ($d = 0, \dot{d} = 0, \theta_e = 0, \dot{\theta}_e = 0$)

Thus, we already know $\bar{x}_2(t) = 0$, $\bar{x}_3(t) = 0$, $\bar{x}_4 = u_1$. In order to find $\bar{x}_1(t)$, one can substitute the known variable in $f_1(x)$

$$\dot{\bar{x}}_1(t) = \frac{\bar{x}_4 \cos(\bar{x}_3)}{1 - \bar{x}_2 \kappa(s)} = v_{ref} = u_1 \tag{11}$$

By integration, one may find :

$$\bar{x}_1(t) = u_1 t + s_0 \tag{12}$$

Moreover, to find $\bar{x}_5(t)$, one may want to use $f_3(t)$ and substitute with the known variable.

$$\begin{aligned} \dot{\bar{x}}_3 = f_3(\bar{x}) &= \frac{\bar{x}_4}{L} \tan\left(\frac{\bar{x}_5}{16}\right) - \frac{\kappa(\bar{x}_1)\bar{x}_4 \cos(\bar{x}_3)}{1 - x_2 \kappa(\bar{x}_1)} \iff \\ 0 &= \frac{v_{ref}}{L} \tan\left(\frac{\bar{x}_5}{16}\right) - \kappa_{ref} v_{ref} \iff \\ \bar{x}_5 &= 16 \tan^{-1}(\kappa_{ref} L) \end{aligned} \tag{13}$$

Finally, one can write the nominal trajectory as :

$$\bar{x} = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \\ \bar{x}_4(t) \\ \bar{x}_5(t) \end{bmatrix} = \begin{bmatrix} u_1 t + s_0 \\ 0 \\ 0 \\ u_1 \\ 16\tan^{-1}(\kappa_{ref} L) \end{bmatrix} \tag{14}$$

Question 0.4 : Linearize the system around such a nominal trajectory.

In this part, we will introduce the linearized model of the system. The linearization is done around a point, or in our case, around a trajectory (equation 14) and for this purpose, we will introduce the small signal linearization, which is the deviation of the state from an nominal point/trajectory.

Small signal linerization :

$$\begin{aligned} \tilde{x} &= x - \bar{x} \\ \tilde{y} &= y - \bar{y} \\ \tilde{u} &= u - \bar{u} \end{aligned} \tag{15}$$

Starting from there, the first step is to find the matrices A and B of the linear system using the first order Taylor expansion of the non-linear model.

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\kappa_{ref} x_4 \cos(x_3)}{(1-x_2\kappa_{ref})^2} & \frac{-x_4 \sin(x_3)}{1-x_2\kappa_{ref}} & \frac{\cos(x_2)}{1-x_2\kappa_{ref}} & 0 \\ 0 & 0 & x_4\cos(x_3) & \sin(x_3) & 0 \\ 0 & \frac{-\kappa_{ref}^2 x_4 \cos(x_3)}{(1-x_2\kappa_{ref})^2} & \frac{\kappa_{ref} x_4 \sin(x_3)}{1-x_2\kappa_{ref}} & \frac{\tan(\frac{x_5}{16})}{L} - \frac{_{ref}\cos(x_3)}{1-x_2\kappa_{ref}} & \frac{x_4}{16L\cos(\frac{x_5}{16})^2} \\ 0 & 0 & 0 & -\sigma_v & 0 \\ 0 & 0 & 0 & 0 & -\sigma_\phi \end{bmatrix} \tag{16}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \vdots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma_v & 0 \\ 0 & \sigma_\phi \end{bmatrix} \tag{17}$$

In a second step, one may evaluate this expansion around the reference trajectory $\bar{x}$ in order to find the matrices $\bar{A}$ and $\bar{B}$.

$$\bar{A} = \begin{bmatrix} 0 & \kappa_{ref} v_{ref} & 0 & 1 & 0 \\ 0 & 0 & v_{ref} & 0 & 0 \\ 0 & -\kappa_{ref}^2 v_{ref} & 0 & 0 & \frac{v_{ref}}{16L\cos(\tan^{-1}(\kappa_{ref}L))^2} \\ 0 & 0 & 0 & -\sigma_v & 0 \\ 0 & 0 & 0 & 0 & -\sigma_\phi \end{bmatrix} \tag{18}$$

$$\bar{B} = B \tag{19}$$

Finally, because state measurement is assumed, C is simply the identity matrix, while $D = 0$. The system can thus be written as :

$$\dot{\tilde{x}} = \bar{A}\tilde{x} + \bar{B}\tilde{u} \tag{20}$$

$$\tilde{y} = \tilde{x} \tag{21}$$

Question 0.5 : Discretize the system using Euler approximation.

With the small signal linear system found previously, one may want to discretize the system. The following formulation uses the euler approximation :

$$
\begin{aligned}
\tilde{x}(k+1) &= \tilde{x}(k) + \Delta t \bar{A}\tilde{x}(k) + \bar{B}\tilde{u} \\
&= \underbrace{\left(I + \Delta t \bar{A}\right)}_{\Phi} \tilde{x}(k) + \underbrace{\Delta t \bar{B}}_{\Gamma} \tilde{u}(k) \\
&= \Phi \tilde{x}(k) + \Gamma \tilde{u}(k)
\end{aligned}
\tag{22}
$$

One may now compute the discretize state space matrix Φ and Γ, with Δt equal to one sampling period τ_s :

$$
\Phi = I + \Delta t \bar{A} = \begin{bmatrix}
1 & \kappa_{ref} v_{ref} \tau_s & 0 & \tau_s & 0 \\
0 & 1 & v_{ref} \tau_s & 0 & 0 \\
0 & -\kappa_{ref}^2 v_{ref} \tau_s & 1 & 0 & \frac{v_{ref} \tau_s}{16L \cos(\tan^{-1}(\kappa_{ref} L))^2} \\
0 & 0 & 0 & 1 - \sigma_v \tau_s & 0 \\
0 & 0 & 0 & 0 & 1 - \sigma_\phi \tau_s
\end{bmatrix}
\tag{23}
$$

$$
\Gamma = \Delta t \bar{B} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\sigma_v \tau_s & 0 \\
0 & \sigma_\phi \tau_s
\end{bmatrix}
\tag{24}
$$

2 Model implementation and simulation

2.1 Provide the arrays Φ and Λ obtained using the Euler approximation method and the Matlab command *c2d*.

Euler approximation

$$\Phi = \begin{bmatrix} 1 & 0 & 0 & 0.1 & 0 \\ 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0.008 \\ 0 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0.5 \end{bmatrix} \tag{25}$$

Matlab *c2d*

$$\Phi = \begin{bmatrix} 1 & 0 & 0 & 0.095 & 0 \\ 0 & 1 & 0.5 & 0 & 0.002 \\ 0 & 0 & 1 & 0 & 0.006 \\ 0 & 0 & 0 & 0.905 & 0 \\ 0 & 0 & 0 & 0 & 0.607 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0.005 & 0 \\ 0 & 0 \\ 0 & 0.002 \\ 0.095 & 0 \\ 0 & 0.393 \end{bmatrix} \tag{26}$$

2.2 Provide a screenshot of your implementation of the Non linear model - continuous time block.

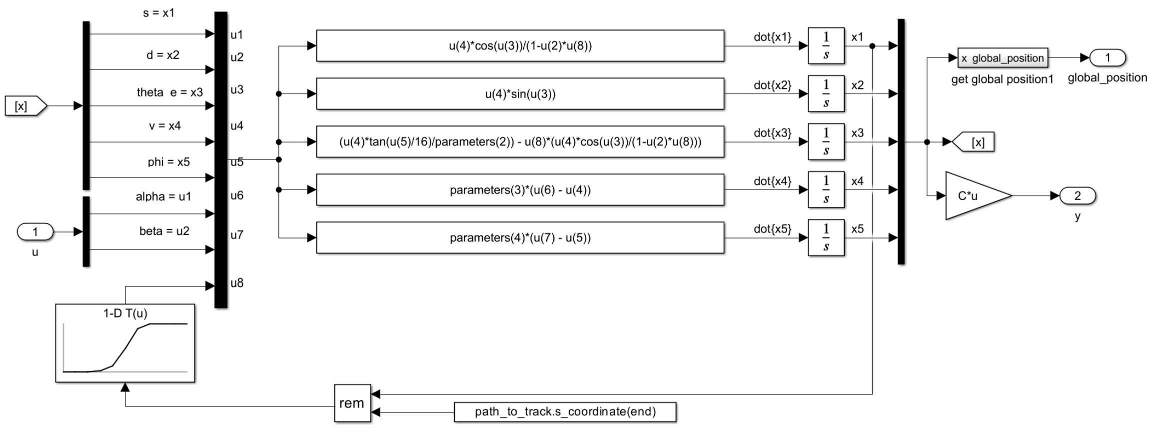


Figure 2: Non linear model in continous time implementation in Simulink

2.3 Provide a screenshot of the block diagram implemented in open loop experiments.slx.

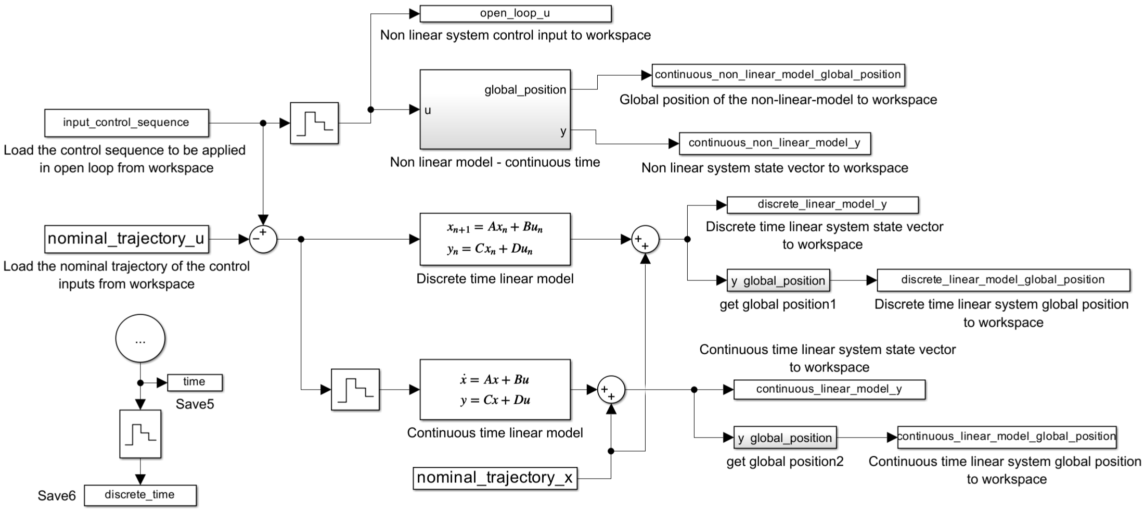
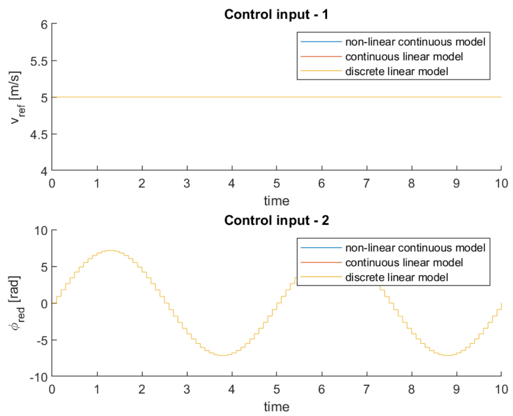


Figure 3: Simulink diagram of exercise 1

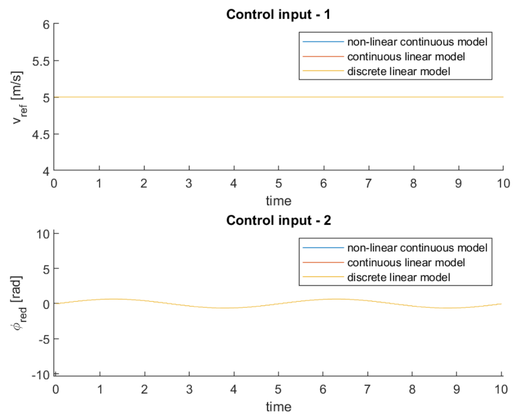
2.4 Report the simulation results, including the information concerning the initial state you used, the sequence of inputs you applied, and the simulated results.

Two experiments were designed to illustrate the divergence or not of the different models developed previously. For this purpose two input sequences were constructed. The vehicle moves at a constant speed v_{ref}, but the steering wheel reference follows a sine function of two periods over the simulation time. It is the amplitude of this sine that will vary between the two experiments. They have been displayed side by side at the same scale in Figure 4.

Experiment 1 was designed to highlight the discrepancies between the non-linear model and the two linear models. Indeed, the latter have been linearized around a point (thanks to the small signals model) and when we deviate too much from this point the approximation is no longer valid and divergences between the models appear. On the other hand, as for experiment 2, as long as we limit ourselves to small deviations around the linearisation point, the approximation is reasonable and we do not observe any notable differences between the results of the different models. The inital state for the two experiments is : $[0, 0, 0, \bar{x}_4(t_0), \bar{x}_5(t_0)]$.

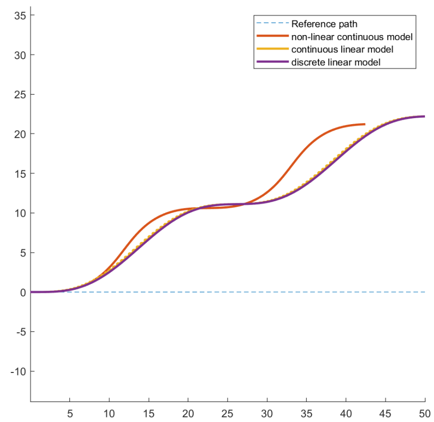


(a) Input sequence 1

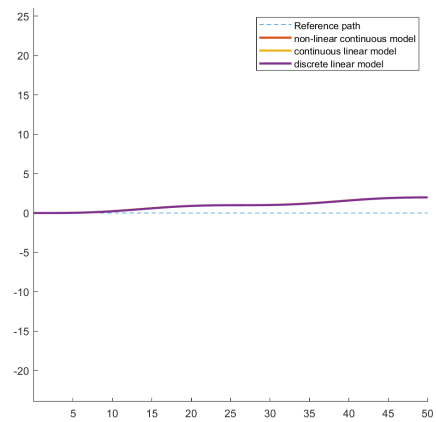


(b) Input sequence 2

Figure 4: Input sequence applied to the three different model used in exercise 1

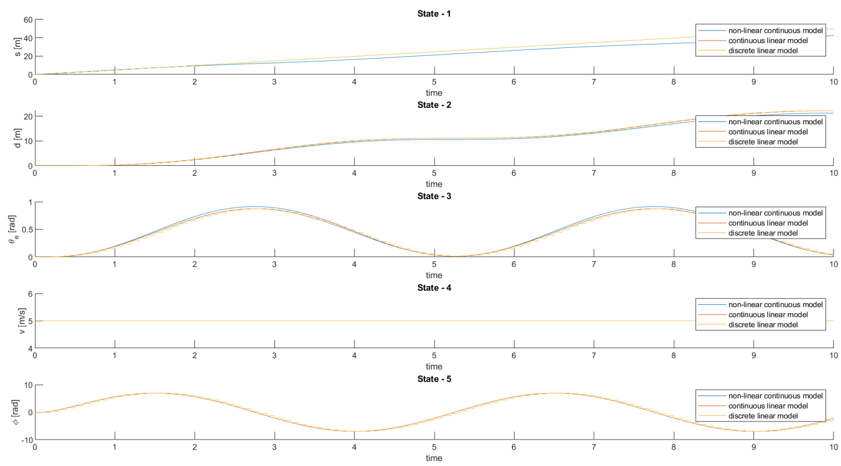


(a) Trajectory for experiment 1

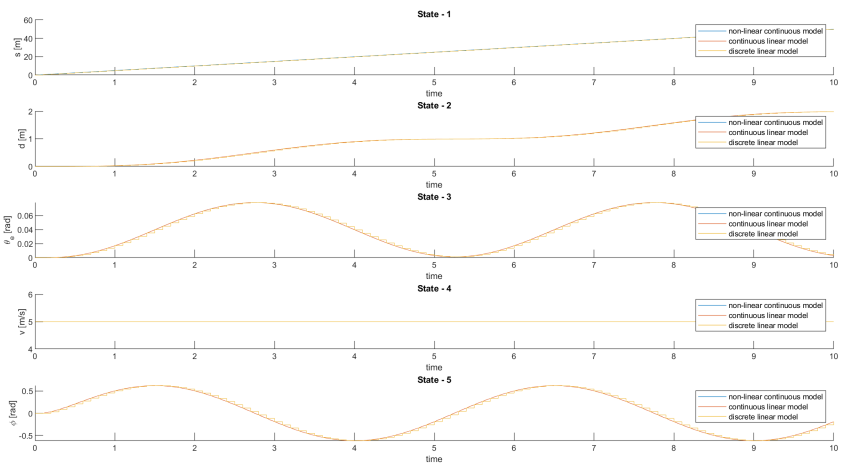


(b) Trajectory for experiment 2

Figure 5: Trajectory of the vehicle according to the three different model used in exercise 1



(a) State for experiment 1



(b) State for experiment 2

Figure 6: State of the vehicle according to the three different model used in exercise 1

2.5 Does the linear model resemble sufficiently well the behavior obtained from the non-linear model? Why?

As observed, there is no simple answer to this question. Indeed, as long as one remains close enough to the linearisation point, the linear model performs as well as the non-linear model, and has the advantage of being much simpler. On the other hand, as soon as one moves away from the linearisation point, as in experiment 1, the linear model gives different results.

The differences between the linear and non-linear model can be explained by the error term (the reminder) associated with the Taylor expansion used for the linearisation. The reminder of the first order can be written in Lagrangian form [1]:

$$R_1(x) = \frac{f^{(2)}(\zeta}{2!}(x-a)^2 \tag{27}$$

It can be seen from the above formula that the errors between the models come from the second or higher order components of the function being linearized.

If we take the non-linear model $f(x)$, we see that $f_1(x)$, $f_2(x)$ and $f_3(x)$ contain them (with the functions sin, cos and tan), while $f_4(x)$ and $f_5(x)$ do not. This is consistent with the observations that can be made. Indeed, the states x_1, x_2 and x_3 diverge between the models, while the states x_4 and x_5 remains the same.

To conclude, the linear model is good as long as it remains close to the linearisation point, but as seen, if the vehicle moves away from it, it is no longer the case. In context of steering an an automated vehicle, susceptible of making a circle, the linear model (as it is) will not be sufficient.

2.6 Do the results obtained using the discrete-time linear model resemble those obtained by the continuous-time linear model? What is then interesting about using the discrete-time linear model instead of the continuous-time linear model to design control strategies?

In the two experiments presented above, the discrete time and continuous time models give very similar results. The discrete time model has the advantage that it corresponds to what a computer can compute in a digital world. Furthermore, it is much less computationally intensive, and can therefore be much more easily implemented in a vehicle, where space and energy are limited.

3 Control and observation

3.1 Report the value of the LQR gain for the proposed Q1 and Q2, as well as the evolution of the singular values of S obtained while solving the Riccati equation

The proposed Q_1 and Q_2 are :

$$Q_1 = \begin{bmatrix} 1e-5 & 0 & 0 & 0 & 0 \\ 0 & 50 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2e-5 \end{bmatrix} \tag{28}$$

The value of the LQR gain obtained by solving the Riccati equation are :

$$K_{LQR} = \begin{bmatrix} 1.5641e-4 & 0 & 0 & 0.2235 & 0 \\ 0 & 199.056 & 722.53 & 0 & 19.474 \end{bmatrix} \tag{29}$$

While the evolution of the singular values of S obtained while while solving the Riccati equation is displayed below :

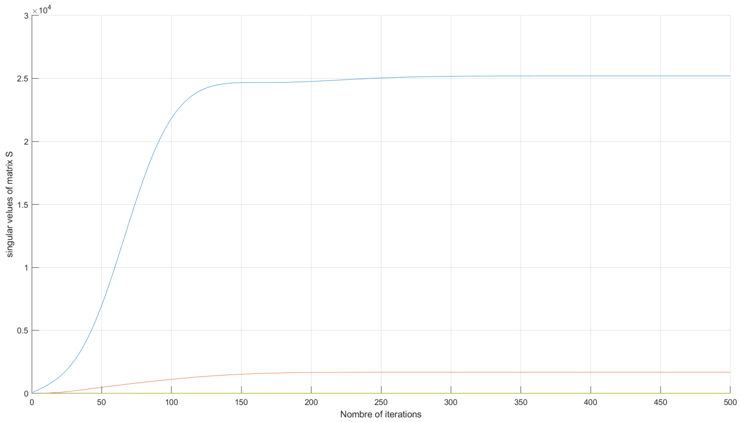


Figure 7: Evolution of singular values of S while solving the Riccati equation

3.2 Report the explicit value of the observation close loop poles, and the observer gain resulting from it.

The poles of the observer were designed to be 99.9% of the poles of the closed loop dynamics and the following values have been found (3 significant digits) :

$$\text{Observer poles} : \begin{bmatrix} 0.999 \\ 0.987 \\ 0.0154 \\ 0.985 + 0.0137i \\ 0.985 - 0.0137i \end{bmatrix} \tag{30}$$

Then by placing the poles of the observer, one may find the following observer gain :

$$
L = \begin{bmatrix} 0.9845 & 0 & 0.01 & 0 \\ 0 & 0.0299 & 0 & 0 \\ 0 & 0.0083 & 0 & 0.0008 \\ 0 & 0 & 0.0032 & 0 \\ 0 & 0 & 0 & -0.0478 \end{bmatrix} \tag{31}
$$

As a reminder, the Observer initial state has been set to zero. This will hold for every simulation.

3.3 Provide a screen shot of your final Simulink scheme and the submodules you had to complete.

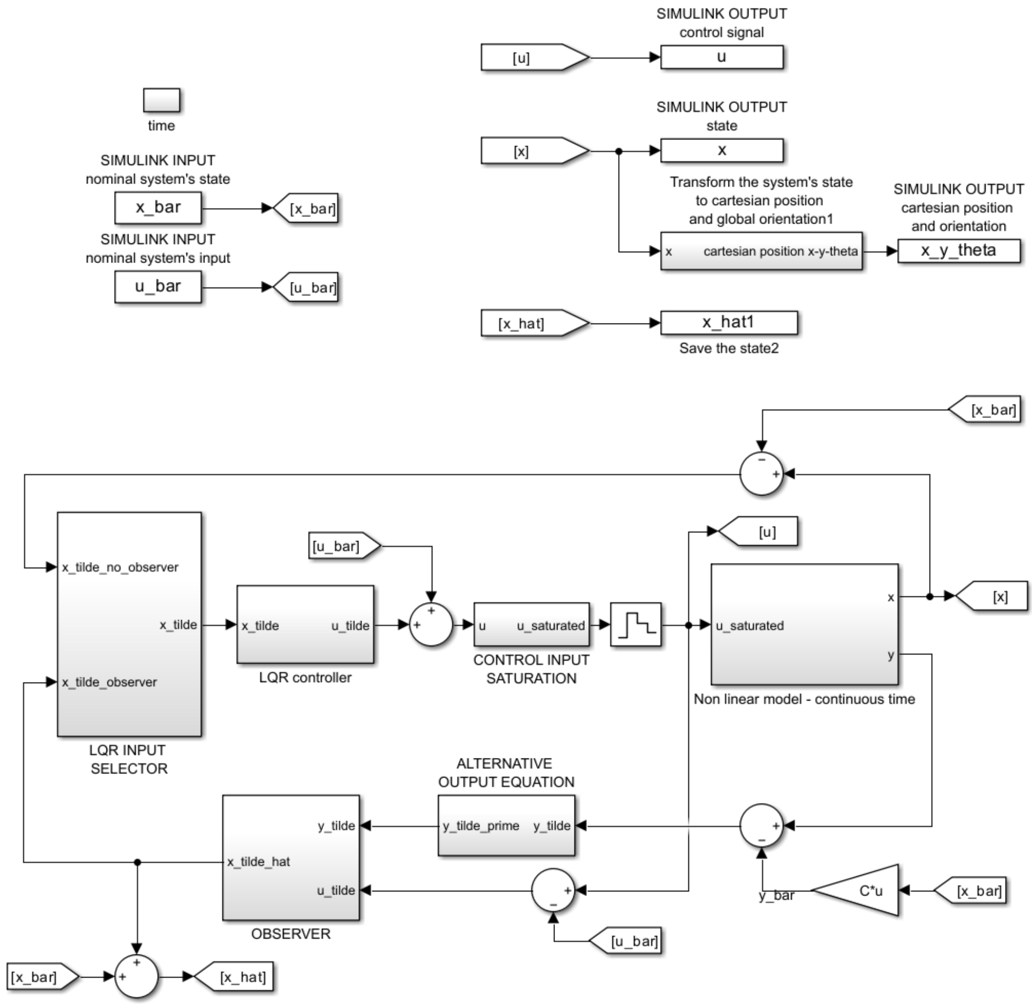
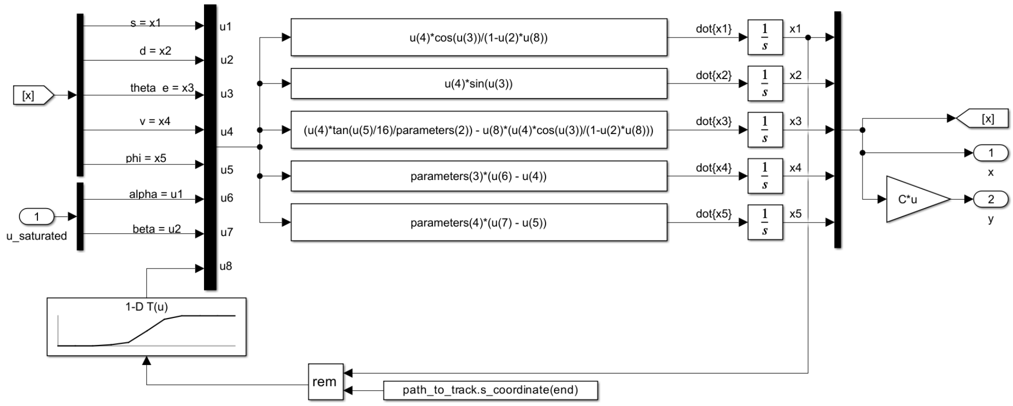
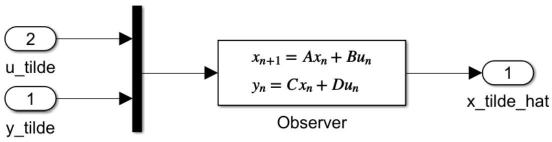


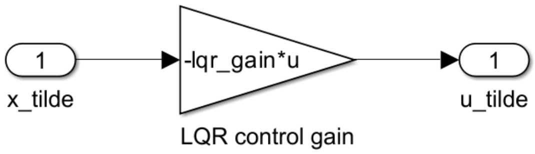
Figure 8: Simulink diagram of exercise 2



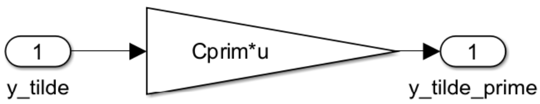
(a) Non linear model in continuous time implementation



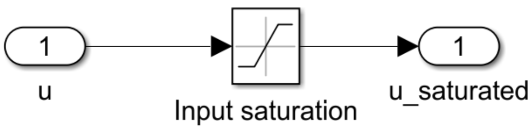
(b) Observer implementation



(c) LQR implementation



(d) Alternative equation implementation

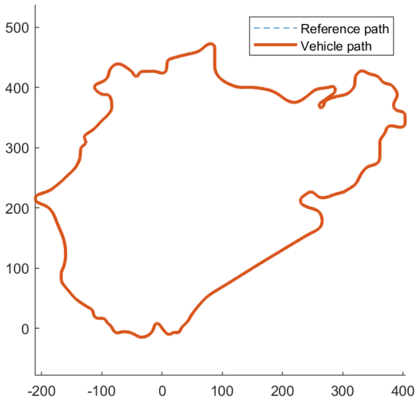


(e) Input saturation implementation

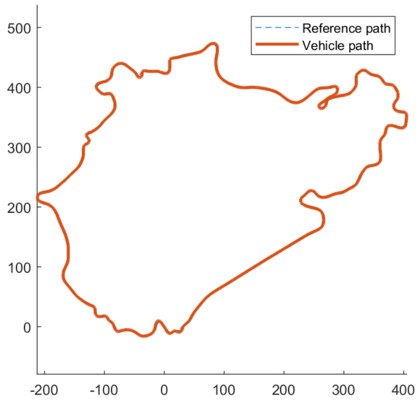
Figure 9: Implementation of the required block in simulink for exercise 2

3.4 Report the simulation results for the proposed values.

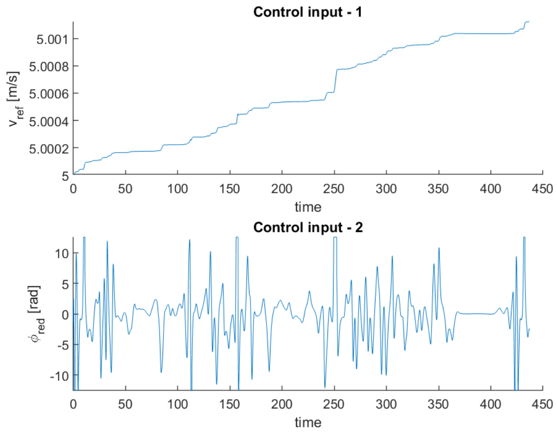
The simulation with the proposed values is very satisfactory, the path is followed almost perfectly, with very little visible errors, in both cases : state measurement or observed state. In the estimation of the x_3 state, there are a tiny bit of irregularities, with the observer, which then leads to some error in control, especially in the heading error. This make sens and was awaited since, its the state variable with can't measure and have to estimate it. However, this could be corrected with a little bit of tuning, but it is not a problem for the overall control, since the vehicle still stay nicely on the track. We can see that the speed is almost constant (less than 1% variation), so the vehicle is on time on the path, which tends to indicate that there is very little error in the other variables.



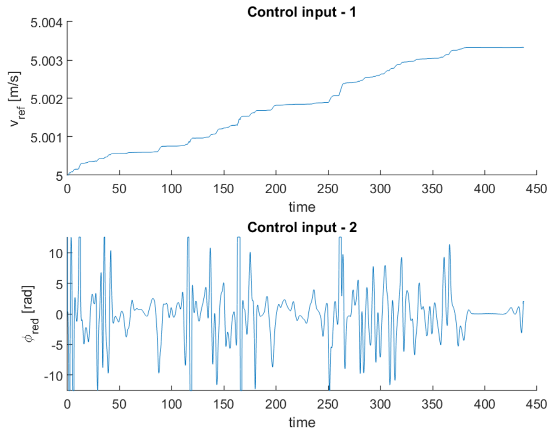
(a) Trajectory with state measurement



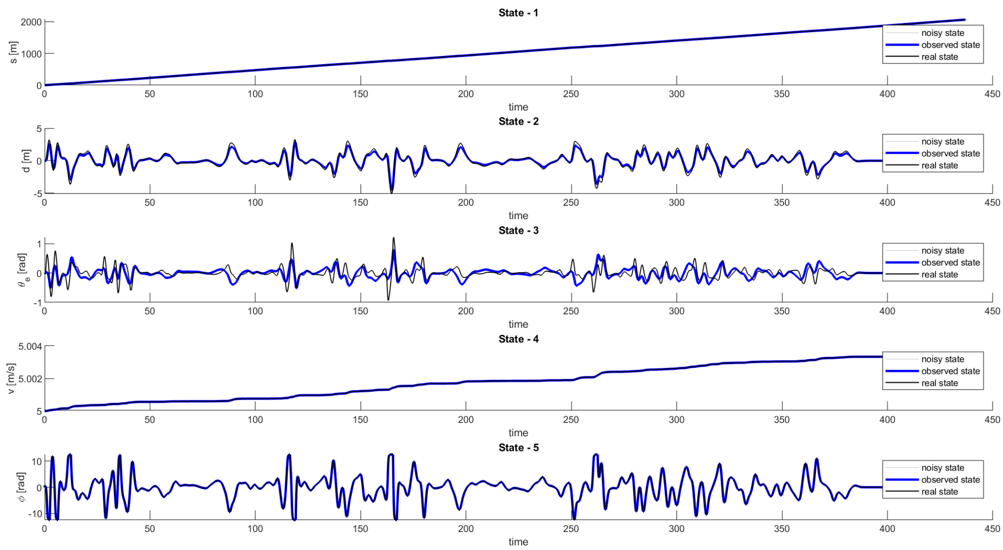
(b) Trajectory with Observed state



(c) Control input with state measurement



(d) Control input with Observed state



(e) Comparison of the states between state measurement and Observed state

Figure 10: Simulation of the LQR regulator with and without an observer

3.5 The proposed value of Q1 intends to assign very little importance to the error deriving from x1. Could you explain why doing this might make sense?

First, x_1 represents s, the distance travelled along the path, which is a function of all the other variables (indirectly for x_5), whereas none of the other variables is a function of x_1. This means that if there is no error (or accumulated error) in the other variables, there is none in x_1 either. Efficiently controlling x_2, x_3, x_4 and x_5 also controls x_1. Moreover, the error in x_1 represents the time advance or delay that the vehicle has on the path, which will result in accelerating or decelerating the vehicle (term $K_{LQR,11}$). One can imagine that if the vehicle accelerates greatly, it becomes increasingly difficult or impossible to control the vehicle. To have a robust controller, it is therefore natural to limit the weight of the error in x_1.

3.6 Propose a different pair Q1 and Q2 such that heading deviation error is highly penalized compared to the other states and report simulation results where the impact of doing so can be clearly observed.

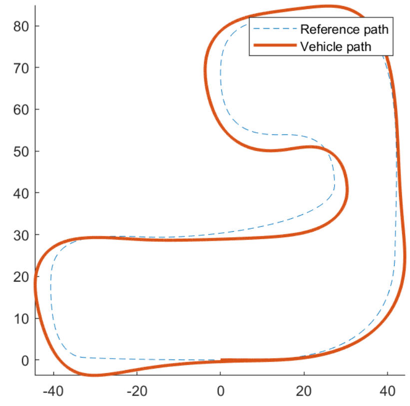
The new proposed Q_1 and Q_2 are :

$$Q_1 = \begin{bmatrix} 1e-5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 50 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2e-5 \end{bmatrix} \tag{32}$$

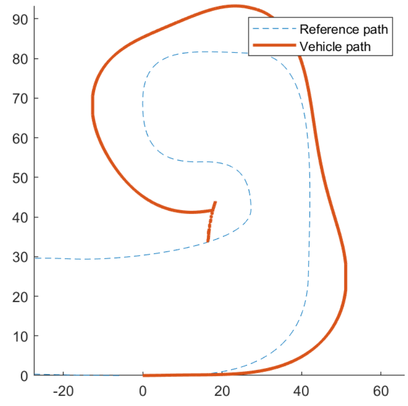
The value of the LQR gain obtained by solving the Riccati equation are :

$$K_{LQR} = \begin{bmatrix} 3.99e-4 & 0 & 0 & 0.2237 & 0 \\ 0 & 20.067 & 304.03 & 0 & 19.308 \end{bmatrix} \tag{33}$$

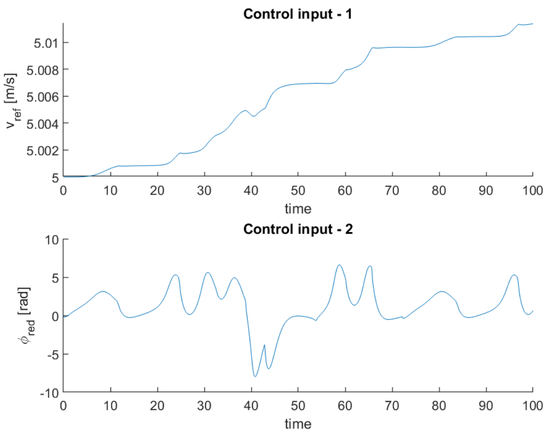
This time, the control problem is more challenging, thus a simpler reference path has been chosen. Even though, the vehicle is no longer tracked on the path and control is completely lost. This is partly due to the fact that the control of x_3 is aggressive (large weight on x_3 in Q_1) while this variable is not measured but estimated. There is a discrepancy between the real and estimated state of x_3. The observer is too slow to correct this error, which then induce more error in the tracking of the path until the control gets finally out of hand. Tuning of the observer dynamics is further needed to achieve control of the vehicle. Indeed, the observer dynamic is almost equal to the dynamic of the system, which is too slow to correct estimation error on x_3.



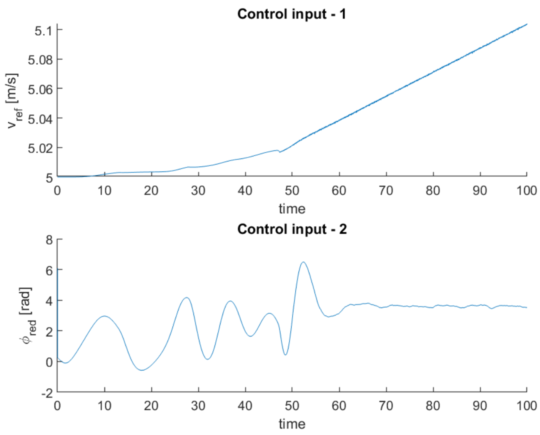
(a) Trajectory with state measurement



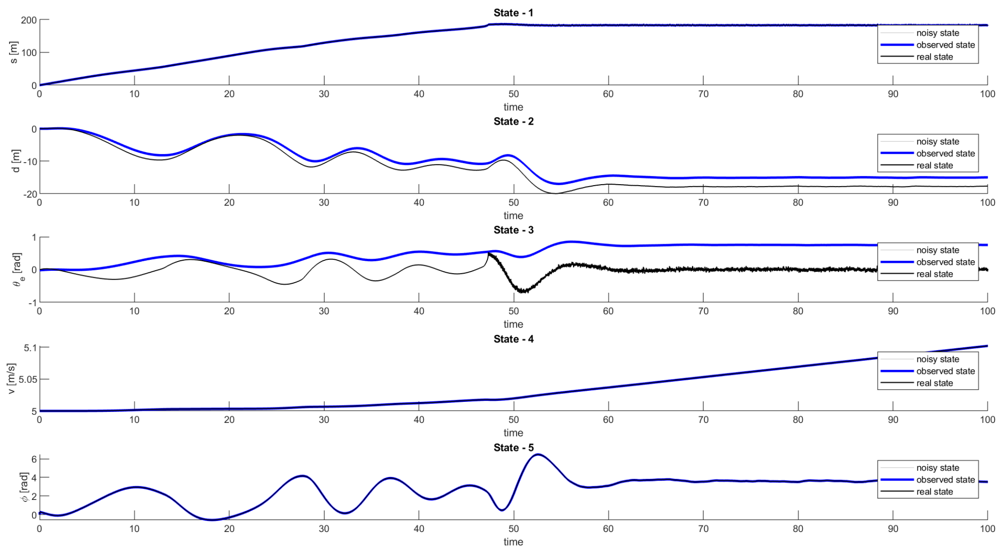
(b) Trajectory with Observed state



(c) Control input with state measurement



(d) Control input with Observed state



(e) Comparison of the states between state measurement and Observed state

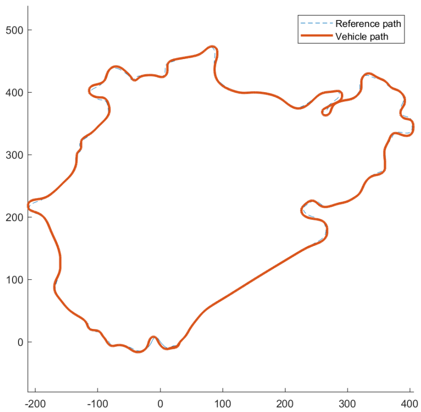
Figure 11: Simulation of the LQR regulator with an observer and new weight matrices Q

3.7 Is the proposed placement for the observation close loop poles appropriate? why? If not,propose a new set of poles to improve the observation performance.

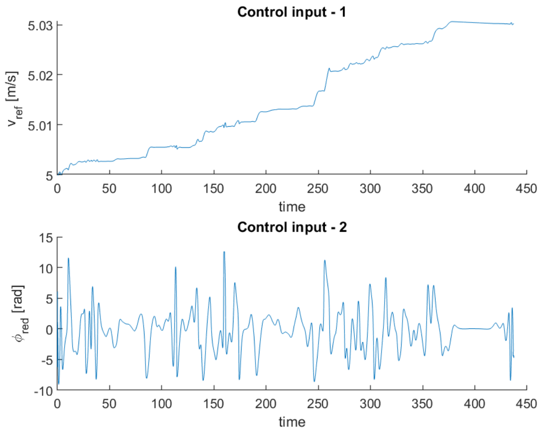
As mentioned above, the control of the vehicle with these new parameters failed because the observer was too slow to correct the error in the estimation of between the states and so the control got carried away and could not catch up. One solution is to impose a more aggressive dynamic on the estimation of the states on the observer. In the results below, the poles of the observer have been chosen to be 1% of the poles of the system dynamics.

$$\text{Observer poles}: \begin{bmatrix} 0.01 \\ 0.00988 \\ 0.000154 \\ 0.00994 + 205e - 5i \\ 0.00994 - 205e - 5i \end{bmatrix} \tag{34}$$

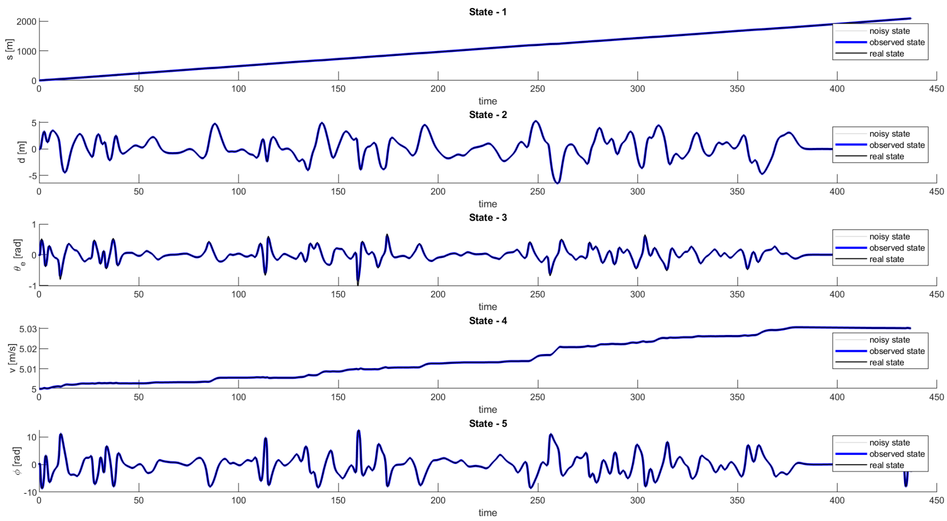
It turns out that with this new observer, the vehicle is again very well controlled and follows the reference path correctly. We can see that the estimated state has no error at all compared to the true state which was not the case with the two previous simulations, which suggest that the observer was correctly tune this time. Since, for this last simulation, its the observer's performances that are evaluated, only its results are presented for visibility purpose.



(a) Trajectory



(b) Control input



(c) Comparison with state measurement

Figure 12: Simulation of the LQR regulator with an observer and new pole placement

References

[1] *Taylor's theorem — Wikipedia, The Free Encyclopedia*. Online; accessed 8-Januar-2023.