02/06/20

Set Theory

Set: collection of things/objects.

Eg: $A = \{1,2,3\}$ —— roaster notation

1 present in A

$1 \in A$

$4 \notin A$

Note:

* When writing 'belongs to' the left side must be object or element and the right side must be a set.
* Same object cannot appear twice in a set. ($\{1,2,2\} = \{1,2\}$)
* Order of elements in set is not important.

It is not considered wrong even if same objects repeat. But the repeatetion doesn't matter

Denoting a set:

$A = \{1, 4, 9, 16, 25, 36, 49, 64, 81\}$: Roaster form

$A = \{x^2 \mid x^2 < 100 \ \& \ x > 0\}$: Set builder form

Cardinality:

Cardinality of a set is no of elements in the set.

Eg: $A = \{1,2,3\}$

Cardinality of set 'A' $= |A| = 3$

Equal sets: (Axiom of extension)

Two set are said to be equal if and only if they have same elements. i.e., $A = B \Leftrightarrow \forall x \, (x \in A \leftrightarrow x \in B)$

i.e., $A \subseteq B$ and $B \subseteq A$

Emptyset / Nullset:

A set is said to empty set if cardinality is zero. It is denoted by - ϕ.

Eg: $\{\ \}$

Subset ($\subseteq$):

'A' is called subset of 'B' iff every element of 'A' is also an element of 'B'.

It is denoted as $A \subseteq B$

$$\forall x \, (x \in A \rightarrow x \in B)$$

Note:
A set with one element is known as a singleton set

* Also 'B' is called superset of A.

Proper subset ($\subset$)

'A' is called proper subset of 'B' iff every element of 'A' is an element of 'B' and A is not equal to B.

$$A \subset B \iff A \subseteq B \text{ and } |A| \neq |B|$$

$$\boxed{\forall x \, (x \in A \rightarrow x \in B) \wedge \exists x \, (x \in B \wedge x \notin A)}$$

Note:

* while writing $\subset$ (or) $\subseteq$ left & Right side must be sets.

* Note:

$A \subseteq B$ and $B \subseteq C \rightarrow A \subseteq C$

$A \subset B$ and $B \subset C \rightarrow A \subset C$

$A \subseteq B$ and $B \subset C \rightarrow A \subset C$

$A \subset B$ and $B \subseteq C \rightarrow A \subset C$

Note:

For every set 'S'

$\phi \subseteq S$

$S \subseteq S$

* $\phi \subset S$ is not true for every set because for $S = \phi$, the relation doesn't hold.

~~Powse~~

Powerset: ($P(A)$ | 2^A)

Powerset is a set of ~~a~~ all ~~set~~ subsets of a given ~~to~~ set.

In powerset each element is a set.

Eg: $A = \{1,2\}$

$$P(A) = 2^A = \{\phi, \{1\}, \{2\}, \{1,2\}\}$$

Cardinality of a power set is

$$|P(A)| = 2^n$$

where $|A| = n$

Eg: Let $A = \{\phi\}$

$$P(A) = \{\phi, \{\phi\}\}$$

Eg: $\phi = \{\}$

$$P(\phi) = \{\phi\}$$

Q1
G-15

For a set A, the power set of A is denoted by 2^A. If $A = \{5, \{6\}, \{7\}\}$ which of the following options are TRUE?

I. $\phi \in 2^A$
II. $\phi \subseteq 2^A$
III. $\{5, \{6\}\} \in 2^A$
IV. $\{5, \{6\}\} \subseteq 2^A$

a) I & III b) II & III c) I, II, III d) I, III, IV

I. $\phi \in 2^A$.

Here ϕ is a set

~~$\therefore \phi \in 2^A$ is false~~ $\therefore$ True

II. power set contains ϕ as an element

$\phi \in 2^A$

But $\phi \subseteq$ to every set (Empty set is subset to all the sets)

$\therefore$ II true

III. $\{5,\{6\}\}$ is subset of A

$\therefore$ It is element of power set

$\{5,\{6\}\} \in 2^A$

IV $\{5,\{6\}\}$ is an element of 2^A

it is not subset

However

$\{\{5,\{6\}\}\} \subseteq 2^A$ is true.

$\therefore$ I, II & III

Q2 Let $A = \{1,\{1\},\{2\}\}$. which of the following are true?

(i) $1 \in A$ — True

(ii) $\{1\} \in A$ — True

(iii) $\{1\} \subseteq A$ — True

(iv) $\{\{1\}\} \subseteq A$ — True

(v) $\{2\} \in A$ — True

(vi) $\{2\} \subseteq A$ — False

(vii) $\{\{2\}\} \subseteq A$ — True

(viii) $\{\{2\}\} \subset A$ — True

Q3 Which of the following statements are true?

a) $\phi \in \phi$ ~~or~~

Here both are sets

$\therefore$ False

b) $\phi \subset \phi$ ($|\phi| = |\phi| \Rightarrow$ False).

ϕ is not a proper subset to ϕ $\because \phi = \phi$

$\therefore$ False

c) $\phi \subseteq \phi$

Every set is subset to itself

$\therefore$ True

d) $\phi \in \{\phi\}$

This is written as

$\{\} \in \{\{\}\}$

$\therefore$ True

e) $\phi \subset \{\phi\}$

ϕ is proper subset to every set but ϕ

$\therefore$ True

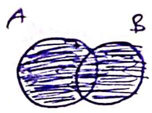
f) $\phi \subseteq \{\phi\}$

ϕ is subset to every set

$\therefore$ True.

Set Operations:

→ $A \cup B = \{x \mid x \in A \vee x \in B\}$



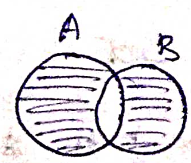
→ $A \cap B = \{x \mid x \in A \wedge x \in B\}$

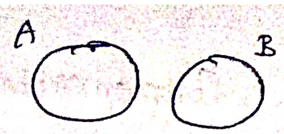
→ $A \Delta B = \{x \mid x \in A \odot x \in B\} = \{x \mid x \in A \cup B \wedge x \notin A \cap B\}$

↓ Symmetric difference ↳ XOR

It ~~can~~ is also written as $A \oplus B$

$$\boxed{\begin{aligned} A \oplus B &= (A-B) \cup (B-A) \\ &= (A \cup B) - (A \cap B) \end{aligned}}$$





Disjoint set:

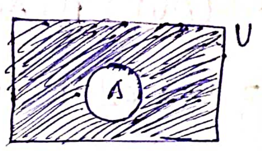
Two sets A & B are said to be disjoint iff $A \cap B = \phi$

set difference:

$A - B = \{x \mid x \in A \wedge x \notin B\}$

* $\boxed{A - B = A - (A \cap B) = A \cap \bar{B}}$

Complement $(\bar{A})$ (A^c):



Complement of set 'A'

$\bar{A} = \{x \mid x \in U \wedge x \notin A\}$

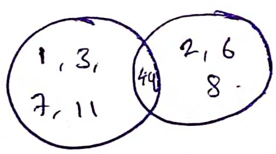
$\bar{A} = U - A = U \cap \bar{A}$

Eg: Let $A - B = \{1, 3, 7, 11\}$

$B - A = \{2, 6, 8\}$

$A \cap B : \{49\}$

Find A & B.



$\therefore A = \{1, 3, 7, 11, 49\}$

$B = \{2, 6, 8, 49\}$.

Note.

→ $A \cap U = A$
$A \cup \phi = A$ } Identity laws

→ $A \cup U = U$
$A \cap \phi = \phi$ } Domination law

→ $A \cup A = A$
$A \cap A = A$ } Idempotent law.

→ $\overline{(\bar{A})} = A$ Complementation law

→ $A \cup B = B \cup A$
$A \cap B = B \cap C$
$A \Delta B = B \Delta A$
} Commutative law

→ $A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$
} Associative law

→ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
} Distributive law

→ $\overline{A \cap B} = \bar{A} \cup \bar{B}$
$\overline{A \cup B} = \bar{A} \cap \bar{B}$
} Demorgan's law

→ $A \cup (A \cap B) = A$
$A \cap (A \cup B) = A$
} Absorption law

→ $A \cup \bar{A} = U$
$A \cap \bar{A} = \phi$
} Complement law

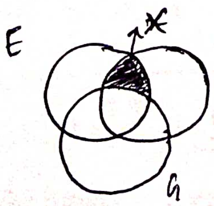
Q4 G-06 E, F & G are finite sets

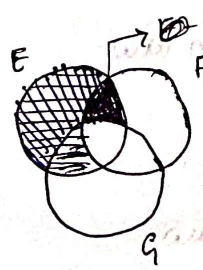
$X = (E \cap F) - (F \cap G)$

$Y = (E - (E \cap G)) - (E - F)$

Find the correct relation

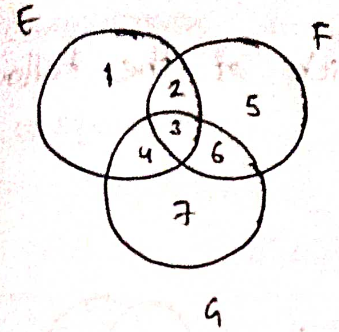
a) $X \subset Y$ b) $X = Y$ c) $X \supset Y$ d) $X - Y \neq \phi$ & $Y - X \neq \phi$





$\therefore X = Y$

Method : 2



$X = (E \cap F) - (F \cap G)$

$= (2,3) - (3,6)$

$= 2$

$Y = (E - (E \cap G)) - (E - F)$

$= ((1,2,3,4) - (3,4)) - ((1,2,3,4) - (2,3,5,6))$

$= (1,2) - (1,4)$

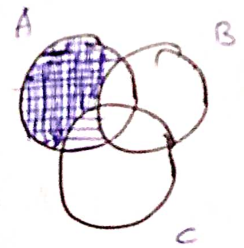
$= 2$

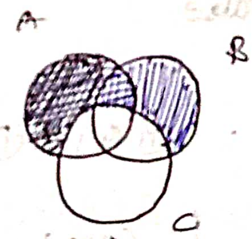
$\therefore X = Y$

Q5 G-05 Let A, B & C be non-empty sets and let $X = (A-B) - C$ and $Y = (A-C) - (B-C)$. which of the following is true?

a) $X = Y$ b) $X \subset Y$ c) $Y \subset X$ d) none

Sol :





$\therefore X = Y$

Finite & Infinite sets :

→ A set with finite no of elements is known as finite set.

→ A set with infinite no of elements is known as infinite set.

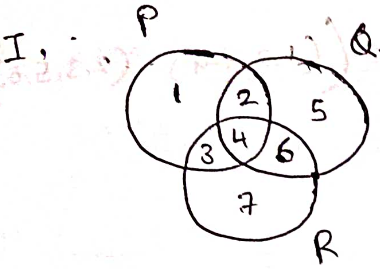
Q6 / G-06

Let P, Q and R be sets. Let Δ denote the symmetric difference operator defined as. $P \Delta Q = (P \cup Q) - (P \cap Q)$.
Using Venn diagram, determine which of the following is/are true?

I) $P \Delta (Q \cap R) = (P \Delta Q) \cap (P \Delta R)$

II) $P \cap (Q \Delta R) = \cancel{P \Delta R}\ (P \cap Q) \Delta (P \cap R)$

I.



$P \Delta (Q \cap R)$

$= (P - (Q \cap R)) - ((Q \cap R) - P)$

$= [(1,2,3,4) - (4,6)] - [(4,6) - (1,2,3,4)]$

$= (1,2,3) - (6)$

$= (1,2,3)$

$(P \Delta Q) \cap (P \Delta R)$

$(1,3,5,6) \cap (1,2,6,7) = (1)$

$P \Delta Q = (1,3,5,6)$

$P \Delta R = (1,2,6,7)$

$\therefore$ False

II. $\cancel{P\Delta Q}$ $P \cap (Q \Delta R) = (P \cap Q) \Delta (P \cap R)$

$P \cap (Q \Delta R)$

$P \cap [(Q-R) \cup (R-Q)]$

$P \cap (2,3,5,7)$

$(1,2,3,4) \cap (2,3,5,7)$

$(2,3)$

$(P \cap Q) \Delta (P \cap R)$

$[(P \cap Q) - (P \cap R)] \cup [(P \cap R) - (P \cap Q)]$

$[(2,4) - (3,4)] \cup [(3,4) - (2,4)]$

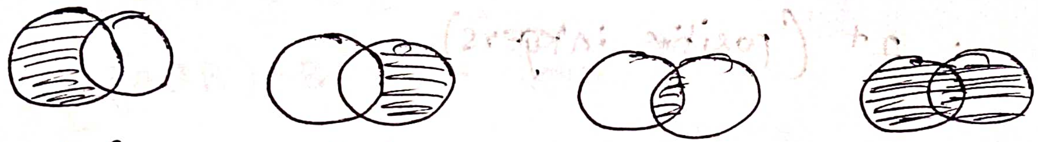
$(2) \cup (3)$

$(2,3)$

$\therefore$ True

17 (1-96) let A and B be sets and let A^c and B^c denote the complements of the set A and B. The $(A-B) \cup (B-A) \cup (A \cap B)$ is equal to

a) $A \cup B$ b) $A^c \cup B^c$ c) $A \cap B$ d) $A^c \cap B^c$



A−B B−A $A \cap B$ $(A-B) \cup (B-A) \cup (A \cap B)$

$= A \cup B$

Q.8 Let $A_i = [-i, i]$ $\forall i \geq 1$ and let domain be the set of integers.

Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.

$A_1 = \{-1, 0, 1\}$

$A_2 = \{-2, -1, 0, 1, 2\}$

⋮

$\bigcup_{i=1}^{\infty} A_i = \{-1, 0, 1\} \cup \{-2, -1, 0, 1, 2\} \cup \dots \cup \{-\infty \dots -2, -1, 0, 1, 2 \dots \infty\}$

$= \{-\infty \dots -2, -1, 0, 1, 2 \dots \infty\}$

$= Z$ (Integers)

$\bigcap_{i=1}^{\infty} A_i = \{-1, 0, 1\} \cap \{-2, -1, 0, 1, 2\} \cap \dots \cap \{-\infty \dots -2, -1, 0, 1, 2 \dots \infty\}$

$= \{-1, 0, 1\}$

Q9 Let $A_i = [i, \infty)$ $\forall i \geq 1$ and let domain be set of integers

Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$

$A_1 = \{1, 2 \ldots \infty\}$

$A_2 = \{2, 3, \ldots \infty\}$

$\bigcup_{i=1}^{\infty} A_i = \{1, 2 \ldots \infty\} \cup \{2, 3 \ldots \infty\} \cup \ldots$

$= \{1, 2, \ldots \infty\}$

$= Z^+$ (positive integers)

$\bigcap_{i=1}^{\infty} A_i = \{1, 2 \ldots \infty\} \cap \{2, 3 \ldots \infty\} \cap \{3, 4 \ldots \infty\} \cap \ldots$

$= \phi$

* If $A \subseteq B$ then. $\bar{A} \cup B = U$

Proof:

~~if x~~ we need to s.t for every $x \in U$

also $x \in \bar{A} \cup B$

Consider $x \in A \Rightarrow x \in B$

$\Rightarrow x \in (\bar{A} \cup B)$

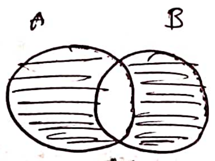
if $x \notin A \Rightarrow x \in \bar{A}$

$\Rightarrow x \in (\bar{A} \cup B)$

$\therefore$ any $x \in \bar{A} \cup B \Rightarrow \bar{A} \cup B = U$

Q10 ~~$A_i = \{$~~

P/5 $X = [1, 2, 3 \ldots 2n]$



For $A \Delta B = \{2, 4, 6 \ldots 2n\}$

the shaded region should contain all these elements

So we have n elements and each element as two ways. $\Rightarrow$ no of ways $= 2^n$

Now the each element $\{1, 3, 5 \ldots 2n-1\}$

Can either fall into intersection part or not

$\Rightarrow 2^n$ ways

total no of ways $= 2^n \times 2^n = 2^{2n}$

P/06 a) $A \oplus B = (A \cup B) - (A \cap B)$

$= (A \cup \emptyset) - (A \cap \emptyset)$

$= A - \emptyset = A$

b) $[\cancel{(A \oplus B) - B}]$

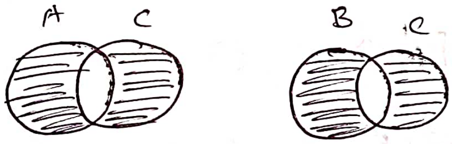
$(A \oplus B) \oplus B$

$[(A \oplus B) - B] \cup [B - (A \oplus B)]$

$A \oplus B$ .. $[A \oplus B] - B$ $B - (A \oplus B)$

$\underbrace{\qquad\qquad}_{\text{union} = A}$

c)



$A \oplus C$ $B \oplus C$

~~From above figure~~ ~~$A \oplus C = B \oplus C$ iff left shaded part is~~ same in both ~~the figures~~

$\therefore$ True $\boxed{\text{explained in P/07 c}}$

$\Rightarrow \cancel{A - C = B - C}$

$\Rightarrow \cancel{A = B}$

d) ~~True~~ False

P/07 a) $A = \{1, 2, 3\}$ $B = \{1, 4, 5\}$ $C = \{1\}$

$\cancel{A = B}$ $A \cap C = \{1\}$ $B \cap C = \{1\}$

But $A \neq B$

$\therefore$ False

b)

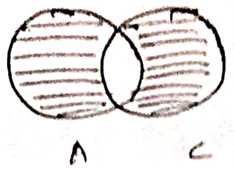
$A = \{1, 2\}$ $B = \{3, 4\}$ $C = \{1, 2, 3, 4\}$

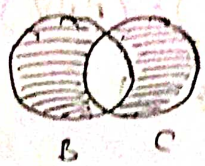
$A \cup C = \{1, 2, 3, 4\}$ $B \cup C = \{1, 2, 3, 4\}$

But $A \neq B$

$\therefore$ False

c) $(A \Delta C) = (B \Delta C)$ then $A = B$





$A = B$

Lets say A has one element which is not present in B.

Then this one element will surely change $A-C$ (or) $C-A$

$\therefore$ A & B must be equal

$\therefore$ True

if the added element is in C also, then $A \Delta C$'s cardinality will reduce, if the added element is not in C, then $A \Delta C$'s cardinarity will raise by 1.

~~d) $A = \{1,2\}$ $B = \{1,5\}$ $C = \{2,3\}$~~

~~$A - C = \{1\}$ $B - C = \{5\}$~~

d) $A = \{1,2\}$ $B = \{2,3\}$ $C = \{1,3\}$

$A - C = \{2\}$ $B - C = \{2\}$

But $A \neq B$

$\therefore$ False

P/01 For every existing subset of n subset

we cand eithe add $\{\}$ (or) $\{a\}$ or $\{b\}$ (or) $\{a,b\}$

$\Rightarrow$ $4 \times n$ new subsets

P/02 S1: $A \in B$ and $B \subseteq C$ then $A \subseteq C$

A ↓ element of B

A ↓ element, C ↓ set

$\therefore$ False

$A \in C$ is Correct

S2: $A \subseteq B$ and $B \in C$ then $A \subseteq C$

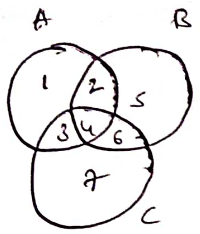
$A \downarrow$ set, $B \downarrow$ set, $B \downarrow$ set, $C \downarrow$

So C must be a set in which set B is an element

So we can't say if $A \subseteq C$ or not

$\therefore$ False

(P/3)



S1: $A \cup (B-C) = (A \cup B) - (C-A)$

$(1,2,3,4) \cup (2,5)$ $\quad$ $(1,2,3,4,5,6) - (6,7)$

$(1,2,3,4,5)$ $\quad$ $(1,2,3,4,5)$

$\therefore$ S1: True

S2: $A \cap (B-C) = (A \cap B) - (A \cap C)$

$(1,2,3,4) \cap (2,5)$ $\quad$ $(2,4) - (3,4)$

(2) $\quad$ (2)

$\therefore$ S2: True

(P/4) If $A = \phi$

B can be choosen in 2^n ways

If $|A| = 1$

B can be choosen in 2^{n-1} ways

we have nC_1 such A's

$\Rightarrow nC_1 \cdot 2^{n-1}$

If $|A| = 2$

B can be choosen in 2^{n-2} ways

we have nC_2 such A's

$\Rightarrow nC_2 \cdot 2^{n-2}$

If $|A|=n$

B can be choosen in 1 way

i.e., $B = S$

$\therefore$ Total no of ways

$$= 2^n + {}^nC_1 \cdot 2^{n-1} + {}^nC_2\, 2^{n-2} + \dots + {}^nC_k\, 2^{n-1} + \dots + {}^nC_n\, 2^{n-n}$$

$$= \sum_{k=0}^{n} {}^nC_k\, 2^{n-k} = (1+2)^n = 3^n$$

03/06/20

Function : (or) Mapping (or) Transformation

→ A function is a rule that assign each input to exactly one output.

(or)

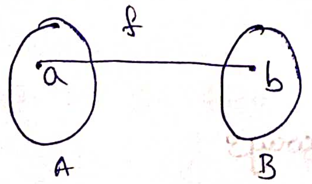
→ A function 'F' form A to B is an assignment of exactly one element of B to each element of A. It is denoted as $F: A \to B$

Here,

Set A is called Domain

set B is called Co-domain

Let $a \in A$ and $b \in B$



$f(a) = b$

b is called image

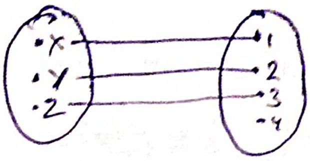
a is called pre-image

Range: Set of all images. It is also called image of the function.

Note:

→ It is not necessary that Range = Co-domain but Range $\subseteq$ Co-domain

Ex: Consider below function



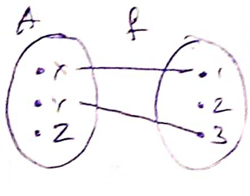
Here domain = $\{x, y, z\}$

Co-domain = $\{1, 2, 3, 4\}$

Range = $\{1, 2, 3\}$

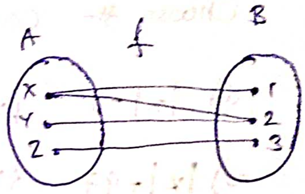
> **Note**:
> If f_1 & f_2 are two functions, then
> $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
> $(f_1 f_2)(x) = f_1(x) f_2(x)$
> $\frac{1}{f}(x) = 1/f(x)$

→ For every elemement in domain, there must exist exactly one image. Otherwise, it won't considered a function.



Here z has no image.

∴ It is not a function.



Here x has two image

∴ It is not a function

~~#Knowlem~~

→ Consider $f : N \to N$, N is set of natural numbers

$f(x) = x/2$

Now for $x = 3$

$f(3) = 1.5$

but $1.5 \notin N$

So there is no image for 3 in N.

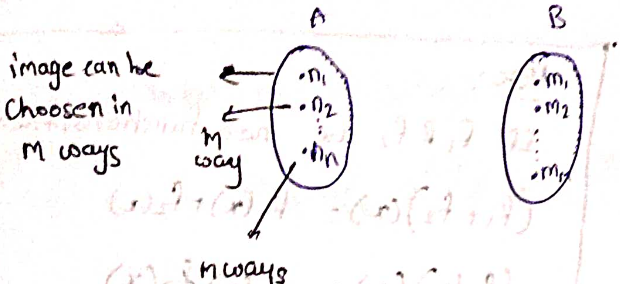
∴ It is not a function.

> Let $f : A \to B$ be a function.
> Let $S \subseteq A$
> $f(S)$ is set of images of S w.r.to f
> $f(S) = \{t \mid \forall s \in S, f(s) = t\}$
> Note that $f(S)$ is a set.

Note:

→ Let $f: A \to B$

$\lvert A\rvert = n$ $\quad$ $\lvert B\rvert = m$	
no of function possible $= m^n$	$= (R \cdot S)^{L \cdot S}$



$$\therefore \underbrace{m \times m \times \cdots \times m}_{n \text{ time}} = m^n$$

Q10 (G-96) Let $f: X \to Y$ be a function.

Let total no of function possible $= 97$

Choose the correct option

a) $\lvert X\rvert = 1$ $\quad$ $\lvert Y\rvert = 97$

b) $\lvert X\rvert = 97$ $\quad$ $\lvert Y\rvert = 1$

c) $\lvert X\rvert = 97$ $\quad$ $\lvert Y\rvert = 97$

d) none

Sol:

no of function $= (\lvert B\rvert)^{\lvert A\rvert} = 97$

$\therefore \lvert B\rvert = 97 \quad \lvert A\rvert = 1$

$\therefore$ opt (a)

Types of functions:

(i) <u>1:1 Function ('Injective)</u>

$f: A \to B$ is Injective iff $\forall a \in A$ and $\forall b \in B$

$f(a) = f(b) \Leftrightarrow a = b$

(or)

$f: A \to B$ is Injective iff $\forall a \in A$ and $\forall b \in B$

$a \neq b \Leftrightarrow f(a) \neq f(b)$

→ In a one-one function, no two elements of the domain should have same image.

Eg: $f: N \to N$
$f(x) = x+1$ is a ~~onea~~-one function

Consider two numbers $x_1 \in N$ and $x_2 \in N$ and $x_1 \neq x_2$

To P.T f is not one-one, we need to P.T

$\cancel{f(x_1) \neq f}$ $f(x_1) = f(x_2)$

$x_1 + 1 = x_2 + 1$

$x_1 = x_2$

which is a contradiction

$\therefore f$ is one-one

Rosen Questions:

1: If A, B & C are sets. Show that

a) $(A \cup B) \subseteq (A \cup B \cup C)$

b) $(A \cap B \cap C) \subseteq (A \cap B)$ (Hint: Use Venn diagrams)

c) $(A-B)-C \subseteq A-C$

d) $(A-C) \cap (C-B) = \phi$

e) $(B-A) \cup (C-A) = (B \cup C) - A$

→ What can you conclude from below statements

a) $A \cup B = A$

b) $A \cap B = A$

c) $A - B = A$

d) $A - B = B - A$

Ans: a) $B \subseteq A$ b) $A \subseteq B$ c) $A \cap B = \phi$ d) $A = B$

04/06/20

Find if given function is one-one or not

Q11 $f: Z \to Z$

$f(x) = x^2 - 5x + 5$

Let $x_1, x_2 \in Z$

$f(x_1) = f(x_2)$

$x_1^2 - 5x_1 + 5 = x_2^2 - 5x_2 + 5$

$x_1^2 - 5x_1 = x_2^2 - 5x_2$

$x_1(x_1 - 5) = x_2(x_2 - 5)$

Put $x_1 = 0$ and $x_2 = 5$

$0(-5) = 5(0)$

$0 = 0$

$\therefore f(x) = x^2 - 5x + 5$ is not one-one

(or)

let $x_1, x_2 \in Z$

$f(x_1) = f(x_2)$

$x_1^2 - 5x_1 + 5 = x_2^2 - 5x_2 + 5$

$x_1^2 - 5x_1 = x_2^2 - 5x_2$

$x_1^2 - x_2^2 = 5x_1 - 5x_2$

$\cancel{(x_1 - x_2)}(x_1 + x_2) = 5\cancel{(x_1 - x_2)}$

$x_1 + x_2 = 5$

we can have infinitely different pairs of x_1, x_2 such that $f(x_1) = f(x_2)$

$\therefore$ not 1-1

Note :

* For a function $F: A \to B$ to be one-one

$|B| \geq |A|$

* let $|B| = n$ and $|A| = m$ and $m \leq n$, then no of 1-1 functions possible from A to B are $\boxed{{}^nP_m = R.S^{P_{L.S}}}$

(ii) Onto (surjective) function :

* A function $F: A \to B$ is surjection, if range = codomain
  i.e., ~~els~~ every element of B has a preimage
  (or)

$$\forall y \in B \; \exists x \in A \; (f(x) = y)$$

* For a function $F: A \to B$

$|A| \geq |B|$ for the function to be onto.

Eg: $f: Z \to Z$ & $f(x) = x+1$ is onto

Eg: $f: Z \to Z$ & $f(x) = x^2$ is not onto

because for element '2' in domain, '2' doesn't have preimage.

* Calculating no of onto functions possible

let $F: A \to B$ be a function & $|A| \geq |B|$

Let $|A| = m$ $|B| = n$

no of onto functions = total no of functions − no of non-onto functions.

* for function ~~two~~ to be non-onto atleast one element in B must not have ~~co-do~~ preimage.

* So no of non-onto functions can be
  = no of non-onto function with 1 element not having ~~co-domain~~ pre-image
  +
  no of non-onto functions with 2 elements not having ~~co-domain~~ pre-image
  +
  ⋮
  +
  no of non-onto functions with n elements not having pre-image

$= \cancel{n^m - [{}^nC_1 (n-1)^m}$

$$\boxed{= {}^nC_0\, n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - {}^nC_3 (n-3)^m + \cdots + (-1)^n\, {}^nC_n (n-n)^m}$$

$$= \boxed{\sum_{i=0}^{n-1} (-1)^i\, {}^nC_i\, (n-i)^m}$$

(inclusion-exlusion principle)

(iii) One to One correspondence (or) Bijection:

1:1 Correspondence = 1:1 + Onto

$F: A \to B$ $\quad |A| = m \quad |B| = n$

for 1:1 $\quad |A| \le |B|$

onto $\quad |A| \ge |B|$

1:1 + onto $\Rightarrow |A| = |B|$

* If $F: A \to B$ is one-one and $|A| = |B|$ then F is bijection.
* If $F: A \to B$ is onto and $|A| = |B|$ then F is ~~an~~ bijection.

* No of 1:1 Correspondences $= n!$

Eg: $f: Z \to Z$

$f(x) = x+1$

f is clearly one-one

also for every element co-domain, it has pre-image

So f is onto.

$\therefore f$ is a bijection

(iv) Identity function:

The function $I_A: A \to A$ is called an identity function, if

$I_A(x) = x, \forall x \in A$

Identity function is a bijection.

Inverse function

For a function $f: A \to B$

inverse function is

$f^{-1}: B \to A$ such that

$f^{-1}(b) = a$ whenever $f(a) = b$

Note:

$f: A \to B$ be a function, $S \subseteq B$

Inverse image of S, is defined as subset of A whose elements are precisely all preimages of S.

$f^{-1}(S) = \{a \in A \mid f(a) \in S\}$

$f^{-1}(y)$, $y \in B$ is defined iff f is invertible

$f^{-1}(S)$, $S \subseteq B$ can be defined for any f

* Inverse of a function exists only if the function is bijection.

Such functions are known as invertible function.

bijection $\Leftrightarrow$ invertible function.

If $f: A \to A$ is a function and also if A is finite then

f is 1-1 $\Leftrightarrow$ f is onto

However this not necessarily the case when A is infinite

Eg: $f: R \to R$

$f(x) = x^2$

Here

$f(-2) = f(2) = 4$

$\therefore$ not one-one

$\therefore$ It is not invertible

Eg: $f: R \to R$

$f(x) = 2x - 3$

Here

$f(x_1) = f(x_2)$

$2x_1 - 3 = 2x_2 - 3$

$\Rightarrow x_1 = x_2$

$\therefore$ 1-1

Note:

A function $f: A \to B$ is said to be a constant function if $f(x) = c \ \forall x$

let $f(x) = y$

$y = 2x - 3$

$\Rightarrow x = \dfrac{y+3}{2}$

Thus for every 'x' we can find preimage

$\therefore$ onto and Hence f is invertible

To find inverse

let $f(x) = y \Rightarrow f^{-1}(y) = x$

$2x - 3 = y$

$2x = y + 3$

$x = \frac{y+3}{2} = f^{-1}(y)$

$f^{-1}(y) = \frac{y+3}{2}$

$\Rightarrow f^{-1}(x) = \frac{x+3}{2}$

Eg: find inverse of

$f: R - \{1/2\} \rightarrow R$

$f(x) = \frac{4x}{2x-1}$

let $f(x) = y \Rightarrow f^{-1}(y) = x$

$\frac{4x}{2x-1} = y$

$4x = y(2x-1)$

$4x = 2xy - y$

$4x - 2xy = -y$

$x(4 - 2y) = -y$

$x = \frac{y}{2y-4}$

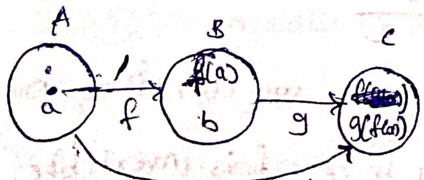
$\Rightarrow f^{-1}(y) = \frac{y}{2y-4}$

$\Rightarrow f^{-1}(x) = \frac{x}{2x-4}$

Composition of a functions:

Consider to functions

$f: A \rightarrow B$ and $g: B \rightarrow C$ $\quad$ $gof: A \rightarrow C$ is called a composition function



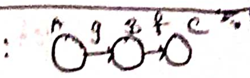
$g \circ f()$ or $g \cdot f()$

* $gof(x) \equiv g(f(x))$

> In gof,
> domain of gof = domain of f
> range of gof = image of range of f w.r.to g

Eg: $f(x) = 2x+3 \quad g(x) = 3x+2$

fog	gof
$fog(x) = f(g(x))$	$gof(x) = g(f(x))$
$= f(3x+2)$	$= g(2x+3)$
$= 2(3x+2)+3$	$= 3(2x+3)+2$
$= 6x+7$	$= 6x+11$

> Note: 
> The composition gof cannot be defined unless range of f is subset of domain of g
> $gof: B \to B$

$\boxed{\therefore gof(x) \neq fog(x)}$

$\boxed{\text{Composition is Associative i.e., } (fog)oh = fo(goh)}$

Note:

* let $f: A \to B$ be a invertible then
  $f^{-1}: B \to A$
  $f^{-1}of: A \to A$ is an identity function I_A
  $fof^{-1}: B \to B$ is an identity funtion I_B

* Theorem 1:

If $f: A \to B$ & $g: B \to C$ are two 1:1 functions then

~~gof is also~~ $gof: A \to C$ is also 1:1

Theorem: 2:

If $f: A \to B$ & $g: B \to C$ are onto, then

$gof: A \to C$ is also onto

Theorem: 3

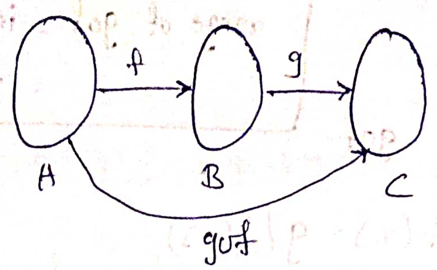
If $f: A \to B$ & $g: B \to C$ are bijections

$gof: A \to C$ is also a bijection

Called a function

Note:

1) If $g\circ f$ is onto then g is onto.



$g\circ f$ is onto $\Rightarrow$ $\forall c \in C$ $\exists a \in A$ such that $g\circ f(a) = c$

$g(f(a)) = c$

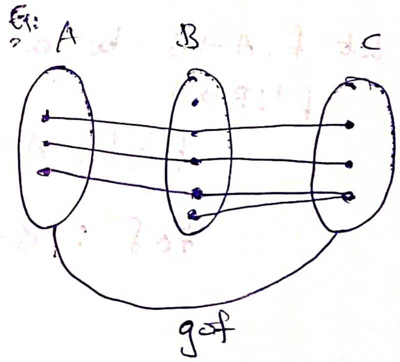
$g(b) = c$ for some $b \in B$

i.e., $\forall c \in C$ $\exists b \in B \mid g(b) = c$

2) If $g\circ f$ is onto, then f need not to be onto

$g\circ f$ is onto $\Rightarrow$ $\forall c \in C$ $\exists a \in A \mid g\circ f(a) = c$.

Eg:



In the figure we can clearly see that $g\circ f$ is onto & f is not onto

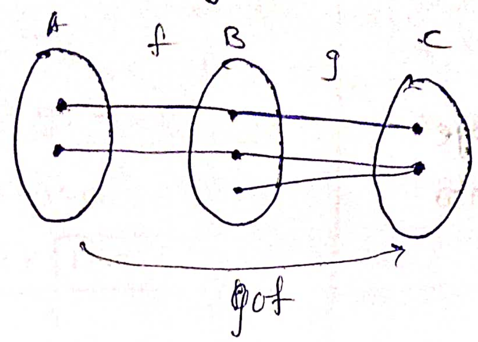
3) If $g\circ f$ is 1:1, then f is 1:1

$g\circ f$ is 1:1 $\Rightarrow$ $\forall a_1 \in A$ & $\forall a_2 \in A$

$g\circ f(a_1) = g\circ f(a_2) \Rightarrow a_1 = a_2$

If we map two element of 'A' to a same element in 'B' then we can never obtain $g\circ f$ such that it is one-one

v) If gof is 1:1 then g need not be 1:1



Note:

If ~~fog~~ gof is a bijection, then f is onto $\Leftrightarrow$ g is 1-1

Above figure is an example to say that gof is 1:1 and g need not to be 1:1

* if onto

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left] & B \arrow[r, "onto"] & C \end{tikzcd}$$

if 1:1

$$\begin{tikzcd} A \arrow[r, "1:1"] \arrow[rr, bend left] & B \arrow[r, leftrightarrow] & C \end{tikzcd}$$

(Q12) Determine if the below functions are one-one &/(or) onto for the two cases $f: \mathbb{Z} \to \mathbb{Z}$ and $f: \mathbb{R} \to \mathbb{R}$

a) $f(x) = x + 7$

$f: \mathbb{Z} \to \mathbb{Z}$

$x_1 + 7 = x_2 + 7$

$x_1 = x_2$

$\therefore$ 1-1

$\forall y \in \mathbb{Z}$ we can determine $x \in \mathbb{Z}$ such that $y = x + 7$

$\therefore$ onto

say $f: \mathbb{R} \to \mathbb{R}$

is also 1-1 & onto

b) $f(x) = 2x - 3$

$f: \mathbb{Z} \to \mathbb{Z}$

1-1

Let $f(x) = y$

$2x - 3 = y \Rightarrow x = \frac{y+3}{2}$

Here for $y = 2$

$x = \frac{2+3}{2} \notin Z$

'2' has no preimage
$\therefore$ not onto

$f: R \to R$

It is clearly 1-1

$f(x) = y$

$x = \frac{y+3}{2}$

for any $y \in R$ we can find x.

$\therefore$ onto

c) $f(x) = -x+5$

$f: Z \to Z$ is 1-1 & onto

$f: R \to R$ is 1-1 & onto

d) $f(x) = x^2$

$f: Z \to Z$

$f(-2) = f(2) = 4$

$\therefore$ not 1-1

$x = 2$ has no preimage in domain

$\therefore$ not onto

$f: R \to R$

$f(-2) = f(2) = 4$

$\therefore$ not 1-1

Let $f(x) = y$

$\forall y < 0, \sqrt{y} \notin R$

$\therefore$ not onto

e) $f(x) = x^2 + x$

$f: Z \to Z$

$x_1^2 + x_1 = x_2^2 + x_2$

$x_1^2 - x_2^2 = x_2 - x_1$

$(x_1 - x_2)(x_2 + x_1) = x_2 - x_1$

$\Rightarrow x_2 + x_1 = -1$

$\therefore$ we can find infinite such (x_1, x_2)

$\therefore$ not 1-1

Let $y = f(x)$

$x^2 + x = y$

put $y = -1$

$x^2 + x + 1 = 0$

$x = \frac{-1 \pm \sqrt{1-4}}{2}$

$x = \frac{-1 \pm 3i}{2} \notin Z$

$\therefore$ not onto

$f: R \to R$

* not 1-1
* $y = -1$ has its preimage in complex numbers

$\therefore$ not onto

f) $f(x) = x^3$

$f: Z \to Z$

$x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$

$\therefore$ one-one

$y = 2$ has no preimage

$\therefore$ not onto

$f: R \to R$

$x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$

$\therefore$ 1-1

$\forall y \in R$ has $\sqrt[3]{y}$ in R

$\therefore$ onto

Q13 / 4-15

Find no of onto functions

$f: X \to Y$

$|X| = m = 4 \quad |Y| = n = 3$

no of onto functions

$= {}^nC_0\, n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - \ldots$

$= {}^3C_0\, 3^4 - {}^3C_1 (2)^4 + {}^3C_2 (1)^4 - {}^3C_3 (0)^4$

$= 3^4 - 3 \cdot 2^4 + 3 \cdot (1) - 0$

$= 81 - 48 + 3 = 84 - 48$

$= 36$

Q14 / G-12

How many onto functions are there from an n-element ($n \geq 2$) set to a 2-element set?

a) 2^n, b) $2^n - 1$, c) $2^n - 2$ d) $2(2^n - 2)$

Sol :

$$2^n - 2C_1(2-1)^n$$

$$2^n - 2\cdot(1)^n = 2^n - 2$$

Q15 / G-96

Let R denote the set of real numbers. Let $f: R\times R \rightarrow R\times R$ be a bijective function defined by $f(x,y) = (x+y, x-y)$. The inverse function of f is given by

a) $f^{-1}(x,y) = \left(\frac{1}{x+y}, \frac{1}{x-y}\right)$ b) $f^{-1}(x,y) = (x-y, x+y)$

c) $f^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$ d) $f^{-1}(x,y) = (2(x-y), 2(x+y))$

Sol :

Let $f(x,y) = (a,b) \Rightarrow f^{-1}(a,b) = (x,y)$

$$x+y = a$$

$$x-y = b$$

$$\Rightarrow 2x = a+b$$

$$x = \frac{a+b}{2}$$

$$\Rightarrow \frac{a+b}{2} + y = a$$

$$y = \frac{a-b}{2}$$

$$\therefore f^{-1}(a,b) = \left(\frac{a+b}{2}, \frac{a-b}{2}\right)$$

$$f^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$$

Q16

Let $f: A \to B$ be a function ; $g: B \to C$ be a function and $h: A \to C$ be a function such that, $h(a) = g(f(a))$ $\forall a \in A$.
which of following statements is always true for all f and g.

a) g is onto $\Rightarrow h$ is onto

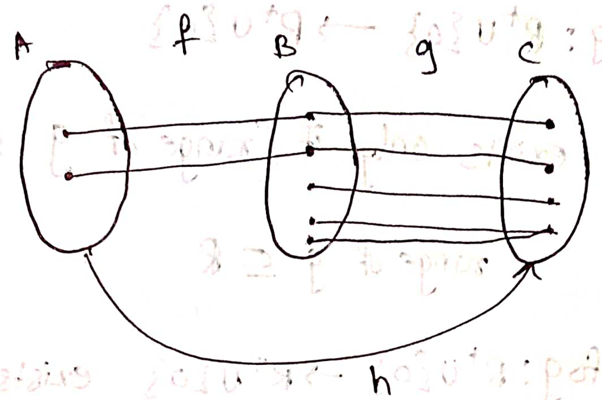
b) h is onto $\Rightarrow f$ is onto

c) h is onto $\Rightarrow g$ is onto

d) h is onto $\Rightarrow f$ and g are onto

Sol:

a)

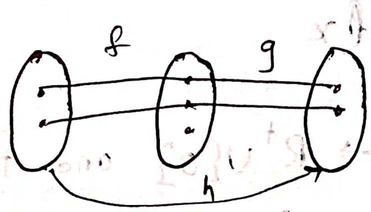


Here g is onto, but h is clearly not

$\therefore$ false

~~Also in the same figure~~

b)



Here h is onto but f is not

$\therefore$ false

Hence we can say (d) is also false

c) true.

Q17) Let $f: R \to R^+ \cup \{0\}$ with $f(x) = x^2$

$g: R^+ \cup \{0\} \to R$ with $g(x) = \sqrt{x}$ ($\sqrt{x}$ is non-negative square root of x).

what is the function $(f \circ g)(x)$?

Sol :

$f \circ g(x) = f(g(x))$

Domain of $f \circ g$ = Domain of $g = R^+ \cup \{0\}$

Co Domain of $f \circ g$ = Co-domain of $f = R^+ \cup \{0\}$

$$f \circ g : R^+ \cup \{0\} \to R^+ \cup \{0\}$$

Composition $f \circ g$ exists only if (range of $g \subseteq$ domain of f)

range of $g \subseteq R$

$\therefore f \circ g : R^+ \cup \{0\} \to R^+ \cup \{0\}$ exists.

Now

$f \circ g(x) = f(g(x))$

$= f(\sqrt{x})$

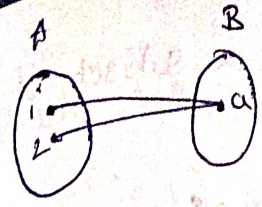
$= (\sqrt{x})^2 = x$

$\therefore f \circ g(x) : R^+ \cup \{0\} \to R^+ \cup \{0\}$ and $f \circ g(x) = x$ is and identity function.

Note :

- If f is function from A to B and S, T are subsets of A, then
  - $\to f(S \cup T) = f(S) \cup f(T)$
  - $\to f(S \cap T) \subseteq f(S) \cap f(T)$

Proof



$A = \{1,2\}$ $B = \{a\}$

Let $S = \{1\}$, $T = \{2\}$

$f(S \cup T) = f(\{1,2\}) = a$

$f(S) \cup f(T) = \{a\} \cup \{a\} = a$

$$\boxed{\therefore f(S \cup T) = f(S) \cup f(T)}$$

$f(S \cap T) = f(\{1\} \cap \{2\}) = f(\phi) = \phi$

$f(S) \cap f(T) = \{a\} \cap \{a\} = \{a\}$

$\therefore$ ~~f(SUT~~ $\boxed{\therefore f(S \cap T) \subseteq f(S) \cap f(T)}$

* If f is 1-1 then :

$$\boxed{f(S \cap T) = f(S) \cap f(T)}$$

Note :

→ If f is a ~~fo~~ function from A to B and $S \subseteq B$, $T \subseteq B$ then

$$\boxed{\begin{aligned} f^{-1}(S \cup T) &= f^{-1}(S) \cup f^{-1}(T) \\ f^{-1}(S \cap T) &= f^{-1}(S) \cap f^{-1}(T) \end{aligned}}$$

Proof :

~~For union it is same as above proof~~

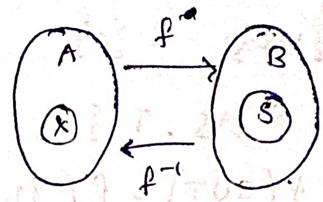
~~For intersection~~

~~f^{-1} exists only if f is bijection~~

~~f^{-1} is 1-1~~

→ Let f be function from A to B. Let S be a subset of B

$$f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$$



Proof:

~~f, f^{-1} are bijection~~

~~Let X be to~~

If X is correspondence to S

$f^{-1}(S) = X$

$f(X) = S$

$$f^{-1}(\overline{S}) = \overline{X} = \overline{(X)} = \overline{f^{-1}(S)}$$

Partial Functions:

A partial function $f: A \to B$ is an assignment to each element a in a subset of A, called the domain definition of f, of a unique element b in B.

We say that f is undefined for elements in A that are not in the domain definition of f. When domain definition of f equals A, we say that f is a total function.

* Partial functions are also denoted as $f: A \to B$ (like total functions)

Eg: $f: \mathbb{Z} \to \mathbb{R}$ where $f(x) = \sqrt{x}$ is a partial function from Z to R

domain definition of f = set of non-negative integers.

Q18) Let $g(x) = \lfloor x \rfloor$. Find

a) $g^{-1}(\{0\}) = [0, 1)$

b) $g^{-1}(\{-1, 0, 1\}) = [-1, 2)$

c) $g^{-1}(\{x \mid 0 < x < 1\}) = \phi$

Note:

$g(x) = \lfloor x \rfloor$ is not a bijection & hence not invertible.

But still we can find inverse image of subset of its range

Q(ii) $f: R \to R$, $f(x) = x^2$. Find

a) $f^{-1}(\{1\})$ b) $f^{-1}(\{x \mid 0<x<1\})$ c) $f^{-1}(\{x \mid x>4\})$

Sol :

$f^{-1}(x) = \sqrt{x}$

a) $f^{-1}(\{1\}) = \{-1, 1\}$

b) $f^{-1}(\{x \mid 0<x<1\}) =$ ~~(-1,0) ∪ (0∪~~ $(-1,0) \cup (0,1)$

(or)

$(-1,1) - \{0\}$

c) $f^{-1}(\{x \mid x>4\}) = (-\infty,-2) \cup (2,\infty)$

(or)

$(-\infty,\infty) - [-2,2]$

Cartesian Products :

* It is one of the operations on sets

→ The cartesian product of two sets A & B is

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

* If $|A| = m$, $|B| = n$

$|A \times B| = mn$

* Cartisian Product is not commutative

$$A \times B \neq B \times A$$

* Carhisian Product is associative

$$\underbrace{A \times B}_{1st} \times C = A \times \underbrace{B \times C}_{2nd}$$

* $(A \times B) \times C \neq A \times B \times C$ ||| slly

Here element are $((a,b),c)$ ||| Here elements are (a,b,c)

$(A \times B) \times (C \times D) \neq A \times (B \times C) \times D$

* Cartisian product of set A with ~~itself~~ itself is denoted as : A^2

$$A^2 = A \times A$$

Slly $A^3 = A \times A \times A$

$A^4 = A \times A \times A \times A$

$\vdots$

Note :

$A \times B = B \times A$
$\Leftrightarrow (A = B \vee A = \phi \vee B = \phi)$

Note :

→ $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof:

$A \times (B \cap C) = \{(a,b) \mid a \in A \wedge (b \in B \cap C)\}$

$= \{(a,b) \mid a \in A \wedge b \in B \wedge b \in C\}$

$= \{(a,b) \mid (a \in A \wedge b \in B) \wedge (a \in A \wedge b \in C)\}$

$= \{(a,b) \mid a \in A \wedge b \in B\} \cap \{(a,b) \mid a \in A \wedge b \in C\}$

$= (A \times B) \cap (A \times C)$

→ $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Proof:

$A \times (B \cup C) = \{(a,b) \mid a \in A \wedge (b \in B \cup C)\}$

$= \{(a,b) \mid a \in A \wedge (b \in B \vee b \in C)\}$

$= \{(a,b) \mid (a \in A \wedge b \in B) \vee (a \in A \wedge b \in C)\}$

$= \{(a,b) \mid a \in A \wedge b \in B\} \cup \{(a,b) \mid a \in A \wedge b \in C\}$

$= (A \times B) \cup (A \times C)$

→ $\left.\begin{array}{l}(A \cap B) \times C = (A \times C) \cap (B \times C) \\ (A \cup B) \times C = (A \times C) \cup (B \times C)\end{array}\right\}$ similar proof

→ $\bar{A} \times \bar{B} \neq \overline{A \times B}$

$\bar{A} \times \bar{B} = (U - A) \times (U - B)$

$\overline{A \times B} = (U \times U) - (A \times B)$

$\bar{A} \times \bar{B} \subseteq \overline{A \times B}$

→ $A \times (B - C) = (A \times B) - (A \times C)$

Note:

* If A, B are two ~~sets~~ finite sets

Jaccard Similarity $J(A,B) = \dfrac{|A \cap B|}{|A \cup B|}$

$J(\phi, \phi) = 1$

Jaccard distance $(d_J(A,B)) = 1 - J(A,B)$