05/07/20

Probobility

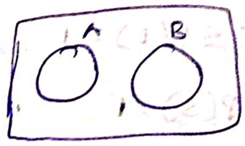
Random Experiment: The outcome of an experiment can't be predicted but it should be any one among several possible outcomes. This experiment is called a random ~~exprem~~ experiment.

Mutually Exclusive (Disjoint) Events:

The occurance of one event prevents the occurance of remaining events, then the events are said to be mutually exclusive. In other words events which cannot occur at the same time. i.e., two are more events that cannot occur simultaneously.

If $A \cap B = \phi$, the A & B are said to be mutually exclusive events

Eg: If a coin is tossed, occurance of head and occurance of tail are mutually exclusive events.



Mutually Independent Events:

Two events are said to be mutually independent if occurrance of one event doesn't affect the result of second event.

Eg: If a coin is tossed twice, the first toss doesn't affect the resut of seccond toss.

Sample Space (S):

The set of all possible outcomes of a random experiment is called sample space and is denoted by S.

Eg: If event is throwing a die, then the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

Probability:

If an event 'E' occurs in m different ways out a total of 'n' cases then the probability of occurance of an event E is defined as

$$P(E) = \frac{\text{Favourable cases}}{\text{Total cases}} = \frac{m}{n}$$

Note: $P(E^c) = P(\bar{E}) = \frac{\text{unfavourable cases}}{\text{total cases}} = 1 - \frac{m}{n}$

$\Rightarrow \boxed{P(E^c) = 1 - P(E)}$

$\Rightarrow \boxed{P(E) + P(\bar{E}) = 1}$

Exhaustive events:

It is a set of events in which atleast one event must occur.
i.e., events in sample space

Axioms of Probability:

1) $0 \leq P(E) \leq 1$
2) $P(S) = 1$
3) If A and B are mutual exclusive events then

$P(A \cup B) = P(A) + P(B)$

Fundamental Rule of Counting:

If a work w_1 can be performed in 'm' different ways and a second work w_2 can be performed (after w_1 has been performed in any one of 'm' different ways) in 'n' different way then the ~~to~~ two works (one after other) can be performed in 'mn' ways.

Eg: If a man has 4 diff shirts and 3 ~~pair~~ pairs of pants then he can select a pair (first shirt, then pants) In $4 \times 3 = 12$ ways

Permutations and Combinations:

1) Permutation (or) Arrangement (order is important)

The number of permutations of n different things taken 'r' at a time = $^nP_r = \frac{n!}{(n-r)!}$

Circular Permutations:

The number of circular permutations of 'n' different things taken all at a time = $(n-1)!$

Eg: Consider 4 letterrs A, B, C, D

the number of circular permutations = $(4-1)! = 6$

Permutations with constraint Repetition:

The number of permutations of n things in which p things are of type I, 'q' things are of type II, 'r' thing are of type III and the rest are different is given by

$$\frac{n!}{p!\,q!\,r!}$$

2) Combination (or) Selection [order is not Important]

The no of combinations of 'n' different things taken 'r' at a time is $^nC_r = \frac{n!}{r!\,(n-r)!}$

Eg: conside 4 letter A, B, C, D

no of ~~ways~~ possible selection of any 2 letter out of 4

$= {}^4C_2 = 6$

Note :

* **Demorgan laws :**

$$\overline{A\cup B} = \bar{A}\cap\bar{B}$$

$$\overline{A\cap B} = \bar{A}\cup\bar{B}$$

* Addition theorem for two event A & B is

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

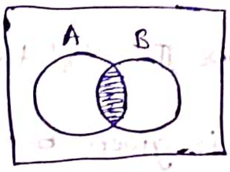
Slly for three events A, B, C is:

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(A\cap C) + P(A\cap B\cap C)$$

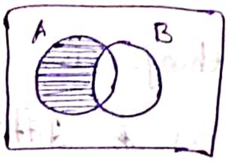
* Atleast = Minimum $\equiv \geq$

Atmost = Maximum $\equiv \leq$

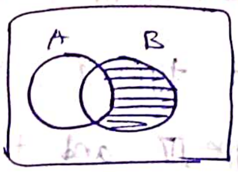
* **Venn Diagrams :**



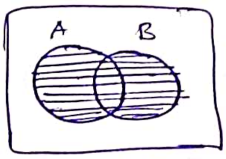
$A\cap B$



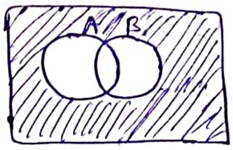
$A\cap\bar{B}$



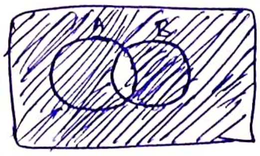
$\bar{A}\cap B$



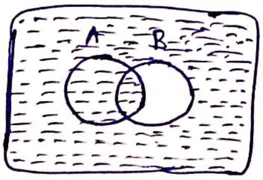
$A\cup B$



$\overline{A\cup B} = \bar{A}\cap\bar{B}$



$(A\cap B)^c = \bar{A}\cup\bar{B}$



$A\cup\bar{B}$

* $P(A-B) = P(A) - P(B)$ iff $B \subseteq A$ (Think why?)

$= P(A) - P(A\cap B)$, otherwise

(Q1) Two dice are rolled. Find the probability that sum is neither 8 not 9 ?

Sol :

Total cases = $6^2 = 36$

Sum = 8	Sum = 9
2,6	3,6
6,2	6,3
3,5	4,5
5,3	5,4
4,4	

$$P(8^c \cap 9^c) = P(\overline{8 \cup 9})$$
$$= 1 - P(8 \cup 9)$$
$$= 1 - \{P(8) + P(9) - \underbrace{P(8 \cap 9)}_{0}\}$$
$$= 1 - \left(\frac{5}{36} + \frac{4}{36} - 0\right)$$
$$= 1 - \frac{9}{36} = \frac{27}{36} = \frac{3}{4}$$

(Q2) A die is rolled four times. Find the probability that

(i) Sum is 22

(ii) Sum is 21

Sol :

Total cases = 6^4

(i) Sum = 22

If all die show up 6's then sum = 24

to obtain 22 we need to subtract 2 from maximum case.

we can subtract 2 from a die → $4C_1 = 4$ ways

or 1, 1 from 2 dies → $4C_2 = 6$ ways

∴ Sum = 22 has 4+6 = 10 ways

Probability of sum = 22 is $\frac{10}{6^4} = \frac{10}{1296} = \frac{5}{648}$

(ii) Sum = 21

we subtract 3

$(6,6,6,3) \rightarrow \frac{4!}{3!} = 4$

$(6,6,4,5) \rightarrow \frac{4!}{2!} = 12$

$(6,5,5,5) \rightarrow \frac{4!}{3!} = 4$

Here alternate method is

(i) $x_1+x_2+x_3+x_4 = 2$

$^{4-1+2}C_2 = {}^5C_2 = 10$

(ii) $x_1+x_2+x_3+x_4 = 3$

$^{4-1+3}C_3 = {}^6C_3 = 20$

$\therefore$ probability $= \frac{4+12+14}{1296} = \frac{20}{1296} = \frac{5}{324}$

(Q3) Three integers are choosen at random without replacement from the set $\{1,2,3,...20\}$, the probability that their product is even is

a) $\frac{2}{19}$ b) $\frac{3}{29}$ c) $\frac{17}{19}$ d) $\frac{4}{19}$

Sol :

Total case $= {}^{20}C_3 = \frac{20 \times 19 \times 18}{3 \times 2} = 1140$

Product is even if atleast one of 3 numbers is even

product is odd if all are odd

$\therefore P(\text{even}) = 1 - P(\text{odd})$

$= 1 - \frac{^{10}C_3}{^{20}C_3} = 1 - \frac{\cancel{10} \times \cancel{9} \times \cancel{8}\, 4\, 2}{\cancel{20}_2 \times 19 \times \cancel{18}_9} = 1 - \frac{2}{19}$

$\therefore P(\text{even}) = \frac{17}{19}$

14) Consider a biased coin that comes up heads with probability 1/3, tails ~~and~~ with probability $\frac{2}{3}$. If the coin is ~~tossed~~ tossed 2n times, then the probability that at sometime during this ~~expre~~ experiment two consecutive ~~coin~~ coin flip come up both heads ~~or~~ both tails is

Sol:

a) $\frac{3^n - 2^n}{3^{2n}}$ b) $\frac{3^{2n} - 2^{n+1}}{3^{2n}}$ c) $\frac{3^{n+1} - 2^n}{3^{2n}}$ d) $\frac{3^n + 2^{n+1}}{3^{2n}}$

Sol:

Given $P(H) = \frac{1}{3}$ & $P(T) = \frac{2}{3}$

put $n=1$

$\therefore$ 2 tosses

$HH \rightarrow \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

$TT \rightarrow \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

$\therefore$ Required probability $= \frac{1}{9} + \frac{4}{9} = \frac{5}{9}$

now in the given option put $n=1$ & verify

a) $\frac{3^1 - 2^1}{3^2} = \frac{1}{9}$ ✗

b) $\frac{3^2 - 2^2}{3^2} = \frac{5}{9}$ ✓

c) $\frac{3^2 - 2^1}{3^2} = \frac{7}{9}$ ✗

d) $\frac{3^1 + 2^2}{3^2} = \frac{7}{9}$ ✗

Method without option verification:

Here we consider ways in which the given condition is not satisfied

i.e., HTHTHT.... HT $\rightarrow \frac{1}{3}\frac{2}{3}\frac{1}{3}\frac{2}{3}\cdots\frac{1}{3}\frac{2}{3} = \frac{2^n}{3^{2n}}$

(or)

THTHTH ... TH $\rightarrow \frac{2}{3}\frac{1}{3}\frac{2}{3}\frac{1}{3}\cdots\frac{2}{3}\frac{1}{3} = \frac{2^n}{3^{2n}}$

~~Requi~~ total $= \frac{2^{n+1}}{3^{2n}}$

Required probability $= 1-\frac{2^{n+1}}{3^{2n}}$

$= \frac{3^{2n}-2^{n+1}}{3^{2n}}$

$\therefore$ opt (b)

(Q5) The probability that a person will get an electric contract is $\frac{2}{5}$ and the probability that he will not get plumbing contract is $\frac{4}{7}$. If the probability of getting atleast one contract is $2/3$, then the probability that he will get both the contracts is

a) $\frac{17}{105}$ b) $\frac{3}{7}$ c) $\frac{4}{7}$ d) $\frac{5}{7}$

Sol:

Electric contract $\rightarrow P(A) = \frac{2}{5}$

Plumbing contract $\rightarrow P(B) = 1-P(\bar{B}) = 1-\frac{4}{7} = \frac{3}{7}$

$P(A\cup B) = 2/3$

$P(A\cap B) = ?$

$P(A\cap B) = P(A)+P(B)-P(A\cup B) = \frac{2}{5}+\frac{3}{7}-\frac{2}{3} = \frac{42+45-70}{105}$

$= 17/105$

(Q6) An integer is selected at random b/w 1 and 1000 (both inclusive) Find the probability that, the integer is not divisible by 12 (or) 15 is ______

Sol :

Total cases = $1000c_1 = 1000$

~~not div by 12 or not div by~~ 15

$$P(\overline{12} \cup \overline{15}) \cancel{= 1 - P(12 \cap 15)}$$

$$= 1 - P(\text{divisibly by } 60)$$

$$= 1 - \frac{\lfloor 1000/60 \rfloor}{1000}$$

$$= 1 - \frac{\cancel{16}}{1000}$$

not (divisible by 12 or 15)

$\overline{P(12 \cup 15)}$

find $P(12 \cup 15)$ first

$$P(12 \cup 15) = P(12) + P(15) - P(12 \cap 15)$$

$$= \frac{83 + 66 - 16}{1000}$$

L.C.M (12, 15) = 60

$$= \frac{133}{1000}$$

$$\overline{P(12 \cup 15)} = 1 - \frac{133}{100}$$

$$= \frac{867}{1000} = 0.867$$

(Q7) Let A and B be events in a sample space 'S' such that

$P(A) = \frac{1}{2} = P(B)$ and $P(\bar{A} \cap \bar{B}) = \frac{1}{3}$. Find

i) $P(A \cup \bar{B})$ ii) $P(\bar{A} \cup \bar{B})$

Sol:

Given

$P(\bar{A} \cap \bar{B}) = \frac{1}{3}$

$P(\overline{(A \cup B)}) = \frac{1}{3}$

$1 - P(A \cup B) = 1/3$

$P(A \cup B) = 2/3$

$P(A) + P(B) - P(A \cap B) = 2/3$

$\frac{1}{2} + \frac{1}{2} - P(A \cap B) = 2/3$

$P(A \cap B) = 1 - 2/3 = \frac{1}{3}$

(i) $P(A \cup \bar{B}) = P(A) + P(\bar{B}) - P(A \cap \bar{B})$

$= P(A) + P(\bar{B}) - [P(A) - P(A \cap B)]$

$= P(A) + 1 - P(B) - [P(A) - P(A \cap B)]$

$= \cancel{\frac{1}{2}} + 1 - \cancel{\frac{1}{2}} - [1/2 - 1/3]$

$= 1 - \frac{1}{2} + \frac{1}{3}$

$= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

(ii) $P(\bar{A} \cup \bar{B}) = P(\overline{(A \cap B)}) = 1 - P(A \cap B)$

$= 1 - 1/3 = 2/3$

Q.8 If seven car accidents occured in a week, what is the probability that

(i) all accidents occur on the same day.

(ii) no two accidents occur on the same day of the week

sol:

(i) The day can be any of 7 days

$$7 \times \frac{1}{7^7} = \frac{1}{7^6}$$

(ii) Each accidents should occur on different days

∴ One accident per day

These accidents can be arranged in 7! ways

total : 7^7

$$\therefore \frac{7!}{7^7} = \frac{6!}{7^6}$$

Q9) A deck of five cards (each carrying a distinct number from 1 to 5) is shuffled. Two cards are then removed from the deck one at a time. what is the probability that the two cards are selected with number of first card being one higher than the number on the second card.

sol :

a) 1/5 b) 4/25 c) 1/4 d) 2/5

sol :

total cases = $5 \times 4 = 20$

favourable cases:

5, 4
4, 3
3, 2
2, 1

$$\therefore \frac{4}{20} = \frac{1}{5}$$

Note :

* If E and F are two events such that

$E \subseteq F$, then $P(E) \le P(F)$

Proof :

$F = E \cup (\bar{E} \cap F)$ (both are mutually exclusive)

$\therefore \boxed{P(F) = P(E) + P(\bar{E} \cap F)}$

$\therefore P(F) \ge P(E)$

07/07/20

Q10 When you throw 3 fair dice, the probability that the sum of the numbers on the top face is 10 is ____

sol :

total cases = 216

1	4	5	$3! = 6$ — 6
1	6	3	$3! = 6$ — 6
2	6	2	3 — 3
2	4	4	3 — 3
3	3	4	— 3
3	2	5	— 6
3	1	6	
4	1	5	
4	2	4	
4	3	3	
5	1	4	
5	2	3	
6	1	3	
6	2	2	

$\left.\right\}$ 27 (first six rows)

Counted already (remaining rows)

$\therefore \frac{27}{216}$

(Q11) A number is selected at random from 1st 200 natural numbers. The probability that is it not divisible by 6 or 8 is

Sol :

not (6 or 8)

$= P(\overline{6 \cup 8})$

$= 1 - P(6 \cup 8)$

$N(6 \cup 8) = P(6) + P(8) - P(6 \cap 8)$

$= P(6) + P(8) - P(24)$

$= 33 + 25 - 8$

$= 25 + 25 = 50$

$P(\overline{6 \cup 8}) = 1 - P(6 \cup 8)$

$= 1 - \frac{50}{200}$

$= 1 - 0.25 = 0.75$

(Q12) A die is loaded in such a way that the probability of the face with j dots turning is proportional to j for j=1,2,3,4,5,6. what is the probability that in one roll of the die, that an odd number of dots will turn up?

Sol :

$1+2+3+4+5+6 = 21$

$P(1) = 1/21$ $P(2) = 2/21$ $P(3) = 3/21$ $P(4) = 4/21$ $P(5) = 5/21$ $P(6) = 6/21$

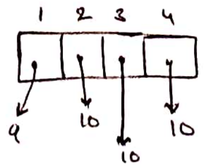
Probability that odd is turned up is

$P(1) + P(3) + P(5) = \frac{1+3+5}{21} = \frac{9}{21} = \frac{3}{7}$

(Q13) A four digit number is choosen at random the probability that there are exactly two zeroes in that number is.

Sol:

Let the four digit number be



total cases = $10 \times 10 \times 10 \times 9 = 9000$

favourable cases

out 2, 3, 4

any 2 places can be filled with two zeroes.

$\therefore$ we can selected the 2 places ~~with~~ in $3C_2$.

The rest two places can be filled with 9×9 ways

$$\therefore \frac{3C_2 \times 9 \times 9}{9000} = 0.027$$

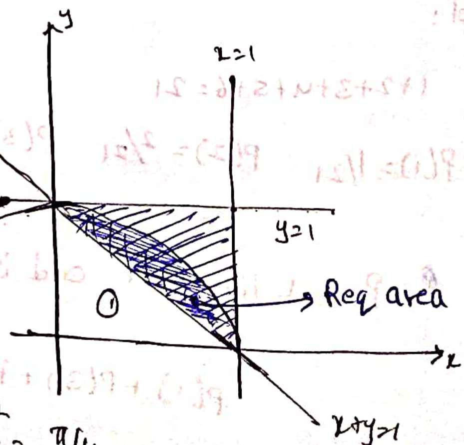
**** (Q14) The probability that the sum of the numbers x & y randomly choosen in the interval [0, 1] greater than 1 while the sum of the squares is less than 1 is equal to

a) $\frac{2}{\pi}$ b) $\frac{\pi}{4}$ c) $\frac{\pi}{6} - \frac{1}{2}$ d) $\frac{\pi}{4} - \frac{1}{2}$

Sol:

Circle $x^2 + y^2 = 1$

and $x + y > 1$



Area of circle in 1st quadrant $= \frac{\pi(1)^2}{4} = \pi/4$

From this subtract area 1 $\frac{\pi}{4} - \frac{1}{2}$

$\therefore$ probability = $\dfrac{\text{Shaded area}}{\text{total area of square}}$

$$= \frac{\frac{\pi}{4} - \frac{1}{2}}{1} = \frac{\pi}{4} - \frac{1}{2}$$

Conditional Probability

Suppose A & B are any two events in the sample space. The probability of occurance of an event 'A' such that the another event 'B' has already occured is given by

$$\underbrace{P(A}_{\text{unknown event}}/\underbrace{B)}_{\text{known even}} = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

P(A/B) is determining what fraction of B is A?

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$$

$$\boxed{P(A/B) + P(\bar{A}/B) = 1}$$ (Try proof)

Q15 A die is thrown 3 times and sum of the numbers is found to be 16. The probability that 5 appears on the third trail is

a) 1/2 b) 1/3 c) 2/5 d) 3/10

Sol:

total = 6 6 4 — B
6 4 6
4 6 6
6 5 5 — ~~3 ways~~
5 6 5
5 5 6

6 ways

favourable - 6 5 5
5 6 5

$\therefore$ Probability = $\frac{2}{6} = \frac{1}{3}$

Q16) Let E and F be any two events with $P(E\cup F) = 0.8$, $P(E) = 0.4$
$P(E/F) = 0.3$ then $P(F)$ is

Sol :

$P(E/F) = 0.3$

$$\frac{P(E\cap F)}{P(F)} = 0.3$$

$P(E\cup F) = P(E) + P(F) - P(E\cap F)$

$\Rightarrow P(F) - P(E\cap F) = 0.8 - 0.4$

$P(F) - P(E\cap F) = 0.4$

divide by $P(F)$

$$1 - \frac{P(E\cap F)}{P(F)} = \frac{0.4}{P(F)}$$

$$1 - 0.3 = \frac{0.4}{P(F)}$$

$$0.7 = \frac{0.4}{P(F)} \Rightarrow P(F) = \frac{4}{7}$$

Q17) P and Q are considering to apply for a job. The probability that P applies for the job is $\frac{1}{4}$, the probability that 'P' applies for the job given that 'Q' applies for the job is $\frac{1}{2}$ and the probability that 'Q' applies for the job given that 'P' applies for the job is $1/3$. Then the probability that 'p' does not apply for the job given that Q does not apply for the job is

a) $4/5$ b) $5/6$ c) $7/8$ d) $11/12$

Sol :

$P(P) = 1/4$

$$P(P/Q) = \frac{P(P\cap Q)}{P(Q)} = 1/2 \Rightarrow P(P\cap Q) = \frac{P(Q)}{2}$$

$$P(Q/P) = \frac{P(P\cap Q)}{P(P)} = 1/3 \Rightarrow P(P\cap Q) \Rightarrow \frac{P(P)}{3}$$

$$\frac{P(Q)}{2} = \frac{P(P)}{3}$$

$$\cancel{P(\bar{P}\cap\bar{Q})} = P(\bar{P}/\bar{Q}) = \frac{P(\bar{P}\cap\bar{Q})}{P(\bar{Q})} = \frac{1-P(P\cup Q)}{1-P(Q)}$$

$$P(P\cup Q) = P(P) + P(Q) - P(P\cap Q)$$

$$= P(P) + \frac{2P(P)}{3} - \frac{P(P)}{3}$$

$$= \frac{4}{3}P(P) = \frac{4}{3}\cdot\frac{1}{4} = \frac{1}{3}$$

$$1 - P(P\cup Q) = \frac{2}{3} \qquad P(Q) = \frac{2}{3}P(P) = \frac{1}{6}$$

$$\frac{1-P(P\cup Q)}{P(\bar{Q})} = \frac{\frac{2}{3}}{\frac{5}{6}} = \frac{4}{5}$$

(Q18) Let E and F be two events with $P(E)>0$, $P(F|E)=0.3$ and $P(E\cap\bar{F})=0.2$. Then $P(E)$ equals

a) $1/7$ b) $2/7$ c) $4/7$ d) $5/7$

Sol :

$$P(F/E) = \frac{P(E\cap F)}{P(E)} = 0.3 \Rightarrow P(E\cap F) = 0.3\,P(E)$$

$$P(E\cap\bar{F}) = P(E-F) = P(E) - P(E\cap F) = 0.2$$

$$\Rightarrow P(E) - 0.3P(E) = 0.2$$

$$\Rightarrow P(E) = 2/7$$

(Q19) An unbalanced die (with six faces numbered from 1 to 6) is thrown. The probability that the face value is odd is 90% of the ~~be~~ probability that face value is even. The probability of getting ~~an~~ an even numbered face is same. If the probability of face is even given that it is greater than 3 is 0.75. what is the probability that the face value exceeds 3?

Sol:

O = odd face   e = even face

g = face value > 3

$$P(o) + P(e) = 1$$

$$0.9\,P(e) + P(e) = 1$$

$$P(e) = \frac{1}{1.9} = \frac{10}{19}$$

getting even numbered faces is same

$$\Rightarrow \cancel{P(e)} \quad P(2) + P(4) + P(6) = \frac{10}{19}$$

$$3P(2) = \frac{10}{19}$$

$$P(2) = \frac{1}{3}\cdot\frac{10}{19}$$

~~Sly~~ $\therefore P(2) = P(4) = P(6) = \frac{1}{3}\cdot\frac{10}{19}$

$$P(e|g) = 0.75$$

$$\frac{P(e \cap g)}{P(g)} = 0.7 \Rightarrow \frac{P(4) + P(6)}{\cancel{P(4)+P(5)}\; P(g)} = \frac{3}{4}$$

$$\Rightarrow \quad P(g) = \frac{\frac{2}{3}\cdot\frac{10}{19}}{\frac{3}{4}}$$

$$\Rightarrow P(g) = \frac{80}{171} = 0.468$$

Let E and F be any two events with $P(E)=0.4, P(F)=0.3$ and $P(F|E) = 3P(F|\bar{E})$ the $P(E/F) =$ ______

sol:

$P(F) = P(F/E) + P(F/\bar{E})$

$= 3P(F/\bar{E}) + P(F/\bar{E})$

$P(F) = 4P(F/\bar{E}) \Rightarrow P(F/\bar{E}) = \frac{3}{10} \times \frac{1}{4}$

$\Rightarrow P(F/\bar{E}) = \frac{3}{40} \Rightarrow P(F/E) = \frac{9}{40}$

$\Rightarrow \frac{P(F \cap E)}{P(E)} = \frac{9}{40}$

$\Rightarrow P(F \cap E) = \frac{4}{10} \times \frac{9}{40} = \frac{36}{400}$

$P(E/F) = \frac{P(E \cap F)}{P(F)}$

$= \frac{\frac{36}{400}}{\frac{3}{10}}$

$= \frac{12}{40}$

$= \frac{3}{10} = 0.3$

Independent Events:

Two events A & B are said to be independent if it satisfies any one of the following:

(i) $P(A|B) = P(A)$

(ii) $P(B|A) = P(B)$

(iii) $P(A \cap B) = P(A)P(B)$

(iv) $P(A|\bar{B}) = P(A)$

(v) $P(B|\bar{A}) = P(B)$

(vi) $P(A|B) = P(A|\bar{B})$

(vii) $P(B|A) = P(B|\bar{A})$

Note:

If A and B are independent then

(i) A and $\bar{B}$ are also independent

$$P(A \cap \bar{B}) = P(A)\,P(\bar{B})$$

(ii) $\bar{A}$ and B are also independent

$$P(\bar{A} \cap B) = P(\bar{A})\,P(B)$$

(iii) $\bar{A}$ and $\bar{B}$ are also independent

$$P(\bar{A} \cap \bar{B}) = P(\bar{A})\,P(\bar{B})$$

The events $A_1, A_2 \ldots A_n$ are said to be indepdent if for every $r \leq n$,

~~P(A_1...)~~

$$P(A_{1'} \cap A_{2'} \cap \cdots \cap A_{r'}) = P(A_{1'})\,P(A_{2'}) \cdots P(A_{r'})$$

For independent events sample spaces are different

If A_1, A_2, A_3 are 3 independent events then A_1 will be independent of any event from A_2 & A_3

Eg: $P(A_1 \cap (A_2 \cup A_3)) = P(A_1)\,P(A_2 \cup A_3)$

(Q24) A problem in probability is given to three students A, B and C whose chances of solving it independently are $1/2$, $1/3$ and $1/4$ respectively. If they try to solve the problem, what is the probability that atleast one will solve the problem?

In other words, set of events are indepedent if every finite & subset of those events are independent

Sol:

$P(A) = 1/2 \quad P(B) = 1/3 \quad P(C) = 1/4$

$P(\bar{A}) = 1/2 \quad P(\bar{B}) = 2/3 \quad P(\bar{C}) = 3/4$

$$P(\text{atleast one}) = 1 - P(\text{none})$$

$$= 1 - P(\bar{A})\,P(\bar{B})\,P(\bar{C})$$

$$= 1 - \left(\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{3}{4}\right)$$

$$= 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

Q22) A six faced unbiased die is thrown until a number greater than four appears. The probability that this occurs on the n^{th} throw where 'n' is an even integer is

a) $1/5$ b) $1/2$ c) $\frac{2}{3}$ d) $\frac{2}{5}$

Sol:

A = number greater than 4

$P(A) = \frac{2}{6} = \frac{1}{3}$

$P(\bar{A}) = \frac{2}{3}$

$P = P(\bar{A})P(A) + P(\bar{A})P(\bar{A})P(\bar{A})P(A) + \cdots$

$= \frac{2}{3}\frac{1}{3} + \left(\frac{2}{3}\right)^3\frac{1}{3} + \left(\frac{2}{3}\right)^5\frac{1}{3} + \cdots$

$= \frac{2}{3}\frac{1}{3}\left(1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^4 + \cdots\right)$

$= \frac{2}{9}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \cdots\right)$ $\boxed{1 + r + r^2 + \cdots = \frac{1}{1-r}}$

$= \frac{2}{9}\left(\frac{1}{1-\frac{4}{9}}\right)$

$= \frac{2}{9}\frac{9}{5} = \frac{2}{5}$

Q23) Let A & B be independent events in the sample space S. It is known that $P(A \cap B) = 0.16$ and $P(A \cup B) = 0.64$. Now $P(A)$ is

Sol:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A) + P(B) = 0.80$

$P(A)P(B) = 0.16$

$(P(A) - P(B))^2 = [P(A) + P(B)]^2 - 4P(A)P(B)$

$= (0.8)^2 - 0.64 = 0$

$\Rightarrow P(A)-P(B) = \boxed{\sqrt{(0.48)^2}} = 0$

$P(A)+P(B) = 0.8$

$\Rightarrow P(A) = 0.4$

(Q24) X and Y are two random independent variables. It is known that $P(X) = 0.4$ & $P(X \cup \bar{Y}) = 0.7$. which of the following is the ~~valte~~ value of $P(X \cup Y)$?

Sol:

a) 0.7 b) 0.5 c) 0.4 d) 0.3

Sol:

$P(X) = 0.4$

$P(X \cup \bar{Y}) = 0.7$

$P(X) + P(\bar{Y}) - P(X \cap \bar{Y}) = 0.7$

$0.4 + P(\bar{y}) - 0.4\,P(\bar{Y}) = 0.7$

$0.6\,P(\bar{Y}) = 0.3$

$P(\bar{Y}) = 1/2 \Rightarrow P(Y) = 1/2$

$P(X \cup Y) = \frac{2}{5} + \frac{1}{2} - \frac{1}{5}$

$= \frac{1}{5} + \frac{1}{2} = \frac{7}{10} = 0.7$

(Q25) An urn contains 5 red & 7 green balls. A ball is drawn at random and its color is noted. The ball is placed back into the urn along with another ball of the same color. The probability of getting red ball in the next draw is

Sol:

② we have 2 case

1st drawn is red | 1st drawn is green
$\frac{5}{12} \cdot \frac{6}{13}$ | $\frac{7}{12} \cdot \frac{5}{13}$

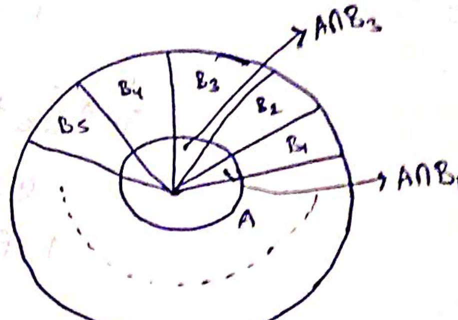
$\therefore$ require probability $= \frac{5\times6 + 7\times5}{12\times13}$

$= \frac{13\times5}{12\times13} = \frac{5}{12}$

Baye's Theorem:

(sequence of events with inverse probability)

If $B_1, B_2 \cdots B_n$ are mutually exclusive events and if A is another event associated with $B_1, B_2 \cdots B_n$ then



$$P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{\sum_{k=1}^{n} P(B_k)\,P(A/B_k)}$$

Q26) For a certain binary communication channel the probability that a transmitted '0' is received as '0' is 0.95 and the probability that a transmitted '1' is recieved as '1' is 0.90. ~~Ob~~ the Probability that a '0' is transmitted is 0.4. Find the probability that

i) a '0' is received

ii) a '1' is received

iii) If '0' is ~~transmitted~~ received then '1' is ~~recieved~~ transmitted

iv) If '1' is ~~transmitted~~ received then '0' is ~~received~~ transmitted

v) If a '0' is transmitted then '0' is received

Sol :

(i) '0' can be received in two ways

~~fast~~ when '0' is transmitted correctly

when '1' is transmitted wrongly

$$\therefore \frac{4}{10} \times \frac{95}{100} + \frac{6}{10} \times \frac{10}{100} = \frac{440}{1000} = 0.44$$

(ii) '1' can be received in two ways

'0' is transmitted wrongly

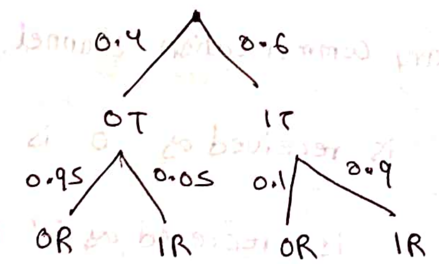
'1' is transmitted correctly

$$\frac{4}{10} \times \frac{5}{100} + \frac{6}{10} \times \frac{90}{100} = \frac{560}{1000} = 0.56$$

(or)

$$1 - P(\text{'0' receiving}) = 1 - 0.44 = 0.56$$

(iii) $P(1\,\text{re}/0\,\text{trans}) = \frac{P(1R \cap 0T)}{P(0T)} = \frac{0.4 \times 0.05}{0.}$



0 is Recieved and Probability that 1 is transmitted

$$\frac{0.6 \times 0.1}{0.4 \times 0.95 + 0.6 \times 0.1} = \frac{6 \times 1}{4 \times 9.5 + 6 \times 1} = \frac{6}{44} = \frac{3}{22}$$

(iv) $P(OT|IR) = \frac{P(OT \cap IR)}{P(IR)}$

$= \frac{0.4 \times 0.05}{0.56} = 0.036$

Note:

* If A and B are two events then

$$\boxed{P(A/B) + P(\bar{A}/B) = 1}$$

Proof :

$P(A|B) + P(\bar{A}/B)$

$= \frac{P(A \cap B)}{P(B)} + \frac{P(\bar{A} \cap B)}{P(B)}$

$= \frac{P(A \cap B) + P(B-A)}{P(B)}$

$= \frac{\cancel{P(A \cap B)} + P(B) - \cancel{P(A \cap B)}}{P(B)} = 1$

* ~~$P(E_1 \cap E_2 \cap \cdots \cap E_n)$~~

$P(E_1 \cap E_2) = P(E_1) P(E_2/E_1) = P(E_2) P(E_1/E_2)$

* $P(E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_n)$

$= P(E_1) P(E_2/E_1) P(E_3/E_1 \cap E_2) P(E_4/E_1 \cap E_2 \cap E_3)$

$\cdots P(E_n/E_1 \cap E_2 \cdots \cap E_{n-1})$

Proof

R.H.S $= P(\cancel{E_1}) \frac{P(E_1 \cap E_2)}{P(\cancel{E_1})} \frac{P(E_1 \cap E_2 \cap E_3)}{P(E_1 \cap E_2)} \cdots \cdots \frac{P(E_1 \cap E_2 \cdots \cap E_n)}{P(E_1 \cap E_2 \cdots E_{n-1})} = P(E_1 \cap E_2 \cap \cdots E_n)$

Q27) Let E and F be any two events with $P(E)=0.4$, $P(F)=0.3$ and $P(F/E)=3P(F/\bar{E})$ then find $P(E/F)$

sol :

$$P(E/F) = \frac{P(E\cap F)}{P(F)} = \frac{P(E\cap F)}{0.3}$$

$$P(F/E) = 3P(F/\bar{E})$$

$$\frac{P(E\cap F)}{P(E)} = 3\,\frac{P(F\cap\bar{E})}{P(\bar{E})}$$

$$\frac{P(E\cap F)}{P(E)} = 3\,\frac{P(F)-P(E\cap F)}{1-P(E)}$$

$$\frac{P(E\cap F)}{0.4} = 3\,\frac{0.3-P(E\cap F)}{0.6}$$

$$0.6\,P(E\cap F) = 1.2\left[0.3-P(E\cap F)\right]$$

$$0.6\,P(E\cap F) = 0.36-1.2\,P(E\cap F)$$

$$1.8\,P(E\cap F) = 0.36$$

$$P(E\cap F) = \frac{0.36}{1.8} = \frac{1}{5}$$

$$P(E/F) = \frac{P(E\cap F)}{0.3} = \frac{1/5}{0.3} = \frac{1/5}{3/10} = \frac{2}{3} = 0.66$$

Q28) Two players A & B alternately keep rolling a fair die. The person to get a six first wins the game. Given that player A starts the game, the probability of A wins the game is

sol :

$\frac{1}{6} + \frac{5}{6}\frac{5}{6}\frac{1}{6} + \frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{1}{6} + \cdots$

$\frac{1}{6}\left(1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \cdots\right)$

$= \frac{1}{6} \cdot \frac{1}{1-\left(\frac{5}{6}\right)^2} = \frac{1}{6} \cdot \frac{1}{1-\frac{25}{36}} = \frac{1}{6} \cdot \frac{1}{\frac{11}{36}}$

$= \frac{6}{11}$

Q29 Let E&G be two events with $P(E)>0$, $P(F|E)=0.3$ and $P(E\cap\bar{F})=0.2$. Then $P(E)$ equals

sol :

a) $\frac{1}{7}$ b) $\frac{2}{7}$ c) $\frac{4}{7}$ d) $\frac{5}{7}$

sol :

$P(F|E) = \frac{P(F\cap E)}{P(E)} = 0.3 \Rightarrow P(E\cap F) = 0.3P(E)$

$P(E\cap\bar{F}) = 0.2 \Rightarrow P(E) - P(E\cap F) = 0.2$

$\Rightarrow P(E) - 0.3P(E) = 0.2$

$\Rightarrow 0.7P(E) = 0.2$

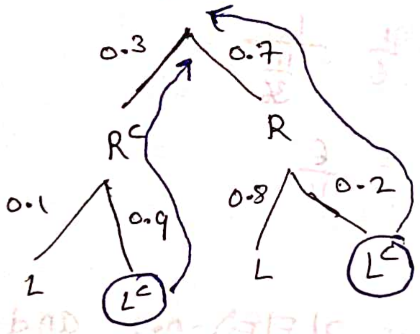
$P(E) = 2/7$

08/07/20

Q30 In a rainy day in the rainy season it may rain 70% of the time. If it rains, chance that village fair will make a loss on that day is 80%. However if it does not rain chance that fair will make a loss on the day is only 10%.. If the fair has not made a loss on a given day in the rainy season, what is the probability that it has not rained on that day?

Not having loss can be either on a rainy day or non-rainy day

$\therefore$ Required probability $P(R^c/L^c) = \dfrac{\cancel{\frac{7}{10}}\ \frac{3}{10}\cdot\frac{9}{10}}{\frac{7}{10}\cdot\frac{2}{10}+\frac{3}{10}\cdot\frac{9}{10}}$



$= \dfrac{3\times 9}{7\times 2+3\times 9} = \dfrac{27}{41} = 0.65$

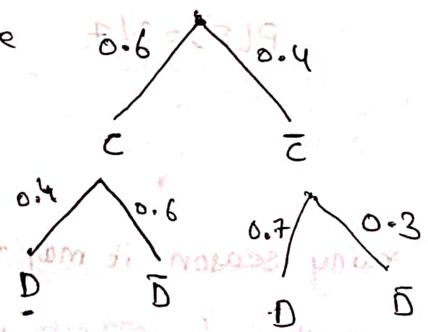
(Q31) The chance that doctor 'A' diagnoses a disease X correctly is 60%. The chances that a patient will die by his treatment after correct diagnosis is 40%. and the chances of death by wrong diagnosis 70%.

i) what is the probability that a patient died with disease X?

ii) If a patient ~~disease~~ died with disease X what is the probability that his disease was diagnosed correctly

Sol:

i) ~~death can be~~



$P(D) = 0.6\times 0.4 + 0.4\times 0.7$

$= 0.24 + 0.28$

$= 0.52$

(ii) $P(C/D) = \dfrac{P(C \cap D)}{P(D)}$

$= \dfrac{0.6 \times 0.4}{0.52} = \dfrac{0.24}{0.52} = \dfrac{6}{13}$

Random Variables

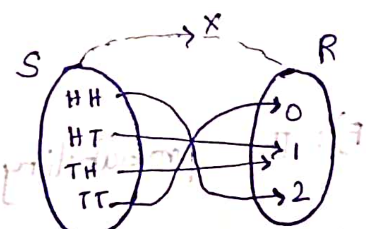
Random Variable (RV) is a function whose domain is the sample space (S) and range is set of real numbers.

Random variable assigns ~~outcomes of the~~ real numbers to outcomes of the random expreriment.

Eg: Suppose two coins are tossed the sample space is

$S = \{HH, HT, TH, TT\}$

Let X = number of heads



There are two types of random variables

(i) Discrete Random variable

(ii) Continuous Random variable.

Discrete Random Variable:

A random variable which takes finite values or countably infinite values is called a discrete random variable.

Continuous Random Variable:

A random variable which takes all values of an interval (or) uncountable values is called continuous random variable.

Probability Mass Function (PMF) or Probability Function (or) Point probability Function:

The probability function corresponding to discrete R.V is called probability mass function (pmf) provided it satisfies the following conditions:

(i) $P(x) \geq 0 \quad \forall x$

(ii) $\sum P(x) = 1$

→ Discrete RV

Eg:

X	0	1	2	3
P(X)	0.1	0.3	0.2	0.4

↓

PMF

Probability Density Function (PDF): The probability function corresponding to continuous RV is called probability density function (PDF) provided the following conditions are satisfied

(i) $f(x) \geq 0 \quad \forall x$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (area under curve b/w $-\infty$ & ∞ is 1)

Mean (or) Average (or) Expected Value of a R.V

$E(X) = \lambda = \mu = \bar{x} = \sum x P(x)$, X is disc

$= \int_{-\infty}^{\infty} x f(x)\,dx$, X is co

$E(x^n) = \sum x_i^n P(x_i)$ is called n^{th} moment of x.

Continuous Random Variable:

A random variable which takes all values of an interval (or) uncountable values is called continuous random variable.

Probability Mass Function (PMF) or Probability Function (or) point probability Function:

The probability function corresponding to discrete R.V is called probability mass function (pmf) provided it satisfies the following conditions:

(i) $P(x) \geq 0 \quad \forall x$

(ii) $\Sigma P(x) = 1$

Eg:

X (Discrete RV)	0	1	2	3
P(X) (PMF)	0.1	0.3	0.2	0.4

Probability Density Function (PDF):

The probability function corresponding to continuous RV is called probability density function (PDF) provided the following conditions are satisfied

(i) $f(x) \geq 0 \quad \forall x$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (area under curve b/w $-\infty$ & ∞ is 1)

Mean (or) Average (or) Expected Value of a R.V

$E(X) = \lambda = \mu = \bar{x} = \Sigma x P(x)$, X is discrete rv

$= \int_{-\infty}^{\infty} x f(x)\,dx$, X is continuous rv

Variance of a Random Variable

The measure of the dispersion or scatter of the values of the random variable (rv) about the mean μ is called variance of rv.

→ If the value tend to be concentrated near the mean, the variance is small

→ If the values tend to be distributed far from the mean, then the variance is large

Mathematically it can be expressed as

$$\mathrm{var}(x) = V(x) = \sigma_x^2 = E\left((x-\mu)^2\right) = \sum (x-\mu)^2 p(x) \quad , x \text{ is Discrete RV}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \,, \quad x \text{ is Cont. RV}$$

Note:

1) $\mathrm{var}(x) = \sigma_x^2 = E\left[(x-\mu)^2\right] = E(x^2) - (E(x))^2$

2) $\sigma_x^2 \geq 0$

$E(x^2) - (E(x))^2 \geq 0$

$\Rightarrow \boxed{E(x^2) \geq (E(x))^2}$

Avg value of square of R.V $\geq$ Square of avg value of random variable

Standard Deviation:

The positive square root of variance is called standard deviation (S.D)

$$\boxed{SD = \sigma = \sqrt{\text{variance}}}$$

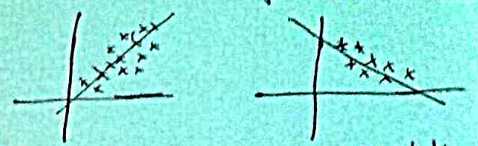
Covariance:

The measure of dispersion of the values of two random variable X & Y about their means μ_x & μ_y is called its covariance.

$$\boxed{cov(X,Y) = E(XY) - E(X)E(Y)}$$

Covariance tells How 2 variables vary w.r.to each other.

Covariance $\begin{cases} +ve : \text{Positive relation} \\ -ve : \text{negative relation} \end{cases}$



+ve relation   -ve relation

$Cov(X,Y) = E((x_i-\mu_x)(y_i-\mu_y))$

Note:

* If X and Y are independent

$E(XY) = E(X)E(Y)$

* If X and Y are independent R

$$\boxed{Cov(X,Y) = E(XY) - E(X)E(Y) = 0}$$

Properties of Mean & Variance:

* $E(k) = k$
* $E(ax) = aE(x)$
* $E(ax \pm b) = aE(x) \pm b$
* $E(ax \pm by) = aE(x) \pm bE(y)$

$$\boxed{E(x^n) = \sum x_i^n P(x)}$$

* $V(k) = 0$
* $V(ax) = a^2 V(x)$
* $V(ax \pm b) = a^2 V(x)$

$$\boxed{\begin{array}{l} Cov(X,X) = Var(X) \\ Cov(aX,Y) = a\,Cov(X,Y) \\ Cov(X,a) = 0 \end{array}}$$

* ~~If X and Y are independent~~

$V(ax \pm by) = a^2 V(x) + b^2 V(y) \pm 2ab\,Cov(X,Y)$

* If X and Y are independent R.Vs then

$V(ax \pm by) = a^2 V(x) + b^2 V(y)$

$$\boxed{\begin{array}{l} V(ax_1 \pm bx_2 \pm cx_3) \\ = a^2 V(x_1) + b^2 V(x_2) + c^2 V(x_3) \end{array}}$$

Covariance:

The measure of dispersion ... variable X & Y about their mea... covariance.

$$\boxed{cov(X,Y) = E(XY) - E(X)E(Y)}$$

Note:

* If X and Y are independent R.Vs then

$$E(XY) = E(X)E(Y)$$

* If X and Y are independent RVs then its covariance is zero.

$$\boxed{Cov(X,Y) = E(XY) - E(X)E(Y) = 0}$$

Properties of Mean & Variance:

* $E(k) = k$
* $E(ax) = aE(x)$
* $E(ax \pm b) = aE(x) \pm b$
* $E(ax \pm by) = aE(x) \pm bE(y)$

$$\boxed{E(x^n) = \sum x_i^n p(x)}$$

* $V(k) = 0$
* $V(ax) = a^2 V(x)$
* $V(ax \pm b) = a^2 V(x)$

$$\boxed{\begin{aligned} &Cov(X,X) = Var(X) \\ &Cov(aX,Y) = aCov(X,Y) \\ &Cov(X,a) = 0 \end{aligned}}$$

* ~~If X and Y are independent~~

$$V(aX \pm bY) = a^2 V(X) + b^2 V(Y) \pm 2ab\,Cov(X,Y)$$

* If X and Y are independent R.Vs then

$$V(aX \pm bY) = a^2 V(X) + b^2 V(Y)$$

$$\boxed{\begin{aligned} &V(aX_1 \pm bX_2 \pm cX_3) \\ &= a^2 V(X_1) + b^2 V(X_2) + c^2 V(X_3) \end{aligned}}$$

(Q32) The two sides of a fair coin are labelled as 0 & 1. The coin is tossed two times independently. Let M & N denote the labels corresponding to the outcomes of those tosses. For a random variable X, defined as X = Min(M, N) the expected value E(X) is ________

Sol:

M	N	X
0	0	0
0	1	0
1	0	0
1	1	1

x	0	1
P(X)	$\frac{3}{4}$	$\frac{1}{4}$

$E(x) = \sum x P(x)$

$= 0 \cdot P(0) + 1 \cdot P(1)$

$= 0 \cdot \frac{3}{4} + 1 \cdot \frac{1}{4} = \frac{1}{4} = 0.25$

(Q33) If X & Y are two random variables such that E(X) = 10 & V(X) = 25. Find the positive values of a & b for which E(Y) = 0 and V(Y) = 1 for Y = ax − b

Sol:

$Y = ax - b$

$V(Y) = 1$

$V(ax-b) = 1$

$a^2 V(x) = 1$

$a^2 = \frac{1}{25} \Rightarrow a = \frac{1}{5}$

$E(Y) = 0 \Rightarrow E(ax-b) = 0 \Rightarrow aE(x) - b = 0$

$b = aE(x)$

$= \frac{1}{5} 10 = 2$

$\therefore a = 0.2, b = 2$

(Q34) Suppose that one word is to selected at random from the sentence "THE GIRL PUT ON HER BEAUTIFUL RED HAT". If x denotes the length of the word that is selected at random then find

(i) $E(x)$ (ii) $Var(x)$

total no of words = 8

X	2	3	4	9
P(x)	1/8	5/8	1/8	1/8

(i) $E(x) = 2\frac{1}{8} + 3\frac{5}{8} + 4\frac{1}{8} + 9\frac{1}{8}$

$= \frac{30}{8} = \cancel{3.75}\ \frac{15}{4} = 3.75$

(ii) $Var(x) = E(x^2) - (E(x))^2$

$E(x^2) = 4\frac{1}{8} + 9\frac{5}{8} + 16\frac{1}{8} + 81\frac{1}{8}$ | $E(x^2) = x^2 P(x)$

$= \frac{4+45+16+81}{8} = \frac{146}{8} = \frac{73}{4}$

$var(x) = \frac{73}{4} - \left(\frac{15}{4}\right)^2 = \frac{73}{4} - \frac{225}{16}$

$= \frac{292-225}{16} = \frac{67}{16}$

(Q35) Passengers try repeatedly to get a seat reservation in any train running b/w two stations until they are successful. If there is 40% of chance of getting reservation in any attempt by a passenger then the average number of attempts needed to make to get a seat reserved is ______

Sol :

Let x = No of attempts needed

x	1	2	3	.....	k	.....
$P(x)$	$\frac{2}{5}$	$\frac{3}{5}\,\frac{2}{5}$	$\left(\frac{3}{5}\right)^2\frac{2}{5}$	...	$\left(\frac{3}{2}\right)^{k-1}\frac{2}{5}$	.....

$$E(x) = 1\cdot\frac{2}{5} + 2\cdot\frac{3}{5}\frac{2}{5} + 3\cdot\left(\frac{3}{5}\right)^2\frac{2}{5} + \cdots$$

$$= \frac{2}{5}\left(1 + 2\cdot\frac{3}{5} + 3\left(\frac{3}{5}\right)^2 + \cdots\right)$$

$$\boxed{(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots}$$

$$= \frac{2}{5}\left(1 - \frac{3}{5}\right)^{-2}$$

$$= \frac{2}{5}\left(\frac{2}{5}\right)^{-2} = \frac{2}{5}\cdot\frac{5}{2}\cdot\frac{5}{2} = \frac{5}{2} = 2.5$$

(36) The random variable x has pdf

$$f(x) = \begin{cases} ax+bx^2, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

If $E(x) = 0.6$ then find

(i) a and b

(ii) mean

(iii) variance

Sol : we know that

$$\int_{-\infty}^{\infty} f(x)\,d(x) = 1$$

But the $f(x)$ is given for $(0,1)$

$$\therefore \int_0^1 f(x)\,dx = 1 \quad = \int_0^1 (ax+bx^2)\,dx = 1$$

$$\Rightarrow \left[a\frac{x^2}{2} + b\frac{x^3}{3}\right]_0^1 = 1$$

$\frac{ax^2}{2} + \frac{bx^3}{3} = 1 \qquad \frac{a}{2} + \frac{b}{3} = 1 \Rightarrow 3a + 2b = 6$ — ①

given $E(x) = 0.6$

$$\int_0^1 x(ax + bx^2)\,dx = 0.6$$

$$\int_0^1 (ax^2 + bx^3)\,dx = 0.6$$

$$\left[\frac{ax^3}{3} + \frac{bx^4}{4}\right]_0^1 = 0.6$$

$$\frac{a}{3} + \frac{b}{4} = 0.6 \Rightarrow 4a + 3b = 7.2 \quad \text{— ②}$$

Frm ① & ②

① × 3 ⇒ $9a + 6b = 18$

② × 2 ⇒ $8a + 6b = 14.4$

$a = 3.8$

$\Rightarrow \frac{3.8}{2} + \frac{b}{3} = 1$

$\frac{b}{3} = -0.9$

$b = -2.4$

$a = 3.8 \quad b = -2.4$

(ii) $P(x < 1/2)$

$= \int_0^{1/2} f(x)\,dx$

$= \int_0^{1/2} (ax + bx^2)\,dx$

$= \left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^{1/2}$

$= \frac{a}{8} + \frac{b}{24}$

$= \frac{3.8}{8} - \frac{2.4}{24}$

$\cancel{= 0.475 - 0.1}$

$\cancel{= 0.375} = 0.35$

(i) mean $= E(x) = 0.6$

(iii) variance $= E(x^2) - (E(x))^2$

$E(x^2) = \int_0^1 x^2(ax + bx^2)\,dx$

Put $b = -2.4$ and solve

$= \left[\frac{ax^4}{4} + \frac{bx^5}{5}\right]_0^1 = \frac{a}{4} + \frac{b}{5}$

$= \frac{18 - 9.6}{20} = \frac{8.4}{20}$

$= 0.42$

$Var(x) = E(x^2) - (E(x))^2$

$= \cancel{\frac{8.2}{20} - 0.36}$

$= 0.42 - 0.36$

$= \cancel{0.03} = 0.06$

Note :

$\rightarrow Var(X) = E[(X-\mu)^2]$

$= \sum_x (x-\mu)^2 P(x)$

$= \sum_x x^2 P(x) + \mu^2 P(x) - 2x\mu P(x)$

$= \sum_x x^2 P(x) + \mu^2 \sum P(x) - 2\mu \sum x P(x)$

$= E(x^2) + \mu^2 - 2\mu^2$

$= E(x^2) - \mu^2$

$= E(x^2) - [E(x)]^2$

$\therefore \boxed{Var(x) = E(x^2) - [E(x)]^2}$

$\rightarrow Var(ax+b) = E((ax+b - a\mu - b)^2)$

$= E(a^2(x-\mu)^2)$

$= a^2 E[(x-\mu)^2]$

$= a^2 Var(x)$

$\therefore \boxed{Var(ax \pm b) = a^2 Var(X)}$

09/07/20

Q37) A man draws 3 balls from a jug containing 5 white balls and 7 ~~bad~~ black balls. He gets Rs.20 for each white ball and Rs.10 for each black ball. What is his expectation?

a) 21.25 B) 42.50 c) 31.25 d) 45.21

Sol:

Let x = profit amount

	3W	3B	1W&2B	2W&1B
x	60	30	40	50
$P(x)$	$\frac{5C_3}{12C_3}$	$\frac{7C_3}{12C_3}$	$\frac{5C_1 \times 7C_2}{12C_3}$	$\frac{5C_2 \times 7C_1}{10C_3}$

$$E(x) = 60 \times \frac{5C_3}{12C_3} + 30 \times \frac{7C_3}{12C_3} + 40\,\frac{5C_1 \times 7C_2}{12C_3} + 50\,\frac{5C_2 \times 7C_1}{12C_3}$$

$$E(x) = 42.50$$

Discrete Probability Distributions:

1) Binomial Distribution:

Conditions for binomial distribution: Parameter: (n, P)

a) Number of trails should be finite

b) There must be only 2 possible outcome (success & failure)

c) Probability of success is each trail is constant

d) All trails are independent.

Binomial distribution can be used only if above four conditions are satisfied

Def: The probability mass function of the binomial distribution is given by

$$P(x) = {}^nC_x \, p^x q^{n-x}, \quad x = 0,1,2,\ldots n \;\&\; q = 1-p$$

where n = no of trails
p = probability of success
q = probability of failure
x = number of successes

Note:

* Mean = np
* variance or $V(x) = npq$
* Standard deviation = $\sqrt{npq}$
* Mean > Variance

~~$Var(x) = E(x^2) - (E(x))^2$~~
~~np^2~~

Q38) Over a large set of inputs a program runs twice as often as it aborts. The probability that of the next 6 attempts 4 or more will run is ______.

Sol :

$$P(R) = 2P(R^c)$$

$$\Rightarrow P(R^c) = 1/3 \Rightarrow P(R) = 2/3$$

$$P(x \geq 4) = \cancel{P(x \geq 4)}\; P(x=4) + P(x=5) + P(x=6)$$

$$= {}^6C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2 + {}^6C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right) + {}^6C_6 \left(\frac{2}{3}\right)^6$$

$$= 15\,\frac{2^4}{3^6} + 6\,\frac{2^5}{3^6} + \frac{2^6}{3^6}$$

$$= \cancel{\frac{80 + 64 + 64}{3^5}} = \cancel{\frac{208}{243}}$$

$$= 0.68$$

(Q39.) Consider an unbiased cubic die, with opposite faces coloured identically and each face coloured red, blue or green such that each color appears only two times on the die. If the die is thrown thrice, the probability of obtaining red colour on the top face of the die atleast twice is.

Sol:

$$P(X \geq 2) = P(X=2) + P(X=3)$$

$$= 3C_2 \left(\frac{2}{6}\right)^2 \left(\frac{4}{6}\right)^1 + 3C_3 \left(\frac{2}{6}\right)^3$$

$$= 3 \cdot \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^3$$

$$= \frac{3 \cdot 2}{3^3} + \frac{1}{3^3} = \frac{7}{3^3} = \frac{7}{27}$$

(Q40) A fair die is rolled three times. Given that 6 appeared atleast once, the conditional probability that 6 appeared exactly twice equals ______

Sol:

$$P(X=2 | X \geq 1) = \frac{P\{(X=2) \cap (X \geq 1)\}}{P(X \geq 1)}$$

$$= \frac{P(X=2)}{P(X \geq 1)}$$

Now $P(X=2) = 3C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^1 = \frac{3 \times 5}{6^3}$

$$P(X \geq 1) = 1 - P(X=0) = 1 - 3C_0 \left(\frac{5}{6}\right)^3 = 1 - \left(\frac{5}{6}\right)^3 = \frac{6^3 - 5^3}{6^3}$$

$$\Rightarrow \frac{P(X=2)}{P(X \geq 1)} = \frac{\frac{3 \times 5}{6^3}}{\frac{6^3 - 5^3}{6^3}} = \frac{15}{91} = 0.1648$$

Q11) A certain type of missile hits the target with probability 0.3.
* The minimum number of missiles that should be fired so that probability of hitting the target atleast once is greater than 75% is ______

Sol:

Let k be the mininum number of missles

Now $P(x \geq 1) \geq 75/100$

$\Rightarrow 1 - P(x=0) \geq 75/100$

$$1 - {}^{k}C_0 \left(\frac{3}{10}\right)^0 \left(\frac{7}{10}\right)^k \geq \frac{75}{100}$$

$$1 - \frac{7^k}{10^k} \geq 0.75$$

$$\Rightarrow \left(\frac{7}{10}\right)^k \leq 0.25$$

for k = 4 is the least value that satisfies the condition $\therefore k=4$.

Q12) A man with n keys wants to open a lock. He tries his key at random. Expected number of attempts for his success is (key are replaced after every attempt)

(a) $n/2$ b) n c) $\sqrt{n}$ d) none

Sol:

X	1	2	3	- - - -
P(x)	$1/n$	$\frac{n-1}{n}\frac{1}{n}$	$\left(\frac{n-1}{n}\right)^2 \frac{1}{n}$	- - - -

$$\therefore E(x) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{n-1}{n}\frac{1}{n} + 3 \cdot \left(\frac{n-1}{n}\right)^2 \frac{1}{n} + \cdots$$

$$= \frac{1}{n}\left(1 + 2\,\frac{n-1}{n} + 3\left(\frac{n-1}{n}\right)^2 + \cdots\right)$$

$$= \frac{1}{n}\left(1 - \frac{n-1}{n}\right)^{-2} = \frac{1}{n}\left(\frac{1}{n}\right)^{-2} = \underline{n}$$

$\therefore$ opt (b)

Poisoson Distribution:

The probability mass function of the poisson distribution is given by

$$P(x) = \frac{\lambda^{x} e^{-\lambda}}{x!}, \quad x=0,1,2\ldots \qquad \begin{pmatrix} n \to \infty \\ p \to 0 \end{pmatrix}$$

λ = Average (or) Mean rate of occurance of an event over the fixed time (or) fixed space.

x = Number of events or occurances within a specified time or space

Note:

* Mean, $E(x) = \lambda$ Parameter: λ
* ~~Variable~~, Variance $V(x) = \lambda$
* $SD = \sqrt{\lambda}$
* Mean = Variance
* If mean of poisson distribution is not given then mean of binomial distribution should be used if
  1) number of trails are large ($n \to \infty$)
  2) Probability of occurance of an event is very small ($p \to 0$)

(Q42) 2000 cashew nuts are throughroughly mixed in a flour. The entire mixature is divided into 1000 equal partes and each part is used to make a biscuit. Assume that no cashew nut is broken in the process. A biscuit is ~~pick at~~ picket a random. The probability that it contains atleast one cashew nut is ______

Sol:

1000 biscuits contain 2000 cashew nuts

∴ on avg each biscuit contains 2 cashew nuts

$\Rightarrow \lambda = \frac{2000}{1000} = 2$ (mean $= np = (2000)\left(\frac{1}{1000}\right) = 2$

Let x = no of cashew nuts | biscuit

$P(X \geq 1) = 1 - P(X=0)$

$= 1 - \frac{\lambda^0 e^{-\lambda}}{0!}$ $P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$= 1 - e^{-2}$

$= 0.865$

(Q43) If a random variable X has a possion distribution with mean 5, then the expectation $E[(X+2)^2]$ equals ______

Sol:

Mean = variance $= \lambda = 5$

Now, $E[(X+2)^2] = E[X^2 + 4X + 4]$ $V(X) = E(X^2) - (E(X))^2$

$= E[X^2] + 4E(X) + 4$ $\Rightarrow E(X^2) = V(X) + (E(X))^2$

$= \lambda + \lambda^2 + 4\lambda + 4$

$= 5 + 25 + 20 + 4$

$= 54$

(Q44) On the average 15 out of state cars pass a certain point on a road per hour. The probability that exactly four out of state cars pass that point in a 12-minute perio is ______.

Sol:

60 min —— 15

12 min —— 3

$\therefore \lambda = 3$

$$P(x=4) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{\lambda^4 e^{-\lambda}}{4!}$$

$$= \frac{3^4 e^{-3}}{4!}$$

$$= 0.168$$

(Q45) A telephone exchange receives an average of 180 calls per hour. The probability that it will receive only two calls in a given minute is ______

Sol :

60 min —— 180

1 min —— 3

$\therefore \lambda = 3$

$$P(x=2) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \frac{3^2 e^{-3}}{2!} = 0.224$$

(Q46) The probability of a resistor being defective is 0.02. There are 50 such resistors in a circuit. The probability of 2 or more defective resistors in the circuit is ______

Sol :

$\lambda = np = 50(0.02) = 1$ (when no mean is given we use mean of binomial distribution)

$$P(x \geq 2) = 1 - P(x=0) - P(x=1)$$

$$= 1 - \frac{\lambda^0 e^{-\lambda}}{0!} - \frac{\lambda^1 e^{-\lambda}}{1}$$

$$= 1 - e^{-1} - e^{-1} = 1 - 2e^{-1} = 0.26$$

Continuous Probability Distributions:

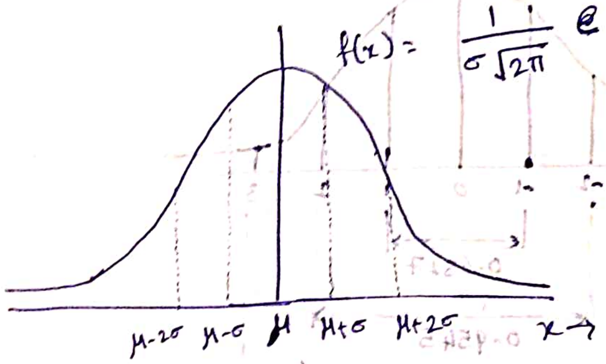
(i) Normal (Gaussian) Distribution:

The probability distribution function of a Random Variable which is normally distributed with mean μ and variance σ^2 is given by

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty, \; -\infty < \mu < \infty, \; \sigma > 0}$$

The below is the graph of normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

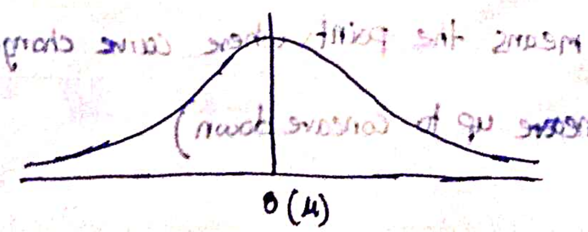


(ii) Standard Normal Distribution (SND):

A RV z is said to have a standard Normal Distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$ if its PDF is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty$$

where $z = \frac{x-\mu}{\sigma}$ is called standard normal Random variable



Note :

Area under the curve b/w

* $\mu-\sigma$ and $\mu+\sigma$ is 0.6827

* $\mu-2\sigma$ and $\mu+2\sigma$ is 0.9545

* $\mu-3\sigma$ and $\mu+3\sigma$ is 0.9973

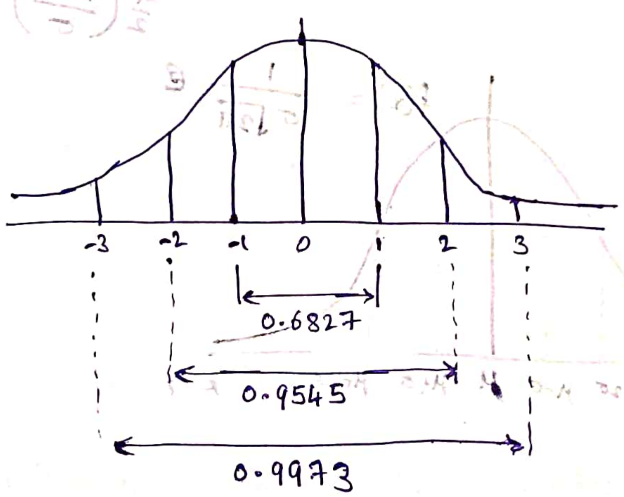
For Standard Normal Distribution Curve

$\mu+\sigma = 0+1=1$

$\mu-\sigma = 0-1=-1$

$\mu+2\sigma = 0+2(1)=2$

⋮



Note:

* Total area under the normal curve is unity

* The curve is symmetric about the line $x=\mu$

∴ The area left to $x=\mu$ is 0.5 and right to $x=\mu$ is 0.5

* Maximum value occurs at $x=\mu$

* Points of inflection of normal curve are $\mu-\sigma$ and $\mu+\sigma$

(point of inflection means the point where curve changes its shape from concave up to concave down)

* If $X_1, X_2 \ldots X_n$ are <u>independent</u> normal random variables then its linear combination $c_1X_1 + c_2X_2 + \cdots + c_nX_n$ is also a normal random variable.

(Q47) A Random variable X has normal distribution with ~~mea~~ mean 100. If $P(100 < X < 120) = 0.3$ then $P(X < 80)$ is

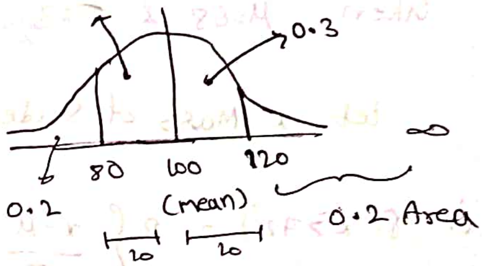
Sol :

Given $P(100 < X < 120) = 0.3$

i.e., Area b/w $x = 100$ and $x = 120$ is 0.3

$\therefore$ Area b/w 100 & 80 will also be 0.3

to the left of 80 area will be 0.2



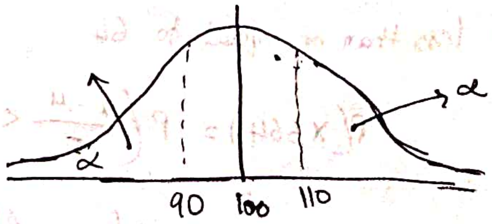
$\therefore P(X < 80) = 0.2$

(Q48) For a RV X $(-\infty < X < \infty)$ following normal distribution the mean is $\mu = 100$. If the probability is $P = \alpha$ for $x \geq 110$ then the probability of x lying b/w 90 & 110 i.e., $P(90 \leq x \leq 110)$ will be equal to ______

Sol :

$\therefore P(90 \leq x \leq 100) = 1 - \alpha - \alpha$

$= 1 - 2\alpha$



Qu9) If the masses of 300 students are normally distributed with mean 68 kg and standard deviation 3 kgs. How many students have masses

(i) Greater than 72 kg

(ii) less than or equal to 64 kg

(iii) b/w 65 & 71 kg (both inclusive)

[Note: Area under normal curve b/w ~~z0~~ ~~z0~~ z=0 & z=1.33 is 0.4082]

Sol:

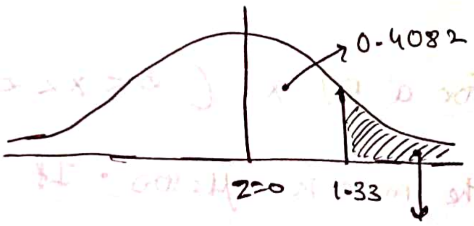
Given $\mu = 68$ & $\sigma = 3$

let x = Mass of students

(i) $P(x > 72) = P\left\{ \frac{x-\mu}{\sigma} > \frac{72-\mu}{\sigma} \right\}$ (Standarizing normal variable)

$= P\left\{ z > \frac{72-68}{3} \right\}$

$= P(z > 1.33)$



$0.5 - 0.4082$

$= 0.0918$

∴ no of students whose mass is greater than ~~28~~ 72 kg is

$300(0.0918) \cong 28$

(ii) less than or equal to 64

$P(x \leq 64) = P\left(\frac{x-\mu}{\sigma} < \frac{64-\mu}{\sigma} \right)$

$= P\left(z < \frac{64-68}{3} \right) = P(z < -1.33)$

$\Rightarrow 300(0.098) \cong 28$ $= 0.0918$

(iii) $P(65 \le X \le 71)$

$= \cancel{P(-1.33 \le X \le} 1.33$

$= P(-1 \le X \le 1)$

$= 0.6827$

$\therefore$ no of students $= 300(0.6827) = 205$

(Q50) A test has 5 multiple choice questions. Each has 4 options. What is the probability that a student will choose 'B' for atleast 3 questions, if he/she leaves no questions blank?

a) $\frac{1}{1024}$ b) $\frac{1}{64}$ c) $\frac{53}{512}$ d) $\frac{29}{128}$

sol:

$P(\text{choosing B}) = 1/4$

$\Rightarrow$ we solve this using binomial distribution

$p = 1/4 \quad q = 3/4$

$P(X \ge 3) = P(X=3) + P(X=4) + P(X=5)$

$= 5C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 + 5C_4 \left(\frac{1}{4}\right)^4 \frac{3}{4} + 5C_5 \left(\frac{1}{4}\right)^5$

$= \frac{10 \cdot 9}{4^5} + \frac{5 \cdot 3}{4^5} + \frac{1}{4^5} = \frac{106}{1024} = \frac{53}{512}$

(Q51) A random variable X follows bionamial distribution with mean $= 2$(variance) and mean + variance $= 3$ then $P(X=3)$ is ______

sol:

mean $= 2$(variance) $\Rightarrow np = 2npq \Rightarrow 2q = 1 \Rightarrow q = 1/2$

$\Rightarrow p = 1/2$

mean + variance = 3

$np + npq = 3$

$\frac{n}{2} + \frac{n}{4} = 3$

$\frac{3n}{4} = 3 \Rightarrow n = 4$

$P(x=3) = {}^4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right) = \frac{4}{2^4} = \frac{1}{4} = 0.25$

(Q52) Let X_1 & X_2 be independent normal random variables with means μ_1 and μ_2 and standard deviation σ_1 & σ_2 respectively consider $Y = X_1 - X_2$; $\mu_1 = \mu_2 = 1$, $\sigma_1 = 1$, $\sigma_2 = 2$. Then

a) Y is normally distribute with mean 0 and variance 1

b) Y is normally distribute with mean 0 and variance 5

c) Y has mean 0 and variance 5 but is not normally distributed

d) Y has mean 0 and variance 1 but is NOT normally distributed

Sol:

Linear combination of independent normally distributed random variable is also a normally distributed random variable.

$E(Y) = E(X_1 - X_2) = E(X_1) - E(X_2)$

$= \mu_1 - \mu_2 = 1 - 1 = 0$

$\text{variance}(Y) = \text{var}(X_1 - X_2)$

$= (1)^2 \text{var}(X_1) + (-1)^2 \text{var}(X_2) \pm 2 \underbrace{\text{Cov}(X_1, X_2)}_{0}$

$= \text{var}(X_1) + (-1)^2 \text{var}(X_2) + 0$

$= \cancel{1 + 4(1)}$

$= \sigma_1^2 + \sigma_2^2$

$= 1 + (2)^2 = 5$

$\therefore$ opt (b)