INDEX

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29/09/20

Combinatorics

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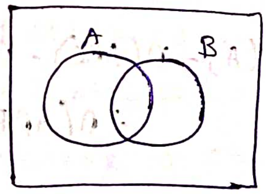
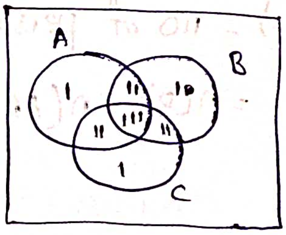
Syllabus

- → Principle of mutual inclusion & exclusion (5Q)
- → Euler function (4Q)
- → Sum rule, product rule (5Q)
- → Permutations & Combinations. (8Q)
- → Pegion hole Principle (3Q)
- → Derangements (2Q)
- → Recurrence Relations (20Q)
- → Generating functions (5Q)

Principle of mutual inclusion & exclusion:

$$n(A \cup B) = \underbrace{n(A) + n(B)}_{\text{inclusion}} - \underbrace{n(A \cap B)}_{\text{exclusion}}$$



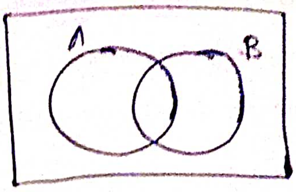


$$n(A \cup B \cup C) = \overbrace{n(A) + n(B) + n(C)}^{\text{inclusion}} - n(A \cap B) \underbrace{- n(B \cap C) - n(A \cap C)}_{\text{exclusion}} + \underbrace{n(A \cap B \cap C)}_{\text{inclusion}}$$

Similarly

$$n(A \cup B \cup C \cup D) = n(A) + n(B) + n(C) + n(D) - n(A \cap B) - n(A \cap C) - n(A \cap D) - n(B \cap C) - n(B \cap D) + n(A \cap B \cap C) + n(A \cap B \cap D) + n(B \cap D \cap C) + n(A \cap C \cap D) - n(A \cap B \cap C \cap D)$$

→ Assume that A & B denote two different skills



$n(A)$ = no of persons with skill A

$n(B)$ = no of persons with skill B

$n(A \cap B)$ = no of persons with both skill A & skill B

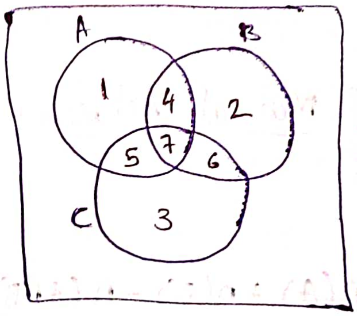
$n(A-B) = n(A) - n(A \cap B)$ = no of persons only with skill A

$n(B-A) = n(B) - n(A \cap B)$ = no of persons with only skill B

$n(A \Delta B)$ = no of persons with exactly one skill

$n(\overline{A \cup B})$ = no of persons having neither skill A nor skill B

→ Assume that A, B, C are 3 different skills



* $n(A \cap \bar{B} \cap \bar{C})$ = no of persons with only skill A

$$= n(A) - n(A \cap C) - n(A \cap B) + n(A \cap B \cap C)$$

* $n(\bar{A} \cap B \cap \bar{C})$ = no of persons with only skill B

$$= n(B) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C)$$

* No of persons with exactly one skill

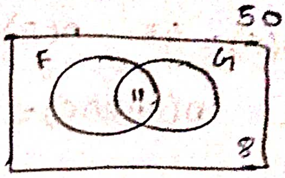
$$= n(A \cup B \cup C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + 2n(A \cap B \cap C)$$

* No of persons having only skill A & skill B

$$= n(A \cap B \cap \bar{C})$$

i.e., Region 4

P/01 $n(F \cap G) = 11$



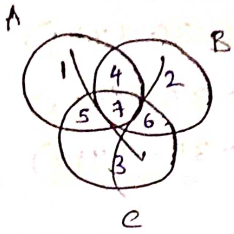
$n(\overline{F \cup G}) = 8$

$n(F \cup G) - n(F \cap G) = ?$

$n(\overline{F \cup G}) = 8 \Rightarrow n(F \cup G) = 50 - 8 = 42$

$n(F \cup G) - n(F \cap G) = 42 - 11 = 31$

P/02



$\{1,2,3,4,5,6,7\}$ — 31

$\{1,4,5,7\}$ — 20

~~$\{2,4,6,7\}$~~ — 16

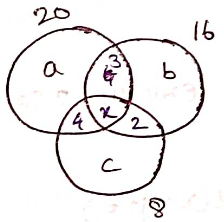
$\{3,5,6,7\}$ — 8

$A \cap B \cap \bar{C} \rightarrow \{4\}$ — 3

~~$A \cap \bar{B} \cap C \rightarrow \{5\}$~~ — 4

~~$\bar{A} \cap B \cap C \rightarrow \{6\}$~~ — 2

P/2



$\Rightarrow 20 = a + 3 + 4 + x \Rightarrow a = 13 - x$

$\Rightarrow 16 = b + 3 + 2 + x \Rightarrow b = 11 - x$

$\Rightarrow 8 = c + 4 + 2 + x \Rightarrow c = 2 - x$

$\Rightarrow$ ~~13-x~~ $a + b + c + 3 + 4 + 2 + x =$ ~~20~~ 31

$13 - x + 11 - x + 2 - x + 9 + x = 31$

$-2x + 35 = 31$

$\Rightarrow x = 2$

P/3 $n(\overline{5 \cup 6 \cup 8}) = 400 - n(5 \cup 6 \cup 8)$

$\Rightarrow n(5) = 80 \quad n(6) = 66 \quad n(8) = 50$

$n(5 \cap 6) = n(30) = 13, \quad n(5 \cap 8) = n(40) = 10, \quad n(6 \cap 8) = n(24) = 16$

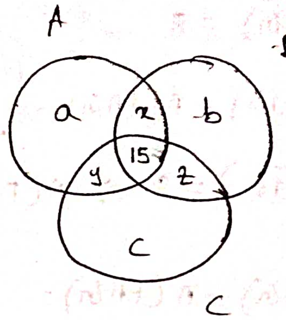
$n(5 \cap 6 \cap 8) = n(120) = 3 \quad \Rightarrow n(5 \cup 6 \cup 8) = 160$

$\Rightarrow$ Ans: 240

P/04

$n(A) = 45 \quad n(B) = 50 \quad n(C) = 50$

$n(A \cap B \cap C) = 15$



a) atleast two skills $= 50$

$x + y + z + 15 = 50$

given $n(A) = 45$

$\Rightarrow a + x + y + 15 = 45$

$\Rightarrow a + x + y = 30 \Rightarrow a = 30 - (x+y)$

Slly $b + x + z = 35 \Rightarrow b = 35 - (x+z)$

$c + y + z = 35 \Rightarrow c = 35 - (y+z)$

$n(A \cup B \cup C) = \underbrace{n(A) + n(B) + n(C)} - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

$\overset{80}{\cancel{a+b+c+x+y+z+15}} = 145 - (x+y+z+45) + 15$

$\cancel{a+b+c+x+y+z+15 = 115 - (x+y+z)}$

$\cancel{a+b+c+2(x+y+z) = 100}$

$\Rightarrow x + y + z = 115 - 80$

$\Rightarrow x + y + z = 35$ (i.e., exactly 2 skills)

$\Rightarrow a + b + c = 30 + 35 + 35 - 2(x+y+z)$

$= 100 - 70$

$= 30$ (i.e., exactly one skill)

$x + y + z + 15 = 35 + 15$

$= 50$ (i.e., atleast 2 skills)

$\cancel{(a+b+c) + (x+y+z}$

$80 - 15 = 65$ (i.e., atmost 2 skills)

P/5

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Prope

→ If

→ If

P/5 $n(A \cup B \cup C \cup D) = 42 + 36 + 28 + 24 - 6(12) + 4(8) - 4$

$= 130 - 72 + 32 - 4$

$= 162 - 76$

$= 86$

Eulers Function

Eulers function, $\phi(n)$ = no of +ve integers which are less than 'n' and coprime to 'n'.

Let 'n' be a positive integer

→ Let $n = p_1^{\alpha_1} \times p_2^{\alpha_2} \times p_3^{\alpha_3} \times \cdots \times p_k^{\alpha_k}$

where $p_1, p_2, \ldots p_k$ are k distinct prime numbers & $\alpha_1, \alpha_2 \ldots \alpha_k \in N$

then
$$\boxed{\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right)}$$

Eg: $\phi(12) = ?$

$12 = 2 \times 2 \times 3$

$= 2^2 \times 3$

$\phi(12) = (12)\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)$

$= 12\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = 2 \times 2 = 4$

→ If $\gcd(a,x) = 1$ & $\gcd(k,x) = 1$ then $\gcd(ak, x) = 1$

30/09/20

Properties of Euler's function

→ If 'p' is a prime number then $\phi(p) = p - 1$

→ If $n = pq$, where p, q are prime numbers then
$\phi(n) = \phi(p)\,\phi(q) = (p-1)(q-1)$

ii) $\rightarrow$ If $n = p^m$, where p is ~~pren~~ prime then

$$\phi(p^m) = p^{m-1}(p-1)$$

(X) no of positive integers ~~what~~ that are less than ~~2104~~ relatively

(P/54) prime to 210 is ______

sol:

$210 = 2 \times 3 \times 5 \times 7$

$\therefore \phi(210) = (2-1)(3-1)(5-1)(7-1)$ $\because$ all are distinct primes

$= (1)(2)(4)(6)$

$= \underline{48}$

$\rightarrow$ If $n = ab$ where a & b are co-primes

i.e., $\gcd(a,b) = 1$

$\Rightarrow \phi(n) = \phi(a).\phi(b)$

(P/55) $1368 = 2^3 \times 3^2 \times 19$

$\phi(1368) = (\cancel{2^3}2^2 \times \cancel{3^2}3 \times \cancel{19})\left(\frac{1}{\cancel{2}}\right)\left(\frac{2}{\cancel{3}}\right)\left(\frac{18}{\cancel{19}}\right)$

$= (4 \times 3)(2)(18)$

$= 432$

(P/56) 317 is a prime number

$\therefore \phi(317) = 316$

(P/57)

$\phi(p^k) = p^k\left(1-\frac{1}{p}\right) = \frac{p^k(p-1)}{p} = p^{k-1}(p-1)$

Note:

If $\phi(n) = x$, then

no of coprimes to n in the interval $[1, kn]$ is kx.

Note:

Euler's Theorem

→ If a, n are +ve integers ~~then~~ such that gcd(a, n) = 1

$$\boxed{a^{\phi(n)} \equiv 1 \pmod n}$$

→ If a is a +ve integer and p is a prime then

Fermat's little theorem:

$$\boxed{a^{p-1} \equiv 1 \pmod p}$$

$$\Rightarrow \boxed{a^{p} \equiv a \pmod p}$$

(~~we can multiply~~ 'a'
~~since gcd(a, p)~~ = 1
~~gcd(ap, p)~~)

(Q1) The value of expression $13^{99} \pmod{17}$ is ____

Sol:

$$a^{p-1} \equiv 1 \pmod p$$

Here a = 13,
p = 17

~~If $\phi(n) = x$, then~~
~~$\phi(kn) = k * x$~~
~~iff gcd(k, n) ≠ 1~~

$$\Rightarrow 13^{16} \equiv 1 \pmod{17}$$

Now $13^{99} \pmod{17} = \left(13^{6\times16} \cdot 13^{3}\right) \bmod 17$

$= \left[(13^{16})^{6} \cdot 13^{3}\right] \bmod 17$

$= \left((1)^{6} \cdot 13^{3}\right) \bmod 17$

$= (2197) \bmod 17$

$= 4$

Method 2:

$13^{16} \equiv 1 \pmod{17}$ gcd(13, 17) = 1

$\sim 13^{96} \equiv 1 \pmod{17}$ ⇒ gcd(13^{16}, 17) = 1

$\sim 13^{96} \cdot 13^{3} \equiv 1 \pmod{17}$

$\sim 13^{3} \equiv 13^{3} \pmod{7}$ ∴ 4

Note :

If $n = P_1^{\alpha_1} \cdot P_2^{\alpha_2} \cdot P_3^{\alpha_3} \cdots P_k^{\alpha_k}$ then

no of distinct positive integral factor for n is given by

$$\boxed{\tau(n) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)}$$

Proof:

$n = P_1^{\alpha_1} \cdot P_2^{\alpha_2} \cdots P_k^{\alpha_k}$

we can select $\underbrace{P_1^0, P_1^1, P_1^2 \ldots P_1^{\alpha_1}}_{\alpha_1+1 \text{ ways}}$ ... α_2+1 ways ... α_k+1 ways

$\Rightarrow$ total no of ways

$= (1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_k)$

(Q2) no of positive integral factors for 2014 = ____

Sol :

$2014 = 2 \times 19 \times 53$

2 ways, 2, 2

$\Rightarrow 2^3 = 8$ ways

(Q3) The exponent of 11 in prime factorization of 300! is ____

Sol :

~~No of~~

$11, 22, \ldots 11\times 27$ lie below 300

also 11×11 & 11×22 adds two more 11's

$\therefore$ exponent of 11 is $27+1+1 = 29$

Method 2 :

$\left\lfloor \frac{300}{11} \right\rfloor = 27$ ... 27 multiples of 11

within 27 we need search for no of multiples of 11 again

$\left\lfloor \frac{27}{11} \right\rfloor = 2$

within 2 we need to search for no of multiples of 11 again

$\left\lfloor \frac{2}{11} \right\rfloor = 0$

$\therefore$ exponent of $11 = 27 + 2 + 0 = 29$

* (Q4) Find exponent of 2 in 100!

Sol :

$\left\lfloor \frac{100}{2} \right\rfloor = 50$ (i.e., ~~1×50, 2×50,~~ ... 2×1, 2×2, 2×3, ... 2×50)

$\left\lfloor \frac{50}{2} \right\rfloor = 25$

$\left\lfloor \frac{25}{2} \right\rfloor = 12$

Understand the logic behind this

$\left\lfloor \frac{12}{2} \right\rfloor = 6$

$\left\lfloor \frac{6}{2} \right\rfloor = 3$

$\left\lfloor \frac{3}{2} \right\rfloor = 1$

$\left\lfloor \frac{1}{2} \right\rfloor = 0$

$\therefore$ exponent of 2 in 100! $= 50 + 25 + 12 + 6 + 3 + 1$

$= 97$

Congruences : (not needed for gate, but it would be helpful for solving ~~queue~~ questions related to ϕ function)

→ we say $a \equiv b \mod n$ iff $n \mid a-b$

$\Rightarrow nk = a-b$ for some integer k

$\Rightarrow a = b+nk$

$\therefore \boxed{a \equiv b \mod n \Leftrightarrow n \mid a-b \Leftrightarrow a = b+nk}$

→ if $a \equiv b \mod n$ and $c \equiv d \mod n$ then

$ac \equiv bd \mod n$ & $(a+c \equiv b+d) \mod n$

Proof :

$a \equiv b \mod n \Rightarrow a = b+nk_1$ --- ①

$c \equiv d \mod n \Rightarrow c = d+nk_2$ --- ②

① + ② $\Rightarrow (a+c) = (b+d) + n(k_1+k_2)$

$\Rightarrow (a+c) = (b+d) + nk$

$\Rightarrow a+c \equiv b+d \pmod n$

① × ② $\Rightarrow ac = bd + bnk_2 + dnk_1 + n^2 k_1 k_2$

$ac = bd + n(bk_2 + dk_1 + nk_1k_2)$.

$ac = bd + nk$

k is a constant

$\Rightarrow ac \equiv bd \pmod n$

→ $a \equiv b \mod n \Rightarrow a^2 \equiv b^2 \mod n$

$\Rightarrow a^3 \equiv b^3 \mod n$ (from above property)

$\Rightarrow a^k \equiv b^k \mod n$

→ If $a \equiv b \pmod n$ then

(i) $ac \equiv bc \pmod{cn}$ where c is a positive integer.

(ii) $ac \equiv bc \pmod n$

→ If $ab \equiv ac \pmod n$ then

$$b \equiv c \left(\mathrm{mod}\ \frac{n}{(a,n)}\right)$$ where (a,n) is gcd of a, n.

if $\gcd(a,n) = 1$ then $b \equiv c \bmod n$

~~Slly if $b \equiv c \bmod n$ then~~

~~$ab \equiv ac \bmod n * (a,n)$~~

Eg: Find $17^{341} \bmod 5$

$17 \bmod 5 = 2$

$17^2 \bmod 5 = 289 \bmod 5 = 4$

$17^2 \equiv 4 \bmod 5$

$\Rightarrow 17^4 \equiv 4^2 \bmod 5$

$17^4 \equiv 16 \bmod 5$

$\Rightarrow 17^4 \equiv 1 \bmod 5$

$\Rightarrow (17^4)^{85} \equiv (1)^{85} \bmod 5$

$17^{340} \equiv 1 \bmod 5$

$17^{341} \equiv 17 \bmod 5$

$\Rightarrow 17^{341} \bmod 5 = 2$

→ $(ab) \bmod p = [(a \bmod p)(b \bmod p)] \bmod p$

let $a \bmod p = r_1 \Rightarrow a = pq_1 + r_1$ --- ①

$b \bmod p = r_2 \Rightarrow b = pq_2 + r_2$ --- ②

① × ② $\Rightarrow ab = r_1 r_2 + p(q_1 r_2 + q_2 r_1 + pq_1 q_2)$

$\Rightarrow ab = r_1 r_2 + pk \Rightarrow ab \bmod p = (r_1 r_2 + pk) \bmod p$

$\Rightarrow ab \bmod p = r_1 r_2 \bmod p \Rightarrow ab \bmod p = [(a \bmod p)(b \bmod p)] \bmod p$

Note :

→ If $\phi(n) = x$ then $\phi(kn) = k * \phi(n) = kx$

iff ~~gcd(k,n)~~ prime factorization of k has subset of same set of primes as the prime factorization of n.

Proof :

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$

$\Rightarrow \phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right)$

~~Since gcd(k,n) ≠ 1~~

since prime factorization of k has ~~same~~ subset set of primes of the prime factorization of n

$\Rightarrow kn = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$

$\phi(kn) = kn\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right)$

$\phi(kn) = k * \phi(n)$

Sum Rule & Product Rule :

Sum Rule

→ Let $E_1, E_2, \ldots E_n$ are n independent events. If we can complete event E_1 in e_1 ways, E_2 in e_2 ways and so on E_n in e_n ways then no of ways to perform E_1 (or) E_2 (or) $E_3 \ldots$ or E_n

$= e_1 + e_2 + e_3 + \cdots + e_n$

Product Rule

→ Let, $E_1, E_2, \cdots E_n$ are n independent events If we can complete event E_1 in e_1 ways, E_2 in e_2 ways and so on E_n in e_n ways then the total no of ways to complete all event one after another (in order $E_1, E_2, \cdots E_n$)

$= e_1 * e_2 * e_3 * \cdots * e_n$

P/09

ET + EH + TH

$6C_1 \cdot 8C_1 + \cancel{8C_1 \cdot 10C_1 +}$

$6C_1 \cdot 10C_1 + 8C_1 \, 10C_1$

$48 + 60 + 80 = 188$

Q5) ~~Let~~ ~~Ap~~ How many bit strings of length n are palindromes?

Sol:

n is even:

let $n = 2k$

we need to fill k entries & remaining k entries are dependent.

$\therefore 2^k$ ways

$= 2^{n/2}$ ways

n is odd:

let $n = 2k+1$

we need to fill k+1 entries & remaining k entries depend on others

$\Rightarrow 2^{k+1}$ ways

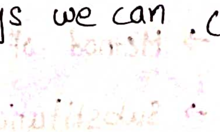
$k+1 = \lceil \frac{2k+1}{2} \rceil = \lceil n/2 \rceil$

$\therefore 2^{\lceil n/2 \rceil}$ ways

$\therefore$ general formaula is $2^{\lceil n/2 \rceil}$

Q6) A set S has n elements. How many ways we can choose subsets of P & Q of S such that $P \cap Q = \phi$?

Sol:



Every elements has to be either in P or in Q or not in any of them. $\therefore$ every element has 3 ways.

$\therefore$ total ways $= 3^n$

Recurrence Relations

→ Let S_n denote sum of first n natural numbers

i.e., $S_n = 1+2+3+\cdots+(n-1)+n$

we can form a recurrence relation for S_n as below

$$S_n = S_{n-1} + n$$

Defn of reccurence relation:

If $\{a_1, a_2, a_3, \ldots a_n\}$ is a sequence of real numbers,

A formula that relates a_n with one or more ~~pre~~ preceding terms of a_n, is called reccurence relation.

Eg: Fibanocci sequence

$$f(n) \begin{cases} 0 & , n=0 \\ 1 & , n=1 \\ f(n-1)+f(n-2) & , n>1 \end{cases}$$

topics:

1. Formation of recurrence relation
2. Solutions to recurrence relations
   - → Master theorem
     - Division
     - Subtraction
   - → Method of characteristic roots.
   - → Method of undetermined coefficients.
   - → Substitution method.

Master Theorem:

If $T(n) = aT\left(\frac{n}{b}\right) + \theta(n^k \log_b^p n)$ and

$a \geq 1, \; b > 1, \; k \geq 0,$

p is real number

(i) If $a > b^k$ then

$$T(n) = \theta(n^{\log_b a})$$

(ii) If $a = b^k$ then

a) if $p > -1$ then

$$T(n) = \theta(n^{\log_b a} \cdot \log_b^{p+1} n)$$

b) if $p = -1$ then

$$T(n) = \theta(n^{\log_b a} \cdot \log_b \log_b n)$$

c) if $p < -1$ then

$$T(n) = \theta(n^{\log_b a})$$

(iii) If $a < b^k$. then

a) if $p > 0$ then $T(n) = \theta(n^k \log_b^p n)$

b) if $p \leq 0$ then

$$T(n) = \theta(n^k)$$

Eg: If $T(n) = 4T(n/2) + n\log n$ then $T(n) =$ ____

Sol:

$\because \theta$ satisfies reflexive property $n\log n = \theta(n\log n)$

$$T(n) = aT(n/b) + \theta(n^k \log_b^p n)$$

Here $a = 4$, $b = 2$, ~~$k=0$~~ $k = 1$, $p = 1$

$(a \geq 1)$ $(b > 1)$ $(k \geq 0)$

$a = 4 = 4$; $b^k = 2^1 = 2$

$a > b^k$

$$\Rightarrow T(n) = \theta(n^{\log_b a}) = \theta(n^{\log_2 4}) = \theta(n^2)$$

Eg: $T(n) = 8T(n/3) + n$

$\Rightarrow T(n) = 8T(n/3) + \theta(n \log_3^0 n)$

$a = 8 \quad b^k = 3^1$

$a > b^k$

$\therefore T(n) = \theta(n^{\log_b a})$

$= \theta(n^{\log_3 8})$

Eg: $T(n) = 2T(n/2) + \sqrt{n}$

$a = 2 \quad b^k = 2^{1/2} = \sqrt{2}$

$a > b^k$

$T(n) = O(n^{\log_b a}) = O(n^{\log_2 2})$

$= O(n).$

Eg: $T(n) = 8T(n/2) + \log n$

$a = 8 \quad b^k = 2^0 = 1$

$a > b^k$

$\therefore T(n) = O(n^{\log_2 8}) = O(n^3)$

Eg: $T(n) = \sqrt{2}T(n/2) + \log n$

$a = \sqrt{2} \quad b^k = 2^0 = 1$

$a > b^k$

$\therefore T(n) = O(n^{\log_2 \sqrt{2}})$

$= O(n^{1/2}) = O(\sqrt{n})$

→ $T(n) = 4T(n/2) + n^2 \log n$

$a = b^k$; $p = 1 > -1$

$\therefore T(n) = \Theta(n^{\log_b a} \log_b^{p+1} n)$

$= \Theta(n^2 \log_2^2 n)$

→ $T(n) = \sqrt{2}\, T(n/2) + \sqrt{n} \log^{-1} n$

$a = \sqrt{2}$ $b = 2$ $k = 1/2$ $p = -1$

$a = b^k$

$\because p = -1 \Rightarrow T(n) = \Theta(n^{\log_b a} \cdot \log_b \log_b n)$

$= \Theta(\sqrt{n} \cdot \log_2 \log_2 n)$

*** → $T(n) = T(n/2) + \log^{-2} n$

$a = 1$ $b^k = 2^0 = 1$

$\therefore$ case (ii)

$p = -2 < -1$

$\therefore T(n) = \Theta(n^{\log_b a})$

$= \Theta(n^{\log_2 1})$

$= \Theta(1)$

→ $T(n) = 4T(n/2) + n^3 \log n$

$a = 4$ $b^k = 2^3 = 8 \Rightarrow a < b^k$

$\therefore$ case (iii)

$p = 1 \geq 0$

$\therefore T(n) = \cancel{\Theta(n^{\log}} \; \Theta(n^k \log_b^p n)$

$= \Theta(n^3 \log_2 n)$

$\rightarrow T(n) = 8T(n/3) + n^2 \log^{-1} n$

$a=8 \quad b^k = 3^2 = 9 \quad p=-1$

$a < b^k$

$\therefore$ case (iii)

$p \le 0$

$\therefore T(n) = \theta(n^k)$

$= \theta(n^2)$

Master Theorem for Subtraction

$\hookrightarrow$ This method doesn't give tight bount

if $T(n) = aT(n-b) + \theta(n^k)$, $a>0, b \ge 1, k \ge 0$

Case (i): If $a>1$, then $T(n) = \theta(n^k \cdot a^{n/b})$

Case (ii): If $a=1$, then $T(n) = \theta(n^{k+1})$

Case (iii): If $a<1$ then $T(n) = \theta(n^k)$

Examples:

$\rightarrow T(n) = 2T(n-1) + n$

$a=2 \quad b=1 \quad k=1$

$a>1$

$\therefore T(n) = \theta(n^k \cdot a^{n/b})$

$= \theta(n \cdot 2^n)$

$= \theta(n \cdot 2^n)$

But here solving with substitution

we, $\theta(2^n)$

$\therefore$ This master theorm gives upper bounds. It may not be tight sometimes.

→ $T(n) = 2T(n-1) + 1$

$a > 1 \Rightarrow T(n) = \Theta(n^k a^{n/b})$

$= \Theta(n^0 2^{n/1})$

$= \Theta(2^n)$

→ $T(n) = T(n-1) + n$

$a = 1$

$\therefore T(n) = \Theta(n^{k+1})$

$= \Theta(n^2)$

→ $T(n) = 0.5\,T(n-1) + n$

$a < 1$

$\therefore T(n) = \Theta(n^k)$

$\therefore T(n) = \Theta(n)$

Solving reccurence relations by using substitution method

Consider $a_n = n a_{n-1}$ where $a_0 = 1$

from this

$a_1 = (1) a_0 = 1$

$a_2 = 2 \cdot a_1 = 2 \cdot 1$

$a_3 = 3 \cdot a_2 = 3 \cdot 2 \cdot 1$

$\vdots$

$a_n = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$

$\therefore a_n = n!$

Eg: $a_n = a_{n-1} + 3^{n-1}$ where $a_1 = 2$ $\rightarrow T(n) = \begin{cases} 2, & n=1 \\ T(n-1)+3^{n-1}, & n>1 \end{cases}$ (P/69)

$a_n = (a_{n-2} + 3^{n-2}) + 3^{n-1}$

$a_n = a_{n-3} + 3^{n-3} + 3^{n-2} + 3^{n-1}$

$a_n = a_{n-k} + 3^{n-k} + 3^{n-(k-1)} + \dots + 3^{n-2} + 3^{n-1}$

$a_1 = 2$ is terminating condition

$\Rightarrow n-k = 1 \Rightarrow k = n-1$

$a_n = a_{n-(n-1)} + 3^{n-(n-1)} + 3^{n-(n-1-1)} + \dots + 3^{n-2} + 3^{n-1}$

$a_n = a_1 + 3^1 + 3^2 + 3^3 + \dots + 3^{n-1}$

$a_n = 2 + 3 + 3^2 + \dots + 3^{n-1}$

$a_n = 2 + \frac{3(3^{n-1}-1)}{2}$

$a_n = 2 + \frac{3^n - 3}{2}$

$a_n = \frac{3^n + 1}{2}$

Method 2:

$a_1 = 2$

$a_2 = a_1 + 3^1 = 2 + 3$

$a_3 = a_2 + 3^2 = 2 + 3 + 3^2$

$a_4 = a_3 + 3^3 = (2 + 3 + 3^2) + 3^3$

$\vdots$

$a_n = 2 + 3 + 3^2 + \dots + 3^{n-1}$

$\Rightarrow a_n = \frac{3^n + 1}{2}$

P/71

$a_n =$

Sol

$n = 1$

$) + 3^{n-1}, n > 1$

-1

P/69 $a_n = a_{n-1} + 3n^2 \quad , \quad a_0 = 7$

$a_n = a_{n-2} + 3(n-1)^2 + 3n^2$

$\vdots$

$a_n = a_{n-k} + 3(n-(k-1))^2 + \dots + 3n^2$

$n-k=0 \Rightarrow k=n$

$a_n = a_0 + 3(n-(n-1))^2 + 3(n-(n-2))^2 + \dots + 3n^2$

$a_n = 7 + 3(1)^2 + 3(2)^2 + \dots + 3n^2$

$a_n = 7 + \cancel{3}\,\dfrac{n(n+1)(2n+1)}{\cancel{6}\,2}$

$\therefore a_{20} = 7 + \dfrac{20(21)(41)}{2}$

$a_{20} = 7 + 10(21)(41)$

$a_{20} = 7 + 8610$

$= 8617$

$$\begin{array}{r} 41 \times 21 \\ \hline 41 \\ 82 \\ \hline 861 \end{array}$$

P/71 $a_n = a_{n-1} + (2n+1)$

sol:

$a_0 = 1$

$a_1 = a_0 + (2(1)+1)$

$= 1 + 2(1)+1$

$a_2 = a_1 + 2(2)+1$

$= 1 + 2(1)+1 + 2(2)+1$

$a_n = a_{n-1} + (2n+1)$

$a_n = a_{n-2} + (2(n-1)+1) + (2n+1)$

$\vdots$

$a_n = a_{n-k} + 2[n + (n-1) + \dots + n-(k-1)] + k$

$n-k=0 \Rightarrow k=n$

$a_n = a_0 + 2[1+2+\dots+n] + n$

$= 1 + \cancel{2}\,\dfrac{n(n+1)}{\cancel{2}} + n$

$= n^2+2n+1 = (n+1)^2$

P/72

$$a_n = a_{n-1} + \frac{1}{n(n+1)} \quad , a_0 = 1$$

$$= a_{n-2} + \frac{1}{(n-1)(n)} + \frac{1}{n(n+1)}$$

$$= a_{n-2} + \left[\frac{1}{n-1} - \frac{1}{n}\right] + \left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$= a_{n-3} + \cancel{\frac{1}{n-2}\ \frac{1}{n}}\ \frac{1}{n-1} - \frac{1}{n+1}$$

$$= a_{n-3} + \left[\frac{1}{n-2} - \frac{1}{n-1}\right] + \left[\frac{1}{n-1} - \frac{1}{n+1}\right]$$

$$= a_{n-3} + \frac{1}{n-2} - \frac{1}{n+1}$$

$$\vdots$$

$$= a_{n-k} + \frac{1}{n-(k-1)} - \frac{1}{n+1}$$

$$\vdots$$

$$= a_0 + \frac{1}{n-(n-1)} - \frac{1}{n+1}$$

$$= 1 + 1 - \frac{1}{n+1}$$

$$= \frac{2(n+1)-1}{n+1}$$

$$= \frac{2n+1}{n+1}$$

P/68

$$a_n = a_{n-1} + 2(n-1) \quad , \quad a_1 = 2$$

$$= a_{n-2} + 2(n-2) + 2(n-1)$$

$$\vdots$$

$$= a_{n-k} + 2\left[(n-k) + (n-(k-1)) + \cdots + (n-1)\right]$$

$$\vdots$$

$$= a_{n-(n-1)} + 2\left[(n-(n-1)) + (n-(n-2)) + \cdots + (n-1)\right]$$

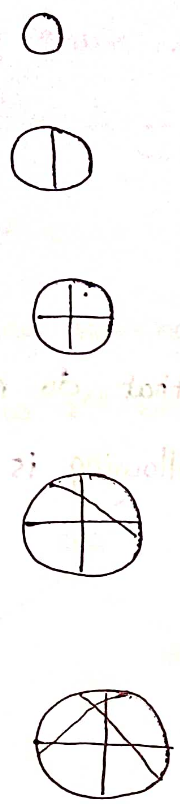
$$= a_1 + 2\left[1 + 2 + \cdots + (n-1)\right]$$

$$= 2 + \cancel{2}\frac{(n-1)(n)}{\cancel{2}} = n^2 - n + 2$$

Formation of recurrence relations:

P/63

n		f(n)
0		1
1		2
2		4
3		7
4		11

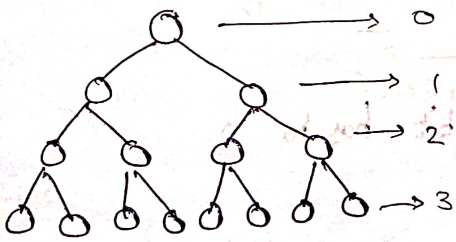


Every i^{th} cut is drawn such that it intersect previous $i-1$ cuts in $i-1$ distinct points. Here every i^{th} cut adds i extra pieces.
In other words n cuts give maximum no of cuts if no 3 cuts are concurrents.

$$\therefore f(n) = f(n-1) + n$$

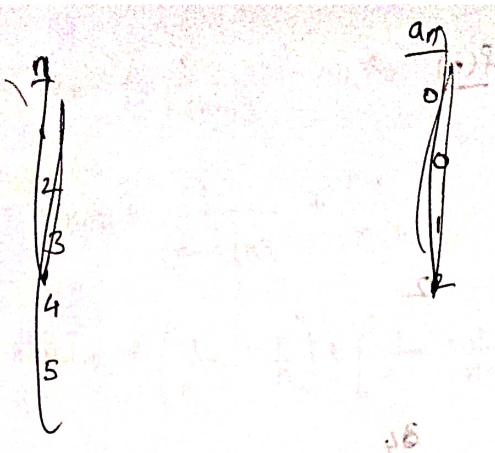
P/64

depth



$n(d)$, $n(d-1)$

$$\Rightarrow n(d) = n(d-1) + 2^d$$



Q. Let a_n denote no of n-bits string that do not contains two consecutive 1's. which of the following is the recurrence relation for a_n?

a) $a_n = a_{n-1} + 2a_{n-2}$

b) $a_n = a_{n-1} + a_{n-2}$ //

c) $a_n = 2a_{n-1} + a_{n-2}$

d) $a_n = 2a_{n-1} + 2a_{n-2}$

Sol:

we consider cases where string starts with 0 or 1.

Case (i): $0 \underbrace{\_\ \_\ \_ \cdots \_}_{a_{n-1}}$

Case (ii): $1\ 0 \underbrace{\_\ \_\ \_ \cdots \_}_{a_{n-2}}$

$\because$ first bit is 1, 2nd bit has to 0

$\therefore a_n = a_{n-1} + a_{n-2}$

P/66

Case (i) :

String starts with '1'.

$\underline{1}\ \underbrace{\_\ \_\ \_\ \_\ \ldots\ \_}_{a_{n-1}}$

Case (ii) :

String starting with '0'

now 2nd bit can be either 0 or 1

00

01 ↳ a_{n-2}

000

001 ↳ a_{n-3}

000 ↓ 2^{n-3}

$\therefore\ a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$

P/65

Case (i) :

first digit is 1/2 (or) 3 ~~(or) 4~~

$\underline{}\ \underbrace{\_\ \_\ \_\ \ldots\ \_}_{a_{n-1}}$

1 (or) 2 (or) 3 ~~(or) 4~~

$\therefore\ 3a_{n-1}$

Case (ii) :

~~first digit is 0.~~

~~00 $\underbrace{\_\ \_\ \ldots}_{a_{n-2}}$~~

~~01 02 03 04~~

Case (ii) :

$\underline{0} \; \underbrace{- - - \cdots -}$

no of ~~od~~ $(n-1)$ bit strings with odd no of 0's.

no of string ~~of~~ of ~~length~~ odd 0's = total - no of strings of even 0's

$$= 4^{n-1} - a_{n-1}$$

$$\therefore a_n = 3a_{n-1} + 4^{n-1} - a_{n-1}$$

$$a_n = 2a_{n-1} + 4^{n-1}$$

(P/67)

Case i :

Starting with 1

$\underline{1} \; \underbrace{- - \cdots -}_{a_{n-1}}$

Case (ii) :

Starting with '0':

If one '0' occur then all the following ~~has~~ bits has to be 0.

$\therefore \; \underline{0} \; \underline{0} \; \underline{0} \cdots \underline{0}$

$\therefore$ 1 ways

$$\therefore a_n = a_{n-1} + 1$$

(a) Recurrence relation for n-bit string to contain 2 consecutive 1's.

Sol :

$\underline{1} \; \underline{\;} \; - - \cdots -$

$\underline{1} \; \underline{0} \; \underbrace{- - \cdots -}_{a_{n-2}}$ $\qquad$ $\underline{1} \; \underline{1} \; \underbrace{\cdots -}_{2^{n-2}}$

$\underbrace{0 \; - \; \cdots \; -}_{a_{n-1}}$

0's.

$\therefore a_n = a_{n-2} + a_{n-1} + 2^{n-2}$

even 0's

Method of Characteristic roots:

Shift operators:

"E" (shift operator) shifts given term by specified no of terms.

Eg: $E^k(a_n) = a_{n+k}$

$E^2(a_n) = a_{n+2}$

Let $\phi(E)$ be a polynomial, E is a shift operator

Here we construct eqn

$$\phi(E)\, a_n = F(n)$$

where F(n) could be of form as shown below

bits has to be a.

F(n)

- $F(n) = 0$ (Homogeneous eqn) → soln $(a_n) = CF$
- $F(n) \neq 0$ ... (non-homogeneous eqn)
  - $F(n) = b^n$
  - $F(n) = n^k$
  - $F(n) = b^n n^k$

  soln $(a_n) = CF + PS$

ve 15.

CF → Complementary function. (found using method of characteristic roots)

PS → Particular solution. (found using method of undetermine coefficients)

Consider the general rec. eqn

$$l_0 a_n + l_1 a_{n-1} + l_2 a_{n-2} + \cdots + l_k a_{n-k} = f(n)$$

Replace n by $n+k$

$$l_0 a_{n+k} + l_1 a_{n+k-1} + l_2 a_{n+k-2} + \cdots + l_k a_n = f(n+k)$$

$$l_0 E^k(a_n) + l_1 E^{k-1}(a_n) + l_2 E^{k-2}(a_n) + \cdots + l_k E^0(a_n) = f(n+k)$$

$$(l_0 E^k + l_1 E^{k-1} + l_2 E^{k-2} + \cdots + l_k) a_n = f(n+k)$$

i.e., $\phi(E) a_n = F(n)$

where $F(n) = f(n+k)$

$$\phi(E) = l_0 E^k + l_1 E^{k-1} + \cdots + l_k$$

Characteristic eqn is $\phi(t) = 0$

$$l_0 t^k + l_1 t^{k-1} + \cdots + l_k = 0$$

By solving above equn we get k roots.
↳ characteristic roots.

Based the roots we construct CF.

(i) If all k roots are distint say $t_1, t_2, t_3 \ldots t_k$ then

$$CF = C_1 t_1^n + C_2 t_2^n + \cdots + C_k t_k^n$$

(ii) If two roots are equal say $t_1 = t_2$ and remaining are distinct then

$$CF = (C_1 + C_2 n) t_1^n + C_3 t_3^n + \cdots + C_k t_k^n$$

(iii) If

then

(iv) If

are

(v) If

Finding P

Co

(i)

(iii) If 3 roots are equal say $t_1 = t_2 = t_3$ and remaining are distinct

then

$$CF = (c_1 + c_2 n + c_3 n^2) t_1^n + c_4 t_4^n + \cdots + c_k t_k^n$$

(iv) If m roots are equal $(m \le k)$ say $t_1 = t_2 = \cdots = t_m$ and remaining are distinct then

$$CF = (c_1 + c_2 n + c_3 n^2 + \cdots + c_m n^{m-1}) t_1^n + c_{m+1} t_{m+1}^n + \cdots + c_k t_k^n$$

(v) If there is a pair of complex roots say $\alpha \pm i\beta$ then

$$CF = r^n (c_1 \cos\theta + c_2 \sin\theta)$$

where $r = \sqrt{\alpha^2 + \beta^2}$

$\theta = \tan^{-1}(\beta/\alpha)$

Finding Particular Soln:

Consider $\phi(E) a_n = F(n)$

(i) If $F(n) = b^n$

* if $\phi(b) \neq 0$ i.e., b is not a characteristic root

then $PS = \frac{1}{\phi(b)} b^n$

* If $\phi(b) = 0$ i.e., b is a characteristic root

then $P.S = {}^nC_k \, b^{n-k}$

$k \to$ multiplicity of root 'b'.

07/10/20

Eg: Find PS for

$$(E-3)^4(E-2)\, a_n = 3^n$$

$b^n = 3^n \Rightarrow b = 3$

& b is root.

$$\therefore PS = \frac{1}{\phi(E)} 3^n$$

$$= \frac{1}{(E-3)^4(E-2)} 3^n$$

$$= \frac{1}{(E-3)^4}\left(\frac{3^n}{E-2}\right)$$

$$= \frac{1}{(E-3)^4} \frac{3^n}{1}$$

$$= \frac{1}{(E-3)^4} 3^n$$

$$PS = {}^nC_4\, 3^{n-4}$$

Eg: Find soln to rec. eqn

$$T(2^k) = 3T(2^{k-1}) + 1 \text{ where } T(0) = 1$$

a) 2^k b) $\frac{3^{k+1}-1}{2}$ c) $2\log_3 k$ c) $3^{\log_2 k}$.

Sol:

Let $T(2^k) = a_k$

$\Rightarrow T(1) = a_0 = 1$

$\Rightarrow a_k = 3a_{k-1} + 1$

$\Rightarrow a_k - 3a_{k-1} = 1$

find

find

Soln,

Finding CF:

replace k by k+1

$a_{k+1} - 3a_k = 1$

$(E-3)\, a_k = 1$

$\phi(E)\, a_k = F(n)$

$\phi(E) = E-3 \qquad F(n) = 1 = b^n$

$\Rightarrow b = 1$

$\phi(t) = 0$

$\Rightarrow t-3 = 0 \Rightarrow t = 3$

$\therefore CF = c_1 t_1^k = c_1 3^k$

Finding PS:

$\phi(b) = \phi(1) = 1-3 = -2 \neq 0$

$PS = \frac{1}{\phi(b)} b^k$

$= \frac{1}{-2}(1)^k$

$= -\frac{1}{2}$

Soln, $a_k = CF + PS$

$a_k = c_1 3^k - \frac{1}{2}$

to find c_1 put $a_0 = 1$

$\Rightarrow a_0 = c_1 3^0 - \frac{1}{2}$

$1 = c_1 - \frac{1}{2} \Rightarrow c_1 = 3/2$

$\therefore a_k = \frac{3}{2} 3^k - 1/2$

i.e., $a_k = \frac{3^{k+1} - 1}{2}$

Eg: $a_n - 2a_{n-1} = \cancel{0}\ -1$; $a_1 = 2$

replace n by $n+1$

$a_{n+1} - 2a_n = -1$

$\Rightarrow (E-2)a_n = -1$

$\Rightarrow$ ~~characteristic~~ root, $t_1 = 2$

$CF = c_1 2^n$

$F(n) = -1 = -1 \cdot 1^n = b^n$

$\Rightarrow b = 1$

$\phi(b) = \phi(1) = 1-2 = -1 \neq 0$

$\therefore PS = \frac{1}{\phi(b)} (-1) 1^n$

$= (-1)\left[\frac{1}{\phi(b)} 1^n\right]$

$= (-1)\left[\frac{1}{(-1)} \cdot 1\right] = 1$

$a_n = CF + PS$

$a_n = c_1 2^n + 1$

given $a_1 = 2$

$\Rightarrow a_1 = c_1 2 + 1$

$2 = 2c_1 + 1$

$\Rightarrow c_1 = 1/2$

$\Rightarrow a_n = 2^{n-1} + 1$

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Q: $\sqrt{a_n} - \sqrt{a_{n-1}} - 2\sqrt{a_{n-2}} = 0$, $a_0 = a_1 = 1$

$\downarrow$

let $\sqrt{a_n} = x_n$ $\qquad x_0 = \sqrt{a_0} = 1 \quad x_1 = \sqrt{a_1} = 1$

$x_n - x_{n-1} - 2x_{n-2} = 0$

replace n by '$n+2$'

$x_{n+2} - x_{n+1} - 2x_n = 0$

$(E^2 - E - 2)x_n = 0$

$\phi(t) = 0$

$t^2 - t - 2 = 0$

$(t-2)(t+1) = 0$

$t = -1, 2$

$\Rightarrow CF = c_1 t_1^n + c_2 t_2^n$

$= \cancel{c_1(1)^n + c_2(2)^n} = c_1(-1)^n + c_2(2)^n$

$\cancel{a_n = c_1 + 2^n c_2}$

$\cancel{a_0 = 1 \Rightarrow a_0 = c_1 + c_2 = 1}$

$\cancel{a_1 = 1 \Rightarrow a_1 = c_1 + 2c_2 = 1}$

$\cancel{-c_2 = 0}$

$\cancel{\Rightarrow c_2 = 0}$

$\cancel{\Rightarrow c_1 = 1}$

$\cancel{\therefore a_n = 1}$

$x_0 = 1 \Rightarrow c_1 + c_2 = 1$

$x_1 = 1 \Rightarrow -c_1 + 2c_2 = 1$

$3c_2 = 2$

$c_2 = 2/3$

$\Rightarrow c_1 = 1/3$

$\Rightarrow x_n = \frac{1}{3}(-1)^n + \frac{2}{3}2^n$

$\Rightarrow a_n = x_n^2$

$\Rightarrow a_n = \left(\frac{1}{3}\right)^2\left[2^{n+1} + (-1)^n\right]^2$

$\Rightarrow a_n = \frac{(2^{n+1} + (-1)^n)^2}{9}$

P/75 $a_n - 2a_{n-1} = 3(2^n)$ → replace 'n' by n+1

$\Rightarrow (E-2)\, a_n = 3.2^{n+1}$

$t = 2$

$C.F = c_1 2^n$

$\phi(2) = 2-2 = 0$

$\Rightarrow PS = \frac{1}{\phi(A)} 3.2^{n+1}$

$= 6 \left[\frac{1}{\phi(A)} \cdot 2^n \right]$

$= 6\, n c_1\, 2^{n-1}$

$= 3n 2^n$

$\therefore a_n = c_1 2^n + 3n 2^n$

P/76 $a_n - 3a_{n-1} + 2a_{n-2} = 2^n.$

$(E^2 - 3E + 2)\, a_n = 2^{n+2}$

$(t-1)(t-2) = 0$

$t = 1 \,\&\, 2$

$CF = c_1 + c_2 2^n$

$PS = \frac{1}{(E-1)(E-2)} F(n)$

$= \frac{1}{(E-1)(E-2)} 2^{n+2}$

$= 4 \cdot \frac{1}{(2-1)(E-2)} 2^n$

$= 4 \left(\frac{1}{E-2} 2^n \right)$

$= 4\, n c_1\, 2^{n-1}$

$= n 2^{n+1}$

$\Rightarrow a_n = c_1 + c_2 \cdot 2^n + n 2^{n+1}$

$$\Rightarrow a_n = c_1 + c_2 \cdot 2^n + (2n)\,2^n$$

$\therefore$ opt (a)

P/[illegible]

$$a_n - 6a_{n-1} + 9a_{n-2} = 3^n$$

$$a_{n+2} - 6a_{n+1} + 9a_n = 3^{n+2}$$

$$\Rightarrow (E^2 - 6E + 9)\,a_n = 3^{n+2}$$

$$(E-3)^2 a_n = 3^{n+2}$$

$$CF = (c_1 + c_2 n)\,3^n$$

$$PS = 9\,\frac{1}{(E-3)^2}\,3^n$$

$$= 9\left({}^{n}C_2\, 3^{n-2}\right)$$

$$\Rightarrow a_n = (c_1 + c_2 n)\,3^n + \frac{n(n-1)}{2}\,3^n$$

$\therefore$ opt (a)

[illegible]

P/[illegible]

$$a_n - 2a_{n-1} + a_{n-2} = 2^n$$

$$(E^2 - 2E + 1)\,a_n = 2^{n+2}$$

$$(E-1)^2 a_n = 2^{n+2} \Rightarrow CF = (c_1 + c_2 n)(1)^n = c_1 + c_2 n$$

$$\Rightarrow \phi(2) \neq 0$$

$$\therefore PS = \frac{1}{\phi(2)}\,2^{n+2}$$

$$= 2^{n+2}$$

$$a_n = c_1 + c_2 n + 2^{n+2}$$

Rule 2 for finding P.S:

If $F(n) = n^k . b^n$

~~Case(i): If b is not a characteristic root~~

~~then p.s = $(A_0 n^k + A_1 n^{k-1} + A_2 n^{k-2} \dots + A_k)$~~

(i) If b is a characteristic root with multiplicity 'm' then

$$a_n = b^n (A_0 n^k + A_1 n^{k-1} + \dots + A_k) n^m$$

(ii) If b is not a characteristic root then

$$a_n = b^n (A_0 n^k + A_1 n^{k-1} + \dots + A_k)$$

Eg: $(E-2) a_n = 2^n . (n + 1)$

$\Rightarrow b = 2,\ k = 1$

'2' is root with multiplicity '1'

$$Ps(a_n) = b^n (A_0 n^k + A_1 n^{k-1} + \dots + A_k) n^m = 2^n (A_0 n + A_1) n^1$$

Eg: $(E-2)^3 a_n = 2^n (n^2 + n + 1)$

$\Rightarrow b = 2 \quad k = 2$

2 is root with multiplicity '3'

$$Ps(a_n) = 2^n (A_0 n^2 + A_1 n + A_2) n^3$$

Eg: $(E-3)^4 a_n = 3^n (n^3 + 1)$

$b = 3 \quad k = 3$

'3' is root wit multiplicity 4

$$PS = 3^n (A_0 n^3 + A_1 n^2 + A_2 n^1 + A_3) n^4$$

Eg: $(E-2)a_n = 3^n(n^3+1)$

$b=3 \quad k=3$

$3'$ is not a char. root

$\Rightarrow PS = 3^n(A_0n^3 + A_1n^2 + A_2n + A_3)$

Q The rec. eqn

$T(1)=1$

$T(n) = 2T(n-1)+n \quad (n\geq 2)$ evaluate to ____

a) $2^{n-1}-n-2$ b) 2^n-n c) $2^{n+1}-2n-2$ d) 2^n+n

Sol:

$a_n = 2a_{n-1}+n \quad a_0=1$

$a_n - 2a_{n-1} = n \longrightarrow (1)$

$(E-2)a_n = n+1$

$CF = C_1 2^n$

PS. $F(n) = n+1$

$b=1 \quad k=1 \longrightarrow$ not a char. root

$\Rightarrow PS = b^n(A_0n^k + A_1n^{k-1} + \cdots + A_k)$

$= (1)^n(A_0n + A_1)$

$PS(a_n) = A_0n + A_1$

Substitute a_n in eqn (1) $\longrightarrow$ Finding undetermined coefficients.

$\Rightarrow A_0n + A_1 - 2(A_0(n-1)+A_1) = n$

$\Rightarrow A_0n - 2A_0n + A_1 + 2A_0 - 2A_1 = n$

$-A_0n + 2A_0 - A_1 = n$

equating coefficient

$A_0 = -1$

$2A_0 - A_1 = 0 \Rightarrow A_1 = -2$

$\Rightarrow a_n = CF + PS$

$a_n = c_1 \cdot 2^n - n - 2$

$a_1 = 1$

$\Rightarrow 1 = 2c - 1 - 2$

$\Rightarrow c = 2$

$\Rightarrow a_n = 2^{n+1} - n - 2$

P/79

$a_n - 3a_{n-1} = n+3 \rightsquigarrow (1)$

$(E-3)a_n = n+4$

$CF = c_1 3^n$

$PS = (1)^n (A_0 n + A_1) n^0 \rightsquigarrow$ multiplicity of $b=1$

$PS(a_n) = A_0 n + A_1$

$(1) \Rightarrow A_0 n + A_1 - 3(A_0(n-1) + A_1) = n+3$

$A_0 n + A_1 - 3A_0 n + 3A_0 - 3A_1 = n+3$

$-2A_0 n + 3A_0 - 2A_1 = n+3$

$\cancel{\Rightarrow A_0 = -1/2}$

$\cancel{\Rightarrow 3A_0 + 1 = 3}$ $\quad \cancel{3(-1/2) - 2A_1 = 3}$

$\cancel{\Rightarrow A_0 = 2/3}$ $\quad \cancel{-2A_1 = 3 + 3/2}$

$\cancel{A_1 = -\frac{9}{4}}$

$\Rightarrow a_n = c_1 3^n + \frac{2}{3} A_0$

Put $n=0 \Rightarrow A_1 + 3A_0 - 3A_1 = 3$

$3A_0 - 2A_1 = 3$

put $n=1 \Rightarrow$ $\cancel{A_0 + A_1 - 3A_0}$

$A_0 + A_1 - 3A_1 = 4$

$A_0 - 2A_1 = 4$

$3A_0 - 2A_1 = 3$

$-2A_0 = 1 \Rightarrow A_0 = -1/2$

$\Rightarrow -\frac{1}{2} - 2A_1 = 4$

$-2A_1 = 9/2$

$A_1 = -9/4$

$$a_n = c_1 3^n + \cancel{3}\left(\frac{-1}{2}\right)^n \cancel{A_0} - \frac{9}{4}$$

$$a_n = c_1 3^n - \cancel{A_0}\frac{n}{2} - \frac{9}{4}$$

P/80

$$a_n - 2a_{n-1} + a_{n-2} = 3n+5$$

$$(E^2 - 2E + 1)a_n = 3(n+2)+5$$

$$(E^2 - 2E + 1)a_n = 3n + 11$$

CF :

roots $\to 1, 1$

$$CF = (c_1 + c_2 n)(1)^n$$

$$= c_1 + c_2 n$$

PS :

$b = 1 \quad k = 1$

$\Rightarrow$ b is root with multiplicity 2

$$\Rightarrow PS\{a_n\} = b^n (A_0 n^1 + A_1) n^m$$

$$= (1)^n (A_0 n + A_1) n^2$$

$$PS(a_n) = A_0 n^3 + A_1 n^2$$

$$\Rightarrow (A_0 n^3 + A_1 n^2) - 2(A_0 (n-1)^3 + A_1 (n-1)^2)$$

$$+ (A_0 (n-2)^3 + A_1 (n-2)^2)$$

$$= 3n + 5$$

$$n=0 \Rightarrow -2(-A_0 + A_1) + (-8A_0 + 4A_1) = 5$$

$$\Rightarrow 2A_0 - 2A_1 - 8A_0 + 4A_1 = 5$$

$$-6A_0 + 2A_1 = 5$$

$$n=1 \Rightarrow (A_0 + A_1) + (-A_0 + A_1) = 8$$

$$\Rightarrow 2A_1 = 8 \Rightarrow A_1 = 4$$

$$\Rightarrow -6A_0 + 2(4) = 5$$

$$-6A_0 = -3$$

$$A_0 = 1/2$$

$$\Rightarrow PS(a_n) = \frac{n^3}{2} + 4n^2$$

$\Rightarrow a_n = c_1 + c_2 n + 4n^2 + \frac{n^3}{2}$

P|82

P|81 $a_n = 4a_{n-1} + 3n2^n$

$a_n - 4a_{n-1} = 3n.2^n$ —— ①

$(E-4)a_n = 3(n+1)2^{n+1} = 3n2^{n+1} + 32^{n+1} = 2^n(6n+6)$

CF = ~~CF~~ $c_1 4^n$

PS : $b=2$ $k=1$

multiplicity $\to 0$

$PS(a_n) = 2^n(A_0 n + A_1)$

① $\Rightarrow 2^n(A_0 n + A_1) - 4(2^{n-1}(A_0(n-1) + A_1)) = 3n2^n$

~~$n=0 \Rightarrow A_1 - 4(-A_0 + A_1) = 0$~~

$\Rightarrow$ ~~$4A_0 - 3A_1 = 0$~~

~~$n=1 \Rightarrow 2A_0 + 2A_1 - 4A_1 = 6$~~

$\Rightarrow$ ~~$2A_0 - 2A_1 = 6$~~

$A_0 - A_1 = 6$

$3A_0 - 3A_1 = 18$

$4A_0 - 3A_1 = 0$

$-A_0 = 18$

$\Rightarrow A_0 = -18$

$\Rightarrow -18 - A_1 = 6$

$\Rightarrow A_1 = -12$

$\Rightarrow PS(a_n) = 2^n(-18n - 12) = -6.2^n(3n+2)$

$a_n = c_1 4^n - 6.2^n(3n+2)$

$a_0 = 4 \Rightarrow 4 = c_1 - 6(2) \Rightarrow c_1 =$

P[82]

$$a_{n+2} - 2a_{n+1} + a_n = n^2 2^n \quad \text{---} ①$$

$$(E - 2E + 1) a_n = n^2 2^n$$

$$CF = c_1 + c_2 n$$

$$PS(a_n) = 2^n (A_0 n^2 + A_1 n + A_2)$$

$$① \Rightarrow \quad 2^{n+2}\left(A_0 (n+2)^2 + A_1 (n+2) + A_2\right) - 2\cdot 2^{n+1}\left(A_0 (n+1)^2 + A_1 (n+1) + A_2\right)$$

$$+ 2^n (A_0 n^2 + A_1 n + A_2) = n^2 \cdot 2^n$$

$$n = -2 \Rightarrow \quad 1\cdot(A_2) - (1)(A_0 - A_1 + A_2) + \frac{1}{4}(4A_0 - 2A_1 + A_2)$$

$$= 4 \cdot \frac{1}{4}$$

$$\Rightarrow \quad \cancel{A_2} - \cancel{A_0} + A_1 - \cancel{A_2} + \cancel{A_0} - \frac{A_1}{2} + \frac{A_2}{4} = 1$$

$$\frac{A_1}{2} + \frac{A_2}{4} = 1$$

$$\Rightarrow 2A_1 + A_2 = 2$$

$$n = -1 \Rightarrow \quad 2(A_0 + A_1 + A_2) - 2(A_2) + \frac{1}{2}(A_0 - A_1 + A_2) = 1/2$$

$$\Rightarrow 2A_0 + 2A_1 + \cancel{2A_2} - \cancel{2A_2} + \frac{A_0}{2} - \frac{A_1}{2} + \frac{A_2}{2} = \frac{1}{2}$$

$$\frac{5A_0}{2} - \frac{3A_1}{2} + \frac{A_2}{2} = 1/2$$

$$\Rightarrow 5A_0 - 3A_1 + A_2 = 1$$

$$n = 0 \Rightarrow \quad 4(4A_0 + 2A_1 + A_2) - 4(A_0 + A_1 + A_2) + (A_2) = 1$$

$$\Rightarrow 16A_0 + 8A_1 + \cancel{4A_2} - 4A_0 - 4A_1 - \cancel{4A_2} + A_2 = 1$$

$$12A_0 - 4A_1 + A_2 = 1$$

$$\begin{bmatrix} 12 & -4 & 1 & 1 \\ 5 & -3 & 1 & 1 \\ 0 & 2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 12 & -4 & 1 & 1 \\ 0 & -16 & 7 & 7 \\ 0 & 2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 12 & -4 & 1 & 1 \\ 0 & -16 & 7 & 7 \\ 0 & 0 & 3 & 5 \end{bmatrix}$$

$\Rightarrow 3A_2 = 5 \Rightarrow A_2 = 5/3$

$-16A_1 + 7A_2 = 7$

$-16A_1 + \frac{35}{3} = 7$

$-16A_1 = -\frac{14}{3}$

$A_1 = \frac{7}{24}$

Eg: ~~$T(n) = T(\sqrt{n}) + 1$, if $n > 2$~~

$= 1$ if $n \leq 2$

Eg: $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$

divide by n on both sides

$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

let $S(n) = \frac{T(n)}{n}$

$S(n) = S(\sqrt{n}) + 1$

$\Rightarrow S(n) = 1 + \log_2 \log_2 n$ → solve it

$\Rightarrow \frac{T(n)}{n} = 1 + \log_2 \log_2 n$

$\Rightarrow T(n) = n + n \log_2 \log_2 n$

Note :

If $T(n) = T(\alpha n) + T((1-\alpha)n) + O(n)$, $0 < \alpha < 1$ & $\alpha < 1-\alpha$

$$\Rightarrow \Omega(n \log_{1/\alpha} n) \leq T(n) \leq O(n \log_{\frac{1}{1-\alpha}} n)$$

~~also asked~~

Eg : If $T(n) = T(n/5) + T(4n/5) + n$

$$\Omega(n \log_{1/\alpha} n) \leq T(n) \leq O(\log_{\frac{1}{1-\alpha}} n)$$

$$\Omega(n \log_5 n) \leq T(n) \leq O(\log_{5/4} n)$$

09/10/20

Permutations & Combinations

Model 1: (dividing objects into groups)

Find no of ways in which we can distribute 10 distinct objects into two groups such that one group contain 4 and other 6?

Ans : $^{10}C_4 \times {}^6C_6 = {}^{10}C_4$

Find no of ways in which we can distribute 20 distinct objects into three groups such that 1st group contain 5, 2nd group contains 8, and 3rd group contain 7,

Ans : $^{20}C_5 \times {}^{15}C_8 \times {}^7C_7 = \frac{20!}{5!\,15!} \times \frac{15!}{8! \times 7!} \times 1 = \frac{20!}{5!\,8!\,7!}$

no of ways to divide 'n' objects b/w k groups such that they contain $r_1, r_2, \ldots r_k$ objects, ~~then~~ $(r_1 + r_2 + \ldots + r_k = n)$ ~~then~~

$$= \frac{n!}{r_1!\, r_2!\, r_3! \ldots r_k!}$$

Model 2 : (dividing objects into groups and distributing it to people)

In How many we can distribute 'n' object b/w two persons A and B such that one person gets 4 and other gets 6.

~~Ans.~~

Ans : $^{10}C_4 \times 2!$

↳ distributing grouped objects among A & B

(Q) Find no of way to distribute 10 distinct objects among three groups of sizes 2, 3 and 5.

$$^{10}C_2 \cdot {}^{8}C_3 \cdot {}^{5}C_5 = \frac{10!}{2!\,3!\,5!}$$

(Q) Find no of ways to distribute 10 distinct objects ~~among~~ into groups of sizes 2, 3 and 5 among 3 persons A, B, C.

Ans : $^{10}C_2 \cdot {}^{8}C_3 \cdot {}^{5}C_5 \times 3!$

$$= \frac{10!}{2!\,3!\,5!} \times 3!$$

Models: (dividing objects into equal sized groups)

Find no of ways in which we can distribute 4 distinct objects b/w two groups such that each group size is two.

Ans: $^4C_2 \times \frac{1}{k!}$

where k is no of groups with equal size

i.e., $^4C_2 \times \frac{1}{2!} = 3$

Eg: Find no of ways in which we can divide 10 objects in 3 groups of sizes 3, 3, 4.

Ans: $^{10}C_3 \cdot {^6C_3} \cdot {^4C_4} \times \frac{1}{2!}$

$\therefore \frac{10!}{3!3!4!2!}$

Eg: no of ways to divide 10 objects in 5 equal size groups

$\frac{10!}{2!2!2!2!2!5!}$

Eg: no of ways to divide 20 objects into 5 group of sizes 4, 4, 5, 5, 2 is

$\frac{20!}{(4!4!2!)(5!5!2!)(2!)}$

Q) Find no of of ways in which we can distribute 52 cards among 4 persons such that each persons gets 13 cards:

sol:

$$\frac{52!}{13!\,13!\,13!\,13!\,4!}(4!) \longrightarrow \text{distributing b/w persons}$$

$$\Rightarrow \frac{52!}{13!\,13!\,13!\,13!}$$

Model 4:

Find no of ways in which we can distribute 8 distinct objects b/w 3 persons and each gets atleast 2.

sol.

way 1: 2+2 2 2 $\cong$ 4, 2, 2 $\longrightarrow \frac{8!}{4!\,2!\,2!\,2!}$

way 2: 2+1 2+1 2 $\cong$ 3, 3, 2 $\longrightarrow \frac{8!}{3!\,3!\,2!\,2!}$

Now we need to distribute among 3 persons

$$\left(\frac{8!}{4!\,2!\,2!\,2!} + \frac{8!}{3!\,3!\,2!\,2!}\right)3!$$

Eg: Find no of ways in which we can distribute 5 objects among 3 persons such that each person gets atleast 1 object

sol:

1+2 1 1 $\longrightarrow \frac{5!}{3!\,1!\,1!\,2!}$

1+1 1+1 1 $\longrightarrow \frac{5!}{2!\,2!\,2!\,1!}$

i.e., $\left(\frac{5!}{3!\,2!} + \frac{5!}{2!\,2!\,2!}\right)3!$

(6) Find no of ways in which 'n' distinct object among 'n' children if exactly one child gets no objects.

Sol:

exactly one child gets 0 objects

$\therefore$ distribution is $\underbrace{2,1,1,1,\cdots 1}_{n-2},0$

$\therefore$ no of ways $\dfrac{n!}{2!\underbrace{(1!1!\cdots1!)}_{(n-2)\text{ times}}(n-2)!} \times n!$

$= \dfrac{n(n-1)\times n!}{2!}$

$= {}^nC_2 \times n!$ → This provide another way to view this problem.

(8) The set $S=\{1,2,3,\cdots 12\}$ is to be partitioned into 3 sets A,B,C of equal size such that $A\cup B\cup C=S$, $A\cap B=B\cap C=C\cap A=\phi$.

The no of way of partition ______

Sol:

$A\cap B=B\cap C=C\cap A=\phi \Rightarrow A\cap B\cap C=\phi$

~~So every object of S can fall into one~~ of the

~~3 set A (or) B (or)~~ C.

$\therefore$ ~~3×3×...×3~~

~~$=3^{12}$ ways~~

Also sets are divided into equal sizes

$\therefore \dfrac{12!}{4!4!4!3!} \times 3! = \dfrac{12!}{(4!)^3}$

10/10/20 50

Model 5:

→ Find no of ways in which we can distribute 10 distinct objects among 3 persons in anyway.

Sol:

Every object has 3 choices $\left\{ \begin{array}{l} O_1 \\ O_2 \\ O_3 \\ O_4 \\ O_5 \\ O_6 \\ O_7 \\ O_8 \\ O_9 \\ O_{10} \end{array} \right.$ $\quad$ P_1, P_2, P_3

$$\left(3C_1 \times 3C_1 \times 3C_1 \times \dots \times 3C_1 \right)_{10 \text{ times}}$$

i.e., 3^{10} → exponent is count of the thing that we are distributing.

→ Find no of ways in which we can distribute 100 distinct letters among 10 letter boxes:

Sol:

Every letter has 10 ways

$$\left(10 \times 10 \times \dots \times 10\right)_{100 \text{ times}}$$

i.e., 10^{100}

Here the event is putting letters in the box. So one by one we put each letter in the box.

→ Find no ...
friends in ...

Sol
Here ...

CH...
Cl...

Doubt
→ Find no o...
among ...
4 objects...

Sol
Here ...
person ...

Now, Re...

→ Find no of ways in which 4 servent can invite your three friends in anyways.

Sol :

Here we consider we invite ~~only~~ one by ~~only~~ one.

1st friend can be invited in 4 ways

2nd " " " " " " "

3rd " " " " " " "

$\therefore 4 \times 4 \times 4 = 4^3$

(Here the event is inviting and we invite one by one choosing a servant)

Doubt

→ Find no of way in which we distribute 10 distinct objects among 4 persons such that ~~only~~ one person gets exactly 4 objects.

Sol :

Here we first choose 4 objects to distribute it to the person who should get exactly 4.

This can be done in ${}^{10}C_4$ way.

Now, Remaining 6 objects have to be distribute among 3 persons.

This can be done in ~~6~~ 3^6 ways

$\therefore {}^{10}C_4 \times 3^6$

52

→ Find no of ways in which we can distribute 10 distinct objects among 4 persons such that ~~each~~ any person gets atmost 9 objects.

Sol:

req no of ways = total − no. of ways in which a person gets 10 objets

$= 4^{10} - 4$

P/15

1st book has 10 ways
2nd book has 9 ways
3rd " " 8 "
4th " " 7 "
5th " " 6 "
6th " " 5 "

$\therefore$ Ans : $^{10}P_6 = 151200$

P/19 Each book has 10 ways

$\Rightarrow 10^6$

P/23

$^{10}C_3 \times {}^7C_2 \times {}^5C_5$ (mode 1)

$$\frac{10!}{3!\,2!\,5!} = \frac{10 \times 9 \times \cancel{8}^{4} \times 7 \times \cancel{6}}{\cancel{3!} \times 2!} = 90 \times 28$$

$= 2520$

P/21

P/22

P/24

P/28

P/27 Max is point

P/25

P/21

$$\frac{10!}{2!\,3!\,5!}$$ (analogous to dividing into groups)

P/22

$$\frac{10!}{7!\,2!} = \frac{10\times9\times8}{2!} = 360$$

P/24

~~10!~~ $\cancel{{}^{10}C_5 \times {}^5C_5 \times \frac{1}{2!}}$

~~↳ 2 equal sized teams~~

$\cancel{\frac{10!}{5!\,5!\,2!}}$

$$\frac{10!}{(2!\,2!\,2!\,2!\,2!)(5!)}$$

P/28

$$\frac{10!}{7!\,3!} = \frac{10\times\cancel{9}^{3}\times\cancel{8}^{4}}{\cancel{3}\times\cancel{2}} = 120$$

P/27 Max is possible when every pair of lines intersect at a point different for other pairs.

$\therefore {}^{10}C_2 = 45$

P/25

R_1	R_2	R_3		
1+2	1	1	$\longrightarrow \frac{5!}{3!\,2!} \Rightarrow 10$	
1+1	1+1	1	$\longrightarrow \frac{5!}{2!\,2!\,2!} \Rightarrow 15$	

$10 + 15 \rightarrow 25$

Since rooms are different

no of ways = $25\times3! = 150$

(This problem can also be ~~explaine~~ solved by calculating no of onto functions)

P/29

_ I _ I _ I _ I _ I _ I _

$$\underbrace{\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow}$$

7 locations for 4 zeroes

$\therefore {}^7C_4 = 35$

P/16

BGBGBGBG → 4!4!

GBGBGBGB → 4!4!

_ B_1 _ B_2 _ B_3 _ B_4 _

${}^5C_4 \times 4! \times 4! = 2880$

P/17

BGBGBGBG → 4!4!

GBGBGBGB → 4!4!

} 1152

P/18

${}^5P_1 + {}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5$

$5 + 20 + 60 + 120 + 120 = 325$ ways

P/26

${}^6C_4 \times {}^6C_4 \times 4!$

$15 \times 15 \times 24$

$= 225 \times 24$

$= 5400$

P/27 P/31 Every person has $2n-2$ ways (except he & his spouse)

no of ways $= \frac{2n(2n-2)}{2} = 2n(n-1)$

Method 2:

Consider forming a graph with $2n$ vertices where each edge represents a handshake

no of way req = total possible edges $- n$

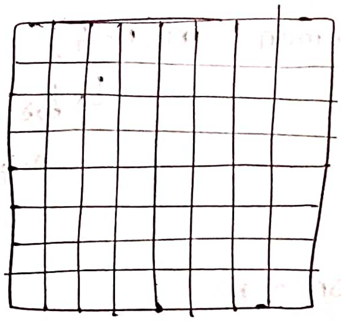
↳ edges b/w vertices corresponding to a couple

$= {}^{2n}C_2 - n$

$= \frac{2n(2n-1)}{2} - n = \frac{2n(2n-1) - 2n}{2}$

$= \frac{2n(2n-2)}{2} = 2n(n-1)$

(P/32)



we obtain a square are rectangle by choosing 2 vertical lines & 2 horizontal lines

this can be done in ${}^9C_2 \times {}^9C_2$ ways.

no of 1x1 squares $\rightarrow 8^2$

no of 2x2 squares $\rightarrow 7^2$

3x3 $\rightarrow 6^2$

⋮

8x8 $\rightarrow 1^2$

$\therefore$ no of rect that are not squares

$$= {}^9C_2 \times {}^9C_2 - \frac{8(8+1)(16+1)}{6}$$

$$= 1092$$

13/10/20

P/33 no of non-working days $= 31 - 5 = 26$

1 2 3 4 .... 25 26

$\uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \ldots \ \uparrow \ \uparrow \ \uparrow$

gaps

$\therefore$ Out of these 27 ~~days~~ we need to choose 5.

$$\therefore {}^{27}C_5$$

P/36 Total no of triangles $= {}^nC_3$

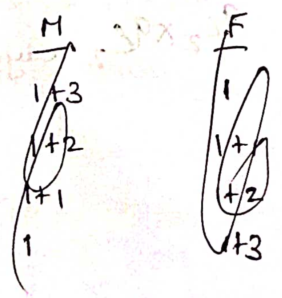
no of triagles with 1 side in common $= n(n-4)$

↳ for each we have $n-4$ triangles.

no of triangles with 2 sides common $= n$

$$\therefore {}^nC_3 - n(n-4) - n$$

P/35



$${}^{20}C_5 - {}^{12}C_5 - {}^8C_5$$

total — all are males — all are females

Distributing Similar objects:

(i) no of ways to distribute 'r' similar objects into 'n' numbered boxes $= {}^{(n-1+r)}C_r$

(ii) no of positive integrals solns for (or) ~~non negative integral~~ $x_1+x_2+\dots+x_n=r$ $= {}^{(r-1)}C_{n-1}$

(iii) no of non-negative integral solns for $x_1+x_2+\dots+x_n=r$ $= {}^{(n-1+r)}C_r$

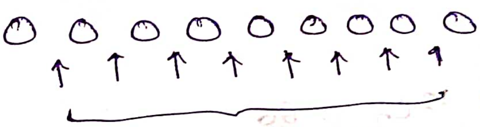
Note:

No of ways to distribute 'r' similar ~~boxes~~ balls in 'n' numbered boxes

$= {}^{(r-1)}C_{n-1}$

Proof:

Consider distribution 10 balls into 3 boxes



we can split in 10-1 i.e., r-1 position.

$\therefore$

in those places, we place n-1 lines denoting boxes.

$\therefore {}^{r-1}C_{n-1}$

P/38 distribute 1 each to each box

12 ball are remaining

$\therefore {}^{(12+4-1)}C_{12} = {}^{15}C_{12} = {}^{15}C_3$

P/39 ~~w+x+y+ω+y+x+z~~ →

(P/39) $w+x+y+z = 14 \rightarrow {}^{14+4-1}C_{14} = {}^{17}C_3$

$w+x+y+z = 13 \rightarrow {}^{13+4-1}C_{13} = {}^{16}C_3$

$w+x+y+z = 12 \rightarrow {}^{12+4-1}C_{12} = {}^{15}C_3$

(P/40)

$x_1+x_2+x_3 = 8$

$x_1 \geq 3, \quad x_2 \geq -2 \quad x_3 \geq 4$

$y_1+3+y_2-2+y_3+4 = 8$

$y_1+y_2+y_3 = 3$

Now we find no of positive integral solutions

$\Rightarrow {}^{3+3-1}C_{3-1} = {}^5C_2 = 10$

(P/41)

$\underline{x_1} \quad \underline{x_2} \quad \underline{x_3}$

$x_1+x_2+x_3 = 10 \Rightarrow {}^{12}C_2 = 66$

Out of this that soln $(10,0,0)$ $(0,10,0)$ $(0,0,10)$ is possible

$\therefore 66-3 = 63$

Derangements:

$D_n = n!\left[\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots + \frac{(-1)^n}{n!}\right]$

Eg: $D_4 = 4!\left[\frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}\right]$

Derangements:

$$\boxed{D_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots + \frac{(-1)^n}{n!}\right]}$$

Eg: $D_4 = 4!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$

$$= 24\left[\frac{12-4+1}{24}\right] = 9$$

$$\boxed{D_n = (n-1)\left[D_{n-1} + D_{n-2}\right]}$$

14/10/20

Q How many 1-1 functions are possible on a set with 6 element so that no element is mapped to itself

P/58

Sol:

$$D_6 = 6!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right]$$

$$= 6!\left[\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right] = 720\left[\frac{360-120+30-6+1}{720}\right]$$

$$= 265$$

P/60

(i) $D_5 = 44$

(ii) total − no letter is correctly placed

$= 5! - D_5 = 120 - 44 = 76$

(iii) ${}^5C_2 \times D_3 = 10 \times 2 = 20$

(iv) $D_5 + ({}^5C_1 \times D_4)$ $\quad 4!\left[\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right] = 24\left(\frac{12-4+1}{24}\right)$

$44 + (5 \times 9) = 89$

(v) total - no letter is wrongly placed

$120 - 1 = 119$

(vi) This arrangement is not possible

$\therefore 0$

P/61

i) $D_5 \times D_5 = 44 \times 44 = 1936$

ii) $5! \times 5! = 120 \times 120 = 14400$

$$\begin{array}{r} 44 \times 44 \\ \hline 176 \\ 176 \\ \hline 1936 \end{array}$$

P/62

$4! \times D_4 = 24 \times 9 = 216$

P/59

$$n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}\right]$$

$$n!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}\right]$$

$$\sum_{i=0}^{n} (-1)^i \frac{n!}{i!}$$

Generating Function:

Transforming problems about sequences into problems about functions.

$\langle a_0, a_1, a_2, \ldots a_i, \ldots \rangle$

a_i acts as coefficient of x^i

Eg: $\langle 1, 1, 1, \ldots, 1 \rangle \cong 1 + x + x^2 + x^3 + \ldots$

$\langle 1, -1, 1, -1, \ldots \rangle \cong 1 - x + x^2 - x^3 + \ldots$

$\langle 2, 0, 2, 0, 2, 0, \ldots \rangle \cong \cancel{2+x}\ 2 + 2x^2 + 2x^4 + 2x^6 + \ldots$

generating functions of given sequences.

1. The generating function for choosing k elements from set $\{a_1\}$ without repetition.

k	element	no of ways
0	$\{\}$	1 $\longrightarrow$ 1
1	$\{a_1\}$	1 $\longrightarrow$ x
2	–	0

~~1+1+x~~

$\therefore$ seq is $\langle 1, 1, 0, 0, \ldots 0 \rangle \cong 1+x$

2. The generating function for choosing k elements from set $\{a_2\}$ is $(1+x)$.

3. The generating function for choosing 'k' elements from set $\{a_1, a_2\}$ is

k	elements	no of way
0	$\{\}$	1
1	$\{a_1\}$ (or) $\{a_2\}$	2
2	$\{a_1, a_2\}$	1
3	–	0
⋮		⋮

$\therefore$ seq is $\langle 1, 2, 1, 0, 0, \ldots \rangle$

$\therefore 1+2x+x^2 = (1+x)^2 = (1+x)(1+x)$

$\therefore$ no of way of choosing 'k' elements from set $\{a_1, a_2\}$ ~~the~~

= no of ways of choosing 'k' elements from set $\{a_1\}$

+

no of ways of choosing 'k' elements from set $\{a_2\}$

4. The generating function for choosing k elements from $\{a_1, a_2, a_3\}$

$$= (1+x)(1+x)(1+x)$$

$$= 1 + \underbrace{3x}_{\text{1 element subsets}} + \underbrace{3x^2}_{\text{2 element subsets}} + \underbrace{1x^3}_{\text{3 element subsets}}$$

$\therefore$ No of ways to choose 2 objects from $\{a_1, a_2, a_3\}$ is
$= $ coefficient of x^2 in $(1+x)^3$

→ The generation function for choosing k elements from $\{a_1, a_2, \dots a_n\}$ without repetition

$= (1+x)(1+x)\dots(1+x)$ (n times)

$= (1+x)^n$ → no of objects in set

$\therefore$ coefficient of x^k in $(1+x)^n$ denotes no of ways in which we can choose k objects from $\{a_1, a_2 \dots a_n\}$

i.e., nC_k

$\therefore (1+x)^n = nC_0 x^0 + nC_1 x^1 + nC_2 x^2 + \dots + nC_k x^k + \dots + nC_n x^n$

$$= \sum_{i=0}^{n} nC_i x^i$$

→ for $x < 1$

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} = (1-x)^{-1}$$

→ for $x < 1$

$$1 - x + x^2 - x^3 + \dots = 1 + (-x) + (-x)^2 + (-x)^3 + \dots = \frac{1}{1-(-x)} = (1+x)^{-1}$$

→ The generation function for choosing k objects from $\{a_1\}$ with repetition is

k	elements	no of ways
0	$\{\}$	1
1	$\{a_1\}$	1
2	$\{a_1, a_1\}$	1
3	$\{a_1, a_1, a_1\}$	1
$\vdots$	$\vdots$	$\vdots$

$\therefore$ The sequence is

$$1 + x + x^2 + x^3 + \dots = (1-x)^{-1}$$

→ The generation function for choosing k objects from $\{a_1, a_2, \dots a_n\}$ with repeatition is $(1-x)^{-n}$

∴ The no of ways to choose k objects from $\{a_1, a_2, \dots a_n\}$ with repeatition is coefficient of x^k in $(1-x)^{-n}$

finding coefficient of x^k in $(1-x)^{-n}$:

Taylor's theorem: about $x_0 = 0$

$$(1-x)^{-n} = f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^k(0)}{k!}x^k + \dots$$

here coefficient of x^k is $\frac{f^k(0)}{k!}$

$$f(x) = (1-x)^{-n}$$

$$f'(x) = -n(1-x)^{-n-1}(-1)$$

$$f'(x) = n(1-x)^{-(n+1)}$$

$$f''(x) = -n(n+1)(1-x)^{-(n+2)}(-1)$$

$$= n(n+1)(1-x)^{-(n+2)}$$

⋮

$$f^k(x) = n(n+1)(n+2)\cdots(n+(k-1))(1-x)^{-(n+k)}$$

$$f^k(0) = n(n+1)(n+2)\cdots(n+(k-1))$$

$$f^k(0) = \frac{1\cdot 2\cdot 3\cdots(n-1)(n)(n+1)\cdots(n+(k-1))}{(n-1)!}$$

$$\frac{f^k(0)}{k!} = \frac{[n+(k-1)]!}{(n-1)!\,k!} = {}^{n+k-1}C_k$$

$\boxed{\therefore \text{coefficient of } x^k \text{ in } (1-x)^{-n} = {}^{(n+k-1)}C_k \cong \text{distributing } k \text{ similar balls into 'n' distinct boxes}}$

Note:

→ G.F for choosing k objects from $\{a_1, a_2, \ldots a_n\}$ without repeatition is $(1+x)^n$.

→ no of ways to choose k objects from $\{a_1, a_2, \ldots a_n\}$ without repeatition = coefficient of x^k in $(1+x)^n$ = ^nC_k

→ G.F for choosing k objects from $\{a_1, a_2, \ldots a_n\}$ with repeatition is $(1-x)^{-n}$

→ no of ways to choose k objects from $\{a_1, a_2, \ldots a_n\}$ with repeatition = coeff of x^k in $(1-x)^{-n}$ = $^{n-1+k}C_k$

$$\boxed{\begin{aligned} \therefore (1+x)^n &= \sum_{k=0}^{n} {}^nC_k x^k \\ (1-x)^{-n} &= \sum_{k=0}^{\infty} {}^{(k+n-1)}C_{n-1} x^k \end{aligned}}$$

If $n=1$.

$$(1-x)^{-1} = \sum_{k=0}^{\infty} {}^kC_0 x^k = 1+x+x^2+\ldots$$

$n=2$,

$$(1-x)^{-2} = \sum_{k=0}^{\infty} {}^{k+1}C_1 x^k = 1+2x+3x^2+4x^3+\ldots$$

$n=3$,

$$(1-x)^{-3} = \sum_{k=0}^{\infty} {}^{k+2}C_2 x^k = 1+3x+6x^2+10x^3+15x^4+21x^5+\ldots$$

(P/85) $\langle 1, -2, 4, -8, 16, \ldots \infty \rangle$

$$1+(-2x)+(-2x)^2+(-2x)^3+(-2x)^4+\ldots$$
$$= [1-(-2x)]^{-1}$$
$$= (1+2x)^{-1}$$

P/86 $x + 3x^2 + 3^2x^3 + 3^3x^4 + \cdots$

$= x\left(\frac{x + 3x^2 + 3^2x^3 + 3^3x^4 + \cdots}{x}\right)$

$= x\left(1 + 3x + (3x)^2 + (3x)^3 + \cdots\right)$

$= x(1-3x)^{-1}$

Pigeon hole Principle:

If there are $kn+1$ ~~pegio~~ pigeons which are to be distributed among 'n' ~~pe~~ pigeon holes

(i) Some pigeon hole contains atleast $\left\lceil \frac{kn+1}{n} \right\rceil$ pigeons.

↳ i.e., $k+1$ pigeons

(ii) Some pigeon hole contains atmost $\left\lfloor \frac{kn+1}{n} \right\rfloor$ pigeons.

↳ i.e., k pigeons

Note: Minimum no of pigeons required to ensure that some pigeon hole contains atleast $k+1$ pigeons $= kn+1$

where 'n' is no of pigeon holes.

P/47 In worst case we may have 8 persons for each month. So we choose one more person.

i.e., $8*12+1 = 97$

P/48 In worst case we may first choose

5 red + 5 blue + 5 blue + 5 white + 5 Yellow

so we add one more balls

$\therefore 5*5+1 = 26$

P/49

$$\begin{array}{ccccc} 6 & 8 & 10 & 15 & 20 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & 8 & 8 & 8 & 8 \end{array}$$

$\therefore (6+8*4)+1 = 39$

P/50

$$\begin{array}{ccc} 12 & 7 & x \\ \downarrow & \downarrow & \downarrow \\ 5 & 5 & k \end{array}$$

$(10+k)+1 = 15$

$\Rightarrow k = 4$

$\therefore \underline{x = 4}$

$\rightarrow$ $1+x+x^2+\cdots+x^n = \dfrac{1-x^{n+1}}{1-x}$

use this while finding coefficients.