**MA8402 - PROBABILITY AND QUEUING THEORY**
**( II YEAR - CSE IV SEM)**

**UNIT I - PROBABILITY AND RANDOM VARIABLES**

**PART - A**

**1. If $A$ and $B$ are independent events then prove that $A$ and $\overline{B}$ are independent.**

Since A and B are independent,

$$P(A\cap B)=P(A)P(B)---1$$

$$P(A\cap \overline{B})=P(A)-P(A\cap B)$$
$$=P(A)-P(A)P(B) \quad \text{[using (1)]}$$
$$=P(A)[1-P(B)]$$
$$P(A\cap \overline{B})=P(A)P(\overline{B}) \quad \therefore A \,\&\, \overline{B} \text{ are independent events}$$

**2. If $A$ and $B$ are two events such that $P(A\cup B)=\frac{3}{4}$, $P(A\cap B)=\frac{1}{4}$, $P(\overline{A})=\frac{2}{3}$, find $P(\overline{A}/B)$.**

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$\frac{3}{4}=\frac{1}{3}+P(B)-\frac{1}{4} \Rightarrow P(B)=\frac{2}{3}$$

$$P(\overline{A}/B)=\frac{P(\overline{A}\cap B)}{P(B)}=\frac{P(B)-P(A\cap B)}{P(B)}=\frac{\frac{2}{3}-\frac{1}{4}}{\frac{2}{3}}=\frac{5}{8}$$

**3. Let A and B be two events such that $P(A)=\frac{1}{3}, P(B)=\frac{3}{4}$, $P(A\cap B)=\frac{1}{4}$. Compute $P(A/B)$ and $P(\overline{A}\cap B)$. (May/June 2019)**

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{4}}{\frac{3}{4}}=\frac{1}{3}.$$

$$P(\overline{A}\cap B)=P(B)-P(A\cap B)=\frac{3}{4}-\frac{1}{4}=\frac{2}{4}=\frac{1}{2}.$$

**4. State Baye's Theorem on Probability.**

If $E_1, E_2 \ldots E_n$ are a set of exhaustive and mutually exclusive events associated with a random experiment and A is any other event associated with $E_i$.

Then $P(E_i/A)=\dfrac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n}P(E_i)P(A/E_i)}$, i=1,2,..n

**5. Let X be a discrete R.V. with probability mass function**

$$P(X=x)=\begin{cases}\frac{x}{10}, & x=1,2,3,4\\ 0, & \text{otherwise}\end{cases}$$

**Compute $P(X<3)$ and $E\left(\frac{X}{2}\right)$. (May/June 2016)**

| $X=x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X=x)$ | 1/10 | 2/10 | 3/10 | 4/10 |

$$P(X<3)=P(X=1)+P(X=2)=\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$$

$$E\left(\frac{X}{2}\right)=\frac{1}{2}E(X)=\frac{1}{2}\sum_{x=1}^{4}xP(X=x)=\frac{1}{2}\left\{1\times\frac{1}{10}+2\times\frac{2}{10}+3\times\frac{3}{10}+4\times\frac{4}{10}\right\}$$

$$=\frac{1}{2}\left\{\frac{1+4+9+16}{10}\right\}=\frac{1}{2}\times\frac{30}{10}=\frac{3}{2}.$$

**6. Let X be the random variable which denotes the number of heads in three tosses of a fair coin. Determine the probability mass function of X. (Nov/Dec 2015)**

A coin is tossed three times then the sample space is

$S=\{HHH,HHT,HTH,HTT,THH,THT,TTH,TTT\}$

Let $X$ be the RV denoting the no. of heads.

The probability mass function is

| $X=x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X=x)$ | 1/8 | 3/8 | 3/8 | 1/8 |

**7. A random variable X has the p.d.f f(x) given by $f(x)=\begin{cases} a(1+x^2), & 2<x<5 \\ 0, & \text{otherwise} \end{cases}$.**

**Find 'a' and P(X < 4). (May/June 2014) (Nov/Dec 2015)**

Since $X$ is a continuous random variable, then the PDF $f(x)\ge 0,\forall x$ and $\int_{-\infty}^{\infty} f(x)\,dx=1$

$$\int_2^5 a(1+x^2)dx=1\Rightarrow a\left[x+\frac{x^3}{3}\right]_2^5=1$$

$$\Rightarrow a\left[\left(5+\frac{5^3}{3}\right)-\left(2+\frac{2^3}{3}\right)\right]=1 \qquad \Rightarrow 42a=1\Rightarrow \boxed{a=\frac{1}{42}}$$

$$P(X<4)=\int_2^4\frac{(1+x^2)}{42}dx=\frac{1}{42}\left[x+\frac{x^3}{3}\right]_2^4=\frac{1}{42}\left[\left(4+\frac{4^3}{3}\right)-\left(2+\frac{2^3}{3}\right)\right]=\frac{31}{63}$$

**8. The CDF of a continuous random variable is given by $F(x)=\begin{cases} 0, & x<0 \\ 1-e^{-\frac{x}{5}}, & x\ge 0 \end{cases}$**

**Find the PDF of $X$ and mean of $X$.**

$$\text{PDF}=f(x)=\frac{d}{dx}[F(x)]=\begin{cases} 0, & x<0 \\ \frac{1}{5}e^{-\frac{x}{5}}, & x\ge 0 \end{cases}$$

$$E(X)=\int_{-\infty}^{\infty}xf(x)\,dx=\int_0^{\infty}\frac{1}{5}xe^{-\frac{x}{5}}\,dx=\frac{1}{5}\left[(x)\left(\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right)-(1)\left(\frac{e^{-\frac{x}{5}}}{\frac{1}{25}}\right)\right]_0^{\infty}=\frac{25}{5}=5$$

**9. Let X be a random variable with E(X)=1, E[X(X-1)]= 4. Find Var(X), Var(3-2X). (April/May 2008)**

E(X)=1,

$E[X(X-1)]=4 \Rightarrow E[X^2-X]=4 \Rightarrow E[X^2]-E[X]=4 \Rightarrow E[X^2]-1=4 \Rightarrow E[X^2]=5$

| | |
|---|---|
| | $Var(X)=E\left[X^2\right]-\left(E[X]\right)^2=5-1=4$<br>$Var(2-3X)=(-3)^2 Var(X)=9\times 4=36 \qquad \because Var(aX+b)=(a)^2 Var(X)$ |
| **10.** | **If the probability density function of a random variable X is $f(x)=\frac{1}{4}$ in $-2<X<2$, find $P(\lvert X\rvert>1)$. (April/May 17)**<br>$P(\lvert X\rvert>1)=1-P(\lvert X\rvert\le 1)=1-\int_{-1}^{1} f(x)\,dx$<br>$=1-\int_{-1}^{1}\frac{1}{4}dx=1-\frac{1}{4}[x]_{-1}^{1}=1-\frac{1}{4}[2]\ =1-\frac{1}{2}=\frac{1}{2}$ |
| **11.** | **Let $M_x(t)=\frac{1}{1-t}$ such that $t\neq 1$, be the mgf of r.v X. Find the mgf of Y = 2X +1.**<br>$M_Y(t)=M_{2X+1}(t)=e^t M_X(2t) \qquad \left[\because M_{aX+b}(t)=e^{bt}M_X(at)\right]$<br>$=e^t\left[\frac{1}{1-t}\right]_{t\to 2t}=\frac{e^t}{1-2t}. \qquad \left[\because M_X(at)=\left[M_X(t)\right]_{t\to at}\right]$ |
| **12.** | **If the random variable has the moment generating function $M_x(t)=\frac{3}{3-t}$, compute $E[X^2]$. (May/June 2016)**<br>$M_X(t)=\frac{3}{3-t}=\frac{3}{3\left(1-\frac{t}{3}\right)}=\left(1-\frac{t}{3}\right)^{-1}$<br>$=1+\left(\frac{t}{3}\right)+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\cdots=1+\frac{1}{3}\left(\frac{t}{1!}\right)+\frac{2}{9}\left(\frac{t^2}{2!}\right)+\frac{2}{9}\left(\frac{t^3}{3!}\right)+\cdots$<br>$E\left(X^r\right)=\mu_r'=\text{coefficient of}\left(\frac{t^r}{r!}\right)\ in\ \ M_X(t)$<br>$E\left(X^2\right)=\mu_2'=\text{coefficient of}\left(\frac{t^2}{2!}\right)\ in\,M_X(t)=\frac{2}{9}.$ |
| **13.** | **A continuous RV X has the pdf $f(x)=\frac{x^2e^{-x}}{2}$, $x>0$ Find the $r^{th}$ moment of X about the origin.**<br>$\mu_r'=E[X^r]=\int_{-\infty}^{\infty}x^r f(x)dx$<br>$=\int_0^{\infty}x^r\frac{x^2e^{-x}}{2}dx=\frac{1}{2}\int_0^{\infty}x^{r+2}e^{-x}dx=\frac{1}{2}\int_0^{\infty}x^{(r+3)-1}e^{-x}dx$<br>$=\frac{1}{2}\Gamma(r+3) \qquad \because \int_0^{\infty}x^{n-1}e^{-x}dx=\Gamma(n)$<br>$=\frac{1}{2}(r+2)! \qquad \because \text{if } n \text{ is positive integer } \Gamma(n)=(n-1)!$ |

| | |
|---|---|
| **14.** | **For a Binomial distribution with mean 6 and standard deviation $\sqrt{2}$, find the first two terms of the distribution. (May/June 2014)**<br>$np = 6$ and $\sqrt{npq} = \sqrt{2} \Rightarrow npq = 2 \Rightarrow 6q = 2 \Rightarrow \boxed{q = \frac{1}{3}} \therefore \boxed{p = \frac{2}{3}}$<br>$n \times \frac{2}{3} = 6 \quad \therefore \boxed{n = 9}$<br>$P(X = x) = nC_x p^x q^{n-x} = 9C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{9-x}, \ x = 0,1,2,3,\ldots 9$<br>$P(X = 0) = 9C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^9 = \left(\frac{1}{3}\right)^9$<br>$P(X = 1) = 9C_1 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^8 = 9 \times \frac{2}{3} \times \frac{1}{3^8} = \frac{2}{3^7}$ |
| **15.** | **If $X$ and $Y$ are independent binomial variates following $B\left(5, \frac{1}{2}\right)$ and $B\left(7, \frac{1}{2}\right)$ respectively find $P[X + Y = 3]$.**<br>Given $X \sim B\left(5, \frac{1}{2}\right)$, $Y \sim B\left(7, \frac{1}{2}\right)$<br>By additive property, $X + Y$ is also a binomial variate with parameters<br>$n_1 + n_2 = 12$ & $p = \frac{1}{2}$<br>$\therefore X + Y \sim B\left(12, \frac{1}{2}\right)$<br>$P(Z = z) = nC_z p^z q^{n-z}; z = 0,1,2,3.....n$<br>$\therefore P[X + Y = 3] = 12C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^9 = \frac{55}{2^{10}}$ |
| **16.** | **One percent of jobs arriving at a computer system need to wait until weekends for scheduling, owing to core-size limitations. Find the probability that among a sample of 200 jobs there are no jobs that have to wait until weekends.**<br>Let X be the Poisson random variable denoting the no. of jobs that have to wait<br>$p = 1\% = 0.01$, $n = 200$, $\lambda = np = (200)(0.01) = 2$,<br>By Poisson distribution, $P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0,1,2,.....$<br>$P(X = 0) = \frac{e^{-2} 2^0}{0!} = e^{-2} = 0.1353$ |
| **17.** | **If X is a Geometric variate, taking values $1,2,3,....,\infty$, find P(X is odd). (April/May 17)**<br>We know that for Geometric Distribution<br>$P(X = x) = q^{x-1}p, \quad x = 1,2,3,\cdots\infty$<br>$P(X \text{ is odd}) = P(X = 1,3,5,\cdots)$<br>$= P(X = 1) + P(X = 3) + P(X = 5) + P(X = 7) + \cdots$<br>$= p + q^2 p + q^4 p + q^6 p + \cdots$<br>$= p\left(1 + q^2 + q^4 + q^6 + \cdots\right)$ |

$$= p(1-q^2)^{-1} = \frac{p}{(1-q^2)} = \frac{(1-q)}{(1-q)(1+q)} = \frac{1}{1+q}.$$

**18. If X is a Uniformly distributed R.V with mean 1 and variance $\frac{4}{3}$ find P(X<0).**

$$\text{Mean} = \frac{a+b}{2} = 1 \Rightarrow a+b = 2 \quad \text{--------------(1)}$$

$$\text{variance} = \frac{(b-a)^2}{12} = \frac{4}{3} \Rightarrow b - a = 4 \text{---------------(2)}$$

(1) + (2) $\Rightarrow$2b = 6 $\Rightarrow$ b = 3

(1) – (2) $\Rightarrow$2a = -2 $\Rightarrow$ a = -1

Probability density function of Uniform distribution is

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases} \Rightarrow f(x) = \begin{cases} \frac{1}{4}, & -1 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

$$P(X<0) = \int_{-1}^{0} f(x)dx = \int_{-1}^{0} \frac{1}{4}dx = \frac{1}{4}[x]_{-1}^{0} = \frac{1}{4}.$$

**19. Suppose the length of life of an appliance has an exponential distribution with mean 10 years. What is the probability that the average life time of a random sample of the appliances is at least 10.5 years?**

Mean of the exponential distribution = E(X) = 1/$\lambda\Rightarrow$10 = 1/$\lambda$

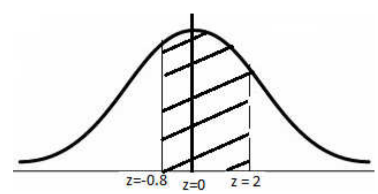
$$\lambda = \frac{1}{10}, \quad f(x) = \lambda e^{-\lambda x}, \ x > 0 \Rightarrow f(x) = \frac{1}{10} e^{-\frac{x}{10}}, x > 0$$

$$P(X > 10.5) = \int_{10.5}^{\infty} f(x)dx = \int_{10.5}^{\infty} \frac{1}{10} e^{-\frac{x}{10}} dx = e^{-1.05} = 0.3499$$

**20. X is a normal variate with mean = 30 and S.D = 5. Find $P[26 \le X \le 40]$**

X follows N(30, 5) $\therefore \mu = 30 \ \& \ \sigma = 5$

Let $Z = \frac{X-\mu}{\sigma}$ be the standard normal variate




$$P[26 \le X \le 40] = P\left[\frac{26-30}{5} \le Z \le \frac{40-30}{5}\right]$$

$$= P[-0.8 \le Z \le 2] = P[-0.8 \le Z \le 0] + P[0 \le Z \le 2]$$

$$= P[0 \le Z \le 0.8] + [0 \le Z \le 2] = 0.2881 + 0.4772 = 0.7653.$$

## PART – B

**1. i) A bag contains 3 black and 4 white balls. Two balls are drawn at random one at a time without replacement.**
**(1) What is the probability that the second ball drawn is white?**
**(2)What is the conditional probability that the first ball drawn is white if the second ball is known to be white? (May/June 2019)**

**Solution:**

$$\text{(1) P(the second ball drawn is white)} = \left(\frac{3}{7} \times \frac{4}{6}\right) + \left(\frac{4}{7} \times \frac{3}{6}\right) = \frac{2}{7} + \frac{2}{7} = \frac{4}{7}$$

(2)P(the second ball is white / first ball drawn is white)

$$= \frac{P(both\ are\ white)}{P(first\ draw\ is\ white)} = \frac{\left(\frac{4}{7}\times\frac{3}{6}\right)}{\frac{4}{7}} = \frac{\left(\frac{2}{7}\right)}{\left(\frac{4}{7}\right)} = \frac{1}{2}$$

**ii)** **There are two boxes $B_1$ and $B_2$. $B_1$ contains two red balls and one green ball. $B_2$ contains one red ball and two green balls.**

**(1) A ball is drawn from one of the boxes randomly. It is found to be red. What is the probability that it is from $B_1$?**

**(2) Two balls are drawn randomly from one of the boxes without replacement. One is red and the other is green. What is the probability that they came from $B_1$?**

**(3) A ball is drawn from one of the boxes is green. What is the prob. that it came from $B_2$?**

**(4) A ball is drawn from one of the boxes is white what is the prob. that it came from $B_2$?**

**Solution:**

Let $B_1$ *and* $B_2$ be the events that the boxes $B_1$ *and* $B_2$ respectively are selected.

$$P(B_1) = \frac{1}{2} \quad P(B_2) = \frac{1}{2}$$

1) Let A be the event that a red ball is selected.

$$P(A/B_1) = \frac{2}{3}, \qquad P(A/B_2) = \frac{1}{3}$$

1) P( ball is from $B_1$, given it is red) $= P(B_1/A) = \dfrac{P(B_1)P(A/B_1)}{P(B_1)P(A/B_1)+P(B_2)P(A/B_2)}$

$$= \frac{\frac{1}{2}\times\frac{2}{3}}{\left(\frac{1}{2}\times\frac{2}{3}\right)+\left(\frac{1}{2}\times\frac{1}{3}\right)} = \frac{2}{3}$$

2) Let C be the event that a red ball and a green ball are selected.

$$P(C/B_1) = \frac{2_{C_1}\times 1_{C_1}}{3_{C2}} = \frac{2}{3}$$

$$P(C/B_2) = \frac{1_{C_1}\times 2_{C_1}}{3_{C2}} = \frac{2}{3}$$

P( $B_1$ was chosen given a red ball and a green ball were selected )

$$= P(B_1/C) = \frac{P(B_1)P(C/B_1)}{P(B_1)P(C/B_1)+P(B_2)P(C/B_2)} = \frac{\frac{1}{2}\times\frac{2}{3}}{\left(\frac{1}{2}\times\frac{2}{3}\right)+\left(\frac{1}{2}\times\frac{2}{3}\right)} = \frac{1}{3}$$

3) Let D be the event that a green ball is selected.

$$P(D/B_1) == \frac{1}{3}$$

$P(D/B_2) = \frac{2}{3}$

$$= P(B_2/D) = \frac{P(B_2)P(D/B_2)}{P(B_1)P(D/B_1) + P(B_2)P(D/B_2)} = \frac{\frac{1}{2}\times\frac{2}{3}}{\left(\frac{1}{2}\times\frac{1}{3}\right)+\left(\frac{1}{2}\times\frac{2}{3}\right)} = \frac{2}{3}$$

4) Let E be the event that a white ball is selected.

The given two boxes does not contained a white ball, hence the prob is 0

**iii) A random variable $X$ has the following probability function:**

| X | : | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| $P(X)$ | : | 0 | K | 2K | 2K | 3K | $K^2$ | $2K^2$ | $7K^2+K$ |

**Find (i) K, (ii) Evaluate $P(X<6), P(X\geq 6)$ and $P(0<X<5)$ (iii) Determine the distribution function of X. (iv) $P(1.5<X<4.5/X>2)$ (v) $E(3X-4)$, Var(3X-4)**

**(vi) If $P[X\leq C]>\frac{1}{2}$, find the minimum value of *C*. (April/May 2015)**

**Solution:**

(i) We know that $\sum_i P(X=x_i)=1$

$\Rightarrow \sum_{x=0}^{7} P(X=x)=1, \Rightarrow K+2K+2K+3K+K^2+2K^2+7K^2+K=1 \Rightarrow$ $10K^2+9K-1=0$

$\Rightarrow K=\frac{1}{10}$ *or* $K=-1$ ( here $K=-1$ is impossible, since $P(X=x)\geq 0$ )

$\therefore K=\frac{1}{10}$

$\therefore$ The probability mass function is

| $X=x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $P(X=x)$ | 0 | $\frac{1}{10}$ | $\frac{2}{10}$ | $\frac{2}{10}$ | $\frac{3}{10}$ | $\frac{1}{100}$ | $\frac{2}{100}$ | $\frac{17}{100}$ |

(ii) $P(X<6)=P(X=0)+P(X=1)+...+P(X=5)=\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}=\frac{81}{100}$

$P(X\geq 6)=1-P(X<6)=1-\frac{81}{100}=\frac{19}{100}$

$P(0<X<5)=P(X=1)+P(X=2)+P(X=3)+P(X=4)=K+2K+2K+3K$

$=8K=\frac{8}{10}=\frac{4}{5}$

(iii) The distribution of X is given by $F_X(x)$ defined by $F_X(x)=P(X\leq x)$

| $X=x$ | $P(X=x)$ | $F_X(x)=P(X\leq x)$ |
|---|---|---|
| 0 | 0 | 0 , $x<1$ |
| 1 | $\frac{1}{10}$ | $0+\frac{1}{10}=\frac{1}{10}$ , $1\leq x<2$ |
| 2 | $\frac{2}{10}$ | $0+\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$, $2\leq x<3$ |

| | | |
|---|---|---|
| 3 | $\frac{2}{10}$ | $0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}=\frac{5}{10}, 3\le x<4$ |
| 4 | $\frac{3}{10}$ | $0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}=\frac{8}{10}=\frac{4}{5}, 4\le x<5$ |
| 5 | $\frac{1}{100}$ | $0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}=\frac{81}{100},\ 5\le x<6$ |
| 6 | $\frac{2}{100}$ | $0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}+\frac{2}{100}=\frac{83}{100},\ 6\le x<7$ |
| 7 | $\frac{17}{100}$ | $0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}+\frac{2}{100}+\frac{17}{100}=\frac{100}{100}=1,\ x\le 7$ |

(iv) $P(1.5<X<4.5/X>2)=\frac{P(X=2,3,4\cap X=3,4,5,6,7)}{P(X=3,4,5,6,7)}=\frac{P(X=3,4)}{P(X=3,4,5,6,7)}$

$=\frac{P(X=3)+P(X=4)}{P(X=3)+P(X=4)+P(X=5)+P(X=6)+P(X=7)}$

$=\frac{2K+3K}{2K+3K+K^2+2K^2+7K^2+K}=\frac{5K}{6K+10K^2}=\frac{5}{6+1}=\frac{5}{7}$

(v) To find $E(3X-4)$, $\mathrm{Var}(3X-4)$

$E(3X-4)=3E(X)-E(4)=3E(X)-4$ -------(1)

$Var(3X-4)=3^2Var(X)-Var(4)=9Var(X)-0=9Var(X)$

$Var(3X-4)=9Var(X)$ ---------- (2)

$E(X)=\sum xP(X=x)$

$=0\times P(X=0)+1\times P(X=1)+2\times P(X=2)+3\times P(X=3)+4\times P(X=4)$
$+5\times P(X=5)+6\times P(X=6)+7\times P(X=7)$

$=0+1\times K+2\times 2K+3\times 2K+4\times 3K+5\times K^2+6\times 2K^2+7\times(7K^2+K)$

$=K+4K+6K+12K+5K^2+12K^2+49K^2+7K=30K+66K^2=\frac{30}{10}+\frac{66}{100}=\frac{366}{100}$

$E(X^2)=\sum x^2P(X=x)$

$=0\times P(X=0)+1^2\times P(X=1)+2^2\times P(X=2)+3^2\times P(X=3)+4^2\times P(X=4)$
$+5^2\times P(X=5)+6^2\times P(X=6)+7^2\times P(X=7)$

$=0+1^2\times K+2^2\times 2K+3^2\times 2K+4^2\times 3K+5^2\times K^2+6^2\times 2K^2+7^2\times(7K^2+K)$

$=K+8K+18K+48K+25K^2+72K^2+343K^2+49K$

$=124K+440K^2=\frac{124}{10}+\frac{440}{100}=\frac{1240+440}{100}=\frac{1680}{100}=\frac{168}{10}=\frac{84}{5}$

(1)$\Rightarrow E(3X-4)=\frac{3\times 366}{100}-4=\frac{1098-400}{100}=\frac{698}{100}=69.8$

(2)$\Rightarrow Var(3X-4)=9Var(X)=9\left[E(X^2)-(E(X))^2\right]=9[16.8-13.3956]=30.6396$

(vi) To find the minimum value of $C$ if $P[X\le C]>\frac{1}{2}$

| $X=x$ | $P(X=x)$ | $P(X\le x)$ |
|---|---|---|
| 0 | 0 | $0<\frac{1}{2}$ |
| 1 | $\frac{1}{10}$ | $0+\frac{1}{10}=\frac{1}{10}<\frac{1}{2}$ |

| | | |
|---|---|---|
| 2 | $\frac{2}{10}$ | $0+\frac{1}{10}+\frac{2}{10}=\frac{3}{10}<\frac{1}{2}$ |
| 3 | $\frac{2}{10}$ | $0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}=\frac{5}{10}=\frac{1}{2}$ |
| 4 | $\frac{3}{10}$ | $0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}=\frac{8}{10}=\frac{4}{5}>\frac{1}{2}$ |
| 5 | $\frac{1}{100}$ | $0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}=\frac{81}{100}>\frac{1}{2}$ |
| 6 | $\frac{2}{100}$ | $0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}+\frac{2}{100}=\frac{83}{100}>\frac{1}{2}$ |
| 7 | $\frac{17}{100}$ | $0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}+\frac{2}{100}+\frac{17}{100}=\frac{100}{100}=1>\frac{1}{2}$ |

$\therefore$ the minimum value of C is 4

**2. i)** **A continuous random variable X has the p.d.f $f(x)=kx^3e^{-x},\ x\ge 0$. Find the $r^{th}$ order moment of X about the origin. Hence find m.g.f, mean and variance of X.**

**Solution:**

Since $\int_0^\infty kx^3e^{-x}dx=1 \Rightarrow k\left[x^3\left(\frac{e^{-x}}{-1}\right)-(3x^2)\left(\frac{e^{-x}}{1}\right)+(6x)\left(\frac{e^{-x}}{-1}\right)-(6)\left(\frac{e^{-x}}{1}\right)\right]_0^\infty=1$

$k\left[-x^3e^{-x}-3x^2e^{-x}-6xe^{-x}-6\right]_0^\infty=1 \Rightarrow k\left[(0)-(-6)\right]=1 \Rightarrow 6k=1 \Rightarrow k=\frac{1}{6}$.

$E(X^r)=\mu_r'=\int_0^\infty x^r f(x)dx=\frac{1}{6}\int_0^\infty x^r x^3 e^{-x}dx=\frac{1}{6}\int_0^\infty x^{r+3}e^{-x}dx \because \Gamma n=\int_0^\infty e^{-x}x^{n-1}dx,\ n>0$

here $n=r+4=\frac{1}{6}\int_0^\infty e^{-x}x^{(r+3+1)-1}dx=\frac{1}{6}\Gamma(r+4)=\frac{(r+3)!}{6} \because \Gamma n=(n-1)!$

Putting $r=1, E(X)=\mu_1'=\frac{4!}{6}=\frac{24}{6}=4$

$r=2, E(X^2)=\mu_2'=\frac{5!}{6}=\frac{120}{6}=20$

$\therefore$ Mean $=E(X)=\mu_1'=4;$ Variance $=E(X^2)-[E(X)]^2=\mu_2'-(\mu_1')^2$

$\mu_2=20-(4)^2=20-16=4$

**To find M.G.F**

$M_X(t)=E(e^{tX})=\int_{-\infty}^{\infty}e^{tx}f(x)dx$

$M_X(t)=\int_{-\infty}^{\infty}e^{tx}\frac{1}{6}x^3e^{-x}dx$

$=\frac{1}{6}\int_0^\infty x^3e^{tx-x}dx=\frac{1}{6}\int_0^\infty x^3e^{-(1-t)x}dx$

$=\frac{1}{6}\left[(x^3)\left(\frac{e^{-(1-t)x}}{-(1-t)}\right)-(3x^2)\left(\frac{e^{-(1-t)x}}{(1-t)^2}\right)+(6x)\left(\frac{e^{-(1-t)x}}{-(1-t)^3}\right)-(6)\left(\frac{e^{-(1-t)x}}{(1-t)^4}\right)\right]_0^\infty$

$$=\frac{1}{6}\left[-x^3\frac{e^{-(1-t)x}}{(1-t)}-3x^2\frac{e^{-(1-t)x}}{(1-t)^2}-6x\frac{e^{-(1-t)x}}{(1-t)^3}-6\frac{e^{-(1-t)x}}{(1-t)^4}\right]_0^\infty$$

$$=\frac{1}{6}\left[(0)-\left(\frac{-6}{(1-t)^4}\right)\right]$$

$$\therefore M_X(t)=\frac{1}{(1-t)^4}$$

**ii)** **A consulting firm rents cars from three rental agencies in the following manner: 20% from agency D, 20% from agency E and 60% from agency F. If 10% cars from D, 12% of the cars from E and 4% of the cars from F have bad tyres, what is the probability that the firm will get a car with bad tyres? Find the probability that a car with bad tyres is rented from agency F. (May/June 2019)**

**Solution:**

The event B represents the consulting firm rent cars have bad tyres
The event D represents the rent car from agency D
The event E represents the rent car from agency E
The event F represents the rent car from agency F
Given P(D) = 0.20, P(E) = 0.20 and P(F) = 0.60
P(Rent cars from agency D have bad tyres ) = P(B/D) = 0.10
P(Rent cars from agency E have bad tyres ) = P(B/E) = 0.12
P(Rent cars from agency F have bad tyres ) = P(B/F) = 0.04

$$P(B)=P(D)P(B/D)+P(E)P(B/E)+P(F)P(B/F)$$

$$P(\text{car had bad tyres}) = (0.2)(0.1) + (0.2)(0.12) + (0.6)(0.4) = 0.068$$

$$P(F/B)=\frac{P(F)P(B/F)}{P(D)P(B/D)+P(E)P(B/E)+P(F)P(B/F)}$$

$$=\frac{.6\times .4}{.068}=3.529$$

**iii)** **The probability of a man hitting a target is $\frac{1}{4}$. If he fires 7 times, what is the probability of his hitting the target atleast twice? And how many times must he fire so that the probability of his hitting the target atleast once is greater than $\frac{2}{3}$?**

**Solution:**

.

Let X be the event of hitting the target.

Then X follows binomial distribution with n = 7 and $p=\frac{1}{4},\ q=\frac{3}{4}$

By Binomial theorem, $P[X=r]=7C_r\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{7-r}$ where $r=0,1,2\ldots 7$ .

$$P[X\geq 2]=1-P(X<2)=1-\left[P(X=0)+P(X=1)\right]$$

| | | |
|---|---|---|
| | | $=1-\left[7C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{7-0}+7C_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{7-1}\right]$ <br> $=1-\left[\left(\frac{3}{4}\right)^7+7\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^6\right]=1-[0.3164+0.175]$ <br> $=0.5086$ <br> $P[X\geq 1] > \frac{2}{3}$ <br> $1-P[X<1]>\frac{2}{3}$ <br> $1-P[X=0]>\frac{2}{3}$ <br> $1-7C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{n-0} > \frac{2}{3}$ <br> $1-\left(\frac{3}{4}\right)^n > \frac{2}{3}$ <br> $-\left(\frac{3}{4}\right)^n > \frac{2}{3}-1$ <br> $-\left(\frac{3}{4}\right)^n > -\frac{1}{3} \quad \Rightarrow\left(\frac{3}{4}\right)^n < \frac{1}{3}\Rightarrow n>3.8$ <br> $\Rightarrow n>4$ <br> $=6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3+6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2+6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)+6C_6\left(\frac{1}{3}\right)^6=0.3196$ <br> $\therefore$ Expected number of times atleast 3 dice to show 5 or 6 $=N\times P[X\geq 3]$=729 X 0.3196 $\cong$ 233. |
| **3.** | **i)** | **A random variable $X$ has the probability mass function $f(x)=\frac{1}{2^x}$, x= 1,2,3,...** <br> **Find its (i) M.G.F (ii) Mean (iii) Variance.** |
| | | **Solution:** <br> M.G.F= $M_X(t)=E(e^{tX})$ <br> $=\sum_{x=1}^{\infty}e^{tx}f(x)=\sum_{x=1}^{\infty}e^{tx}\frac{1}{2^x}=\sum_{x=1}^{\infty}\left(\frac{e^t}{2}\right)^x$ <br> $=\frac{e^t}{2}+\left(\frac{e^t}{2}\right)^2+\left(\frac{e^t}{2}\right)^3+\left(\frac{e^t}{2}\right)^4+\ldots$ <br> $=\frac{e^t}{2}\left[1+\left(\frac{e^t}{2}\right)+\left(\frac{e^t}{2}\right)^2+\left(\frac{e^t}{2}\right)^3+\ldots\right]$ <br> $=\frac{e^t}{2}\left[1-\frac{e^t}{2}\right]^{-1}$ <br> M.G.F= $M_X(t)=\frac{e^t}{2-e^t}$ …………….. (1) |

Mean = $E(X) = \frac{d}{dx}[M_X(t)]_{t=0} = \frac{d}{dx}\left[\frac{e^t}{2-e^t}\right]_{t=0} = \left[\frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2}\right]_{t=0} = 2$

Variance = $Var(X) = E(X^2) - (E(X))^2$

Where

$E(X^2) = \frac{d}{dx}[M'_X(t)]_{t=0} = \frac{d}{dx}\left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0} = \left[\frac{(2-e^t)^2 e^t - e^t 2(2-e^t)(-e^t)}{(2-e^t)^4}\right]_{t=0} = 6$

Variance = $Var(X) = E(X^2) - (E(X))^2$ = 6-4 = 2.

**ii)** **A component has an exponential time to failure distribution with mean of 10,000 hours.**

**(i) The component has already been in operation for its mean life. What is the probability that it will fail by 15,000 hours?**

**(ii) At 15,000 hours the component is still in operation. What is the probability that it will operate for another 5000 hours? (Nov/Dec 2015)**

**Solution:**

Let $X$ be the random variable denoting the time to failure of the component following exponential distribution with $\text{Mean} = 10000$ hours. $\therefore \frac{1}{\lambda} = 10,000 \Rightarrow \lambda = \frac{1}{10,000}$

The p.d.f. of $X$ is $f(x) = \begin{cases} \frac{1}{10,000} e^{-\frac{x}{10,000}}, & x \ge 0 \\ 0, & otherwise \end{cases}$

(i) Probability that the component will fail by 15,000 hours given that it has already been in operation for its mean life $= P[X < 15,000 / X > 10,000]$

$= \frac{P[10,000 < X < 15,000]}{P[X > 10,000]} \quad --------(1)$

$P[10,000 < X < 15,000] = \int_{10,000}^{15,000} \frac{1}{10000} e^{-\frac{x}{10000}} dx$

$= \frac{1}{10000}\left[\frac{e^{-\frac{x}{10000}}}{\frac{-1}{10000}}\right]_{10000}^{15000} = -\left[e^{-\frac{x}{10000}}\right]_{10000}^{15000}$

$= -\left[e^{-\frac{15000}{10000}} - e^{-\frac{10000}{10000}}\right] = -\left[e^{-\frac{3}{2}} - e^{-1}\right] = e^{-1} - e^{-1.5} \quad ----(2)$

$P[X > 10,000] = \int_{10,000}^{\infty} \frac{1}{10000} e^{-\frac{x}{10000}} dx = \frac{1}{10000}\left[\frac{e^{-\frac{x}{10000}}}{\frac{-1}{10000}}\right]_{10000}^{\infty} = -\left[e^{-\frac{x}{10000}}\right]_{10000}^{\infty}$

$= -\left[e^{-\infty} - e^{-1}\right] = e^{-1} \quad --------(3)$

Sub (2) & (3) in (1)

$(1) \Rightarrow P[X < 15,000 / X > 10,000] = \frac{e^{-1} - e^{-1.5}}{e^{-1}} = \frac{0.3679 - 0.2231}{0.3679} = 0.3936$.

(ii) Probability that the component will operate for another 5000 hours given that it is in operation 15,000 hours $= P[X > 20,000 / X > 15,000]$

$= P\left[X > 5000\right]$ [By memoryless property]

$$= \int_{5000}^{\infty} f(x)\,dx = \int_{5000}^{\infty} \frac{1}{10000} e^{-\frac{x}{10000}} dx = \frac{1}{10000}\left[\frac{e^{-\frac{x}{10000}}}{\frac{-1}{10000}}\right]_{5000}^{\infty} = -\left[e^{-\frac{x}{10000}}\right]_{5000}^{\infty} = e^{-0.5} = 0.6065$$

**4. i) If the density function of a continuous random variable X is given**

$$\textbf{by } f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

**(i) Find the value of a Find the c.d.f of X Find $P(X \le 1.5)$. (April/May 17)**

**Solution:**

(i) Since $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$$

$$\int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{3} (3a - ax)dx = 1$$

$$a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^2 + a\left[3x - \frac{x^2}{2}\right]_2^3 = 1 \Rightarrow a = \frac{1}{2}$$

**(ii) CDF**

If $0 \le x \le 1$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{x} \frac{x}{2}dx = \left[\frac{x^2}{4}\right]_0^x = \frac{x^2}{4}$$

*If* $1 \le x \le 2$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{x} \frac{1}{2}dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^x = \frac{x}{2} - \frac{1}{4}$$

*If* $2 \le x \le 3$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{2} \frac{1}{2}dx + \int_{2}^{x} \left(\frac{3}{2} - \frac{x}{2}\right)dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^2 + \left(\frac{3x}{2} - \frac{x^2}{4}\right)_2^x$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$$

| | | |
|---|---|---|
| | | (ii) $F_X(x)=\begin{cases}0, & x<0\\ \frac{x^2}{2}, & 0\le x\le 1\\ \frac{x}{2}-\frac{1}{4}, & 1\le x\le 2\\ \frac{3x}{2}-\frac{x^2}{4}-\frac{5}{4}, & 2\le x\le 3\\ 1, & x>3\end{cases}$ <br> $(iii) P(X\le 1.5)=F(1.5)$ $[F(x)=P(X\le x)]$ <br> $=\frac{1.5}{2}-\frac{1}{4}=0.5$ |
| | **ii)** | **Trains arrive at a station at 15 minutes interval starting at 4 a.m. If a passenger arrive at a station at a time that is uniformly distributed between 9.00 a.m. and 9.30 a.m., find the probability that he has to wait for the train for (i) less than 6 minutes (ii) more than 10 minutes. (May/June 2014)** |
| | | **Solution:** <br> Let X denotes number of minutes past 9.00 a.m. that the passenger arrives at the stop till 9.30a.m. <br> X ~U[0,30] $\Rightarrow f(x)=\frac{1}{30},\ 0<x<30$ <br> $(i)$**P(that he has to wait for the train for less than 6 minutes)** <br> $=P[(9<x<15)\cup(24<x<30)]$ <br> $=\int_9^{15} f(x)dx+\int_{24}^{30} f(x)dx=\int_9^{15}\frac{1}{30}dx+\int_{24}^{30}\frac{1}{30}dx=\frac{1}{30}\{[x]_9^{15}+[x]_{24}^{30}\}=\frac{12}{30}=0.4$ <br> $(ii)$**P(that he has to wait for the train for** *more* **than 10 minutes)** <br> $=P[(0<x<5)\cup(15<x<20)]$ <br> $=\int_0^{5} f(x)dx+\int_{15}^{20} f(x)dx=\int_0^{5}\frac{1}{30}dx+\int_{15}^{20}\frac{1}{30}dx=\frac{1}{30}\{[x]_0^{5}+[x]_{15}^{20}\}=\frac{10}{30}=0.3333$ |
| **5.** | **i)** | **In a normal population with mean 15 and standard deviation 3.5, it is found that 647 observations exceed 16.25. What is the total number of observations in the population? (April / May 17)** |
| | | **Solution:** <br> *Given that* $\mu=15,\ \sigma=3.5,\ n=647$ <br> $P(X>16.25)=P\left(\frac{x-\mu}{\sigma}>\frac{16.25-\mu}{\sigma}\right)$ <br> $=P\left(z>\frac{16.25-15}{3.5}\right)=P(z>3.6)$ <br> $=0.5-P(0\le z\le 0.36)=0.5-0.1406=0.3594$ <br> $P(X>16.25)\times N=647$ <br> $N=\frac{647}{0.3594}$ <br> $=1800$ |

| | | |
|---|---|---|
| | ii) | **Derive the MGF, mean and variance of Geometric distribution and also state and prove the special property of it. ( May/June 16)** |
| | | **Solution:**<br>$P(X=x)=pq^{x-1}, x=1,2,3,\cdots$<br>**Moment Generating Function**<br>$M_X(t)=E(e^{tX})=\int_0^{\infty} e^{tx} f(x)dx = pe^t[1+qe^t+(qe^t)^2+\cdots]=pe^t[1+qe^t]^{-1}=\frac{pe^t}{1+qe^t}$<br>**Mean and Variance**<br>$\mu_1^{'}=M_X^{'}(0)=\left[\frac{d}{dt}\left(\frac{pe^t}{(1+qe^t)}\right)\right]_{t=0}=\left[\left(\frac{pe^t}{(1+qe^t)^2}\right)\right]_{t=0}=\frac{1}{p}$<br>$\mu_2^{'}=M_X^{''}(0)=\left[\frac{d^2}{dt^2}\left(\frac{pe^t}{(1+qe^t)}\right)\right]_{t=0}=\left[\frac{d}{dt}\left(\frac{pe^t}{(1+qe^t)^2}\right)\right]_{t=0}=\frac{1+q}{p^2}$<br>$Mean=\mu_1^{'}=\frac{1}{p}$<br>$Variance=\mu_2^{'}-(\mu_1^{'})^2=\frac{1+q}{p^2}-\left(\frac{1}{p}\right)^2=\frac{q}{p^2}$<br>**Memoryless property of geometric distribution.**<br>Statement:<br>If $X$ is a random variable with geometric distribution, then $X$ lacks memory, in the sense that $P[X>s+t/X>s]=P[X>t] \quad \forall s,t>0$.<br>Proof:<br>The p.m.f of the geometric random variable $X$ is $P(X=x)=q^{x-1}p \;, \quad x=1,2,3,....$<br>$P[X>s+t/X>s]=\frac{P[X>s+t\cap X>s]}{P[X>s]}=\frac{P[X>s+t]}{P[X>s]}$ -------(1)<br>$\therefore P[X>t]=\sum_{x=t+1}^{\infty} q^{x-1}p=q^t p+q^{t+1}p+q^{t+2}p+....=q^t p\left[1+q+q^2+q^3+....\right]$<br>$=q^t p(1-q)^{-1}=q^t p(p)^{-1}=q^t$<br>Hence $P[X>s+t]=q^{s+t}$ and $P[X>s]=q^s$<br>$(1)\Rightarrow P[X>s+t/X>s]=\frac{q^{s+t}}{q^s}=q^t=P[X>t]\Rightarrow P[X>s+t/X>s]=P[X>t]$ |
| **6.** | **i)** | **Let X be a Uniformly distributed R.V over [-5,5]. Determine (i) P(X ≤ 2) (ii) P(\|X\|≤ 2) (iii) Cumulative distribution function of X (iv) Var ( X) (May/June 2016)** |
| | | **Solution:**<br>**The R.V X ~ U[-5,5].**<br>**The p.d.f**<br>$f(x)=\begin{cases}\frac{1}{10} & for\ -5\le x\le 5\\ 0 & otherwise\end{cases}$ |

| | | |
|---|---|---|
| | | $(1)\ P(X \le 2) = \int_{-5}^{2} f(x)dx = \int_{-5}^{2} \frac{1}{10} dx = \frac{1}{10}\int_{-5}^{2} dx = \frac{1}{10}[x]_{-5}^{2}$ <br> $= \frac{1}{10}[2+5] = \frac{7}{10}$ <br> $(2)\ P(\lvert X\rvert > 2) = 1 - P(\lvert X\rvert \le 2) = 1 - P(-2 \le X \le 2)$ <br> $\therefore P(-2 \le X \le 2) = \int_{-2}^{2} f(x)dx = \int_{-2}^{2} \frac{1}{10} dx = \frac{1}{10}\int_{-2}^{2} dx = \frac{1}{10}[x]_{-2}^{2}$ <br> $= \frac{1}{10}[2+2] = \frac{4}{10}$ <br> $(3)$ *Cumulative distribution function of* $X$ <br> *If* $x < -5$ <br> $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} 0\,dx = 0$ <br> *If* $-5 \le x < 5$ <br> $F(x) = \int_{-5}^{x} f(x)dx = \int_{-5}^{x} \frac{1}{10} dx = \frac{1}{10}[x]_{-5}^{x} = \frac{x+5}{10}$ <br> *If* $x \ge 5$ <br> $F(x) = \int_{-5}^{5} f(x)dx + \int_{5}^{x} f(x)dx = \int_{-5}^{5} \frac{1}{10} dx + 0 = \frac{1}{10}[x]_{-5}^{5} = \frac{5+5}{10} = 1$ <br> $f(x) = \begin{cases} 0 & for\ x < -5 \\ \frac{x+5}{10} & for -5 \le x \le 5 \\ 1 & for\ x > 5 \end{cases}$ <br> $(4) Var(X) = \frac{(b-a)^2}{12} = \frac{(5-(-5))^2}{12} = \frac{100}{12} = \frac{25}{3}.$ |
| | **ii)** | **Let $P(X=x) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{x-1}$, $x = 1,2,3,....$ be the probability mass function of the R.V. X. Compute (1) P(X > 4) (2) P(X > 4/ X >2) (3) E(X) (4) Var ( X) (May/June 2016)** |
| | | **Solution:** <br> $(1)\ P(X > 4) = P(X=5) + P(X=6) + P(X=7) + \cdots$ <br> $= \sum_{X=5}^{\infty} P(X=x) = \sum_{X=5}^{\infty} \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{x-1} = \left(\frac{3}{4}\right)\left[\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \left(\frac{1}{4}\right)^6 + \cdots\right]$ |

$$=\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4\left[1+\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2+\cdots\right]$$

$$=\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4\left[1-\frac{1}{4}\right]^{-1}=\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^{-1}=\left(\frac{1}{4}\right)^4$$

$(2)\, P(X>4/X>2)=P(X>2)$

$$=P(X=3)+P(X=4)+P(X=5)+\cdots$$

$$=\sum_{X=3}^{\infty}P(X=x)=\sum_{X=3}^{\infty}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{x-1}$$

$$=\left(\frac{3}{4}\right)\left[\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4+\cdots\right]$$

$$=\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2\left[1+\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2+\cdots\right]$$

$$=\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2\left[1-\frac{1}{4}\right]^{-1}=\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{-1}=\left(\frac{1}{4}\right)^2$$

$(3)\;\; E(X)=\frac{1}{p}=\frac{1}{\left(\frac{3}{4}\right)}=\frac{4}{3}$

$4)Var(X)=\frac{q}{p^2}=\frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{4}\right)^2}=\frac{1}{4}\times\frac{16}{9}=\frac{4}{9}.$

## UNIT – II TWO DIMENSIONAL RANDOM VARIABLES

### PART – A

**1. Determine the value of the constant c if the joint density function of two discrete random variables X and Y is given by p(x,y) = cxy, x = 1,2,3 and y = 1,2,3.**

| X/Y | 1 | 2 | 3 | p(y) |
|---|---|---|---|---|
| 1 | c | 2c | 3c | 6c |
| 2 | 2c | 4c | 6c | 12c |
| 3 | 3c | 6c | 9c | 18c |
| p(x) | 6c | 12c | 18c | 36c |

Since p(x,y) is the joint pdf of X and Y

$p(x,y) \geq 0$ , for all x ,y

$\sum_m \sum_n p(x,y) = 1$

$\Rightarrow 36c = 1$

$\Rightarrow c = \frac{1}{36}$

**2.** **The following table gives the joint probability distribution of X and Y, find the marginal distribution function of X and Y.**

| X/Y | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 0.1 | 0.1 | 0.2 |
| 2 | 0.2 | 0.3 | 0.1 |

| X/Y | 1 | 2 | 3 | p(y) |
|---|---|---|---|---|
| 1 | 0.1 | 0.1 | 0.2 | 0.4 |
| 2 | 0.2 | 0.3 | 0.1 | 0.6 |
| p(x) | 0.3 | 0.4 | 0.3 | 1 |

The marginal distribution of X is

| Y | 1 | 2 |
|---|---|---|
| p(y) | 0.4 | 0.6 |

The marginal distribution of Y is

| X | 1 | 2 | 3 |
|---|---|---|---|
| p(x) | 0.3 | 0.4 | 0.3 |

**3.** **The joint probability mass function of the discrete random variable (X , Y) is given by the following table Find the conditional probability P ( X = 2 / Y = 3)**

| Y \ X | 2 | 4 |
|---|---|---|
| 1 | 1/10 | 1.5/10 |
| 3 | 2/10 | 3/10 |
| 5 | 1/10 | 1.5/10 |

| Y \ X | 2 | 4 | $P_Y(y)$ |
|---|---|---|---|
| 1 | 1/10 | 1.5/10 | 2.5/10 |
| 3 | 2/10 | 3/10 | 5/10 |
| 5 | 1/10 | 1.5/10 | 2.5/10 |
| $P_X(x)$ | 4/10 | 6/10 | 1 |

$$P(X = 2 / Y = 3) = \frac{P(X=2, Y=3)}{P_Y(3)} = \frac{2/10}{5/10} = \frac{2}{5}$$

**4.** **The joint probability mass function of X and Y is**

| X\Y | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0.1 | 0.04 | 0.02 |
| 1 | 0.08 | 0.2 | 0.06 |
| 2 | 0.06 | 0.14 | 0.3 |

**Check if X and Y are independent.**

| X\Y | 0 | 1 | 2 | $p_X(x)$ |
|---|---|---|---|---|
| 0 | 0.1 | 0.04 | 0.02 | 0.16 |
| 1 | 0.08 | 0.2 | 0.06 | 0.34 |
| 2 | 0.06 | 0.14 | 0.3 | 0.5 |
| $p_Y(y)$ | 0.24 | 0.38 | 0.38 | 1 |

$p_X(0) \cdot p_Y(0) = (0.16)(0.24) \neq 0.1 = p(0,0)$

$\therefore$ X and Y are not independent.

**5.** **The joint pdf of a two dimensional random variable (X,Y) is given by $f(x,y) = kxe^{-y}, 0 \le x \le 2, y > 0$. Find the value of k.**

Given that f(x,y) is pdf of (X,Y)

$\therefore$ f(x,y) $\ge$ 0 , for all x ,y

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1 \Rightarrow \int_0^{\infty}\int_0^{2} kxe^{-y}dxdy = 1 \Rightarrow k\int_0^2 xdx.\int_0^{\infty} e^{-y}dy = 1 \Rightarrow k\left[\frac{x^2}{2}\right]_0^2.\left[-e^{-y}\right]_0^{\infty} = 1$$

$$\Rightarrow k(2)(1) = 1 \Rightarrow k = \frac{1}{2}$$

**6.** **Find the value of k, if the joint density function of (X , Y) is given by**

$$\mathbf{f(x,y) = \begin{cases} k(1-x)(1-y) & ,0 < x < 4, 1 < y < 5 \\ 0 & ,otherwise \end{cases}}$$

Given the joint pdf of (X , Y) is f(x , y) = k (1 – x) (1 – y), 0 < x < 4, 1 < y < 5

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1 \Rightarrow \int_1^{5}\int_0^{4} k(1-x)(1-y)dxdy = 1$$

$$\Rightarrow k\int_1^5\left[x - \frac{x^2}{2} - yx + y\frac{x^2}{2}\right]_0^4 dy = 1 \Rightarrow k\left[-4y + 4\frac{y^2}{2}\right]_1^5 = 1$$

$$\Rightarrow k\int_1^5(-4+4y)dy = 1 \Rightarrow k(30+2) = 1 \Rightarrow 32k = 1 \Rightarrow k = \frac{1}{32}$$

**7.** **Given the joint probability density function of X and Y as**

$$\mathbf{f(x,y) = \begin{cases} \frac{1}{6}, & 0 < x < 2, 0 < y < 3 \\ 0, & otherwise \end{cases}}$$ **, determine the marginal density functions.**

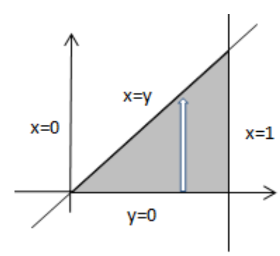
The marginal function of X is $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^3 \frac{1}{6}dy = \left[\frac{y}{6}\right]_0^3 = \frac{1}{2},\ 0 < x < 2$

The marginal function of Y is $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^2 \frac{1}{6}dx = \left[\frac{x}{6}\right]_0^2 = \frac{1}{3},\ 0 < y < 3$

**8.** **If $\mathbf{f(x,y) = \begin{cases} 8xy & ,0 < x < 1, 0 < y < x \\ 0 & ,elsewhere \end{cases}}$ is the joint probability density function of X and Y, find f(y/x).**

$$f_X(x) = \int_y f(x,y)dy = \int_{y=0}^{x} 8xydy = \left[8x\frac{y^2}{2}\right]_0^x = 4x^3,\ 0 < x < 1$$

$$f(y/x) = \frac{f(x,y)}{f_X(x)}, = \frac{8xy}{4x^3} = \frac{2y}{x^2},\ 0 < y < x, 0 < x < 1$$




**9.** **The joint p.d.f. of R.V. (X,Y) is given as $f(x,y) = \begin{cases} \frac{1}{x} & ,0 < y < x \le 1 \\ 0 & ,elsewhere \end{cases}$.**

**Find the marginal p.d.f. of Y.**

The marginal pdf of $Y$ *is*

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_y^1 \frac{1}{x}dx = [\log x]_y^1 = \log 1 - \log y = -\log y,\ 0 < y < 1$$

**10.** **The joint probability density function of bivariate random variable (X , Y) is given by** $\mathbf{f(x,y)=\begin{cases}4xy & ,0<x<1,0<y<1\\ 0 & ,elsewhere\end{cases}}$ **Find P (X + Y<1 )**

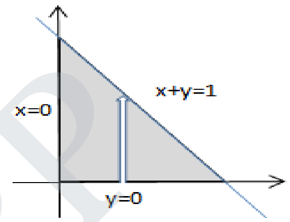
Given the joint pdf of (X , Y) is $f(x,y)=\begin{cases}4xy,0<x<1,0<y<1\\ 0 \quad ,elsewhere\end{cases}$.

$$\therefore P(X+Y<1)=\int_0^1\int_0^{1-x}4xy\,dy\,dx=4\int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x}dx$$

$$=2\int_0^1 x(1-x)^2dx=2\int_0^1 x(1-2x+x^2)dx$$

$$=2\int_0^1(x-2x^2+x^3)dx=2\left[\frac{x^2}{2}-2\frac{x^3}{3}+\frac{x^4}{4}\right]_0^1$$

$$=2\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]=\frac{1}{6}$$




**11.** **Let X and Y be two random variables having joint density function.** $\mathbf{f(x,y)=\frac{3}{2}(x^2+y^2),0\le x\le 1,0\le y\le 1.}$ **Determine** $\mathbf{P\left(X<\frac{1}{2},Y>\frac{1}{2}\right)}$

$$P\left(X<\frac{1}{2},Y>\frac{1}{2}\right)=\int_{x=-\infty}^{\frac{1}{2}}\int_{y=\frac{1}{2}}^{\infty}f(x,y)dydx=\int_{x=0}^{\frac{1}{2}}\int_{y=\frac{1}{2}}^{1}\frac{3}{2}\left(x^2+y^2\right)dydx=\frac{3}{2}\int_0^{\frac{1}{2}}\left[x^2y+\frac{y^3}{3}\right]_{\frac{1}{2}}^{1}dx$$

$$=\frac{3}{2}\int_0^{\frac{1}{2}}\left[x^2\left(1-\frac{1}{2}\right)+\frac{1}{3}\left(1-\frac{1}{8}\right)\right]dx=\frac{3}{2}\int_0^{\frac{1}{2}}\left(\frac{x^2}{2}+\frac{7}{24}\right)dx=\frac{3}{2}\left[\frac{x^3}{6}+\frac{7x}{24}\right]_0^{\frac{1}{2}}$$

$$=\frac{3}{2}\left[\frac{1}{6}.\frac{1}{8}+\frac{7}{24}.\frac{1}{2}\right]$$

$$=\frac{3}{2}\left[\frac{8}{48}\right]=\frac{1}{4}$$

**12.** **If the joint cumulative distribution function of X and Y is given by** $\mathbf{F(x,y)=(1-e^{-x})(1-e^{-y}),x>0,y>0}$ **, find P( 1 < X < 2 , 1 < Y < 2 )**

The joint pdf is

$$f(x,y)=\frac{\partial^2F}{\partial x\partial y}=\frac{\partial^2}{\partial x\partial y}\left(1-e^{-x}\right)\left(1-e^{-y}\right)=\frac{\partial}{\partial x}\left(1-e^{-x}\right).e^{-y}=e^{-x}.e^{-y}=e^{-(x+y)},x>0,y>0$$

$$P(1<X<2,1<Y<2)=\int_1^2\int_1^2 f(x,y)dxdy=\int_1^2\int_1^2 e^{-(x+y)}dxdy=\int_1^2\int_1^2 e^{-x}.e^{-y}dxdy$$

$$=\int_1^2 e^{-x}dx.\int_1^2 e^{-y}dy=\left[-e^{-x}\right]_1^2.\left[-e^{-y}\right]_1^2=\left(e^{-1}-e^{-2}\right)^2=\left(\frac{1}{e}-\frac{1}{e^2}\right)^2=\left(\frac{e-1}{e^2}\right)^2$$

$$=0.054$$

**13.** **If X has mean 4 and variance 9, while Y has mean –2 and variance 5 and the two are independent find (a) E[XY] (b) E[XY²]**

Given $E[X]=4,\ E[Y]=-2,\ X$ and $Y$ are independent.

$(a)\ E[XY]=E[X]\,E[Y]=4(-2)=-8$

$(b)\ E[XY2]=E[X]\,E[Y2]$

| | |
|---|---|
| | $\because \sigma_Y^2 = E[Y^2]-[E[Y]]^2 \Rightarrow 5 = E[Y^2]-4$ <br> $\Rightarrow E[Y^2]=9 \quad \therefore E[XY^2]=4(9)=36$ |
| **14.** | **Let X and Y be two independent R.Vs with Var(X) = 9 and Var(Y) = 3. Find Var(4X – 2Y + 6).** <br> Var(4X – 2Y + 6) = 16 Var(X) + 4 Var(Y) = 16(9) + 4(3) = 156 |
| **15.** | **If Y = -2X + 3, find Cov (X , Y).** <br> $\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)=E(X(-2X+3))-E(X)\{E(-2X+3)\}$ <br> $=\left[E(-2X^2+3X)-E(X)\right]\{-2E(X)+3\}$ <br> $=-2E(X^2)+3E(X)+2(E(X))^2-3E(X)$ <br> $=2(E(X))^2-2E(X^2)=-2\,\mathrm{var}(X)$ |
| **16.** | **The correlation coefficient of two random variables X and Y is $-\frac{1}{4}$ while their variances are 3 and 5. Find the covariance.** <br> Given $r_{xy}=-\frac{1}{4},\ \sigma_X^2=3, \sigma_Y^2=5 \quad r_{xy}=\frac{Cov(X,Y)}{\sigma_X\sigma_Y},\ \sigma_X\neq 0, \sigma_Y\neq 0$ <br> $\mathrm{Cov}(X,Y)=r_{xy}\,\sigma_X\sigma_Y = -\frac{1}{4}\sqrt{3}\,.\,\sqrt{5} = -0.968$ |
| **17.** | **The lines of regression in a bivariate distribution are X + 9Y = 7 and Y + 4X = $\frac{49}{3}$. Find the coefficient of correlation.** <br> $x+9y-7=0$ ----------- (1) $\qquad y+4x-\frac{49}{3}=0$ ----------- (2) <br> Let (1) be the regression line of Y on X and let (2) be the regression line of X on Y. <br> $y=-\frac{1}{9}x+\frac{7}{9}\Rightarrow b_1=-\frac{1}{9}$ <br> $x=-\frac{1}{4}y+\frac{49}{12}\Rightarrow b_2=-\frac{1}{4}$ <br> $r=\pm\sqrt{b_1b_2}=\sqrt{-\frac{1}{9}.-\frac{1}{4}}$ <br> $=\sqrt{\frac{1}{36}}=\frac{1}{6}<1$ <br> Since both regression coefficients are negative, correlation coefficient is negative. |
| **18.** | **The regression equations of X on Y and Yon X are respectively 5x – y = 22 and 64x – 45y = 24. Find the mean values of X and Y.** <br> Regression lines pass through the mean values of X and Y. Solving the two equations we get the mean values. <br> Let $5x-y=22$ --------------(1) <br> $64x-45y=24$ --------------(2) <br> Multiply equation (1) by 45 and subtract equation ( 2) <br> $225x-45y = 990$ <br> $-\ 64x+45y = -\ 24$ <br> $161x = 966 \Rightarrow x=6$ <br> Substitute in equation 1 <br> $5(6)-y=22\Rightarrow y=8.$ <br> $\therefore$ mean value of X = 6 and mean value of Y = 8 |

**19.** **The two lines of regression are 4x – 5y + 33 = 0 and 20x – 9y = 107. Calculate the coefficient of correlation between X and Y.**

4x – 5y + 33 = 0 ----------- (1) 20x – 9y = 107 ----------- (2)

Let (1) be the regression line of Y on x and let (2) be the regression line of X on Y.

$$\therefore y = \frac{4}{5}x + \frac{33}{5} \Rightarrow b_1 = \frac{4}{5}$$

$$x = \frac{9}{20}y + \frac{107}{20} \Rightarrow b_2 = \frac{9}{20} \quad \therefore r = \sqrt{b_1 b_2} = \sqrt{\frac{4}{5}\cdot\frac{9}{20}} = \sqrt{\frac{9}{25}} = \frac{3}{5} = 0.6 < 1$$

**20.** **Given the joint density function of X and Y as f(x , y) = $\begin{cases} xe^{-y}, 0 < x < 2, y > 0 \\ 0 \quad ,\text{elsewhere} \end{cases}$.**

**Find the range space for the transformation X + Y.**

Let the auxiliary random variable be V = Y.

The transformation functions are u = x + y, v = y, y > 0 ⇒ v > 0 and

0 < x < 2 ⇒ 0 < u – v < 2 ⇒ v < u < v + 2

**PART – B**

**1.** **(i) The joint pdf of the random variables X and Y is defined as $f(x,y) = \begin{cases} 25e^{-5y}, 0 < x < 0.2, y > 0 \\ 0 \quad ,\text{elsewhere} \end{cases}$.**

**(a) Find the marginal PDFs of X and Y**

**(b) COV (X,Y)**

**Solution:**

The marginal PDF of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^{\infty} 25e^{-y}dy = 25\left[-e^{-y}\right]_0^{\infty} = 25\left[-e^{-\infty} + e^0\right] = 25(1) = 25,\ 0 < x < 0.2$$

The marginal PDF of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^{0.2} 25e^{-y}dx = 25e^{-y}\left[x\right]_0^{0.2} = 25e^{-y}\left[0.2\right] = 5e^{-y},\ 0 < y < \infty$$

$$E(X) = \int_{-\infty}^{\infty} xf_X(x)dx = \int_0^{0.2} x(25)dx = 25\left[\frac{x^2}{2}\right]_0^{0.2} = 25\left[\frac{0.04}{2}\right] = 0.5$$

$$E(Y) = \int_{-\infty}^{\infty} yf_Y(y)dy = \int_0^{\infty} y(5e^{-y})dx = 5\left[-ye^{-y} - e^{-y}\right]_0^{\infty} = 5\left[0+1\right] = 5$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy = \int_0^{\infty}\int_0^{0.2} xy(25e^{-y})dxdy = \int_0^{\infty} 25ye^{-y}dy.\int_0^{0.2} xdx$$

$$= 25\left[-ye^{-y} - e^{-y}\right]_0^{\infty}.\left[\frac{x^2}{2}\right]_0^{0.2} = 25\left[0+1\right].\left[\frac{0.04}{2}\right] = (25)(0.02) = 0.5$$

Cov(x,y) = E(XY) – E(X)E(Y) = 0.5 – (0.5)(5) = –2

**(ii)Find the constant k such that $f(x,y) = \begin{cases} k(x+1)e^{-y}, 0 < x < 1, y > 0 \\ 0 \quad ,\text{otherwise} \end{cases}$ is a joint p.d.f. of the continuous random variable (X,Y). Are X and Y independent R.Vs? Explain.**

**Solution:**

To find k :

| | |
|---|---|
| | Given that f(x,y) is pdf of (X,Y)<br>$\therefore f(x,y) \geq 0$ , for all x ,y<br>$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1 \Rightarrow \int_0^{\infty}\int_0^1 k(x+1)e^{-y}dxdy = 1 \Rightarrow k\int_0^1 (x+1)dx.\int_0^{\infty} e^{-y}dy = 1$<br>$\Rightarrow k\left[\frac{x^2}{2}+x\right]_0^1.\left[-e^{-y}\right]_0^{\infty} = 1$<br>$\Rightarrow k\left(\frac{3}{2}\right)(1) = 1 \Rightarrow k = \frac{2}{3}$<br>$f(x,y) = \begin{cases} \frac{2}{3}(x+1)e^{-y}, 0 < x < 1, y > 0 \\ 0 \quad , otherwise \end{cases}$<br>The marginal PDF of X is<br>$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^{\infty}\frac{2}{3}(x+1)e^{-y}dy = \frac{2}{3}(x+1)\left[-e^{-y}\right]_0^{\infty} = \frac{2}{3}(x+1)\left[-e^{-\infty}+e^0\right] = \frac{2}{3}(x+1)(1) = \frac{2}{3}(x+1), 0$<br>The marginal PDF of Y is<br>$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 \frac{2}{3}(x+1)e^{-y}dx = \frac{2}{3}e^{-y}\left[\frac{x^2}{2}+x\right]_0^1 = \frac{2}{3}e^{-y}\left[\frac{1}{2}+1\right] = \frac{3}{2}.\frac{2}{3}e^{-y} = e^{-y},\ 0 < y < \infty$<br>Consider $f_X(x).\ f_Y(y) = \frac{2}{3}(x+1).\ e^{-y} = f(x,y)$<br>$\therefore$ X and Y are independent |
| 2. | **(i). Let the joint p.d.f. of R.V. (X,Y) be given as $f(x,y) = \begin{cases} 4xy & ,0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & ,elsewhere \end{cases}$, find the marginal densities of X and Y and the conditional densities of X given Y = y.**<br>The marginal density function of X is<br>$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 4xy\,dy = 4\left[x\left(\frac{y^2}{2}\right)_0^1\right] = 2x,\ 0 < x < 1$<br>The marginal density function of Y is<br>$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 4xy\,dx = 4\left[y\left(\frac{x^2}{2}\right)_0^1\right] = 2y, 0 < y < 1$<br>The conditional densities function of X given Y = y is<br>$f\left(x/y\right) = \frac{f(x,y)}{f_Y(y)} = \frac{4xy}{2y} = 2x,\ 0 < x < 1$ |
| | **(ii). The joint density function of two random variable X and Y is given by**<br>$\mathbf{f(x,y) = \begin{cases} \frac{6}{7}\left(x^2+\frac{xy}{2}\right), 0 \leq x \leq 1,\ 0 \leq y \leq 2 \\ 0 \quad , elsewhere \end{cases}}$.<br>**(a) Compute the marginal p.d.f of X and Y ? (b) Find E(X) & E(Y) (c) $P\left(X < \frac{1}{2}, Y > \frac{1}{2}\right)$**<br>**Solution:**<br>The marginal pdf of X is |

| | |
|---|---|
| | $f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\int_0^2 \frac{6}{7}\left(x^2+\frac{xy}{2}\right)dy=\frac{6}{7}\left[x^2y+\frac{x}{2}\frac{y^2}{2}\right]_0^2=\frac{6}{7}\left[2x^2+x\right],0\le x\le 1$<br>The marginal pdf of Y is<br>$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_0^1 \frac{6}{7}\left(x^2+\frac{xy}{2}\right)dx=\frac{6}{7}\left[\frac{x^3}{3}+\frac{x^2}{2}\frac{y}{2}\right]_0^1=\frac{6}{7}\left[\frac{1}{3}+\frac{y}{4}\right],0\le y\le 2$<br>E(X) =<br>$\int_{-\infty}^{\infty} xf_X(x)dx=\int_0^1 x\frac{6}{7}\left[2x^2+x\right]dx=\frac{6}{7}\left[2x^3+x^2\right]_0^1=\frac{6}{7}\left[\frac{2x^4}{4}+\frac{x^2}{2}\right]_0^1=\frac{6}{7}\left[\frac{1}{2}+\frac{1}{2}\right]=\frac{6}{7}$<br>E(Y) = $\int_{-\infty}^{\infty} yf_Y(y)dy=\int_0^2 y\frac{6}{7}\left[\frac{1}{3}+\frac{y}{4}\right]dy=\frac{6}{7}\left[\frac{y}{3}+\frac{y^2}{4}\right]_0^2=\frac{6}{7}\left[\frac{y^2}{6}+\frac{y^3}{12}\right]_0^2=\frac{6}{7}\left[\frac{2}{3}+\frac{2}{3}\right]=\frac{8}{7}$<br>$P(X<\frac{1}{2},Y>\frac{1}{2})=\int_{-\infty}^{\frac{1}{2}}\int_{\frac{1}{2}}^{\infty} f(x,y)\,dy\,dx=\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{2}\frac{6}{7}\left(x^2+\frac{xy}{2}\right)dy\,dx=\int_0^{\frac{1}{2}}\frac{6}{7}\left[x^2y+\frac{x}{2}\frac{y^2}{2}\right]_{\frac{1}{2}}^{2}dx$<br>$=\int_0^{\frac{1}{2}}\frac{6}{7}\left[2x^2+x-\frac{x^2}{2}-\frac{x}{16}\right]dx=\int_0^{\frac{1}{2}}\frac{6}{7}\left[\frac{3x^2}{2}+\frac{15x}{16}\right]_0^{\frac{1}{2}}dx=\frac{6}{7}\left[\frac{x^3}{2}+\frac{15x^2}{32}\right]_0^{\frac{1}{2}}$<br>$=\frac{6}{7}\left[\frac{1}{16}+\frac{15}{128}\right]=\frac{6}{7}.\frac{23}{128}=\frac{69}{448}$ |
| **3.** | **(i). If X,Y and Z are uncorrelated random variables with zero means and standard deviations 5 , 12 and 9 respectively and if U = X + Y, V = Y + Z, find the correlation coefficient between U and V.**<br>**Solution:**<br><br>Given E(X) = E(Y) = E(Z) = 0<br><br>$Var(X) = E(X^2) – (E(X))^2 = 25$ $\therefore E(X^2) = 25$<br>$Var(Y) = E(Y^2) – (E(Y))^2 = 144$ $\therefore E(Y^2) = 144$<br>$Var(Z) = E(Z^2) – (E(Z))^2 = 81$ $\therefore E(Z^2) = 81$<br><br>Also given that X, Y and Z are uncorrelated<br>$\therefore r_{XY} = 0$, i.e., E(XY) – E(X) . E(Y) = 0 $\Rightarrow$ E(XY) = 0<br>$\therefore r_{YZ} = 0$, i.e., E(YZ) – E(Y) . E(Z) = 0 $\Rightarrow$ E(YZ) = 0<br>$\therefore r_{XZ} = 0$, i.e., E(XZ) – E(X) . E(Z) = 0 $\Rightarrow$ E(XZ) = 0<br><br>Now, E(U) = E(X+Y) = E(X) + E(Y) = 0 and E(V) = E(Y+Z) = E(Y) + E(Z) = 0<br>$E(U^2) = E((X+Y)^2) = E(X^2 + Y^2 + 2XY) = E(X^2) + E(Y^2) + 2E(XY) = 25 + 144 + 0 = 169$<br>$E(V^2) = E((Y+Z)^2) = E(Y^2 + Z^2 + 2YZ) = E(Y^2) + E(Z^2) + 2E(YZ) = 144 + 81 + 0 = 225$<br><br>$\sigma_U^2 = E(U^2) – (E(U))^2 = 169$ and $\sigma_V^2 = E(V^2) – (E(V))^2 = 225$<br>$E(UV) = E\{(X+Y)(Y+Z)\} = E(XY) + E(XZ) + E(Y^2) + E(YZ) = 0 + 0 + 144 + 0 = 144$<br>$r_{UV}=\frac{E(UV)-E(U)E(V)}{\sigma_U.\sigma_V}=\frac{144}{(13)(15)}=\frac{48}{65}$ |

**(ii) The joint pdf of the continuous R.V (X,Y) is given as** $f(x,y)=\begin{cases} e^{-(x+y)}, x>0, y>0 \\ 0 \quad , elsewhere \end{cases}$.

**Find the pdf of the random variable** $U=\frac{X}{Y}$.

**Solution:**

The transformation functions are $u=\frac{x}{y}$ *and* $v=y$

Solving for x , we get $u=\frac{x}{v} \Rightarrow x=uv$

The Jacobian of the transformation is J = $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$

The joint density of U and V is $f_{UV}(u,v)=|J|\, f_{XY}(x,y)=|v|\, e^{-(x+y)}=v\, e^{-v(u+1)}$

The range space of (U , V) is obtained from the range space of (X , Y) and the transformations x = uv, y = v

$\therefore x>0$ and $y>0$ we have $uv>0$ and $v>0$

$\Rightarrow u>0$ and $v>0$

$$f_{UV}(u,v)=\begin{cases} v\,e^{-v(u+1)}, \ u>0, v>0 \\ 0 \ , \ elsewhere, \end{cases}$$

The pdf of U is the marginal density function of U,

$$f_U(u)=\int_{-\infty}^{\infty} f(u,v)dv=\int_0^{\infty} v\,e^{-v(u+1)}\,dv$$

$$=\left[v.\frac{e^{-v(u+1)}}{-(u+1)}-1.\frac{e^{-v(u+1)}}{(u+1)^2}\right]_0^{\infty}$$

$$=0+\frac{1}{(u+1)^2}=\frac{1}{(u+1)^2}\ , u>0$$

**(iii) The life time of a certain brand of an electric bulb may be considered as a random variable with mean 1200h and standard deviation 250h. Find the probability, using central limit theorem, that the average lifetime of 60 bulbs exceeds 1250hours.**

Let $X_i$ (i=1,2,...60) denote the life time of the bulbs.

Here $\mu$=1200, $\sigma^2=250^2$

Let $\overline{X}$ denote the average life time of 60 bulbs.

By Central limit theorem,

$$\overline{X} \ \ follows \ N\left(\mu, \frac{\sigma^2}{n}\right).$$

$$Let \ Z=\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$$

$$P\left(\overline{X}>1250\right)=P\left(Z>1.55\right)$$
$$=0.0606$$

**4.** **(i). Obtain the equation of the regression line Y on X from the following data.**

| X | 3 | 5 | 6 | 8 | 9 | 11 |
|---|---|---|---|---|---|---|
| Y | 2 | 3 | 4 | 6 | 5 | 8 |

**Solution:**

| X | Y | U = X – 6 | V = Y – 6 | $U^2$ | $V^2$ | UV |
|---|---|---|---|---|---|---|
| 3 | 2 | –3 | –4 | 9 | 16 | 12 |
| 5 | 3 | –1 | –3 | 1 | 9 | 3 |
| 6 | 4 | 0 | –2 | 0 | 4 | 0 |
| 8 | 6 | 2 | 0 | 4 | 0 | 0 |
| 9 | 5 | 3 | –1 | 9 | 1 | –3 |
| 11 | 8 | 5 | 2 | 25 | 4 | 10 |
| | | 6 | –8 | 48 | 34 | 22 |

$$n=6, \sum U = 6, \sum V = -8, \sum U^2 = 48, \sum V^2 = 34, \sum UV = 22\ \ \overline{U} = \frac{\sum U}{n} = \frac{6}{6} = 1,$$

$$\overline{V} = \frac{\sum V}{n} = \frac{-8}{6} = -1.33,\ \sigma_U^2 = \frac{\sum U^2}{n} - \left(\overline{U}\right)^2 = \frac{48}{6} - 1 = 7,\ \sigma_U = 2.646,$$

$$\sigma_V^2 = \frac{\sum V^2}{n} - \left(\overline{V}\right)^2 = \frac{34}{6} - (-1.33)^2 = 3.898,\ \sigma_V = 1.974$$

$$\text{Cov}(U, V) = \frac{\sum UV}{n} - \overline{U}\ \overline{V} = \frac{22}{6} - (1)(-1.33) = 4.997$$

$$\therefore r_{UV} = \frac{Cov(U,V)}{\sigma_U \cdot \sigma_v} = \frac{4.997}{(2.646)(3.898)} = 0.484$$

$$\therefore r_{XY} = 0.484$$

$$\overline{X} = \overline{U} + 6 \Rightarrow \overline{X} = 1 + 6 = 7$$

$$\overline{Y} = \overline{V} + 6 \Rightarrow \overline{Y} = -1.33 + 6 = 4.67$$

$$\sigma_X = \sigma_U \Rightarrow \sigma_X = 2.646\ ;\ \sigma_Y = \sigma_V \Rightarrow \sigma_Y = 3.898$$

The regression line of Y on X is

$$Y - \overline{Y} = \frac{r\,\sigma_Y}{\sigma_X}\left(X - \overline{X}\right) \Rightarrow Y - 4.67 = \frac{(0.484)(3.898)}{(2.646)}\left(X - 7\right) \Rightarrow Y = 0.713X - 0.321$$

**(ii) X and Y are two random variables having the joint probability mass function $f(x, y) = k(3x + 5y);\ x = 1,2,3 : y = 0,1,2$. Find the marginal distributions and conditional distribution of X, $P(X = x_i / Y = 2)$, $P(X \leq 2\ / Y \leq 1)$.** **(April/May 2019)**

**Solution:**

| Y \ X | 1 | 2 | 3 | $P_Y(y)$ |
|---|---|---|---|---|
| 0 | 3k | 6k | 9k | 18k |
| 1 | 8k | 11k | 14k | 33k |
| 2 | 13k | 16k | 19k | 48k |
| $P_X(x)$ | 24k | 33k | 42k | 99k |

To find k:

We know that Total probability = 1

$\Sigma\Sigma P(x,y) = 1 \Rightarrow 99k = 1$

$\Rightarrow k = 1/99$

The marginal distribution of X is

| X | 1 | 2 | 3 |
|---|---|---|---|
| p(x) | 24/99 | 33/99 | 42/99 |

The marginal distribution of Y is

| Y | 0 | 1 | 2 |
|---|---|---|---|
| p(y) | 18/99 | 33/99 | 48/99 |

Conditional distribution of X given Y = 2

$P(X = x_i / Y = 2)$

| X | 1 | 2 | 3 |
|---|---|---|---|
| $P(X = x_i / Y = 2) = [P(X = x_i, Y = 2)] / P(Y = 2)$ | 13/48 | 16/48 | 19/48 |

$P(X \le 2 \ / \ Y \le 1) = P(X=1, Y=0) + P(X=1, Y=1) + P(X=2, Y=0) + P(X=2, Y=1)$

$= 3k + 8k + 6k + 11k$

$= 28k = 28/99$

5. **(i) In a partially destroyed laboratory record only the lines of regressions and variance of X are available. The regression equations are 8x – 10y + 66 = 0 and 40x – 18y = 214 and variance of X = 9. Find (a) the correlation coefficient between X and Y (b) Mean values of X and Y (c) variance of Y.**

**Solution:**

iven $8x - 10y = -66$ ……(1)

$40x - 18y = 214$ ……(2)

Let (1) be the regression line of y on x and (2) be the regression line of x on y.

$\therefore 10y = 8x + 66 \Rightarrow y = \frac{8x}{10} + \frac{66}{10}$ $\therefore$ the regression coefficient of y on x is $b_1 = \frac{8}{10} = \frac{4}{5}$

$\therefore 40x = 18y + 214 \Rightarrow x = \frac{18y}{40} + \frac{214}{40}$ $\therefore$ the regression coefficient of x on y is $b_2 = \frac{18}{40} = \frac{9}{20}$

$\therefore b_1 b_2 = \left(\frac{4}{5}\right)\left(\frac{9}{20}\right) = \frac{9}{25} < 1$

Let r be the correlation between x and y.

$\therefore r = \sqrt{b_1 b_2} = \sqrt{\frac{9}{25}} = \frac{3}{5} = 0.6$ [Since both regression coefficients are positive, r is positive]

Let $(\bar{x}, \bar{y})$ be the point of intersection of the two regression lines.

Solving (1) and (2) we get $\bar{x}, \bar{y}$

5 x (1) $\Rightarrow$ $40x - 50y = -330$

$40x - 18y = 214$

Subtracting $-32y = -544$

$\therefore y = 17$

Now, $8x - 10y = -66 \Rightarrow 8x - 10(17) = -66 \Rightarrow 8x = 170 - 66 \Rightarrow 8x = 104 \Rightarrow x = 13$

$\therefore (\bar{x}, \bar{y}) = (13, 17)$ is the mean of X and Y.

We know, $\dfrac{\sigma_Y^2}{\sigma_X^2} = \dfrac{b_1}{b_2} \Rightarrow \sigma_Y^2 = \dfrac{b_1}{b_2}\sigma_X^2 \Rightarrow \sigma_Y^2 = \dfrac{\frac{4}{5}}{\frac{9}{20}}.(9) \quad \Rightarrow \sigma_Y^2 = 16$

$\therefore$ Variance of Y is 16

**(ii) The joint probability density function of a two dimensional random variable (X,Y) is given by $f(x,y) = xy^2 + \dfrac{x^2}{8}, 0 \le x \le 2, 0 \le y \le 1$.**

**Compute (i) P(X > 1), (ii) P(Y < ½ ), (iii) P(X < Y) (iv) Are X and Y independent?**

**Solution:**

The marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \left(xy^2 + \frac{x^2}{8}\right)dy = \left[x\frac{y^3}{3} + \frac{x^2}{8}y\right]_0^1$$

$$= \frac{x}{3} + \frac{x^2}{8} = \frac{x}{24}(8+3x), 0 \le x \le 2$$

The marginal pdf of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \left(xy^2 + \frac{x^2}{8}\right)dx = \left[\frac{x^2}{2}y^2 + \frac{x^3}{24}\right]_0^2 = 2y^2 + \frac{1}{3}, 0 \le y \le 1$$

(i) $\displaystyle\int_1^{\infty} f_X(x)\,dx = \int_1^2 \left(\frac{x}{3} + \frac{x^2}{8}\right)dx = \left[\frac{x^2}{6} + \frac{x^3}{24}\right]_1^2 = \frac{1}{6}(4-1) + \frac{1}{24}(8-1) = \frac{3}{6} + \frac{7}{24} = \frac{19}{24}$

$$P\left(Y < \frac{1}{2}\right) = \int_{-\infty}^{1/2} f_Y(y)\,dy = \int_0^{1/2}\left(2y^2 + \frac{1}{3}\right)dy = \left[\frac{2y^3}{3} + \frac{1}{3}\right]_0^{1/2}$$

$$= \frac{2}{3}.\frac{1}{8} + \frac{1}{3}.\frac{1}{2} = \frac{3}{12} = \frac{1}{4}$$

$$P(X < Y) = \int_0^1\int_0^y \left(xy^2 + \frac{x^2}{8}\right)dx\,dy = \int_0^1 \left(\frac{x^2y^2}{2} + \frac{x^3}{24}\right)_0^y dy$$

$$= \int_0^1 \left(\frac{y^4}{2} + \frac{y^3}{24}\right)dy = \left[\frac{y^5}{10} + \frac{y^4}{96}\right]_0^1 = \frac{1}{10} + \frac{1}{96} = \frac{53}{480}$$

Consider $f_X(x).\ f_Y(y) = \dfrac{x}{24}(8+3x).\ 2y^2 + \dfrac{1}{3} \ne f(x, y)$

$\therefore$ X and Y are not independent.

## UNIT – III RANDOM PROCESSES

### PART – A

**1. What is a random process? When do you say a random process is a random variable?**

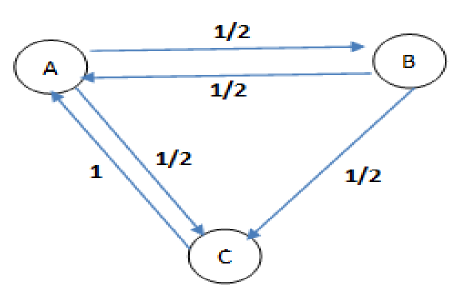
A random process is an infinite indexed collection of random variables { X(t) : t ε T }, defined over a common probability space. The index parameter t is typically time, but can also be a spatial dimension. Random processes are used to model random experiments that evolve in time such as Daily price of a stock. If the time ‘t’ is fixed, random process is called a random variable.

**2. Define (a) Continuous-time random process (b) Discrete state random process. (May 2011)**

Consider a random process {X(t), t$\in$T}, where T is the index set or parameter set. The values assumed by X(t) are called the states, and the set of all possible values of the state’s forms the state space E of the random process.

(a) If the state space E and index set T are both continuous, then the random process is called

| | |
|---|---|
| | continuous-time random process.<br>(b) If the state space E is discrete and the index set T is continuous, then the random process is called discrete state random process |
| **3.** | **Define Wide sense stationary process. (May 2012, 2013)**<br>A random process $\{X(t)\}$ is called wide-sense stationary if the following conditions hold:<br>$(i)\ E\left[X(t)\right] = \text{a constant}$<br>$(ii)\ R_{XX}(t_1,t_2) = E\left[X(t_1)X(t_2)\right] = R_{XX}(t_1 - t_2)$<br>= function of time difference. |
| **4.** | **Examine whether the Poisson process {X(t)} is stationary or not. (Dec2010, 2012, Apr 2015)**<br>A random process to be stationary in any sense, its mean must be a constant. We know that the mean of a Poisson process with rate $\lambda$ is given by $E\{X(t)\} = \lambda t$ which depends on the time t. Thus the Poisson process is not a stationary process. |
| **5.** | **What are the properties of Poisson process? (May 2012)**<br>(a)The Poisson process is not a Stationary process.<br>(b)The Poisson process is a Markov process.<br>(c) The sum of two independent Poisson process is again a Poisson process.<br>(d) The difference of two independent Poisson process is not a Poisson process. |
| **6.** | **Define a $k^{th}$ order stationary process. When will it become a strict sense stationary process? (April 2017)**<br>The process is said to be $k^{th}$ order stationary if all its finite dimensional distributions are invariant under translation of time for all $t_1, t_2, \ldots, t_n$ and for n=1,2,3……k only and not for all n>k, then the process is called the $k^{th}$ order stationary process. It becomes a strict sense stationary proess when $k \to \infty$. |
| **7.** | **Prove that first order stationary random process has a constant mean. (DEC 2013)**<br>Let X(t) be a first-order stationary process. Then the first-order probability density function of X(t) satisfies $f_X(x_1 ; t_1) = f_X(x_1 ; t_1 + \varepsilon)$ …..(A) for all $t_1$ and $\varepsilon$.<br>Now, consider any two time instants $t_1$ and $t_2$, and define the random variable $X_1 = X(t_1)$ and $X_2 = X(t_2)$. By definition, the mean values of $X_1$ and $X_2$ are given by<br>$E(X_1) = E\left[X(t_1)\right] = \int_{x_1=-\infty}^{\infty} x_1 f_X(x_1;t_1)\,dx_1 \quad \cdots(1)$<br>$E(X_2) = E\left[X(t_2)\right] = \int_{x_2=-\infty}^{\infty} x_2 f_X(x_2;t_2)\,dx_2 \quad \cdots(2)$<br>$Let\, t_2 = t_1 + \varepsilon \quad (2) \Rightarrow E\left[X(t_1+\varepsilon)\right] = \int_{x_2=-\infty}^{\infty} x_2 f_X(x_2;t_1+\varepsilon)\,dx_2$<br>$\text{Using (A)},\ E\left[X(t_1+\varepsilon)\right] = \int_{x_2=-\infty}^{\infty} x_2 f_X(x_2;t_1)\,dx_2$<br>$= \int_{x_1=-\infty}^{\infty} x_1 f_X(x_1;t_1)\,dx_1 = E\left[X(t_1)\right]$<br>which shows first order stationary random process has a constant mean. |
| **8.** | **Define transition probability matrix. (Dec 2011)**<br>The transition probability matrix (TPM) of the process $\{X_n, n \geq 0\}$ is defined by |

| | |
|---|---|
| | $P = [p_{ij}] = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots \\ p_{21} & p_{22} & p_{23} & \cdots \\ p_{31} & p_{32} & p_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$ <br> Where the transition probabilities (elements of P) satisfy $p_{ij} \geq 0$, & $\sum_{j=1}^{\infty} p_{ij} = 1,\ i = 1, 2, 3, \cdots$ |
| **9.** | **Define Markov process.** **(DEC 2011)** <br> A random process or Stochastic process X(t) is said to be a Markov process if given the value of X(t), the value of X(v) for v > t does not depend on values of X(u) for u < t. In other words, the future behavior of the process depends only on the present value and not on the past value. |
| **10.** | **Define Markov Chain. When do you say that a Markov chain is irreducible? (April 2013)** <br> A Markov process is called Markov chain if the states $\{X_i\}$ are discrete no matter whether t is discrete or continuous. The Markov chain is irreducible if all states communicate with each other at some time. |
| **11.** | **When do you say the Markov chain is regular? When do you say that state 'i' is periodic and aperiodic? (Dec 2013,April 2011)** <br> A regular Markov chain is defined as a chain having a transition matrix P such that for some power of P, it has only non-zero positive probability values. <br> Let A be the set of all positive integers n such that $p_{ii}^{(n)} > 0$ and 'd' be the Greatest Common Divisor of the set A. We say state 'i' is periodic if d>1 and aperiodic if d = 1. |
| **12.** | **When is a Markov chain, called homogeneous? (Dec 2010, 2015)** <br> If the one-step transition probability is independent of n, i.e., $p_{ij}(n, n+1) = p_{ij}(m, m+1)$ then the Markov chain is said to have stationary transition probabilities and the process is called as homogeneous Markov chain. |
| **13.** | **Is a Poisson process a continuous time Markov chain? Justify your answer (May2010)** <br> We know that Poisson process has the Markovian property. Therefore, it is a Markov chain as the states of Poisson process are discrete. Also, the time't' in a Poisson process is continuous. Therefore, the Poisson process a continuous time Markov chain. |
| **14.** | **Consider the Markov chain consisting of the three states 0, 1, 2 and transition probability matrix** $P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$. **Draw the state transition diagram. (Dec 2015)** <br>   |

**15. State Chapman-Kolmogrov Equation.** **(May 2017)**

The Chapman-Kolmogrov equation provides a method to compute the n-step transition probabilities. The equation can be represented as $P_{ij}^{n+m} = \sum_{k=0}^{\infty} p_{ik}^{n} p_{kj}^{m} \; \forall\, n, m \geq 0$.

**16. If the initial state probability distribution of a Markov chain is $P^{(0)} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix}$ and the transition probability matrix of the chain is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, find the probability distribution of the chain after 2 steps.** **(May 2012)**

Probability distribution after 2 steps = $P^{(2)} = P^{(1)}P = \{P^{(0)}P\}P$

Now $P^{(1)} = P^{(0)}P = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1/12 & 11/12 \end{pmatrix}$

$\therefore\ P^{(2)} = P^{(1)}P = \begin{pmatrix} \frac{1}{12} & \frac{11}{12} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 11/24 & 23/24 \end{pmatrix}$

**17. If the transition probability matrix of a Markov chain is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, find the steady state distribution of the chain.** **(May 2013)**

The steady state distribution of the chain is given by $A = \begin{bmatrix} a & b \end{bmatrix}$, where A.P = A

$$A.P = A \Rightarrow \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}$$

$$\Rightarrow b/2 = a \quad \cdots (1)$$

$$a + b/2 = b \quad \cdots (2)$$

We know that $a + b = 1 \quad \cdots (3)$

substituting (1) in (3),

we get 3b/2 = 1 ⇒ b = 2/3

$$\therefore\ (1) \Rightarrow a = 1/3$$

$\therefore$ Steady state distribution of the chain $= A = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$

**18. Consider the Markov chain consisting of the three states 0, 1, 2 and transition probability matrix P = $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$. Is it irreducible? Justify.** **(May 2010)**

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.25 & 0.25 \\ 0 & 0.33 & 0.67 \end{bmatrix} \quad \text{and} \quad P^2 = \begin{bmatrix} 0.5 & 0.375 & 0.125 \\ 0.375 & 0.395 & 0.23 \\ 0.165 & 0.304 & 0.531 \end{bmatrix}$$

Here $P_{ij}^{(n)} > 0, \forall\, i, j$. So, P is irreducible.

**19. Define Ergodic Random process.**

A random process {X(t)} is called ergodic if all its ensemble averages are equal to appropriate time averages.

**20. A radioactive source emits particles at a rate of 5 per min in accordance with Poisson process. Each particle emitted has a probability 0.6 of being recorded. Find the probability that 10 particles are recorded in 4 min period. (Dec 2014)**

The number of particles N(t) emitted is Poisson with parameter $\lambda = np = 5(0.6) = 3$

$$P(N(t)=m)=\frac{e^{-3t}(3t)^m}{m!}$$

$$P(N(4)=10)=\frac{e^{-3(4)}(3(4))^{10}}{10!}$$

$$=0.1048.$$

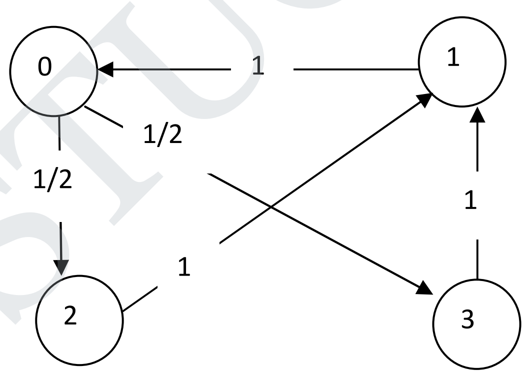
**PART – B**

**1. i) Let the Markov Chain consisting of the states 0, 1, 2, 3 have the transition probability matrix $P=\begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. Determine which states are transient and which are recurrent by defining transient and recurrent states. (MAY 2010)**

**Solution:**

Transient state: A state ‘a’ is transient if $F_{aa} < 1$.Recurrent state: A state ‘a’ is recurrent if $F_{aa}=1$.

Here $F_{aa}=\sum_{n=0}^{\infty} f_{aa}^{(n)}$, where $f_{aa}^{(n)}$ = first time return probability of state 'a' after n steps.




$$\text{Here } P_{00}^3>0,\ P_{01}^2>0,\ P_{02}^1>0,\ P_{03}^1>0$$

$$P_{10}^1>0,\ P_{11}^3>0,\ P_{12}^2>0,\ P_{13}^2>0$$

$$P_{20}^2>0,\ P_{21}^1>0,\ P_{22}^3>0,\ P_{23}^3>0$$

$$P_{30}^2>0,\ P_{31}^1>0,\ P_{32}^3>0,\ P_{33}^3>0$$

Therefore, the Markov chain is irreducible. And also it is finite.

So, all the states are of same nature.

Consider the state ‘0’

$f_{00}^{1}=0;\ \ f_{00}^{2}=0;\ \ f_{00}^{3}=\frac{1}{2}+\frac{1}{2}=1;\ \ f_{00}^{4}=0$ and so on.

Therefore, the state '0' is recurrent.
Since, the chain is irreducible, all the states are recurrent.

**ii)** **The transition probability matrix of a Markov chain { X(t)}, n = 1,2,3,….} having three states 1, 2 and 3 is** $\mathbf{P}=\begin{bmatrix}0.1 & 0.5 & 0.4\\ 0.6 & 0.2 & 0.2\\ 0.3 & 0.4 & 0.3\end{bmatrix}$ **and the initial distribution is** $\mathbf{P^{(0)}=(0.7\quad 0.2\quad 0.1)}$**.Find (i)** $\mathbf{P[X_2=3]}$ **(ii)** $\mathbf{P[X_3=2, X_2=3, X_1=3, X_0=2]}$**.**

**(May 2012,2014, Dec 2013)**

**Solution:**

We have $P^2=P.P=\begin{pmatrix}0.43 & 0.31 & 0.26\\ 0.24 & 0.42 & 0.34\\ 0.36 & 0.35 & 0.29\end{pmatrix}$

(I) $P(X_2=3)=\sum_{i=1}^{3}P(X_2=3/X_0=i)P(X_0=i)$

$=P(X_2=3/X_0=1)P(X_0=1)+P(X_2=3/X_0=2)P(X_0=2)\ \ +P(X_2=3/X_0=3)P(X_0$

$=P_{13}^2P(X_0=1)+P_{23}^2P(X_0=2)+P_{33}^2P(X_0=3)=0.26\times0.7+0.34\times0.2+0.29\times0.1=0.279$

(II) $P(X_3=2, X_2=3, X_1=3, X_0=2)$

$=P[X_0=2, X_1=3, X_2=3]P[X_3=2/X_0=2, X_1=3, X_2=3]$

$=P[X_0=2, X_1=3, X_2=3]P[X_3=2/X_2=3]$

$=P[X_0=2, X_1=3]P[X_2=3/X_0=2, X_1=3]P[X_3=2/X_2=3]$

$=P[X_0=2, X_1=3]P[X_2=3/X_1=3]P[X_3=2/X_2=3]$

$=P[X_0=2]P[X_1=3/X_0=2]P[X_2=3/X_1=3]P[X_3=2/X_2=3]$

$=(0.2)(0.2)(0.3)(0.4)=0.0048$

**iii)** **Three boys A,B and C are throwing a ball to each other . A always throws the ball to B and B always throws the ball to C , but C is just likely to throw the ball to B as to A .Show that the process is Markovian. Find the transition matrix and classify the states.** **(April 2017)**

**Solution:**

**TPM=** $P=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0.5 & 0.5 & 0\end{bmatrix}$ Since states of $X_n$ depend only on states of $X_{n-1}$ .{ $X_n$} is a Markov chain.

Now $P^2=\begin{bmatrix}0 & 0 & 1\\ 0.5 & 0.5 & 0\\ 0 & 0.5 & 0.5\end{bmatrix}$; $P^3=\begin{bmatrix}0.5 & 0.5 & 0\\ 0 & 0.5 & 0.5\\ 0.25 & 0.25 & 0.5\end{bmatrix}$

Therefore $P_{11}^{(3)}>0$, $P_{13}^{(2)}>0$, $P_{21}^{(2)}>0$, $P_{22}^{(2)}>0$, $P_{33}^{(2)}>0$ and all other $P_{ij}(1)>0$.
Therefore chain is irreducible.

$$P^4 = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \\ 0.25 & 0.5 & 0.25 \end{bmatrix}; P^5 = \begin{bmatrix} 0.25 & 0.25 & 0.5 \\ 0.25 & 0.5 & 0.25 \\ 0.125 & 0.375 & 0.5 \end{bmatrix}$$ and so on.

We note that $P_{ii}^{(2)}, P_{ii}^{(3)}, P_{ii}^{(5)}$ …. are greater than zero for i=2,3,.. and GCD of 2,3,5,6,….=1

Therefore the state 1 is aperiodic. Since the chain is finite and irreducible , all its states are non–null Persistant.

**iv)** **Let $\{X_n : n \geq 0\}$ be a Markov chain having state space $S = \{1, 2, 3\}$ and one step TPM**

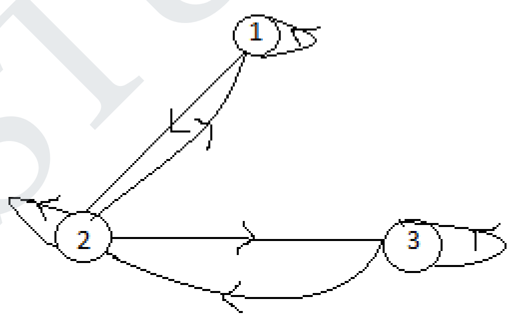
$$\mathbf{P} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

**(i) Draw a transition diagram for this chain.**
**(ii) Is the chain irreducible? Explain.**
**(iii) Is the state 3 ergodic? Explain.** **(April/May 2019)**
**Solution:**
The TPM of the given Markov chain is

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix} \end{array}$$

(i) Let $X_n = \{1, 2, 3\} \Rightarrow$ finite

(ii)




State 1 and state 2 are communicate with each other.
State 2 and state 3 are communicate with each other.
State 1 and state 3 are not communicate with each other.
The Markov chain is not irreducible.

(iii) The Markov chain is not irreducible.
The states are not non-null persistent.
Hence the chain is not Ergodic.

**2. i) On the average a submarine on patrol sights 6 enemy ships per hour. Assuming that the number of ships sighted in a given length of time is a Poisson variate. Find the probability of sighting 6 ships in the next half-an-hour , 4 ships in the next 2 hours and at lest 1 ship in the next 15 minutes. (April 2017)**

**Solution:**

Mean arrival rate = mean number of arrivals per minute (unit time) = $\lambda$=6/hr

Given $\lambda$ = 6. $P\{X(t)=k\}=\dfrac{e^{-\lambda t}(\lambda t)^k}{\lfloor k}$

(i) $P\left\{X\left(\frac{1}{2}\right)=6\right\}=\dfrac{e^{-3}3^6}{\lfloor 6}=0.0504$

(ii) $P\{X(2)=4\}=\dfrac{e^{-12}(12)^4}{\lfloor 4}=0.053$

(iii) $P\left\{X\left(\frac{1}{4}\right)\geq 1\right\}=1-P\left\{X\left(\frac{1}{4}\right)=0\right\}1-\left\{\dfrac{e^{-\frac{3}{2}}\left(\frac{3}{2}\right)^0}{\lfloor 0}\right\}=1-e^{-\frac{3}{2}}=0.776.$

**ii) Prove that (i) difference of two independent Poisson processes is not a Poisson process and (ii) Poisson process is a Markov process. (MAY 2013)**

(i) Let $X(t)=X_1(t)-X_2(t)$ where $X_1(t)$ and $X_2(t)$ are Poisson processes with $\lambda_1$ and $\lambda_2$ as the parameters

$$E[X(t)]=E[X_1(t)]-E[X_2(t)]=(\lambda_1-\lambda_2)t$$

$$E[X^2(t)]=E\left\{[X_1(t)-X_2(t)]^2\right\}=E[X_1^2(t)]+E[X_2^2(t)]-2E[X_1(t)X_2(t)]$$

$$=E[X_1^2(t)]+E[X_2^2(t)]-2E[X_1(t)]E[X_2(t)]$$

$$=(\lambda_1^2t^2+\lambda_1t)+(\lambda_2^2t^2+\lambda_2t)-2(\lambda_1t)(\lambda_2t)=(\lambda_1+\lambda_2)t+(\lambda_1^2+\lambda_2^2)t^2-2\lambda_1\lambda_2t^2$$

$$=(\lambda_1+\lambda_2)t+(\lambda_1-\lambda_2)^2t^2\neq(\lambda_1-\lambda_2)t+(\lambda_1-\lambda_2)^2t^2$$

$\therefore X_1(t)-X_2(t)$ is not a Poisson process.

(ii)

Consider $P[X(t_3)=n_3/X(t_2)=n_2,X(t_1)=n_1)]=\dfrac{P[X(t_1)=n_1,X(t_2)=n_2,X(t_3)=n_3]}{P[X(t_1)=n_1,X(t_2)=n_2]}$

$$=\dfrac{\dfrac{e^{-\lambda t_3}\lambda^{n_3}t_1^{n_1}(t_2-t_1)^{n_2-n_1}(t_3-t_2)^{n_3-n_2}}{n_1!(n_2-n_1)!(n_3-n_2)!}}{\dfrac{e^{-\lambda t_2}\lambda^{n_2}t_1^{n_1}(t_2-t_1)^{n_2-n_1}}{n_1!(n_2-n_1)!}}=\dfrac{e^{-\lambda(t_3-t_2)}\lambda^{n_3-n_2}(t_3-t_2)^{n_3-n_2}}{(n_3-n_2)!}$$

$$=P[X(t_3)=n_3/X(t_2)=n_2]$$

$$\therefore P[X(t_3=n_3/X(t_2)=n_2,X(t_1)=n_1)]=P[X(t_3)=n_3/X(t_2)=n_2]$$

This means that the conditional probability distribution of $X(t_3)$ given all the past values $X(t_1)=n_1$, $X(t_2)=n_2$ depends only on the most recent values $X(t_2)=n_2$.

i.e., The Poisson process possesses Markov property.

**3. i)** **A salesman territory consists of three cities A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day he sells in city B. However if he sells in either city B or city C, the next day he is twice as likely to sell in city A as in the other city. In the long run how often does he sell in each of the cities? (MAY 2012,DEC 2013)**

**Solution:**

States: A, B and C

The transition probability matrix is given by $P = \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix}$

The long run probability is given by $\pi = \begin{bmatrix} a & b & c \end{bmatrix}$, where $\pi P = \pi$.

Now $\pi P = \pi \Rightarrow \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}$

$\therefore\ 0.a + \frac{2}{3}b + \frac{2}{3}c = a \quad \Rightarrow \quad -a + \frac{2}{3}b + \frac{2}{3}c = 0 \cdots (1)$

$1.a + 0.b + \frac{1}{3}c = b \quad \Rightarrow \quad a - b + \frac{1}{3}c = 0 \cdots (2)$

$0.a + \frac{1}{3}b + 0.c = c \quad \Rightarrow \quad \frac{1}{3}b - c = 0 \cdots (3)$

Also, we know that $a + b + c = 1 \cdots (4)$

From (3), $c = b/3$

From (2), $a - b + b/9 = 0 \Rightarrow a = 8b/9$

From (4), $8b/9 + b + b/3 = 1 \Rightarrow 20b/9 = 1 \Rightarrow \boxed{b = 9/20}$

$\therefore \boxed{c = 3/20 \text{ and } a = 8/20}$

$\therefore$ Long run probability $= \begin{bmatrix} 8/20 & 9/20 & 3/20 \end{bmatrix}$

**ii)** **A gambler has Rs. 2. He bets Re. 1 at a time and wins Re. 1 with probability ½. He stops playing if he loses Rs. 2 or wins Rs. 4. i) What is the tpm of the related Markov chain? ii) What is the probability that he has lost his money at the end of 5 plays?**

**(MAY 2013)**

**Solution:**

Let $X_n$ denote the amount with the player at the end of the $n^{th}$ round of the play.

The possible values of $X_n$ = State space = {0, 1, 2, 3, 4, 5, 6}

Initial probability distribution= $P^{(0)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

(i) The transition probability matrix is =

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

(ii) Probability that he has lost his money at the end of 5 plays = $P\left[X_5 = 0\right]$

To find this we need $P^{(5)}$

$$P^{(1)} = P^{(0)}P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$

$$P^{(2)} = P^{(1)}P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \end{pmatrix}$$

$$P^{(3)} = P^{(2)}P = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{3}{8} & 0 & \frac{1}{8} & 0 \end{pmatrix}$$

$$P^{(4)} = P^{(3)}P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{3}{8} & 0 & \frac{1}{8} & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \frac{3}{8} & 0 & \frac{5}{16} & 0 & \frac{1}{4} & 0 & \frac{1}{16} \end{pmatrix}$$

$$P^{(5)} = P^{(4)}P = \begin{pmatrix} \frac{3}{8} & 0 & \frac{5}{16} & 0 & \frac{1}{4} & 0 & \frac{1}{16} \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \frac{3}{8} & \frac{5}{32} & 0 & \frac{9}{32} & 0 & \frac{1}{8} & \frac{1}{16} \end{pmatrix}$$

$$\therefore \boxed{P(X_5 = 0) = 3/8}$$

**4. i)** **Show that the random process $X(t) = A\cos(w_0 t + \theta)$ is wide-sense stationary, if A and $w_0$ are constants and $\theta$ is uniformly distributed random variable in $(0,2\pi)$ (DEC 2011,MAY 2014,APR 2017)**

**Solution:**

Since $\theta$ is uniformly distributed in $(0,2\pi)$ p.d.f. is $f(\theta) = \frac{1}{2\pi}\ (0 \le 0 \le 2\pi)$

$$E[X(t)] = \int_0^{2\pi} \frac{1}{2\pi} A\cos(\omega_0 t + \theta)\, d\theta = \frac{A}{2\pi}\left[\sin(\omega_0 t + \theta)\right]_0^{\pi} = \frac{A}{2\pi}\left[\sin(\omega_0 t + 2\pi) - \sin(\omega_0 t)\right]$$

$$= \frac{A}{2\pi}\left[\sin\omega_0 t - \sin\omega_0 t\right] = 0$$

$$R(t_1,t_2) = E\left[X(t_1)X(t_2)\right] = E\left[A\cos(\omega_0 t_1+\theta)A\cos(\omega_0 t_2+\theta)\right]$$

$$= E\left[A^2\cos(\omega_0 t_1+\theta)\cos(\omega_0 t_2+\theta)\right] = \frac{A^2}{2}E\left[\cos\left[\omega_0(t_1+t_2)+2\theta\right]+\cos\omega_0(t_1-t_2)\right]$$

$$= \frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}\left[\cos\left(\omega_0(t_1+t_2)+2\theta\right)+\cos\left(\omega_0(t_1-t_2)\right)\right]d\theta$$

$$= \frac{A^2}{4\pi}\left\{\left[\frac{\sin\left(\omega_0(t_1+t_2)+2\theta\right)}{2}\right]_0^{2\pi} + \cos\left(\omega_0(t_1-t_2)\right)[\theta]_0^{2\pi}\right\}$$

$$= \frac{A^2}{4\pi}\left\{\frac{\sin\left(\omega_0(t_1+t_2)+4\pi\right)}{2} - \frac{\sin\left(\omega_0(t_1+t_2)\right)}{2} + \cos\left(\omega_0(t_1-t_2)\right)(2\pi)\right\}$$

$$= \frac{A^2}{4\pi}\left[\frac{\sin\omega_0(t_1+t_2)}{2} - \frac{\sin\omega_0(t_1+t_2)}{2} + 2\pi\cos\omega_0(t_1-t_2)\right]$$

$= \frac{A^2}{4\pi}.2\pi\cos\omega_0(t_1-t_2) = \frac{A^2}{2}\cos\omega_0(t_1-t_2)$ =a function of $t_1-t_2$

$\therefore$ The process $X(t)$ is W.S.S.

**ii)** **A random process has sample function of the form $X(t) = A\cos(w_0 t+\theta)$ where $w_0$ is a constant. 'A' is a R.V that has a magnitude of +1 and -1 with equal probability and $\theta$ is a R.V that is uniformly distributed over $[0, 2\pi]$. Assume that the random variable A and θ are independent. Is X(t) a wide-sense stationary process? (April/ May 2019)**

**Solution:**

$X(t) = A\cos(w_0 t+\theta)$

| | | |
|---|---|---|
| A | 1 | -1 |
| P(A) | 1/2 | 1/2 |

$$E(A) = \sum_a ap(a) = 1\times\frac{1}{2}+(-1)\times\frac{1}{2} = 0$$

$$E(A^2) = \sum_a a^2 p(a) = 1^2\times\frac{1}{2}+(-1)^2\times\frac{1}{2} = 1$$

$$E[X(t)] = E[a\cos(\omega_0 t+\theta)] = E[A]E(\cos(\omega_0 t+\theta)) = 0. \quad \because E(A) = 0$$

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

$$= E[A^2\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)]$$

$$= E(A^2)E[\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)]$$

$$= (1)\ E\left[\frac{\cos(\omega_0 t+\theta+\omega_0 t+\omega_0\tau+\theta)+\cos(\omega_0 t+\theta-\omega_0 t-\omega_0\tau-\theta)}{2}\right]$$

$$= \frac{1}{2}E[\cos(2\omega_0 t+\omega_0\tau+2\theta)] + \frac{1}{2}E[\cos(-\omega_0\tau)] ----------(1)$$

$$E[\cos(2\omega_0 t+\omega_0\tau+2\theta)]=\int_0^{2\pi}\cos(\cos(2\omega_0 t+\omega_0\tau+2\theta)\frac{1}{2\pi}d\theta$$

$$=0. \quad \because \int_0^{2\pi}\cos(\cos(\omega t+n\theta)d\theta=0,\ n\,is\ an\ integer,\ n\neq 0$$

From (1),

$R_{XX}(\tau)=0+\frac{1}{2}\cos(\omega_0\tau)=\frac{1}{2}\cos(\omega_0\tau)=a$ function of $t$. $\therefore$ The process is W.S.S.

**iii)** **A random process is given as $X(t)=U+V\cos(wt+\theta)$ where U is a R.V with $E(U)=0$ and $Var(U)=3$, V is a R.V with $E(V)=0$ and $Var(V)=4$, $\omega$ is a constant and $\theta$ is a R.V with p.d.f $f(\theta)=\frac{1}{2\pi}$, $\frac{-\pi}{2}\le\theta\le\frac{3\pi}{2}$. It is further assumed that U,V and $\theta$ are independent R.Vs. Is the process $X(t)$ stationary in the wide-sense? (April 2019)**

**Solution:**

$X(t)=U+V\cos(wt+\theta)$

$E(U)=0\ \&\ Var(U)=3;\ E(v)=0\ \&\ Var(V)=4$

$E(UV)=E(U)E(V)=0 \quad \because U\ \&\ V$ *are independent.*

$Var(U)=E(U^2)-(E(U))^2$

$3=E(U^2)-0\Rightarrow \quad E(U^2)=3$

$ll^{ly}\ E(V^2)=4$

$E[X(t)]=E[U+V\cos(\omega t+\theta)]$

$E[X(t)]=E(U)+E(V)\cos(\omega t+\theta)=0+0=0$

$R_{XX}(\tau)=E[X(t)X(t+\tau)]$

$=E\{U+V\cos(\omega t+\theta)\ \ U+V\cos(\omega(t+\tau)+\theta)\}$

$=E\{U^2+UV\cos(\omega t+\theta)\ +UV\cos(\omega(t+\tau)+\theta)+V^2\cos(\omega t+\theta)\cos(\omega(t+\tau)+\theta)\}$

$=E(U^2)+E(UV)\cos(\omega t+\theta)\ +E(UV)\cos(\omega(t+\tau)+\theta)$

$+E(V^2)\cos(\omega t+\theta)\cos(\omega(t+\tau)+\theta)$

$=3+0+0+4E(\cos(\omega t+\theta)\cos(\omega(t+\tau)+\theta))$

$=3+4E\left[\frac{1}{2}\{\cos(\omega t+\theta+\omega t+\omega\tau+\theta)+\cos(\omega t+\theta-\omega t-\omega\tau-\theta)\}\right]$

$=3+\frac{4}{2}E[c\,os(2\omega t+\omega\tau+2\theta)+\cos(-\omega\tau)]$

$=3+2\int_{-\frac{\pi}{2}}^{\frac{-3\pi}{2}}\cos(2\omega t+\omega\tau+2\theta)\frac{1}{2\pi}d\theta+2\cos\omega\tau------(1)$

$$2\int_{-\frac{\pi}{2}}^{\frac{-3\pi}{2}}\cos(2\omega t+\omega\tau+2\theta)\frac{1}{2\pi}d\theta=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{-3\pi}{2}}\cos(2\omega t+\omega\tau+2\theta)d\theta=\frac{1}{\pi}\left[\frac{\sin(2\omega t+\omega\tau+2\theta)}{2}\right]_{-\frac{\pi}{2}}^{\frac{-3\pi}{2}}$$

$$=\frac{1}{2\pi}[\sin(2\omega t+\omega\tau+3\pi)-\sin(2\omega t+\omega\tau-\pi)]=\sin(2\omega t+\omega\tau)-in(2\omega t+\omega\tau)=0$$

Hint : $\sin(n\pi+\theta)=(-1)^n\sin\theta;\ \sin(n\pi-\theta)=-(-1)^n\sin\theta$

| | | |
|---|---|---|
| | | From (1), $R_{XX}(\tau) = 3 + 0 + 2\cos\omega\tau = 3 + 2\cos\omega\tau.$ $\therefore$ The process is $W.S.S.$ |
| 5. | i) | **A man either drives a car (or) catches a train to go to office each day. He never goes two days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair die and drove to work if and only if a 6 appeared. Find (1) the probability that he takes a train on the third day and (2) the probability that he drives to work in the long run. (DEC 2011)** |

**Solution:**

i) Here train (T) and car(C) are the states. The tpmis $P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Initial state probability distribution $P^{(1)} = \left[\frac{5}{6}, \frac{1}{6}\right]$,

as P[traveling by car]=P[getting 6]=$\frac{1}{6}$; P[traveling by train]=$\frac{5}{6}$

$$P^{(2)} = P^1 P = \left[\frac{5}{6}, \frac{1}{6}\right]\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \left[\frac{1}{12}, \frac{11}{12}\right]; P^{(3)} = P^2 P = \left[\frac{1}{12}, \frac{11}{12}\right]\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \left[\frac{11}{24}, \frac{13}{24}\right]$$

Probability that the man travels by train on 3rd day $= \frac{11}{24}$

ii) Let $\pi = (\pi_1, \pi_2)$ be the stationary state distribution of the Markov chain. By property of $\pi P = \pi$

$$(\pi_1, \pi_2)\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = (\pi_1, \pi_2) \Rightarrow \frac{1}{2}\pi_2 = \pi_1 \text{ and } \pi_1 + \frac{1}{2}\pi_2 = \pi_2 \Rightarrow 2\pi_1 = \pi_2$$

Also $\pi_1 + \pi_2 = 1$ ( $\because$ Since $\pi$ is the probability distribution )

$\therefore \pi_1 + 2\pi_1 = 1 => \pi_1 = \frac{1}{3}$ $\therefore \pi_2 = \frac{2}{3}$; $\therefore$ Probability that he travels by car in the long run $= \frac{2}{3}$.

| | | |
|---|---|---|
| | ii) | **The process {X(t)} whose probabilitydistribution under certain condition is given by** $P[X(t) = n] = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1, 2, 3, \cdots \\ \frac{at}{1+at}, & n = 0. \end{cases}$ **Show that $\{X(t)\}$ is not stationary. (MAY 2012, DEC 2013)** |

**Solution:**

$$\text{Given } P[X(t) = n] = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1,2,3,\cdots \\ \frac{at}{1+at}, & n = 0. \end{cases}$$

$$E[X(t)] = \sum_{n=0}^{\infty} nP\{X(t) = n\} = 0 + 1P\{X(t) = 1\} + 2P\{X(t) = 2\} + 3P\{X(t) = 3\} + \cdots$$

$$=1\left(\frac{1}{(1+at)^2}\right)+2\left(\frac{at}{(1+at)^3}\right)+3\left(\frac{(at)^2}{(1+at)^4}\right)+\cdots$$

$$=\frac{1}{(1+at)^2}\left[1+2\frac{at}{1+at}+3\left(\frac{at}{1+at}\right)^2+\cdots\right]=\frac{1}{(1+at)^2}\left[1-\frac{at}{1+at}\right]^{-2}$$

$$=\frac{1}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-2}=1=a\text{ constant}$$

$$E\left[X^2(t)\right]=\sum_{n=0}^{\infty}n^2P\{X(t)=n\}=\sum_{n=0}^{\infty}\{n(n+1)-n\}P\{X(t)=n\}$$

$$=\sum_{n=0}^{\infty}n(n+1)P\{X(t)=n\}-nP\{X(t)=n\}$$

$$=\sum_{n=0}^{\infty}n(n+1)P\{X(t)=n\}-1 \qquad \left\{\because \sum_{n=0}^{\infty}nP\{X(t)=n\}=1\right\}$$

$$=\left[0+1.2P\{X(t)=1\}+2.3P\{X(t)=2\}+3.4P\{X(t)=3\}+\cdots\right]-1$$

$$=\left\{2\left(\frac{1}{(1+at)^2}\right)+6\left(\frac{at}{(1+at)^3}\right)+12\left(\frac{(at)^2}{(1+at)^4}\right)+\cdots\right\}-1$$

$$=\frac{2}{(1+at)^2}\left[1+3\frac{at}{1+at}+4\left(\frac{at}{1+at}\right)^2+\cdots\right]-1=\frac{2}{(1+at)^2}\left[1-\frac{at}{1+at}\right]^{-3}-1$$

$$=\frac{1}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-3}-1=1+at-1=at,\text{ not a constant}$$

So, X(t) is not a stationary process.

## UNIT – IV QUEUEING MODELS

### PART – A

**1.** **Write down the Kendall's notations for queueing model.**

**(OR) What do the letters in the symbolic representation *(a / b / c) : (d / e)* of a queuing model represent?** **(April/May'15), (April/May'11) ,(Nov/Dec'11)**

$(a/b/c):(d/e)$ is the Kendall's notation.

$a \to$ Arrival Pattern

$b \to$ Service pattern

$c \to$ No. of servers

$d \to$ Capacity of the system

$e \to$ Service discipline.

**2.** **What are the basic characteristics of queueing system?** **(May/June'13)( April/May'19)**

The basic characteristics of a queueing system are

1. Arrival pattern of customer
2. Service pattern of customer
3. Number of service channels
4. System capacity
5. Queue discipline

**3. Define Balking and Reneging of the customers in the queueing system.**
**(April/May'15) (May/June2016)**

Balking: It is queue behavior where in the customer does not enter the queue due to some unwillingness.
Reneging: It refers to the queue behavior where in the customer leave a queue without receiving service after waiting for some time due to impatience.

**4. Which queue is called to be the queue with discouragement?**

If a customer is discouraged to join the queue expecting a long waiting time or having the impatience in getting the service, the queuing model is said to be the queue with discouragement.

**5. What do you mean by transient state and steady state queuing system? (May/June'12)**

A queuing system is in transient state when its operating characteristics are depending on time. It is in steady state when the characteristics are independent of time.

**6. Define Markovian queueing models. (Nov/Dec'13), (May/June'12)**

Queueing models in which both inter arrival time and service time which are exponentially distributed are called Markovian Queueing models

**7. State the relationship between expected number of customers in the queue and in the system. (Nov/Dec'14)**

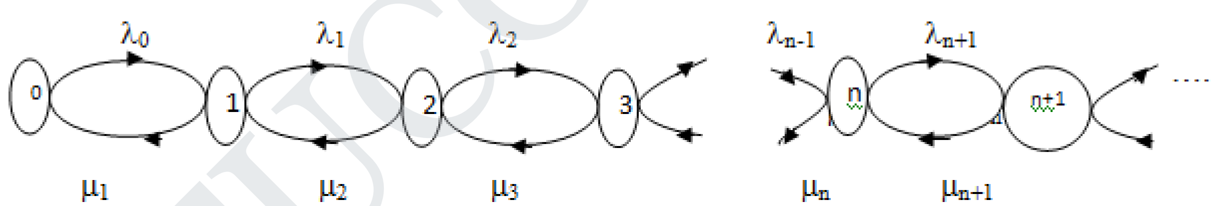
The relationship between expected number of customers in the queue and in the system is

$$L_s = \frac{\lambda}{\mu} + L_q$$

**8. What is the steady state condition for $M/M/C$ queuing model? (Nov/Dec'14)**

The traffic intensity must be less than 1 $(i.e)\ \frac{\lambda}{\mu c} < 1$

**9. Draw the state transition diagram for M/M/1 queueing model. (Nov/Dec'11)**




This model is based on the birth and death process. Arrivals are births and departures are deaths. When the system is in state $n$ if an arrival comes, it goes to state $n+1$ and if a departure is there then the system goes to state $n$-1. The number of customers in the system is the state of the system.

**10. What is the probability that a customer has to wait more than 15 min. to get his service completed in $(M/M/1):(\infty/FIFO)$ queue system if $\lambda = 6/hr$ and $\mu = 10/hr$? (May/June'13)**

$$P(w_s > 15\,\text{min}) = P\left(w_s > \frac{1}{4} hr\right)$$

Where $w_s$ is exponential random variable with parameter $\mu - \lambda$.

$$P(w_s > 15\,\text{min}) = \ = \int_{1/4}^{\infty} (\mu-\lambda)e^{-(\mu-\lambda)\omega}\,d\omega$$

$$= \left[-e^{-(\mu-\lambda)\omega}\right]_{1/4}^{\infty}$$

$$= e^{\frac{-(\mu-\lambda)}{4}} = e^{\frac{-(10-6)}{4}} = e^{-1}$$

| | |
|---|---|
| **11.** | **Suppose that customers arrive at a poisson rate of one per every 12 minutes and that the service is exponential at a rate of one service per 8 minutes.**<br>**a) What is the average number of customers in the system?**<br>**b) What is the average time of a customer spends in the system? (Nov/Dec'13)**<br>This problem is of the model $(M/M/1):(\infty/FCFS)$<br>Arrival rate $\lambda = \frac{1}{12}$ per min $\Rightarrow \lambda = \frac{60}{12}/hr \Rightarrow \lambda = 5$/hr<br>Service rate $\mu = \frac{1}{8}$ per min $\Rightarrow \mu = \frac{60}{8}/hr \Rightarrow \mu = 7.5$/hr<br>a) The average number of customers in the system is $L_s = \frac{\lambda}{\mu-\lambda} = \frac{5}{7.5-5} = 2$ customers<br>b) The average time, a customer spends in the system is $W_s = \frac{1}{\mu-\lambda} = \frac{1}{7.5-5} = \frac{1}{2.5} = 0.4$ hrs |
| **12.** | **Arrival rate of telephone calls at a telephone booth is according to Poisson distribution with an average time of 9 minutes between two consecutive arrivals. The length of a telephone call is assumed to be exponentially distributed with mean 3 minutes. Determine the probability that a person arriving at the booth will have to wait.**<br>This problem is under the model $(M/M/1):(\infty/FIFO)$<br>Arrival rate $\lambda = \frac{1}{9}$ customers per minute<br>Service rate $\mu = \frac{1}{3}$ customers per minute<br>$\therefore$ The probability that the booth is empty = The probability that there is no customer in the booth<br>$\therefore P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{3}{9} = \frac{2}{3}$<br>The probability that the person arriving at the booth will have to wait = $= 1 - P_0 = 1 - \frac{2}{3} = \frac{1}{3}$ |
| **13.** | **In a railway marshaling yard, goods trains arrive at a rate of 30 trains per day: Assume that the inter arrival time follows an exponential distribution and the service time distribution is also exponential with an average of 36 minutes. Calculate the probability that the yard is empty. (Nov/Dec'11)**<br>This problem is under the model $(M/M/1):(\infty/FCFS)$<br>Arrival rate $\lambda = 30$ trains per day<br>Service rate $\mu = \frac{1}{36}$ per min<br>$\Rightarrow \mu = \frac{60\times 24}{36}/day \Rightarrow \mu = 40$ trains / day<br>$\therefore$ The probability that the yard is empty = The probability that there is no train in the yard<br>$\therefore P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{30}{40} = \frac{1}{4}$ |
| **14.** | **In an $(M/M/1):(\infty/FIFO)$ queueing model if $\lambda = 8/hr$, $\mu = 10/hr$, what is the average waiting time in the queue?**<br>$\lambda = 8/hr$, $\mu = 10/hr$<br>$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{8}{10(10-8)} = 0.4$ hr |

**15.** **Find the traffic intensity for an $M/M/C$ queue with $\lambda = 10$ per hour, $\mu = 15$ per hour and two servers.** **(April/May 11)**

For $M/M/C$ model, the traffic intensity is $\rho = \dfrac{\lambda}{\mu c} = \dfrac{10}{15\times 2} = \dfrac{10}{30} = \dfrac{1}{3}$

**16.** **Write down Little's formula for the queuing model $(M/M/S):(\infty/FCFO)$** **(April/May'2017)**

$$L_s = L_q + \frac{\lambda}{\mu};\quad L_q = L_s - \frac{\lambda}{\mu};\quad W_s = \frac{L_s}{\lambda};\quad W_q = \frac{L_q}{\lambda}.$$

**17.** **What is the effective arrival rate for $(M/M/1):(4/FCFS)$ queueing model?**

The effective arrival rate is $\lambda' = \mu(1-P_0)$, where
$$P_0 = \begin{cases} \dfrac{1}{4+1} & \text{if } \lambda = \mu \\[2ex] \dfrac{1-\dfrac{\lambda}{\mu}}{1-\left(\dfrac{\lambda}{\mu}\right)^{4+1}} & \text{if } \lambda \neq \mu \end{cases}$$

**18.** **What is the probability that an arrival to an infinite capacity 3 server Poisson queue, with $\dfrac{\lambda}{c\mu} = \dfrac{2}{3}$ and $p_0 = \dfrac{1}{9}$ enters the service without waiting?**

Arriving customer shall enter the service without waiting if

No. of customer in the system $\leq$ No. of server (=3)

$$\therefore\ P(n<3) = P_0 + P_1 + P_2 = P_0 + \frac{1}{1}\frac{\lambda}{\mu}P_0 + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 P_0 \qquad \therefore\ P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$$

Given $\dfrac{\lambda}{c\mu} = \dfrac{2}{3} \Rightarrow \dfrac{\lambda}{3\mu} = \dfrac{2}{3} \Rightarrow \dfrac{\lambda}{\mu} = 2$ and $c = 3$ $\quad \therefore\ P(n<3) = \dfrac{1}{9} + 2.\dfrac{1}{9} + \dfrac{1}{2}4\dfrac{1}{9} = \dfrac{5}{9}$

**19.** **What are the values of $P_0$ & $P_n$ for the queueing model $(M/M/1):(K/FIFO)$ when $\lambda = \mu$?**

When $\lambda = \mu \ \Rightarrow\ P_0 = P_n = \dfrac{1}{K+1}$. **(April/May'17)**

**20.** **In an $(M/M/2):(5/FIFO)$ queueing model if $\lambda = 3$, $\mu = 4$, and $P_0$=0.4564 find the effective arrival rate $\lambda'$.**

Given : $\lambda = 3,\ \mu = 4$ and $P_0$=0.4564

Number of servers $c = 2$

Capacity = $k = 5$

Effective arrival rate

$$\lambda' = \mu\left[c - \sum_{n=0}^{c-1}(c-n)P_n\right]$$

$$= 3\left[2 - \sum_{n=0}^{1}(2-n)P_n\right]$$

$$\lambda' = 3\left[2 - \left(2P_0 + P_1\right)\right] \text{ where } P_1 = \left(\frac{\lambda}{\mu}\right)P_0$$

here $\dfrac{\lambda}{\mu} = \dfrac{3}{4} = 0.75$

$$\lambda' = 3\left[2 - (2\times 0.4564) + (0.75\times 0.4564)\right] = 3(1.4295) = 4.2885$$

**PART – B**

**1.** **(i) A supermarket has 2 girls running up sales at the counters. If the service time for each customer is exponential with mean 4 minutes and if people arrive in Poisson fashion at the rate of 10 per hour, find the following:**

**(i) What is the probability of having to wait for service?**
**(ii) What is the expected percentage of idle time for each girl?**
**(iii) What is the expected length of customer's waiting time?**
**(iv) What is the expected number of idle girls at any time?**

**(April/May'15) (Ap/May'19)**

**Solution:**

Model identification: Since there are two girls and infinite capacity. Hence this problem comes under the model (M/M/c);($\infty$/FCFS).

Given data:

Arrival rate $\lambda = 10$ per hr

Service rate $\mu = \frac{1}{4}$ per min (i.e) $\mu = \frac{60}{4} = 15$ per hr.

Number of servers $c = 2$

w.k.t $P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\mu c}{c!(\mu c - \lambda)}\left(\frac{\lambda}{\mu}\right)^c\right]^{-1}$

$$P_0 = \left[\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{10}{15}\right)^n + \frac{2\times 15}{2!((2\times 15)-10)}\left(\frac{10}{15}\right)^2\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{1}{n!}(0.6667)^n + \frac{30}{40}(0.6667)^2\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{1}{n!}(0.6667)^n + 0.3333\right]^{-1}$$

$$P_0 = \left[(1+0.6667)+0.3333\right]^{-1} = 2^{-1} = 0.5$$

i) The probability that a customer has to wait for the service is

$$P(N_s \geq 2) = \frac{\left(\frac{10}{15}\right)^2}{2!\left(1-\frac{10}{15\times 2}\right)}(0.5) \qquad \because P(N_s \geq c) = \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!\left(1-\frac{\lambda}{\mu c}\right)} P_0$$

$$P(N_s \geq 2) = \frac{(0.6667)^2}{1.3333}(0.5) = \frac{1}{6} = 0.1667$$

ii) Probability of time that a girl is busy $= \rho = \frac{\lambda}{c\mu} = \frac{10}{2\times 15} = \frac{1}{3}$

$\therefore$ Probability of time when a girl is idle = 1- Probability of time that a girl is busy

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

$\therefore$ Percentage of idle time of for each girl $= \frac{2}{3}\times 100 = 67\%$

iii) Expected waiting time of customer = $W_s = \frac{L_q}{\lambda} + \frac{1}{\mu}$

Where $L_q = \frac{1}{c.c!}\left(\frac{\lambda}{\mu}\right)^{c+1}\left(1-\frac{\lambda}{\mu c}\right)^{-2} P_0$

$$= \frac{1}{(2)2!}(0.6667)^3\left(1-\frac{10}{30}\right)^{-2}(0.5) = 0.083$$

$\therefore W_s = \frac{L_q}{\lambda}+\frac{1}{\mu} = \frac{0.083}{10}+\frac{1}{15} = 0.0083$ hrs

iv) The expected idle no of girl:

E(idle no of girl)=?

| No.of idle girls: | 2 | 1 | 0 |
|---|---|---|---|
| Probability | $P_0$ | $P_1$ | $P_2$ |

E(idle time for each girl)=2 $P_0$ +1 $P_1$ +0 $P_2$

Now, $P_0$=0.5

w.k.t $P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$ , $0 \le n < c$

$$P_1 = \frac{1}{1!}\left(\frac{10}{15}\right)^1 (0.5) = 0.333 = \frac{1}{3}$$

$$P_2 = \frac{1}{2!}\left(\frac{10}{15}\right)^2 (0.5) = 0.1111 = \frac{1}{9}$$

E(idle no of girl)=2 $P_0$ +1 $P_1$ +0 $P_2 = 2\times\frac{1}{9}+1\times\frac{1}{3}+0 = \frac{5}{9}$

$\therefore$ The expected percentage idle time for each girl = 45%

**(ii) A small mail –order business has one telephone line and a facility for call waiting for two additional customers. Orders arrive at the rate of one per minute and each order requires 2 minutes and 30 seconds to take down the particulars. What is the expected number of calls waiting in the queue? What is the mean waiting time in the queue?**

**Solution:**

Model identification:

since there is only one telephone line and the capacity of the system is finite. Hence this problem comes under the model (M/M/1);(k /FCFS).

Given Data:

Arrival rate $\lambda = 1/\min$

Service rate $\frac{1}{\mu} = \frac{5}{2} \Rightarrow \mu = \frac{2}{5}$ per min , k=3 $\rho = \frac{\lambda}{\mu} = \frac{1}{2/5} = \frac{5}{2} = 2.5$

The expected number of calls waiting in the queue = $L_q = L_s - \frac{\lambda'}{\mu}$

Where $L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}, if\ \lambda \ne \mu$ and $\lambda' = \mu(1-P_0)$

$$L_s = \frac{2.5}{1-2.5} - \frac{4(2.5)^4}{1-(2.5)^4} = 0.3333+4.105 = 4.4383$$

$P_0 = \frac{(1-\rho)}{1-\rho^{k+1}}$, if $\lambda \ne \mu$

$$P_0 = \frac{1-2.5}{1-(2.5)^4} = \frac{-1.5}{-38.06} = 0.0394$$

$$\lambda' = \frac{2}{5}(1-0.0394) = 0.3842$$

The expected number of calls waiting in the queue = $L_q = 4.4383 - \dfrac{0.3842}{0.4} = 3.4778$

The mean waiting time in the queue= $W_q = \dfrac{L_s}{\lambda'} - \dfrac{1}{\mu} = \dfrac{4.4383}{0.3842} - 2.5 = 11.55 - 2.5 = 9.052$

**2.** **(i) Find the system size probabilities for an $M/M/S$ : $FCFS/\infty/\infty$ queueing system under steady state conditions. Also obtain the expression for average number of customers in the system and waiting time of a customer in the system.**
**(Nov/Dec'11), (April/May'11)**

**Solution:**

Assume $S = c \Rightarrow (M/M/c);(\infty/FCFS)$

This model represents a queueing system with poisson arrivels, exponential service time, multiple servers, infinite capacity and FCFS queue service from a single queue.

If $n < c$, then only $n$ of the c servers will be busy and others are idle and hence mean service rate will be $n\mu$.

If $n \ge c$, all $c$ servers will be busy and hence the mean service rate is $c\mu$

$$\therefore \mu_n = \begin{cases} n\mu & ,0 \le n < c \\ c\mu & ,n \ge c \end{cases} \quad \text{and } \lambda_n = \lambda \ \ \forall n$$

By birth and death process

$$P_n = \frac{\lambda_0\lambda_1\lambda_2\lambda_3\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\mu_4\ldots\mu_n} P_0$$

Case i: When $0 \le n < c$

$$P_n = \frac{\lambda\lambda\lambda\lambda\ldots\lambda\,(n\,\text{times})}{1\mu\,2\mu\,3\mu\,4\mu\ldots\text{n}\,\mu} P_0$$

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right) P_0$$

Case ii: when $n \ge c$

$$P_n = \frac{\lambda\lambda\lambda\lambda\ldots\ldots(n\,\text{times})}{\mu_1\mu_2\ldots\mu_c\ldots\mu_{c+1}\ldots\mu_n} P_0$$

$$= \frac{\lambda^n}{\{1\mu\,2\mu\,3\mu\ldots(c-1)\mu\}\{c\mu\,c\mu\ldots(c-(c-1))\,\text{times}\}} P_0$$

$$= \frac{\lambda^n}{(c-1)!\,\mu^{c-1}c^{n-c+1}\mu^{n-c+1}} P_0$$

$$= \frac{\lambda^n}{(c-1)!\,c\,\mu^{c-1}c^{n-c}\mu^{c-1}\mu^{-(c-1)}} P_0$$

$$= \frac{\lambda^n}{(c-1)!\,\mu^{c-1}c^{n-c+1}\mu^{n-c+1}} P_0$$

$$P_n = \frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n P_0, n \geq c$$

$$P_n = \begin{cases} \frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n P_0 & , n \geq c \\ \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 & , 0 \leq n < c \end{cases}$$

To find $P_0$

Since $\sum_{n=0}^{\infty} P_n = 1$

$$\Rightarrow \sum_{n=0}^{c-1} P_n + \sum_{n=c}^{\infty} P_n = 1$$

$$\Rightarrow \sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 + \sum_{n=c}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$\Rightarrow P_0\left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c}^{\infty} \frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n\right] = 1 \quad \text{---------(1)}$$

Consider

$$\Rightarrow \sum_{n=c}^{\infty} \frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n = \frac{1}{c!c^{-c}} \sum_{n=c}^{\infty}\left(\frac{\lambda}{\mu c}\right)^n$$

$$= \frac{1}{c!c^{-c}}\left[\left(\frac{\lambda}{\mu c}\right)^c + \left(\frac{\lambda}{\mu c}\right)^{c+1} + \left(\frac{\lambda}{\mu c}\right)^{c+2} + \ldots\right]$$

$$= \frac{1}{c!c^{-c}}\left(\frac{\lambda}{\mu c}\right)^c\left[1 + \left(\frac{\lambda}{\mu c}\right)^1 + \left(\frac{\lambda}{\mu c}\right)^2 + \left(\frac{\lambda}{\mu c}\right)^3 + \ldots\right]$$

$$= \frac{1}{c!c^{-c}}\left(\frac{\lambda}{\mu c}\right)^c\left[1 - \left(\frac{\lambda}{\mu c}\right)\right]^{-1}$$

$$= \frac{1}{c!c^{-c}}\left(\frac{\lambda}{\mu c}\right)^c\left[\frac{\mu c - \lambda}{\mu c}\right]^{-1}$$

$$\Rightarrow \sum_{n=c}^{\infty} \frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n = \frac{1}{c!c^{-c}}\left(\frac{\lambda}{\mu c}\right)^c\left[\frac{\mu c}{\mu c - \lambda}\right]$$

$$(1) \Rightarrow P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c}^{\infty} \frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}$$

$$\Rightarrow P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{c!c^{-c}}\left(\frac{\lambda}{\mu c}\right)^c\left(\frac{\mu c}{\mu c - \lambda}\right)\right]^{-1}$$

To find the average number of customer in the system: $(L_s)$

$$L_s = L_q + \frac{\lambda}{\mu}$$

Where

$L_q =$ Average number of customer in the queue

$$L_q = \sum_{n=c}^{\infty}(n-c)P_n$$

$$L_q = \sum_{n=c}^{\infty}(n-c)\frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n P_0$$

$$= \frac{1}{c!c^{-c}} P_0 \sum_{n=c}^{\infty}(n-c)\left(\frac{\lambda}{\mu c}\right)^n$$

$$= \frac{1}{c!c^{-c}} P_0 \left[0+(c+1-c)\left(\frac{\lambda}{\mu c}\right)^{c+1} + (c+2-c)\left(\frac{\lambda}{\mu c}\right)^{c+2} + (c+3-c)\left(\frac{\lambda}{\mu c}\right)^{c+3} + \ldots\right]$$

$$= \frac{1}{c!c^{-c}} P_0 \left(\frac{\lambda}{\mu c}\right)^{c+1}\left[1+(2)\left(\frac{\lambda}{\mu c}\right)^{1} + (3)\left(\frac{\lambda}{\mu c}\right)^{2} + \ldots\right]$$

$$= \frac{1}{c!c^{-c}} P_0 \left(\frac{\lambda}{\mu c}\right)^{c+1}\left[1-\frac{\lambda}{\mu c}\right]^{-2}$$

$$L_q = \frac{1}{c\ c!}\left(\frac{\lambda}{\mu}\right)^{c+1}\left[1-\frac{\lambda}{\mu c}\right]^{-2} P_0$$

$$\therefore L_s = L_q + \frac{\lambda}{\mu} = \frac{1}{c\ c!}\left(\frac{\lambda}{\mu}\right)^{c+1}\left[1-\frac{\lambda}{\mu c}\right]^{-2} P_0 + \frac{\lambda}{\mu}$$

**(ii) In a cinema theatre people arrive to purchase tickets at the average rate of 6 per minute and it takes 7.5 seconds on the average to purchase a ticket. If a person arrives just 2 minutes before the picture starts and it takes exactly 1.5 minutes to reach the correct seat after purchasing the ticket.**

**i) Can he expect to be seated for the start of the picture?**

**ii) What is the probability that he will be seated when the film starts?**

**iii) How early must he arrive in order to be 99% sure of being seated for the start of the picture?**

**(Nov/Dec'14)**

**Solution:**

Model identification:

since there is only one counter and the arrival of persons are infinite, capacity of the system is infinie. Hence this problem comes under the model (M/M/1);($\infty$/FCFS).

Given Data:

Arrival rate $\lambda = 6$ per min

Service rate $\mu = \frac{1}{7.5}$ per sec $\Rightarrow \mu = \frac{60}{7.5} = 8$ per min

i) Expected total time required to purchase the ticket and to reach the seat

=waiting time in the system + time to reach the seat = $W_s + 1.5$.

Where $W_s = \frac{1}{\mu-\lambda} = \frac{1}{8-6} = \frac{1}{2} = 0.5$

Expected total time required to purchase the ticket and to reach the seat =0.5+1.5=2 min

ii) P(he will be the seated for the start of the picture)

=P(Total time < 2 min)

$$= P\left(W_s < \frac{1}{2}\right) = 1 - P\left(W_s > \frac{1}{2}\right)$$

$$= 1 - e^{-(\mu-\lambda)t}$$

$$= 1 - e^{-(8-6)\frac{1}{2}} = 0.63$$

iii) Suppose t minutes be the time of arrival so that he is seated 99%, then

$P(W \le t) = 0.99$

$\Rightarrow 1 - P(W > t) = 0.99$

$\Rightarrow P(W > t) = 1 - 0.99 = 0.01$

$e^{-(\mu-\lambda)t} = 0.01 \Rightarrow e^{-2t} = 0.01 \Rightarrow -2t = \log(0.01) \Rightarrow t = 2.3$

This is the waiting time in the system. that is to purchase ticket.
He takes 1.5 minutes to reach the seat after purchasing ticket.
$\therefore$ total time = 2.3+1.5=3.8
Hence he must arrive atleast 3.8 minutes earlier so as to be 99% sure of seeing the start of the film.

**3. (i) There are three typists in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour,**
**(a) What fraction of the time all the typists will be busy?**
**(b) What is the average number of letters waiting to be typed?**
**(c) What is the average time a letter has to spend for waiting and for being typed.**
**(d) What is the probability that a letter will take longer than 20 min waiting to be typed and being typed? Assume that arrival and service rates follow poisson distribution.**
**(May-June'13), (May/June'12), (Nov/Dec'11), (Nov/Dec'10) (April/May'17)**

**Solution:**

This is an $(M/M/3) : (\infty/FIFO)$ model.

$$\lambda = 15 \ / \ hr. \ \& \ \mu = 6/hr, \ s = 3, \ \therefore \frac{\lambda}{\mu} = 2.5 \ \& \ \frac{\lambda}{s\mu} = \frac{2.5}{3} = 0.833$$

(a) All the typists will be busy if there are at least 3 customers (letters) in the system

$$p(n \ge 3) = p(3) + p(4) + p(5) + \ldots\ldots = 1 - [p_0 + p_1 + p_2]$$

$$p_0 = \left( \sum_{n=0}^{s-1} \frac{1}{n\,!} \frac{\lambda^n}{\mu^n} + \frac{\lambda^s}{\mu^s \ s\,!} \frac{1}{\left(1 - \frac{\lambda}{s\mu}\right)} \right)^{-1} = \left[ 1 + 2.5 + \frac{2.5^2}{2} + \frac{(2.5)^3}{6 \times \left(1 - \frac{5}{6}\right)} \right]^{-1} = \frac{1}{22.25} = 0.0449$$

$$p_1 = \frac{\lambda}{1!\,\mu} p_0 = 2.5 p_0, \qquad p_2 = \frac{\lambda^2}{2!\,\mu^2} p_0 = \frac{1}{2}(2.5)^2 p_0$$

$P(n \ge 3) = 1 - [1 + 2.5 + (2.5^2/2)].(.0449) = 1 - 0.2974625 = 0.7025375 = 0.7025$

(b) Waiting to be typed (queue)

$$L_q = \frac{\lambda^{s+1}}{(s)(s!)\mu^{s+1}} p_0 \frac{1}{\left(1 - \frac{\lambda}{s\mu}\right)^2} = \frac{1}{3 \times 6}(2.5)^4 p_0 = 3.5078$$

$$\text{(c)} \ W_s = \frac{1}{\lambda}\left[L_q + \frac{\lambda}{\mu}\right] = \frac{1}{15}[3.5078 + 2.5] = 0.4005 \ \text{hr.}$$

$$d)\ \ P(W>t)=e^{-\mu t}\left\{1+\frac{\left(\frac{\lambda}{\mu}\right)^{s}\left[1-e^{-\mu t\left(s-1-\frac{\lambda}{\mu}\right)}\right]P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)\left(s-1-\frac{\lambda}{\mu}\right)}\right\}$$

$$P(W>t)=e^{-6\times\frac{1}{3}}\left\{1+\frac{(2.5)^{3}\left[1-e^{(-2)(-0.5)}\right](0.0449)}{6\left(1-\frac{2.5}{3}\right)(-0.5)}\right\}$$

$$=0.4616$$

**(ii) A TV repairman finds that the time spend on his job has an exponential distribution with mean 30 minutes. If he repair sets in the order in which they come in and if the arrival of sets is approximately Poisson with an average rate of 10 per 8 hour day.**
**a) Find the repairman's expected idle time on each day?**
**b) How many jobs are ahead of average set just brought? (Nov/Dec'13) (May/June'12)**

**Solution:**

<u>Model identification</u>:

since there is only one repair man and the capacity of the system is infinity. Hence this problem comes under the model (M/M/1);($\infty$/FCFS).

<u>Given Data:</u>

Arrival rate $\lambda=10$ per 8hr day

Service rate $\frac{1}{\mu}=30 \Rightarrow \mu=\frac{1}{30}$ per min $\Rightarrow (i.e)\ \mu=8\times 2=16$ per 8hr day

i) The repairman's idle time $=P_0=1-\frac{\lambda}{\mu}=1-\frac{10}{16}=\frac{6}{16}=\frac{3}{8}$ / day

The Expected idle time $=8\times\frac{3}{8}=3\ hrs$

ii) The average number of jobs in the system $=L_s=\frac{\lambda}{\mu-\lambda}=\frac{10}{16-10}=1.667\simeq 2$ jobs.

**Another method:**

Arrival rate $\lambda=\frac{10}{8}=\frac{5}{4}$ per hr

Service rate $\mu=\frac{1}{30}$ per min $(i.e)\ \mu=\frac{60}{30}=2/hr$

i) The repairman's idle time $=1-\frac{\lambda}{\mu}=1-\frac{10}{16}=\frac{6}{16}=\frac{3}{8}$

The Expected idle time each day $=8\times\frac{3}{8}=3\ hrs$

ii) Number of the jobs ahead of the average set brought in = The average number of jobs in the system

$$=L_s=\frac{\lambda}{\mu-\lambda}=\frac{\frac{5}{4}}{2-\frac{5}{4}}=1.667\simeq 2\text{ jobs.}$$

**4.** **i) A group of engineers has two terminals available to aid their calculations. The average computing job requires 20 minutes of terminal time and each engineer requires some computation one in half an hour. Assume that these are distributed according to an exponential distribution. If there are 6 engineers in the group, find**
**a) the expected number of engineers waiting to use the terminals in the computing center.**
**b) the total time lost per day.**

**Solution:**

Since there are 2 terminals, also since there are 6 engineers in the group, the capacity of the system is finite.

Hence this problem comes under the model (M/M/c);(k/FCFS)

Given data:

Arrival rate $\lambda = \dfrac{1}{1/2}$ =2 per hr

Service rate $\mu = \dfrac{1}{20}$ per min (i.e) $\mu = \dfrac{60}{20} = 3$ per hr.

Number of servers $c = 2$

Capacity = $k$ = $6$

Expected number of engineers waiting to use in the computing center = $L_s$

$$L_s = L_q + \frac{\lambda'}{\mu}$$

Where $P_0 = \left[ \sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c + \sum_{n=c}^{k}\left(\frac{\lambda}{\mu c}\right)^{n-c} \right]^{-1}$

$$P_0 = \left[ \sum_{n=0}^{1} \frac{1}{n!}\left(\frac{2}{3}\right)^n + \frac{1}{2!}\left(\frac{2}{3}\right)^2 + \sum_{n=2}^{6}\left(\frac{2}{3\times 2}\right)^{n-2} \right]^{-1}$$

$$= \left[ 1 + \frac{2}{3} + \frac{1}{2}\times\left(\frac{2}{3}\right)^2 \left\{ 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 \right\} \right]^{-1} = 0.5003$$

And $\rho = \dfrac{\lambda}{\mu c} = \dfrac{1}{3}$ $\qquad L_q = \left(\dfrac{\lambda}{\mu}\right)^c \dfrac{\rho}{c!(1-\rho)^2}\left\{1-\rho^{k-c} - (k-c)(1-\rho)\rho^{k-c}\right\}P_0$

$$L_q = \left(\frac{2}{3}\right)^2 \frac{\frac{1}{3}}{2!\left(1-\frac{1}{3}\right)^2}\left\{1-\left(\frac{1}{3}\right)^{6-2} - (6-2)\left(1-\frac{1}{3}\right)\left(\frac{1}{3}\right)^{6-2}\right\}(0.5003) = 0.0796$$

Effective arrival rate $\lambda' = \mu\left[c - \sum_{n=0}^{c-1}(c-n)P_n\right] = 3\left[2 - \sum_{n=0}^{1}(2-n)P_n\right]$

$\lambda' = 3\left[2 - \left(2P_0 + P_1\right)\right]$ where $P_1 = \left(\dfrac{\lambda}{\mu}\right)P_0$

$$\lambda' = 3\left[2 - \left(2\times 0.5003 + \frac{2}{3}\times 0.5003\right)\right] = 1.9976$$

Expected number of engineers waiting to use in the computing center = $L_s$

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| | $L_s = L_q + \frac{\lambda'}{\mu}$ <br> $= 0.0796 + \frac{1.9976}{3} = 0.7455$ |
| | **ii) A car service station has 2 bays offering service simultaneously. Because of space constraints, only 4 cars are accepted for servicing. The arrival pattern with 12cars per day. The service time in both the bays is exponentially distributed with $\mu = 8$ cars per day .Find the average number of cars in the service station, the average number of cars waiting for service and the average time a car spends in the system. (April/May'19)** |
| | **Solution:** <br> This is a multiple server model with finite capacity. Arrival rate λ=12/day and Service rate μ=8/day ,S=2, K=4 <br> $p_0 = \left[\sum_{n=0}^{s-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k}\left(\frac{\lambda}{\mu}\right)^{n-s}\right]^{-1} = \left[1+\frac{1.5}{1}+\frac{1}{2}(1.5)^2\left[1+(0.75)+(0.75)^2\right]\right]^{-1}$ <br> $P_0 = 0.1960 , \rho = \frac{\lambda}{s\mu}, L_q = P_0\left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{s!(1-\rho)^2}\left[1-\rho^{k-s}-(k-s)(1-\rho)\rho^{k-s}\right]$ <br> $L_q = 0.1960(1.5)^2\left(\frac{0.75}{2(0.25)^2}\right)\left[1-(0.75)^2-2(0.25)(0.75)^2\right] = 0.4134 \text{ car.}$ <br> $L_s = L_q + s\sum_{n=0}^{s-1}(s-n)p_n = 0.4134 + 2\sum_{n=0}^{1}(2-n)p_n = 2.4134 - 2(2p_0+p_1) = 1.73 \text{ cars.}$ <br> $W_s = \frac{L_s}{\lambda'}$ where $\lambda' = \left[s - \sum_{n=0}^{s-1}(s-n)p_n\right] = 8\left[2-(2p_0+p_1)\right] = 10.512$ <br> $W_s = \frac{1.73}{10.512} = 0.1646$ day <br> Average time that a car has to spend in the system = 0.16 |
| **5.** | **(i) Patients arrive at a clinic according to Poisson distribution at the rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with a mean rate of 20 per hour. (April/May'19)** <br> **(a) Find the effective arrival rate at the clinic.** <br> **(b)What is the probability that an arriving patient will not wait?** <br> **(c)What is the expected waiting time until a patient is discharged from the clinic?** |
| | **Solution:** <br> Arrival rate $\lambda = 30/\text{hr.}$, Service rate $\mu = 20/\text{hr.}$, $M/M/1:K/FIFO$ model. $\because \lambda \neq \mu$, $p_0 = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{k+1}} = \frac{1-3/2}{1-(3/2)^{16}} = 0.00076$ <br> (a)The effective arrival rate is $\lambda' = \mu(1-p_0) = 20(1-0.00076) = 19.98$hr. <br> (b) $P(\text{a patient will not wait}) = p(n=0) = p_0 = 0.00076$ |

(c) $L_s = \frac{\lambda}{\mu-\lambda} - \frac{(K+1)\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}} = (-3) - \frac{16\left(\frac{3}{2}\right)^{16}}{1-\left(\frac{3}{2}\right)^{16}} = 13\text{patients}$

$W_s = \frac{L_s}{\lambda'} = \frac{13}{19.98} = 0.65\text{hr or } 39\text{min.}$

**(ii) A one-person barber shop has 6 chairs to accommodate people waiting for a haircut. Assume that customers who arrive when all the 6 chairs are full leave without entering the barber shop. Customers arrive at the rate of 3 per hour and spend an average of 15 minutes in the barber's chair. Compute**

**1) the probability that a customer can go to the barber's chair without waiting.**

**2) the average waiting time in the queue and in the system.**

**3) the average number of customers in the system and in the queue.**

**4) the probability that there are seven customers in the system. (May/June 2016)**

**Solution:**

Model identification:

since there is only one barber and the capacity of the system is finite. Hence this problem comes under the model (M/M/1);(k /FCFS).

Given Data:

Arrival rate $\lambda = 3/hr$

Service rate $\mu = \frac{1}{15}\times 60 = 4/hr$ per min , k=6+1=7 $\rho = \frac{\lambda}{\mu} = \frac{3}{4} = 0.75$

1) The probability that a customer can go to the barber's chair without waiting

$$= P_0 = \frac{(1-\rho)}{1-\rho^{k+1}}, \text{ if } \lambda \neq \mu$$

$$P_0 = \frac{1-\frac{3}{4}}{1-\left(\frac{3}{4}\right)^8} = 0.2778$$

2) The average waiting time in the queue $W_s = \frac{L_s}{\lambda'}$

Where $L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}, if\ \lambda \neq \mu$ and $\lambda' = \mu(1-P_0)$

$\lambda' = 4(1-0.2778) = 2.89$

$L_s = \frac{0.75}{1-0.75} - \frac{8(0.75)^8}{1-(0.75)^8} = 3-0.89 = 2.11$

The average waiting time in the System $W_s = \frac{2.11}{2.89} = 0.73$

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| | The average waiting time in the System $W_q = \frac{L_s}{\lambda'} - \frac{1}{\mu} = 0.73 - \frac{1}{3} = 0.417$<br>3) The average number of customers in the system= $L_s = 2.11 \cong 2$<br>The average number of customers in the queue $= L_q = L_s - \frac{\lambda'}{\mu} = 2.11 - \frac{2.89}{4} = 1.387$<br>4) The probability that there are seven customers in the system<br>$P_n = \rho^n \left( \frac{(1-\rho)}{1-\rho^{k+1}} \right) = \rho^n P_0$ if $\lambda \neq \mu$<br>$P_7 = (0.75)^7 (0.2778) = 0.0371$ |
| | **UNIT V ADVANCED QUEUEING MODELS** |
| | **PART - A** |
| **1.** | **What are the differences between Markovian and non Markovian queueing model?**<br>Markovian queueing model: Inter-arrival and inter-service times follows exponential distribution. Non Markovian queueing model: Service time distribution follows any general distribution. |
| **2.** | **Explain M/G/1 model.**<br>It is a non Markovian queueing model, where the arrival pattern follows the Poisson distribution, the service time distribution G is any general distribution and the number of servers is one. |
| **3.** | **Write down Little's formula in M/G/1 model.**<br>$L_s = \lambda W_s; L_q = \lambda W_q; W_s = W_q + \frac{1}{\mu}$ |
| **4.** | **Write down Pollaczek-khintchin formula.**<br>**(June 2014, Nov / Dec 2015, Nov / Dec 2016, April / May 2017)**<br>$L_s = \lambda E(T) + \frac{\lambda^2 \left[ Var(T) + E(T)^2 \right]}{2[1-\lambda E(T)]}$ |
| **5.** | **When a M/G/1 queuing model will become a classic M/M/1 queuing model?**<br>**(May/June 2012)**<br>If the service time distribution is exponential then a M/G/1 queuing model will become a classic M/M/1 queuing model. |
| **6.** | **An M/D/1 queue has an arrival rate of 10 customers per second and a service rate of 20 customers per second. Compute the mean number of customers in the system.**<br>**(Apr/May 2019)**<br>$Var(T) = 0; \lambda = 10 \text{ cust/sec}; \mu = 20 \text{ cust/sec } \mu = \frac{1}{E(T)} \Rightarrow E(T) = \frac{1}{\mu}$<br>$L_s = \lambda E(T) + \frac{\lambda^2 \left[ Var(T) + E(T)^2 \right]}{2[1-\lambda E(T)]}$<br>$= 10\left(\frac{1}{20}\right) + \frac{10^2 \left[ 0 + \left(\frac{1}{20}\right)^2 \right]}{2\left[1 - 10\left(\frac{1}{20}\right)\right]} = \left(\frac{1}{2}\right) + \frac{\left(\frac{1}{4}\right)}{2\left(\frac{1}{2}\right)} = \left(\frac{1}{2}\right) + \left(\frac{1}{4}\right) = \frac{3}{4}$ |

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| **7.** | **Write down Pollaczek-khintchin formula for the case when service time distribution is Erlang distribution with k phases. (May 2011)**<br>$E(T)=\frac{1}{\mu}, Var(T)=\frac{1}{k\mu^2}$ ,<br>$L_s=\frac{\lambda}{\mu}+\frac{\lambda^2[1+k]}{2k\mu(\mu-\lambda)}$ |
| **8.** | **Write short notes on queue networks.**<br>Networks of queues are systems in which a number of queues are connected by customer routing. When a customer is serviced at one node, they can join another node and queue for service, or leave the network. For a network of m queues, the state of the system can be described by an m–dimensional vector $(x_1, x_2, ..., x_m)$ where $x_i$ represents the number of customers at each node. The customers may enter the system at some node, can traverse from node to node in the system and finally can leave the system from any node. Also customers can return to the nodes already visited.<br>Examples<br>• Customers go from one queue to another in post office, bank, supermarket etc<br>• Data packets traverse a network moving from a queue in a router to the queue in another router |
| **9.** | **Define series queues (Tandem queues) with an example.**<br>In series queues , there are series of service stations through which each calling unit must progress prior to leaving the system. Ex. A physical examination for a patient where the patient under goes a series of stages, lab tests, ECG , X ray etc. |
| **10.** | **What do you mean by bottle neck of a network? (Nov/Dec 2010, 2015,2016)**<br>The service station for which the utilization factor is maximum among all the other service stations of the network is called the bottle neck of a network. |
| **11.** | **A company's repair section has 2 sequential stations with respective service rates of 4/hour and 5/ hour. The cumulative failure rate of all the machines is 1/hour. Assuming the system behavior can be approximated by the two stage tandem queue find the bottleneck of the repair facility.**<br>$\frac{\lambda}{\mu_1}=\frac{1}{4}, \frac{\lambda}{\mu_2}=\frac{1}{5}$, since $\frac{\lambda}{\mu_1}>\frac{\lambda}{\mu_2}$<br>the service station 1 is the bottle neck of the repair facility. |
| **12.** | **What is the joint probability that there are m customers at station 1 and n customers at station 2 for a 2 stage series queue?**<br>$P_{mn}=\left(\frac{\lambda}{\mu_1}\right)^m\left(1-\frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_2}\right)^n\left(1-\frac{\lambda}{\mu_2}\right)$ |
| **13.** | **Consider a service facility with two sequential stations with respective service rates of 3/min and 4/min. The arrival rate is 2/min. What is the average waiting time of the system if the system could be approximated by a two series Tandem queue?**<br>$\mu_1=3/\text{min.}, \mu_2=4/\text{min.}, \lambda=2/\text{min.}$<br>Average waiting time of customers in the system =Ws(station 1) + Ws(station 2)<br>$=\frac{1}{\mu_1-\lambda}+\frac{1}{\mu_2-\lambda}=\frac{1}{3-2}+\frac{1}{4-2}=1.5\,\text{min.}$ |

**14.** **A TVS company in Chennai containing a repair station shared by a large number of machines has 2 sequential stations with respective service rates of 3/hr and 4/hr. The failure rate of all the machines is 1 per hour. Assuming that the system behavior can be approximated by a 2 stage tandem queue find the probability that the service stations are idle.**

$\mu_1 = 3/\text{hr}$, $\mu_2 = 4/\text{hr}$., $\lambda = 1/\text{hr}$.,

$$P_{mn} = \left(\frac{\lambda}{\mu_1}\right)^m \left(1 - \frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^n \left(1 - \frac{\lambda}{\mu_2}\right)$$

$$P_{00} = \left(\frac{1}{3}\right)^0 \left(1 - \frac{1}{3}\right) \left(\frac{1}{4}\right)^0 \left(1 - \frac{1}{4}\right) = 0.5$$

**15.** **Consider a two-station tandem Markovian queueing network with customers arrival rate of $\lambda = 2$ /min and service rates $\mu_1$ = 4/min at station-1 and $\mu_2$ = 6/min at station-2. Compute the waiting time of a customer in the system and the probability that the both the servers are idle. (April/May 2019)**

$\mu_1 = 4/\text{min.}$, $\mu_2 = 6/\text{min.}$, $\lambda = 2/\text{min.}$

Average waiting time of customers in the system =Ws(station 1) + Ws(station 2)

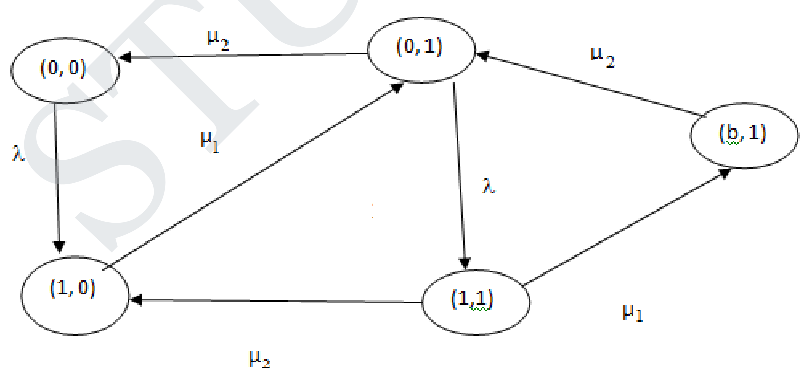
$$= \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = \frac{1}{4-2} + \frac{1}{6-2} = 0.75\,\text{min}$$

$$P_{mn} = \left(\frac{\lambda}{\mu_1}\right)^m \left(1 - \frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^n \left(1 - \frac{\lambda}{\mu_2}\right)$$

$$= \left(\frac{2}{4}\right)^0 \left(1 - \frac{2}{4}\right) \left(\frac{2}{6}\right)^0 \left(1 - \frac{2}{6}\right) = \frac{1}{3}$$

**16.** **Write the steady state equations for 2 stage series queues with blocking.**

Let $\lambda$ be the arrival rate at station 1 which follows Poisson distribution and service time is exponential with parameters $\mu_1$ and $\mu_2$ respectively. $P_{n_1 n_2}$ represents the probability that there are $n_1$ customers in station 1 and $n_2$ customers in station 2.




$$-\lambda P_{0,0} + \mu_2 P_{0,1} = 0$$

$$-\mu_1 P_{1,0} + \mu_2 P_{1,1} + \lambda P_{0,0} = 0$$

$$-(\lambda + \mu_2) P_{0,1} + \mu_1 P_{1,0} + \mu_2 P_{b,1} = 0$$

$$-(\mu_1 + \mu_2) P_{1,1} + \lambda P_{0,1} = 0, \quad -\mu_2 P_{b,1} + \mu_1 P_{1,1} = 0$$

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| **17.** | **Define Jackson networks.**<br>Networks preserving the following characteristics are called Jackson networks.<br>1. Arrivals from outside through node i follow a Poisson process with mean arrival rate $r_i$.<br>2. Service times at node i are independent and exponentially distributed with parameter $\mu_i$.<br>3.The probability that a customer who has completed service at node i will go to next node j is Pij , i = 1,2,...k , j= 0,1,2,...k. |
| **18.** | **Write a expression for the traffic equation of open Jackson queueing network.** **(May/June 2016, April / May 2017)**<br>$\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij}\ , j = 1,2,...k$<br>where $r_j$ is the arrival rate to node j , $\lambda_j$ is the total arrival rate of customers to node j and $P_{ij}$ is the probability that a departure from server i joins the queue at server j. |
| **19.** | **Write a short note on Finite resource model.**<br>Models in which arrivals are drawn from a small population are called Finite resource model. In this case the arrival rate may greatly depend on the state of the system. For example the case of machine repairman model. |
| **20.** | **A police department has 5 patrol cars. A patrol car breaks down and requires service once every 30 days. The department has two repair workers, each of whom takes an average of 3 days to repair a car. Breakdown and repair times are exponential. Identify the queuing model of the problem.**<br>Since the server here is the repair man and the customer is the cars [only 5 cars] that require service from the repairman, this problem comes under machine interference model which is one of the finite resource models. |
| | **PART – B** |
| **1.** | **(i) A Laundromat has 5 washing machines. A typical machine breaks down once every 5 days. A repairer takes an average of 2.5 days to repair a machine. Currently, there are three repair workers on duty. The owner has the option of replacing them with a super worker, who can repair a machine in an average of (5/6) day. The salary of the super worker equals the pay of the three repair workers. Break down time and repair time are exponential. Should the Laundromat replace the three repairers with a super worker?** **(Nov/Dec 2015)** |
| | **Solution:**<br>This problem comes under the machine interference problem and let K = No. of machines; R = No. of repairman, the steady state probabilities.<br>$P_j = \begin{cases} KC_j \rho^j P_0, & j = 0,1,2,3...R \\ \dfrac{KC_j \rho^j j! P_0}{R! R^{j-R}} & j = R+1, R+2, ....K \end{cases}$<br>Part 1 : Three repair Workers K = 5 and R = 3:<br>$\lambda = (1/5)\,\text{mch/day}, \mu = (1/2.5)\,\text{mch/day},\ \rho = 1/2$<br>$P_j = \begin{cases} 5C_j (1/2)^j P_0, & j = 0,1,2,3 \\ \dfrac{5C_j (1/2)^j j! P_0}{3! 3^{j-3}} & j = 4,5 \end{cases}$<br>$L_S = \sum_{j=0}^{5} jP_j = 1.7149$ with $P_0 = 0.1292$<br>Part 2 : One Super Workers K = 5 and R =1 : |

$\lambda = (1/5)\,mch/day, \mu = (6/5)\,mch/day,\ \rho = 1/6$

$$P_j = \begin{cases} 5C_j\,(1/6)^j\,P_0, & j = 0,1, \\ \dfrac{5C_j\,(1/6)^j\,j!P_0}{1!1^{j-1}} & j = 2,3,4,5 \end{cases}$$

$$L_S = \sum_{j=0}^{5} jP_j = 1.162 \quad with \quad P_0 = 0.3604$$

Therefore One Super Worker is better option than the three repair workers as Ls is lesser in this case.

**(ii) Discuss the Jackson open queueing network system and hence obtain the corresponding product form solution of the system size probabilities. (April/May 2019)**

A queueing network is called as Open Jackson network if it has the following characteristics.
1. It is a system with *m* nodes where each node *i* (i = 1,2,…,m) has an infinite queue.
2. Customers arriving from outside the system to node *i* follow a poisson process with mean rate $r_i$.
3. Service times at each channel at node *i* are independent and exponentially distributed with parameter $\mu_i$
4. A customer who has completed serice at node *i* (with ci servers) is routed next to node *j* with probability $r_{ij}$ (independent of the system state) or leaves the system from node *i* with probability $r_{i0}$, where *i* = 1,2,3,…,m , *j* = 1,2,3,…,m and $r_{i0} = 1 - \sum_{j=1}^{m} r_{ij}$

Open Jackson network has the following key property, called traffic equations.

For each node *i*, its input processes from various sources are independent poisson processes and so their sum or aggregate is a Poisson process with parameter $\lambda_i$. Now the probability for a customer going from node *i* to node *j* is $r_{ij}$ i.e. the proportion of customers going from node *i* to node *j* is $r_{ij}$. Therefore, the rate at which customers from node *i* arrive at node *j* is $\lambda_i\, r_{ij}$.

$\therefore \sum_i \lambda_i r_{ij}$ is the total rate from all node *i* to node *j*. The rate of arrival from outside the network to node *j* is $r_j$. Hence the total arrival rate to node *j* from all sources is $\lambda_j = r_j + \sum_{i=1}^{m} \lambda_i r_{ij}$,

*j* = 1,2,3,…,m where $\lambda_j < c_j \mu_j$.

The equations $\lambda_j = r_j + \sum_{i=1}^{m} \lambda_i r_{ij}$, j = 1,2,3,…,m is called traffic equations or flow balance equations.

If $N_i$,i = 1,2,3,…,m be the number of customers in the node i, then the steady state joint probability $P[N_1=n_1, N_2=n_2,\ldots,N_m=n_m] = P[N_1=n_1].P[N_2=n_2]\ldots P[N_m=n_m]$ where $P[N_i=n_i]$ is the steady state probability mass function of an $M/M/c_i$ system with arrival rate $\lambda_i$ and service rate $\mu_i$

i.e. the joint probability $p_{n_1,n_2,\ldots,n_m} = p_{n_1}.p_{n_2}\cdots p_{n_m}$

Thus the joint probability is the product of marginal probabilities.

**2.** **(i) Consider a queuing system where arrivals are according to a Poisson distribution with mean 5 per hour. Find the expected waiting time in the system if the service time distribution is (1) a uniform distribution between t = 5 minutes and t = 15 minutes.**
**(2) Normal distribution with mean 3 minutes and standard deviation 2 minutes.**

This is M/G/1 queueing model, here arrival rate λ = 5 per/hr = 5/60min'
i)Let T be a continuous R.V denoting the service time, then by Uniform distribution
E(T)=(a+b)/2 =10; μ = 1/ E(T) = 1/10,Var(T) = (b-a)2 /12 =100/12 = 25/3.
By the Pollaczek-khintchin formula.

$$L_s = \lambda E(T) + \frac{\lambda^2\left[Var(T)+E(T)^2\right]}{2(1-\lambda E(T))} = 3.09023,$$

$$W_S = \frac{L_S}{\lambda} = \frac{(3.09023)}{\left(\frac{1}{12}\right)} = 37.08276\text{min.},$$

ii) E(T) =3 minutes and S .D = 2 i.e Var(T) = 4,
therefore μ = 1/ E(T) = μ = 1/ 3 .
By the Pollaczek-khintchin formula.

$$L_s = \lambda E(T) + \frac{\lambda^2\left[Var(T)+E(T)^2\right]}{2(1-\lambda E(T))},$$

Ls= 0.310185.

$$W_S = \frac{L_S}{\lambda} = \frac{(0.310185)}{\left(\frac{1}{12}\right)} = 3.7222\text{min.},$$

**(ii) A one-man barber shop takes exactly 25 minutes to complete one hair-cut. If customers arrive at the barber shop in a Poisson fashion at an average rate of one every 40 minutes, how long on the average a customer in the spends in the shop. Also, find the average time a customer must wait for service? (Nov/Dec 2015)**

**Solution:**

The service time is constant. Hence it is a $(M/D/1)$ queuing model.

$E(T)=25 \qquad Var(T)=0$

Arrival rate $\lambda=\frac{1}{40}$/min.

$$L_S = \lambda E(T) + \frac{\lambda^2\left[E^2(t)+Var(T)\right]}{2\left[1-\lambda E(T)\right]} = \frac{1}{40}(25) + \frac{\left(\frac{1}{40}\right)^2\left[25^2+0\right]}{2\left[1-\frac{1}{40}(25)\right]} = \frac{55}{48}\text{customers.}$$

$$W_S = \frac{L_S}{\lambda} = \frac{\left(\frac{55}{48}\right)}{\left(\frac{1}{40}\right)} = 45.8\text{min.},\quad W_q = W_S - \frac{1}{\mu} = W_S - \frac{1}{\frac{1}{E(T)}} = 45.8-25 = 20.8 \text{ min.}$$

**3.** **(i) A patient goes to a single doctor clinic for a general checkup has to go through 4 phases. The doctor takes on the average 4 minutes for each phase of the check up and the time taken for each phase is exponentially distributed. If the arrivals of the patients at the clinic are approximately Poisson at the average rate of 3 per hour, what is the average time spent by a patient (i) in the examination (ii) waiting in the clinic?**

**Solution:**

The clinic has 4 phases ( each having different service nature) in series as follows:
Considering all the phases (each with exponential service time) together as "one server" we shall take it as a server with Erlang service time. So this is a $(M/E_k/1)$ model with

$E(T) = \frac{k}{\theta} = \frac{4}{\left(\frac{1}{4}\right)} = 16\,\text{min. and Var(T)} = \frac{k}{\theta^2} = \frac{4}{\left(\frac{1}{16}\right)} = 64\,\text{min.}$

The arrival rate $\lambda = 3\, per\, \text{hr.} = \frac{3}{60}\, per \min = \frac{1}{20}\, per \min.$

$$L_s = \lambda E(T) + \frac{\lambda^2\left[E^2(t) + Var(T)\right]}{2\left[1-\lambda E(T)\right]} = \frac{1}{20}(16) + \frac{\left(\frac{1}{20}\right)^2\left[16^2+64\right]}{2\left[1-\frac{1}{20}(16)\right]} = \frac{14}{5}$$

$$W_s = \frac{L_s}{\lambda} = \frac{\left(\frac{14}{5}\right)}{\left(\frac{1}{20}\right)} = 56\,\text{min}, \qquad W_q = W_s - \frac{1}{\mu} = W_s - \frac{1}{\frac{1}{E(T)}} = 56 - 16 = 40\,\text{min.}$$

**(ii) Discuss an M/G/1/∞:FCFS queuing system and hence obtain the Pollaczek-Khintchine (P-K) mean value formula. Deduce also the mean number of customers for the M/M/1/∞: FCFS queuing system from the P-K mean value formula. (April/May 2019)**

**Proof:**

Let N and $N'$ be the number of customers in the system at times t and t+T, when two consecutive customers have just left the system after getting service. Thus T is the random service time, which is a continuous random variable.

Let $f(t), E(T), V(T)$ be the probability density function, mean and variance of the service time T. Let M be the number of customers arriving in the system during service time T, then

$$N' = \begin{cases} M, & if \quad N = 0 \\ N-1+M, & if \quad N > 0 \end{cases}$$

where M is a discrete random variable taking the values 0,1,2,…

The same can be written as

$$N' = N-1+M+\delta .... (1), \text{ where } \delta = \begin{cases} 1, & \text{if} \quad N=0 \\ 0, & \text{if} \quad N>0 \end{cases}$$

Note that by the definition of $\delta$, $\delta^2 = \delta$ and $N\delta = 0$

Squaring both sides of (1), we have,

$$N'^2 = N^2 + (M-1)^2 + \delta^2 + 2N(M-1) + 2N\delta + 2(M-1)\delta = N^2 + (M-1)^2 + \delta + 2N(M-1) + 2(M-1)\delta$$

$$= N^2 + M^2 - 2M + 1 + 2N(M-1) + (2M-2+1)\delta$$

$$2N(1-M) = N^2 - N'^2 + M^2 - 2M + 1 + (2M-1)\delta .......(2)$$

Taking Expectation on both sides of (1) we get,

$$E(N') = E(N) - 1 + E(M) + E(\delta)$$

In steady state, the probability that the number of customers in the system is constant

$$E(N') = E(N) \quad \text{and} \quad E\left(N^2\right) = E\left(N'^2\right) ........(3)$$

and substituting in previous equation we have $E(\delta) = 1 - E(M) .....(4)$

Taking Expectation on the both sides of (2) we get

$$2E\left[N(1-M)\right] = E\left(N^2\right) - E\left(N'^2\right) + E\left(M^2\right) - 2E(M) + 1 + \left(2E(M)-1\right)E(\delta)$$

Using (3) we have

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| | $2E[N(1-M)] = E(M^2)-2E(M)+1+(2E(M)-1)E(\delta)$.. <br> Using (4) we have <br> $2E[N(1-M)] = E(M^2)-2E(M)+1+(2E(M)-1)(1-E(M))$ <br> $= E(M^2)-2E(M)+1+2E(M)-2E(M)^2-1+E(M) = E(M^2)-2E(M)^2+E(M)$ <br> Since the number of arrivals M to a system is independent of the number of customers already in the system N, we have <br> $2E[N]E[(1-M)] = E(M^2)-2E(M)^2+E(M) = E(M^2)-2E^2(M)+E(M)$ <br> $E(N) = \dfrac{E(M^2)-2E^2(M)+E(M)}{2[1-E(M)]}$ ......(5) <br> Since the arrivals in service time T follows Poisson process with parameter $\lambda$, <br> $E(M/T)=\lambda T$ , $E(M^2/T)=\lambda^2T^2+\lambda T$ , $\sigma^2_{M/T}=\lambda T$ <br> $E(M) = E(\lambda T) = \lambda E(T)$ <br> $E(M^2) = E(\lambda^2T^2+\lambda T) = \lambda^2E(T^2)+\lambda E(T) = \lambda^2[\mathrm{var}(T)+E^2(T)]+\lambda E(T)$ <br> $E^2(M) = \lambda^2E^2(T)$ <br> substituting in (5) we have, <br> $E(N) = \dfrac{\lambda^2[\mathrm{var}(T)+E^2(T)]+\lambda E(T)-2\lambda^2E^2(T)+\lambda E(T)}{2[1-\lambda E(T)]}$ <br> $= \dfrac{\lambda^2[\mathrm{var}(T)+E^2(T)]+2\lambda E(T)-2\lambda^2E^2(T)}{2[1-\lambda E(T)]}$ <br> $= \dfrac{\lambda^2[\mathrm{var}(T)+E^2(T)]+2\lambda E(T)(1-E(T))}{2[1-\lambda E(T)]}$ <br> $= \dfrac{\lambda^2[\mathrm{var}(T)+E^2(T)]}{2[1-\lambda E(T)]}+\dfrac{2\lambda E(T)(1-E(T))}{2[1-\lambda E(T)]}$ <br> $= \dfrac{\lambda^2[\mathrm{var}(T)+E^2(T)]}{2[1-\lambda E(T)]}+\lambda E(T)$ <br> This is called P-K formula. <br> If the average service time is $\frac{1}{\mu}$ then $E(T)=\frac{1}{\mu}$ <br> $\therefore \lambda E(T)=\frac{\lambda}{\mu}=\rho$. If $Var(T)=\sigma^2$, then the above formula can be written as <br> $L_S = \rho+\dfrac{\lambda^2\left[\sigma^2+\frac{1}{\mu^2}\right]}{2(1-\rho)} = \rho+\dfrac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}$ |

In M/M/1 model, $\sigma^2 = \dfrac{1}{\mu^2}$

$$L_S = \rho + \frac{\lambda^2\left(\frac{1}{\mu^2}\right)+\rho^2}{2(1-\rho)} = \rho + \frac{\rho^2+\rho^2}{2(1-\rho)} = \rho + \frac{2\rho^2}{2(1-\rho)} = \rho + \frac{\rho^2}{(1-\rho)} = \frac{\rho}{1-\rho}$$

**4.** **(i) A repair facility by a large number of machines has two sequential stations with respective rates one per hour and two per hour. The cumulative failure rate of all the machines is 0.5 per hour. Assuming that the system behavior may be approximated by the two-stage tandem queue, determine the average repair time. (May/June 2016)**

**Solution:**

$\lambda = 0.5,\ \mu_1 = 1,\ \ \mu_2 = 2,\ \rho_1 = 0.5,\ \rho_2 = 0.25$

Each station is a (M/M/1) queue model

The average length of the queue at station 1 $L_{S1} = \dfrac{\rho_1}{1-\rho_1} = \dfrac{0.5}{1-0.5} = 1$

$$W_{S_1} = \frac{L_{S1}}{\lambda} = \frac{1}{0.5} = 2$$

The average length of the queue at station 2 $L_{S2} = \dfrac{\rho_2}{1-\rho_2} = \dfrac{0.25}{1-0.25} = \dfrac{1}{3}$

$$W_{S_2} = \frac{L_{S2}}{\lambda} = \frac{1/3}{0.5} = \frac{2}{3}$$

The total repair time of the network is $= W_{s1} + W_{s2} = 2 + \dfrac{2}{3} = \dfrac{8}{3}$ hours

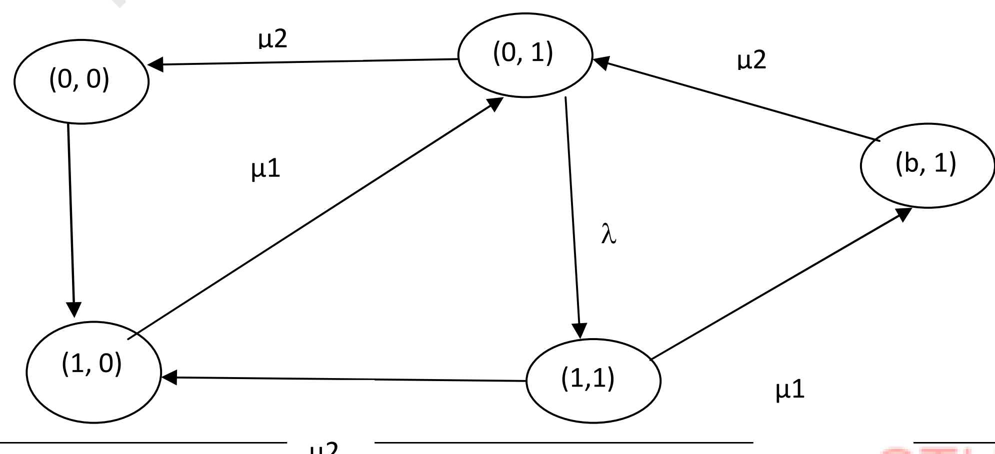
**(ii) There are 2 salesmen in a super market. Out of the 2 salesman, one is in charge of billing and receiving payment while other salesman is in charge of weighing and delivering the items. Due to lack of space, Only one customer is allowed to enter the shop, only if the billing clerk is free. The customer who has finished is billing job has to wait until the delivery section becomes free. If customer arrive according to Poisson process at a 1/hour and the service times of 2 clerks are independent and have exponential rates of 3 and 2/hour. Find (i) The proportion of customers who enter the supermarket.**

**(ii)The average number of customer in the system.**

**(iii)The average amount of time a customer spends in the shop. (Nov / Dec 2013)**

**Solution:**

This problem is 2 stage sequential queuing model with blocking



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| | From the balance equations with $\lambda=1;\ \mu_1=3;\ \mu_2=2$<br>we have $p_{0,0}=2p_{0,1}$ .....(1)<br>$3p_{1,0}=p_{0,0}+2p_{1,1}$ .......(2)<br>$3p_{0,1}=3p_{1,0}+2p_{b,1}$ .......(3)<br>$5p_{1,1}=p_{0,1}$ .......(4)<br>$2p_{b,1}=3p_{1,1}$ .. .....(5)<br>From the boundary condition<br>$p_{0,0}+p_{1,0}+p_{0,1}+p_{1,1}+p_{b,1}=1$.....(6)<br>$\Rightarrow$ P0,1 = (1\2)P0,0; P1,0 = (2\5) P0,0; P1,1= (1\10) P0,0; Pb,1 = (3\20) P0,0<br>Solve the above equations we get $p_{0,0}=\frac{20}{43}, p_{1,0}=\frac{8}{43}, p_{0,1}=\frac{10}{43}, p_{1,1}=\frac{2}{43}, p_{b,1}=\frac{3}{43}$<br>(i)Proportion of customers entering the shop is equal to $p_{0,0}+p_{0,1}=\frac{20}{43}+\frac{10}{43}=\frac{30}{43}$<br>(ii) $L_s=\sum_m\sum_n (m+n)p_{m,n}=1\left(p_{1,0}+p_{0,1}\right)+2\left(p_{1,1}+p_{b,1}\right)=\frac{28}{43}$<br>(iii) $W_s=\frac{L_s}{\lambda_A}, \lambda_A=$ Average rate entering of super market $\lambda_A=\frac{30}{43}(1)$,<br>$W_s=\frac{28/43}{30/43}=\frac{14}{15}$ hrs. = 56 min. |
| **5.** | **(i) For an open queuing network with three nodes 1, 2 and 3., let customers arrive from outside the system to node j according to a Poisson input process with parameters $r_j$ and let $P_{ij}$ denote the proportion of customers departing from facility i to facility j.**<br>**Given $(r_1, r_2, r_3)$ = ( 1, 4, 3 ) and $P_{ij}=\begin{bmatrix}0 & 0.6 & 0.3\\0.1 & 0 & 0.3\\0.4 & 0.4 & 0\end{bmatrix}$**<br>**Determine the average arrival rate λj to the node j for j = 1,2,3. (June2012)** |
| | **Solution:**<br>The traffic equations for the arrival rates are $\lambda_j=r_j+\sum_{i=1}^{k}\lambda_i P_{ij}$ , $j=1,2,\ldots k$<br>$\lambda_1=1+0.1\lambda_2+0.4\lambda_3;\lambda_2=4+0.6\lambda_1+0.4\lambda_3;\lambda_3=3+0.3\lambda_1+0.3\lambda_2$<br>$\lambda_1-0.1\lambda_2-0.4\lambda_3=1----------(1)$<br>$-0.6\lambda_1+\lambda_2-0.4\lambda_3=4----------(2)$<br>$-0.3\lambda_1-0.3\lambda_2+\lambda_3=3----------(3)$<br>(1) – (2) $\Rightarrow 1.6\lambda_1-1.1\lambda_2=-3------(4)$<br>(2) + 0.4(3) $\Rightarrow -0.72\lambda_1-0.88\lambda_2=2.8------(5)$<br>0.8(4) $\Rightarrow 1.28\lambda_1-0.88\lambda_2=-2.64-------(6)$<br>(4) + (6) $\Rightarrow 0.56\lambda_1=0.16 \quad \Rightarrow \lambda_1=0.2857$<br>From (4) $\lambda_2=3.1428$<br>From (3) $\lambda_3=4.0286$ |

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| | **(ii) Consider a system of two servers where customers from outside the system arrive at server 1 at a Poisson rate 4 and at server 2 at a Poisson rate 5. The service rates for server 1 and 2 are 8 and 10 respectively. A customer upon completion of service at server 1 is likely to go server 2 or leave the system; whereas a departure from server 2 will go 25 percent of the time to server 1 and will depart the system otherwise. Determine the limiting probabilities, $L_s$ and $W_s$.** **(June 2013, Nov/Dec 2015, Nov/Dec 2016)** |
| | **Solution:**<br>The traffic equations for the arrival rates are $\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij}$ , $j = 1, 2, \ldots k$<br>$\Rightarrow \lambda_1 = 4 + \frac{1}{4}\lambda_2; \lambda_2 = 5 + \frac{1}{2}\lambda_1$<br>$4\lambda_1 - \lambda_2 = 16 ------(1)$<br>$-\lambda_1 + 2\lambda_2 = 10 ------(2)$<br>Solving (1) and (2) we get, $\lambda_1 = 6$ and $\lambda_2 = 8$<br>$P(n_1, n_2) = (1-\rho_1)(1-\rho_2)\rho_1^{n_1} \quad \rho_2^{n_2} = \left(\frac{3}{4}\right)\left(\frac{4}{5}\right)\left(\frac{3}{4}\right)^{n_1}\left(\frac{4}{5}\right)^{n_2}$<br>$L_{s_1} = \frac{\lambda_1}{\mu_1 - \lambda_1} = \frac{6}{2}, \quad L_{s_2} = \frac{\lambda_2}{\mu_2 - \lambda_2} = \frac{8}{2}, \; L_s = 3 + 4 = 7$<br>$W_s = \frac{L_s}{\lambda} = \frac{7}{9}, \; \lambda = 4 + 5 = 9$ |