Calculus

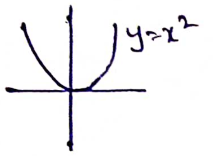
(2M)

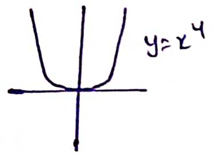
Funtion : A relation $f: A \to B$ is called a function if for each element of set A i.e., $x \in A$ there exists a unique element in set B. i.e., $y \in B$

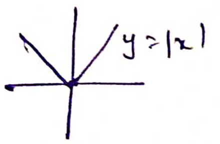
Eg:

Even Function: If $f(-x) = f(x)$ then $f(x)$ is called an even function

Eg : x^2, $\cos x$, $|x|$



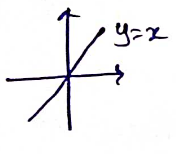


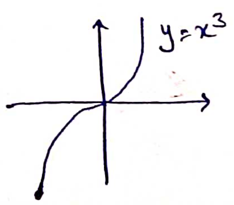


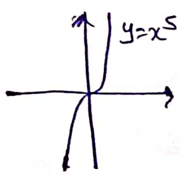
The curve of an even function is symmetric above y-axis

Odd function : If $f(-x) = -f(x)$, then $f(x)$ is called an odd function.

Eg : x, x^3, $\sin x$







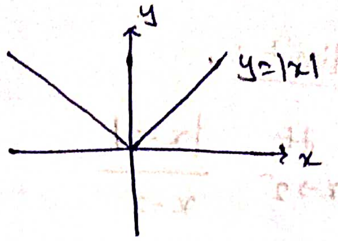
Every odd function is symmetrical about origin.

Note :

1) The derivative of an odd function is even function and vice-versa.
2) Product of two odd functions is even function.
3) Product of two even functions is even function.
4) Product of even and odd functions is odd function.

Modulus function:

$$y = f(x) = |x| = x, \quad x > 0$$
$$= -x, \quad x < 0$$
$$= 0, \quad x = 0$$



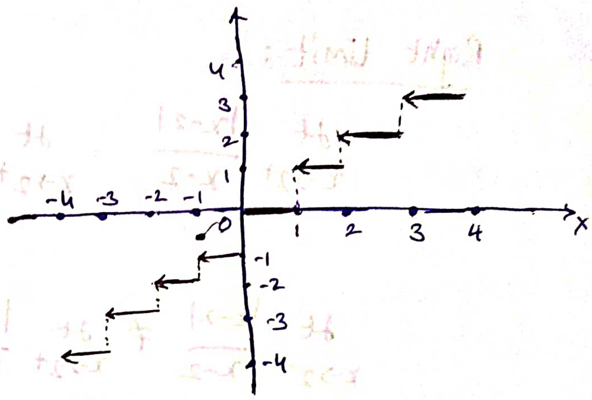
Step function: (or) Greatest Integer Function (or) Bracket (or) Stair Case Function:

For any integer n,

$$y = [x] = n, \quad n \le x < n+1$$

Eg: $[1.5] = 1$

$[-1.5] = -2$



Limit:

Suppose f(x) is defined when x is near the number 'a' (This means that f(x) is defined on some open interval that contains 'a' except possibly at 'a' itself). Then we can write

$$\lim_{x \to a} f(x) = L \quad \text{i.e., } f(x) \text{ approaches } L$$

Note:

$\lim_{x \to a} f(x)$ exists iff

$$\boxed{\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)}$$

left limit — Right limit

Q1 Find

$$\lim_{x \to 2} \frac{|x-2|}{x-2} = ?$$

a) 1 b) 2 c) −1 d) does not exist

Sol :

Left limit :

$$\lim_{x \to 2^-} \frac{|x-2|}{x-2}$$

$|x-2| = x-2 , x>2$
$= -(x-2) , x<2$
$= 0 , x=2$

$$= \lim_{x \to 2^-} \frac{-(x-2)}{x-2} = -1$$

Right limit :

$$\lim_{x \to 2^+} \frac{|x-2|}{x-2} = \lim_{x \to 2^+} \frac{x-2}{x-2} = 1$$

$$\lim_{x \to 2^-} \frac{|x-2|}{x-2} \neq \lim_{x \to 2^+} \frac{|x-2|}{x-2}$$

$\therefore$ Limit does not exist

Q2) $\lim_{x \to 4} [x] = ?$

a) 4 b) 3 c) -4 d) does not exist

Sol :

$\lim_{x \to 4^-} [x] = 3$ $\lim_{x \to 4^+} [x] = 4$

Left $\lim$ $\neq$ Rigth $\lim$

$\therefore$ limit does not exist

Some Standard Limits

1) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

2) $\lim_{x \to 0} \frac{x}{\sin x} = 1$

3) $\underset{x \to 0}{Lt} \dfrac{\sin(ax)}{x} = a$

4) $\underset{x \to 0}{Lt} \dfrac{\sin(ax)}{\sin(bx)} = \dfrac{a}{b}$

5) $\underset{x \to 0}{Lt} \dfrac{\tan x}{x} = 1$

6) $\underset{x \to 0}{Lt} \dfrac{x}{\tan x} = 1$

7) $\underset{x \to 0}{Lt} \dfrac{\tan(ax)}{x} = a$

8) $\underset{x \to 0}{Lt} \dfrac{\tan(ax)}{\tan(bx)} = \dfrac{a}{b}$

9) $\underset{x \to \infty}{Lt} \left(1+\dfrac{a}{x}\right)^x = e^a$

10) $\underset{x \to \infty}{Lt} (1+ax)^{1/x} = e^a$

11) $\underset{x \to \infty}{Lt} x^n e^{-ax} = 0$

Indeterminate Forms :

$\underbrace{\dfrac{0}{0}, \dfrac{\infty}{\infty}}_{\text{L'Hospital rule}}, \underbrace{0 \times \infty, \infty - \infty, 0^0, \infty^0, 1^\infty}_{\text{Reduce to } \frac{0}{0} \text{ (or) } \frac{\infty}{\infty} \text{ forms}}$

Model I : $\frac{0}{0}$ form

Q3 Evaluate the following limits

(i) $\lim\limits_{x \to 1} \dfrac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$ (ii) $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x \sin x}$

Sol :

(i) $\lim_{x\to 1} \dfrac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$ $\left(\frac{0}{0} \text{ form}\right)$

Apply L'Hospital rule

$\lim_{x\to 1} \dfrac{7x^6 - 10x^4}{3x^2 - 6x} = \dfrac{7-10}{3-6} = 1$

(ii) $\lim_{x\to 0} \left(\dfrac{1-\cos x}{x\sin x}\right)$

$\lim_{x\to 0} \left[\dfrac{1-\cos x}{x^2\left(\frac{\sin x}{x}\right)}\right] = \lim_{x\to 0} \left[\dfrac{1-\cos x}{x^2(1)}\right]$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 0} \dfrac{\sin x}{2x}$ L' Hospital

$= \lim_{x\to 0} \frac{1}{2} = 1/2$

Method 2 :

$\lim_{x\to 0} \left[\dfrac{1-\cos x}{x\sin x}\right]$ $\left(\frac{0}{0} \text{ form}\right)$

$\lim_{x\to 0} \dfrac{\sin x}{x\cos x + \sin x}$ $\left(\frac{0}{0} \text{ form}\right)$

$\lim_{x\to 0} \dfrac{\cos x}{-x\sin x + \cos x + \cos x} = \dfrac{1}{0+1+1} = 1/2$

Q4) $\lim_{x\to 0} \left(\dfrac{\tan 2x - \sin 2x}{x^3}\right)$

Sol :

$\lim_{x\to 0} \left\{\dfrac{\tan(2x) - \sin(2x)}{x^3}\right\} = \lim_{x\to 0} \left(\underbrace{\dfrac{\tan(2x)}{x}}_{2} \cdot \left[\dfrac{1-\cos 2x}{x^2}\right]\right)$

$$= 2 \lim_{x\to 0} \frac{1-\cos 2x}{x^2} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= 2 \lim_{x\to 0} \frac{\sin 2x\,(\cancel{2})}{\cancel{2}x}$$

$$= 2(2) = 4$$

(Q5) If $\lim_{x\to 0} \frac{\sin(2x) + a\sin x}{x^3} = b$, where b is finite then the values of ~~and~~ a & b are ?

Sol

$$\lim_{x\to 0} \frac{\sin(2x) + a\sin x}{x^3} = b$$

L's Hospital Rule

$$\lim_{x\to 0} \frac{2\cos 2x + a\cos x}{3x^2} = b \quad \text{——} \; (1)$$

$$\frac{2+a}{0} = b$$

$$\Rightarrow a+2=0 \Rightarrow \boxed{a=-2}$$

$$(1) \Rightarrow \lim_{x\to 0} \frac{2\cos 2x - 2\cos x}{3x^2} = b$$

$$\lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x} = b$$

$$\lim_{x\to 0} \frac{-8\cos 2x + 2\cos x}{6} = b$$

$$\Rightarrow \frac{-8+2}{6} = b \Rightarrow \boxed{b=-1}$$

$$\therefore a=-2,\; b=-1$$

Model : II ($\frac{\infty}{\infty}$ form)

(Q6) Evaluate the following limits

(i) The value of $\lim_{x\to 0} \frac{\log x}{\cot x}$ is

a) $\sqrt{3}$ b) $\sqrt{2}$ c) 0 d) 1

Sol :

$$\lim_{x\to 0} \frac{\log x}{\cot x} \quad \left(\frac{-\infty}{\infty}\right)$$

L' Hospital rule

$$\lim_{x\to 0} \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x} = -\lim_{x\to 0} \frac{\sin^2 x}{x}$$

$$= -\lim_{x\to 0} \frac{\sin x}{x} \cdot \sin x = -\lim_{x\to 0} \sin x = 0$$

(Q7) $\lim_{\theta\to \pi/2} \frac{\log(\theta - \pi/2)}{\tan\theta}$ 6

Sol :

$$\lim_{\theta\to \pi/2} \frac{\log(\theta-\pi/2)}{\tan\theta} \quad \left(\frac{-\infty}{\infty}\right) \quad (\because \log 0 = -\infty)$$

L' Hospital rule

$$\lim_{\theta\to \pi/2} \frac{\frac{1}{\theta-\pi/2}}{\sec^2\theta} = \lim_{\theta\to\pi/2} \frac{\cos^2\theta}{\theta-\pi/2}$$

$$= \lim_{\theta\to\pi/2} \frac{\sin^2(\pi/2-\theta)}{\theta-\pi/2}$$

$$= \lim_{\theta-\pi/2\to 0} \frac{\sin^2(\theta-\pi/2)}{\theta-\pi/2} = \lim_{\theta-\pi/2\to 0} \sin(\theta-\pi/2)$$

$$= 0$$

(Q8) $\lim_{x \to 0} \dfrac{\ln[\sin(2x)]}{\ln[\sin x]}$ $\left(\dfrac{-\infty}{-\infty}\right)$

$$\lim_{x \to 0} \frac{\frac{1}{\sin 2x} 2\cos(2x)}{\frac{1}{\sin x} \cos x} = 2 \lim \frac{\sin x}{\sin 2x} = 2 \frac{1}{2} = 1$$

Model III : ($0 \times \infty$ form)

$$\underset{x \to a}{\mathrm{Lt}} f(x)\, g(x) = 0 \times \infty \quad \begin{cases} \frac{1}{\infty} \times \infty = \frac{\infty}{\infty} \\ 0 \times \frac{1}{0} = \frac{0}{0} \end{cases} \text{ Apply L'Hospital}$$

we first change to $0/0$ (or) $\frac{\infty}{\infty}$ form and apply L'Hospital

- Still if we get $0/0$ (or) $\frac{\infty}{\infty}$ then we change it to $\frac{\infty}{\infty}$ (or) $\frac{0}{0}$ respectively.

Note : If either of $f(x)$ or $g(x)$ is a logarithmic function then it should be kept in the ~~numb~~ numerator and other function should be brought to the denominator such that it becomes either $0/0$ (or) $\frac{\infty}{\infty}$ and it can be evaluated by applying L'Hospital rule.

(Q9) Find $\underset{x \to a}{\mathrm{Lt}} (a-x) \tan\left(\dfrac{\pi x}{2a}\right)$ ($0 \times \infty$ ~~fa~~ form)

Sol :

Let us try to solve by bringing a-x to the denominator

$$\Rightarrow \lim_{x \to a} \frac{\tan\left(\frac{\pi x}{2a}\right)}{\frac{1}{a-x}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\lim_{x\to a} \frac{\sec^2 \frac{\pi x}{2a}\left(\frac{\pi}{2a}\right)}{\frac{+1}{(a-x)^2}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

∴ we now bring $\tan\frac{\pi x}{2a}$ to the denominator and try to solve

$$\lim_{x\to a} \frac{a-x}{\cot\frac{\pi x}{2a}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$\lim_{x\to a} \frac{-1}{-\operatorname{cosec}^2\frac{\pi x}{2a}\left(\frac{\pi}{2a}\right)} = \lim_{x\to a} \frac{\sin^2\frac{\pi x}{2a}}{\frac{\pi}{2a}}$$

$$= \frac{\sin^2\frac{\pi a}{2a}}{\frac{\pi}{2a}} = \frac{1}{\frac{\pi}{2a}} = \frac{2a}{\pi}$$

Q10 $\lim_{x\to 0} x \ln(\sin x)$

Sol:

$\lim_{x\to 0} x\ln(\sin x)$ $[0\times\infty \text{ form}]$

$$\lim_{x\to 0} \frac{\ln(\sin x)}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

('ln' must be kept in ~~den~~ numerator)

$$\lim_{x\to 0} \frac{\frac{1}{\sin x}\cos x}{\frac{-1}{x^2}} = -\lim_{x\to 0} \frac{x}{\sin x}\cdot x\cos x$$

$$= -\lim_{x\to 0} x\cos x = 0$$

$$= -\lim_{x\to 0} \frac{x}{\tan x}\cdot x = 0$$

Model - IV :

($\infty - \infty$ form)

$$\lim_{x\to a}\left[f(x)-g(x)\right] = \infty - \infty$$

$$\left[\text{Take LMC of } f(x)-g(x)\right] \rightarrow \left.\begin{array}{l} 0/0 \\ \frac{\infty}{\infty} \end{array}\right\} \text{Apply L'Hospital rule}$$

(Q11) $\underset{x\to\pi/2}{Lt}(\tan x - \sec x)$

sol :

$$\lim_{x\to\pi/2}(\tan x - \sec x) \quad (\infty - \infty \text{ form})$$

$$\lim_{x\to\pi/2}\left[\frac{\sin x}{\cos x} - \frac{1}{\cos x}\right]$$

$$= \lim_{x\to\pi/2}\left[\frac{\sin x - 1}{\cos x}\right] \quad (0/0 \text{ form})$$

$$= \lim_{x\to\pi/2}\frac{\cos x}{-\sin x}$$

$$= -\frac{\cos \pi/2}{\sin \pi/2} = \frac{-0}{1} = 0$$

~~Q12 lim x→~~

Model V : (0^0, ∞^0, 1^∞ forms)

$$\lim_{x\to a}\left[f(x)\right]^{g(x)} = 0^0 \text{ (or) } \infty^0 \text{ (or) } 1^\infty$$

Procedure :

$$\text{Let } y = \lim_{x\to a}\left[f(x)\right]^{g(x)}$$

Taking logarithms on both sides

$$\ln y = \lim_{x \to a} \ln (f(x))^{g(x)}$$

$$\ln y = \lim_{x \to a} g(x) \ln(f(x))$$

↳ This will be in $0 \times \infty$ form

Now solve the above as model III

(Q12) Evaluate the following limits:

$$\lim_{x \to 0} (\cos x)^{1/x^2}$$

Sol:

$$\lim_{x \to 0} (\cos x)^{1/x^2} \quad (1^\infty \text{ form})$$

let $y = \lim_{x \to 0} [\cos x]^{1/x^2}$

$$\ln y = \lim_{x \to 0} \ln\left((\cos x)^{1/x^2}\right)$$

$$\ln y = \lim_{x \to 0} \frac{1}{x^2} \ln(\cos x) \quad (\infty \times 0 \text{ form})$$

$$\ln y = \lim_{x \to 0} \frac{\ln(\cos x)}{x^2}$$

$$= \lim_{x \to 0} \frac{\frac{1}{\cos x} -\sin x}{2x}$$

$$= \frac{-1}{2} \lim_{x \to 0} \frac{\tan x}{x} = -1/2$$

$\ln y = -1/2 \Rightarrow y = e^{-1/2}$

(Q13) $\lim_{x\to\infty} (1+x^2)^{e^{-x}}$

Sol:

$\lim_{x\to\infty} (1+x^2)^{e^{-x}}$ (∞^0 form)

$$y = \lim_{x\to\infty} (1+x^2)^{e^{-x}}$$

$$\ln y = \lim_{x\to\infty} e^{-x} \ln[1+x^2] \quad (0\times\infty \text{ form})$$

$$= \lim_{x\to\infty} \frac{\ln(1+x^2)}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+x^2}\,2x}{e^x}$$

$$= \lim_{x\to\infty} \frac{2x}{e^x(1+x^2)}$$

$$= 2\lim_{x\to\infty} \frac{1}{e^x(2x)+(1+x^2)e^x}$$

$$= 2\lim_{x\to\infty} \frac{1}{e^x(x^2+2x+1)}$$

$$= 2(0) = 0$$

$$\ln y = 0$$

$$\Rightarrow y = e^0 = 1$$

Continuity and Differentiability

Continuous Functions:

A function $f(x)$ is said to be continuous at $x=a$ if

(i) $f(a)$ is defined (i.e., $f(a)$ is a finite number)

(ii) $\lim_{x\to a} f(x)$ exist

$$\left(\text{i.e., } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \right)$$

(iii) $\lim_{x\to a} f(x) = f(a)$

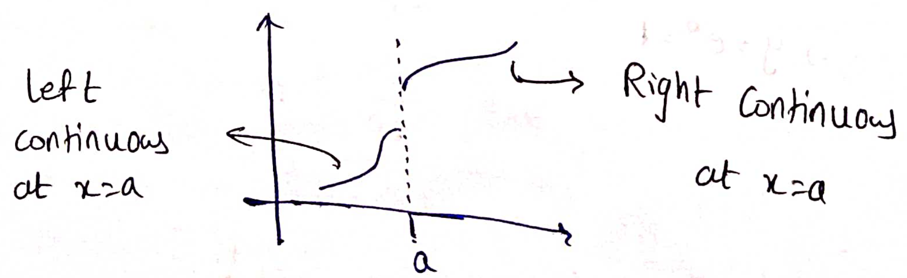
Note:

A function $f(x)$ is said to be continuous iff

$$\boxed{\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)}$$ [i.e., all are finite and equal]

→ If $\lim_{x\to a^-} f(x) = f(a)$ then $f(x)$ is left continuous at $x=a$

→ If $\lim_{x\to a^+} f(x) = f(a)$ then $f(x)$ is right continuous at $x=a$



Differentiable Function :

A function $f(x)$ is differentiable at $x=a$ if

$$\boxed{\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}}$$

(or)

$$\boxed{\lim_{h\to 0} \frac{f(a)-f(a-h)}{h} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}$$

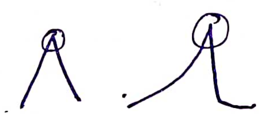
(or)

$$\boxed{f'(a^-) = f'(a^+)}$$

Note :

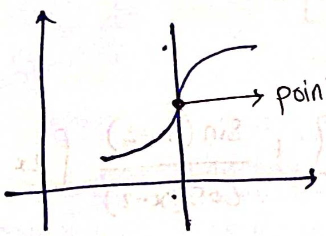
1) A function fails to be differentiable in the following cases:

(i) Corner Point (i.e., point at which curve changes its direction rapidly. So we can't draw a tangent at that point)

Eg : 

(ii) Point of vertical tangent

(i.e., point on curve at which the tangent is parallel to y-axis)



→ point of vertical tangent

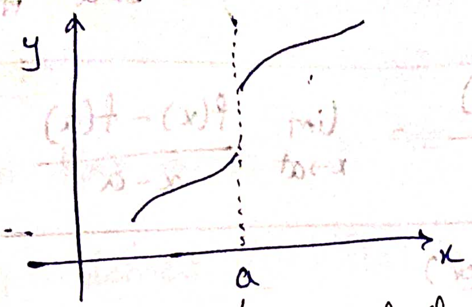
i.e., $\frac{dy}{dx} = \infty$

↓

It means $\frac{dy}{dx}$ is not defined

∴ not differentiable.

(iii) Point of Discontinuity :



↳ point of discontinuity

(Q14) If $\lim_{x\to 2} \dfrac{\tan(x-2)\,[x^2+(k-2)x-2k]}{x^2-4x+4} = 5$, then $k =$ ______

Sol :

$$\lim_{x\to 2} \frac{\tan(x-2)\,[x^2+(k-2)x-2k]}{x^2-4x+4} = 5 \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$\lim_{x\to 2} \frac{\sec^2(x-2)\,[x^2+(k-2)x-2k] + \tan(x-2)\,[2x+k-2]}{2x-4} = 5$$

$$\cancel{\frac{\sec^2(0)\,[4+(k-2)(2)-2k] + \tan 0}{\sec^2(1)\,[4+2k-4-2k]}} = 5 \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$\frac{1}{2} \lim_{x\to 2} \frac{\frac{1}{\cos^2(x-2)}\,[x^2+(k-2)x-2k] + \frac{\sin(x-2)}{\cos(x-2)}\,[2x+k-2]}{x-2} = 5$$

$$\Rightarrow \lim_{x\to 2} \frac{(x+k)\cancel{(x-2)}}{[\cos^2(x-2)]\cancel{(x-2)}} + \lim_{x\to 2} \frac{\tan(x-2)[2x+k-2]}{x-2} = 10$$

$$\Rightarrow \quad 2+k + \lim_{x\to 2} (2x+k-2) = 10$$

$$\Rightarrow \quad \cancel{2}+k+4+k-\cancel{2} = 10$$

$$\Rightarrow \quad 2k = 6$$

$$\Rightarrow \quad k = 3$$

Q15 $\lim_{x\to 0} \left\{ \frac{\tan(2x)-\sin(2x)}{x\sin x} \right\} =$ ______

Sol :

$$\lim_{x\to 0} \frac{\tan(2x)-\sin(2x)}{x\sin x}$$

$$= \lim_{x\to 0} \frac{\sin 2x \left(\frac{1}{\cos x} - 1\right)}{x \sin x}$$

$$= \lim_{x\to 0} (2) \frac{\frac{1-\cos x}{\cos x}}{\sin x} = \lim_{x\to 0} (2) \frac{1-\cos x}{\cos x \sin x}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{\sin x} \quad (0/0 \text{ form})$$

$$= \lim_{x\to 0} (2) \frac{\sin x}{\cos x}$$

$$= (2)\frac{(0)}{1}$$

$$= 0$$

Q.16

$$\lim_{x \to \pi/2} \frac{\tan x}{\ln(\cos x)} = \_\_\_\_$$

Sol:

$$\lim_{x \to \pi/2} \frac{\tan x}{\ln(\cos x)} \quad \left(\frac{\infty}{-\infty}\right)$$

$$= \lim_{x \to \pi/2} \frac{\sec^2 x}{\frac{1}{\cos x} -\sin x}$$

$$= -\lim_{x \to \pi/2} \frac{\cancel{\cos x}}{\cos^{\cancel{2}} x \sin x}$$

$$= -\lim_{x \to \pi/2} \frac{1}{\sin x \cos x} \quad \infty$$

$$= -\frac{1}{0} = -\infty$$

Q17

$$\lim_{x \to 0} \frac{\ln(\sin x)}{\cot x} = ?$$

Sol:

$$\lim_{x \to 0} \frac{\ln(\sin x)}{\cot x} \quad \left(\frac{-\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}$$

$$= -\lim_{x \to 0} \frac{\frac{\cos x}{\cancel{\sin x}}}{\frac{1}{\sin^{\cancel{2}} x}}$$

$$= -\lim_{x \to 0} \sin x \cos x = 0$$

Q18 $\lim_{x\to 0} \sin x(\ln x) = ?$

Sol :

$\lim_{x\to 0} \sin x(\ln x)$ $(0 \times -\infty)$

$\lim_{x\to 0} \frac{\ln x}{\operatorname{Cosec} x}$ $\left(\frac{-\infty}{\infty} \text{ form}\right)$

$\lim_{x\to 0} \frac{\frac{1}{x}}{-\operatorname{Cosec} x \cot x}$

$= -\lim_{x\to 0} \frac{\sin x}{x \cdot \cot x} = -\lim_{x\to 0} \frac{1}{\cot x} = -\lim_{x\to 0} \tan x = 0$

Q19 $\lim_{x\to 0} x \ln x =$ ______

Sol :

$\lim_{x\to 0} x(\ln x)$ $(0 \times -\infty)$

$\lim_{x\to 0} \frac{\ln x}{\frac{1}{x}}$ $\frac{-\infty}{\infty}$

$\lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = -\lim_{x\to 0} x = 0$

Q20 $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\tan x}\right) =$ ______

Sol :

$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\tan x}\right)$ $(\infty - \infty \text{ form})$

$$\lim_{x\to 0}\left(\frac{\tan x - x}{x\tan x}\right) \quad \left(\frac{0}{0}\text{ form}\right)$$

$$\lim_{x\to 0} \frac{\sec^2 x - 1}{x\sec^2 x + \tan x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$\lim_{x\to 0} \frac{2\sec x\,(\sec x \tan x)}{x(2\sec x \sec x \tan x) + \sec^2 x + \tan x}$$

$$\phi = \frac{0}{0+1+0} = \frac{0}{1} = 0$$

Q21 $$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) \quad (\infty - \infty)$$

$$\lim_{x\to 0}\left(\frac{\sin x - x}{x\sin x}\right) \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{\cos x - 1}{x\cos x + \sin x}$$

$$\lim_{x\to 0} \frac{-\sin x}{-x\sin x + \cos x + \cancel{\sin x}\cos x} = \frac{-0}{0+1+1} = 0$$

Q22 $$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) \quad (\infty - \infty)$$

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x(e^x - 1)} \quad \frac{0}{0}\text{ form}$$

$$\lim_{x\to 0} \frac{e^x - 1}{x(e^x) + (e^x - 1)} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{e^x}{xe^x + e^x + e^x} = \frac{1}{0+1+1} = \frac{1}{2}$$

Q23) $\lim_{x\to\infty} x^{1/x}$ (∞^0 form)

Sol :

Let $y = \lim_{x\to\infty} x^{1/x}$

$$\ln y = \lim_{x\to\infty} \frac{1}{x}\ln x \quad (0 \times \infty \text{ form})$$

$$\ln y = \lim_{x\to\infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\cancel{\ln y = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{-1}{x^2}}}$$

$$\cancel{\ln y = \lim_{x\to\infty} (-x)}$$

$$\cancel{\ln y =}$$

$$\ln y = \lim_{x\to\infty} \frac{\frac{1}{x}}{(1)}$$

$$\ln y = \frac{1}{\infty} = 0$$

$$\Rightarrow y = e^0 = 1$$

Q24) $\lim_{x\to\pi/2} (\tan x)^{\cos x}$ (∞^0)

$$\ln y = \lim_{x\to\pi/2} \cos x \ln(\tan x) \quad 0 \times \infty$$

$$\ln y = \lim_{x\to\pi/2} \frac{\ln(\tan x)}{\sec x}$$

$$= \lim_{x\to\pi/2} \frac{\frac{1}{\tan x}\sec^2 x}{\sec x \tan x}$$

$$= \lim_{x\to\pi/2} \frac{\sec x}{\tan^2 x} = \lim_{x\to\pi/2} \frac{\cos x}{\sin^2 x} = 0$$

$$\ln y = 0 \Rightarrow y = e^0 = 1$$

Q25

$$\lim_{x\to 0} (\sin x)^{\tan x} \qquad 0^0$$

sol:

$$\ln y = \lim_{x\to 0} \tan x \ln(\sin x)$$

$$= \lim_{x\to 0} \frac{\ln(\sin x)}{\cot x}$$

$$= \lim_{x\to 0} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x\to 0} -\cos x \sin x$$

$$\ln y = 0$$

$$y = e^0 = 1$$

Q26

$$\lim_{x\to a} (x-a)^{x-a} \qquad (0^0)$$

$$\ln y = \lim_{x\to a} (x-a)\ln(x-a)$$

$$\cancel{= \lim_{x\to a} \frac{\ln(x-a)}{x-a}}$$

$$\cancel{= \lim_{x\to a} x-a}$$

$$\cancel{= \lim_{x\to 0}}$$

$$= \lim_{x\to a} \frac{\ln(x-a)}{\frac{1}{x-a}}$$

$$= \lim_{x\to a} \frac{\frac{1}{x-a}}{\frac{-1}{(x-a)^2}}$$

$$= \lim_{x\to a} -(x-a)$$

$$= 0$$

$$\ln y = 0$$

$$\Rightarrow y = e^0 = 1$$

Note !

1) Every differentiable function is continuous but the reverse need not to be true.

2) ∴ Discontinuos functions are non - differentiable functions.

(not differentiable means no rate of change)

3) $\sin x, \cos x, \sinh x, \cosh x, e^{x}, e^{-x}$ and every polynomial function is continuous and differentiable everywhere.

(Q27) Which of the following are continuous at $x=2$

(a) $f(x) = \begin{cases} 3 & x=2 \\ 2x-1 & , x>2 \\ \frac{x+7}{3} & , x<2 \end{cases}$

c) $f(x) = \begin{cases} x+2 & , x \leq 2 \\ x-4 & , x>2 \end{cases}$

d) $f(x) = \dfrac{1}{x^2-8} \quad , x \neq 2$

b) $f(x) = \begin{cases} 2 & , x=2 \\ 8-x & , x \neq 2 \end{cases}$

Sol :

opt (a)

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x+7}{3} = \frac{9}{3} = 3$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2x-1 = 4-1 = 3$$

$$f(2) = 3$$

∴ continuous

opt (b)

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (8-x) = 6$$

$f(2) = 2$

$\therefore$ not cont.

opt (c)

$f(2) = 2+2 = 4$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x+2) = 2+2 = 4$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x-4) = 2-4 = -2$$

not cont.

opt (d)

not defined for $x=2$

$\therefore$ not cont.

Note :

* $y = f(x) = |ax-b|$ is both continuous and differentiable everywhere except at $x = b/a$

(Q28) $y = |x|$

It is cont & diff everywhere except at $x=0$

(Q29) $y = |2-3x|$

It is both cont & diff everywhere except at $x = \frac{2}{3}$

(Q30) $y = |x| + |x+1| + |x-3|$

~~It~~ Is cont. & diff everywhere except at

$x = 0, -1, 3$

(Q31) $f(x) = \begin{cases} 2x+3 & , x \le 4 \\ 7 + \frac{16}{x} & , x > 4 \end{cases}$

a) cont but not diff

b) diff but not cont

c) cont and diff

d) neither cont nor diff

Sol :

$f(4) = 2(4)+3 = 11$

& $\lim_{x \to 4^-} f(x) = \cancel{7 + \frac{16}{4}}\ 2(4)+3 = 11$

$\lim_{x \to 4^+} f(x) = 7 + \frac{16}{4} = 7+4 = 11$

$\therefore f(x)$ is cont.

Diff :

$f'(x) = \begin{cases} 2 & , x \le 4 \\ -16/x^2 & , x > 4 \end{cases}$

$f'(4^-) = 2$

$f'(4^+) = \frac{-16}{4^2} = -1$

$f'(4^-) \neq f(4^+)$

$\therefore f(x)$ is not differentiable.

$\therefore$ opt (a)

(Q32) A function defined as

$f(x) = \begin{cases} e^x & , x < 1 \\ \ln x + ax^2 + bx & , x \ge 1 \end{cases}$ when $x \in R$

which of the following stmts is true?

a) $f(x)$ is not diff at $x=1$ for any values of $a \& b$

b) $f(x)$ is diff at $x=1$ for ~~unique~~ unique values of $a \& b$

c) $f(x)$ is diff at $x=1$ for all values of $a \& b$ such that $a+b=e$

d) $f(x)$ is diff at $x=1$ for all values of $a \& b$

Sol :

$$f'(x) = \begin{cases} e^x, & x<1 \\ \frac{1}{x} + 2ax + b, & x \geq 1 \end{cases}$$

$$f'(1^-) = e^{(1)} = e$$

$$f'(1^+) = \frac{1}{1} + 2a + b = 2a+b+1$$

$f(x)$ is diff iff $f'(1^-) = f'(1^+)$ at $x=1$

$$\Rightarrow e = 2a+b+1$$

$$\Rightarrow 2a+b = e-1 \quad \text{—} \ (1)$$

~~since~~ If $f(x)$ is diff then it is cont.

$$\Rightarrow \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$e^1 = \ln(1) + a + b$$

$$\Rightarrow a+b = e \quad \text{—} \ (2)$$

from (1) & (2)

$$a = -1, \ b = e+1$$

$\therefore$ opt (b)

12/07/20

Q22

$$f(x) = \begin{cases} \frac{1}{|x|} & \text{if } |x| > 2 \\ a+bx^2 & \text{if } |x| \le 2 \end{cases}$$

If $f(x)$ is differentiable at $x=2$ then the values of a+b are

Sol:

$$\boxed{\begin{array}{l} |x| \le a = -a \le x \le a \\ |x| > a = (-\infty < x < -a) \cup (a < x < \infty) \end{array}}$$

$$f(x) = \begin{cases} \frac{1}{|x|} & \text{if } |x| > 2 = (-\infty < x < -2) \cup (2 < x < \infty) \\ a+bx^2 & \text{if } |x| \le 2 = -2 \le x \le 2 \end{cases}$$

$$f(x) = \begin{cases} \frac{-1}{x}, & -\infty < x < -2 \\ a+bx^2, & -2 \le x \le 2 \\ \frac{1}{x}, & 2 < x < \infty \end{cases}$$

given $f(x)$ is diff at $x=2$

$\Rightarrow f(x)$ is continuous at $x=2$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$\lim_{x \to 2^-} (a+bx^2) = \lim_{x \to 2^+} \frac{1}{x}$$

$$a+4b = 1/2 \quad \text{--- (1)}$$

$$f'(x) = \begin{cases} \frac{1}{x^2}, & -\infty < x < -2 \\ 2bx, & -2 \le x \le 2 \\ \frac{-1}{x^2}, & 2 < x < \infty \end{cases}$$

$$f'(2^-) = f'(2^+)$$

$$2b(2) = -\frac{1}{2^2} \Rightarrow \boxed{b = \frac{-1}{16}} \quad \text{from (1)} \quad a - \frac{1}{4} = 1/2$$

$$\boxed{a = 3/4}$$

Q34) If the function $f(x) = ax + bx^2, 0 \le x \le 1$

$= c + 3x, 1 < x \le 2$

is such that $f(1) = 1$, $f(2) = 4$ and $f(x)$ is diff at $x = 1$. Find ~~a+b~~ . a, b, c

Sol :

$f(1) = 1$

$a + b = 1$ — (1)

$f(2) = 4$

$c + 6 = 4$

$\boxed{c = -2}$

$$f'(x) = \begin{cases} a + 2bx, & 0 \le x \le 1 \\ 3, & 1 < x \le 2 \end{cases}$$

$f'(1^-) = f(1^+)$

$a + 2b = 3$ — (2)

(1) ⇒ $a + b = 1$

$\boxed{b = 2}$ ⇒ $\boxed{a = -1}$

Q35) Let $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ mx + b & \text{if } x > 2 \end{cases}$

if $f(x)$ is diff at $x = 2$ then

a) $m = 4, b = -4$ b) $m = 4, b = 4$

c) $m = -4, b = -4$ d) $m = -4, b = 4$

Sol :

diff ⇒ cont

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ ⇒ $\boxed{4 = 2m + b}$ — (1)

$\cancel{f'(2)}$ $f'(2^-) = f'(2^+)$

$2(2) = m$

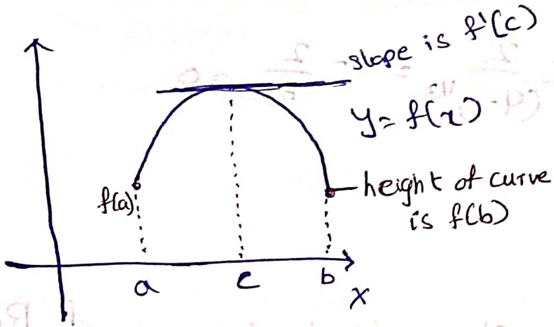
$\Rightarrow \boxed{m=4}$

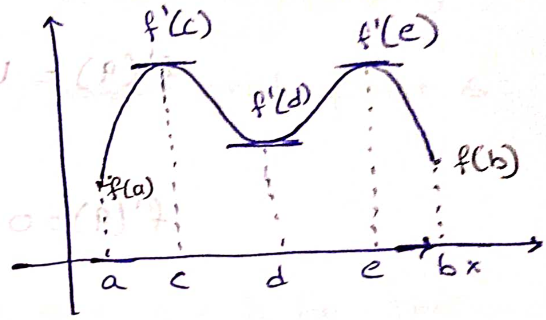
$\Rightarrow \boxed{b=-4}$

$\therefore$ opt (a)

Mean Value Theorems :

I. Rolle's Theorem (or) Fundamental Mean Value theorem of Calculus





Let $f(x)$ be defined on $[a,b]$ such that

1) $f(x)$ is cont on $[a,b]$

2) $f(x)$ is diff on (a,b)

3) $f(a) = f(b)$ then there exist atleast one point $c \in (a,b)$ such the $f'(c) = 0$

Note :

1) There exists atleast one point (one or more points) b/w a & b where the tangent drawn to the curve is parallel to x-axis. This is the geometrical significance of rolle's theorem.

2) Converse of Rolle's theorem need ~~to~~ not to be true.

for example

$$f(x) = x - 3(x-1)^{2/3} \text{ in } [0,10]$$

$$f'(x) = 1 - 3\left(\frac{2}{3}\right)(x-1)^{-1/3}$$

$$f'(x) = 1 - 2(x-1)^{-1/3}$$

$$f'(x) = 1 - \frac{2}{(x-1)^{1/3}}$$

$f'(1) = \infty$ i.e., $f(x)$ is not diff at $x=1$

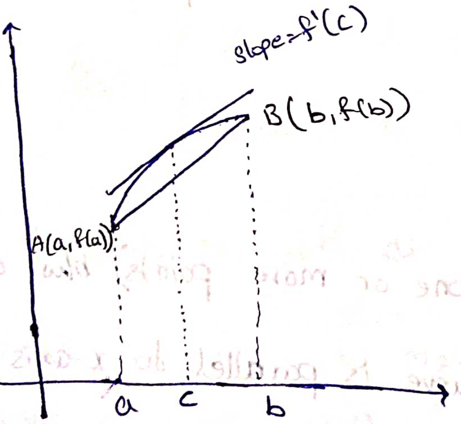
Now consider $f'(9)$

$$f'(9) = 1 - \frac{2}{(9-1)^{1/3}} = 1 - \frac{2}{2} = 0$$

$$f'(9) = 0$$

Thus the above example shows that converse of Rolle's theorem is not true.

II. Mean Value Theorem (or) Lagrange's Mean Value Theorem (or) First Mean Value Theorem of Calculus:



Slope of the curve at $x=c$

$= f'(c)$

Slope of the line $\overline{AB} = \dfrac{f(b)-f(a)}{b-a}$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

Instantaneous rate of change at $x=c$

Average rate of change in $[a,b]$

Let $f(x)$ be defined on $[a,b]$ such that

i) $f(x)$ is continuous on $[a,b]$

ii) $f(x)$ is differentiable on (a,b) then there exists

atleast one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Note:

* The geometrical significance is that

there exist atleast one point (one or more) b/w

a & b where tangent to ~~to~~ the curve at that point is

parallel to the chord

* Converse of Mean value theorem need not to be true.

* If $f(a)=f(b)$ then mean value theorem reduces to

Rolle's theorem.

III. Cauchy's Mean Vaule Theorem (or) Second Mean value theorem of Calculus:

Let $f(x)$ & $g(x)$ be two functions such that

i) $f(x)$ and $g(x)$ are continous on $[a,b]$

ii) $f(x)$ & $g(x)$ are diff on (a,b)

iii) $g'(x) \neq 0 \quad \forall x \in (a,b)$

then there exists atleast one point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Q36 Find the point on the curve $f(x) = 2\sin x + \cos(2x)$ at $\underset{\bar{a}}{(0,1)}\ \underset{f(a)}{}$ and $\underset{\bar{b}}{\left(\frac{\pi}{2}, 1\right)}\ \underset{f(b)}{}$ such that tangent at 'c' is parallel to x-axis is

Sol :

Given $f(x) = 2\sin x + \cos 2x$

$f'(x) = 2\cos x - 2\sin 2x$

$f'(c) = 0$

$\Rightarrow 2\cos c - 2\sin 2c = 0$

$2\cos c - 4\sin c\cos c = 0$

$2\cos c(1 - 2\sin c) = 0$

$\cos c = 0$ (or) $1 - 2\sin c = 0$

$c = \pi/2$ $\quad\quad$ $\sin c = 1/2$

$c = \pi/6$

$c \in (a, b)$

$c \in (0, \pi/2)$

$\therefore c = \pi/6$

Q37 The number c satisfies the conclusion of mean value theorem for $f(x) = x + \frac{4}{x}$ in $[1, 8]$ is ______

Sol :

$f(x) = x + \frac{4}{x}$ $\quad$ Mean value theorem means Langrange's mean value theorem

$f'(x) = 1 - \frac{4}{x^2}$

$\cancel{f'(c) = 0}$

$\cancel{1 - \frac{4}{c^2} = 0}$

$\cancel{c^2 - 4 = 0} \Rightarrow \cancel{c = \pm 2 \text{ and } c \in (1, 8)}$

$\cancel{\therefore c = 2}$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$1-\frac{4}{c^2} = \frac{8.5-5}{8-1}$$

$$1-\frac{4}{c^2} = \frac{3.5}{7} \Rightarrow 1-\frac{4}{c^2} = \frac{1}{2}$$

$$\frac{c^2-4}{c^2} = \frac{1}{2} \Rightarrow 2c^2-8 = c^2$$

$$c^2-8=0$$

$$c = \pm\sqrt{8}$$

and $c \in (1, 8)$

$$\therefore c = 2\sqrt{2}$$

*

(Q38) * If $f(0)=2$ and $f'(x) = \frac{1}{5-x^2}$ then lower and upper bounds

*

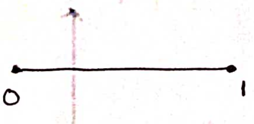
of $f(1)$ estimated by the mean value theorem are

Sol :

No interval is given. So we need to choose one

from the question we

choose interval $[0, 1]$ ($a = 0$, $b = 1$)



$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{5-c^2} = \frac{f(1)-f(0)}{1-0} \Rightarrow \frac{1}{5-c^2} = f(1)-2 \quad \text{—} \; (1)$$

also from mean value theorem

w.k.T $\quad 0 < c < 1$

$$0 < c^2 < 1$$

$$0 > -c^2 > -1$$

$$5 > 5-c^2 > 4$$

$$\frac{1}{5} < \frac{1}{5-c^2} < \frac{1}{4}$$

$$\frac{1}{5} < f(1) - 2 < \frac{1}{4}$$

$$2 + \frac{1}{5} < f(1) < 2 + \frac{1}{4}$$

$$\frac{11}{5} < f(1) < \frac{9}{4}$$

$$2.2 < f(1) < 2.25$$

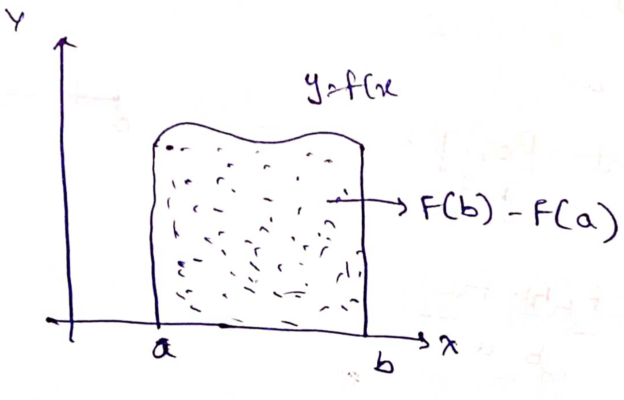
$$\therefore f(1) = (\underbrace{2.2}_{\text{lower bound}}, \underbrace{2.25}_{\text{upper bound}})$$

Definite Integrals [Areas under curves]:

the integral

$$\int_a^b f(x)\,dx = \left(F(x)\right)_a^b = F(b) - F(a)$$

is called a definite integral



Properties of definite integrals:

1) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2) $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even func} \\ 0, & \text{if } f(x) \text{ is odd func} \end{cases}$$

3) $$\int_0^{2a} f(x)\,dx \begin{cases} 2\int_0^a f(x)\,dx, & \cancel{\text{if } f(x) \text{ is even func}}\ f(2a-x) = f(x) \\ 0, & \cancel{\text{if } f(x) \text{ is odd func}}\ f(2a-x) = -f(x) \end{cases}$$

4) $$\int_0^a f(x)\,dx = \begin{cases} 2\int_0^{a/2} f(x)\,dx, & \text{if } (a-x) = f(x) \\ 0, & \text{if } f(a-x) = -f(x) \end{cases}$$

5) $$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

6) $$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$$

7) $$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{1}{2}\cdot\frac{\pi}{2}, & n \text{ is even} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}\times 1, & n \text{ is odd} \end{cases}$$

8) $$\int_0^{\pi/2} \sin^m x \cos^n x = \frac{\{(m-1)(m-3)(m-5)\cdots 1 \text{ or } 2\}\{(n-1)(n-3)(n-5)\cdots(1 \text{ or } 2)\}}{(m+n)(m+n-2)(m+n-4)\cdots(1 \text{ or } 2)}\,K$$

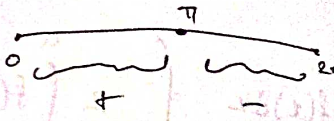
where $K = \pi/2$ if m & n are even

$= 1$ otherwise

Q39) If $\int_0^{2\pi} |x\sin x|\,dx = k\pi$ then $k =$ ______

Sol :

$$\int_0^{2\pi} |x\sin x|\,dx$$



$$= \int_0^{\pi} |x\sin x|\,dx + \int_{\pi}^{2\pi} |x\sin x|\,dx$$

$$= \int_0^{\pi} x\sin x\,dx - \int_{\pi}^{2\pi} x\sin x\,dx$$

$$= \Big[-x\cos x + \sin x\Big]_0^{\pi} - \Big[-x\cos x + \sin x\Big]_{\pi}^{2\pi}$$

$$= \Big[(-\pi\cos\pi + \sin\pi) - 0\Big] - \Big\{\big[-2\pi\cos 2\pi + \sin 2\pi\big] - \big[-\pi\cos\pi + \sin\pi\big]\Big\}$$

$$= 2(-\pi\cos\pi + \sin\pi) + 2\pi\cos 2\pi - \sin 2\pi$$

$$= 2(\pi + 0) + 2\pi(+1) - 0$$

$$= 2\pi + 2\pi = 4\pi$$

$$\boxed{\therefore k = 4}$$

Q40) $\int_1^4 \left(|x-1| + |x-2| + |x-3|\right) dx =$ ______

Sol :

$$\int_1^2 \{(x-1) - (x-2) - (x-3)\}\,dx + \int_2^3 \{(x-1) + (x-2) - (x-3)\}\,dx$$

$$+ \int_3^4 \{(x-1) + (x-2) + (x-3)\}\,dx$$

$$= \int_1^2 (-x+4)\,dx + \int_2^3 x\,dx + \int_3^4 (3x-6)\,dx$$

$$= \left(4x - \frac{x^2}{2}\right)_1^2 + \left(\frac{x^2}{2}\right)_2^3 + \left(\frac{3x^2}{2} - 6x\right)_3^4$$

$$= \left\{(8-2) - \left(4 - \frac{1}{2}\right)\right\} + \left\{\frac{9}{2} - \frac{4}{2}\right\} + \left\{\left(\frac{48}{2} - 24\right) - \left(\frac{27}{2} - 18\right)\right\}$$

$$= 6 - 4 + \frac{1}{2} + \frac{5}{2} - \frac{27}{2} + 18 = 23 - \frac{27}{2}$$

$$= 19/2$$

(Q41) $\int_0^{10} [x]\,dx =$ ______

Sol :

$$\int_0^{10} [x]\,dx = \int_0^1 (0)\,dx + \int_1^2 (1)\,dx + \int_2^3 (2)\,dx + \cdots + \int_9^{10} (9)\,dx$$

$$= (x)_1^2 + 2(x)_2^3 + \cdots + 9(x)_9^{10}$$

$$= 1 + 2 + \cdots + 9 = \frac{9(9+1)}{2} = 45$$

(Q42) The value of $\int_0^{1.5} x[x^2]\,dx$ is ______

Sol :

$$\int_0^{1.5} x[x^2]\,dx = \int_0^1 x(0)\,dx + \int_1^{\sqrt{2}} x(1)\,dx + \int_{\sqrt{2}}^{1.5} x(2)\,dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_1^{\sqrt{2}} + \cancel{2}\left[\frac{x^2}{\cancel{2}}\right]_{\sqrt{2}}^{1.5}$$

$$= \left(\frac{2}{2} - \frac{1}{2}\right) + \left[(1.5)^2 - 2\right]$$

$$= \frac{1}{2} + 0.25 = 3/4 = 0.75$$

(Q43) $\int_0^{\pi} x \sin^8 x \cos^6 x\,dx$ is ______

Sol :

Let $I = \int_0^{\pi} x \sin^8 x \cos^6 x \, dx$ —— ①

$$\left(\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \right)$$

$I = \int_0^{\pi} (\pi - x) \sin^8(\pi - x) \cos^6(\pi - x) \, dx$

$I = \int_0^{\pi} (\pi - x) \sin^8 x \cos^6 x \, dx$ —— ②

$\cancel{I = \int_0^{\pi} \pi \sin^8 x \cos^6 x \, dx}$

Adding ① & ②

$2I = \int_0^{\pi} \pi \sin^8 x \cos^6 x \, dx$

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \quad f(2a-x) = f(x)$$

$\cancel{2}I = \pi(\cancel{2}) \int_0^{\pi/2} \sin^8 x \cos^6 x \, dx$

$$I = \pi \frac{(8-1)(8-3)(8-5)(8-7)\cdot(6-1)(6-3)(6-5)}{14(12)(10)(8)(6)(4)(2)}$$

$$I = \pi \frac{(7)(5)(3)(1)(5)(3)(1)}{(14)(12)(10)(8)(6)(4)(2)} \times \frac{\pi}{2} = \frac{5\pi}{2048} \quad \frac{5\pi^2}{4096}$$

2 4 2 2

($\because$ m, n are even)

Note

Le

*

Q44 $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x} dx = ?$

Q45

Sol: $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x} = \int_0^{\pi/2} \frac{\cos^3 x}{\sin^3 x + \cos^3 x} dx$

let $I = \int_0^{\pi/2} \frac{\cos^3 x}{\sin^3 x + \cos^3 x} dx$ — ①

$$I = \int_0^{\pi/2} \frac{\cos^3(\pi/2 - x)}{\sin^3(\pi/2 - x) + \cos^3(\pi/2 - x)} dx$$

This is f form

$$\int_a^b \frac{f(x)}{f(x) + f(a+b-x)} dx = \frac{b-a}{2}$$

$$I = \int_0^{\pi/2} \frac{\sin^3(x)}{\sin^3(x) + \cos^3 x} dx \quad — ②$$

Adding ① & ②

$$2I = \int_0^{\pi/2} dx$$

$$2I = (x)_0^{\pi/2}$$

$$I = \frac{\pi/2 - 0}{2} \Rightarrow I = \pi/4$$

Note:

Leibnitz's Rule:

$$* \quad \frac{d}{dx}\left[\int_{U(x)}^{V(x)} f(t)\,dt\right] = f(V(x)) \frac{d\,V(x)}{dx} - f(U(x)) \frac{d\,U(x)}{dx}$$

(Q45) If $x \sin(\pi x) = \int_0^{x^2} f(t)\,dt$, where f is a continuous function then $f(4) = ?$

a) $\pi/2$ b) $\pi^2/4$ c) $\pi/4$ d) $\pi^2/2$

Sol:

$$x\sin\pi x = \int_0^{x^2} f(t)\,dt$$

$$\frac{d}{dx}\left(x\sin\pi x\right) = \frac{d}{dx}\int_0^{x^2} f(t)\,dt$$

$$x\cos\pi x(\pi) + \sin\pi x = f(x^2)(2x) - f(0)(0) \quad (\because \text{Leibnitz's Rule})$$

$$x\cos\pi x(\pi) + \sin\pi x = 2x\, f(x^2)$$

put $x=2$

$$2(\cos 2\pi)(\pi) + \sin 2\pi = 4\, f(4)$$

$$2\pi(1) + (0) = 4\, f(4)$$

$$f(4) = \pi/2$$

(Q46) A function $f(x) = 1-x^2+x^3$ in $[-1,1]$. The value of x in the open interval $(-1,1)$ for which ~~wheat~~ mean value theorem is satisfied is ?

Sol :

$$f(x) = 1-x^2+x^3 \quad , [-1,1]$$

$\downarrow$ a, $\downarrow$ b

$$f'(x) = -2x+3x^2$$

$f(1) = 1-1+1 = 1$

$f(-1) = 1-1-1 = -1$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$-2c+3c^2 = \frac{1+1}{1+1}$$

$$-2c+3c^2 = 1 \Rightarrow 3c^2-2c-1=0$$

$$3c^2-3c+c-1=0 \Rightarrow 3c(c-1)+1(c-1)=0$$

$c=1$ (or) $c = 1/3$ but $c\in(-1,1)$ $\therefore c=1/3$

Q47 Find the point on the curve $f(x) = x^2$, at which tangent is parallel to the chord joining points $(1,1)$ & $(3,4)$

Sol :-

$y = x^2$ $(1,1)$ $(3,4)$

$a\ f(a)$ $b\ f(b)$

$f'(x) = 2x$

$f'(c) = \dfrac{f(b) - f(a)}{b-a}$

$2c = 3$

$c = 3/4$

$y = x^2$

$f'(x) = 2x$

parallel $\Rightarrow$ equal slopes

$2x_1 = \dfrac{4-1}{3-1}$

$x_1 = \dfrac{3}{4}$

$\Rightarrow y_1 = \left(\dfrac{3}{4}\right)^2$

$\therefore$ The req point is $\left(\dfrac{9}{16}, \dfrac{3}{4}\right)$

Q48 If $f'(x) = \dfrac{1}{1+x^2}$ for all x and $f(0) = 0$ then an interval in which $f(2)$ lies is

Sol :

Consider curve $f(x)$ in the interval $[0,2]$

$f'(c) = \dfrac{f(2) - f(0)}{2-0}$

$$\frac{1}{1+c^2} = \frac{f(2)}{2} \Rightarrow f(2) = \frac{2}{1+c^2}$$

W.K.T $c \in (0,2)$

$$0 < c < 2$$

$$0 < c^2 < 4$$

$$1 < 1+c^2 < 5$$

$$1 > \frac{1}{1+c^2} > \frac{1}{5}$$

$$2 > \frac{2}{1+c^2} > \frac{2}{5}$$

$$2 > f(2) > \frac{2}{5}$$

$$f(2) \in (2/5, 2)$$

(Q49) If $f'(x) = e^x$ and $f(0) = 5$, using mean value theorem $f(1)$ lies b/w

Sol:

Consider interval $[0,1]$

$$f'(x) = e^x$$

There exists $c \in (0,1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1-0}$$

$$e^c = f(1) - 5$$

$$f(1) = e^c + 5$$

W.K.T

$$0 < c < 1$$

$$\Rightarrow e^0 < e^c < e$$

$$\Rightarrow 1+5 < e^c + 5 < e+5$$

$$\Rightarrow f(1) \in (6, e+5) \Rightarrow f(1) \in (6, 7.718)$$

Q.50) The value of c of cauchy mean value theorem for $f(x) = e^x$ and $g(x) = e^{-x}$ in the interval $(2,3)$ is ______

Sol:

$f(x) = e^x \quad g(x) = e^{-x} \quad (2,3)$

$f'(x) = e^x \quad g'(x) = -e^{-x}$

$$\frac{f'(x)}{g'(x)} = \frac{e^x}{-e^{-x}} = -e^{2x}$$

There exists atleast one '$c \in (2,3)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$-e^{2c} = \frac{e^3-e^2}{e^{-3}-e^{-2}}$$

$$-e^{2c} = \frac{e^3-e^2}{\frac{1}{e^3}-\frac{1}{e^2}} \Rightarrow -e^{2c} = \frac{e^3-e^2}{\frac{1-e}{e^3}}$$

$$\Rightarrow -e^{2c} = \frac{e^5(e-1)}{1-e}$$

$$\Rightarrow e^{2c} = e^5 \Rightarrow c = 2.5$$

Q51) $\int_0^3 |3-2x|\,dx$ ______

Sol:

$3-2x=0 \Rightarrow x = 1.5$

$$\cancel{\int_0^{1.5} (2x-3)\,dx + \int_{1.5}^{3} (3-2x)\,dx = \left(x^2-3x\right)_0^{1.5} + \left(3x-x^2\right)_{1.5}^{3}}$$

$$\cancel{= (2.25-4.5) + ((0)-(2.25-4.5))}$$

$$\int_0^{1.5}(3-2x)dx + \int_{1.5}^{3}(2x-3)\,dx$$

$$\left(3x-x^2\right)_0^{1.5} + \left(x^2-3x\right)_{1.5}^{3}$$

$$(4.5-2.25)+\left(0-(2.25-4.5)\right)$$

$$= 2(4.5-2.25) = 2(2.25)$$

$$= 4.5$$

(Q52) $\int_{-1}^{2} |x^3-x|\,dx$

~~$|x^3-x|$ is +ve~~

$x^3-x=0$

$x^2(x-1)=0$

$x=0 \mid x=1$

$$|x^3-x| = \begin{cases} x^3-x, & -1<x<0 \\ x-x^3, & 0<x<1 \\ x^3-x, & 1<x<2 \end{cases}$$

$$\therefore \int_{-1}^{2}|x^3-x|\,dx = \int_{-1}^{0}(x^3-x)\,dx + \int_0^1 (x-x^3)\,dx + \int_1^2 (x^3-x)\,dx$$

$$= \left(\frac{x^4}{4}-\frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2}-\frac{x^4}{4}\right)_0^1 + \left(\frac{x^4}{4}-\frac{x^2}{2}\right)_1^2$$

$$= \left[0-\left(\frac{1}{4}-\frac{1}{2}\right)\right] + \left[\left(\frac{1}{2}-\frac{1}{4}\right)-0\right] + \left[\left(\frac{16}{4}-\frac{4}{2}\right)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]$$

$$= \frac{1}{4}+\frac{1}{4}+\left[(2)-(-1/4)\right]$$

$$= \frac{1}{4}+\frac{1}{4}+\frac{9}{4} = 11/4$$

(Q53) $\int_{-4}^{7}|x|\,dx$

$$= \int_{-4}^{4}|x|\,dx + \int_4^7 |x|\,dx = 2\int_0^4 x\,dx + \int_4^7 x\,dx$$

$$\cancel{2}\left(\frac{x^2}{\cancel{2}}\right)_0^{4} + \left(\frac{x^2}{2}\right)_4^7$$

$$16 + \frac{49}{2} - \frac{16}{2}$$

$$16 + \frac{33}{2} = 16 + 16.5 = 32.5$$

Q53) $\int_0^{\pi/2} \ln(\tan x)\, dx =$ ______

sol :

let $I = \int_0^{\pi/2} \ln(\tan x)\, dx$ — ①

$I = \int_0^{\pi/2} \ln(\cot x)\, dx$ — ②

$\cancel{I^2 = \int_0^{\pi/2} \ln(\tan x) \ln(\cot x)\, dx}$

① + ②

$2I = \int_0^{\pi/2} [\cancel{\log} \ln(\tan x) + \ln(\cot x)]\, dx$

$2I = \int_0^{\pi/2} \ln[\tan x \cdot \cot x]\, dx$

$2I = \int_0^{\pi/2} \ln(1)\, dx$

$2I = 0 \Rightarrow I = 0$

Q54) $\int_0^{\pi/2} \ln(\sin x)\, dx$

sol :

let $I = \int_0^{\pi/2} \ln(\sin x)\, dx$ — ①

$I = \int_0^{\pi/2} \ln(\cos x)\, dx$ — ②

Adding ① & ②

$$2I = \int_0^{\pi/2} \ln\left(\sin x \cos x\right) dx$$

$$2I = \int_0^{\pi/2} \ln\left(\tfrac{1}{2}\sin 2x\right) dx$$

$$2I = \int_0^{\pi/2} \ln(\sin 2x)\, dx - \int_0^{\pi/2} \ln(2)\, dx$$

first let us comput

$$\int_0^{\pi/2} \ln(\sin 2x)\, dx$$

$\cancel{\text{Put } 2x = t \Rightarrow 2dx =} dt$

$$\Rightarrow \cancel{\frac{1}{2}\int_0^{\pi} \ln(\sin t)}\, dt$$

$$\text{let } \cancel{a = \frac{1}{2}\int_0^{\pi} \ln(\sin t)}\, dt$$

$$= \int_0^{\pi/4} \ln(\sin 2x)\, dx + \int_{\pi/4}^{\pi/2} \ln(\sin 2x)\, dx$$

let $t = 2x$

$\Rightarrow$ $dt = 2dx$

$$\Rightarrow \frac{1}{2}\int_0^{\pi/2} \ln(\sin t)\, dt$$

$$\Rightarrow \frac{1}{2} I$$

$t = 2x - \pi/2 \Rightarrow dt = 2x$

$$\frac{1}{2}\int_0^{\pi/2} \ln\left(\sin\left(t + \pi/2\right)\right) dt$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi/2} \ln(\cos t)\, dt$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi/2} \ln(\sin t)\, dt$$

$$= \tfrac{1}{2} I$$

$$\Rightarrow \int_0^{\pi/2} \ln(\sin 2x)\,dx = \frac{1}{2}I + \frac{1}{2}I = I$$

$$\Rightarrow 2I = I - \int_0^{\pi/2} \ln(2)\,dx$$

$$I = -(x)_0^{\pi/2} \ln 2$$

$$I = -(\ln 2)\left(\frac{\pi}{2}\right)$$

Q55 $\displaystyle\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx =$ __________

Sol:

let

$$I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$$

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx$$

$$\Rightarrow 2I = \int_0^a dx$$

$$\Rightarrow 2I = (x)_0^a \Rightarrow I = a/2$$

Q56 $\displaystyle\int_3^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{8-x}}\,dx =$ __________

Sol:

$$I = \int_3^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{8-x}}\,dx$$

$$\Rightarrow I = \int_3^5 \frac{\sqrt{8-x}}{\sqrt{8-x}+\sqrt{x}}\,dx \Rightarrow 2I = \int_3^5 (1)\,dx$$

$$\Rightarrow I = 1$$

Q57) $\displaystyle\int_{-\pi}^{\pi} \frac{e^{\sin x}}{e^{\sin x}+e^{-\sin x}}\,dx$

Sol :

Let $I = \displaystyle\int_{-\pi}^{\pi} \frac{e^{\sin x}}{e^{\sin x}+e^{-\sin x}}\,dx$ —— ①

$I = \displaystyle\int_{-\pi}^{\pi} \frac{e^{\sin(\pi-\pi-x)}}{e^{\sin(\pi-\pi-x)}+e^{-\sin(\pi-\pi-x)}}\,dx$

$I = \displaystyle\int_{-\pi}^{\pi} \frac{e^{-\sin x}}{e^{-\sin x}+e^{\sin x}}\,dx$ —— ②

① + ②

$\Rightarrow 2I = \displaystyle\int_{-\pi}^{\pi} dx$

$2I = (x)_{-\pi}^{\pi}$

$I = \dfrac{\pi+\pi}{2}$ $\Rightarrow \underline{I = \pi}$

Q58) If $f(x) = \displaystyle\int_{\sin x}^{\cos x} e^{-t^2}\,dt$ then $f'(\pi/4)$ is

Sol :

given $f(x) = \displaystyle\int_{\sin x}^{\cos x} e^{-t^2}\,dt$

diff on b.s $\qquad F(x) = e^{-x^2}$

$f'(x) = F(\cos x)\dfrac{d}{dx}(\cos x) - F(\sin x)\dfrac{d}{dx}(\sin x)$

$$f'(x) = e^{-\cos^2 x}(-\sin x) - e^{-\sin^2 x}\cos x$$

$$f'(\pi/4) = e^{-1/2}\left(-1/\sqrt{2}\right) - e^{-1/2}\, 1/\sqrt{2}$$

$$= -\frac{1}{e^{1/2}}\left(\frac{2}{\sqrt{2}}\right) = \frac{-\sqrt{2}}{e^{+1/2}} = -\sqrt{\frac{2}{e}}$$

Calculus

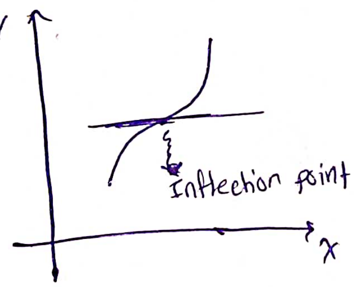
Maxima & Minima:

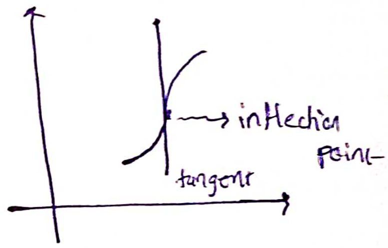
Stationary point: The point where $f'(x)=0$, is called stationary point.

Critical point: The point where $f'(x)=0$ (or) $f'(x)$ is undefined ($f'(x)=\infty$ i.e., $f(x)$ is not differentiable) is called critical point.

Inflection point: The point where the graph of the function has a tangent line and where the curve changes its shape from concave up to concave down (or) vice versa is called inflection point.

→ A graph neither has maxima nor minima at inflection point.





the point where $f''(x)$ is undefined (or) $f''(x)=0$ and $f'''(x)\neq 0$ is called and inflection point.

Global Maxima & Minima (or) Absolute Maxima & Minima:

To find global maxima and minima of a function defined on $[a,b]$ we proceed as follows

1) find the interior critical points of the function and let critical points be x_0, x_1, x_2 etc.
2) Let the end points of function be a & b.
3) Find $f(x)$ at critical points and at end points i.e., find $f(a)$, $f(b)$, $f(x_0)$, $f(x_1)$, $f(x_2)$ etc.

4) Global Maximum = Largest $\{f(a), f(b), f(x_0), f(x_1), \ldots\}$

5) Global Minimum = Smallest $\{f(a), f(b), f(x_0), f(x_1) \ldots\}$

Q59) Find the maxima and minima of the below

$$f(x) = x^3 - 3x^2 - 24x + 100 \quad \text{in } [-3,3]$$

Sol:

finding interior critical points:

$$f'(x) = 3x^2 - 6x - 24 = 0$$

for no value of x $f'(x)$ can be ∞

$\therefore f'(x) = 0$

$$3x^2 - 6x - 24 = 0$$

$$\Rightarrow x^2 - 2x - 8 = 0$$

$x = 4, -2$ are critical points.

However -2 is the interior critical point

$$\therefore f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 100$$

$$= -8 - 12 + 48 + 100$$

$$= 128$$

$$f(3) = 27 - 27 - 24(+3) + 100 = 28$$

$$f(-3) = 27 - 3(-3)^2 - 24(-3) + 100 = 118$$

largest $(f(-2), f(3), f(-3)) = f(-2) = 128$ is global maximum

min $(f(-2), f(3), f(-3)) = f(3) = 28$ is global minimum

Local Maxima & Local Minima:

1) Find the critical points of the function and let these critical points be x_0, x_1, x_2 etc.

2) Find $f''(x)$ at Critical points.

3) If $f''(x_0) > 0$ then $f(x)$ has local minimum at $x = x_0$

and $f_{min} = f(x_0)$

4) If $f''(x_0) < 0$ then $f(x)$ has local maximum at $x = x_0$

and $f_{max} = f(x_0)$

5) If $f''(x_0) = 0$ then $f(x)$ may or may not have maxima (or) minima

6) If $f''(x_0) = 0$ & $f'''(x_0) \neq 0$ then x_0 is inflection point.

7) If $f''(x_0) = 0$ & $f'''(x_0) = 0$ then find $f''''(x_0)$ and repeat ~~step ③ & step~~ from step ③

$\underline{f'''(x_0) = 0}$ → Take this as $\underline{f'(x)}$

(Q60) A point on the curve is said to be an extremum if it is a local minima (or) a local maxima. The number of distinct extrema for curve $3x^4 - 16x^3 + 24x^2 + 37$ is

extremum → (maxima or minima)

a) 0 b) 1 c) 2 d) 3

Sol:

$f(x) = 3x^4 - 16x^3 + 24x^2 + 37$

$f'(x) = 12x^3 - 48x^2 + 48x$

$\Rightarrow f'(x) = 0$

$\Rightarrow 12x(x^2 - 4x + 4) = 0$

$\Rightarrow x(x-2)^2 = 0$

$\Rightarrow x = 0 \mid x = 2$

$f''(x) = 36x^2 - 96x + 48$

Now $f''(0) = 48 > 0$ $\therefore$ $f(x)$ has Maximum at $x = 0$

$f''(2) = 36(4) - 96(2) + 48$

$= 0$

Now we need to find $f'''(x)$

$f'''(x) = 72x - 96$

$f'''(2) = 72(2) - 96 = 48$

we have $f''(2) = 0$ & $f'''(2) \neq 0$

$\therefore$ $x = 2$ is an inflection ~~pont~~ point.

i.e., no maxima & minima

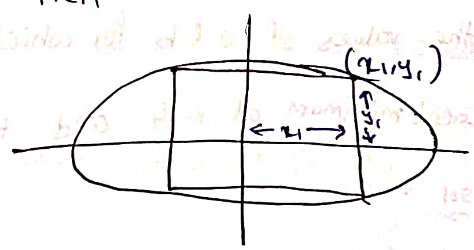
$\therefore$ Ans : 1

(Q61) The maximum area of a rectangle whose vertices lie on the ellipse $x^2 + 4y^2 = 1$ is

Sol :

Let (x_1, y_1) be point on the ellipse. Now we have 4 smaller rectangles as shown in the figure.

If a is the area of rectangle then



$\therefore a = 4x_1y_1$

(x_1, y_1) ~~lies~~ lies on ellipse

$\Rightarrow x_1^2 + 4y_1^2 = 1$

$\Rightarrow 4y_1^2 = 1 - x_1^2$

$a^2 = 16x_1^2y_1^2$

$= 4x_1^2(4y_1^2)$

$= 4x_1^2(1 - x_1^2)$

$a^2 = 4x^2(1 - x^2)$

let $f(x) = a^2 = 4x^2 - 4x^4$

$f'(x) = 8x - 16x^3$

$f'(x) = 0 \Rightarrow 8x - 16x^3 = 0 \Rightarrow x(8 - 16x^2) = 0 \Rightarrow x = 0 \mid 8 - 16x^2 = 0$

$\Rightarrow x=0$ or $x=\pm\frac{1}{\sqrt{2}}$

$f''(x) = 8-48x$

$f''(0) = 8-0 = 8 > 0$

$\therefore$ $f(x)$ has minimum at $x=0$

$f''(1/\sqrt{2}) = 8-48\left(\frac{1}{\sqrt{2}}\right) < 0$

$\therefore f(x)$ has Maximum at $x=1/\sqrt{2}$

$f''(-1/\sqrt{2}) = 8-48(-1/\sqrt{2}) > 0$ $\therefore f(x)$ has Minimum at $x=-1/\sqrt{2}$

Now $f(x) = 4x^2(1-x^2)$

$= 4\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)$

$= 4\,\frac{1}{2}\cdot\frac{1}{2}$

$= 1$

Maximum value of a^2 is 1

$\Rightarrow$ maximum value of a is 1.

(Q52) Find the values of a & b for which the function $f(x) = x^3+ax^2+bx$ has local minimum at $x=4$ and the point of inflection at $x=1$?

Sol

$f(x) = x^3+ax^2+bx$

$f'(x) = 3x^2+2ax+b$

$f''(x) = 6x+2a$

$f'''(x) = 6$

$f'(4) = 0 \Rightarrow 3(4)^2+2a(4)+b=0$

$8a+b+48=0$

$x=1$ is point of inflection

$\Rightarrow f''(x)=0 \Rightarrow 6(1)+2a=0 \Rightarrow \boxed{a=-3}$

$\Rightarrow 8(-3)+b+48=0 \Rightarrow$ $b=-24$

Maxima & Minima (in two variables):

* A function $f(x,y)$ is said to have a maximum value at the point (a,b) if $f(a,b) > f(a+h, b+k)$ for small values of h and k, positive (or) negative

* A function $f(x,y)$ is said to have a minimum value at the point (a,b) if $f(a,b) < f(a+h, b+k)$ for small values of h and k positive (or) negative.

* A maximum (or) minimum value of a function is called an extreme value.

Rule to find Extreme values of a function $f(x,y)$:

1) find $p = \frac{\partial f}{\partial x}$ and $q = \frac{\partial f}{\partial y}$

2) Solve $p=0$ and $q=0$ simultaneously for x & y

   let (x_1,y_1) (x_2,y_2) --- be the solutions of these equations

   these (x_1,y_1) (x_2,y_2) .. are called stationary points.

3) For each solution of step (2) find

$$r = \frac{\partial^2 f}{\partial x^2}, \quad s = \frac{\partial^2 f}{\partial x \partial y} \quad \text{and} \quad t = \frac{\partial^2 f}{\partial y^2}$$

4) If $rt - s^2 > 0$ and $r < 0$ at (x_1,y_1) then $f(x,y)$ has maximum value at (x_1,y_1) and $f_{max} = f(x_1,y_1)$

5) If $rt - s^2 > 0$ and $r > 0$ at (x_1,y_1) then $f(x,y)$ has minimum value at (x_1,y_1) and $f_{min} = f(x_1,y_1)$

6) If ~~$rt-s^2$~~ $rt-s^2<0$ at (x_1,y_1) then $f(x,y)$ has neither a maximum nor a minimum at (x_1,y_1). Such (x_1,y_1) is called a saddle point.

7) If $rt-s^2=0$ at (x_1,y_1) the case is doubtful and further investigation is needed.

Q63) The maximum value of the determinant among all 2x2 real symmetric matrices with trace 10 is ______

Sol:

Let x & y be eigen values

~~$tr(A)=\lambda_1+\lambda_2$~~

$tr(A)=\lambda_1+\lambda_2$ (Sum of eigen values)

$x+y=10$ — (1)

$det(A)=\lambda_1\lambda_2$

$det(A)=xy$

$det(A)=x(10-x)$ ($\because$ from (1))

$det(A)=10x-x^2$

Let $f(x)=10x-x^2$

$f'(x)=10-2x$

$f''(x)=-2<0$

~~for~~ $\therefore f(x)$ has maximum value

~~$f'(x)$~~ $f'(x)=0 \Rightarrow 10-2x=0$

$\Rightarrow x=5$ is the critical point

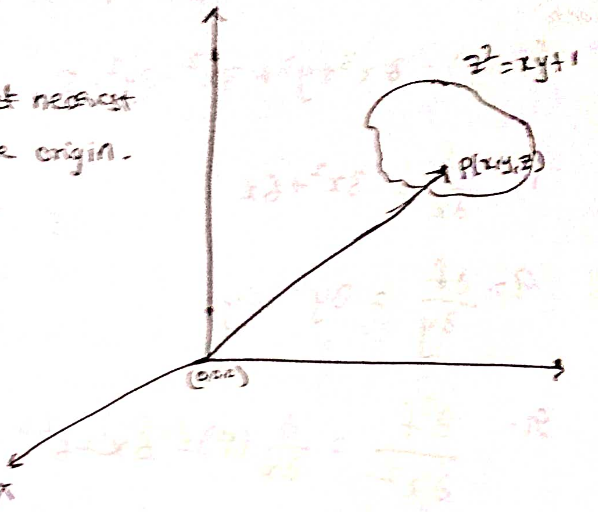
$f''(5)<0 \Rightarrow f(x)$ has max value at $x=5$

$f_{max}=f(5)=10(5)-5^2=25$

(864) Find the distance b/w origin and a point ~~nearset~~ nearest to it on the surface $z^2 = xy+1$?

Sol:

Let $P(x,y,z)$ be the ~~nearest~~ nearest point on the surface from the origin.



$d = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$

$= \sqrt{x^2+y^2+z^2}$

$= \sqrt{x^2+y^2+xy+1}$

Let $f(x,y) = x^2+y^2+xy+1$

$p = \frac{\partial f}{\partial x} = 2x+y,$

$q = \frac{\partial f}{\partial y} = 2y+x$

Solve $p=0$ & $q=0$

$2x+y=0$

$x+2y=0$

$2x+y=0$

$2x+4y=0$

$-3y=0 \Rightarrow y=0$

$\Rightarrow x=0$

Stationary point is $(0,0)$

In question it is already mentioned that it has min value and we have only one stationary point.

$\therefore f_{min} = f(0,0) = \sqrt{0+0+0+1} = 1$

$d = \sqrt{f_{min}} = \sqrt{1} = 1$ ($\because$ distance is positive)

(Q65) Find the maxima & minima of following function

$$f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 5$$

Sol :

Given

$$f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 5$$

$$p = \frac{\partial f}{\partial x} = 3x^2 + 6x$$

$$q = \frac{\partial f}{\partial y} = 3y^2 - 6y.$$

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(p) = 6x + 6$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(q) = 6y - 6$$

finding stationary points :

$$p = 0 \Rightarrow 3x^2 + 6x = 0 \Rightarrow 3x(x+2) = 0 \Rightarrow x = 0, -2$$

$$q = 0 \Rightarrow 3y^2 - 6y = 0 \Rightarrow 3y(y-2) = 0 \Rightarrow y = 0, 2$$

Now we have 4 stationary points

i.e., $(0,0)$ $(0,2)$ $(-2,0)$ $(-2,2)$

i) At $(0,0)$

$$rt - s^2 = (6)(-6) - (0)^2 = -36$$

$rt - s^2 < 0 \Rightarrow f(0,0)$ has neither maxima nor minima

$\therefore$ $(0,0)$ is a saddle point.

ii) At $(0,2)$

$$rt - s^2 = (6)(6) - (0)^2 = 36$$

$rt - s^2 > 0$ and $r > 0$

$\therefore f(x,y)$ has minimum at $(0,2)$

i.e., $f_{min} = f(0,2) = (0)^3 + (2)^3 + 3(0)^2 - 3(2)^2 - 5$

$= 8 - 12 - 5 = -9$

(iii) At $(-2,0)$

$rt - s^2 = (-6)(-6) - 0^2 = 36$

$rt - s^2 > 0$ and $r = -6 < 0$

$\therefore f(x,y)$ has maximum at $(-2,0)$

i.e., $f_{max} = f(-2,0) = (-2)^3 + (0)^3 + 3(-2)^2 - 3(0)^2 - 5$

$f_{max} = -1$

(iv) At $(-2,2)$

$rt - s^2 = (-6)(6) - 0^2 = -36$

$rt - s^2 < 0$

$\Rightarrow f(x,y)$ has neither maxima nor minima at $(-2,2)$

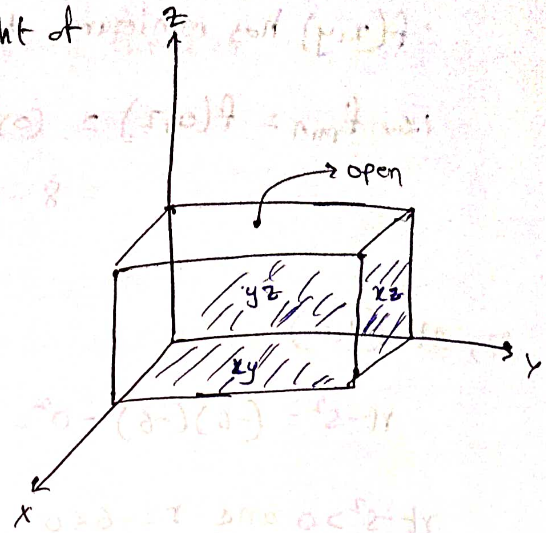
$(-2,2)$ is a saddle point.

$\therefore f_{max} = -1$ $\quad f_{min} = -9$

Q66) A rectangular box open at the top is to have a volume of 32 cubic feet. Find the ~~dim~~ dimensions of the box requiring least material for its construction.

Sol :

let x, y, z be length, breadth and height of the rectangular box.



volume $v = xyz$

$xyz = 32$

$\Rightarrow z = \frac{32}{xy}$

we need to construct a rectangular box with minimum surface area.

$S = xy + 2xz + 2yz$

$= xy + 2x\left(\frac{32}{xy}\right) + 2y\left(\frac{32}{xy}\right)$

$= xy + \frac{64}{y} + \frac{64}{x}$

let $f(x,y) = xy + 64\left(\frac{1}{x} + \frac{1}{y}\right)$

$p = \frac{\partial f}{\partial x} = y - \frac{64}{x^2}$

$q = \frac{\partial f}{\partial y} = x - \frac{64}{y^2}$

$p = 0 \Rightarrow y - \frac{64}{x^2} = 0$ $\qquad$ $q = 0 \Rightarrow x - \frac{64}{y^2}$

$\Rightarrow x^2y - 64 = 0$ ① $\qquad$ $xy^2 - 64 = 0$ ②

from ① & ②

$x^2y - xy^2 = 0$

$xy(x-y) = 0$

$x = 0 \mid y = 0 \mid x = y$

These two are not possible cuz length and breadth can't be zero.

from ①

$$x^2y - 64 = 0$$

$$x^3 - 64 = 0 \Rightarrow x = 4$$

$$\Rightarrow y = 4$$

and $z = \frac{32}{xy} = \frac{32}{16} = 2$

$\therefore$ The dimensions of the box are

$$x = y = 4 \ \& \ z = 2$$

(Q67) If $f(x) = a\ln|x| + bx^2 + x$ has extreme values at $x=1$ & $x=3$ then find a & b?

Sol :

$$f(x) = a\ln|x| + bx^2 + x$$

$$f'(x) = a\frac{1}{\cancel{|x|}}\frac{\cancel{|x|}}{x} + 2bx + 1$$

$$= \frac{a}{x} + 2bx + 1$$

$f'(1) = 0$ $\qquad$ $f'(3) = 0$

$a + 2b + 1 = 0$ $\qquad$ $\frac{a}{3} + 6b + 1 = 0$

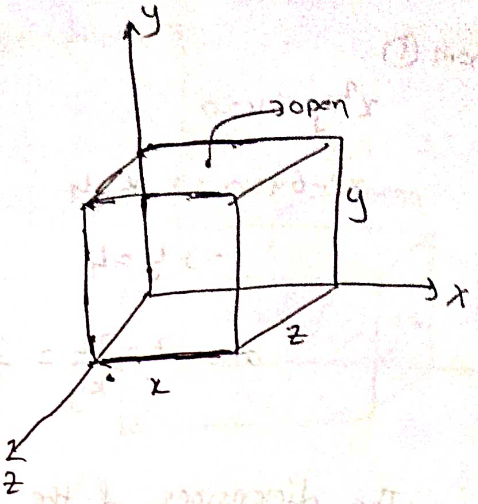
$$a + 18b = -3$$

$$a + 2b = -1$$
$$a + 18b = -3$$
$$\underline{- \quad - \quad +}$$
$$-16b = 2 \Rightarrow \boxed{b = -1/8} \quad \boxed{a = -3/4}$$

(Q68) A rectangular box without top cover having a square base is to be made from a sheet of 108 square meters. Then the largest possible volume of the box in cubic meters is ______.

Sol :

square base $\Rightarrow x = z$



Surface area $= xz + 2yz + 2xy = 108$

$\Rightarrow x^2 + 2yx + 2xy = 108$

$\Rightarrow x^2 + 4xy = 108$ — ①

volume, $V = xyz$

$= x^2 y$

$= x^2\left(\frac{108 - x^2}{4x}\right)$

from ①

$y = \frac{108 - x^2}{4x}$

$V = \frac{1}{4}(108x - x^3)$

$V' = \frac{1}{4}(108 - 3x^2)$, $\Rightarrow V'' = -6x < 0$ ($\because x > 0$)

$\therefore$ has maximum

$V' = 0 \Rightarrow \frac{1}{4}(108 - 3x^2) = 0$

$\Rightarrow 3x^2 = 108 \Rightarrow x = \pm 6$ (Stationary point)

$\therefore$ But x is length (positive)

$\therefore x = 6 \Rightarrow z = 6$

from ① $(6)^2 + 4(6)y = 108$

$\Rightarrow y = 3$

$\therefore$ ~~volu~~ maximum volume

$= xyz$

$= (6)(3)(6)$

$= 108$ cubic meters.

Calculus

Material Problems

(P1) $\lim_{x\to 1} \left(\frac{1+x}{2+x}\right)^{\left(\frac{1-\sqrt{x}}{1-x}\right)} = \left(\frac{2}{3}\right)^{(0/0)}$

So we apply limit only to power

$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$ $\left(\frac{0}{0}\right)$

$\lim_{x\to 1} \frac{\left(\frac{-1}{2\sqrt{x}}\right)}{-1} = 1/2$

$\Rightarrow \lim_{x\to 1} \left(\frac{1+x}{2+x}\right)^{1/2} = \sqrt{\frac{2}{3}} = 0.8165$

(P2) $\lim_{R\to 0} i = \lim_{R\to 0} \frac{E(1-e^{-Rt/L})}{R}$ (0/0 form)

$= E \lim_{R\to 0} \frac{-e^{-Rt/L}\left(\frac{-t}{L}\right)}{(1)}$

$= E \; -e^{0}\left(\frac{-t}{L}\right) = \frac{Et}{L}$

$\therefore$ opt (a)

(P3) $\lim_{x\to a} \frac{\sqrt{2a^3x-x^4} - a(a^2x)^{1/3}}{a-(ax^3)^{1/4}}$ $\left(\frac{0}{0}\text{ form}\right)$

$\lim_{x\to a} \frac{\frac{1}{2\sqrt{2a^3x-x^4}}(2a^3-4x^3) - a\frac{1}{3}(a^2x)^{-2/3}(a^2)}{0-\frac{1}{4}(ax^3)^{-3/4}(a)3x^2}$

$$= \frac{\frac{1}{2\sqrt{2a^4-a^4}}(2a^3-4a^3) - \frac{a}{3}(a^2)^{-2/3}(a^2)}{-\frac{1}{4}(a^4)^{-3/4}(a)\,3a^2}$$

$$= \frac{\frac{1}{2a^2}(-2a^3) - \frac{a}{3}}{-\frac{3}{4}} = \frac{a+\frac{a}{3}}{\frac{3}{4}} = \left(\frac{4a}{3}\right)\left(\frac{4}{3}\right) = \frac{16a}{9}$$

P/4

a) $f(x) = |x|$

$f'(x) = \frac{|x|}{x}$

$f'(0^-) = \frac{-x}{x} = -1$ $\quad f'(0^+) = \frac{x}{x} = 1$

$f'(0^-) \neq f'(0^+)$

$\therefore$ not diffrentiable at $x=0$

b) $f(x) = \cot x$

~~f(0) is undefi~~

$f'(x) = -\operatorname{cosec}^2 x$

$f'(0)$ is undefined

$\therefore$ not diff at $x=0$

c) $f(x) = \sec x$

$f'(x) = \sec x \tan x$

$\therefore$ differentiable

d) $f'(x) = -\operatorname{cosec} x \cot(x)$

$f'(0)$ is undefined

$\therefore$ not diff at $x=0$

P/5 $f(x) = \begin{cases} x & , x \le 1 \\ 2x-1 & , x > 1 \end{cases}$

$\lim_{x \to 1^+} f(x) = 2x-1 = 2-1 = 1$

$\lim_{x \to 1^-} f(x) = 1$

$\therefore$ continuous

$f'(x) = \begin{cases} 1, & x \le 1 \\ 2, & x > 1 \end{cases}$

$f'(1^-) = 1$

$f'(1^+) = 2$

$\therefore$ not diff at $x = 1$

$\therefore$ cont but not diff

opt (a)

P/6 $f'(x) = \begin{cases} 2x & , \text{ if } x \le 2 \\ m & , \text{ if } x > 2 \end{cases}$

$f'(2^+) = f(2^-)$

$m = 2(2) \Rightarrow m = 4$

diff $\Rightarrow$ cont

$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

$(2)^2 = m(2) + b$

$4 = 8 + b \Rightarrow b = -4$

$\therefore m = 4, b = -4$

P/7 $f(x) = x + \frac{4}{x}$ $[1, 8]$

$f'(x) = 1 - \frac{4}{x^2}$

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$$1-\frac{4}{c^2}=\frac{\left(8+\frac{4}{8}\right)-\left(1+\frac{4}{1}\right)}{8-1}$$

$$\frac{c^2-4}{c^2}=\frac{\cancel{7}/2}{\cancel{7}}$$

$2c^2-8=c^2 \Rightarrow c^2=8 \Rightarrow c=\pm 2\sqrt{2}$

$c \in (1,8)$

$\therefore c=2\sqrt{2}$

P/8

$f(x)=3x^2+4x-5$

$f'(x)=6x+4$

$f(1)=3+4-5=2$

$f(3)=3(3)^2+4(3)-5=27+12-5=34$

$f(1)\neq f(3)$

$\therefore$ we cannot use rolle's theorem

So we go for lagrange's theorem

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$=\frac{34-2}{3-1}=\frac{32}{2}=16$$

$\therefore$ for atleast one point

$f'(x)=16$

P/9

$f(x)=e^x \Rightarrow f'(x)=e^x$

$g(x)=e^{-x} \Rightarrow g'(x)=-e^{-x}$ $(2,3)$

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)} \Rightarrow \frac{e^c}{-e^{-c}}=\frac{e^3-e^2}{e^{-3}-e^{-2}}=\frac{e^3-e^2}{\frac{1}{e^3}-\frac{1}{e^2}}$$

$$\Rightarrow -e^{2c}=\frac{e^3-e^2}{e^2-e^3}\cdot e^5 \Rightarrow e^{2c}=e^5$$

$\Rightarrow c=2.5$

P/10

$f(x) = \frac{1}{x} \Rightarrow f'(x) = \frac{-1}{x^2}$ $[2,3]$

$g(x) = \frac{1}{x^2} \Rightarrow g'(x) = \frac{-1}{x^4} 2x = \frac{-2}{x^3}$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\frac{-1}{c^2}}{\frac{-2}{c^3}} = \frac{\frac{1}{3}-\frac{1}{2}}{\frac{1}{9}-\frac{1}{4}}$$

$$\Rightarrow \frac{c}{2} = \frac{\frac{2-3}{6}}{\frac{4-9}{36}} \Rightarrow \frac{c}{2} = \frac{-1}{\frac{-5}{6}} = \frac{6}{5}$$

$$c = 2.4$$

Note:

→ $\sinh x = \frac{e^x - e^{-x}}{2}$

→ $\cosh x = \frac{e^x + e^{-x}}{2}$

→ $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

→ $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

→ $\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$

→ $\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$

P/11

$f(x) = \cosh x + \cos x$

$f'(x) = \sinh x - \sin x$

$= \frac{e^x - e^{-x}}{2} - \sin x \qquad \Rightarrow f'(0) = \frac{1-1}{2} - 0 = 0$

$\therefore$ 0 is a critical point

$f''(x) = \cosh x - \cos x$

$f''(x) = \frac{e^x + e^{-x}}{2} - \cos x$

$\Rightarrow f''(0) = \frac{1+1}{2} - 1 = 0$

Now we need to find $f'''(0)$ also

$f'''(x) = \sinh x + \sin x$

$= \frac{e^x - e^{-x}}{2} + \sin x$

$\Rightarrow f'''(0) = \frac{1-1}{1} + 0 = 0$

$f''(0) = 0$ & $f'''(0) = 0$

$\therefore$ '0' is not an inflection point

~~Now we~~ Now we need to repeat the process taking $f'''(x)$ as $f'(x)$

$f'(x) = \sinh x + \sin x$

$\Rightarrow f'(x) = 0$

$f''(x) = \cosh x + \cos x$

$= \frac{e^x + e^{-x}}{2} + \cos x$

$\Rightarrow f''(0) = \frac{1+1}{2} + 1 = 2 > 0$

$\therefore f(x)$ has minimum at $x = 0$

(P/12) $f(x) = x^4 - 2x^3 + x^2 + 3$

$f'(x) = 4x^3 - 6x^2 + 2x$

$f'(x) = 0$

$\Rightarrow x(4x^2 - 6x + 2) = 0$

$\Rightarrow x = 0 \quad | \quad 2x^2 - 3x + 1 = 0$

$2x^2 - 2x - x + 1 = 0$

$2x(x-1) - (x-1) = 0$

$(2x-1)(x-1) = 0$

$x = 1/2 \quad | \quad x = 1$

$f''(x) = 12x^2 - 12x + 2$

$f''(0) = 2 > 0 \quad \therefore$ min at $x = 0$

$f''(1/2) = 12\frac{1}{4} - 12\frac{1}{2} + 2 = 3 - 6 + 2 = -1 < 0$

$\therefore$ max $x = 1/2$

$f''(1) = 12 - 12 + 2 = 2 > 0$

$\therefore$ min at $x = 1$

req area

$$= \int_0^1 (x^4 - 2x^3 + x^2 + 3)\,dx$$

$$= \left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3} + 3x\right]_0^1$$

$$= \frac{1}{5} - \frac{2}{4} + \frac{1}{3} + 3$$

$$= \frac{12 - 30 + 20 + 180}{60} = \frac{182}{60} = \frac{91}{30}$$

P/13

$$\int_0^{\pi/4} \frac{\sin 2x}{\cos^4 x + \sin^4 x}\, dx$$

$$(\sin^2 x + \cos^2 x)^2 = \sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x$$

$$1 = \sin^4 x + \cos^4 x + \frac{\sin^2 2x}{2}$$

$$\Rightarrow \sin^4 x + \cos^4 x = 1 - \frac{\sin^2 2x}{2}$$

$$\int_0^{\pi/4} \frac{\sin 2x}{1 - \frac{\sin^2 2x}{2}}\, dx$$

$$2\int_0^{\pi/4} \frac{\sin 2\left(\frac{\pi}{4} - x\right) dx}{2 - \sin^2 2\left(\frac{\pi}{4} - x\right)}$$

$$2\int_0^{\pi/4} \frac{\cos 2x}{2 - \cos^2 2x}\, dx$$

$$2\int_0^{\pi/4} \frac{\cos 2x}{1 + \sin^2 2x}\, dx$$

Put

$\sin 2x = t$

$2\cos 2x\, dx = dt$

$$= \int_0^1 \frac{1}{1+t^2}\, dt$$

$$\because \int \frac{1}{a^2 + x^2} = \frac{1}{a}\tan^{-1}(x/a)$$

$$= \left[\tan^{-1}(t)\right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0 = \pi/4$$

$$= 0.785$$

P/14 The equation can be defined only for +ve values of x.

Also it is clear that curve is symmetric about x-axis

$$y^2 = x(x-1)^2$$

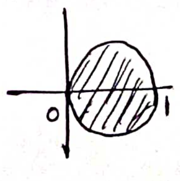
$$\cancel{y = \sqrt{x}(x-1)}$$

$$y = \pm\sqrt{x}(x-1)$$

$$\pm\sqrt{x}(x-1) = 0$$

$$\Rightarrow x = 0 \text{ (or) } x = 1$$

$\therefore$ Curve is



$$\therefore \text{Req area} = \int_0^1 \sqrt{x}(x-1)$$

$$= \left(\frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}\right)_0^1$$

$$= \frac{2}{5} - \frac{2}{3} = \frac{-4}{15}$$

since area is positive $\Rightarrow \frac{4}{15}$

However we have 2 such area

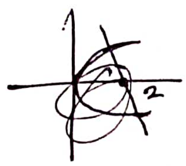
$\therefore$ total area $= 2\left(\frac{4}{15}\right) = 0.53$

P/15

$\cancel{y^2 = 4x}$

$\cancel{y = 2x - 4} = 0$

$\Rightarrow x = 2$



P/17

$$x \sin(\pi x) = \int_0^{x^2} f(t)\,dt$$

$$\frac{d}{dx}(x \sin \pi x) = \frac{d}{dx}\left(\int_0^{x^2} f(t)\,dt\right)$$

$$\pi x \cos \pi(x) = f(x^2)\frac{d}{dx}(x^2) - 0$$

$$\pi x \cos \pi x = 2x f(x^2)$$

Put $x = 2$

$\cancel{4\pi(\cos 4\pi)}$

$$\cancel{2}\pi \cos(2\pi) = \cancel{2}(2) f(4)$$

$$f(4) = \pi/2$$

P/20

$$\int_{-\infty}^{0} e^{x+e^x}\,dx = \int_{-\infty}^{0} e^x \cdot e^{e^x}\,dx$$

$t = e^x \Rightarrow dt = e^x dx$

$$= \int_0^1 e^t\,dt = (e^t)_0^1 = e^1 - e^0 = e - 1$$