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Relations:

A binary relation from A to B is a subset of $A \times B$. If $A = B$, we say R is a relation on A.

Let $|A| = m$ $|B| = n$

$|A \times B| = mn$

Total no of relation possible = 2^{mn} (no of subsets possible)

*. If $|A| = n$, no of relations possible on A = 2^{n^2}

Reflexive relation:

→ A relation on a set A is said to be reflexive $\Leftrightarrow$ aRa $\forall a \in A$

Eg: Let $A = \{1,2,3\}$

$R_1 = \{(1,1)\ (2,2)\ (3,3)\}$ is a reflexive relation

$R_2 = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (2,1)\}$ is a reflexive relation

$R_3 = \{(1,1)\ (2,2)\}$ is not a reflexive relation.

* size of reflexive relation is

$$n \leq |R| \leq n^2$$

Note:

~~R={}~~

Relation, $R = \{\}$ on set S is reflexive $\Leftrightarrow S = \phi$

Eg: Let $A = \{1,2,3,4\}$

$R = \{(a,b) \mid a+b \leq 4 \text{ and } a,b \in A\}$

R is not reflexive because $(3,3) \notin R$

* Let R_1 & R_2 be two relations on set A, ~~then~~

~~$R_1 \cup R_2$ is~~ If R_1, R_2 are reflexive relations, then

$R_1 \cup R_2$ is a reflexive relation

$R_1 \cap R_2$ is a reflexive relation

* Total no of reflexive relations possible on a set A with $|A| = n$ is: 2^{n^2-n}

we need to include every (a,a) $\forall$ $a \in A$ in Relation

we may or may not include rest of the n^2-n ordered pairs.

$\therefore$ no of reflexive relations possible $= 2^{n^2-n}$

Symmetric Relation:

A relation R on set A is said to be symmetric

if xRy then yRx $\forall x,y \in A$

i.e., $\forall x \forall y \, (xRy \rightarrow yRx)$

(or)

$\forall x \forall y \, ((x,y) \in R \rightarrow (y,x) \in R)$

Eg:

$R_1 = \{(1,2)\}$ is not a symmetric relation

$R_2 = \{(1,2)(2,1)\}$ is a symmetric relation

$R_3 = \{(1,1)\}$ is a symmetric relation

$R_3 = \{\}$ is a symmetric relation

- Eg: Let $R = \{(a,b) \mid (a+b) \leq 4 \;\&\; a,b \in A\}$ $A = \{1,2,3\}$

This is a symmetric relation because addition is commutative.

Eg: Let $R = \{(a,b) \mid a = b+1\}$

$a = b+1 \Rightarrow a-b = 1$

Subtraction is not commutative

Hence not symmetric.

Note:

In a relation R, Domain of relation $= \{x \mid (x,y) \in R\}$

Range of relation $= \{y \mid (x,y) \in R\}$

* If R_1 & R_2 are symmetric relations on A, then

  $R_1 \cup R_2$ is a symmetric relation

  $R_1 \cap R_2$ is a symmetric relation

Finding no of symmetric relation

* If A is a set with n elements

  $\forall a \in A$ (a,a) may or may not belong to symmetric relation

  Out of remaining elements we get $\frac{n^2-n}{2}$ symmetric pairs.

  For each pair we may or may not include it in the relation

$$\therefore \boxed{2^{n+\frac{n^2-n}{2}} \quad : \quad 2^{\frac{n^2+n}{2}}}$$

* No of relations that are both symmetric & Reflexive is

  $\forall a \in A$ (a,a) must be include

  Remaining elements are considered $\frac{n^2-n}{2}$ symmetric pairs

  Out of which each pair may or may not be included

$$\therefore 2^{\frac{n^2-n}{2}}$$

* Cardinality of a ~~refle~~ symmetric relation ranges from 0 to n^2.

Q20 / G-06 Let X, Y, Z be sets of sizes x, y and z respectively. Let $W = X \times Y$ and E be set of all subsets of W. The no of functions from Z to E is

a) $2^{2^{xy}}$ b) 2×2^{xy} c) $2^{2^{x+y}}$ d) 2^{xyz}

Sol:

$|X| = x$ $|Y| = y$ $|Z| = z$ $|W| = xy$ $|E| = 2^{xy}$

no of function from Z to E $= (2^{xy})^z = 2^{xyz}$

Q21 / G-15 Let R be a relation on the set of ordered pairs of positive integers such that $((p,q),(r,s)) \in R^2$ if and only if $p-s = q-r$. which of the following is true about R?

a) Both reflexive & symmetric

b) Reflexive but not symmetric

c) Not reflexive but symmetric

d) Neither reflexive nor ~~sym~~ symmetric

Sol:

Consider $a,b \in \mathbb{Z}^+$

~~$\forall a,b \in R$~~ $\forall a,b \ ((a,b),(a,b)) \notin R$

$\because a-b \neq b-a$

$\therefore$ not reflexive

Let $a,b,c,d \in \mathbb{Z}^+$

Let $((a,b),(c,d)) \in R$

$\Rightarrow a-d = b-c$

Consider $((c,d),(a,b))$ (p, q, r, s)

$c-b = d-a$

$\Rightarrow b-c = a-d$

$\therefore$ Symmetric

$\therefore$ R is not reflexive but symmetric.

P/8 X can be choosen in 2^n ways

for $X = \phi$ we can't define (x,X)

for $|X|=1$ we can define one (x,X)

we have nc_1 such X.

for $|X|=2$ we can define two (x,X)

we have nc_2 such X

∴ No. of ways = $nC_1 \cdot (1) + nC_2 \cdot (2) + nC_3 (3) + \cdots + nC_n (n)$

$$= \sum_{k=1}^{n} k \, nC_k$$

$$= n 2^{n-1} \quad \text{(By option verification)}$$

∴ Both I & II

Material questions on functions:

(P/43) a) $f(x) = x^2 \quad f: R \to R$

$f(-2) = f(2) = 4$

∴ not 1-1 ⇒ not bijection

b) $g(x) = |x|$

$g(-1) = g(1) = 1$

∴ not bijection

c) $h(x) = [x]$

$h(2.5) = h(2.3) = 2$

∴ not bijection

d) ~~$g(x) = x^3$~~ $\phi(x) = x^3$

1-1 & onto

∴ bijection

(P/44) Given $f(A) = g(B) = h(C)$

$f: A \to S \quad g: B \to S \quad h: C \to S$

$|A| = |B| = |C| = k \quad |S| = n, \quad k \leq n$

f, g, h are 1-1

given images of A under f = B under g = C under h

since they are 1-1, $|f(A)| = |g(B)| = |h(C)| = k$

This k from S can be choosen in ^nC_k ways

for function 'f' no of one-one possible = $k!$

" " 'g' " " " " = $k!$

" " 'h' " " " " = $k!$

$\therefore$ no of ways = $^nC_k (k!)^3$

P/46 Here $|A| = |B| = 50$

$f: A \to B$ is one-one

$\Rightarrow$ f must be onto

Hence f is bijection

$\therefore f^{-1}$ exists

P/47 $f: X \to Y$ is bijection

$\Rightarrow |X| = |Y|$

$f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

$f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$ } Its intuitive

P/48 $f(x,y) = (2x-y, x-2y) = (a,b)$ (say)

$f^{-1}(a,b) = (x,y)$

$2x - y = a$

$x - 2y = b$

$2x - 4y = 2b$

$y + 3y = 3y = a - 2b \Rightarrow y = \frac{a-2b}{3}$

$2x - \frac{a-2b}{3} = a \Rightarrow 2x = a + \frac{a-2b}{3} = \frac{4a-2b}{3}$

$x = \frac{2a-b}{3}$

$\Rightarrow f^{-1}(x,y) = \left(\frac{2x-y}{3}, \frac{x-2y}{3}\right)$

(P/49) let string be 00110

Here position of '0' has 3 values

$\therefore$ 'i' is not function

number 1's in a string give a unique value

$\therefore$ function

(P/50) S1: $f(x) = x^3$

1-1

'2' has no preimage

$\therefore$ not onto

$\therefore$ True

S2: $f(n) = \left\lceil \frac{n}{2} \right\rceil$

for

$f(5) = f(6) = 3$

$\therefore$ not 1-1

for every y we can define its preimage as 2y

$\therefore$ onto

$\therefore$ True

(P/51) $f(x_1) = f(x_2) \quad x_1, x_2 \in A$

$$\frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3} \Rightarrow x_1x_2 - 2x_2 - 3x_1 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$$

$$\Rightarrow x_1 - x_2 = 0$$

$$\Rightarrow x_1 = x_2$$

$\therefore$ 1-1

Let $f(x)=y$

$$\frac{x-2}{x-3}=y \Rightarrow x-2=xy-3y$$

$$\Rightarrow x-xy=2-3y$$

$$\Rightarrow x=\frac{3y-2}{y-1}$$

Thus onto

∴ Bijection

P/52

$$f(x)=\frac{x}{x+1} \quad g(x)=\frac{x}{1-x}$$

$$(fog)^{-1}x=(g^{-1}of^{-1})x$$

$$=g^{-1}(f^{-1}(x))$$

$$=g^{-1}\left(\frac{x}{1-x}\right)$$

$$=\frac{\frac{x}{1-x}}{1+\frac{x}{1-x}}$$

$$=\frac{\frac{x}{1-x}}{\frac{1}{1-x}}=x$$

$$f(x)=\frac{x}{x+1}=y$$

$$\Rightarrow x=xy+y$$

$$x=\frac{y}{1-y}$$

$$f^{-1}(x)=\frac{x}{1-x}$$

$$g(x)=\frac{x}{1-x}=y$$

$$\Rightarrow x=y-xy$$

$$x=\frac{y}{1+y}$$

$$\Rightarrow g^{-1}(x)=\frac{x}{1+x}$$

P/54

a) $f: A \to B$ $\quad f^{-1}: B \to A$

$fof^{-1}=I_A$

$f(f^{-1}(b))$ let $f(a)=b$

$f(a)=b$

$\therefore fof^{-1}=I_B$

∴ False

b) $f^{-1}of$ ✗

$f^{-1}(f(a))$

$f^{-1}(b) = a$

$\therefore f^{-1}of = I_A$

$\therefore$ True

c) $I_B of^{-1}$

$I_B(f^{-1}(b))$

$I_B(a)$

not defined

d) foI_A

$f(I_A(a))$

$f(a) = b$

$\therefore foI_A = f$

$\therefore$ True

$\therefore$ a & c are wrong

P/55

$f: A \to B \quad g: B \to A$

$gof: I_A$

f is identity

$\Rightarrow A = B$

$\therefore f: A \to A \quad g: A \to B$

$gof: I_A \Rightarrow g = f^{-1}$

f is identity ~~f⁻¹ exits~~

$\therefore$ f is one-one & f is onto

P/53

a) Every function can be represented graphically

$\therefore$ True

b) $f(x) = x$, $f(2) = 2$

& $g(x) = \sqrt{x^2}$, $g(2) = \sqrt{4} = \pm 2$

$\therefore$ false

c) $f(x) = \log x^2$, $f(-2) = \log 4$

$g(x) = 2\log x$, $g(-2) = 2\log(-2)$ i.e., undefined $\therefore$ false

d) $f(x) = \frac{1}{\sqrt{|x|-x}}$

$\sqrt{|x|-x} \neq 0$ & $|x|-x \geq 0$

$|x|-x \neq 0$ $\quad |x| \geq x$

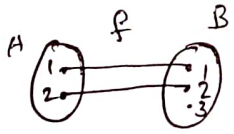
$|x| \neq x$ $\quad x \in (-\infty, 0]$

$\Rightarrow x \in (-\infty, 0)$

$x \in (-\infty, 0)$

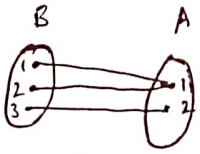
$\therefore$ True

P/45 Let $A = \{1,2\}$ & $B = \{1,2,3\}$



1-1

$\therefore$ S1: True



onto

$\therefore$ S2: True

06/05/20

Antisymmetric Relation:

$\forall a \forall b \, (aRb \wedge bRa \rightarrow a=b)$

(or)

$\forall a \forall b \, ((a,b) \in R \wedge (b,a) \in R \rightarrow a=b)$

Eg: $R_1 = \{(1,2)\}$ is an antisymmetric relation.

$R_2 = \{(1,2),(2,1)\}$ is not an antisymmetric relation

$R_3 = \{(1,2)(1,1)\}$ is an antisymmetric relation

$R_4 = \{\}$ is an antisymmetric relation

Eg: $R_1 = \{(a,b) \mid a \leq b \text{ & } a,b \in Z\}$ is an antisymmetric relation

Eg: $R = \{(a,b) \mid a = b+1\}$ is an antisymmetric relation.

Here a never equals to b

$a \neq b$

$\therefore$ antisymmetric

Eg: $R = \{(a,b) \mid a > b\}$ is antisymmetric

Eg: $R = \{(a,b) \mid a = b \text{ (or) } a = -b\}$ is not antisymmetric

because $5R{-5}$ & $-5R5$

for an integer x

$xR(-x)$ & $(-x)Rx$

$\therefore$ not antisymmetric

Eg: $R_5 = \{(a,b) \mid a+b \leq 3\}$ is not antisymmetric

$2R_1$ & $1R_2$

* If R_1 & R_2 are two antisymmetric relations

$R_1 \cup R_2$ need not to be an antisymmetric relation

$R_1 \cap R_2$ is an antisymmetric relation.

Proof:

→ Let $R_1 = \{(1,2)\}$ $R_2 = \{(2,1)\}$ are antisymmetric relations

$R_1 \cup R_2 = \{(1,2),(2,1)\}$ is not antisymmetric

→ $R_1 \cap R_2 = \{(x,y) \mid (x,y) \in R_1 \wedge (x,y) \in R_2\}$

for $R_1 \cap R_2$ to be antisymmetric $(y,x) \notin R_1 \cap R_2$

@ $(x,y) \in R_1$

because R_1 is antisymmetric $\Rightarrow (y,x) \notin R_1$

Silly $(y,x) \notin R_2$

$\therefore (y,x) \notin R_1 \cap R_2 \Rightarrow R_1 \cap R_2$ is antisymmetric

* Total no of antisymmetric relations possible on set A, with $|A|=n$ is:

~~$\forall a\,(a,a)\in A$~~

$\forall a \in A$ (a,a) may ~~are~~ or may not be included $\Rightarrow 2^n$ ways

$\forall a \forall b \wedge a \neq b$ we may include (a,b) or (b,a) ~~not~~ or not include both $\Rightarrow$ $3^{\frac{n^2-n}{2}}$ ways

$\Rightarrow$ $\boxed{2^n \cdot 3^{\frac{n^2-n}{2}}}$

* No of relation such that it is both symmetric and antisymmetric $= 2^n$

* No of relations such that it is both reflexive & antisymmetric $= 3^{\frac{n^2-n}{2}}$

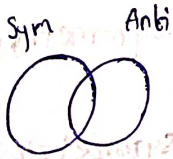
* No of relation such that it is ~~both~~ reflexive, symmetric & antisymmetric is only 1.

i.e. $\{(a_1,a_1),(a_2,a_2)\dots(a_n,a_n)\}$, $a_i \in A$

* Size of an antisymmetric relation ranges ~~from~~ from

0 to $n+\frac{n^2-n}{2}$

* No of relations that are only symmetric but not antisymmetric



~~|Sym − Anti|~~ $|\text{symmetric}| - |\text{sym} \cap \text{Anti}|$

$= 2^n \cdot 2^{\frac{n^2-n}{2}} - 2^n$

$= 2^n\left(2^{\frac{n^2-n}{2}} - 1\right)$

Asymmetric Relation:

$\forall a \forall b \; (aRb \rightarrow b \not R a)$

(or)

$\forall a \forall b \; ((a,b) \in R \rightarrow (b,a) \notin R)$

Eg: $R = \{(1,1)(1,2)\}$ is not assymetric relation

$R = \{(1,3)(3,1)\}$ is not assymetric relation

$R = \{(1,2)(2,3)(1,3)\}$ is assymetric relation

$R = \{\}$ is assymetric relation

$R = \{(a,b) \mid a \leq b\}$ is not assymetric relation

$R = \{(a,b) \mid (a+b) \leq 3\}$ is not assymmetric

$R = \{(a,b) \mid a < b\}$ is assymetric relation.

* If R_1, R_2 are two assymmetric relations

$R_1 \cup R_2$ is ~~also assym~~ not assymmetric relation

$R_1 \cap R_2$ is an assymmetric relation

* No of assymmetric relations possible $= 3^{\frac{n^2-n}{2}}$

* Size of an assymmetric relation ranges from

0 to $\frac{n^2-n}{2}$

* Every assymmetric relation is antisymmetric relation

* No of relations that are both symmetric & assymetric $= 1$. i.e, $\{\}$

* ~~|Reflexive & Assymetric| = 1 (relation that are both assymetric and~~

Irreflexive Relation:

$\forall a \in A \; a \not R a$ (or) $\forall a \in A \; \{(a,a) \notin R\}$

Let $A = \{1,2,3\}$

Eg: $R_1 = \{(1,1),(1,2)\}$ is not reflexive and not irreflexive.

$R_2 = \{(1,2)\}$ is irreflexive and not reflexive

$R = \{(1,1)(2,2)(3,3)\}$ is not irreflexive but reflexive

~~$R = \{(1,2)\}$~~ $R = \{\}$ is irreflexive.

* There is no relation such that it is both reflexive & irreflexive
* But there are relations that are ~~no~~ neither reflexive ~~nor~~ nor irreflexive.

* No of irreflexive relations possible on set A, $|A| = n$ is

$$2^{n^2-n}$$

* If R_1, R_2 are two irreflexive relations, then

$R_1 \cup R_2$ is irreflexive

$R_1 \cap R_2$ is irreflexive

* Size of an irreflexive ~~element~~ relation ranges from

0 to n^2-n.

* The below table shows no of relations such that both are possible.

	Reflexive	Irreflexive	Symmetric	Antisymmetric	Asymmetric
Asymmetric	0	$3^{\frac{n^2-n}{2}}$	1	$3^{\frac{n^2-n}{2}}$	$3^{\frac{n^2-n}{2}}$
Antisymmetric	$3^{\frac{n^2-n}{2}}$	$3^{\frac{n^2-n}{2}}$	2^n	$2^n 3^{\frac{n^2-n}{2}}$	
Symmetric	$2^{\frac{n^2-n}{2}}$	$2^{\frac{n^2-n}{2}}$	$2^n \cdot 2^{\frac{n^2-n}{2}}$		
Irreflexive	0	2^{n^2-n}			
reflexive	2^{n^2-n}				

(Q22) If R is a relation such that

$R = \{(A,B) \mid A \cap B = \phi\}$, where A, B are sets.

is R reflexive & symmetric?

Sol:

for any non-empty set A

$A \cap A = A \neq \phi$

$\therefore$ not reflexive

for any two disjoint sets A, B

$A \cap B = \phi$

$\Rightarrow B \cap A = \phi$.

$\therefore$ Symmetric

(Q23) $R = \{(a,b) \mid gcd(a,b) \neq 1 \wedge a \neq b\}$

Find reflexive, symmetric & antisymmetric nature of R

Sol:

$\forall a, (a,a) \notin R$

because it is given if $(a,b) \in R$ then $a \neq b$

$\therefore$ not reflexive

$\forall a,b$, if $a \neq b$,

and $gcd(a,b) \neq 1$

then $(a,b) \in R$

$\Rightarrow b \neq a \wedge gcd(b,a) \neq 1$

$\Rightarrow (b,a) \in R$

$\therefore$ symmetric

$\forall a,b$ if $gcd(a,b) \neq 1 \wedge a \neq b$ then $(a,b) \in R$

then $(b,a) \in R$

$\therefore$ not antisymmetric.

from R to R

(Q24) Let $R_1, R_2, R_3, R_4, R_5, R_6$ be relations defined as follows

$R_1 = \{(a,b) \mid a > b\}$ $\quad R_4 = \{(a,b) \mid a \leq b\}$

$R_2 = \{(a,b) \mid a \geq b\}$ $\quad R_5 = \{(a,b) \mid a = b\}$

$R_3 = \{(a,b) \mid a < b\}$ $\quad R_6 = \{(a,b) \mid a \neq b\}$.

Talk about below relations.

a) $R_2 \cup R_4$

$R_2 \cup R_4 = \{(a,b) \mid a \geq b\} = R^2$

i.e., Reflexive & symmetric & Transitive

b) $R_3 \cup R_6$

$R_3 \cup R_6 = \{(a,b) \mid a < b \vee a > b\} = R_6$

i.e., irreflexive, symmetric,

c) $R_3 \cap R_6$

$R_3 \cap R_6 = \{(a,b) \mid a < b\} = R_3$

i.e., irreflexive, antisymmetric, assymmetric, transitive

d) $R_4 \cap R_6$

$R_4 \cap R_6 = \{(a,b) \mid a < b\} = R_3$

i.e., irreflexive, antisymmetric, assymmetric, transitive

e) $R_3 - R_6$

$R_3 - R_6 = \{\} = \phi$

i.e., irreflexive, symmetric, antisymmetric, assymmetric, transitive

f) $R_6 - R_3 = \{(a,b) \mid a > b\} = R_1$

i.e., ireflexive, antisymmetric, assymmetric, transive

g) $R_2 \oplus R_6$

$(R_2 \cup R_6) - (R_2 \cap R_6)$

$\{(a,b) \mid \forall a, \forall b\} - \{(a,b) \mid a > b\}$

$\{(a,b) \mid a \leq b\} = R_4$

i.e., Reflexive, Antisymmetric, Trasitive

h) $R_3 \oplus R_5$

$(R_3 \cup R_5) - (R_3 \cap R_5)$

$\{(a,b) \mid a \leq b\} - \{\}$

$\{(a,b) \mid a \leq b\} = R_4$

i.e., Reflexive, Antisymmetric, Transitive.

09/06/20

Transitive Relation:

$\forall a \forall b \forall c \; (aRb \wedge bRc \longrightarrow aRc)$

Let $A = \{1,2,3\}$ and the below relations be defined on A.

Eg: $R = \{\}$ is a transitive relation

$R = \{(1,1)(2,2)\}$ is a transitive relation.

$R = \{(1,2)\}$ is a transitive relation

$R = \{(1,2)(2,1)\}$ is not transitive relation

$R = \{(1,2)(2,1)(1,1)\}$ is a transitive relation

Eg: $R_1 = \{(a,b) \mid a \leq b\}$

Consider $a,b,c \in A$

$a \leq b$, $b \leq c$

$\Rightarrow a \leq c$

$\therefore$ transitive

$R_2 = \{(a,b) \mid a = b+1\}$

aRb & bRc

$a = b+1$ $\quad b = c+1$

$a = c+1+1$

~~$a = c+2$~~

~~$\therefore aRc$~~

$\therefore aRc$

$\therefore$ not transitive

$R_3 = \{(a,b) \mid a+b \leq 3\}$

aRb $\quad bRc$

$a+b \leq 3$ $\quad b+c \leq 3$

Eg: $2+1 \leq 3$ $\quad 1+2 \leq 3$ $\quad (a=2, b=1, c=2)$

$a+c = 2+2 = 4$

$\therefore a \not R c$

$\therefore$ not transitive

Eg:

$R_4 = \{(a,b) \mid \gcd(a,b) \neq 1$ & a,b are distinct$\}$

$a = 10$ $\quad b = 2$ $\quad c = 10$

aRb & bRc

$a \not R c$

$\therefore$ not transitive

$R_5 = \{(A,B) \mid A \cap B = \phi\}$

let $A = X$, $B = \bar{X}$, $C = X$

$A R B$ & $B R C$

but $A \cap C = A = \phi$

$\therefore$ not transitive

Q25 / G-16 A binary relation R on NxN is defined as follows:

$(a,b) R (c,d)$ if $a \leq c$ or $b \leq d$. Consider the following propositions:

P: R is reflexive

Q: R is transitive

which one of the following statements is TRUE?

a) P & Q b) only P c) only Q d) none

Sol:

~~$(a,b) R (c,d) \Rightarrow a \leq c$ & $b \leq d$~~

~~Consider~~

~~$(c,d) R (a,b) \Rightarrow c \leq a$ & $d \leq b$~~

~~clearly not~~

$(a,b) R (a,b) \Rightarrow a \leq a$ or $b \leq b$

which is true

$\therefore$ reflexive

~~Consider~~

~~$(a,b) R (c,d)$ $(c,d) R (e,f)$~~

~~$a \leq c$ & $b \leq d$ $c \leq e$ & $d \leq f$~~

~~We can conclude $a \leq e$ & $b \leq f$~~

Let $(a,b)R(c,d)$ and $a \leq c$, $b \geq d$

Let $(c,d)R(e,f)$ and $c \geq e$, $d \leq f$

$(a,b)R(e,f) \Rightarrow (a \leq e)$ or $(b \leq f)$

we don't know about these

$\therefore$ not transitive

Ex:

$(1,5)\ R\ (3,2)$

$(3,2)\ R\ (-5,3)$ because $2 \leq 3$

but $(1,5)\ \not R\ (-5,3)$

$\therefore$ not transitive

So only P is true

(or)

To P.T it is transitive we need to P.T

$(a,b)R(d) \wedge (c,d)R(x,y) \rightarrow (a,b)R(x,y)$ is tautology

i.e., $\underbrace{\underbrace{[(a \leq c) \vee (b \leq d)]}_{F \quad T} \wedge \underbrace{[(c \leq x) \vee (a \leq y)]}_{T \quad F}}_{T} \rightarrow \underbrace{[(a \leq x) \vee (b \leq y)]}_{F \quad F}$

$\therefore$ not valid $\Rightarrow$ not transitive

Note:

→ If R_1, R_2 are two transitive relations, then

* $R_1 \cup R_2$ need not to be transitive

Eg: $R_1 = \{(1,2)\}$ is transitive

$R_2 = \{(2,3)\}$ is transitive

$R_1 \cup R_2 = \{(1,2)(2,3)\}$ is not transitive

* $R_1 \cap R_2$ is transitive

Proof:

$$R_1 \cap R_2 = \{(a,b) \mid (a,b) \in R_1 \ \& \ (a,b) \in R_2\}$$

Now let $(a,b), (b,c) \in R_1 \cap R_2$

$\Rightarrow (a,b) \& (b,c) \in R_1 \quad \& \quad (a,b) \& (b,c) \in R_2$

$\Rightarrow \quad (a,c) \in R_1 \qquad\qquad (a,c) \in R_2$

i.e., $(a,c) \in R_1 \cap R_2$

$\therefore$ transitive

Diagonal relation:

Diagonal relation on set A is

$\forall a \forall b \ (aRb \leftrightarrow a = b)$

Eg. $A = \{1,2,3\}$

diagonal relation $\Delta_A = \{(1,1)(2,2)(3,3)\}$

→ If R is a reflexive relation

$$\Delta_A \subseteq R$$

Inverse relation: (R^{-1}):

Inverse relation of R from A to B

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$

Eg: $R_1 = \{(a,b),(c,d)\}$

$R^{-1} = \{(b,a),(d,c)\}$

Complementary relation:

If R is a relation from A to B

Complementary relation

$$\bar{R} = \{(a,b) \mid (a,b) \notin R\}$$

i.e., $\bar{R} = A \times B - R$

Ex: Let $R = \{(a,b) \mid a \text{ divides } b\}$

$$R^{-1} = \{(a,b) \mid b \text{ divides } a\}$$

$$\bar{R} = \{(a,b) \mid a \text{ does not divide } b\}$$

Composition of Relation:

Let R be a relation from A to B

and S be a relation from B to C

Composite relation $S \circ R$ is defined as

$$S \circ R = \{(a,c) \mid (a,b) \in R \ \& \ (b,c) \in S\}$$

(Q26) Consider below relations

$R_1: a > b$ $\quad$ $R_4: a \leq b$

$R_2: a \geq b$ $\quad$ $R_5: a = b$

$R_3: a < b$ $\quad$ $R_6: a \neq b$

Find

a) $R_2 \circ R_1$

Let $(a,b) \in R_1$ & $(b,c) \in R_2$

$\therefore a > b$ & $b \geq c$

i.e., $a > b \geq c \Rightarrow a > c = R_1$

b) $R_3 \circ R_5$

$(a,b) \in R_5 \quad (b,c) \in R_3$

$a = b \quad b < c$

i.e., $a < c$

$R_3 \circ R_5 = R_3$

R_5 is diagonal relation

$R_5 \circ R_3 = R_3$

$R_3 \circ R_5 = R_3$

c) $R_5 \circ R_3$

$(a,b) \in R_3 \quad (b,c) \in R_5$

$a < b \quad b = c$

i.e., $a < c$

$\therefore R_5 \circ R_3 = R_3$

> If d is a diagonal relation, and if R is any relation
>
> $R \circ d = R$
>
> $d \circ R = R$

↓

Diagonal relation is nothing but an identity function

d) $R_1 \circ R_4$

$(a,b) \in R_4 \quad (b,c) \in R_1$

$a \leq b \quad b > c$

$a \leq b < c$

In this case sometimes

$a < c$

$\therefore R_1 \circ R_4 = R_3$

$\therefore R_1 \circ R_4 =$ Domain $\times$ Domain

e) $R_3 \circ R_6$

$(a,b) \in R_6 \quad (b,c) \in R_3$

$a \neq b \quad b < c$

↓

Here we have 2 cases

$a > b$ or $a < b$

↓ ↓

$a > b \;\&\; b < c$ $a < b < c$

↓ ↓

any (a,c) $a < c$

$\therefore R_3 \circ R_6 =$ Domain $\times$ Domain

Note:

* R is symmetric $\Rightarrow$ R^{-1} is symmetric
* R is reflexive $\Rightarrow$ R^{-1} is reflexive
* R is reflexive $\Rightarrow$ $\overline{R}$ is irreflexive
* RUR^{-1} is symmetric.
* R_1, R_2 are symmetric then $R_1 \circ R_2$ need not to be symmetric

If R is transitive, R^{-1} is also transitive

Proof:

$R_1 = \{(1,2)(2,1)\}$ $\quad R_2 = \{(2,3), (3,2)\}$

$R_1 \circ R_2 = \{(3,1)\}$ is not symmetric

$R_2 \circ R_1 = \{(1,3)\}$ is not symmetric

Note that if R is symmetric, then R^2 is symmetric

* If R_1, R_2 are antisymmetric then their composition need not to be antisymmetric

Eg:

$R_1 = \{(1,2), (3,2)\}$ $\quad R_2 = \{(2,3), (2,1)\}$

$R_2 \circ R_1 = \{(1,3), (3,3), (3,1)\}$

$\therefore$ not antisymmetric

If R_1 & R_2 are asymetric $R_1 \circ R_1$ need not to be asymmetric

* If R_1, R_2 are transitive, then $R_1 R_2$ need not to be transitive
* If R_1, R_2 are irreflexive, then $R_1 \circ R_2$ need not to be irreflexive.

Closure:

If R is a relation then

closure of a property on R is R^+ such that

(i) property holds on R

(ii) Contain R

(iii) It is minimal possible

If R_1 & R_2 on set A are reflexive then $R_1 \circ R_2$ is reflexive.

(i) Reflexive closure:

Let $R = \{(1,2)\}$ be a relation on set $A = \{1,2,3\}$ then

reflexive closure of R^* is

R^+ which is Reflexive
contains R.
minimal

$\therefore$ reflexive closure, $R^+ = \{(1,2)(1,1)(2,2)(3,3)\}$

Reflexive closure: $R^+ = R \cup \Delta_A$

(ii) Symmetric closure:

Symmetric closure of a relation R^* is

R^+ which is Symmetric
contains R
minimal

Eg: If $R = \{(1,2)(1,1)\}$

$R^+ = \{(1,2)(2,1)(1,1)\}$

Symmetric closure, $R^+ = R \cup R^{-1}$

(iii) Transitive closure:

transitive closure of a relation R is

R^+ which is transitive
contains R
minimal

Eg: Let R be relation on

$A = \{1,2,3,4,5\}$

when asked to find both transitive and symmetric closures, changing order might effect the result.

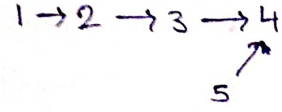
Eg:

$\{(1,2)\} \xrightarrow{\text{transit}} \{(1,2)\} \downarrow \text{sym} \{(1,2)(2,1)\}$

$\{(1,2)\} \xrightarrow{\text{sym}} \{(1,2)(2,1)\} \downarrow \text{transitive} \{(1,2)(2,1)(1,1)(2,2)\}$

Let $R = \{(1,2)(2,3)(3,4)(5,4)\}$

Draw a diagram as below



Now from each vertex find all reachable vertices

for 1:	for 2:	for 3:	for 5:
(1,2)	(2,3)	(3,4)	(5,4)
(1,3)	(2,4)		
(1,4)			

∴ Transitive closure $= \{(1,2)(1,3)(1,4)(2,3)(2,4)(3,4)(5,4)\}$

* Let a relation R be represented by a directed graph in way that whenever aRb a directed edge is drawn from a to b.

Now, if

$(a,c) \in R \circ R$

⇒ there is 2-edge path from a to c.

Sily: for two vertice a & b, there exists an n-length path from a to b iff $(a,b) \in R^n$

Note:

→ If R is a relation,

a composite relation from R to itself is denoted as R^2.

Sily $R \circ R \circ R = R^3$

$R \circ R \circ \cdots \circ R$ (n times) $= R^n$

* The standard procedure to calculate transitive closure is

$$R^* = R \cup R^2 \cup R^3 \cup R^4 \cdots \cup R^n$$

where n is a number such that

$$R^n = R^{n+1}$$

$$\therefore R^* = \bigcup_{n=1}^{\infty} R^n$$

Note:

→ If R is a transitive relation, then

$$R^n \subseteq R \quad \forall n \geq 1$$

also R^n is transitive too.

* Irreflexive closure can be defined iff the relation on which it is to be defined is irreflexive.

Also the set itself is the irreflexive closure

* The closure of relation R w.r.to a property, if exists, is the intersection of all the relations with the property P that contain R.

* When asked to find a

Symmetric, reflexive & transitive closure on a relation

such that all 3 properties hold at the same time, then the order of finding closures is.

(i) Reflexive

(ii) Symmetric

(iii) Transitive

~~other ord~~

also

~~tran~~ transitive

reflexive

symmetric.

works but may produce redundant ~~relati~~ pairs

But other order would also work but gives redundant pairs

Q27 Let relation R be defined on real numbers as

$$R = \{(a,b) \mid a = b+1\}$$

what is the transitive closure of R?

Let aRb bRc cRd .....

$a = b+1$ $b = c+1$ $c = d+1$

By finding transitive closure we obtain

(a,b) (a,c) (a,d)....

(b,c) (b,d) ...

Thus, transitive closure = $\{(a,b) \mid a > b\}$

Q28 / G-98 The binary relation

$R = \{(1,1)(2,1)(2,2)(2,3)(2,4)(3,1)(3,2)(3,3)(3,4)\}$ on the set $A = \{1,2,3,4\}$ is

a) Reflexive, symmetric & transitive

b) Neither reflexive, nor irreflexive but transitive

c) Irreflexive, symmetric & transitive

d) Irreflexive & antisymmetric.

Sol:

$(4,4) \notin R$

∴ not reflexive

$(1,1) \in R$

∴ not irreflexive

Thus we can eliminate opt @ © ⓓ

∴ b

Q29 (A-01) Consider the following relations

R_1: (a,b) iff (a+b) is even over set of integers

R_2: (a,b) iff (a+b) is odd over set of integers

R_3: (a,b) iff $a \cdot b > 0$ over set of non-zero rational numbers

R_4: (a,b) iff $|a-b| \leq 2$ over the set of natural numbers

Which of the following ~~is correct~~? are equivalence relations?

a) R_1 & R_2

b) R_1 & R_3

c) R_1 & R_4

d) all.

Sol:

R_1: $\forall a \in Z$ a+a is even

∴ reflexive

$\forall a, \forall b \in Z$ a+b is even ⇒ b+a is even

∴ symmetric

a+b is even & b+c is even

⇒ (a+b)+(b+c) is also even.

a+c+2b is even

↓

even

∴ a+c must be even

∴ transitive

Hence R_1 is equivalence relation

R_2: $\forall a \in Z$ a+a is even

∴ not reflexive

Hence not equivalence relation

R_3: $\forall a \in R \quad a^2 > 0$

$a \neq 0$

$\therefore$ reflexive

$ab > 0 \Rightarrow ba > 0$

$\therefore$ symmetric

$ab > 0$ & $bc > 0$

$\Rightarrow (ab)(bc) > 0$

$(ac)(b^2) > 0$

$\downarrow$

> 0

$\Rightarrow ac > 0$

$\therefore R_3$ is equivalence relation

R_4: $\forall a \in N$

$|a-a| = 0 \leq 2$

$\therefore$ reflexive

$\forall a, b \in N$

$|a-b| \leq 2 \Rightarrow |b-a| \leq 2$

$\therefore$ symmetric

$a, b, c \in N$

Let $a = 1 \quad b = 3 \quad c = 5$

$|a-b| \leq 2 \qquad |b-c| \leq 2$

$|a-c| = 4 \nleq 2$

$\therefore$ not transitive

$\therefore R_1$ & R_3 are equivalence relations.

Q30 / A-04 Consider the binary relation

$S = \{ (x,y) \mid y = x+1 \text{ and } x, y \in \{0, 1, 2 \ldots\} \}$

The reflexive transitive closure of S is

a) $\{(x,y) \mid y > x \text{ and } x, y \in \{0, 1, 2 \ldots\}\}$

b) $\{(x,y) \mid y \geq x \text{ and } x, y \in \{0, 1, 2 \ldots\}\}$

c) $\{(x,y) \mid y \in x \text{ and } x,y \in \{0,1,2\cdots\}\}$

d) $\{(x,y) \mid y \leq x \text{ and } x,y \in \{0,1,2\cdots\}\}$

Sol:

Finding reflexive closure

$$\forall x \in \{0,1,2\cdots\}$$
$$(x,x) \in R$$

Consider the graph

$$0 \to 1 \to 2 \to 3 \to \cdots$$

By applying transitive closure

$$(x,y) \in R \mid y > x$$

∴ Reflexive transitive closure is

$$\{(x,y) \mid y \geq x \text{ and } x,y \in \{0,1,2\cdots\}\}$$

Q31 / G-04 The number of different $n \times n$ symmetric matrices with each element being either 0 or 1 is

a) 2^n b) 2^{n^2} c) $2^{\frac{n^2+n}{2}}$ d) $2^{\frac{n^2-n}{2}}$

Sol:

If we represent a relation on set A with n elements by a matrix such that if xRy then the corresponding cell of matrix is 1 and 0 otherwise.

Thus the question reduces to no of symmetric relations possible on set with n elements

i.e., $2^n \cdot 2^{\frac{n^2-n}{2}} = 2^{\frac{n^2+n}{2}}$

P/13 Let $R = \{(1,2),(2,1)\}$. ~~not~~ (irreflexive)

$R^* = \{(1,1),(2,2)\}$ (not irreflexive)

$\therefore$ S_1 is false

Let $(a,c) \in$ transitive closure of R

then

$(a,b) \in R$ $(b,c) \in R$ for some $b \in$ Domain

$\Rightarrow$ $(b,a) \in R$ $(c,b) \in R$

$\Rightarrow$ $(c,a) \in$ transitive closure of R

$\therefore$ S_2 is true

P/14 xRy iff $|x-y| < 1$

$\forall x$ xRx

$\therefore$ Reflexive

$\forall x \forall y$ $|x-y| < 1 \Rightarrow |y-x| \geq 1$

$\therefore$ symmetric

Let $a = 1$ $b = 1.5$ $c = 2$

$|a-b| = 0.5$ $\therefore aRb$ $|b-c| = 0.5$ $\therefore bRc$

$|a-c| = 1$

$\therefore a \not R c$

$\therefore$ not transitive

$\therefore$ not an equivalence relation

$\therefore$ option (a)

P/15 No of relations such that

symmetric & Anti symmetric is 2^n

This kind of relation includes on (x,x)

∴ It is also transitive

$\therefore 2^n$ ways

P/16 $\forall a \quad \frac{a}{a} = 1 = 2^0$

∴ reflexive

$\forall a \; \forall b \quad a \neq b$

$\frac{a}{b} = 2^i$

$\therefore \quad i > 0 \quad (\because a \neq b)$

Now $\frac{b}{a} = 2^{-i} \neq 2^i$

∴ antisymmetric

and not symmetric

$\frac{a}{b} = 2^i \qquad \frac{b}{c} = 2^j$

$\frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = 2^{i+j}$

∴ transitive

∴ opt (d)

P/22 Sol: $(R \cap R^{-1}) \subseteq \Delta_A$

if R was symmetric then R^{-1} ~~contains~~ is also symmetric which contains also symmetric pairs

if R is symmetric

$R = R^{-1}$

$\Rightarrow R \cap R^{-1} = R$

But it is given that

$$(R \cap R^{-1}) \subseteq I_A$$

it means

$$(a,b) \notin R \cap R^{-1} \text{ for } a \neq b$$

$\Rightarrow$ (a,b) either belongs to R or R^{-1} but not both for $a \neq b$

i.e., antisymmetric

$\therefore$ S1 is true

S2: R is transitive

Let $(a,c) \in R \circ R$

$\Rightarrow (a,b) \in R$ & $(b,c) \in R$ for some $b \in$ Domain

Since R is transitive

$(a,c) \in R$

$\therefore \forall a \forall c \left((a,c) \in R \circ R \rightarrow a \right)$

$\therefore \forall a \forall c \left((a,c) \in R \circ R \rightarrow (a,c) \in R \right)$

$\therefore R \circ R \subseteq R$

$\therefore$ S2 is true

(P/22)

$R = \{(a,b) \mid a \text{ divides } b\}$

$R^{-1} = \{(a,b) \mid b \text{ divides } a\}$

Symmetric closure $R \cup R^{-1} = \{(a,b) \mid a \text{ divides } b \text{ (or) } b \text{ divides } a\}$

10/06/20

Equivalence Relation:

A relation R is equivalence relation if it is Reflexive, symmetric, transitive.

Eg: $R=\{(P_1,P_2) \mid P_1, P_2$ born in same month$\}$

P_1 & P_1 born in same month $\therefore$ reflexive

$P_1RP_2 \Rightarrow P_2RP_1$ $\therefore$ symmetric

$P_1RP_2 \wedge P_2RP_3 \rightarrow P_1RP_3$ $\therefore$ transitive

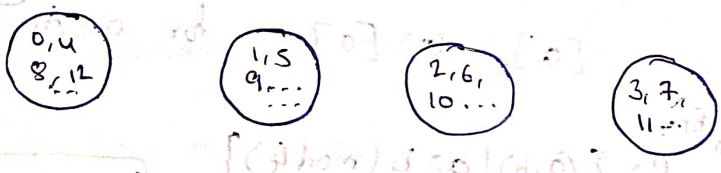
$\therefore$ equivalence relation

Eg: $R=\{(a,b) \mid a \equiv b \pmod n\}$ is equivalence relation

* Equivalence relation creates partition in domain. These partitions are called equivalence classes

Eg: $R=\{(a,b) \mid a \equiv b \pmod 4\}$

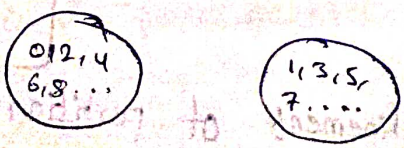
Now this is an equivalence relation and we obtain 4 equivalence classes.



→ Every element of same class is related to each other

→ No element is related to an element of different class.

Eg: $R=\{(a,b) \mid a+b$ is even$\}$ is an equivalence relation and it forms two equivalence ~~classes~~ classes.



Eg: Let $A = \{0,1,2,3,4,5\}$

Let R be an equivalence relation which partitions A into 3 equivalence classes

$\{0\}$ $\{1,2\}$ $\{3,4,5\}$.

Find R.

$R = \{(0,0)\ (1,1)\ (2,2)\ (1,2)\ (2,1)\ (3,3)\ (4,4)\ (5,5)\ (3,4)\ (4,3)\ (3,5)\ (5,3)\ (4,5)\ (5,4)\}$

* Let an equivalence relation partitioned set A as

$A_1, A_2 \ldots A_n$ then

* $A_1 \cup A_2 \cup \ldots \cup A_n = A$
* $A_1 \cap A_2 \cap \ldots \cap A_n = \phi$
* $x R y \quad \forall x \in A_i \quad \forall y \in A_j \quad i \neq j$

A partition on a set A is defined as $A_1, A_2 \ldots A_n$ such that

$A_1 \cup A_2 \cup \ldots \cup A_n = A$

$A_i \cap A_j = \phi$ $\forall i \neq j \leq n$

Thus equivalent classes forms a partition

* An equivalence class A_i is represent using one of its element

A_i is represent as $[a]_R$ (or) $[a]$ for $a \in A_i$

$[a]_R = \{s \mid (a,s) \in R\}$

Eg:

$R = \{(a,b) \mid a \equiv b \pmod 4\}$

The equivalence classes are

$[0] = [0,4,8,12 \ldots]$

$[1] = [1,5,9 \ldots]$

$[2] = \{2,6,10 \ldots\}$

$[3] = \{3,7,11 \ldots\}$

In an equivalent relation if aRb then we say a is equivalent to b. It is denoted as $a \sim b$

In an equivalence relation the below 3 stmts are same

i) aRb (ii) $[a]=[b]$ (iii) $[a] \cap [b] \neq \phi$

Refinement:

→ Partition P_1 is called refinement of partition P_2 if every set P_1 is a subset one of set in P_2

Q:

mod 6 (R1)

$[0]_6 = \{0, 6, 12 \cdots\}$

$[1]_6 = \{1, 7, 13 \cdots\}$

$[2]_6 = \{2, 8, 14 \cdots\}$

$[3]_6 = \{3, 9, 15 \cdots\}$

$[4]_6 = \{4, 10, 16 \cdots\}$

$[5]_6 = \{5, 11, 17 \cdots\}$

mod 3 (R2)

$[0]_3 = \{0, 3, 6, 9 \cdots\}$

$[1]_3 = \{1, 4, 7, 10 \cdots\}$

$[2]_3 = \{2, 5, 8, 11 \cdots\}$

Here

$[0]_6 \subseteq [0]_3$ $\quad [1]_6 \subseteq [1]_3$ $\quad [2]_6 \subseteq [2]_3$

$[3]_6 \subseteq [0]_3$ $\quad [4]_6 \subseteq [1]_3$ $\quad [5]_6 \subseteq [2]_3$

∴ we say mod 6 is a refinement of mod 3

~~R1 is a refine~~

i.e., partition formed from congruence class mod 6 is a refinement of partition formed from congruence class mod 3.

* If R_1 & R_2 are two equivalence relations

$R_1 \cup R_1$ is ~~equival~~ not an equivalence relation (∵ union of 2 transitive relation is not transitive)

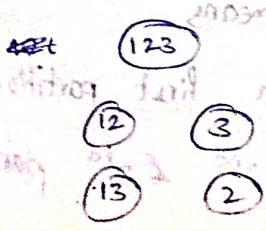
$R_1 \cap R_2$ is an equivalence relation

Finding no of equivalence relations possible:

→ Here we can find no of way we can partition. This will be equal to no of equivalent classes

Let $A = \{1, 2, 3\}$

different types of ~~equiva~~ partition are



(23) (1)

(1) (2) (3)

∴ 5 equivalence relations.

this no of equivalence relations is called bell number.

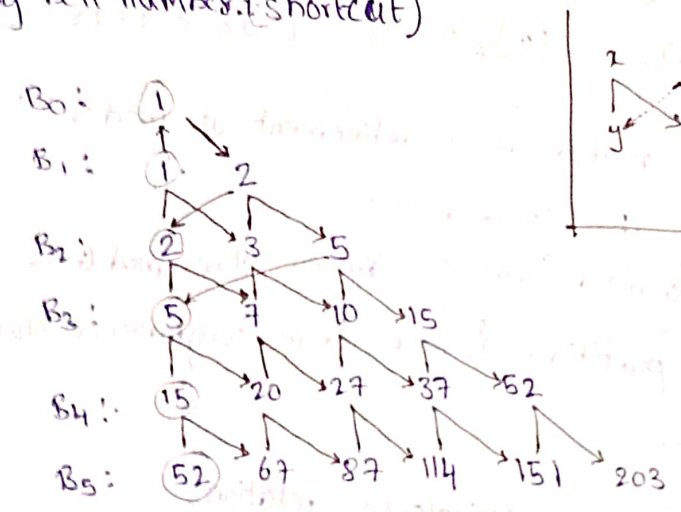
$B_3 = 5$

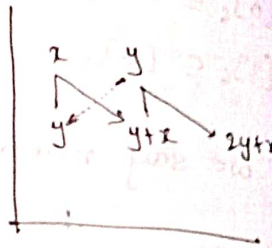
Bell Number :

Bell number

B_n = no of possible partitions of a set with n elements

finding bell number: (shortcut)





Thus from figure

$B_0 = 1$ $B_1 = 1$ $B_2 = 2$ $B_3 = 5$

$B_4 = 15$ $B_5 = 52$ $B_6 = 203$

Proof (This is not needed for gate):

We calculate no of equivalence relations of $A = \{a, b, c, d\}$ by considering no of onto function from A to $B = \{1, 2, 3\}$

This will give us no of equivalence relation with 3 partitions

consider

a mapping to 1 means

a falls in first partition

silly 2 & 3 are 2nd & 3rd partitions

But we get duplicates as shown below

$[ab]_1$ $[c]_2$ $[d]_3$
$[ab]_1$ $[d]_2$ $[c]_3$
$[c]_1$ $[ab]_2$ $[d]_3$
$[c]_1$ $[d]_2$ $[ab]_3$
$[d]_1$ $[ab]_2$ $[c]_3$
$[d]_1$ $[c]_2$ $[ab]_3$

These are calculated as 6 onto function but they actual mean the same. So we divide by 3!

∴ no of equivalence relation on $A=\{a,b,c,d\}$ with 3 partition is

$$\frac{\text{no of onto function for A to B(size 3)}}{3!}$$

$$= \frac{36}{6} = 6$$

(This is called sterling 2nd kind number)

Sterling 2nd kind number

$$S(m,n) = \frac{\text{no of onto function from A to B}}{n!} \quad (|A|=m \ |B|=n)$$

$$= \frac{1}{n!}\sum_{i=0}^{n}(-1)^i \, {}^nC_i \, (n-i)^m$$

* To calculate total number of equivalence relation on A, we need to find

no of equivalence relation with 4 partitions $S(4,4)$
+
no of equivalence relations with 3 partition i.e., $S(4,3)$
+
- - - - - - - - - - with 2 partitions $S(4,2)$
+
- - - - - - - - - - with 1 partition $S(4,1)$

$$S(4,2) = \frac{1}{2!}\sum_{i=0}^{2}(-1)^i\, {}^4C_i\,(n-i)^m$$

$$= \frac{1}{2!}\left(2^4 - {}^2C_1(2-1)^4\right)$$

$$= \frac{16-2}{2} = 7$$

Silly $S(4,1) = 1$

$S(4,4) = 1$

∴ no of equivalence relations on $A=\{a,b,c,d\}$ is

$$S(4,1) + S(4,2) + S(4,3) + S(4,4)$$

$$1 + 7 + 6 + 1$$

$$= 15$$

∴ No of equivalence relations on a set with n elements is

Bell number $B_n = S(n,1) + S(n,2) + \cdots + S(n,n)$

$$\boxed{B_n = \sum_{k=1}^{n} S(n,k)}$$

where

$$\boxed{S(m,n) = \frac{1}{n!}\sum_{i=0}^{n}(-1)^i\, {}^nC_i\,(n-i)^m}$$

Partial Order Relation:

* The relation which is reflexive
  antisymmetric
  transitive

  is called a partial order relation

Eg: $R_1 = \{(a,b) \mid a \leq b\}$ is a partial order relation

$R_2 = \{(a,b) \mid a < b\}$ is not a partial order relation.

$R_3 = \{(a,b) \mid a \text{ divides } b\}$ on domain Z^+ or $Z - \{0\}$

a/a ∴ reflexive

$a/b \wedge b/a \rightarrow a = b$ ∴ antisymmetric

$a/b \wedge b/c \rightarrow a/c$ ∴ transitive

∴ partial order relation

R_3 on Z is not a partial order relation

because

for 0

$0/0$ is not true

∴ not reflexive

∴ not a partial order relation.

If (S, R) is a poset, (S, R^{-1}) is called dual of the poset. (S, R^{-1}) is also a poset

* If R is a relation and bRa or aRb then we say a, b are comparable

* Eg $R = \{(a,b) \mid a \text{ divides } b\}$ on Z

$3 \not R 2 \quad 2 \not R 3$

∴ 3, 2 are not comparable.

Total ordered set or Linear ordered set:

* If relation R is such that every pair of elements are comparable, then it is called total ordered set.

Eg: $R = \{(a,b) \mid a \leq b\}$ on Z

This relation is called total ordered set relation.

* Some elements are comparable ⇒ partial order set and relation is called partial order relation.

* Total ordered set is called toset
  Partial ordered set is called poset.
  Every toset is poset.

* If R_1, R_2 are two partial order relations,
  $R_1 \cup R_2$ is need not ~~a poset~~ to be a partial order relation
  $R_1 \cap R_2$ is a partial order relation.

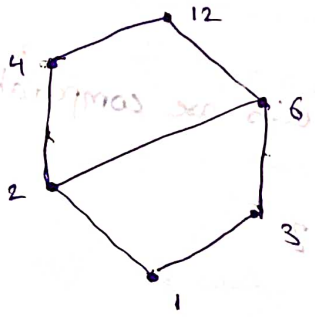
Hasse Diagrams:

Let $A = \{1, 2, 3, 4, 6, 12\}$

Let R be a relation on A

$R = \{(a,b) \mid a \text{ divides } b\}$

* This represented by Hasse diagram



$1R_2$ is represented as

Reflexivity is not represent in hasse diagram
transitivity is not represented in hasse diagram.
However, we can obtain transitive pairs from the diagram.

* The set of Divisors of a ~~no~~ number n is represented as D_n.

Eg: $D_{12} = \{1, 2, 3, 4, 6, 12\}$

Let $(S, \leq)$ be a poset. we say $y \in S$ cover an element $x \in S$ if $x < y$ and there is no element $z \in S$ such that $x < z < y$. The set of pairs $x < y$ such that y covers x is called the covering relation of $(S, \leq)$. Infact, applying reflexive-transitive closure of its covering relation, we can obtain the corresponding Poset. The edges in Hasse diagram are elements from the covering relation

→ ~~Every relation on~~ D_n

→ $(D_n, /)$ is always a poset.

Q32 $R = \{(f,g) \mid \text{for some } c \in Z,\ \forall x \in Z,\ f(x) - g(x) = c\}$

where f, g are functions.

is above relation an equivalence relation?

Sol:

In other

$(f,g) \in R$, if $\forall x \in Z$ ~~$f(x)$~~ difference b/w $f(x)$ and $g(x)$ is ~~constant~~ integer

Let $(g,h) \in R$

$\therefore$ difference b/w $g(x)$ & $h(x)$ is ~~constant~~ integer for all x

$\Rightarrow$ difference b/w $f(x)$ & $h(x)$ is ~~constant~~ integer for all x

$\therefore$ transitive.

Also it is clearly symmetric & reflexive

$\therefore$ Equivalence relation.

* In a poset if aRb we denote it as $a \leq b$

Note that '$\leq$' doesn't mean ~~less~~ logical comparion $\leq$

The notation $a < b$ is used when $a \leq b$ but $a \neq b$

* If A is a set and R is partial order set, then set A along with R is called poset. It is denoted as (A, R)

P/9

∴ 3 equivalence class

P/10

R is an antisymmetric relation

i.e., $(aRb \wedge bRa) \rightarrow a = b$

$R \cap S = \{(a,b) \mid (a,b) \in R \wedge (a,b) \in S\}$

∴ antisymmetry holds

$R - S = R \cap \bar{S}$

∴ antisymmetric

R^{-1} is also antisymmetric

Let $A = \{1,2\}$

$R = \{\}$ is antisymmetric

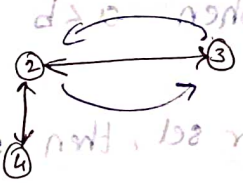
$\bar{R} = A \times A$ is not antisymmetric

P/11

find symmetric closure

$\{(1,1)\ (2,2)\ (4,4)\ (2,3)\ (3,2)\ (4,2)\ (2,4)\}$

finding transitive closure



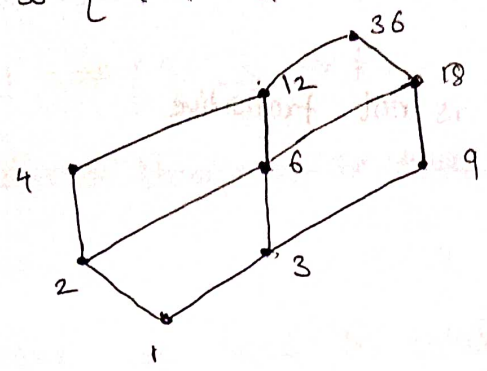
→ Here we can also obtain (3,3)

so the pairs (4,3) (3,4) can be added

∴ $\{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (2,3)(3,2)\ (4,2)(2,4)\ (4,3)(3,4)\}$

∴ ~~9 pairs~~ 10 pairs

P/12 $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



∴ 12 edges

P/17 n is no of elements

no of pairs in equivalence relation is

no of pairs due to reflexivity
+
no of pairs due to symmetricity
+
no of pairs due to transitivity

no of reflexive pairs = n

no of rest of pairs = even (cuz they all follow symmetry)

∴ if n is odd, then no of pairs = odd

even ——— even

∴ Both S_1 & S_2 are true

P/18 S_1: Reflexive ∪ anything is reflexive

S_2: if $(a,b) \in R \cup S$

$(a,b) \in R \vee (a,b) \in S$

$\Rightarrow (b,a) \in R \vee (b,a) \in S$

$\Rightarrow (b,a) \in R \cup S$

∴ symmetric

S3: $\{(1,2)\} \cup \{(2,3)\}$

transitive transitive

~~(1,3)~~ $\{(1,2)\ (2,3)\}$ is not transitive

$\therefore S1 \& S2$

P/19 $P_1 \cup P_2 \cup \cdots P_n = S$

$P_1 \cap P_2 \cap \cdots P_n = \phi$

P/20 Same as (19) P/19

P/21 $\{\{1,2,3\}\ \{4,5\}\ \{6\}\}$

no of equivalence relations possible on 3 elements $B_3 = 5$

no of equivalence relations on 2 elements $B_2 = 2$

$B_1 = 1$

$\therefore$ no of refinements $= 5 \times 2 \times 1$

$= 10$

P/24 ~~we need to find~~

R

$\{[1,2]\ [3,4]\ [5]\}$

S

$\{[1]\ [2]\ [3]\ [4,5]\}$

to find smallest equivalence relation which contains R & S, that relation should have both R & S as its refinements

$\therefore \{\{1,2\}\ \{3,4,5\}\}$

P/25 $[x] = [y] \Rightarrow xRy$

$[x] \cap [y] = \{\} \Rightarrow x \not R y$

∴ either of them must be true

P/26 It means we need to satisfy both antisymmetric and symmetric property at the same time.

It possible only when aRb iff $a=b$

∴ $R = \{(1,1)(2,2)(3,3)\}$ is the only relation

P/27

$\forall x \quad x-x=0$ is even

∴ reflexive

$\forall x \in Z \quad \forall y \in Z$

if $x-y$ = even then $y-x$ is also even

∴ symmetric

$\forall x, \forall y, \forall z \in Z$

$x-y$ = even & $y-z$ = even

$x-y+y-z$ = even + even

$x-z$ = even

i.e., transitive

∴ Equivalence

∴ only S1

P/28 not reflexive $\Rightarrow \exists x \, (x,x) \notin R$

not irreflexive $\Rightarrow \exists x \, (x,x) \in R$

∴ no of way for choosing (x,x) in Relation is $2^n - 2$

Since relation is symmetric

no of way for rest of the elements is $2^{\frac{n^2-n}{2}}$

$\therefore$ total req no of ways $= 2^n \cdot (2^n - 2)\, 2^{\frac{n^2-n}{2}}$

$= (8-2)\, 2^3$

$= 6 \times 8 = 48$

P129 Δ_A is reflexive
antisymmetric
transitive
$\therefore$ partial order

for Δ_A $\forall a, b$ $(a \neq b \Rightarrow a \not R b)$

$\therefore$ not total order

Δ_A is reflexive
Symmetric
transitive
$\therefore$ equivalence

$\therefore S_1 \not\subseteq S_3$

P130 A Reflexive $\cup$ Reflexive = Reflexive
Reflexive $\cap$ Reflexive

Antisymmetric $\cup$ Antisymmetric may or may not be antisymmetric
Antisymmetric $\cap$ Antisymmetric = Antisymmetric

transitive $\cup$ transitive may or may not be transitive
transitive $\cap$ transitive is transitive

$\therefore$ R $\cup$ S need not to be partial order
R $\cap$ S is partial order.

$\therefore$ opt (b)

(P131)

if we include (

as (1,1) & (2,2) are

∴ (1,2) ∉ R (2,1) ∉ R

∴ the only relation is {(1,1) (2,2)}

∴ Now this is also transitive

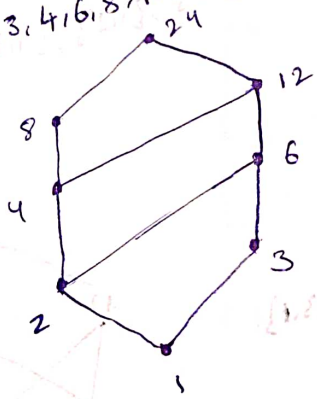
~~To P.T not not transitive we need to~~

∴ no of relations possible = 0.

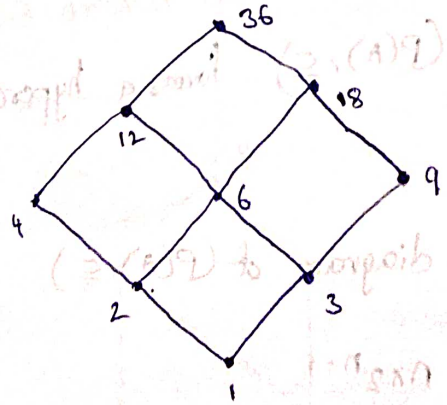
11/06/20

→ $(D_{24}, /)$

$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$



→ $(D_{36}, /)$ $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



symmetric

(P/31) (1,1) (2,2) must be included to be reflexive

if we include (1,2) we must include (2,1)

as (1,1) & (2,2) are present transitivity holds

∴ (1,2) ∉ R (2,1) ∉ R

∴ ~~the only relations~~ {(1,1) (2,2)}

∴ Now this is also transitive

~~To f(not transitive we need to~~

∴ no of relations possible = 0.

11/06/20

→ $(D_{24}, |)$

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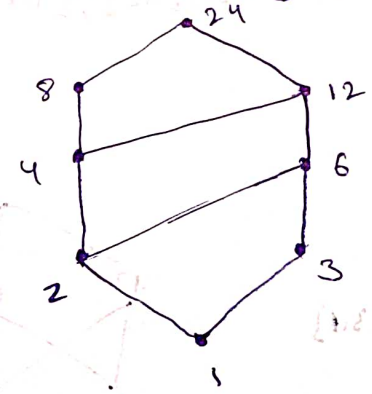
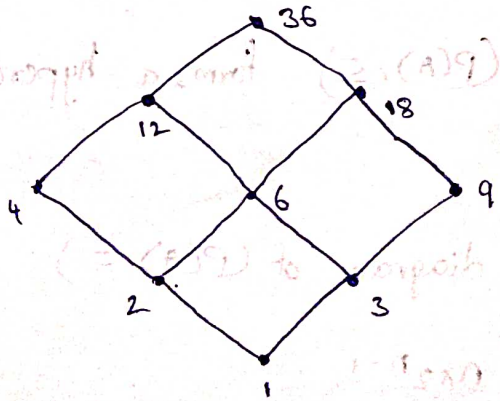
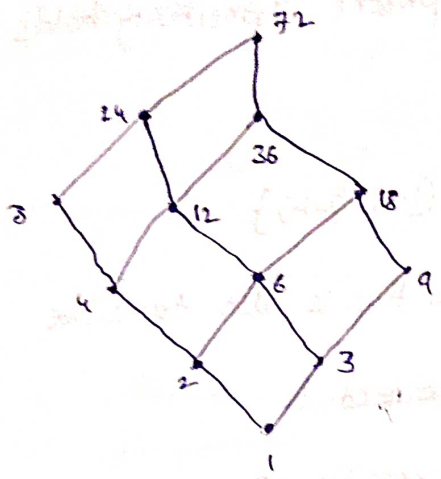


→ $(D_{36}, |)$ $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



→ $(D_{72}, |)$

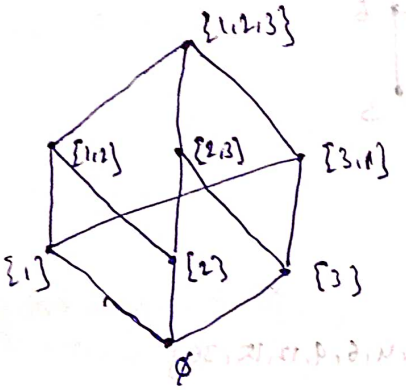
$D_{72} = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$

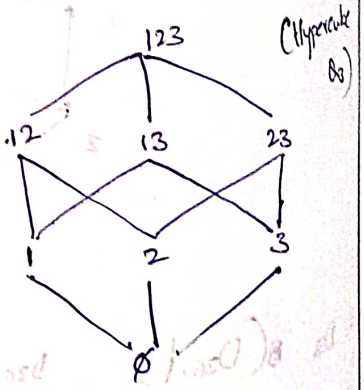


→ $A = \{1, 2, 3\}$

Draw Hasse diagram $(P(A), \subseteq)$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$





(Hypercube of 3)

The hasse diagram of any $(P(A), \subseteq)$ forms a hypercube.

Thus if $|A| = n$

no of edges in hasse diagram of $(P(A), \subseteq)$

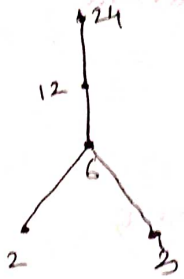
is $\frac{2^n \times n}{2} = n \times 2^{n-1}$

for $n = 3$

$3 \times 2^{3-1} = 12$ edges.

(Q33) G-96 Let $X = \{2, 3, 6, 12, 24\}$. Let $\leq$ be the partial order defined by $x \leq y$ if x divides y. The number of edges in Hasse diagram of $(X, \leq)$ is

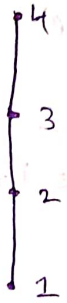
a) 2 b) 4 c) 9 d) None



~~∴ 5 edges~~ ∴ 4 edges

* Hasse diagram of a toset is a chain.

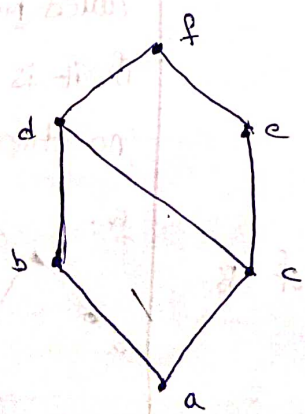
Eg.: $(\{1, 2, 3, 4\}, \leq)$



Greatest element (or) Maximum element:

x is called greatest element ~~if $a \leq x$~~ of poset $(A, \leq)$ if $a \leq x \ \forall a \in A$

Eg:



Here f is the greatest element

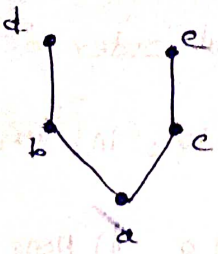
Maximal element:

An element of poset is called maximal element if it is less than no other element

Eg:

e, d are maximal elements

Eg:



This diagram has no greatest element

~~Case~~ for d: $c \not\leq d$

for e: $d \not\leq e$

$\therefore$ not greatest element

Note:

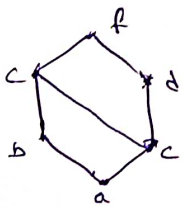
→ Thus, it is not neccessary that a poset has greatest element. But if there is a greatest element then it is unique.

Least element or Minimimum element:

In a poset (A, R)

$x \in A$ is called a least element if ~~$a \leq x \ \forall a \in A$~~ $x \leq a \ \forall a \in A$

Eg:



Here 'a' is the least element.

Here we don't have any least element.

Note:

→ Least element need not to exist every time. If exists, it is unique.

Upper bound(UB)

If (A, R) is poset and $B \subseteq A$

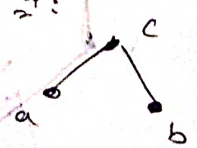
$x \in A$ is called upper bound of B

$\forall b \in B \quad b \leq x$

Minimal element

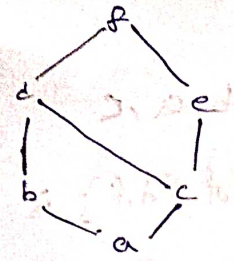
An element of a poset is called minimal element if it is greater than no other element

Eg:



a, b are minimal elements

Eg:



Let $B=\{b,c\}$

for d

$b \leq d$ & $c \leq d$

$\therefore$ d is upper bound of B

for f

$b \leq f$ & $c \leq f$

$\therefore$ f is upper bound of B

$b \not\leq e$ & $c \leq e$

$\therefore$ e is not upper bound of B

Let $B=\{a,b\}$

b is also upper bound of B

d, f are also upper bounds of B

> Every finite non-empty poset has atleast one minimal and atleast one maximal element

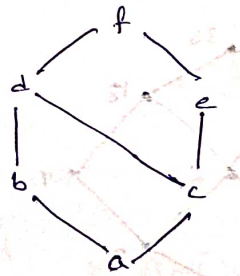
Lower Bound (LB)

If (A,R) is a poset and $B \subseteq A$

$x \in A$ is called lower bound of B if

$\forall b \in B \quad x \leq b$

Eg:



Lower bounds of $\{d,c\}$ is c, a

Lower bounds of $\{f,e\}$ are e, c, a

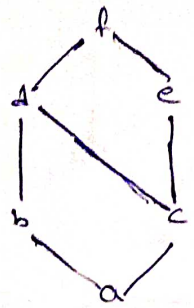
Greatest Lower Bound (GLB):

(A,R) is poset & $B \subseteq A$

let X be ~~lower bo~~ set of lower bounds of B

then GLB, g is $g \in B$ & $\forall x \in X \quad x \leq g$

Eg:



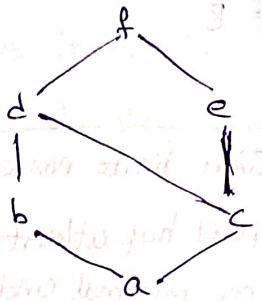
Lower bounds of $\{d, f\}$ are d, b, a, c

But the greatest lowerbound of $\{d, f\}$ is d

Least Upper Bound: (LUB)

it is lowest among all the upper bounds.

Eg:



for $\{b, c\}$

upper bounds are d, f

The least upper bound is d.

for $\{a, b\}$

upper bounds are b, d, f

the least upper bound is b

Eg: In $(D_{36}, |)$

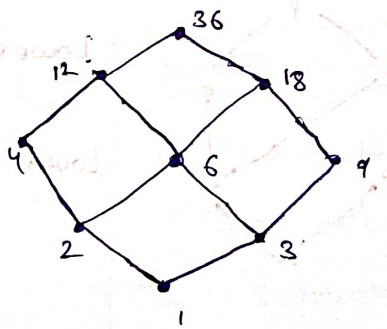
find a) $glb(3, 9)$

b) $glb(4, 12)$

c) $lub(3, 9)$

b) $lub(4, 12)$

$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



a) ~~$3 | 9$~~ $\therefore$ ~~$glb(3,9) = 9$~~

a) $glb(3, 9) = 9$

b) $glb(4, 12) = 12$

c) $lub(3, 9) = 3$

d) $glb(4, 12) = 4$

Since the relation is divides

lub is nothing but L.C.M

glb is nothing but G.C.D

→ Let $(A, \subseteq)$ A be a set and

$(P(A), \subseteq)$ be a power set.

~~Let $P \in P(A)$~~ Let $x \in P(A)$, $y \in P(A)$

$glb(x,y)$ is $x \cap y$

$lub(x,y)$ is $x \cup y$

Sly $P_1 \in P(A)$, $P_2 \in P(A) \cdots P_n \in P(A)$

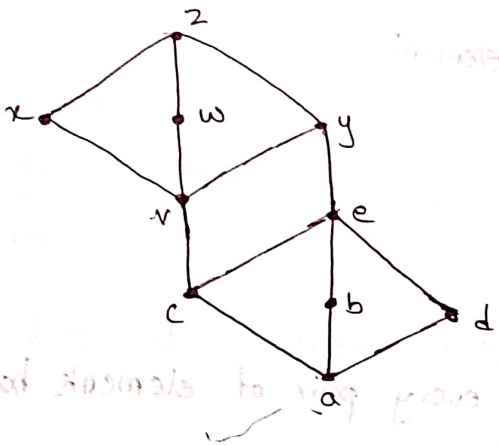
$glb(P_1, P_2, \cdots P_n) = P_1 \cap P_2 \cap \cdots \cap P_n$

$lub(P_1, P_2 \cdots P_n) = P_1 \cup P_2 \cup \cdots \cup P_n$.

→ Thus glb is also represented by operator $\wedge$ (or) .

lub is also represented by operator $\vee$ (or) +

(Q34) Consider below hasse diagram



Find a) $glb(b,c)$ b) $glb(b,w)$ c) $glb(e,x)$

d) $lub(c,b)$ e) $lub(d,x)$ f) $lub(c,e)$ g) $lub(a,x)$

Sol:

a) $glb(b,c) = e$ ~~d) $lub(c,b) = e$~~

b) $glb(b,w) = z$

c) $glb(e,x) = z$

a) glb(b,c) = a

b) glb (b,w) = a

c) glb (e,x) = c

d) lub(c,b) = e

e) lub (d,x) = z

f) lub(c,e) = e

g) lub(a,x) = x

Note:

→ If $a \leq b$

lub (a,b) = b

glb (a,b) = a

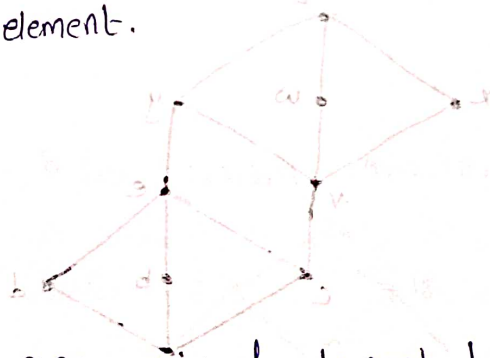
→ lub (greatest element , a) = greatest element

glb (greatest element , a) = a

lub(least element , a) = a

glb (least element , a) = least element

where a is any element.



12/06/20

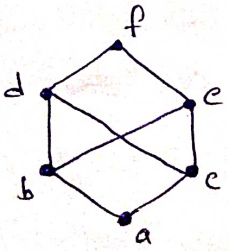
Lattice (L):

Lattice is a poset in which every pair of elements has both glb and lub

lub(a,b) = a∨b is called join of a,b

glb (a,b) = a∧b is called meet of a,b

Eg:



~~This is a lattice because lub and glb exists for every pair of elements~~

upperbounds of b,c are d,e,f

but {d,e,f} has not least element

Thus {b,c} has no lub

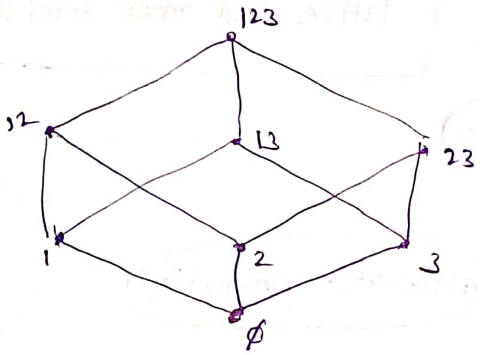
∴ This is not a lattice.

A Lattice is generally represented as [L, ∨, ∧]

Eg: A={1,2,3}

$(P(A), \subseteq)$ is a poset.

find whether the poset is lattice or not.



This is a lattice because every pair has lub & glb

Eg: Determine whether $(D_n, |)$ is lattice or not.

Sol:

we know that

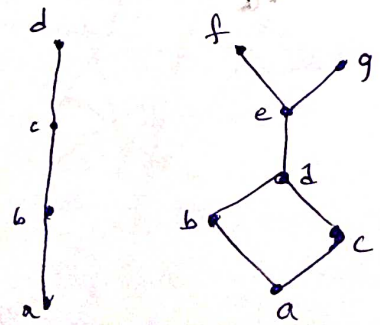
glb is gcd

lub is lcm

gcd, lcm always exists and hence it is always a lattice.

Every finite lattice has a greatest element and a least element

Eg:



Lattice

not lattice

lub(f,g) not exists

Lattice

has both glb

exists lub

Note:

→ $a \vee b = b \Leftrightarrow a \leq b$

$lub(a,b) = b$

→ $a \wedge b = a \Leftrightarrow a \leq b$

→ $a \wedge b = a \Leftrightarrow a \vee b = b$

Join Semi Lattice:

A poset $[A; R]$ in which each pair of element a & b of A have a least upper bound is called join semi lattice.

Meet Semi Lattice:

A poset $[A; R]$ in which each pair of element a & b of A have a glb is called meet semi-lattice.

Thus a lattice is both join semi lattice and meet semi lattice

Properties of Lattices:

Let L be a lattice

1) $a \vee a = a$ (Idempotent)
$a \wedge a = a$

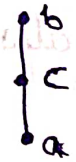
2) $a \vee b = b \vee a$ (Commutative)
$a \wedge b = b \wedge a$

3) $a \vee (b \vee c) = (a \vee b) \vee c$
$a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (associative properties)

4) $a \vee (a \wedge b) = a$
$a \wedge (a \vee b) = a$ (Absorption law)

5) If a, b, c are elements in L

(i) $a \leq b \Rightarrow a \vee c \leq b \vee c$
$a \leq b \Rightarrow a \wedge c \leq b \wedge c$
→ verify with this example



(ii) $(a \leq c)$ and $(b \leq c)$
iff
$(a \vee b) \leq c$

(iii) $c \leq a$ and $c \leq b$
iff
$c \leq (a \wedge b)$

(iv) If $a \leq b$ and $c \leq d$

$\therefore a \vee c \leq b \vee d$

$a \wedge c \leq b \wedge d$

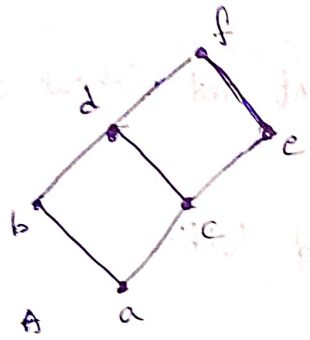
Sublattice:

B is called sublattice of a lattice A when it satisfies

(i) $B \subseteq A$

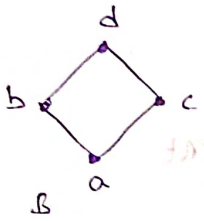
(ii) B is also a lattice.

Eg:



is a lattice

consider

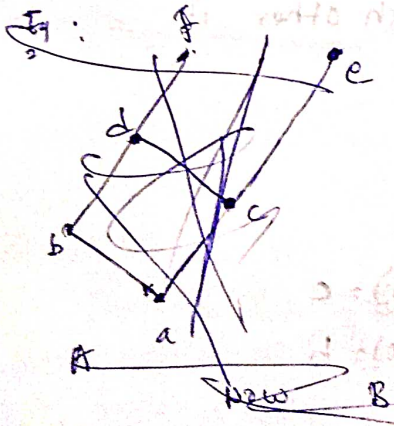


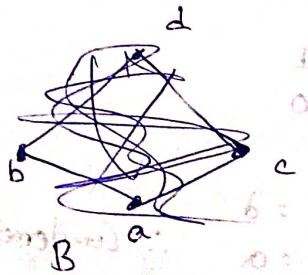
every pair in B has lub & glb

$\therefore B$ and $B \subseteq A$.

$\therefore$ B is sublattice of A.

Eg:





$B \subseteq A$

for ~~every pair in B, lub & glb exists~~

Bounded lattice:

Every lattice which has a greatest element (1) and a least element (0) is called bounded lattice.

Eg: 

is also bounded lattice

Note:

* $(Z^+, |)$ is a lattice with least element 1 and greatest element undefined.
* $(Z, \leq)$ is a lattice without least element and without greatest element.
* Thus, every finite lattice is a bounded lattice.

$\forall a \in A, \quad 0 \leq a \leq 1$

0 - least element

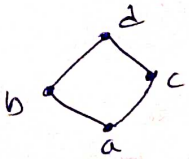
1 - greatest element

Complement Lattice:

Lattice A is said to be complement lattice if every element has a complement.

a & b are said to be complements of each other if

$a \vee b = 1$
$a \wedge b = 0$

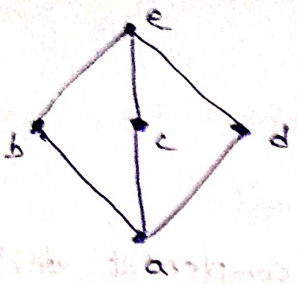


$b \vee c = d$
$b \wedge c = a$
$\therefore$ Complement(b) = c
Complement(c) = b

$a \vee d = d$
$a \wedge d = a$
$\bar{a} = d, \bar{d} = a$

$\therefore$ The above lattice is a complement lattice

Eg:



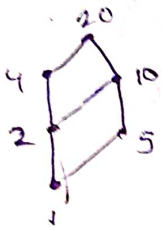
$a' = e \quad e' = a$

$b' = e \quad c' = b \quad , b' = d \quad , d' = b \quad , d' = c \quad , c' = d$

∴ Bounded lattices.

Eg: $(D_{20}, |)$

$D_{20} = \{1, 2, 4, 5, 10, 20\}$



$2' = 5 \quad , 5' = 2$

$4' = 10 \quad , 10' = 4$

$1' = 20 \quad , 20' = 1$

∴ Complement lattice

$1' = 20 \quad , 20' = 1$

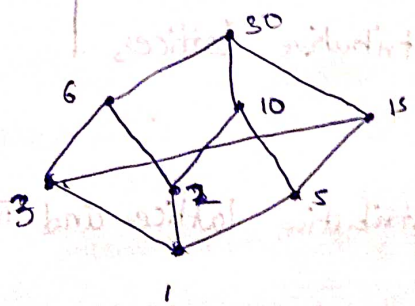
$4' = 5 \quad , 5' = 1$

But we don't have complement for 2 & 10

∴ not a complement lattice.

Eg: $(D_{30}, |)$

~~$D_{12} = \{1, 2, 3, 4, 6, 12\}$~~ $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$



$1' = 30$

$3' = 10$

$5' = 6$

$2' = 15$

∴ Complement lattice.

Eg: $(P(S), \subseteq)$

Since glb is ~~union~~ intersection and

lub is union

For every subset of A (say B), we can find its complement which is A-B

~~A ∪ A = B ∪ (B − A)~~ $B \cup (A - B) = A$

$B \cap (A - B) = \emptyset$

∴ Complement lattice.

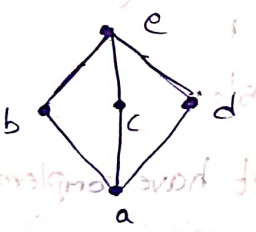
Distributive lattice:

A lattice L is called distributive lattice, if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

are satisfied $\forall a, b, c \in L$

Eg:



Now consider

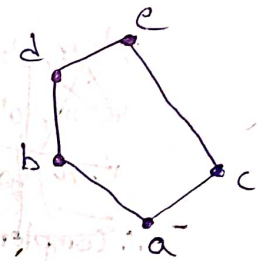
$b \vee (c \wedge d) = (b \vee c) \wedge (b \vee d)$

$b \vee (a) = e \wedge e$

$b = e$

∴ not true

∴ not a distributive lattices



$b \vee (c \wedge d) = (b \vee c) \wedge (b \vee d)$

$b \vee (a) = e \wedge d$

$b = d$

∴ not a distributive lattice

Note:

If L is a bounded distributive lattice and if complement exists the it must be unique.

Eg: $[D_n; |]$ is a distributive lattice.

Eg: $(P(S), \subseteq)$

~~Has every element has a unique complement~~

In boolean algebra every element has exactly one complement

∴ ~~distributive~~

glb → ∩ lub → ∪

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

∴ It is a distributive lattice.

In a distributive lattice each element can have atmost one complement

Thus if you find an element with more than one complement then it is not a distributive lattice

But this is only a sufficient condition.

Boolean Algebra:

A lattice which is both distributive and complement is known as boolean algebra.

Boolean lattice contains 2^n elements for $n \geq 0$
This lattice is isomorphic to poset $[P(S); \subseteq]$ where S is a set with n element. This lattice is a hypercube Q_n

→ It is called boolean algebra because when it is both distributive & complement, it satisfies all the properties of a boolean algebra.

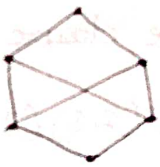
→ If n is a square free number, $[D_n; /]$ is a boolean algebra.

for $x \in D_n$ $\bar{x} = \frac{n}{x}$

GO35 4-08 Consider the following hasse diagrams.

i) 

ii) 

iii) 

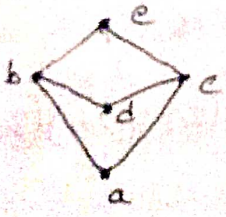
iv) 

~~Sol~~ which of the above are lattices?

a) (i) and (iv) b) (ii) & (iii) c) (iii) only d) (i) (ii) and (iv)

Sol:

(i) is clearly a lattice

(ii) 

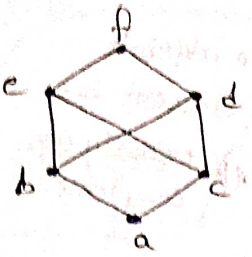
lower bounds of $[d, a] = \phi$

∴ d, a has not ~~lub~~ glb

∴ not a lattice

also glb(b, c) doesn't exist.

(iii)



upper bounds of $\{b,c\} = e, d, f$

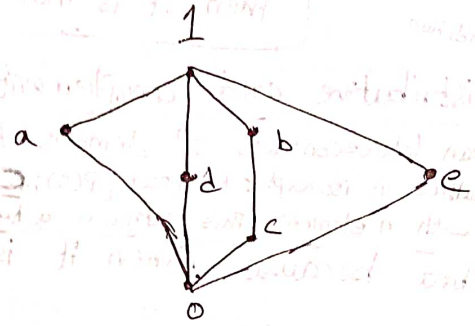
but (b,c) has no lub

$\therefore$ not a lattice.

(iv) is clearly a lattice

$\therefore$ opt (a)

(Q) **Q36 / 4-88** the complement(s) of the element 'a' in the lattice shown in the fig is (are):



$a \vee e = 1$ $a \wedge e = 0$	$a \vee d = 1$ $a \wedge d = 0$	$a \vee b = 1$ $a \wedge b = 0$	$a \vee b \vee c = 1$ $a \wedge c = 0$

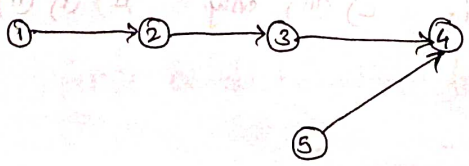
$\therefore$ Complements of 'a' are : b, c, d, e

Q37 / 4-89 The transitive closure of the relation

$\{(1,2)(2,3)(3,4)(5,4)\}$

on the set $A = \{1,2,3,4,5\}$ is _____.

Sol :



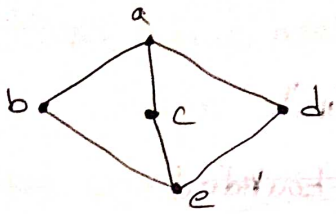
(1,2) (1,3) (1,4)	(2,3) (2,4)	(3,4)	(5,4)

$\therefore$ transitive closure is

$\{(1,2)(1,3)(1,4)(2,3)(2,4)(3,4)(5,4)\}$

Q38 / 9-05 The following is the hasse diagram of the poset $[\{a,b,c,d,e\}, \leq]$

The poset is:



a) not a lattice

b) a lattice but not a distributive lattice

c) a distributive lattice but not a boolean atgebra

d) a boolean algebra

Sol:

Every pair has lub & glb

$\therefore$ lattice

Consider

$$b \vee (c \wedge d) = (b \vee c) \wedge (b \vee d)$$

$$b \vee (e) = a \wedge a$$

$b = a$ (false)

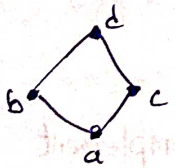
$\therefore$ not a distributive lattice

$\therefore$ opt (b)

Note:

→ Sublattice of a complement lattice need not to be a complement

proof:

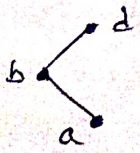


is complement lattice in whice

$a' = d$ & $d' = a$

$b' = c$ & $c' = b$

Now



is a sublattice in which 'b' has no complement.

→ Sublattice of a bounded lattice is also bounded

proof:

~~Bounded lattice is finite~~

~~∴ Its sublattice is also finite~~

~~and every finite lattice is bounded~~

~~∴ The sublattice is also bounded~~

→ Any linear order is a distributive lattice

proof:

Let a, b, c belong to the toset.

let $a \le b \le c$

Now

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$a \vee (b) = b \wedge c$

$b = b$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

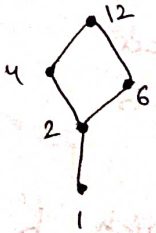
$a \wedge c = a \vee a$

$a = a$

∴ It is a distributive lattices.

P/32 Given $(D_{12}; |)$

$(D_n; |)$ is always a distributive lattice



2 has not complement

lattice is bounded

∴ opt. (b)

P/33 Consider $(a,b) \in A \times A$

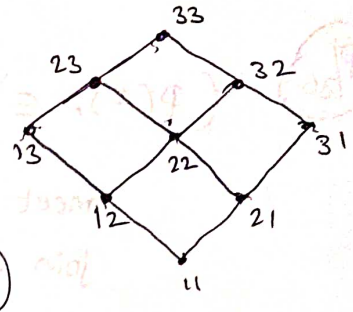
$(a,b)\ R\ (a,b) \Leftrightarrow (a \le a \wedge b \le b)$

this is true and hence

reflexive

It is also clear that it is antisymmetric & transitive

$\therefore$ partial order.



Also for (a,b) & $(c,d) \in A \times A$

ever pair has $lub = (\max(a,c), \max(b,d))$

$glb = (\min(a,c), \min(b,d))$

Thus it is a lattice

$\therefore$ opt ©

P/34

a) ~~$2 \vee 18 = lcm(2,18) = 18$~~ ~~true~~

b)

P/34

a) $2 \vee 18 = lcm(2,18) = 18 \neq 36$

$\therefore$ false

b) $3 \vee 12 = lcm(3,12) = 12 \neq 36$

$\therefore$ false

c) $4 \vee 9 = lcm(4,9) = 36$

$4 \wedge 9 = gcd(4,9) = 1$

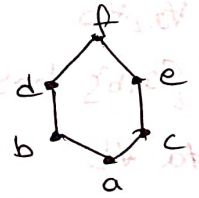
$\therefore$ True

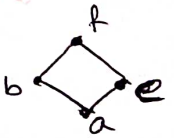
d) $6 \vee 1 = 6$

$6 \wedge 1 = 1$

Note:

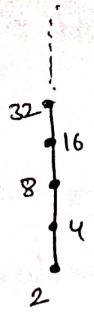
For lattice





is not a sublattice

P/35



It is a distributive lattice

Since it is infinite, we don't have greatest element. Hence not a bounded lattice.

P/36

$(P(A), \subseteq)$ is the poset with

- meet as intersection and
- join as union.

$B = \{2, 3, 5, 7\}$

$\bar{B} = A - B = \{1, 4, 6, 8, 9, 10\}$

P/37

Every non-empty subset of S has a minimum element

⇒ for every subset of 2 elements we have minimum element

i.e., $\forall a, \forall b$

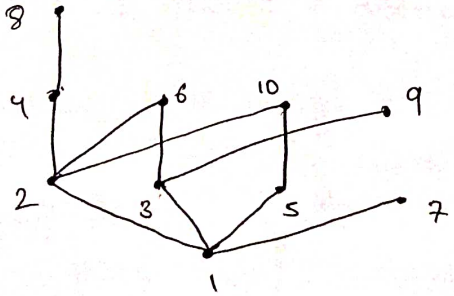
$\{a, b\}$ has minimum

i.e., $\forall a \; \forall b$

a & b are comparable

i.e., total ordered set

P/38



∴ no of edges = 11

P/39 Given $RUR^{-1} = A \times A$

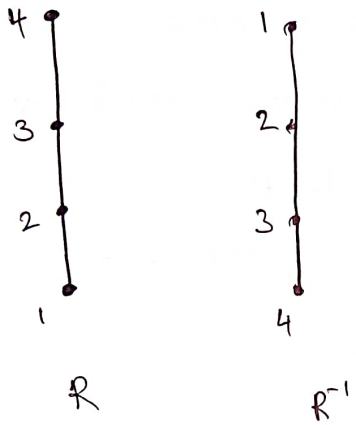
Given R is partial ordering

$\therefore \forall (a,b) \in R \Rightarrow$ ~~$(a,b) \notin R$~~ $(b,a) \notin R$

Thus $(b,a) \in R^{-1}$

Since $RUR^{-1} = A \times A$

we can say the partial ordering is a toset



clearly $[A:R]$ is a distributive lattice

Complement doesnot exist for 2,3

$\therefore$ not a Complemented lattice.

P/40 Distributive $\Rightarrow$ atmost 1 complement

atmost 1 complement $\not\Rightarrow$ Distributive.

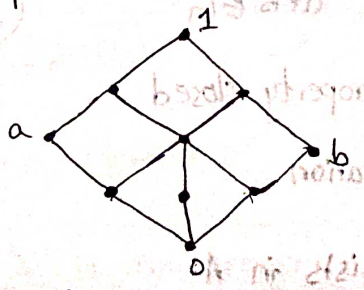
$\therefore$ the statement is false.

Proof:

Here we prove this with an example.

In this example sublattice ◈ is included to make it not distributive.

Also we design the example such ~~that~~ a way that few elements have no complement.

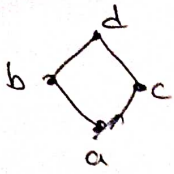


Here the only complement pairs are (0,1) & (a,b) so the complement of every element is atmost one and still not a distributive lattice.

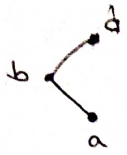
Think about this example for a while

$\therefore$ S1: false

S2:



is a complemented lattice



is not a complemented lattice.

∴ false

P|41 $(x \wedge y) \vee y = y$

∵ let $a = x \wedge y$

Now aRy

$\therefore a \vee y = y$

P|42 $x \vee (y \wedge z) = (x \vee y) \wedge z$

$x \vee (0) = I \wedge z$

$x = z$

∴ S1 is false

S2 is false because the lattice is not distributive

13/06/20

Groups:

$(G, *)$ is called group when it satisfies following properties.

(i) $a \in G$, & $b \in G$ then $a * b \in G$

we call this property closed

(ii) * is associative operation

(iii) Identity element exists in A.

(iv) Every element has inverse. $i.e \in A$.

* is any ~~operation~~ binary operation

Eg: Integers are closed under addition, subtraction & multiplication

Integer are not closed under division.

⇒ If 'e' is an identity element, then

$$\forall a \in G \quad a * e = a$$

and identity element, if exists, is unique.

→ In k is inverse of a

$$a \times k = e$$

where e is identity element

we denote inverse as

$$a^{-1} = k$$

also $k^{-1} = a$

Eg: Determine whether $(Z, +)$ is a group or not

$(Z, +)$

(i) $a \in Z, b \in Z \Rightarrow a+b \in Z$

∴ closed

(ii) $a+(b+c) = (a+b)+c$

∴ associative

(iii) $\forall a \in A \quad a+0 = a$

∴ Identity element exists.

(iv) $\forall a \in Z$

$a+(-a) = 0$

∴ Every element has inverse

∴ $(Z, +)$ is a group.

Eg: $(G,*)$, where $a*b = \frac{ab}{2}$

Find whether it is group or not if G is set of real number.

1) $a \in G,\ b \in G \Rightarrow \frac{ab}{2} \in G$

∴ closed

2) $a*(b*c) = (a*b)*c$

$a*\left(\frac{bc}{2}\right) = \left(\frac{ab}{2}\right)*c$

$\frac{abc}{4} = \frac{abc}{4}$

∴ Associative

3) $\forall a \in R$

$a*2 = \frac{a(2)}{2} = a$

∴ 2 is identity element

4) $\forall a \in R$

Let b be inverse

$a*b = 2$

$\frac{ab}{2} = 2$

$b = \frac{4}{a} \in R$ $\quad \therefore a^{-1} = \frac{4}{a}$

∴ Every element has inverse

∴ $(G,*)$ where $a*b = \frac{ab}{2}$ and G is real number is a group.

Eg: Consider $(Z, \times)$ $\times$ is multiplication

(i) closed

(ii) Associative

(iii) 1 is identity

(iv) $5^{-1} = \frac{1}{5} \notin Z \rightarrow$ ∴ not a group

Eg: Let $\mathbb{Q}$ be rational numbers

$(\mathbb{Q}, \times)$

(i) closed

(ii) Associative

(iii) 1 is identity element

(iv) $a \times \frac{1}{a} = 1$

But for 0 inverse doesn't exist

$\therefore (\mathbb{Q}, \times)$ is not a group

Sly: $(\mathbb{R}, \times)$ is not a group.

But $(\mathbb{Q} - \{0\}, \times)$ is a group.

$(\mathbb{R} - \{0\}, \times)$ is also a group.

→ Now let us see about finite groups

Eg: ~~$(\{2, 4, 6, 8\}, \text{gcd})$~~ $(\{0,1,2,3,4,5\}, \oplus_6)$

↳ Addition modulo m, $\oplus_m$

Finite groups can be represented with a table (cayley's table)

$\oplus_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

For every finite group, If we draw its cayley's table, every element is present in every row and column exactly once

(i) closed

(ii) $2 \oplus (3 \oplus 5) = (2 \oplus 3) \oplus 5$

$2 \oplus (2) = 5 \oplus 5$

$4 = 4$

Sly for any other elements

$\therefore$ associative.

(iii) we have column under '0' as 0, 1, 2, 3, 4, 5

$\therefore$ 0 is identity element

(iv) Every row has 0

$\therefore$ every element has inverse

$\therefore$ group.

Eg: $5 \oplus 1 = 0$

$4 \oplus 2 = 0$

$3 \oplus 3 = 0$

:

Q39 / G

The set $\{1, 2, 4, 7, 8, 11, 13, 14\}$ is a group under multiplication mod 15. what is inverse of 4 & 7 respectively?

a) 3, 13 b) 2, 11 c) 4, 13 d) 8, 14

Sol:

It is clear that '1' is identity element.

opt a: ~~4 × 13~~ $= 4 \times 3 = 12 \bmod 15 = 12$

$\therefore 4^{-1} \neq 3$

opt b: $4 \times 2 = 8 \bmod 15 = 8$ ✗

opt c: $4 \times 4 = 16 \bmod 15 = 1$

$7 \times 13 = 91 \bmod 15 = 1$

$\therefore$ opt c

Eg: $(\{e, b, a\}, *)$ be a group

Let e be the identity element. (find)

Now fill the table

*	e	b	a
e	e	b	a
b	b	a	e
a	a	e	b

row and column with e can be filled easily

Consider the cell [b, a]

it must be fill with e because we have a & b in it column and row

Now cell [b, b] has only 1 possible i.e., a

(same like sudoko)

Q40 / 9-04

Consider

Fill the below table which is commutative

*	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b				
c				

a) c a e b

b) e b a e

c) c b e a

d) c e a b

- what will be the last row.

Sol:

observing table we can say 'e' is identity element.

·	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

filling like sudoko

given $a \cdot b = c$

$\therefore b \cdot a = c$

also ~~b · c = e~~ $a \cdot c = e$

$\therefore$ ~~e~~ $c \cdot a = e$

$\therefore$ opt (d) is true

because it starts with ee

Subgroup:

H is called subgroup of G when it satisfies below properties:

i) $H \subseteq G$

ii) H should also be a group.

Eg: Consider $(\{0,1,2,3,4,5\}, \oplus_6)$ is a group G

H be $(\{1,3,5\}, \oplus_6)$ is not a sub group of G.

i) $H \subseteq G$

ii) $1 * 3 = 1+3 = 4 \bmod 6$

$= 4 \in H$

∴ H is not a group

∴ It is not a subgroup

Note:

→ If G is a group with identity element e.

then for any H which subgroup of G, $e \in H$.

i.e., Every subgroup contains identity element of its original group.

Eg: Now consider $H = \{0,3,5\}$

i) $H \subseteq G$

ii) H should be group

a) $0 \oplus_6 3 = 3$, $0 \oplus_6 5 = 5$

b) $3 \oplus_6 5 = 8 \bmod 6 = 2 \notin H$

∴ not closed

∴ not group, Hence not subgroup.

Eg: Consider $H = \{0, 2, 4\}$

	0	2	4
0	0	2	4
2	2	4	0
4	4	0	2

It is clear that $H \subseteq G$

(ii) Closed
associative
0 is identity.

Every row has 0.

∴ every element has inverse

$0^{-1} = 0$, $2^{-1} = 4$, $4^{-1} = 2$

∴ group

∴ H is subgroup of G.

Note:

* Every group consits of 2 trivial subgrougps:

(i) $\{e\}$ is a subgroup of G where e is identity element of G.

(ii) G is also a subgroup of G.

Lagrange's theorem:

If H is a subgroup of G, then $|H|$ divides $|G|$.
(reverse need not to be true).

Eg: $(\{0,1,2,3,4,5\}, \oplus_6)$ is a group G, $|G| = 6$

$H_2 (\{0,2,4\}, \oplus_6)$ is a subgroup of G., $|H| = 3$

∴ $3 | 6$

Order of a group:

* Cardinality of a group is also known as order of a group.

Eg: Let G be a group.

Let $|G|=84$.

What is the maximum size of proper subgroup of G.

Sol:

$H \subset G$

$|H| \neq 84$

$|H| / |G|$

$\therefore |H| = 42$ (maximum size possible)

Eg: Let G be a group with 15 elements. Let L be a subgroup of G and $L \neq G$. Also size of L is atleast 4. Find size of L.

Sol:

$|G|=15 \quad L \neq G$

$|L| \geq 4$

$5/15$ & $15/15$

$\therefore |L| = 5 \quad (\because |L| \geq 4)$

Exponential:

→ $(G, *)$ is a group and let $a \in G$

$a^1 = a$

$a^2 = a * a$

$a^3 = a^2 * a$

$\forall n \geq 1, \; a^n \in G$

Eg: If $(\{0,1,2,3,4,5\}, \oplus_6)$ be a group

$3^1 = 3$

$3^2 = 3 \oplus_6 3 = 0$

$3^3 = 3^2 \oplus_6 3 = 0 \oplus_6 3 = 3$

$\vdots$

Now exponential of 3 generates ~~$\{0,3\}$~~ $\{0,3\}$

we write it as

$$\langle 3 \rangle = \{0,3\}$$

Consider $\langle 2 \rangle$

$2^1 = 2$

$2^2 = 2 \oplus_6 2 = 4$

$2^3 = 4 \oplus_6 2 = 0$

$\vdots$

$\langle 2 \rangle = \{0,2,4\}$

Consider $\langle 1 \rangle$

$1^1 = 1$

$1^2 = 1 \oplus_6 1 = 2$

$1^3 = 2 \oplus_6 1 = 3$

$1^4 = 4$

$1^5 = 5$

$1^6 = 0$

$\vdots$

$\therefore \langle 1 \rangle = \{0,1,2,3,4,5\}$

Consider $\langle 5 \rangle$

$5^1 = 5$

$5^2 = 5 \oplus_6 5 = 4$

$5^3 = 4 \oplus_6 5 = 3$

$5^4 = 3 \oplus_6 5 = 2$

$5^5 = 2 \oplus_6 5 = 1$

$5^6 = 1 \oplus_6 5 = 0$

$\therefore \langle 5 \rangle = \{0,1,2,3,4,5\}$

→ Here 1, 5 have generated all the ~~gener~~ elements of the group

Such elements are called generators.

Here 1, 5 are generators of the group

Cyclic group:

A group with atleast one generator element is called a cyclic group.

Thus the whole group can be represented with powers of generator element.

Ex: $(\{0, 1, 2, 3, 4, 5\}, \oplus_6)$

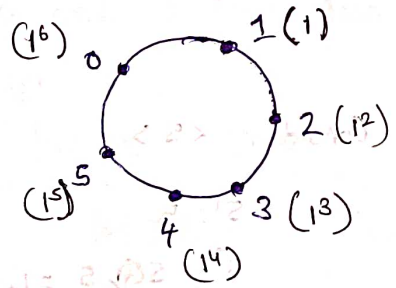
5, 1 are generators

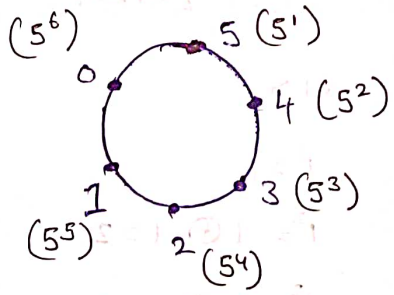
$(\{1^6, 1, 1^2, 1^3, 1^4, 1^5\}, \oplus_6)$

(or)

$(\{5^6, 5^5, 5^4, 5^3, 5^2, 5^1\}, \oplus_6)$

we can represent the both with cycles



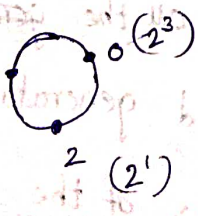


we can see that both the cycles are exactly in reverse directions. It is because 5 and 1 are inverse to each other.

Note:

Thus if 'a' is a generator, then a^{-1} is also a generator.

$\langle 2 \rangle = \{0, 2, 4\}$



Note:

→ Subgroup of a cyclic group is also a cyclic group.

→ A set and operation following
  a) closed then binary structure or binary operation
  b) closed & associative is called semigroup.
  c) closed & associative & identity is called monoid
  d) closed, associative, identity, inverse is called group.

→ A group whose operation is commutative is called abelian group.

Groups Revised:

Binary operation (closed operation):

The binary operator * is said to be a binary operation on a non empty set A, if $(a*b) \in A \quad \forall a,b \in A$.

Algebraic Structure (or) Binary structure:

A non empty set 'A' is called a algebraic structure with respect to a binary operation *

if $(a*b) \in A \quad \forall a,b \in A$

It is denoted as $(A,*)$

Eg: $(N,+)$, $(N,\times)$, $(Z,+)$ $(Z,\times)$, $(Z,-)$ are algebraic structures

$(N,-)$, $(N,\div)$, $(Z,\div)$, $(Q,\div)$ are not algebraic structures

($1/0$ are not defined)

→ $(Q^*, /)$ is an algebraic structure

Q^* is set of non-zero rational numbers.

Semi Group:

An algebraic system $(A, *)$ is said to be a semi group if

1. $*$ is closed operation on A.
2. $*$ is an associative property.

Monoid:

An algebraic system $(A, *)$ is said to be a monoid if:

1) $*$ is closed operation
2) $*$ is an associative property
3) Identity element exists in A.

Group:

An algebraic system $(A, *)$ is said to be a group if the following conditions are satisfied

1) $*$ is a closed operation
2) $*$ is an associative property
3) Identity element exists in A.
4) Every element of A has inverse.

Eg: $(N, \times)$ is a monoid but not a group

$(R - \{0\}, \times)$ is a group i.e., $(R^*, \times)$ is a group.

Slly $(Q^*, \times)$ is a group

Finite group:

A group containing finite number of elements is known as a finite group.

Eg: $(\{0\}, +)$ $(\{1\}, \times)$ $(\{-1, 1\}, \times)$ are examples of finite groups.

→ The only finite group of real numbers with respect to addition is $\{0\}$

→ The only finite group of real numbers w.r.to multiplication is $\{1\}$, $\{-1, 1\}$

→ Cube roots of unity is also a group of order 3. w.r.to multiplication

Note:

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$1^{-1} = 1$

$\omega^{-1} = \omega^2$

$(\omega^2)^{-1} = \omega$

→ In any group (G) of order 2

$$a^{-1} = a, \quad \forall a \in G$$

→ Set $S = \{0, 1, 2 \cdots, m-1\}$ is a group w.r.to addition modulo m

i.e., $(\{0, 1, 2 \cdots m-1\}, \oplus_m)$ is a group

→ Set S_n be set of positive integers which are less than n and relatively prime to n. then

$(S_n, \otimes_n)$ is group where $\otimes_n$ is multiplication modulo n.

$S_6 = (\{1, 5\}, \otimes_6)$ is a group

→ For the previous statement we can conclude that, if p is a prime, then

$(\{1,2,3,\ldots p-1\}, \otimes_p)$ is a group.

Abelian Group:

A group $(G, *)$ is said to be abelian or commutative if $a*b = b*a \quad \forall a,b \in G$

Eg: $(\mathbb{Z}, +)$, $(\mathbb{R}^*, \times)$ are abelian groups

Properties of groups:

→ The identity element of a group is unique.

Proof:

Let us assume there are 2 identity elements e_1, e_2.

$e_2 = e_1 * e_2 = e_1$

(e_2 since e_1 is identity; $\because e_2$ is identity)

$\therefore e_1 = e_2$

→ Inverse of every element is unique.

Proof:

Let $a, b, c \in G$, and

b, c are 2 different inverse of a.

now $b = b \cdot e$ (e is identity element)

$= b \cdot (ac)$ ($\because c = a^{-1}$)

$= (ba)c$

$= e \cdot c = c \Rightarrow \boxed{b = c}$

→ Left cancellation property

let $a, b, c \in G$, then

$$\boxed{ab = ac \Rightarrow b = c}$$

Proof :

$$ab = ac$$

$$a^{-1}(ab) = a^{-1}(ac)$$

$$(a^{-1}a)b = (a^{-1}a)c$$

$$e \cdot b = e \cdot c$$

$$b = c$$

→ Right cancellation property

$$\boxed{ba = ca \Rightarrow b = c}$$

→ Due to right cancellation & left cancellation properties, every row & column in cayley table do not contain repeatition of any element.

In every row & column all elements are appeared exactly once.

→ If $(G, \circ)$ & $(H, *)$ are two groups.

$(G \times H, \cdot)$ is a group where '$\cdot$' is defined as

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \circ g_2, h_1 * h_2)$$

This group is known as direct product of G and H.

Identity element : (e_1, e_2) where e_1 is identity element of G
e_2 is identity element of H.

Inverse of element (g, h) is (g^{-1}, h^{-1})

Grimaldi Questions :

① P.T $G = \{q \in \mathbb{Q} \mid q \neq -1\}$ with respect to binary operation $\circ$ on G by $x \circ y = x + y + xy$ is an abelian group.

② P.T set $\mathbb{Z}$ is not a group under subtraction.

③ P.T set of all one-one functions $g: A \to A$, where $A = \{1, 2, 3, 4\}$ under function composition is a group.

Note:

→ If G is a group, $a, b \in G$

then

$$\boxed{(ab)^{-1} = b^{-1}a^{-1}}$$

Proof:

Let ~~$(ab)^{-1} b^{-1} a^{-1} = x$~~

~~$(ab)^{-1} b^{-1} a^{-1} a = x$~~

~~$(ab)^{-1} (ab) = x$~~

~~$(ab)^{-1} ab\, b^{-1} = x b^{-1}$~~

~~$ab (ba)^{-1} =$~~

$ab(ab)^{-1} = e$

$a^{-1} ab(ab)^{-1} = a \cdot e$

$b(ab)^{-1} = a^{-1}$

$b^{-1} b(ab)^{-1} = b^{-1} a^{-1}$

$(ab)^{-1} = b^{-1} a^{-1}$

$\therefore$ true.

→ Thus a group is said to be abelian if

$$(ab)^{-1} = a^{-1} b^{-1} \quad \forall a, b \in G$$

Order of a group:

It is cardinality of the group.

Order of an element:

The smallest positive integer n such such that $a^n = e$ is called order of 'a'.

Note:

→ $a^0 = e$

→ $a^n = a * a * a * \ldots * a$ (n times)

→ The order of an element of a finite group is a divisor of the order of the group.

→ If no n exists such that $a^n = e$, we say order of 'a' is infinite.

→ Order of identity element, $e = 1$

→ $a^{-n} = (a^{-1})^n = b^n$ where b is inverse of a.

→ Order(a) = Order(a^{-1})

Proof:

Let order of $a = n$

we need to P.T

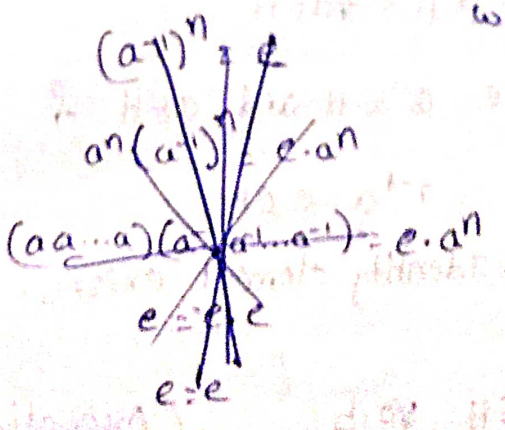
w.k.T

$a^n = e$

$a^n a^{-n} = a^{-n}$

$a^{-n} = e$

$(a^{-1})^n = e$



Subgroup:

A non-empty subset H of a group $(G, *)$ is called subgroup of G, if $(H, *)$ is a group.

Eg: for group $(\{0,1,2,3,4,5\}, \oplus_6)$

$(\{0,2,4\}, \oplus_6)$ is a subgroup

→ For every group G, $\{e\}$ and G are called trivial subgroups
→ Others are called non-trivial or proper subgroups.
→ Every subgroup contain identity element of its parent group

Properties of subgroups:

→ If $H \subseteq G$, then H is called subgroup of G iff

a) $(a*b) \in H \quad \forall a,b \in H$ (H is non-empty)

b) $a^{-1} \in H \quad \forall a,b \in H$

Proof:

for H to be group

(i) $(a*b) \in H \quad \forall a,b \in H$

(ii) Associative (since G is group, binary operation of H which is same as G is also associative)

(iii) $a^{-1} \in H \quad \forall a \in H$

Since $a^{-1} \in H$ and $a \in H$ & $a*b \in H \ \forall a,b$

$a^{-1}a = e \in H$

∴ Identity element exists.

∴ Having $(a*b) \in H \quad \forall a,b$

$a^{-1} \in H \quad \forall a \in H$

is enough to say H is subgroup

(Associativity is inherited from G; e's existence is proved by 2nd stmt)

→ If G is a group and $H \subseteq G$ $(H \neq \emptyset)$

If H is finite, then

H is subgroup of G iff

$(a * b) \in H \quad \forall a, b \in H$

This is theorem 16.3 in Grimaldi. The proof involves cosets.

Note that this applies only if H is finite.

Eg: $(\mathbb{Z}, +)$ is a group

Yet $(\mathbb{N}, +)$ is not a subgroup

because $\mathbb{N}$ is not finite.

if $S \neq \emptyset$ & $S \subseteq G$, $(G, *)$ is group, then S is subgroup $\Leftrightarrow$ $\forall a, b \in S$ $a * b^{-1}$ es

→ If $(G, *)$ is a group then

$\langle a \rangle$ is a subgroup where $a \in G$ and order(a) is finite

Eg: Consider $(\{0,1,2,3,4,5\}, \oplus_6)$

$\langle 1 \rangle$ is calculated as

$1^1 = 1$, $1^2 = 1 \oplus_6 1 = 2$, $1^3 = 1^2 \oplus 1 = 2 \oplus_6 1 = 3$, $1^4 = 4$, $1^5 = 5$, $1^6 = 0$, $1^7 = 1$

$\therefore \langle 1 \rangle = \{0,1,2,3,4,5\}$

$= \{1^0, 1^1, 1^2, 1^3, 1^4, 1^5\}$

$\langle 2 \rangle$

$2^1 = 2$ $2^2 = 2 \oplus_6 2 = 4$ $2^3 = 2^2 \oplus_6 2 = 0$ $2^4 = 2$

$\langle 2 \rangle = \{0, 2, 4\}$.

$\langle 3 \rangle$

$3^1 = 3$, $3^2 = 0$, $3^3 = 3$...

$\langle 3 \rangle = \{0, 3\}$

Here $\langle 0\rangle, \langle 2\rangle, \langle 3\rangle, \langle 1\rangle, \langle 5\rangle$ are called subgroups.

Eg: $\langle 2\rangle = \{0,2,4\}$ is a subgroup

$\langle 3\rangle = \{0,3\}$ is a subgroup

⋮

→ The subgroups of form $\langle a\rangle$ are called generating sets.
But these are not the only possible subgroups.

→ If generating set of 'a' $\langle a\rangle$ is equal to the group G, where $a \in G$, then a is called generator.

Eg: $(\{0,1,2,3,4,5\}, \oplus_6)$

For above group 1, 5 are generators.

→ If 'a' is a generator, then a^{-1} is also a generator.

Eg: In the previous example 1, 5 are inverses to each other.

→ If H is a subgroup of a group G, then

$|H| \mid |G|$ (Lagrange's Theorem)

The converse of lagrange's theorem holds for abelian group

→ If G is a group with $|G| = n$, then number of generators possible to G are $\phi(n)$

where ϕ is euler phi function.

$$\phi(n) = n \cdot \frac{(P_1-1)(P_2-1)\cdots(P_k-1)}{P_1 \cdot P_2 \cdot \cdots \cdot P_k}$$

where $P_1, P_2 \cdots P_k$ are distinct prime divisors of n.

Cyclic group:

→ If H and K are two subgroups of G, then

$H \cap K$ is also a subgroup

$H \cup K$ need not to be a subgroup.

proof:

(i) $(a*b) \in H \cap K \quad \forall a,b \in H \cap K$

$a \in H$ & $b \in H$	$a \in K$ & $b \in K$
$a*b \in H$	$a*b \in K$

$\therefore a*b \in H \cap K$

(ii) $a^{-1} \in H \cap K \quad \forall a \in H \cap K$

$a \in H$	$a \in K$
$a^{-1} \in H$	$a^{-1} \in K$

since a^{-1} is unique for a

$a^{-1} \in H \cap K$

$\therefore H \cap K$ is subgroup

since we are proving $H \cap K$ is subgroup, proving $(a*b) \in H \cap K, \forall a,b \in H \cap K$ $a^{-1} \in H \cap K \quad \forall a \in H \cap K$ is enough.

Cyclic groups:

A group G is called cyclic if $\exists a \in G$ such that every element of G can be written as an integral power of a.

i.e., G is called cyclic group if it contains a generator element.

Eg: $G = \{1, \omega, \omega^2\}$ is a cyclic group w.r.to multiplication

ω, ω^2 are generators.

$G = \{-1, 1, i, -i\}$ is a cyclic group w.r.to multiplication

$i, -i$ are generators.

→ $(Z_n, \oplus_n)$ is a cyclic group $\forall n \geq 2$

$1, n-1$ are generators

→ All subgroups of a cyclic group are cyclic.

→ If G is a cyclic group of g with generator a, then for every divisor d of |a| there exists exactly one subgroup of order d.

This subgroup is generated by $a^{|a|/d}$.

Eg: $G = \{0,1,2,3,4,5\}$ is group under $\oplus_6$

$|a| = 6$

3 is divisor of 6 ($|G|$)

∴ we have a unique subgroup of order 3 which is generated by $a^{|a|/d} = a^{6/3} = a^2$

Here a is 1 or 5 (generators)

∴ The subgroup is generated by 1^2.

$(1^2)^1 = 1^2 = 2$

$(1^2)^2 = 1^4 = 4$

$(1^2)^3 = 1^6 = 0$

$(1^2)^4 = 1^8 = 2$

⋮

$\langle 1^2 \rangle = \{0,2,4\}$ is a unique subgroup of size 3.

→ If G is a group of compositie order, then G has a non-trivial subgroups.

Homomorphism:

→ If $(G,*)$ and $(G',\oplus)$ are two groups than a function $f: G \to G'$ is called a homomorphism

if $f(a*b) = f(a) \oplus f(b)$

If f is a bijection, then the homomorphism f is called an isomorphism between G and G'.

we write it as $G \equiv G'$

i.e., G is isomorphic to G'.

→ If two groups are isomorphic, it means that both the group are same with different names.

Ex: $(\{0,1,2,3\}, \oplus_4)$ is isomorphic to $(\{1,-1,i,-i\}, \times)$

The isomorphism f is defined as

$f(0) = 1$

$f(1) = i$

$f(2) = -1$

$f(3) = -i$

we can see that cayley table's for both are same and have one to one correspondence.

$\oplus_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Here generators are 1 & 3

×	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

Here generator are i, -i

i.e., f(1) & f(3)

Problems on Groups:-

P/57 $A * B = A \oplus B$

$A \in P(S) \quad B \in P(S)$

$\Rightarrow A \subseteq S, \; B \subseteq S$

$\therefore A \oplus B \subseteq S$ i.e., closed

$(A \oplus B) \oplus C \equiv A \oplus (B \oplus C)$ i.e., Associative

$\forall A \in P(S)$

$A \oplus \phi = A$

$\therefore \phi$ is identity element

$\forall A \in P(S)$

$A \oplus A = \phi$

$\therefore A^{-1} = A$

i.e., Every element has inverse

$\therefore$ group

P/58 $a * b = \frac{ab}{2} \quad (R - \{0\}, *)$

Let e be identity element

$\frac{ae}{2} = a$

$e = 2$

to find invere of 4

$4^{-1} = x$

$4x = e$

$4 * x = \frac{4x}{2} = 2$

$4x = e = 2 \Rightarrow x = 2/4 = 1/2$

$2x = 2 \Rightarrow x = 1$

$\therefore 4^{-1} = 1$

P/59 $a * b = 2ab$

Let e be identity element

$a * e = a$

$2ae = a$

$e = 1/2$

To find inverse of $2/3$

$\left(\frac{2}{3}\right)^{-1} = x$

$\frac{2}{3} * x = e$

$2\frac{2}{3} x = \frac{1}{2}$

$x = \frac{3}{8}$

$\therefore \left(\frac{2}{3}\right)^{-1} = \frac{3}{8}$

P/60 Given binary operation

$a * b = a + b + ab$

a) Finding identity

$a * e = a$

$a + e + ae = a$

$e + ae = 0$

$e = 0$

$\therefore 0$ is the identity

b) finding invere

let $a^{-1} = b$

$a * b = e$

$a + b + ab = 0$ → $b = \frac{-a}{a+1}$ for $a = -1$, $b = \frac{1}{0}$

for $a = -1$, $-1 + b - b = 0$

$-1 = 0$ ∴ inverse doesn't exists

∴ b is false

c) $a*b = 0 \quad a^{-1} = b$

$a + b + ab = 0$

$b = \frac{-a}{a+1}$

true

d) As -1 has no inverse

R is not a group

P/61 a) Let group of order 3 be $\{e, a, b\}$

possible groups are

i) $e^{-1} = e \quad a^{-1} = a \quad b^{-1} = b$

ii) $e^{-1} = e \quad a^{-1} = b \quad b^{-1} = a$

i)

	e	a	b
e	e	a	b
a	a	e	b
b	b	a	e

Here

$ab = b$

$ba = a$

∴ ~~not commutative~~

∴ ~~not abelian~~

ii)

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Here $ab = e$

$ba = e$

It is commutative

Every group of order less than or equal to 5 is abelian

. Here (i) itself is not a group because associativity doesn't hold

$(ab)a = a(ba)$

$(b)a = a(a)$

$a = e$

∴ not associative

∴ every group of order 3 is abelian

b) $\{1,-1\}$ under $\times$ is a group

c) let $(\{a,b\},*)$ be group

let 'a' be identity element

Now $a^{-1}=a$

since every element has inverse

b^{-1} must be b

$\therefore$ true

d) $(\{1,-1,i,-i\},\times)$ is a group under multiplication.

$\therefore$ false

P/62 a) $a*a=e$

also w.k.t $a*e=e$

$a*a=a*e$

$a=e$ ($\because$ left cancellation property)

b) let $a,b\in G$

$a^{-1}=a$ $b^{-1}=b$, $ab\in G$

$(ab)^{-1}=ab$

Consider $ab=(ab)^{-1}$

$=b^{-1}a^{-1}$

$=ba$

i.e., $ab=ba$

$\therefore$ abelian group

c) Let $(\{e, a_1, a_2, a_3, a_4, a_5\}, *)$ be a group

Now we have

$e^{-1} = e$

Let us assume that

$a_1^{-1} = a_2$

$a_3^{-1} = a_4$

Now a_5^{-1} must be a_5 only

∴ true

d) from problem (61) option (a)

it is false.

(P/63) $2 \oplus_6 4 = 6 \equiv 0$

$3 \oplus 3 = 6 \equiv 0$

$5 \oplus 2 = 7 \equiv 1$

$1 \oplus 5 = 6 \equiv 0$

∴ opt (c)

(P/64) $(-i)^2 = i^2 = -1$

$(-i)^3 = (-1)(-i) = i$

$(-i)^4 = i(-i) = 1$ (Identity element)

$\therefore O(-i) = 4$

(P/65) (a) & (b) are already proved

a is false.

c) It is set of multiples of k

$nk + mk = (n+m)k$ ∴ close

Addition is associative

0 is identity

for nk, -nk is inverse

∴ group

d) Sub group of abelian group is abelian because commutativity always holds.

(P/66) a) ~~Ex~~ If 'a' is generator every element can be represented as a^k.

Consider

$$bc = cb$$

$$a^m \cdot a^n = a^n \cdot a^m$$

$$a^{m+n} = a^{m+n}$$

∴ commutative

Hence abelian

b) If G is a group of order n and 'a' is a generator

$$a^k = a^k \cdot e$$

$$= a^k \cdot e^{-1}$$

$$= a^k \cdot (a^n)^{-1}$$

$$= a^k \cdot a^{-n}$$

$$= a^{k-n}$$

$$= (a^{-1})^{n-k}$$

∴ a^{-1} is also a generating element

c) If order of generation element is not n then it cannot generate all the elements ($\because a^k = e$, $k < n$)

d) Subgroup of a cycle group is always cyclic

∴ ~~fa~~ false

P/67

a) Every element of a prime order group is a generator.

proof:

Let G be a group of prime order

Let $a \in G$

Let $\langle a \rangle = H$

Now H is subgroup of G

$|H| / |G|$

since $|G|$ is prime

$|H| = 1$ or $|H| = |G|$

for elements other than identity

$\langle a \rangle = H$ then $|H| \geq 2$

because H contains both a & e

$\therefore |H| = |G|$

Hence the group is cyclic.

$\therefore$ true.

b) Same as P/62-(b)

c) True (every group of order less than or equal 5 is abelian)

d)

$2^1 = 2$

$2^2 = 4$

$2^3 = 8 \equiv 1$

$2^4 = 16 \equiv 2$

2 is not a generator.

$\therefore$ opt-(d) is false

P/68 a) $4 \oplus_6 4 = 8 = 2 \notin \{0,4\}$

∴ not a subgroup

b) $\{1,5\}$ has no identity

∴ not a subgroup

c) $\{0,2,4\}$ is a subgroup

d) $\{1,3,5\}$ has no identity element

∴ not subgroup

P/69 order of Group = 10

no of generators = $\phi(10)$

$10 = 2 \times 5$

$\phi(10) = 10 \dfrac{(2-1)(5-1)}{2 \cdot 5}$

$= 4$

P/70

$\otimes_{10}$	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

(i) from table it is closed

(ii) Multiplication is associative

(iii) 6 is identity

(iv) $2^{-1} = 8 \quad 4^{-1} = 4 \quad 6^{-1} = 6$

$8^{-1} = 2$

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

$2^4 = 16 \equiv 6$

$8^1 = 8$

$8^2 = 64 \equiv 4$

$8^3 = 512 \equiv 2$

$8^4 = 4096 \equiv 6$

∴ 2, 4 are generators

(i) is false

P/71

*	a	b	c	d
a	b	d	a	c
b	d	c	b	a
c	a	b		
d	c	a		

every 4-element group is abelian

From above figure only 'c' has chance to be identity element thus $[c,c]=c$ $[c,d]=d$ and from that

$[d,c]=d$ $[d,d]=b$

*	a	b	c	d
a	b	d	a	c
b	d	c	b	a
c	a	b	c	d
d	c	a	d	b

identity is c

$a^{-1}=d$ $b^{-1}=b$ $c^{-1}=c$

$d^{-1}=a$

$\therefore$ opt(a) is true

Questions:

1) S.T in any boolean algebra, for any a and b, $a \le b$ iff $\bar{b} \le \bar{a}$

2) S.T in a boolean algebra, for any a and b

$$a=b \Leftrightarrow (a \wedge \bar{b}) \vee (\bar{a} \wedge b) = 0$$

3) S.T in any boolean algebra for any a, b, c

i) $a \le b \Rightarrow a \vee c \le b \vee c$

ii) $a \le b \Rightarrow a \wedge c \le b \wedge c$