# ~~Linea~~ Linear Algebra:

1. System of linear equations
2. Determinants and their Properties
3. Eigen values and Eigen Vectors
4. LU Decomposition (only for CSE)

Solve the following system of linear equations

$2x + y + z = 1$

$6x + 2y + z = -1$

$-2x + 2y + z = 7$

$3y + 2z = 8$

$8y + 4z = 20$

$2y = 4 \Rightarrow y = 2$

$z = 0$

$x = -1$

using matrices:

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 6 & 2 & 1 & -1 \\ -2 & 2 & 1 & 7 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{matrix} \quad \begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 3 & 2 & 8 \end{pmatrix}$$

$R_3 \to R_3 + 3R_2$

Row Echolan form

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & -4 & -4 \end{pmatrix} \Rightarrow -4z = -4$$

basic colums & Non basic column

O → pivot elements.

$\boxed{z = 1}$

$-y - 2z = -4$

$-y - 2 = -4$

$\boxed{y = 2}$

$2x + y + z = 1$

$\boxed{x = -1}$

Zero row: All elements in row are ze

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

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Eg: 2

~~Now consider~~ Solve the following system of linear equation

$$x_1+2x_2+x_3+3x_4+3x_5=5$$

$$2x_1+4x_2+4x_4+4x_5=6$$

$$x_1+2x_2+3x_3+5x_4+5x_5=9$$

$$2x_1+4x_2+4x_4+7x_5=9$$

Augumented matrix

$$[A\ b]=\begin{pmatrix}1&2&1&3&3&5\\2&4&0&4&4&6\\1&2&3&5&5&9\\2&4&0&4&7&9\end{pmatrix}$$

$R_2\to R_2-2R_1$

$R_3\to R_3-R_1$

$R_4\to R_4-2R_1$

$$\sim\begin{pmatrix}1&2&1&3&3&5\\0&0&-2&-2&-2&-4\\0&0&2&2&2&4\\0&0&-2&-2&1&-1\end{pmatrix}$$

$R_3\to R_3+R_2$

$R_4\to R_4-R_2$

$$\begin{pmatrix}1&2&1&3&3&5\\0&0&2&2&2&4\\0&0&0&0&0&0\\0&0&0&0&3&3\end{pmatrix}$$

$R_3\leftrightarrow R_4$ (interchanging)

$$\begin{pmatrix}1&2&1&3&3&5\\0&0&2&2&2&4\\0&0&0&0&3&3\\0&0&0&0&0&0\end{pmatrix}$$

pattern present

from row 3 $\quad 3x_5=3\Rightarrow \boxed{x_5=1}$

from row 2 $\quad 2x_3+2x_4+2x_5=4\Rightarrow x_3+x_4+x_5=2$

$$\boxed{x_3+x_4=1}$$

Let $x_4 = 1 \Rightarrow x_3 = 0$

from row 1

$$x_1 + 2x_2 + x_3 + 3x_4 + 3x_5 = 5$$

$$x_1 + 2x_2 = -1$$

$$x_2 = 0 \Rightarrow x_1 = -1$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

general solution

$$x_5 = 1$$

$$x_3 + x_4 = 1$$

let $x_4 = k_1$

$$\Rightarrow x_3 = 1 - k_1$$

from row 1

$$x_1 + 2x_2 + x_3 + 3x_4 + 3x_5 = 5$$

$$x_1 + 2x_2 + 1 - k_1 + 3k_1 + 3 = 5$$

$$x_1 + 2x_2 = 1 - 2k_1$$

$$x_2 = k_2$$

$$x_1 = 1 - 2k_1 - 2k_2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1-2k_1-2k_2 \\ k_2 \\ 1-k_1 \\ k_1 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

In above G.S substitute any values for $k_1$ & $k_2$ and it will be a solution.

Solve the following system of linear equations

$$\left.\begin{aligned} x_1+2x_2+x_3+3x_4+3x_5&=0\\ 2x_1+4x_2+4x_4+4x_5&=0\\ x_1+2x_2+3x_3+5x_4+5x_5&=0\\ 2x_1+4x_2+4x_4+7x_5&=0 \end{aligned}\right\}$$ homogeneous equations

The general solution of the above homogeneous system is given as

$$\begin{pmatrix} x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = k_1\begin{pmatrix}-2\\0\\-1\\1\\0\end{pmatrix} + k_2\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}$$

for $k_1=k_2=0$

Solution is $\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \rightarrow$ trivial solution

$k_1=1\ k_2=1 \Rightarrow \begin{pmatrix}-4\\1\\-1\\1\\0\end{pmatrix}$ Non trivial Solution

## Consistent System:

→ A linear system is said to be consistent if it has atleast one solution

## Inconsistent System:

→ A linear system with no solution

## * Summary (observations):

$$\begin{pmatrix} 2 & 1 & 1 & 1\\ 0 & -1 & -2 & -4\\ 0 & 0 & -4 & -4 \end{pmatrix}$$

Non basic column

n = total no of variables

r = Rank(A)

$\boxed{\text{no of free variables} = n - r}$

$n=3$ $r=3$ Rank$(A|b)=3$

no free variables : Unique Solution

$\boxed{\text{The columns which doesn't contain pivot elements correspond to free variable}}$

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$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 & 5 \\ 0 & 0 & -2 & -2 & -2 & -4 \\ 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$n=5$

$Rank(A)=3$

$Rank(A/b)=3$

no of free variables $=5-3=2$

$\therefore$ infinit no of solutions

Non basic columns

↳ Colums that ~~contain~~ corresponds to free variables

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 & 5 \\ 0 & 0 & -2 & -2 & -2 & -4 \\ 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & \alpha \end{pmatrix}$$

$n=5$

$Rank(A)=3$ $\quad rank(A/b)=4$

$rank(A) \neq rank(A/b)$

Inconsistent ($\therefore$ no solution)

Non basic columns

Basic colums

↓

if $b$ is a basic column then solution doesn't exist.

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'm' linear equations in 'n' unknowns is given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

The above ~~equated~~ equations in matrix form can be written as

$$AX = b$$

Let $S$ be the linear system with 'm' eq in 'n' unknowns

$$S = \begin{Bmatrix} E_1 \\ E_2 \\ \vdots \\ E_i \\ E_j \\ \vdots \\ E_m \end{Bmatrix}$$

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for a linear system S each of the following 3 elementary operations results in an equivalen system S'.

① Interchange ith & jth equation

$$S' \left\{ \begin{matrix} E_1 \\ E_2 \\ \vdots \\ E_j \\ E_i \\ \vdots \\ E_m \end{matrix} \right\}$$

② Replace the ith equation by non-zero multiple of itself

$$S' = \left\{ \begin{matrix} E_1 \\ E_2 \\ \vdots \\ \alpha E_i \\ \vdots \\ E_m \end{matrix} \right\} \quad \alpha \neq 0$$

③ Replace the ith equation by combination of itself + multiple of jth equation.

$$S' = \left\{ \begin{matrix} E_1 \\ E_2 \\ \vdots \\ E_i + \alpha E_j \\ \vdots \\ E_j \\ \vdots \\ E_m \end{matrix} \right\}$$

*

$$\boxed{\begin{matrix} R_2 \to 2R_2 - R_1 \ \times \\ R_2 \to R_2 - \frac{1}{2} R_1 \ \checkmark \end{matrix}}$$

while solving linear eq above two eq can be used but in determinants only 2nd is correct.

→ A typical structure of a matrix in row echolon form is given below with pivots encircled

$$\begin{pmatrix} x & x & x & x & x & x & x & x \\ 0 & 0 & x & x & x & x & x & x \\ 0 & 0 & 0 & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

~~Homogeneous System of Linear Equations~~

~~let $A_{m\times n}$ be the co-efficient matrix~~

Homogeneous System of Linear Equation :

→ A system of 'm' linear equations in 'n' unknowns in which the right hand side consists ~~of~~ entirely of zeroes is said to be homogeneous system.

Let $A_{m\times n}$ be the coefficient matrix.

Suppose $\text{Rank}(A) = r$

→ the unknowns that correspond to the positions of the basic columns are, called the basic variables.

→ the unknowns corresponding to positions of the non-basic columns are called free variables.

→ There are exactly 'r' number of basic variables and 'n-r' free variables.

→ To describe all solutions reduce A to row echolan form using Gaussian Elimination and then use back substitution to solve for basic variables in terms of free variables.

→ This produces the general solution that has the form

$$x = x_{f_1}h_1 + x_{f_2}h_2 + \dots + x_{f_{n-r}}h_{n-r}$$

where $x_{f_1}, x_{f_2} \dots x_{f_{n-r}}$ are free variables

$h_1, h_2 \dots h_{n-r}$ are $n\times 1$ columns

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→ A homogeneous system posses a unique solution (trivial solution) $\Leftrightarrow n - r = 0$

→ whenever rank(A) is less than n system has non-trival solution or non-zero solutions (infinite solutions)

→ For a homogeneous system we have $n-r$, linearly independent solutions (i.e, $h_1, h_2 \ldots h_{n-r}$)

i.e., free variables

Non-Homogeneous System of Linear Equations:

A system of 'm' linear equations in 'n' unknowns is said to be non-homogeneous whenever $b_i \neq 0$ for atleast one 'i'.

→ Unlike homogeneous system, a non-homogeneous system may be inconsistent

→ To describe all possible solutions of a consistent non-homogeneous system construct a general solution by exactly the same used for homogeneous systems as follows

① Use gaussian elimination to reduce $[A/b]$ to row echolon form.

② Identify basic & free variables.

③ Use back substitution and solve for the basic variables in terms of free variables

④ write the result in the form of

$$x = x_{f_1} h_1 + x_{f_2} h_2 + \cdots + x_{f_{n-r}} h_{n-r} + P$$

P is called particular solution (n×1)

Eg:

$$x = k_1 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

→ The general solution of non-homogeneous system is given by general solution of Associated homogeneous system + particular solution.

→ Each of the following is equivalent to say that $[A/b]$ is consistent.

(i) $\text{Rank}(A) = \text{Rank}(A/b)$

(ii) A row of the following form will never appear in the row echolan form $[A/b]$

$$(0\ 0\ 0 \ldots 0\ \alpha) \quad \text{where } \alpha \neq 0$$

(iii) b is a non-basic column

(iv) b is linear combination of basic columns of A

Eg: $[A/b]$

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 6 & 2 & 1 & -1 \\ -2 & 2 & 1 & 7 \end{pmatrix}$$

$$-1\begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$$

## Problems:

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$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{pmatrix} \Rightarrow \begin{matrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & \lambda- & \mu- \end{matrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & \lambda-1 & \mu-6 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

Also for eq ② & eq ③

if $\lambda = 6$ ② will be ||el to ③

Now if $\mu \neq 20$ then they don't have a solution

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & \lambda-6 & \mu-20 \end{pmatrix}$$

Here for $\lambda = 6$, $\mu \neq 20$

$\therefore$ we get row of form $(0\ 0 \ldots \alpha)$

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$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 2 & 5 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & -3 \\ 0 & -3 & 0 & -3 \end{pmatrix}$$

identical so directly write row 3 0's

$R_3 \to R_3 - R_2$

free variable & hence infinite solutions.

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$r = \text{Rank}(A) = 2$

$\text{Rank}(A/b) = 2$

$n =$ no of variables $= 3$

$n - r = 1 \neq 0$

$\therefore$ free variables exist

hence infinite solutions

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$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Homogeneous $\Rightarrow$ Consistent

$R_2 \to R_2 - \frac{1}{2} R_1$

$$\begin{pmatrix} 2 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$n = 2 \quad \text{Rank}(A) = 1$

no of free variables $= 2 - 1 = 1$

$\therefore$ infinite solutions

Q2

$$\begin{pmatrix} 2 & 1 & 3 & 5 \\ 3 & 0 & 1 & -4 \\ 1 & 2 & 5 & 14 \end{pmatrix}$$

$\boxed{\text{A linear system can never have exactly 2 solutions}}$

$R_2 \to R_2 - \frac{3}{2}R_1$

$R_3 \to R_3 - \frac{1}{2}R_1$

$$\begin{pmatrix} 2 & 1 & 3 & 5 \\ 0 & -\frac{3}{2} & -\frac{7}{2} & -\frac{23}{2} \\ 0 & \frac{3}{2} & \frac{7}{2} & \frac{23}{2} \end{pmatrix} \to \text{infinite solutions}$$

$\boxed{\text{zero row can never form infinite solutions only free variables give infinite solutions}}$

To avoid more calculations

$R_1 \leftrightarrow R_3$

$$\begin{pmatrix} 1 & 2 & 5 & 14 \\ 3 & 0 & 1 & -4 \\ 2 & 1 & 3 & 5 \end{pmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 1 & 2 & 5 & 14 \\ 0 & -6 & -14 & -46 \\ 0 & -3 & -7 & -23 \end{pmatrix} \times 2 \to$$

Rank$(A) = 2$

Rank$(A|b) = 2$

$n - r = 3 - 2 = 1$

infinite solutions

Q3

$$\begin{pmatrix} 1 & 2 & 2 & b_1 \\ 5 & 1 & 3 & b_2 \end{pmatrix}$$

$R_2 \to R_2 - 5R_1$

$$\begin{pmatrix} 1 & 2 & 2 & b_1 \\ 0 & -9 & -7 & b_2 - 5b_1 \end{pmatrix}$$

Rank$(A) = 2$ Rank$(A|b) = 2$ (irrespective of $b_1$ & $b_2$ values)

$n - r = 3 - 2 = 1$

$\therefore$ infinite irrespective of $b_1$ & $b_2$

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$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 4 & 0 & 7 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & -2 & 7 & 0 \end{pmatrix}$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - R_1$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & -1 & -3 & -8 \\ 0 & 1 & -1 & -2 \end{pmatrix} \simeq \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & -4 & 3 & -11 \\ 0 & 1 & -1 & -2 \end{pmatrix}$$

$R_3 \to R_3 - 4R_2$

$R_4 \to R_4 + R_2$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & -4 & -10 \end{pmatrix} \quad R_4 \to R_4 + \frac{4}{15} R_3$$

$-10 + \frac{4}{15} \times 21 +$

$$\begin{matrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 0 & \end{matrix}$$

(wrong calculation)

$\downarrow$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank$(A) = 3$

Rank$(A|b) = 3$

$\therefore n - r = 3 - 3 = 0$

$\therefore$ unique solution

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$AX = 0$

$r (0 < r < n) \Rightarrow n - r > 0$

hence infinite solutions of form

$x = xf_1 h_1 + xf_2 h_2 + \cdots + xf_{n-r} h_{n-r}$

$\therefore n-r$

$h_1, h_2 \ldots h_{n-r}$ linearly independent solutions.

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(215)

$$\begin{pmatrix} 2 & 3 & 5 \\ 3 & p & 10 \end{pmatrix} R_2 \to R_2 - \frac{3}{2} R_1$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 0 & p-\frac{9}{2} & \frac{5}{2} \end{pmatrix}$$

To have no solution row 2 should be of form

$$\left(0, 0, \frac{5}{2}\right)$$

$$\Rightarrow p - \frac{9}{2} = 0 \Rightarrow p = \frac{9}{2}.$$

(227) Take option (c)

$p+q+r=0$ (or) $p=q=r$

if $p=q=r$

Coefficient matrix

$$\begin{pmatrix} p & p & p \\ p & p & p \\ p & p & p \end{pmatrix} \underset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}}{\simeq} \begin{pmatrix} p & p & p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Rank}(A) = 1$$

$n - r$

$3 - 1 = 2$ free variable

$\Rightarrow$ non-trivial

(244)

||el lines $\Rightarrow$ zero solutions

(268) $\to$ Home Work (277)

(277)

$$\begin{bmatrix} k & & \cancel{2k} \\ k^2 & \cancel{k} & k^2 \end{bmatrix}$$

$k^2 - k = k$

$k^2 = 2k$

$k^2 = 2k$

$k = \sqrt{k}$

# Determinants & their Properties

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2} \qquad |A| = ad - bc$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} + 1(-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$

Minor

Cofactor

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Cofactor of $-1 = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = -3$

Cofactor of $-2 = (-1)^{1+2}\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -6$

Cofactor matrix of $A = \begin{pmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{pmatrix}$

Det using 1st row

$$(-1)(-3) + (-2)(-6) + (-2)(-6)$$

$$3 + 12 + 12 = 27$$

Det using 2nd row

$$2(6) + 1(3) + (-2)(-6) = 27$$

Det using 3rd row

$$2(6) + (-2)(-6) + 1(3) = 27$$

Det using 1st column

$$(-1)(-3) + 2(6) + 2(6) = 27$$

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→ Determinant of a square matrix A is defined as sum of product of elements of a row / column with their cofactors.

→ ~~If elements fo of a row are multipli~~

→ Sum of product of elements of a row with cofactors of another row is always zero. Slly with columns too.

Eg:

row ① × row ②

$(-1)(6) + (-2)(3) + (-2)(-6) = 0$

## Effects of row operation on Determinants:

Let $B_{n\times n}$ be a matrix obtained from $A_{n\times n}$ by one of the three elementary row operations.

Type 1: Interchange rows i & j

Type 2: Multiply row 'i' by 'α', $\alpha \neq 0$

Type 3: Add α times row 'j' to row 'i'.

The determinant B is as follows

Type 1: $|B| = -|A|$

Type 2: $|B| = \alpha|A|$ → (if all ~~add~~ row are multiplied by α then $|B| = \alpha^n |A|$)

$\boxed{\text{Type 3: } |B| = |A|}$

Let Type 3:

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ x & y & z \end{pmatrix}$

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$$|A| = x(-1)^{3+1}\begin{vmatrix} b & c \\ e & f\end{vmatrix} + y(-1)^{3+2}\begin{vmatrix} a & c \\ d & f\end{vmatrix} + z(-1)^{3+3}\begin{vmatrix} a & b \\ d & e\end{vmatrix}$$

$R_3 \to R_3 + \alpha R_1$

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ x+\alpha a & y+\alpha b & z+\alpha c \end{pmatrix}$$

$$|B| = (x+\alpha a)(-1)^{3+1}\begin{vmatrix} b & c \\ e & f\end{vmatrix} + (y+\alpha b)(-1)^{3+2}\begin{vmatrix} a & c \\ d & f\end{vmatrix} + (z+\alpha c)(-1)^{3+3}\begin{vmatrix} a & b \\ d & e\end{vmatrix}$$

$$= |A| + \alpha\left(a(-1)^{3+1}\begin{vmatrix} b & c \\ e & f\end{vmatrix} + b(-1)^{3+2}\begin{vmatrix} a & c \\ d & f\end{vmatrix} + c(-1)^{3+3}\begin{vmatrix} a & b \\ d & e\end{vmatrix}\right)$$

$$= |A| + \alpha(0) = |A|$$

Now slly even if $R_3 \to R_3 + \alpha R_1 + \beta R_2$

$$|B| = |A|$$

we can extend it to any extent

$R_1 \to R_1 + R_2 + R_3 + R_4 + R_5$

Properties of determinants:

→ If each element of a row or a column of square matrix is zero then the determinant is zero.

→ If two rows or two columns of square matrix are identical then determinant is zero.

→ The determinant of a triangular of diagonal matrix is product of principle diagonal elements.

Eg:
$$\begin{vmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{vmatrix} = t_{11}t_{22}t_{33}$$

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→ Transpose does not change value of determinant

$|A^T| = |A|$

→ $|AB| = |A|\,|B|$

$|A+B| \neq |A| + |B|$

observation:

$|A^2| = |A|\,|A| = |A|^2$

$|A^n| = |A|^n$

→ The determinant of skew symmetric $(A^T = -A)$ matrix of odd order is always zero.

$A^T = -A$ ($\alpha$ multiplied to each row $|B| = \alpha^n |A|$)

$|A^T| = |(-1)A| = (-1)^n |A|$

$|A| = (-1)^n |A| \because (n \text{ is odd})$

$|A| = -|A|$

$\Rightarrow |A| = 0$

→ The determinant of an orthogonal matrix $(AA^T = A^TA = I)$ is either 1 or -1

Problems:

171) $|I_m + AB| = |I_n + BA|$

$$\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}_{R_1 \to R_1+R_2+R_3+R_4} = 5 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - R_1$

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$$\approx 5\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$ UT

i.e., upper triangular matrix

$\approx 5(1\times1\times1\times1) = 5$

(180) $P = I_6 + \alpha J_6$

$|P| = 0$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

To get det = 0 one of conditions is two rows are equal

So put $\alpha = 1$

(181) $|A| = 5 \quad |B| = 40$

$|AB| = |A|\,|B| = 5\times40 = 200$

(183) let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$|A| = ac - b^2$

$trace(A) = a + c$

to maximize $|A|$ i.e, $ac - b^2$ $b^2$ is zero

$ac$

$|A| = ac = a(14-a)$

$\frac{d}{da}|A| = 14 - 2a = 0 \Rightarrow a = 7$

$\frac{d^2}{da}|A| = -2 < 0$

$\therefore \max = 7(14-7) = 7\times7 = 49$

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(191)

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = -12$$

$$\begin{vmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{vmatrix}$$ (every row is multiplied by 2)

$\therefore (2^3)(-12) = -96$

(225) Given operation do not change det, so det is same.

Now find det

$$\begin{bmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{bmatrix} \times 15 = \begin{bmatrix} 45 & 4 & 45 \\ 105 & 9 & 105 \\ 195 & 2 & 195 \end{bmatrix}$$ Same col $\therefore$ det $= 0$

(229) upper triangular matrix

$\therefore$ ~~det = 100 = a×5×2×b =~~ 10 ab

~~a+5+2~~

$$A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{bmatrix} \quad |A| = b(-1)^{4+4} \begin{vmatrix} a & 0 & 3 \\ 2 & 5 & 1 \\ 0 & 0 & 2 \end{vmatrix}$$

$= b\ 2(5a-0)$

$= 10ab = 100$

$ab = 10$

also $a+5+2+b = 14$ — ~~a=5 b=2~~

$a+b = 7$

~~|a-b| = 3~~

$(a-b)^2 = (a+b)^2 - 4ab$

$= 49 - 40 = 9 \Rightarrow |a-b| = 3$

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## Inverse of Square Matrix :

→ The inverse of a non-singular matrix A is given by

$$A^{-1} = \frac{Adj(A)}{|A|}$$

$$Adj(A) = \begin{pmatrix} \text{Cofactor matrix} \\ \text{of } A \end{pmatrix}^{T}$$

Find inverse of the following matrix

$$A = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \qquad |A| = 27$$

$$\text{Cofactor matrix of } A = \begin{pmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{pmatrix}$$

$$Adj(A) = \begin{pmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{27}\begin{pmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{pmatrix}$$

$$AA^{-1} = A^{-1}A = I$$

$$\frac{1}{27}\begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(297)

$$\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix} \to (1)$$

a) $\begin{pmatrix} -10 & \cdot & \cdot \\ 15 & \cdot & \cdot \\ -5 & \cdot & \cdot \end{pmatrix}$ b) $\begin{pmatrix} 2 & \cdot & \cdot \\ -3 & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}$ (c) $\begin{pmatrix} -2 & -4/5 & \cdot \\ 3 & 4/5 & \cdot \\ 1 & -1/5 & \cdot \end{pmatrix}$ d) $\begin{pmatrix} 10 & \cdot & \cdot \\ -15 & \cdot & \cdot \\ 5 & \cdot & \cdot \end{pmatrix}$

∴ C ✓

Some more Properties of Determinants :

$\rightarrow$ $AA^{-1} = I$

$|AA^{-1}| = |I|$

$|A||A^{-1}| = 1$

$$\boxed{|A^{-1}| = \frac{1}{|A|}}$$

$\rightarrow$ $A^{-1} = \frac{\text{Adj } A}{|A|}$

$\text{Adj } A = |A| A^{-1}$

$|\text{Adj}(A)| = \left| |A| A^{-1} \right|$

$= |A|^n |A^{-1}|$

$$\boxed{|\text{Adj}(A)| = |A|^{n-1}}$$

Problems

(210) $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

$|A^T A^{-1}| = |A^T| |A^{-1}|$

$= |A| \frac{1}{|A|}$

$= 1$

(284) $|A^{-1}| = \frac{1}{|A|} = \frac{1}{|kak|} = \frac{1}{4} = 0.25$

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