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Linear Algebra

(1-3M)

Topics:

- → Determinants
- → Inverse
- → Rank, linearly independent & dependent vectors
- → Solution of system of linear equations.
- → Eigen values, Eigen vectors & Cayley Hamilton theorem.

Determinants:

Matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

(or)

$A = [a_{ij}]_{m \times n}$ (or) $A = [a_{ij}]$ where $1 \le i \le m$, $1 \le j \le n$

2nd order determinant:

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

the expression $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is called a 2nd order determinant of $A_{2\times 2}$. It is denoted by $|A|$ (or) $\det(A)$. It is given by

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

3rd order determinant :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$(a_{ij}) \rightarrow (-1)^{i+j}$$

Method 2 :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

rearrange as shown below

$$\begin{array}{c|ccc|c} a_{13} & a_{11} & a_{12} & a_{13} & a_{11} \\ & a_{21} & a_{22} & a_{23} & \\ a_{33} & a_{31} & a_{32} & a_{33} & a_{31} \end{array}$$

Now we draw 6 lines as shown below.

④ ⑤ ⑥

$$\begin{array}{c|ccc|c} a_{13} & a_{11} & a_{12} & a_{13} & a_{11} \\ & a_{21} & a_{22} & a_{23} & \\ a_{33} & a_{31} & a_{32} & a_{33} & a_{31} \end{array}$$

① ② ③

① ② ③ ④ ⑤ ⑥ indicate ~~sum~~ product of all terms in that line.

i.e., ① ... $a_{13}\, a_{21}\, a_{32}$

⋮

$$|A| = [① + ② + ③] - [④ + ⑤ + ⑥]$$

4th order determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

$$- a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

Elementary operations / Transformations:

i) $R_i \leftrightarrow R_j$: Interchanging two rows.

ii) $R_i \to kR_i$; Multiplying all elements of row R_i with k. $(k \neq 0)$

iii) $R_i \to R_i + kR_j$ $(k \neq 0)$

Similar operations can be done on columns too.

Note :

$R_i \to R_i + kR_j$ can be applied on matrix or on ~~not~~ determinant as shown below

$R_i \to R_i + kR_j$

A (matrix) | $|A|$ (det)

$R_i \to R_i + kR_j$ | $R_i \to lR_i + kR_j$ | $R_i \to R_i + kR_j$

↳ Apply this on $|A|$ changes the value of $|A|$

(Q1) The value of the determinant of a matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -3 & 3 \\ 0 & 2 & 4 \end{bmatrix}$ is

Sol :

$$\begin{array}{c} \qquad\quad 0 \quad 0 \quad 12 \\ + \begin{vmatrix} 2 & 0 & 1 \\ 4 & -3 & 3 \\ 0 & 2 & 4 \end{vmatrix} \begin{matrix} 2 \\ \\ 0 \end{matrix} \\ \cdot 4 \\ \qquad 8 \quad -24 \quad 0 \end{array}$$

$\therefore |A| = [8-24+0] - [0+0+12]$

$= -18-12 = \cancel{30}\ -28$

(Q2) The value of the determinant of the matrix

$$A = \begin{vmatrix} 1 & 4 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 2 \end{vmatrix}$$

$$(1)\begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 2 \end{vmatrix} - 4\begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & -1 & 2 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{vmatrix} - (0)\begin{vmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{vmatrix}$$

$(1)(-1) - 4(0) + 1[-2(0) + 1(-2)] - 0$

$-1 - 0 - \cancel{1} = \cancel{4}\ -3$

Method 2 :

(*) Select a row or column with more no of zeroes and expand along that row.

$\therefore$ we select row 3.

$$(-1)^{3+2}(1)\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{vmatrix}$$

↳ expand along 2nd row

$(-1)[(-1)(-1-2)] = -3$

Q3 $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$. Find $|A|$

Sol :

Expand along 2nd row

$$(-1)(1)\begin{array}{c|ccc|c} 3 & 1 & 2 & 3 & 1 \\ & 3 & 0 & 1 & \\ 2 & 0 & 1 & 2 & 0 \end{array} + (-1)(3)\begin{array}{c|ccc|c} 3 & 0 & 1 & 3 & 0 \\ & 2 & 3 & 1 & \\ 2 & 3 & 0 & 2 & 3 \end{array}$$

$$= (-1)\left[(9+0+0)-(12+0+1)\right] - (3)\left[(0+0+3)-(4+27+0)\right]$$

$$= (-1)[-4] - 3[-28]$$

$$= +4 + 84 = 88$$

★ Method 2 :

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$$

Generally to make row elements 0's we apply column operations and to make column elements 0's we apply row operations.

$C_3 \to C_3 - 3C_1$

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 3 & -6 & 1 \\ 3 & 0 & -8 & 2 \end{vmatrix} = (-1)^{2+1}(1)\begin{array}{c|ccc|c} 3 & 1 & 2 & 3 & 1 \\ & 3 & -6 & 1 & \\ 2 & 0 & -8 & 2 & 0 \end{array}$$

$$= (-1)\left[(-72-12+0)-(12+0-8)\right]$$

$$= (-1)[-84-4] = 88$$

Q4) Let A be a sequare matrix of order $(n-1)$. The elements of A are defined by $a_{ij} = \begin{cases} n-1, & \text{for } i=j \\ -1 & \text{for } i \neq j \end{cases}$

a) n^{n-1} b) n^{n-2} c) $(n-1)^{n-2}$ d) $(n-1)^{n-3}$

Sol :

we choose matrix of order 2

i.e., $n-1=2 \Rightarrow n=3$

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-(1) = 3$$

option verification

a) $n^{n-1} = 3^2 = 9$

b) $n^{n-2} = 3^1 = 3$

c) $(n-1)^{(n-2)} = 2^1 = 2$

d) $(n-1)^{n-3} = 2^0 = 1$

$\therefore$ opt (b)

Q5) If A is $m \times n$ matrix & B is $n \times P$ matrix then the no of multiplications ~~involve~~ & additions involved in computing product AB are ___

a) mnp, $mnp+1$ b) $m(n-1)p$, $mn(p+1)$

c) $(m-1)np$, mnp d) none of these

Sol :

$A_{m \times n} \times B_{n \times p} = C_{m \times p}$

$\therefore$ C will have mp no of element.

each element of C is computed with n multiplications and n-1 additions.

$\therefore$ no of multiplications $= n(mp)$

no of additions $= (n-1)mp$

$\therefore$ opt (d)

Properties of determinants:

→ $|AB| = |A||B|$

→ $|A^k| = |A|^k \quad \forall k \in N$

Note:

$|A^k| = |A|^k \quad \forall k \in I \; : |A| \neq 0$ (for negative numbers)

→ $|A^T| = |A|$

→ In a matrix A, if atleast one row or atleast one column is all zeroes, then $|A| = 0$.

→ In a matrix A, if two rows or two colums are same or proportional then $|A| = 0$

→ If any two rows are two columns are interchanged, then determinant value of new matrix will be negative of determinant of previous matrix.

Thus odd no of interchanges ---- sign of det changes

even no of interchanges ---- sign of det remains same.

→ $k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix} = \cdots$

Slly we can take out a scalar value as common from either a row or a column.

Note:

when we multiply scalar with matrix, we need to multiply it with every element of the matrix.

i.e., $k\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$

$\rightarrow \left| k A_{n \times n} \right| = k^n \left| A_{n \times n} \right|$

$\rightarrow R_i \rightarrow R_i + k R_j$

Applying this operation doesn't change the value of determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = \begin{vmatrix} 1+4k & 2+5k & 3+6k \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix}$$

$R_1 \rightarrow R_1 + k R_2$

Note :

$R_i \leftrightarrow R_j$: changes sign of determinant

$R_i \rightarrow k R_i$: raises value of determinant k times.

$R_i \rightarrow R_i + k R_j$: value of determinant remains unchanged.

$\rightarrow$ Determinant of an upper triangular or a lower triangular (or) diagonal matrix is equal to product of it diagonal elements.

i.e., $\begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} = adf$

$\begin{vmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{vmatrix} = acf$

$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ ---- diagonal matrix

$\begin{vmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix} = k^3$ ----- scalar matrix

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$ -------- identity matrix

$$\rightarrow \begin{vmatrix} a_1+x_1+y_1 & b_1 & c_1 \\ a_2+x_2+y_2 & b_2 & c_2 \\ a_3+x_3+y_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} y_1 & b_1 & c_1 \\ y_2 & b_2 & c_2 \\ y_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1+x_1+y_1 & a_2+x_2+y_2 & a_3+x_3+y_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}'$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Inverse of a square matrix:

Minor & cofactor of an element:

If $A = (a_{ij})_{n \times n}$ is a square matrix then,

(i) minor of an element "a_{ij}" is

$M_{ij} = (n-1)^{th}$ order determinant which remains after deleting ith row and jth column in the matrix.

(ii) the cofactor of an element "a_{ij}" is $= (-1)^{i+j} M_{ij}$

Eg: $A = \begin{bmatrix} 1 & 2 & 7 \\ 4 & 3 & 5 \\ 0 & 6 & -1 \end{bmatrix}$

Here $a_{23} = 5$

Minor of $a_{23} = \begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix} = 6 - 0 = 6$

Cofact of $a_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix} = -6$

Cofactor matrix:

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Let A_{ij} be cofactor of a_{ij}

th. let $B = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Here B is called cofactor matrix of A.

Adjoint matrix:

If $B_{n\times n}$ is a cofactor matrix of $A_{n\times n}$ then adjoint matrix of $A_{n\times n}$ is denoted by adj(A) and it is defined as

$$\text{adj}(A) = B^T$$

Singular matrix

→ A is called singluar matrix, if $|A| = 0$

non-singular matrix:

→ A is called non-singular matrix, if $|A| \neq 0$

Inverse (or) reciprocal of a square matrix:

If $AB = BA = I_n$ for $|A| \neq 0$ & $|B| \neq 0$ then B is called inverse of matrix A. It is denoted as A^{-1}.

Note:

→ If $|A| \neq 0$ then

$$\boxed{A^{-1} = \frac{1}{|A|}\text{adj}(A)} \quad \& \quad \boxed{|A^{-1}| = \frac{1}{|A|}}$$

The matrix for which A^{-1} exists is called invertible matrix.

→ If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

→ If A is any square matrix of order n then

(i) $A \cdot adj(A) = adj(A) \cdot A = |A| \cdot I_n$

(ii) $|adj(A)| = |A|^{n-1}$ (n is order)

(iii) $|adj(adj(A))| = |A|^{(n-1)^2}$

Slly $\underbrace{|adj(adj(\ldots adj(A)))\ldots)|}_{k \text{ times}} = |A|^{(n-1)^k}$

→ If $|A| \neq 0$ then

$adj(A) = |A| \cdot A^{-1}$

$adj(adj(A)) = |A|^{n-2} \cdot A$

→ Symmetric matrix ------ $A^T = A$ (or) $a_{ij} = a_{ji}$

Skew symmetric matrix ----- $A^T = -A$ (or) $a_{ij} = -a_{ji}$

Orthogonal matrix ----- $AA^T = A^TA = I_n$ (or) $A^{-1} = A^T$

$a_{ii} = -a_{ii}$

$\Rightarrow a_{ii} = 0$

$\therefore$ diagonal element of a skew symmetric matrix are 0's

Eg: $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 8 & -5 \\ 4 & -5 & 3 \end{bmatrix}$ ---- symmetric matrix

$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix}$ ---- skew symmetric matrix

$A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$ ... orthogonal matrix

→ Every odd order determinant of a skew symmetric matrix is ~~add~~ always 0.

Proof: $A^T = -A \Rightarrow |A^T| = |-A| \Rightarrow |A| = (-1)^n |A|$

$\Rightarrow |A| = -|A| \Rightarrow |A| = 0$

i.e., Every odd ordered skew symmetric matrix is singular.

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Q6) The inverse of matrix $A = \begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$ where $i^2 = -1$ is ______

Sol:

First we check whether matrix is singular or not

$$\begin{vmatrix} 4+3i & -i \\ i & 4-3i \end{vmatrix} = (4+3i)(4-3i) - (-i^2)$$

$$= 25 - 1 = 24 \neq 0$$

∴ inverse exist

$$A^{-1} = \frac{1}{24}\begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$$

Q7) The inverse of the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ is ______

Sol:

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix} = 2(10) - 3(15) + 4(5)$$

$$= 20 - 45 + 20 = -5 \neq 0$$

∴ inverse exists

Cofactor matrix is $\begin{bmatrix} 10 & -15 & 5 \\ -4 & 4 & -1 \\ -9 & 14 & -6 \end{bmatrix}$

$$\text{adj}(A) = \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{-5}\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

Method 2: (Shortcut for adj A)

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

~~start~~ Start from 2nd row middle element and form the 1st column of below matrix

Now do the same for row 3.

Now do the same for row 1

Now again do the same for row 2.

2nd row of adj A

$$\begin{bmatrix} 3 & 2 & 3 & 3 \\ 1 & 4 & 4 & 1 \\ 4 & 1 & 2 & 4 \\ 3 & 2 & 3 & 3 \end{bmatrix}$$

1st row of adj A

3rd row of adj A

Now from this 4x4 matrix we find adj(A)

Cofactor - write as rows.
adj A - write as columns

$$adj(A) = \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{-1}{5} \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

(Q8) If $adj(A) = \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$ then the absolute value of the determinant of A is ______

Sol:

$$|adj(A)| = |A|^{n-1}$$

$$\begin{array}{c|ccc|c} -10 & -18 & -11 & -10 & -18 \\ & 2 & 14 & -4 & \\ -8 & 4 & 5 & -8 & 4 \end{array}$$

$\Rightarrow |adj A| = (-160 + 2016 + 176) - (176 - 560 + 360)$

$= 2116$

$\therefore |A|^2 = 2116$

$\Rightarrow |A| = \pm 46$

absolutive value of $|A|$ is 46

(Q9) If $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 3 \\ 1 & 3 & -1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ then $|AB - BA| =$ ______

Sol:

Here A & B are symmetric matrices $\Rightarrow A^T = A,\ B^T = B$

Consider $(AB - BA)^T = (AB)^T - (BA)^T$

$= B^T A^T - A^T B^T$

$= BA - AB$

$(AB - BA)^T = -(AB - BA)$

$\therefore (AB - BA)$ is skew symmetrice

$\therefore |AB - BA| = 0$

(Q10) If A & B are symmetric matrices of order 3x3, then consider the following $S_1: AB = BA$

$S_2: AB - BA$ is singular

which of the following is true ?

a) S_1 is true & S_2 is false

b) S_1 is false & S_2 is true

c) both S_1 & S_2 are true

d) both S_1 & S_2 are false.

Sol:

$A^T = A \quad B^T = B$

Consider $(AB - BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA - AB$

$= -(AB - BA)$

$\Rightarrow$ ~~AB-B~~ is AB - BA is skew symmetric

$\Rightarrow |AB - BA| = 0$

$\Rightarrow$ AB - BA is singular

$\therefore S_2$ is true

S_1 : $AB = A^T B^T = (BA)^T$

$BA = B^T A^T = (AB)^T$

$(BA)^T \neq (AB)^T$

$\therefore S_1$ is false

$\therefore$ opt (b)

(Q11) The number of different $n \times n$ symmetric matrices with each element being either 0 or 1 is ——

sol :

total no of elements $= n^2$

Every diagonal element has 2 ways .... 2^n

now remaining elements form $\frac{n^2-n}{2}$ pair

$\Rightarrow 2^{\frac{n^2-n}{2}}$

$\therefore 2^n \cdot 2^{\frac{n^2-n}{2}} = 2^{\frac{n^2+n}{2}}$

(Q12) The no of $n \times n$ symmetric matrices possible with entries choosen from the set $\{0, 1, 2 \ldots (q-1)\}$ is

sol :

$$\underbrace{q^n}_{\text{diagonal element}} \; \underbrace{q^{\frac{n^2-n}{2}}}_{\text{remaining elements}} = q^{\frac{n^2+n}{2}}$$

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Rank, Linearly Independent & dependent vector:

Echelon form:

A matrix $A_{m\times n}$ is said to be in echelon form if

(i) the no of 0's before the first non-zero element in a row is less than the no of such zeroes in the next non-zero row.

(ii) the zero rows must be below the non-zero rows.
↳ rows with all elements as zeroes.

→ The rank of matrix $A_{m\times n}$ is denoted by $\rho(A)$ (or) r

→ If $A_{m\times n}$ is in echelon form then

Rank, $\rho(A_{m\times n})$ = no of non-zero rows in the echelon form

Eg:

$$A = \begin{bmatrix} 0 & 4 & 7 & 2 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

no of 0's in row i before a non-zero element be z_i

Here $z_1 < z_2 < z_3$

$\therefore$ A is in echelon form

$\therefore$ Rank $(A) = 3$

Eg:

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 7 & 3 \\ 0 & 0 & 8 \end{bmatrix}$$

~~Here~~ A is in echelon form

$\therefore$ Rank$(A) = 3$

Eg: $A = \begin{bmatrix} 0 & 4 & 7 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$

A is in echelon form

no of non-zero rows = 2.

$\therefore$ Rank(A) = 2

Eg: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

A is in echelon form

no of non-zero rows = 0

$\therefore$ Rank(A) = 0

Note:

Rank(A) = 0 $\iff$ A is a null matrix

Eg: $A = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, Rank(A) = 1

→ If we need to find rank of a matrix which is not in echelon form, we first reduce the matrix in to echelon form using row operation. Now we find the rank.

→ Echelon form obtained by applying row operations is called row echelon form.

Slly we have column echelon form.

Eg: $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ -4 & -2 & 7 & 9 & 5 & 6 \\ 3 & 10 & 2 & -9 & -3 & -2 \\ 6 & 7 & 13 & 14 & 2 & 1 \end{bmatrix}$

Here A is not in echelon form.

Note :

→ $\rho(O_{m\times n}) = 0$

→ $\rho(A^T) = \rho(A)$

→ if $A \neq O$, then $\rho(A) \geq 1$

→ If $A_{mn} \neq 0$ then $\rho(A_{m\times n}) \leq \min\{m, n\}$

→ ~~If~~ $|A_{n\times n}| \neq 0$ ~~then~~ $\Longleftrightarrow$ $\rho(A_{n\times n}) = n$

→ $|A_{n\times n}| = 0 \Longleftrightarrow \rho(A_{n\times n}) < n$

→ $\rho(I_n) = n \quad (\because |I_n| = 1 \neq 0)$

*→ $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

*→ $\rho(A+B) \leq \rho(A) + \rho(B)$

*→ $\rho(A-B) \geq \rho(A) - \rho(B)$

*→ If

(i) $\rho(A_{n\times n}) = n$ then $\rho(adj(A)) = n$ & $\rho(A^{-1}) = n$

(ii) $\rho(A_{n\times n}) = n-1$ then $\rho(adj(A)) = 1$

↳ All the rows of adj A are proportional if $\rho(A) = n-1$

(iii) $\underbrace{\rho(A_{n\times n}) \leq (n-2)}_{\text{↳ adj A is null matrix}}$ then $\rho(adj(A)) = 0$

→ If $A = (a_{ij})_{m\times n}$ where $a_{ij} = k$

i.e., all elements are same

then $\rho(A) = 1$

→ If all the rows of a matrix A are proportional then $\rho(A) = 1$

(Q13) The rank of a matrix $A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$ is ____

Sol :

$$\begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & -3 & 3 & 9 \end{bmatrix}$$

$R_3 \to 2R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank $(A) = 2$

(Q14) Let $A = \begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$. If rank of A is 1 then $p =$ ____

Sol :

A is square matrix & $\rho(A) = 1$

$\therefore |A| = 0$

$$\begin{array}{c} p \\ \\ 3 \end{array} \begin{vmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{vmatrix} \begin{array}{c} 3 \\ \\ p \end{array} = 0$$

$\Rightarrow \cancel{(p^3 + 27 + p^3)} - \cancel{(3p^2 + 3p^2 + 3p^2)} = 0$

$\cancel{2p^3 - 9p^2 + 27 = 0}$

value of determinant is 0 if two rows are same or proportionate

$\therefore p = 3$

(Q15) Let $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ where a, b, c are non-zero real numbers.

**

Then find rank of A.

Sol :

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$A = \begin{bmatrix} -a & 0 & c \\ 0 & a & b \\ -b & -c & 0 \end{bmatrix} \quad R_3 \to aR_3 - bR_1$$

$$= \begin{bmatrix} -a & 0 & c \\ 0 & a & b \\ 0 & -ac & -bc \end{bmatrix} \quad R_3 \to R_3 + cR_2$$

$$= \begin{bmatrix} -a & 0 & c \\ 0 & a & b \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \text{Rank}(A) = 2$

(Q16) Suppose that $A_{n \times n}$ is upper triangular matrix such that $a_{ii} = 0$, $i = 1, 2, 3, \dots n$. Then the rank of $A^n =$ ____

a) 0 b) 1 c) $n-1$ d) n

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$\therefore$ ~~Rank(A) = n-1~~

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore \text{Rank}(A^2_{2\times 2}) = 0$

Slly $\text{Rank}(A^n) = 0$

let $A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{Rank}(A^3) = 0$

Slly we get $A^n = O_{n \times n}$

$\therefore \text{Rank}(A^n) = 0$

Linearly independent & dependent vector

A vector $x\hat{i} + y\hat{j} + z\hat{k}$ can be represented as

(i) $[x \ y \ z]$ ... row vector

(ii) $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ..... column vector

consider below pairs of vectors

$x_1 = [1 \ -4 \ 7]$ $x_2 = [2 \ 5 \ 6]$	$x_1 = [1 \ 2 \ 3]$ $x_2 = [2 \ 4 \ 6]$ $x_2 = 2x_1$	$x_1 = [2 \ 4 \ 7]$ $x_2 = [2 \ 4 \ 7]$ $x_1 = x_2$
here x_1 & x_2 are called linearly independent vectors	Here x_1 & x_2 are said to be linearly dependent	

→ If $x_1, x_2, \ldots x_n$ are row vectors of same order n

(i) $\rho(A)$ = no of given vectors or $= n$ then

$x_1, x_2, \ldots x_n$ are linearly Independent (LI) vectors

(ii) $\rho(A) \neq$ no of given vectors (n) then

$x_1, x_2, \ldots x_n$ are linearly dependent (L.D) vectors

where

$$A = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times m} \quad \text{(or)} \quad A = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}_{m \times n}$$

↳ Here $x_1, x_2 \ldots x_n$ are column vectors.

Eg: Check whether the following vectors are L.I (or) L.D ?

$x_1 = [1 \ -2 \ 4]$ $x_2 = [-2 \ 1 \ 0]$ & $x_3 = [-1 \ 2 \ 7]$

Sol:

Let $A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 1 & 0 \\ -1 & 2 & 7 \end{bmatrix}$

If all the vectors are same or proportionate then the vectors are L.D. Otherwise we need to follow the procedure

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & -3 & 8 \\ 0 & 0 & 11 \end{bmatrix}$$

Rank is same as no of independent vectors

Rank$(A) = 3$ = no of given vectors

$\therefore$ the vectors are linearly independent.

Q17) Check whether the following vectors are LI or LD ?

$x_1 = [1 \ -2 \ 4]$ $x_2 = [2 \ 0 \ 5]$ $x_3 = [-2 \ 4 \ -8]$

Sol.

Let $A = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 0 & 5 \\ -2 & 4 & -8 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + 2R_1$

$$= \begin{bmatrix} 1 & -2 & 4 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank$(A) = 2 \neq$ no of given vectors → Here out 3 vectors(rows), 2 rows are independent.

$\therefore x_1, x_2$ & x_3 are L.D vectors.

Note :

→ $\rho(A_{m\times n}) = r$ = no of LI vectors

→ If $|A_{n\times n}| \neq 0$ then all the vectors are LI

→ If $|A_{n\times n}| = 0$ then the vectors are linearly dependent.

→ Every subset of a LI set is also a LI set.

→ Every subset of a LD set may or may not be a LD set.

Q18 Consider the following statements

S1: If $\{x_1\ x_2\ x_3\ x_4\}$ is a LI set of vectors then the set $\{x_1, x_2, x_3\}$ is a LI set of vectors.

S2: If $\{x_1\ x_2\ x_3\ x_4\}$ is a LD set of vectors then the set $\{x_1\ x_2\ x_3\}$ is a LD set of vectors.

which of the following is true?

a) only S_1 b) only S_2

b) both S_1 & S_2 d) none

Q19 Let A & B be two n×n matrices over real numbers. Let rank(M) & det(M) denote the rank & determinant of a matrix M respectively. Consider the following statements.

I: rank(AB) = rank(A)· rank(B)

II: det(AB) = det(A) det(B)

III: rank(A+B) ≤ rank(A) + rank(B)

IV: det(A+B) ≤ det(A) + det(B)

which of the above are true

a) I & II b) I & IV c) II & III d) III & IV

Sol:

opt ⓒ (from properties)

Solution of system of linear equations:

→ The equation in which degree of all terms is same is called homogeneous equation.

Eg : $2x+4y=0$ - linear homogeneous equation

$2x+3y+4=0$ -- linear non-homogeneous equation.

Non-homogeneous system:

Consider
$$\left.\begin{aligned} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n&=b_1\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n&=b_2\\ &\vdots\\ a_{m1}+a_{m2}x_2+\cdots+a_{mn}x_m&=b_m \end{aligned}\right\}\text{ Collection of } m \text{ equations in } n \text{ variables.}$$

This is represented as

$$AX=B$$

where

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}_{m\times n} \quad X=\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}_{n\times 1} \quad B=\begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_m\end{bmatrix}_{m\times 1}$$

Coefficient matrix

variable matrix (or) soln matrix

Constant matrix

Method of finding the solution of the system

Step 1: Express the given system of algebraic linear equations in matrix form as $AX=B$

Step 2: Consider the augumented matrix $(A|B)$ & reduce it to echelon form by applying only row opererations.

Step 3: Find $\rho(A)$, $\rho(A|B)$ & n = no of variables

(i) if $\rho(A) = \rho(A|B) = n$ then

$AX = B$ is consistent & it has unique soln.

(ii) if $\rho(A) = \rho(A|B) \neq n$ then

$AX = B$ is consistent & it has infinite no of solutions

(iii) If $\rho(A) \neq \rho(A|B)$ then

$AX = B$ is inconsistent and hence no solution.

Step 4: If the soln exists then rewrite the system of linear equations and find the soln by backward substitution.

Here augmented matrix

$$(A|B) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}_{m \times (n+1)}$$

Note:

$AX = B$

$\downarrow$

$A_{n \times n}$

- $|A| \neq 0$
  - unique soln
- $|A| = 0$
  - no soln
  - unique soln

Homogeneous System:

If $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ then B is zero vector and

$AX = B$ is called a homogeneous system.

It is denoted as $AX = 0$

~~we the solution vector is~~

$X = \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}}$

~~Here X is called trivial solution.~~

Eg: Consider $x - 2y = 0$ --- (1)
$2x + y = 0$ --- (2)

from (1) $x = 2y$

(2) $\Rightarrow 4y + y = 0 \Rightarrow y = 0$

from (1) $x - 2(0) = 0 \Rightarrow x = 0$

$\therefore X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Here x is called trivial soln (or) zero soln or unique soln

Eg: Consider $x + 2y = 0$
$2x + 4y = 0$

Here the two equations are proportional

This means the system represents only one equation

from (1) $x = -2y$

Put $x = 0$

$$\Rightarrow X_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Big\} \text{ zero/trivial soln}$$

Put $y = -1$

$$\Rightarrow X_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Put $y = 2 \Rightarrow X_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

$\Big\}$ non-zero soln (or) non-trivial soln

Here we get infinite no of soln.

<u>Note</u> :

→ If $\rho(A) = r$ & n = no of variables in $AX = 0$ then $AX = 0$ will have $p = n - r$ linearly independent solutions.

→ $AX = 0$

↓

$A_{n \times n}$

$\lvert A \rvert \neq 0$	$\lvert A \rvert = 0$
unique soln (trivial)	trivial & non-trivial soln.

→ $AX = 0$

- $\rho(A) = n$ (or) $|A| \neq 0$ ..... trivial soln & unique soln
- $\rho(A) \neq n$ (or) $|A| = 0$ .... trivial & non-trivial soln

Q20) Consider the linear equations

$$x - 2y + z = 3$$

$$2x + \alpha z = -2$$

$$-2x + 2y + \alpha z = 1$$

Inorder to have a unique soln for this linear system of eq. the value of α should not be equal to

a) $-2/3$ b) $2/3$ c) $4/3$ d) $-4/3$

Sol:

$AX = B$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & \alpha \\ -2 & 2 & \alpha \end{bmatrix}$$

Here A is a square matrix

$\therefore$ unique soln $\Rightarrow |A| \neq 0$

$$\begin{array}{c} 1 \\ \\ \alpha \end{array} \begin{vmatrix} 1 & -2 & 1 \\ 2 & 0 & \alpha \\ -2 & 2 & \alpha \end{vmatrix} \begin{array}{c} 1 \\ \\ -2 \end{array} \neq 0$$

$$(4 + 0 + 4\alpha) - (-4\alpha + 0 + 2\alpha) \neq 0$$

$$4\alpha + 4 + 2\alpha \neq 0$$

$$6\alpha \neq 4$$

$$\Rightarrow \alpha \neq -2/3$$

$\therefore$ opt (a)

(Q21) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 4 & 3 & 10 \end{bmatrix}$ then which of the following is not true?

a) $P(A) = 2$

b) $AX = 0$ has infinitely many solutions.

c) $AX = B$ has a unique solution

d) A^{-1} does not exist.

Sol:

finding rank of matrix A

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 4 & 3 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & \cancel{-4}-2 \\ 0 & -5 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2 \Rightarrow |A| = 0$

$\therefore$ inf solns

$\therefore$ inverse does not exist

~~AXB~~ $AX = B$

$\text{rank}(A) = \cancel{3}\,2 \neq$ no of ~~vectors~~ equations

$\therefore$ no unique soln.

$\therefore$ opt (c) is wrong

(Q22) For what values of α & β the following system of equations has an infinite no of solutions.

$x+y+z=5$, $x+3y+3z=9$, $x+2y+\alpha z=\beta$

a) 2, 7 b) 3, 8 (c) 8, 3 d) 7, 2

Sol :

Augmented matrix

$$A/B = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha-1 & \beta-5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 2\alpha-4 & 2\beta-14 \end{bmatrix}$$

infinite soln $\Rightarrow$ $Rank(A) = Rank(A|B) \neq 3$

~~from the figure it is clear that~~

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 2 & \alpha \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2\alpha-4 \end{bmatrix}$$

$Rank(A) \neq 3$

$\Rightarrow 2\alpha - 4 = 0$

$\alpha = 2$

$\because Rank(A|B) \neq 3$

$2\beta - 14 = 0$

$\Rightarrow \beta = 7$

Q23 If a matrix A of order 4 has a rank 2 then $AX = 0$ has

a) one linearly independent solutions

b) two linearly independent solutions

c) three " " "

d) only trivial solution

Sol:

$$AX = 0$$
$$\downarrow$$
$$A_{4\times4}$$

$r = \rho(A_{4\times4}) = 2$ --- System has inf no of solns.

no of independent solns $= n - r = 4 - 2 = 2$

$\therefore$ opt (b)

Q24) let $AX=B$ be a system of 3 equations in 3 variables x,y,z.

The augmented matrix of the system is given by $(A|B)$

$$(A|B) = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

which of the following is not true?

a) $\rho(A|B)=2$ b) $\rho(A)=2$

c) $AX=B$ has many solns

d) $AX=B$ has a unique soln

Sol:

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(A) = 2$

$\text{Rank}(AB) = 2$

$\therefore \text{Rank}(A) = \text{Rank}(AB) \neq 3$

$\therefore AX=B$ has inf no of solns

$\therefore$ opt (d)

Q25) Consider the matrix $A = \begin{bmatrix} k & k & k \\ 0 & k-1 & k-1 \\ 0 & 0 & k^2-1 \end{bmatrix}$. If the system $AX=0$

has only one independent soln, then $k=$ ____

a) $0,-1$ b) $-1,1$ c) $0,1$ d) $0,1,-1$

Sol:

no of independent solns = $n - r$

$$3 - r = 1$$
$$\Rightarrow r = 2$$

$$\Rightarrow \cancel{k^2 + 1} = 0$$

$$\Rightarrow \cancel{k = \pm 1}$$

i.e., the echelon form has 2 non-zero rows

put $k=0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\therefore k = 0$ ✓

put $k = -1$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank} = 2$$

$\therefore k = -1$

put $k=1$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Rank $= 1$

$\therefore k \neq 1$

Hence opt (a)

(Q26) Let $AX = B$ be a system of 3 equations in 3 variables x, y, z. If A has 3 linearly independent columns & B is a linear combination then which of the following is true?

a) system has a unique soln.

b) System has no soln

c) $AX=0$ has a non-zero soln

d) The system has many solns.

Sol :

given linearly independent colums

$\Rightarrow \rho(A) = 3$

$\because$ A is square matrix of order 3

no ~~of uniqu~~

$AX = B$ has a unique soln

$\therefore$ opt (a)

Eigen values, Eigen vectors & Cayley-Hamilton Theorem

Eigen matrix, Eigen polynomial, Eigen equation & eigen values

If $A = (a_{ij})_{n\times n}$ is square matrix of order 'n' .. 'I_n' is a unit matrix of order 'n' & 'λ' is a scalar then

i) the matrix $A - \lambda I$ is called a characteristic (or eigen) matrix of a matrix A.

ii) the expression $|A - \lambda I|$ is called a characteristic (or eigen) polynomial of matrix.

iii) the equation $|A - \lambda I| = 0$ is called a characteristic (or eigen) equation of a matrix $A_{n\times n}$

iv) the roots of a characteristic equation of a matrix A are called eigen values (or) characteristic roots of matrix A.

Eg: Consider $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$ is a charecteristic or eigen matrix

$|A-\lambda I| = (4-\lambda)^2 - 4$

$= \lambda^2 + 16 - 8\lambda - 4$

$= \lambda^2 - 8\lambda + 12$ is a eigen polynomial of A

$|A-\lambda I| = 0$

$\Rightarrow \lambda^2 - 8\lambda + 12 = 0$ is eaigen ~~equat of~~ equation of A

$\lambda^2 - 8\lambda + 12 = 0$

$\lambda^2 - 6\lambda - 2\lambda + 12 = 0$

$(\lambda - 6)(\lambda - 2) = 0$

$\Rightarrow \lambda = 6$ (or) 2

6, 2 are eigen values of A

~~Rank(A) = no of non-zero eigen values of A~~

Eigen vector :

A non-zero column vector $X_{n\times 1}$ is said to be an eigen vector of matrix $A_{n\times n}$ if $AX = \lambda X$ for some scalar λ.

Note :

→ $AX = \lambda X$

$AX - \lambda X = 0$

$(A - \lambda I)X = 0$ — trivial soln

— non-trivial soln ↓ Eigen vector.

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→ For a square matrix A

(i) $|A - \lambda I| = 0$ .... λ Characteristic roots

(ii) $(A - \lambda I)X = 0$ .... $X \neq 0$ ↓ eigen vector

→ If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the characteristic equation of A is

$$\lambda^2 - (a+d)\lambda + |A| = 0$$

→ If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the characteristic equation of A is

$$\lambda^3 - (a_{11}+a_{22}+a_{33})\lambda^2 + \underbrace{\left[\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right]}_{\text{Sum of minors of diagonal elements.}} \lambda - |A| = 0$$

Orthogonal vectors (Perpendicular vectors):

Two vectors $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are said to be an orthogonal vectors if $\underbrace{X \cdot Y}_{\text{dot product}} = 0$ (or) $x_1y_1 + x_2y_2 + x_3y_3 = 0$

$$X \cdot Y = X^T Y = 0$$

Eg: $X = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ & $Y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$X \cdot Y = X^T Y = Y^T X$$

$$X^T Y = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= 1(0) + 0(1) - 2(0) = 0$$

Properties of Eigen values & Eigen vectors:

→ Eigen values of upper triangular matrix or lower triangular matrix or triangular matrix, eigen values are the diagonal elements of the matrix.

Eg: $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \Rightarrow \lambda = a, d, f$ $\quad \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \Rightarrow \lambda = a, c, f$

Eg: $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \Rightarrow \lambda = a, b, c$

→ If $\lambda_1, \lambda_2 \cdots \lambda_n$ are the eigen values of a matrix $A_{n\times n}$ then

(i) $\lambda_1 + \lambda_2 + \cdots + \lambda_n = tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

(ii) $\lambda_1 \lambda_2 \cdots \lambda_n = |A_{n\times n}|$

(iii) the eigen values of a matrix A^m are $\lambda_1^m, \lambda_2^m, \cdots \lambda_n^m$, $m \in I$

(iv) the eigen value of a matrix KA are $K\lambda_1, K\lambda_2, \cdots K\lambda_n$

(v) the eigen values of matrix A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots \frac{1}{\lambda_n}$

(vi) the eigen value of matrix adj A are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \cdots \frac{|A|}{\lambda_n}$

(vii) the eigen values of matrix

$a_0 I + a_1 A + a_2 A^2$ are

$a_0 + a_1\lambda_1 + a_2\lambda_1^2,\ a_0 + a_1\lambda_2 + a_2\lambda_2^2, \cdots a_0 + a_1\lambda_n + a_2\lambda_n^2$

→ If $a+ib$ or $a+\sqrt{b}$ is an eigen value of a real matrix A then $a-ib$ or $a-\sqrt{b}$ is also another eigen value of same matrix.

→ The eigen values of a real symmetric matrix are always real. and the eigen values of a real skew-symmetric matrix are either 0 or purely imaginary.

Eg: $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \lambda_1 = 2 \quad \lambda_2 = 6$

A is real symmetric matrix and hence λ is real.

$A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$

Characteristic eqn $\lambda^2 - (a+d)\lambda + |A| = 0$

$\lambda^2 + 16 = 0 \Rightarrow \lambda = \pm 4i$

A is real skew symmetric matrix and hence λ is 0 or purely imaginary

→ The eigen vectors of a real symmetric matrix are always pairwise orthogonal.

Eg: $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

$$\lambda^2 - (3-3)\lambda + (-9-16) = 0$$

$$\lambda^2 = 25 \Rightarrow \lambda = \pm 5$$

$$(A - \lambda I)X = 0$$

$\lambda = 5$

$$\begin{bmatrix} 3-5 & 4 \\ 4 & -3-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -2x_1 + 4x_2 \\ 4x_1 - 8x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} -2x_1 + 4x_2 = 0 \\ 4x_1 - 8x_2 = 0 \end{array}\right\} x_1 - 2x_2 = 0$$

$$x_1 = 2 \Rightarrow x_2 = 1$$

$$X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda = -5$

$$\begin{bmatrix} 3+5 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 8x_1 + 4x_2 \\ 4x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 + x_2 = 0$$

$$x_1 = 1 \Rightarrow x_2 = -2$$

$$\Rightarrow X = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\therefore \lambda_1 = 5 \Rightarrow X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \lambda_2 = -5 \Rightarrow X_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$X_1 \cdot X_2 = 2(1) + 1(-2) = 0$ $\quad \therefore X_1$ & X_2 are pairwise orthogonal.

→ If the nth order matrix A has n different eigen values $\lambda_1, \lambda_2, \dots \lambda_n$ then the nth order matrix $A_{n\times n}$ will have 'n' linearly independent eigen vectors. $X_1, X_2, \dots X_n$.

→ The no of linearly independent eigen vectors of a matrix $A_{n\times n}$ corresponding to any eigen value 'λ' is given by

$p = n - r$ where $r = \text{rank}(A - \lambda I)$

n = no of variables in eigen vectors
(or)
order of the matrix.

Q27) If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ then the eigen values of A are

a) 8, 7, 3 b) 0, 3, 15 c) 1, 2, 3 d) 1, -1, 2

Sol:

Sum of eigen values = tr(A)

a) ~~8+7+3~~ $tr(A) = 8+7+3 = 18$

a) $8+7+3 = 18$ ✓
b) $0+3+15 = 18$ ✓
c) $1+2+6 = 6 \neq 18$
d) $1-1+2 = 2 \neq 2$

also product of eigen values = determinant

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

2 8.
3 2

$= (48 + 168 + 48) - (108 + 28 + 128)$
$= (264) - (264)$
$= 0$

∴ opt (b)

Q28) The no. of distinct eigen values of the matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n\times n}$$ are

a) 0, n b) 1, n-1 c) 0, n-1 d) 1, n

Sol :

Put n=3

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \begin{matrix} 1-\lambda \\ 1 \\ 1 \end{matrix}$$

$$(1 + (1-\lambda)^3 + 1) - (1-\lambda + 1-\lambda + 1-\lambda) = 0$$

$$(1-\lambda)^3 + 2 + 3\lambda - 3 = 0$$

$$(1-\lambda)^3 + 3\lambda - 1 = 0$$

$$\cancel{1} - \lambda^3 - \cancel{3\lambda} + 3\lambda^2 + \cancel{3\lambda} - \cancel{1} = 0$$

$$-\lambda^3 + 3\lambda^2 = 0$$

$$\lambda^2(\lambda - 3) = 0$$

$$\lambda = 0 \text{ \& } \lambda = 3$$

$$\therefore \lambda = 0, n$$

Put n=2

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\lambda_1 + \lambda_2 = tr(A) = 2$$

$$\lambda_1 + \lambda_2 = 2$$

$$\lambda_1 \lambda_2 = |A| = 0$$

$$\Rightarrow \lambda_1 = 0 \text{ (or) } \lambda_2 = 0$$

$$\Rightarrow \lambda_2 = 2 \quad \lambda_1 = 2$$

$$\therefore \lambda = 0, 2$$

i.e, $\lambda = 0, n$ $\therefore$ opt (a)

(Q29) Consider the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ whose eigen values are

1, -1, 3. then the trace of $A^4 - 3A^3$ is ______

Sol :

$\lambda_1 = 1 \quad \lambda_2 = -1 \quad , \lambda_3 = 3$

eigen values of $A^4 - 3A^3$

are $\lambda_1' = \lambda_1^4 - 3\lambda_1^3 = 1 - 3 = -2$

$\lambda_2' = \lambda_2^4 - 3\lambda_2^3 = 1 + 3 = 4$

$\lambda_3' = \lambda_3^4 - 3\lambda_3^3 = 27 - 27 = 0$

now trace of $A^4 - 3A^3 = \lambda_1' + \lambda_2' + \lambda_3'$

$= -2 + 4 + 0$

$= \underline{2}$

(Q30) Let A be a non-zero upper triangular matrix all of whose eigen values are zero. Then I+A is

a) singular b) invertable c) symmmetric d) skew symmetric

Sol :

Let n be order of A

given eigen values are all zeroes

$\Rightarrow$ ~~λ_1~~ $\lambda_1 = 0, \lambda_2 = 0, \cdots \lambda_n = 0$

$\therefore \lambda_1 + \lambda_2 + \cdots \lambda_n = 0$

$\lambda_1 \lambda_2 \cdots \lambda_n = 0$

now eigen values of I+A is $\lambda + 1$

$\therefore$ eigen vaule of I+A are

$\lambda_1 + 1, \lambda_2 + 1, \cdots, \lambda_n + 1$

i.e., $1, 1, 1, \ldots 1$

$\Rightarrow \quad |I+A| = (1)(1)\cdots(1) = 1 \neq 0$
(n times)

$\therefore$ I+A is invertible.

I+A is uppertriangular and hence not symmetric and not skew symmetric.
($\because$ A is non-zero)

$\therefore$ opt (b)

Method 2 :

for upper triangular matrix eigen values are its diagonal elements.

$$\Rightarrow I+A = \begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 1 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

determinant of ~~upper~~ triangular matrix
= product of diagonal elements.

$\therefore |I+A| = 1$

$\therefore$ invertable.

(Q31) If the characteristic polynomial of a 3x3 matrix M over R (real numbers) is $\lambda^3 - 12\lambda^2 + a\lambda - 32$, $a \in R$ and one of the eigen value of M is 2 then the largest among the absolute values of the eigen values of M is ______

Sol :

let $\lambda_1 = 2$

$\lambda_1 \lambda_2 \lambda_3 = |A| = 32 \quad \Rightarrow \lambda_2 \lambda_3 = 16$

also $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33} = 12$

$\Rightarrow \lambda_2 + \lambda_3 = 10$

now $(\lambda_2+\lambda_3)^2 = (\lambda_2-\lambda_3)^2 + 4\lambda_2\lambda_3$

$$100 = (\lambda_2-\lambda_3)^2 + 4(16)$$

$$(\lambda_2-\lambda_3)^2 = 36$$

$$\lambda_2-\lambda_3 = \pm 6$$

$$\lambda_2+\lambda_3 = 10$$

$$\lambda_2-\lambda_3=6$$
$$\lambda_2+\lambda_3=10$$
$$\Rightarrow 2\lambda_2=16$$
$$\lambda_2=8$$
$$\Rightarrow \lambda_3=2$$

$$\lambda_2-\lambda_3=-6$$
$$\lambda_2+\lambda_3=10$$
$$2\lambda_2=4$$
$$\Rightarrow \lambda_2=2$$
$$\Rightarrow \lambda_3=8$$

$\therefore$ ~~large~~ $\lambda_1=2 \quad \lambda_2=2 \quad \lambda_3=8$

$\therefore$ largest possible eigen value $= \underline{8}$

<u>Method 2:</u>

put $\lambda=2$ in the eqn

$$2^3-12(2)^2+2a-32=0$$

$$\Rightarrow 8-48+2a-32=0$$

$$2a=72 \Rightarrow a=36$$

now characteristic eqn is

$$\lambda^3-12\lambda^2+36\lambda-32=0$$

we have $\lambda=2$

$$\begin{array}{c|cccc} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$$\therefore (\lambda-2)(\lambda^2-10\lambda+16)=0$$

$\Rightarrow \lambda^2 - 10\lambda + 16 = 0$

$\Rightarrow (\lambda-8)(\lambda-2) = 0$

$\Rightarrow \lambda = 8, 2$

$\therefore$ largest possible eigen value $= 8$

(Q32) An eigen vector corresponding to the smallest eigen value of the matrix $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is

a) $[1\ 0\ 0\ 0]^T$ b) $[1\ 1\ 0\ 0]^T$

c) $[1\ 0\ 2\ 0]^T$ c) $[-1\ -1\ 2\ 2]^T$

Sol:

given matrix is upper triangular

$\therefore$ eigen values are diagonal elements

i.e., $-1, 0, 1, 2$

smallest eigen value $= -1$

Let X be eigen vector corresponding to $\lambda = -1$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & -1 & 2 \\ 0 & 2-\lambda & 0 & 1 \\ 0 & 0 & -1-\lambda & 1 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = -1$

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce matrix A into echelon form

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 2x_1 + x_2 - x_3 + 2x_4 = 0 \quad \cdots (1)$

$3x_2 + x_4 = 0 \quad \cdots (2)$

$x_4 = 0 \quad \cdots (3)$

$(2) \Rightarrow 3x_2 + 0 = 0 \Rightarrow x_2 = 0$

$(1) \Rightarrow \quad 2x_1 - x_3 = 0$

Put $x_1 = k$

$\Rightarrow x_3 = 2k$

$$\therefore \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 2k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$\therefore$ opt (c)

Method 2 :

wkT

$AX = \lambda k$

opt a :

$$AX = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \lambda = 1$

opt (b):

$$AX = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \lambda = 2$

Slly do for opt (c) & opt (d)

Q33 Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. If -3 & -3 are two eigen values of A then the eigen vector corresponding to the 3rd eigen value is

a) $[1\ 2\ 1]^T$ b) $[1\ 2\ -1]^T$ c) $[1\ -2\ 1]^T$ d) $[-1\ 2\ 1]^T$

Sol :

$$\lambda_1 + \lambda_2 + \lambda_3 = tr(A)$$

$$-6 + \lambda_3 = -2 + 1 + 0$$

$$\Rightarrow \lambda_3 = 5$$

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} -7 & 2 & -3 \\ 0 & -24 & -48 \\ 0 & -16 & -32 \end{bmatrix}$$

$$\sim \begin{bmatrix} -7 & 2 & -3 \\ 0 & -24 & -48 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow -24x_2 - 48x_3 = 0$$

$$\Rightarrow x_2 + 2x_3 = 0$$

$$x_3 = k \quad \Rightarrow x_2 = -2k$$

$$-7x_1 + 2x_2 - 3x_3 = 0$$

$$-7x_1 - 4k - 3k = 0$$

$$\Rightarrow x_1 = -k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ -2k \\ k \end{bmatrix} = -k \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\therefore$ opt (b)

Method 2: (option verification)

$AX = \lambda X$

$AX = 5X$

(a) opt a:

$$AX = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -5 \end{bmatrix} \not\propto$$

Here $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is even a eigen vector

opt b:

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Here $\lambda = 5$

$\therefore$ opt (b)

(Q34) If $[2 \ -2 \ 1]^T$ is an eigen vector ~~of~~ the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & x & -4 \\ 2 & -4 & 3 \end{bmatrix}$

then $x =$ ______

Sol:

$AX = \lambda X$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & x & -4 \\ 2 & -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ -2x-16 \\ 15 \end{bmatrix} = \lambda \begin{bmatrix} 2\lambda \\ -2\lambda \\ \lambda \end{bmatrix}$$

$\Rightarrow 30 = 2\lambda \Rightarrow \lambda = 15$

$-2x - 16 = -2\lambda$

$2x + 16 = 2(15)$

$2x = 14 \Rightarrow x = 7$

also ~~$\lambda = 15$~~ $\lambda = 15$ is eigen value corresponding to the eigen ~~vat~~ vector $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

(Q35) The no of linearly independent eigen vector of a matrix

$$A = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 2 & -4 \\ 0 & 0 & 9 \end{bmatrix} \text{ is } \_\_\_\_$$

Sol :

A is triangualr Matrix

$\therefore$ eigen values are 1, 2, 9

$\because$ we have 3 distinct eigen values

we have 3 linearly independent eigen vectors

$\therefore$ Ans : 3

***

(Q36) The number of linearly independent eigen vectors of a matrix

**

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \text{ is } \_\_\_\_$$

Sol :

Matrix is triangular

eigen values are 3, 2, 2

$\therefore$ no of linearly independent eigen vectors $= n - r$

$\hookrightarrow$ Rank$(A - \lambda I)$

$\Rightarrow$ ~~Rank(A), r=3~~

$\therefore$ ~~no of linearly independent eigen vectors = 3-3=0~~

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$\lambda = 3, \; 2, 2$$

$\swarrow$ $\downarrow$

x_1 $\quad$ $\underbrace{x_2 \text{ (or) } x_2, x_3}$

$\hookrightarrow \rho = n - r = 3 - \rho(A - \lambda I)$

Put $\lambda = 2$

$$A - 2I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$Rank(A - 2I) = 2$

$\Rightarrow$

no of linearly independent vectors corresponding to eigen value 2 $= n - r$
$= 3 - 2 = 1$

Slly corresponding to eigen value 3,

we have one linearly independent vector.

$\therefore$ no of linearly independent vectors $= 1 + 1 = 2$

(Q37) If A is a skew symmetric matrix of order n, then the no of lineary independent eigen vectors of $A + A^T$ is ____

a) 0 b) 1 c) $n-1$ d) n

Sol:

A is skew symmetric

$\Rightarrow A + A^T$ is null matrix

eigen values of $A + A^T$ are $0, 0, \ldots 0$ (n times)

$\therefore$ no of LI vectors $=$ ~~$A-\lambda I$~~ $n - \rho(A - \lambda I)$ $\quad n - \rho(A + A^T - \lambda I)$

~~$\lambda = 0$~~ $\quad$ Here $\lambda = 0$

~~$\Rightarrow n - \rho(A)$~~ $\quad \Rightarrow n - \rho(A + A^T)$

$\Rightarrow \quad = n - 0$

$= n$

Cayley Hamilton Theorem:

Stmt:
Every square matrix satisfies its own characteristic equation.

Eg: If $\lambda^2-8\lambda+12=0$ is a characteristic eqn for matrix $A=\begin{bmatrix}4&2\\2&4\end{bmatrix}$

then by Cayley Hamilton theorem, we have

$$A^2-8A+12I=0$$

(Q38) If 1, 2 & 3 are the eigen values of matrix $A_{3\times3}$ then

$6A^{-1}=$ ____

a) $A^2+6A-11I$

b) $A^2-6A-11I$

c) $A^2-6A+11I$

d) $A^2+6A+11I$

Sol:

The characteristic eqn is

$$(\lambda-1)(\lambda-2)(\lambda-3)=0$$

$$(\lambda^2-3\lambda+2)(\lambda-3)=0$$

$$(\lambda^3-3\lambda^2-3\lambda^2+9\lambda+2\lambda-6)=0$$

$$\lambda^3-6\lambda^2+11\lambda-6=0$$

from Cayley Hamilton theorem

$$A^3-6A^2+11A-6I=0$$

$$A^3-6A^2+11A=6I$$

Multiplying A^{-1} on both sides

$$6A^{-1}=A^2-6A+11I$$

$\therefore$ opt (c)

(Q39) Let $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $B = A^3 - A^2 - 4A + 6I$, where I is 3×3 identity matrix. The determinant of B is ______

Sol:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & -1 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{bmatrix}$$

Characteristic eqn is

$|A - \lambda I| = 0$

$$\begin{array}{c|ccc|c} -1 & 1-\lambda & 0 & -1 & 1-\lambda \\ & -1 & 2-\lambda & 0 & \\ -2-\lambda & 0 & 0 & -2-\lambda & 0 \end{array}$$

$$\left[(1-\lambda)(2-\lambda)(-2-\lambda)\right] - \left[0+0+0\right] = 0$$

$$(\lambda-1)(\lambda-2)(\lambda+2) = 0$$

$$= \lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

from cayley hamilton theorem

$$A^3 - A^2 - 4A + 4I = 0$$

$$\Rightarrow \quad A^3 - A^2 - 4A + 4I + 2I = 2I$$

$$\Rightarrow \left|A^3 - A^2 - 4A + 6I\right| = |2I| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2^3 = 8$$

LU Decomposable matrix (or) LU Factorizable matrix:

If a square matrix $A_{n \times n}$ is written as $A = L \cdot U$, where 'L' is a lower triangular matrix & 'U' is a upper triangular matrix then the square matrix $A_{n \times n}$ is called an LU decomposable matrix.

→ Methods of finding the matrices L & U :

Method I : (Doo little's method (or) Doolittle's LU decomposition method) :

If $A = L \cdot U$ then → unit lower triangular matrix

Consider $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$

and $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Here we find product of L & U and solve the equations to find elements of L & U.

Method II ~~(Crowd's method)~~ (Crout's method) :

If $A = LU$ the consider

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$

↳ unit upper triangular matrix

slly $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

shortcut : we can find echelon forms to find L & U matrices.

Q40 In the LU decomposition of the matrix $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$, if the diagonal elements of U are both 1 then lower diagonal entry l_{22} of L is ______

sol :

Let $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} \\ l_{21} & l_{21}u_{12}+l_{22} \end{bmatrix}$

$\Rightarrow l_{11} = 2 \Rightarrow u_{12} = 1$

$l_{21} = 4$ $\quad l_{21}u_{12} + l_{22} = 9 \Rightarrow 4 + l_{22} = 9 \Rightarrow l_{22} = 5$

* * * *
* Method 2 (Shortcut) :
* * *

Consider $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$

we convert this matrix into column echelon form

$$\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} \quad C_2 \to C_2 - C_1$$

$$\begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix}$$

$\therefore l_{22} = 5$

** Shortcut for finding L, U

* Q41 Suppose $A = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix}$. If $A = LU$, where L is lower triangular matrix
* *

with diagonal elements as unity then U & L are ____

Sol:

~~Now we app reduce~~

to find U we reduce matrix to row echelon form

to find L we reduce matrix to column echelon form

$$A = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & -8 & 4 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix} = U$$

to find L:

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix}$$

$C_2 \to 3C_2 - 5C_1$

$C_3 \to 3C_3 - 2C_1$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 24 & 6 \\ 6 & -24 & 12 \end{bmatrix}$$

$C_3 \to 4C_3 - C_2$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 24 & 0 \\ 6 & -24 & 72 \end{bmatrix}$$

However we need unit lower triangular matrix

$C_1 \to C_1/3$

$C_2 \to C_2/24$

$C_3 \to C_3/72$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} = L$$

$$\therefore U = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$