

NAME: K. Goutham STD.: _____ SEC.: _____ ROLL NO.: _____ SUB.: _____

Linear Algebra

(1-3 M)

Topics:

- Determinants
- Inverse
- Rank, linearly independent & dependent vectors
- Solution of system of linear equations.
- Eigen values, Eigen vectors & Cayley Hamilton theorem.

Determinants:

Matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$$A = [a_{ij}]_{m \times n} \quad (\text{or}) \quad A = [a_{ij}] \quad \text{where } 1 \leq i \leq m, 1 \leq j \leq n$$

2nd order determinant:

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

the expression $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is called a 2nd order determinant

of $A_{2 \times 2}$. It is denoted by $|A|$ (or) $\det(A)$. It is given by

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

3rd order determinant:

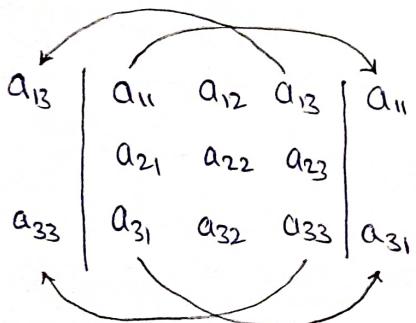
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$(a_{ij}) \rightarrow (-1)^{i+j}$

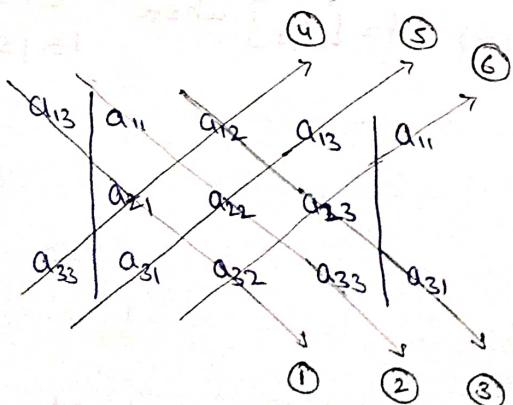
Method 2: \rightarrow Rearrange matrix to convert to a triangular matrix.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Rearrange as shown below



Now we draw 6 lines as shown below.



① ② ③ ④ ⑤ ⑥

Product
indicate ~~sum~~ of all terms in
that line.

i.e., ① $\dots a_{13} a_{21} a_{32}$

$$|A| = [① + ② + ③] - [④ + ⑤ + ⑥]$$

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4th order determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

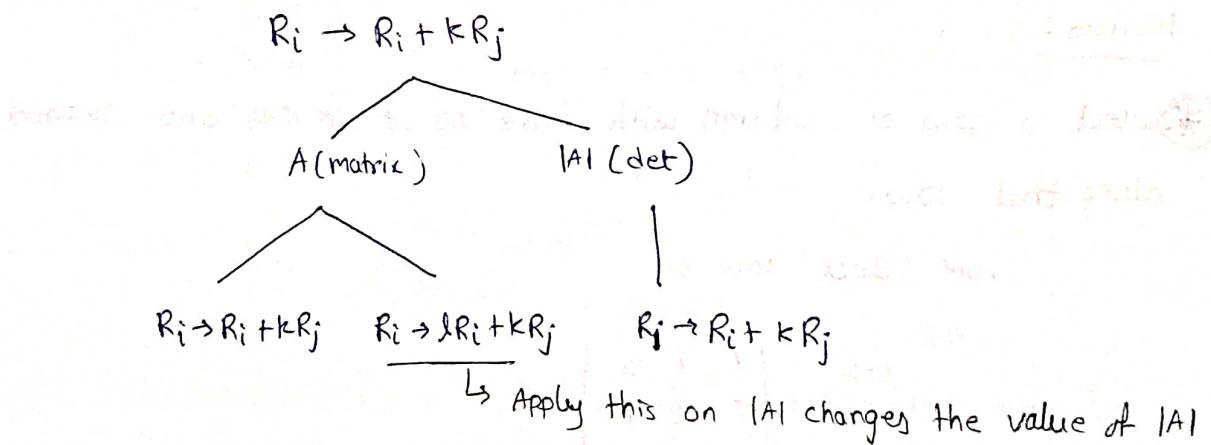
Elementary operations / Transformations:

- i) $R_i \leftrightarrow R_j$: Interchanging two rows.
- ii) $R_i \rightarrow kR_i$: Multiplying all elements of row R_i with k . ($k \neq 0$)
- iii) $R_i \rightarrow R_i + kR_j$ ($k \neq 0$)

Similar operations can be done on columns too.

Note:

$R_i \rightarrow R_i + kR_j$ can be applied on matrix or ~~on~~ determinant by shown below



Q1) The value of the determinant of a matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 3 & 3 \\ 0 & 2 & 4 \end{bmatrix}$ is

Sol :

$$\begin{array}{|ccc|} \hline & 2 & 0 & 1 \\ & 4 & -3 & 3 \\ & 0 & 2 & 4 \\ \hline & 8 & -24 & 0 \\ \hline \end{array}$$

$$\begin{aligned} \therefore |A| &= [8-24+0] - [0+0+12] \\ &= -16-12 = -28 \end{aligned}$$

Q2) The value of the determinant of the matrix

$$A = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 2 \end{bmatrix}$$

$$(1) \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & -1 & 2 \end{bmatrix} + 1 \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} - (0) \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned} (1) (-1) - 4(0) + 1[-2(0) + 1(-2)] - 0 \\ -1 - 0 - 2 = -3 \end{aligned}$$

Method 2 :

* Select a row or column with more no of zeroes and expand along that row.

∴ we select row 3.

$$(-1)^{3+2} (1) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{bmatrix}$$

expand along 2nd row

$$(-1) [(-1)(-1-2)] = -3$$

(Q3) $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$. Find $|A|$

Sol:

Expand along 2nd row

$$\begin{aligned} & (-1)(1)^3 \begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix} + (-1)(3)^3 \begin{vmatrix} 0 & 1 & 3 \\ 2 & 3 & 1 \\ 2 & 3 & 0 \end{vmatrix} \\ &= (-1)[(9+0+0)-(12+0+1)] - (3)[(0+0+3)-(4+27+0)] \\ &= (-1)[-4] - 3[-28] \\ &= +4 + 84 = 88 \end{aligned}$$

* Method 2:

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$$

Generally to make row elements 0's we apply column operations and to make column elements 0's we apply row operations.

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 3 & -6 & 1 \\ 3 & 0 & -8 & 2 \end{vmatrix}$$

$$= (-1)^{2+1} (1) \begin{vmatrix} 1 & 2 & 3 \\ 3 & -6 & 1 \\ 2 & 0 & -8 \end{vmatrix}$$

$$= (-1)[(-72 - 12 + 0) - (12 + 0 - 8)]$$

$$= (-1)[-84 - 4] = 88$$

Q4 Let A be a square matrix of order $(n-1)$. The elements of A are defined by $a_{ij} = \begin{cases} n-1, & \text{for } i=j \\ -1, & \text{for } i \neq j \end{cases}$

Pro

- a) n^{n-1}
- b) n^{n-2}
- c) $(n-1)^{n-2}$
- d) $(n-1)^{n-3}$

Sol:

We choose matrix of order 2

$$\text{i.e., } n-1=2 \Rightarrow n=3$$

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - (1) = 3$$

option verification

- a) $n^{n-1} = 3^2 = 9$
- b) $n^{n-2} = 3^1 = 3$
- c) $(n-1)^{(n-2)} = 2^1 = 2$
- d) $(n-1)^{n-3} = 2^0 = 1$

\therefore opt (b)

Q5 If A is $m \times n$ matrix & B is $n \times p$ matrix then the no of multiplications ~~involve~~ & additions involved in computing product AB are —

- a) $mnp, mnp+1$
- b) $m(n-1)p, mn(p+1)$
- c) $(m-1)np, mnp$
- d) none of these

Sol:

$$A_{m \times n} \times B_{n \times p} = C_{m \times p}$$

$\therefore C$ will have mp no of element.

each element of C is computed with n multiplications and $n-1$ additions.

$$\therefore \text{no of multiplications} = n(mp)$$

$$\text{no of additions} = (n-1)mp$$

\therefore opt (d)

Properties of determinants:

$$\rightarrow |AB| = |A||B|$$

$$\rightarrow |A^k| = |A|^k \quad \forall k \in \mathbb{N}$$

Note:

$$|A^k| = |A|^k \quad \forall k \in \mathbb{I} : |A| \neq 0 \quad (\text{for negative numbers})$$

$$\rightarrow |AT| = |A|$$

\rightarrow In a matrix A, if atleast one row or atleast one column is all zeroes, then $|A|=0$.

\rightarrow In a matrix A, if two rows or two columns are same or proportional then $|A|=0$

\rightarrow If any two rows or two columns are interchanged, then determinant value of new matrix will be negative of determinant of previous matrix.

Thus odd no of interchanges ---- sign of det changes

even no of interchanges ---- sign of det remains same.

$$\rightarrow k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix} = \dots$$

So we can take out a scalar value as common from either a row or a column.

Note:

when we multiply scalar with matrix, we need to multiply it with every element of the matrix.

$$\text{i.e., } k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$$

$$\rightarrow |kA_{n \times n}| = k^n |A_{n \times n}|$$

$$\rightarrow R_i \rightarrow R_i + kR_j$$

Applying this operation doesn't change the value of determinant.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = \begin{vmatrix} 1+4k & 2+5k & 3+6k \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix}$$

$$R_1 \rightarrow R_1 + kR_2$$

Note:

$R_i \leftrightarrow R_j$: changes sign of determinant

$R_i \rightarrow kR_i$: raises value of determinant k times.

$R_i \rightarrow R_i + kR_j$: value of determinant remains unchanged.

\rightarrow Determinant of an upper triangular or a lower triangular (or) diagonal matrix is equal to product of its diagonal elements.

i.e., $\begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} = adf$

$$\begin{vmatrix} a & 0 & 0 \\ b & c & 0 \\ 0 & e & f \end{vmatrix} = acf$$

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc \quad \text{--- diagonal matrix}$$

$$\begin{vmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix} = k^3 \quad \text{--- scalar matrix}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad \text{--- identity matrix}$$

Inverse

Minor & co.

If

Eg.: $A =$

$$\rightarrow \begin{vmatrix} a_1+x_1+y_1 & b_1 & c_1 \\ a_2+x_2+y_2 & b_2 & c_2 \\ a_3+x_3+y_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} y_1 & b_1 & c_1 \\ y_2 & b_2 & c_2 \\ y_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1+x_1+y_1 & a_2+x_2+y_2 & a_3+x_3+y_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Inverse of a square matrix:

Minor & cofactor of an element:

If $A = (a_{ij})_{m \times n}$ is a square matrix then,

i) minor of an element "a_{ij}" is

$M_{ij} = (n-1)^{\text{th}}$ order determinant which

remains after deleting ith row and
jth column in the matrix.

ii) the cofactor of an element "a_{ij}" is $= (-1)^{i+j} M_{ij}$

$$\text{Eg: } A = \begin{bmatrix} 1 & 2 & 7 \\ 4 & 3 & 5 \\ 0 & 6 & -1 \end{bmatrix}$$

Here $a_{23} = 5$

$$\text{Minor of } a_{23} = \begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix} = 6 - 0 = 6$$

$$\text{Cofact of } a_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix} = -6$$

Cofactor matrix:

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Let A_{ij} be cofactor of a_{ij}

then let $B = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Here B is called cofactor matrix of A .

Adjoint matrix:

If $B_{n \times n}$ is a cofactor matrix of $A_{n \times n}$ then adjoint matrix of $A_{n \times n}$ is denoted by $\text{adj}(A)$ and it is defined as

$$\text{adj}(A) = B^T$$

Singular matrix

→ A is called singular matrix, if $|A| = 0$

non-singular matrix:

→ A is called non-singular matrix, if $|A| \neq 0$

Inverse (or) reciprocal of a square matrix:

If $AB = BA = I_n$ for $|A| \neq 0$ & $|B| \neq 0$ then

B is called inverse of matrix A . It is denoted as A^{-1} .

Note:

→ If $|A| \neq 0$ then

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

$$\& |A^{-1}| = \frac{1}{|A|}$$

The matrix for which A^{-1} exists is called invertible matrix.

→ If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

→ If A is any square matrix of order n then

$$(i) A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = |A| \cdot I_n$$

$$(ii) |\text{adj}(A)| = |A|^{n-1} \quad (\text{n is order})$$

$$(iii) |\text{adj}(\text{adj}(A))| = |A|^{(n-1)^2}$$

Similarly $|\text{adj}(\text{adj}(\dots \text{adj}(A)))\dots| = |A|^{(n-1)^k}$

$\xleftarrow[k \text{ times}]{}$

→ If $|A| \neq 0$ then

$$\text{adj}(A) = |A| \cdot A^{-1}$$

$$\text{adj}(\text{adj}(A)) = |A|^{n-2} \cdot A$$

→ \Rightarrow Symmetric matrix $A^T = A$ (or) $a_{ij} = a_{ji}$

Skew Symmetric matrix $A^T = -A$ (or) $a_{ij} = -a_{ji}$

Orthogonal matrix $A A^T = A^T A = I_n$ (or) $A^{-1} = A^T$

$$a_{ii} = -a_{ii} \\ \Rightarrow a_{ii} = 0$$

Eg: $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 8 & -5 \\ 4 & -5 & 3 \end{bmatrix}$... symmetric matrix

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix} \dots \text{skew symmetric matrix}$$

∴ diagonal elements of a skewsymmetric matrix are 0's

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \dots \text{orthogonal matrix}$$

→ Every odd order determinant of a skew symmetric matrix is always 0.

$$\text{Proof: } A^T = -A \Rightarrow |A^T| = |-A| \Rightarrow |A|^n = (-1)^n |A| \\ \Rightarrow |A| = -|A| \Rightarrow |A| = 0$$

i.e., Every odd ordered skew symmetric matrix is singular.

17/09/20

Q6 The inverse of matrix $A = \begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$ where $i^2 = -1$ is _____

Sol:

First we check whether matrix is singular or not

$$\begin{vmatrix} 4+3i & -i \\ i & 4-3i \end{vmatrix} = (4+3i)(4-3i) - (-i^2) \\ = 25 - 1 = 24 \neq 0$$

∴ inverse exist

$$A^{-1} = \frac{1}{24} \begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$$

Q7 The inverse of the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ is _____

Sol:

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix} = 2(10) - 3(15) + 4(5) \\ = 20 - 45 + 20 = -5 \neq 0$$

∴ inverse exists

Q8 Cofactor matrix is $\begin{bmatrix} 10 & -15 & 5 \\ -4 & 4 & -1 \\ -9 & 14 & -6 \end{bmatrix}$

$$\text{adj}(A) = \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

$$\therefore \text{Q9 } A^{-1} = \frac{1}{-5} \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

Method 2: (Shortcut for adj A)

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

start from 2nd row middle element
and form the 1st column of below matrix

2nd row of adj A

$$\rightarrow \left[\begin{array}{cccc} 3 & 2 & 3 & 3 \\ 1 & 4 & 4 & 1 \\ 4 & 1 & 2 & 4 \\ 3 & 2 & 3 & 3 \end{array} \right]$$

Now do the same for row 3.
Now do the same for row 1.
Now again do the same for row 2.

1st row of adj A
3rd row of adj A

Now from this 4×4 matrix we find adj(A)

$$\text{adj}(A) = \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

Cofactor - write as rows.
adj A - write as columns

$$\therefore A^{-1} = \frac{1}{5} \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

- Q8 If $\text{adj}(A) = \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$ then the absolute value of the determinant of A is _____

Sol:

$$|\text{adj}(A)| = |A|^{n-1}$$

$$\begin{array}{|ccc|} \hline -10 & -18 & -11 & -10 \\ \hline 2 & 14 & -4 & \\ -8 & 4 & 5 & -8 \\ \hline \end{array} \Rightarrow |\text{adj}A| = (-100 + 2016 + 176) - (176 - 560 + 360) = 2116$$

$$\therefore |A|^2 = 2116$$

$$\Rightarrow |A| = \pm 46$$

∴ absolute value of $|A|$ is 46

Q9 If $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 3 \\ 1 & 3 & -1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ then $|AB - BA| =$

Sol:

Here A & B are symmetric matrices $\Rightarrow A^T = A, B^T = B$

$$\text{Consider } (AB - BA)^T = (AB)^T - (BA)^T \\ = B^T A^T - A^T B^T \\ = BA - AB$$

$$(AB - BA)^T = -(AB - BA)$$

$\therefore (AB - BA)$ is skew symmetric

$$\therefore |AB - BA| = 0$$

Q10 If A & B are symmetric matrices of order 3×3 , then consider

the following S1: $AB = BA$

S2: $AB - BA$ is singular

which of the following is true?

- a) S1 is true & S2 is false
- b) S1 is false & S2 is true
- c) Both S1 & S2 are true
- d) Both S1 & S2 are false.

Sol:

$$A^T = A \quad B^T = B$$

$$\text{Consider } (AB - BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA - AB$$

$$\therefore (AB - BA)^T = -(AB - BA)$$

$\Rightarrow AB - BA$ is skew symmetric

$$\Rightarrow |AB - BA| = 0$$

$\Rightarrow AB - BA$ is singular

$\therefore S2$ is true

$$\text{S1: } AB = A^T B^T = (BA)^T$$

$$BA = B^T A^T = (AB)^T$$

$$(BA)^T \neq (AB)^T$$

$\therefore S_1$ is false

$\therefore \text{opt (b)}$

- (Q11) The number of different $n \times n$ symmetric matrices with each element being either 0 or 1 is —

Sol:

Total no of elements = n^2

Every diagonal element has 2 ways $\dots 2^n$

now remaining elements form $\frac{n^2-n}{2}$ pair

$$\Rightarrow 2^{\frac{n^2-n}{2}}$$

$$\therefore 2^n \cdot 2^{\frac{n^2-n}{2}} = 2^{\frac{n^2+n}{2}}$$

- (Q12) The no of $n \times n$ symmetric matrices possible with entries chosen from the set $\{0, 1, 2, \dots, (q-1)\}$ is

Sol:

$$q^n \cdot q^{\frac{n^2-n}{2}} = q^{\frac{n^2+n}{2}}$$

↙ ↘
diagonal remaining
element elements

18/09/20

Rank, Linearly Independent & dependent vectors

Echelon form:

A matrix $A_{m \times n}$ is said to be in echelon form if

- the no of 0's before the first non-zero element in a row is less than the no of such zeroes in the next non-zero row.
- the zero rows must be below the non-zero rows.
↳ rows with all elements as zeroes.

→ The rank of matrix $A_{m \times n}$ is denoted by $P(A)$ or r

→ If $A_{m \times n}$ is in echelon form then

Rank, $P(A_{m \times n})$ = no of non-zero rows in the echelon form

Eg: $A = \begin{bmatrix} 0 & 4 & 7 & 2 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

no of 0's in row i
before a non-zero element be z_i

Here $z_1 < z_2 < z_3$

∴ A is in echelon form

∴ Rank $(A) = 3$

Eg: $A = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 7 & 3 \\ 0 & 0 & 8 \end{bmatrix}$

Here A is in echelon form

∴ Rank $(A) = 3$

$$\text{Ex: } A = \begin{bmatrix} 0 & 4 & 7 \\ 0 & 0 & 8 \\ 0 & 0 & 6 \end{bmatrix}$$

A is in echelon form

no of non-zero rows = 2

$$\therefore \text{Rank}(A) = 2$$

$$\text{Ex: } A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A is in echelon form

no of non-zero rows = 0

$$\therefore \text{Rank}(A) = 0$$

Note:

$$\text{Rank}(A) = 0 \iff A \text{ is a null matrix}$$

$$\text{Ex: } A = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{Rank}(A) = 1$$

→ If we need to find rank of a matrix which is not in echelon form, we first reduce the matrix in to echelon form using row operation. Now we find the rank.

→ Echelon form obtained by applying row operations is called row echelon form.

Also we have column echelon form.

$$\text{Ex: } A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 8 & 9 & 10 & 11 & 12 \\ -4 & 7 & 9 & 5 & 6 & \\ 3 & 10 & 2 & -9 & -3 & -2 \\ 6 & 7 & 13 & 14 & 2 & 1 \end{bmatrix}$$

Here A is not in echelon form.

Note:

$$\rightarrow P(0_{m \times n}) = 0$$

$$\rightarrow P(A^T) = P(A)$$

$$\rightarrow \text{if } A \neq 0, \text{ then } P(A) \geq 1$$

$$\rightarrow \text{If } A_{mn} \neq 0 \text{ then } P(A_{m \times n}) \leq \min\{m, n\}$$

$$\rightarrow \boxed{\exists |A_{n \times n}| \neq 0 \Leftrightarrow P(A_{n \times n}) = n}$$

$$\rightarrow |A_{n \times n}| = 0 \Leftrightarrow P(A_{n \times n}) < n$$

$$\rightarrow P(I_n) = n \quad (\because |I_n| = 1 \neq 0)$$

$$*\rightarrow P(AB) \leq \min\{P(A), P(B)\}$$

$$*\rightarrow P(A+B) \leq P(A) + P(B)$$

$$*\rightarrow P(A-B) \geq P(A) - P(B)$$

* → If

$$\text{i) } P(A_{n \times n}) = n \text{ then } P(\text{adj}(A)) = n \text{ & } P(A^{-1}) = n$$

$$\text{ii) } P(A_{n \times n}) = n-1 \text{ then } P(\text{adj}(A)) = 1$$

$$\text{iii) } \underbrace{P(A_{n \times n})}_{\text{adj } A \text{ is non zero}} \leq (n-2) \text{ then } P(\text{adj}(A)) = 0 \quad \begin{matrix} \text{All the rows of adj } A \text{ are proportional} \\ \text{if } P(A) = n-1 \end{matrix}$$

$$\rightarrow \text{If } A = (a_{ij})_{m \times n} \text{ where } a_{ij} = k$$

i.e., all elements are same

$$\text{then } P(A) = 1$$

→ If all the rows of a matrix A are proportional then

$$P(A) = 1$$

Q13) The rank of a matrix $A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$ is _____

Sol:

$$\begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 4R_1$$

$$\begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & 3 & 3 & 9 \end{bmatrix}$$

$$R_3 \rightarrow 2R_3 - 3R_2$$

$$\begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \text{Rank}(A) = 2$$

Q14) Let $A = \begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$. If rank of A is 1 then $p = \underline{\hspace{2cm}}$

Sol:

A is square matrix $\therefore |A| = 0$

$$\therefore |A| = 0$$

$$\begin{array}{|ccc|c} p & 3 & p & 3 \\ p & p & 3 & p \\ 3 & p & p & 3 \end{array} = 0$$

$$\Rightarrow (p^3 + 2p^2 + p^2) - (p^2 + p^2 + p^2) = 0$$

$$2p^3 - 3p^2 + 1 = 0$$

Value of determinant is 0 if two rows are same or proportional

$$\therefore p = 3$$

(Q15) Let $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ where a, b, c are non-zero real numbers.

Then find rank of A .

Sol:

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$A = \begin{bmatrix} -a & 0 & c \\ 0 & a & b \\ -b & -c & 0 \end{bmatrix} \quad R_3 \rightarrow aR_3 - bR_1$$

$$= \begin{bmatrix} -a & 0 & c \\ 0 & a & b \\ 0 & -ac & -bc \end{bmatrix} \quad R_3 \rightarrow R_3 + CR_2$$

$$= \begin{bmatrix} -a & 0 & c \\ 0 & a & b \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \text{Rank}(A) = 2$$

Linearly

A vector

(Q16) Suppose that $A_{n \times n}$ is upper triangular matrix such that $a_{ii} \neq 0$,
 $i = 1, 2, 3, \dots, n$. Then the rank of $A^n = \underline{\hspace{2cm}}$

- a) 0 b) 1 c) $n-1$ d) n

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$\therefore \text{Rank}(A) = n-1$$

consid

$x_1 =$

$x_2 =$

Here x_1

linearly

vector

let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

→ If

$$\therefore \text{Rank}(A_{2 \times 2}^2) = 0$$

∴ $\text{Rank}(A^n) = 0$

Let $A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Rank}(A^3) = 0$$

So we get $A^n = 0_{n \times n}$

$$\therefore \text{Rank}(A^n) = 0, 0$$

Linearly independent & dependent vector

A vector $x\hat{i} + y\hat{j} + z\hat{k}$ can be represented as

i) $\begin{bmatrix} x & y & z \end{bmatrix}$... row vector

ii) $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ column vector

Consider below pairs of vectors

$$x_1 = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 2 & 5 & 6 \end{bmatrix}$$

Here x_1, x_2 are called

linearly independent
vectors

$$x_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$$

$$x_2 = 2x_1$$

$$x_1 = \begin{bmatrix} 2 & 4 & 7 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 2 & 4 & 7 \end{bmatrix}$$

$$x_1 = x_2$$

Here

Here x_1, x_2 are said to be linearly dependent.

→ If x_1, x_2, \dots, x_n are row vectors of same order n

if $P(A) = \text{no of given vectors } \leq n$ then

x_1, x_2, \dots, x_n are linearly Independent (LI) vectors

(ii) $P(A) \neq \text{no of given vectors (n)}$ then

x_1, x_2, \dots, x_n are linearly dependent (L.D.) vectors

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times m}$$

$$(or) A = [x_1 \ x_2 \ \dots \ x_n]_{m \times n}$$

↳ Here x_1, x_2, \dots, x_n are column vectors.

Q: Check whether the following vectors are L.I (or) L.D?

$$x_1 = [1 \ -2 \ 4] \quad x_2 = [-2 \ 1 \ 0] \quad x_3 = [-1 \ 2 \ 7]$$

Sol:

$$\text{Let } A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 1 & 0 \\ -1 & 2 & 7 \end{bmatrix}$$

$$\begin{aligned} R_2 &\rightarrow R_2 + 2R_1 \\ R_3 &\rightarrow R_3 + R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & -3 & 8 \\ 0 & 0 & 11 \end{bmatrix}$$

$$\text{Rank}(A) = 3 = \text{no of given vectors}$$

If all the vectors are same or proportionate then the vectors are L.D. otherwise we need to follow the procedure

Rank is same as no of independent vectors

∴ the vectors are linearly independent.

Q17 Check whether the following vectors are L.I or L.D?

$$x_1 = [1 \ -2 \ 4] \quad x_2 = [2 \ 0 \ 5] \quad x_3 = [-2 \ 4 \ -8]$$

Sol.

$$\text{let } A = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 0 & 5 \\ -2 & 4 & -8 \end{bmatrix}$$

$$\begin{aligned} R_2 &\rightarrow R_2 - 2R_1 \\ R_3 &\rightarrow R_3 + 2R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank(A) = 2 → Here out 3 vectors (rows), 2 rows are independent.
 $\therefore x_1, x_2 \& x_3$ are L.D vectors.

Note :

$\rightarrow P(A_{m \times n}) = r = \text{no of LI vectors}$

\rightarrow If $|A_{n \times n}| \neq 0$ then all the vectors are LI

\rightarrow If $|A_{n \times n}| = 0$ then the vectors are linearly dependent.

\rightarrow Every subset of a LI set is also a LI set.

\rightarrow Every subset of a LD set, may or may not be a LD set.

(Q18) Consider the following statements

S1: If $\{x_1, x_2, x_3, x_4\}$ is a LI set of vectors then the set

$\{x_1, x_2, x_3\}$ is a LI set of vectors.

S2: If $\{x_1, x_2, x_3, x_4\}$ is a LD set of vectors then the set

$\{x_1, x_2, x_3\}$ is a LD set of vectors.

which of the following is true?

- a) only S1
- b) only S2
- c) both S1 & S2
- d) none

(Q19) Let A & B be two $n \times n$ matrices over real numbers. Let $\text{rank}(M)$ & $\det(M)$ denote the rank & determinant of a matrix M respectively.

Consider the following statements.

I: $\text{rank}(AB) = \text{rank}(A) \cdot \text{rank}(B)$

II: $\det(AB) = \det(A) \det(B)$

III: $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$

IV: $\det(A+B) \leq \det(A) + \det(B)$

which of the above are true

- a) I & II
- b) I & IV
- c) II & III
- d) III & IV

Sol:

opt C (from properties)

Solution of system of linear equations:

→ The equation in which degree of all terms is same is called homogeneous equation.

E.g.: $2x + 4y = 0$ -- linear homogeneous equation

$2x + 3y + 4 = 0$ -- linear non-homogeneous equation.

Non-homogeneous system:

Consider $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$,
 $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$,
 \vdots
 $a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_m = b_m$

Collection of m equations
in n variables.

This is represented as

$$AX = B$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m \times n}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n \times 1}$$

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m \times 1}$$

variable matrix (or) soln matrix constant matrix

Method of finding the solution of the system

Step 1: Express the given system of algebraic linear equations in matrix form as $AX = B$

Step 2: Consider the augmented matrix $(A|B)$ & reduce it to echelon form by applying only row operations.

Step 3: Find

(i) if

(ii) if

(iii) if

Step 4: If the equations are

Here Augmented

Note :

$$AX =$$

$$\downarrow$$

$$A \underset{n \times n}{\sim}$$

$$|A| \neq 0$$

• unique soln

Step 3: Find $P(A)$, $P(A|B)$ & $n = \text{no of variables}$

(i) if $P(A) = P(A|B) = n$ then

$AX = B$ is consistent & it has unique soln.

(ii) If $P(A) = P(A|B) \neq n$ then

$AX = B$ is consistent & it has infinite no of solutions

(iii) If $P(A) \neq P(A|B)$ then

$AX = B$ is inconsistent and hence no solution.

Step 4: If the soln exists then rewrite the system of linear equations and find the soln by backward substitution.

Here augmented matrix

$$(A|B) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right] \text{ mx}(n+1)$$

Note :

$$AX = B$$

\downarrow
 $A_{n \times n}$

$$|A| \neq 0$$

• unique soln

$$|A| = 0$$

no soln unique soln

Homogeneous System:

If $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ then B is zero vector and

$AX=B$ is called a homogeneous system.

It is denoted as $AX=0$

we the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Here x is called trivial solution

Ex: Consider $x - 2y = 0 \dots \textcircled{1}$

$$2x + y = 0 \dots \textcircled{2}$$

$$\text{from } \textcircled{1} \quad x = 2y$$

$$\textcircled{2} \Rightarrow 4y + y = 0 \Rightarrow 5y = 0$$

$$\text{from } \textcircled{1} \quad x - 2(0) = 0 \Rightarrow x = 0$$

$$\therefore x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Here x is called trivial soln or zero soln or unique soln

Ex: Consider $x + 2y = 0$

$$2x + 4y = 0$$

Here the two equations are proportional

This means the system represents only one equation

$$\text{from } \textcircled{1} \quad x = -2y$$

put $x=0$

$$\Rightarrow x_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left. \begin{array}{l} \text{zero/trivial soln} \\ \text{non-zero soln} \end{array} \right\}$$

put $y=1$

$$\Rightarrow x_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \left. \begin{array}{l} \text{non-zero soln} \\ \text{(or) non-trivial soln} \end{array} \right\}$$

$$\text{put } y=2 \Rightarrow x_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

Here we get infinite no of soln.

Note :

→ If $P(A) = r_1$ & $n = \text{no of variables in } AX=0$ then $AX=0$ will have $P=n-r_1$ linearly independent solutions.

→

$$AX = 0$$



$$A_{n \times n}$$

$$|A| \neq 0$$

unique soln
(trivial)

$$|A| = 0$$

trivial &
non-trivial soln.

→

$$AX = 0 \quad \begin{array}{l} P(A) = n \text{ (or) } |A| \neq 0 \quad \text{--- trivial soln & unique soln} \\ P(A) \neq n \text{ (or) } |A| = 0 \quad \text{--- trivial & non-trivial soln} \end{array}$$

Q20

Consider the linear equations

$$x - 2y + z = 3$$

$$2x + \alpha z = -2$$

$$-2x + 2y + \alpha z = 1$$

Inorder to have a unique soln for this linear system of eq. the value of α should not be equal to

- a) $-2/3$ b) $2/3$ c) $4/3$ d) $-4/3$

Sol:

$$AX=B$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & \alpha \\ -2 & 2 & \alpha \end{bmatrix}$$

Here A is a square matrix.

\therefore Φ unique soln $\Rightarrow |A| \neq 0$

$$\left| \begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 2 & 0 & \alpha & 0 \\ -2 & 2 & \alpha & -2 \end{array} \right| \neq 0$$

$$(4 + 0 + 4\alpha) - (-4\alpha + 0 + 2\alpha) \neq 0$$

$$4\alpha + 4 + 2\alpha \neq 0$$

$$6\alpha \neq 4$$

$$\Rightarrow \alpha \neq -2/3$$

\therefore opt-a

- (Q21) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 4 & 3 & 10 \end{bmatrix}$ then which of the following is not true?

a) $P(A) = 2$

b) $AX=0$ has infinitely many solutions.

c) $AX=B$ has a unique solution

d) A^{-1} does not exist.

Sol:

finding rank of matrix A

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 4 & 3 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & -5 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore P(A) = 2 \Rightarrow |A| = 0$$

\therefore inf solns

\therefore inverse does not exist

$$\cancel{Ax = B}$$

rank(A) = 2 \neq no of ~~vectors~~ equations

\therefore no unique soln.

\therefore opt C is wrong

Q22 For what values of α & β the following system of equations has an infinite no of solutions.

$$x + y + z = 5, x + 3y + 3z = 9, x + 2y + \alpha z = \beta$$

- a) 2,7 b) 3,8 c) 8,3 d) 7,2

Sol:

Augmented matrix

$$A/B = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha-1 & \beta-5 \end{bmatrix}$$

$$\sim \left[\begin{array}{cccc} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 2\alpha-4 & 2\beta-14 \end{array} \right]$$

infinite soln $\Rightarrow \text{Rank}(A) = \text{Rank}(A/R) \neq 3$

~~from the figure it is clear that~~

$$A = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 2 & \alpha \end{array} \right]$$

$$\sim \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2\alpha-4 \end{array} \right]$$

$$\text{Rank}(A) \neq 3$$

$$\Rightarrow 2\alpha - 4 = 0$$

$$\alpha = 2$$

$$\therefore \text{Rank}(A/R) \neq 3$$

$$2\beta - 14 = 0$$

$$\Rightarrow \beta = 7$$

(Q23) If a matrix A of order 4 has a rank 2 then $AX=0$ has

- a) one linearly independent solutions
- b) two linearly independent solutions
- c) three " "
- d) only trivial solution

syl: $AX=0$

$$\downarrow$$

 $A_{4 \times n}$

$r = P(A_{4 \times n}) = 2$ --- system has inf no of solns.

no of independent solns = $n - r = 4 - 2 = 2$

ans : opt (b)

(Q24) Let $AX=B$ be a system of 3 equations in 3 variables x, y, z .

The augmented matrix of the system is given by $(A|B)$

$$(A|B) = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

which of the following is not true?

- a) $\rho(A|B)=2$
- b) $\rho(A)=2$
- c) $AX=B$ has many solns
- d) $AX=B$ has a unique soln

Sol:

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Rank}(A) = 2$$

$$\text{Rank}(AB) = 2$$

$$\therefore \text{Rank}(A) \neq \text{Rank}(AB) \neq 3$$

$\therefore AX=B$ has inf no of solns

\therefore opt (d)

(Q25) Consider the matrix $A = \begin{bmatrix} k & k & k \\ 0 & k-1 & k-1 \\ 0 & 0 & k^2-1 \end{bmatrix}$. If the system $AX=0$

has only one independent soln, then $k = \underline{\hspace{2cm}}$

- a) 0, -1
- b) -1, 1
- c) 0, 1
- d) 0, 1, -1

Sol:

no of independent cols = $n - r$

$$3 - r = 1$$

$$\Rightarrow r = 2$$

$$\Rightarrow \cancel{1^2} + \cancel{0} = 0$$

$$\Rightarrow \cancel{1} \cancel{0} = 1$$

i.e., the echelon form has 2 non-zero rows

put $k=0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore k=0 \checkmark$$

put $k = -1$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank} = 2$$

$$\therefore k = -1$$

put $k=1$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Rank = 1

$$\therefore k \neq 1$$

Hence opt (a)

- (826) Let $AX=B$ be a system of 3 equations in 3 variables x, y, z .
If A has 3 linearly independent columns & B is a linear combination then which of the following is true?

a) System has a unique soln.

b) System has no soln

c) $AX=0$ has a non-zero soln

d) The system has many solns.

so:

given linearly independent columns

$$\Rightarrow P(A) = 3$$

$\therefore A$ is square matrix of order 3

~~no of rows~~

$AX = B$ has a unique soln

\therefore opt(a)

Eigen values, Eigen vectors & Cayley-Hamilton Theorem

Eigen matrix, Eigen polynomial, Eigen equation & eigen values

If $A = (a_{ij})_{n \times n}$ is square matrix of order 'n' .. 'In' is a unit matrix of order 'n' & ' λ ' is a scalar then

i) the matrix $A - \lambda I$ is called a characteristic (or eigen) matrix of a matrix A .

ii) the expression $|A - \lambda I|$ is called a characteristic (or eigen) polynomial of matrix.

iii), the equation $|A - \lambda I| = 0$ is called a characteristic (or eigen) equation of a matrix $A_{n \times n}$

iv) the roots of a characteristic equation of a matrix A are called eigen values (or) characteristic roots of matrix A .

Eg: Consider $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$ is a characteristic or eigen matrix

$$\begin{aligned}
 |A - \lambda I| &= (4-\lambda)^2 - 4 \\
 &= \lambda^2 - 16 + 8\lambda - 4 \\
 &= \lambda^2 - 8\lambda + 12 \text{ is a eigen polynomial of } A
 \end{aligned}$$

$$|A - \lambda I| = 0$$

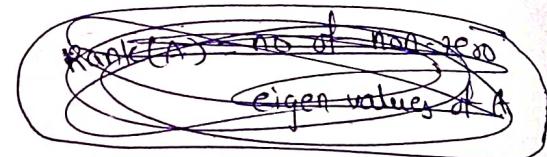
$\Rightarrow \lambda^2 - 8\lambda + 12 = 0$ is eigen ~~equation~~ equation of A

$$\lambda^2 - 8\lambda + 12 = 0$$

$$\lambda^2 - 6\lambda - 2\lambda + 12 = 0$$

$$(\lambda - 6)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 6 \text{ or } 2$$



6, 2 are eigen values of A

Eigen vector:

A non-zero column vector $x_{n \times 1}$ is said to be an eigen vector of matrix $A_{n \times n}$ if $AX = \lambda X$ for some scalar λ .

Note:

$$\rightarrow AX = \lambda X$$

$$AX - \lambda X = 0$$

$$(A - \lambda I)X = 0$$

trivial soln
non-trivial soln

↓
Eigen vector.

19/09/20

\rightarrow For a square matrix A

i) $|A - \lambda I| = 0 \dots \lambda$ characteristic roots

ii) $(A - \lambda I)X = 0 \dots X \neq 0$

↓
eigen vector

\rightarrow If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the characteristic equation of A is

$$\lambda^2 - (a+d)\lambda + |A| = 0$$

→ If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the characteristic equation of A is

$$\lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + \underbrace{\left[\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right]}_{\text{sum of minors of diagonal elements.}} \lambda - |A| = 0$$

Orthogonal vectors (Perpendicular vectors):

Two vectors $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are said to be orthogonal vectors if $x \cdot y = 0$ or $x_1y_1 + x_2y_2 + x_3y_3 = 0$

dot product

$$x \cdot y = x^T y = 0$$

$$\text{Ex: } x = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x \cdot y = x^T y = y^T x$$

$$x^T y = [1 \ 0 \ -2] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= 1(0) + 0(1) - 2(0) = 0$$

Properties of Eigen values & Eigen vectors:

→ Eigen values of upper triangular matrix or lower triangular matrix or triangular matrix, eigen values are the diagonal elements of the matrix.

$$\text{Ex: } \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \Rightarrow \lambda = a, d, f \quad \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ 0 & 0 & f \end{bmatrix} \Rightarrow \lambda = a, c, f$$

$$Ex: \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \Rightarrow \lambda = a, b, c$$

The eigen
orthogonal.

→ If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigen values of a matrix $A_{n \times n}$ then

$$(i) \lambda_1 + \lambda_2 + \dots + \lambda_n = \text{tr}(A) = a_{11} + a_{22} + \dots + a_{nn}$$

$$(ii) \lambda_1 \lambda_2 \dots \lambda_n = |A_{n \times n}|$$

(iii) the eigen values of a matrix A^m are $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$

(iv) the eigen value of a matrix kA are $k\lambda_1, k\lambda_2, \dots, k\lambda_n$

(v) the eigen values of matrix A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}$

(vi) the eigen value of matrix $\text{adj } A$ are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \dots, \frac{|A|}{\lambda_n}$

(vii) the eigen values of matrix

$$a_0 I + a_1 A + a_2 A^2$$

$$a_0 + a_1 \lambda_1 + a_2 \lambda_1^2, a_0 + a_1 \lambda_2 + a_2 \lambda_2^2, \dots, a_0 + a_1 \lambda_n + a_2 \lambda_n^2$$

→ If $a+ib$ or $a+i\bar{b}$ is an eigen value of a real matrix A then $a-ib$ or $a-i\bar{b}$ is also another eigen value of same matrix.

→ The eigen values of a real symmetric matrix are always real. and the eigen values of a real skew-symmetric matrix are either 0 or purely imaginary.

$$Ex: A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \lambda_1 = 2, \lambda_2 = 6$$

A is ~~real~~ symmetric matrix and hence λ is real.

$$A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

$$\text{characteristic eqn } \lambda^2 - (\text{ad})\lambda + |A| = 0$$

$$\lambda^2 + 16 = 0 \Rightarrow \lambda = \pm 4i$$

A is real ~~or~~ skew symmetric matrix and hence λ is 0 (~~or~~) purely imaginary

$$x_1, x_2$$

→ The eigen vectors of a real symmetric matrix are always pairwise orthogonal.

$$\therefore A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

$$\lambda^2 - (3+3)\lambda + (-9+16) = 0$$

$$\lambda^2 = 25 \Rightarrow \lambda = \pm 5$$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 3-5 & 4 \\ 4 & -3-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\lambda = 5 \quad \begin{bmatrix} 3-5x_1 & 4 \\ 4 & -3-5x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -(3-5x_1)x_1 + 4x_2 \\ 4x_1 - (3+5x_2)x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2x_1 + 4x_2 \\ 4x_1 - 8x_2 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -2x_1 + 4x_2 = 0 \\ 4x_1 - 8x_2 = 0 \end{cases} \quad \begin{cases} x_1 - 2x_2 = 0 \\ x_1 - 2x_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -4x_1 + 8x_2 = 0 \\ 4x_1 - 8x_2 = 0 \end{cases} \quad \begin{cases} x_1 = 2 \\ x_1 = 2 \end{cases} \Rightarrow x_2 = 1$$

$$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda = -5$$

$$\begin{bmatrix} 3+5 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 8x_1 + 4x_2 \\ 4x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 + x_2 = 0$$

$$x_1 = 1 \Rightarrow x_2 = -2$$

$$\Rightarrow x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\therefore \lambda = 5 \Rightarrow x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda = -5 \Rightarrow x_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$x_1 \cdot x_2 = 2(1) + 1(-2) = 0 \quad \therefore x_1 \text{ and } x_2 \text{ are pairwise orthogonal.}$$

→ If the n th order matrix A has n different eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ then the n th order matrix $A_{n \times n}$ will have n linearly independent eigen vectors x_1, x_2, \dots, x_n .

→ The no of linearly independent eigen vectors of a matrix $A_{n \times n}$ corresponding to any eigen value ' λ ' is given by $p = n - g_1$ where $g_1 = \text{rank}(A - \lambda I)$
 $n = \text{no of variables in eigen vectors}$
 (or)
 order of the matrix.

(Q27) If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ then the eigen values of A are

- a) 8, 7, 3 b) 0, 3, 15 c) 1, 2, 3 d) 1, -1, 2

Sol:

$$\text{sum of eigen values} = \text{tr}(A)$$

$$a) \cancel{8+7+3} = 8 + 7 + 3 = 18$$

$$b) 0+3+15 = 18 \quad \checkmark$$

$$c) 1+2+6 = 9 \neq 18$$

$$d) 1-1+2 = 2 \neq 2$$

also product of eigen values = determinant

$$|A| = \frac{2}{3} \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} 8.$$

$$= (48 + 168 + 48) - (108 + 28 + 128)$$

$$= (264) - (264)$$

$$= 0$$

∴ opt (b)

(Q28) The ~~no.~~ distinct eigen values of the matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n \times n}$$

are

- a) 0, n b) 1, n-1 c) 0, n-1 d) 1, n

Sol:

Put $n=3$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$(A - \lambda I) \geq 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \geq 0$$

$$(1 + (1-\lambda)^2 + 1) - (1-\lambda + 1-\lambda + 1-\lambda) \geq 0$$

$$(1-\lambda)^2 + 2 + 3\lambda - 3 \geq 0$$

$$(1-\lambda)^2 + 3\lambda - 1 \geq 0$$

$$\lambda^2 - \lambda - 3\lambda + 3\lambda^2 + 3\lambda - 1 = 0$$

$$-\lambda^3 + 3\lambda^2 = 0$$

$$\lambda^2(\lambda - 3) = 0$$

$$\text{Therefore } \lambda = 0 \text{ or } \lambda = 3$$

$$\therefore \lambda = 0, n$$

Put $n=2$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\lambda_1 + \lambda_2 = \text{tr}(A) = 2$$

$$\lambda_1 + \lambda_2 = 2$$

$$\lambda_1 \lambda_2 = |A| = 0$$

$$\Rightarrow \lambda_1 = 0 \text{ or } \lambda_2 = 0$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 2$$

$$\therefore \lambda = 0, 2$$

$$\text{i.e., } \lambda = 0, n \quad \therefore \text{opt(a)}$$

(Q29) Consider the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ whose eigen values are 1, -1, 3. Then the trace of $A^4 - 3A^3$ is _____

Sol :

$$\Rightarrow \lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 3$$

Eigen values of $A^4 - 3A^3$

$$\text{are } \lambda'_1 = \lambda_1^4 - 3\lambda_1^3 = 1 - 3 = -2$$

$$\lambda'_2 = \lambda_2^4 - 3\lambda_2^3 = 1 + 3 = 4$$

$$\lambda'_3 = \lambda_3^4 - 3\lambda_3^3 = 27 - 27 = 0$$

$$\begin{aligned} \text{Now trace of } A^4 - 3A^3 &= \lambda'_1 + \lambda'_2 + \lambda'_3 \\ &= -2 + 4 + 0 \\ &= 2 \end{aligned}$$

(Q30) Let A be a non-zero upper triangular matrix all of whose eigen values are zero. Then I+A is

- a) singular b) invertible c) symmetric d) skew symmetric

Sol :

Let n be order of A

given eigen values are all zeroes

$$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_n = 0$$

$$\therefore \lambda_1 + \lambda_2 + \dots + \lambda_n = 0$$

$$\lambda_1 \lambda_2 \dots \lambda_n = 0$$

Now eigen values of I+A is $\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_n + 1$

\therefore eigen values of I+A are

$$\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_n + 1$$

\Rightarrow i.e., 1, 1, 1, ..., 1

$$\Rightarrow |I+A| = (1)(1) \cdots (1) + (-1)^n = 1 \neq 0$$

(n times)

∴ $I+A$ is invertible.

$I+A$ is uppertriangular and hence not symmetric and
not skew symmetric.
 $\therefore A$ is non-zero (∴ A is non-zero)

∴ opt ⑥

Method 2:

for uppertriangular matrix eigen values are its diagonal elements.

$$\Rightarrow I+A = \begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 1 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

determinant of upper triangular matrix

> product of diagonal elements.

$$\therefore |I+A| = 1$$

∴ A invertible.

- Q31 If the characteristic polynomial of a 3×3 matrix M over \mathbb{R} (real numbers) is $\lambda^3 - 12\lambda^2 + a\lambda - 32$, $a \in \mathbb{R}$ and one of the eigen value of M is 2 then the largest among the absolute values of the eigen values of M is _____

sol:

let $\lambda_1 = 2$

$$\lambda_1 \lambda_2 \lambda_3 = |A| = 32 \Rightarrow \lambda_2 \lambda_3 = 16$$

$$\text{also } \lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33} = 12$$

$$\Rightarrow \lambda_2 + \lambda_3 = 10$$

$$\text{now } (\lambda_1 + \lambda_3)^2 = (\lambda_2 - \lambda_3)^2 + 4\lambda_2\lambda_3$$

$$100 = (\lambda_2 - \lambda_3)^2 + 4(16)$$

$$(\lambda_2 - \lambda_3)^2 = 36$$

$$\lambda_2 - \lambda_3 = \pm 6$$

$$\lambda_2 + \lambda_3 = 10$$

$$\begin{array}{l} \lambda_2 - \lambda_3 = 6 \\ \lambda_2 + \lambda_3 = 10 \\ \hline \Rightarrow 2\lambda_2 = 16 \\ \lambda_2 = 8 \\ \Rightarrow \lambda_3 = 2 \end{array}$$

$$\begin{array}{l} \lambda_2 - \lambda_3 = -6 \\ \lambda_2 + \lambda_3 = 10 \\ \hline 2\lambda_2 = 4 \\ \Rightarrow \lambda_2 = 2 \\ \Rightarrow \lambda_3 = 8 \end{array}$$

$$\therefore \text{largest } \lambda_1 = 2 \quad \lambda_2 = 8 \quad \lambda_3 = 2$$

\therefore largest possible eigen value = 8

Method 2:

put $\lambda = 2$ in the eqn

$$2^3 - 12(2)^2 + 2a - 32 = 0$$

$$\Rightarrow 8 - 48 + 2a - 32 = 0$$

$$2a = 72 \Rightarrow a = 36$$

now characteristic eqn is

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

we have $\lambda = 2$

$$\begin{array}{c|cccc} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & 0 \end{array}$$

$$\therefore (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$$

$$\Rightarrow \lambda^2 - 10\lambda + 16 = 0$$

$$\Rightarrow (\lambda-8)(\lambda-2) = 0$$

$$\Rightarrow \lambda = 8, 2$$

\therefore largest possible eigen value = 8

(Q32) An eigen vector corresponding to the smallest eigen value

of the matrix $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is

a) $[1 \ 0 \ 0 \ 0]^T$ b) $[1 \ 1 \ 0 \ 0]^T$

c) $[1 \ 0 \ 2 \ 0]^T$ d) $[-1 \ -1 \ 2 \ 2]^T$

Sol:

given matrix is upper triangular

\therefore eigen values are diagonal elements

i.e., $-1, 0, 1, 2$

smallest eigen value = -1

let x be eigen vector corresponding to $\lambda = -1$

Consider $(A - \lambda I)x = 0$

$$\begin{bmatrix} 1-\lambda & 1 & -1 & 2 \\ 0 & 2-\lambda & 0 & 1 \\ 0 & 0 & -1-\lambda & 1 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = -1$$

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce matrix A into echelon form

$$\left[\begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right]$$

$$\Rightarrow 2x_1 + x_2 - x_3 + 2x_4 = 0 \quad \dots \textcircled{1}$$

$$3x_2 + x_4 = 0 \quad \dots \textcircled{2}$$

$$x_4 = 0 \quad \dots \textcircled{3}$$

$$\textcircled{2} \Rightarrow 3x_2 + 0 = 0 \Rightarrow x_2 = 0$$

$$\textcircled{1} \Rightarrow 2x_1 - x_3 = 0$$

$$\text{Put } x_1 = k$$

$$\Rightarrow x_3 = 2k$$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 2k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

\therefore opt \textcircled{1}

Method 2 :

wk T

$$AX = \lambda k$$

opt a :

$$AX = \left[\begin{array}{cccc} 1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right] = \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right] = \lambda \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right]$$
$$\Rightarrow \lambda = 1$$

opt \textcircled{b} :

$$AX = \left[\begin{array}{cccc} 1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right] = \left[\begin{array}{c} 2 \\ 2 \\ 0 \\ 0 \end{array} \right] = \lambda \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right]$$
$$\Rightarrow \lambda = 2$$

Similarly do for opt \textcircled{c} & opt \textcircled{d}

(Q33) Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ - If -3 & -3 are two eigen values of A

eigen values of A then the eigen vector corresponding to the 3rd eigen value is

- a) $[1 \ 2 \ 1]^T$ b) $[1 \ 2 \ -1]^T$ c) $[1 \ -2 \ 1]^T$ d) $[-1 \ 2 \ 1]^T$

Sol:

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(A)$$

$$-6 + \lambda_3 = -2 + 1 + 0$$

$$\Rightarrow \lambda_3 = 5$$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} -7 & 2 & -3 \\ 0 & -24 & -48 \\ 0 & -16 & -32 \end{bmatrix}$$

$$\sim \begin{bmatrix} -7 & 2 & -3 \\ 0 & -24 & -48 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow -24x_2 - 48x_3 = 0$$

$$\Rightarrow 2x_2 + 2x_3 = 0$$

$$x_3 = k \Rightarrow x_2 = -2k$$

$$-7x_1 + 2x_2 - 3x_3 = 0$$

$$-7x_1 - 4k - 3k = 0$$

$$\Rightarrow x_1 = -k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ -2k \\ k \end{bmatrix} = -k \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

\therefore opt (b)

Method 2: (option verification)

$$AX = \lambda X$$

$$AX = 5X$$

Q2 opt a:

$$AX = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -5 \end{bmatrix}$$

Here $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is even a eigen vector

opt b:

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Here $\lambda = 5$

\therefore opt(b)

Q3) If $\begin{bmatrix} 2 & -2 \end{bmatrix}^T$ is an eigen vector of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & x & -4 \\ 2 & -4 & 3 \end{bmatrix}$

then $x = \underline{\hspace{2cm}}$

Sol:

$$AX = \lambda X$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & x & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ -2x-16 \\ 15 \end{bmatrix} = \lambda \begin{bmatrix} 2\lambda \\ -2\lambda \\ \lambda \end{bmatrix}$$

$$\Rightarrow 30 = 2\lambda \Rightarrow \lambda = 15$$

$$-2x-16 = -2\lambda$$

$$2x+16 = 2(15)$$

$$2x = 14 \Rightarrow x = 7$$

also ~~A~~ $\lambda = 15$ is eigen value corresponding to
the eigen vector $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

(Q35) The no of linearly independent eigen vector of a matrix

$$A = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 2 & -4 \\ 0 & 0 & 9 \end{bmatrix} \text{ is } \underline{\quad}$$

Sol:

A is triangular Matrix

\therefore eigen values are $1, 2, 9$

\because we have 3 distinct eigen values

we have 3 linearly independent eigen vectors

$\therefore \underline{\text{Ans: 3}}$

*** Q36 The number of linearly independent eigen vectors of a matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \text{ is } \underline{\quad}$$

Sol: A is triangular Matrix

\therefore Matrix is triangular

eigen values are $3, 2, 2$

\therefore no of linearly independent eigen vectors = $n - g$

\hookrightarrow Rank($A - \lambda I$)

~~$$A - \lambda I = \begin{bmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$~~

\Rightarrow Rank($A - \lambda I$)

~~no of linearly independent eigen vectors = $3 - 3 = 0$~~

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$\lambda = 3, 2, 2$$

$$\lambda_1, \lambda_2, \lambda_3$$

$$\hookrightarrow p = n - g = 3 - P(A - \lambda I)$$

Put $\lambda = 2$

$$A - 2I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Rank}(A - 2I) = 2$$

~~rank~~

no of linearly independent vectors = $n - r$

corresponding to eigen value 2 $= 3 - 2 = 1$

Similarly corresponding to eigen value 3,

we have one linearly independent vector.

\therefore no of linearly independent vectors = $1 + 1 = 2$

Q37) If A is a skew symmetric matrix of order n, then the no of linearly independent eigen vectors of $A + A^T$ is —

- a) 0 b) 1 c) $n-1$ d) n

Sol:

A is skew symmetric

$\Rightarrow A + A^T$ is null matrix

eigen values of $A + A^T$ are $0, 0, \dots, 0$ (n times)

\therefore no of LI vectors = ~~$n - P(A - \lambda I)$~~ $n - P(A + A^T - \lambda I)$

$$\begin{aligned} &\text{Here } \lambda = 0 \\ &\Rightarrow n - P(A + A^T) \\ &\Rightarrow n - 0 \end{aligned}$$

$$\Rightarrow n$$

Cayley Hamilton Theorem:

stmt:

Every square matrix satisfies its own characteristic equation.

Eg: If $\lambda^2 - 8\lambda + 12 = 0$ is a characteristic eqn for matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

then by Cayley Hamilton theorem, we have

$$\underline{A^2 - 8A + 12I = 0}$$

(Q38) If 1, 2 & 3 are the eigen values of matrix $A_{3 \times 3}$ then

$$6A^{-1} = \underline{\quad}$$

- a) $A^2 + 6A - 11I$
- b) $A^2 - 6A + 11I$
- c) $A^2 - 6A - 11I$
- d) $A^2 + 6A + 11I$

Sol:

The characteristic eqn is

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

$$(\lambda^2 - 3\lambda + 2)(\lambda - 3) = 0$$

$$(\lambda^3 - 3\lambda^2 - 3\lambda^2 + 9\lambda + 2\lambda - 6) = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

from Cayley Hamilton theorem

$$A^3 - 6A^2 + 11A - 6I = 0$$

$$A^3 - 6A^2 + 11A = 6I$$

Multiplying A^{-1} on both sides

$$6A^{-1} = A^2 - 6A + 11I$$

$\therefore \text{opt (c)}$

(Q39) Let $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $B = A^3 - A^2 - 4A + 6I$, where I is 3×3

identity matrix. The determinant of B is _____

Sol.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & -1 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{bmatrix}$$

Characteristic eqn is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$[(1-\lambda)(2-\lambda)(-2-\lambda)] - [0+0+0] = 0$$

$$(1-\lambda)(2-\lambda)(-2-\lambda) = 0$$

$$= \lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

From Cayley Hamilton theorem

$$A^3 - A^2 - 4A + 4I = 0$$

$$\Rightarrow A^3 - A^2 - 4A + 4I + 2I = 2I$$

$$\Rightarrow |A^3 - A^2 - 4A + 6I| = |2I| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2^3 = 8$$

LU Decomposable matrix (or) LU Factorizable matrix:

If a square matrix A_{nn} is written as $A = L \cdot U$, where

'L' is a lower triangular matrix & 'U' is a upper triangular matrix

then the square matrix A_{nn} is called an LU decomposable matrix.

→ Methods of finding the matrices L & U:

Method I = (Doolittle's method or) Doolittle's LU decomposition method):

If $A = L \cdot U$ then \rightarrow unit lower triangular matrix

$$\text{consider } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

$$\text{and } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Here we find product of L & U and solve the equations to find elements of L & U.

Method II (Gauss's method
(Crout's method)):

If $A = LU$ then consider

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$$

↳ unit upper triangular matrix

$$\text{then } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

shortcut: we can find echelon forms to find L & U matrices.

- Q40 In the LU decomposition of the matrix $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$, if the diagonal elements of U are both 1 then lower diagonal entry l_{22} of L is _____

Sol:

$$\text{let } \begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} \\ l_{21} & l_{21}u_{12} + l_{22} \end{bmatrix}$$

$$\Rightarrow l_{11}=2 \Rightarrow u_{12}=1$$

$$l_{21}=4 \quad l_{21}u_{12}+l_{22}=9 \Rightarrow 4+l_{22}=9 \Rightarrow l_{22}=5$$

* * *
Method 2 (shortcut) :
 * * *

Consider $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$

we convert this matrix into column echelon form

$$\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} \xrightarrow{C_2 \rightarrow C_2 - 2C_1} \begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix} \xrightarrow{\downarrow} \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$$

$$\therefore d_{22} = 5$$

* * * Shortcut for finding L, U and diagonal matrix of a matrix
 * * * Suppose $A = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix}$. If $A = LU$, where L is lower triangular matrix

with diagonal elements as unity then U & L are _____

Sol:

Now we app reduce

to find U we reduce matrix to row echelon form

to find L we reduce matrix to column echelon form

$$A = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_1} \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & -8 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 \rightarrow R_3 + R_2} \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix} = U$$

To find L:

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix}$$

$$3C_2 \rightarrow 3C_2 - 5C_1$$

$$C_3 \rightarrow 3C_3 - 2C_1$$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 24 & 6 \\ 6 & -24 & 12 \end{bmatrix} \quad C_3 \rightarrow 4C_3 - C_2$$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 24 & 0 \\ 6 & -24 & 72 \end{bmatrix}$$

However we need unit lower triangular matrix

$$C_1 \rightarrow C_1/3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} = L \quad C_2 \rightarrow C_2/24$$

$$C_3 \rightarrow C_3/72$$

$$\therefore U = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$