INDEX

NAME: K.Gowtham STD.: ______ SEC.: ______ ROLL NO.: ______ SUB.: ______

S. No.	Date	Title	Page No.	Teacher's Sign / Remarks
		Graph Theory	7 to 69	
		Logic	70 to 154	

26/04/20

Discrete Mathematics

- → Mathematical logic (2-4 M)
- → Combinatorics (2-4)
- → Set Theory (2-4)
- → Graph Theory ~~(4-5)~~ (4-6)

~~SHgt~~ Syllabus :

Mathematical logic (2M - 4M)

Propositional logic
logical equivalence
Inference rules
} Basics

Quantifier
Nested Quantifier Imp → (2 marks)
Quantifier with inference rules
} Advanced

Set Theory: (2m - 4m)

Basics
functions
Relations
} basic part

Types of relations
Imp
posets & tosets
lattices
Group theory Imp
} Advanced

Graph Theory: (4m - 6m)

Basics of graph
types of graphs
} basic part (1Q)

Planarity has been removed from syllabus

* connectivity
* coloring
Matching
Covering
} Advanced part (1Q)

4m is sure
but if 3 Q is asked it ~~would~~ would be asked along with some other subject

Combinatorics (2m - 4m)

- → Product rule & sum rule
- → permutations & combinations (repetations)
- → Inclusion & exclusion
- → pigeon hole principle
- → Binomial coefficient
- ★ → Generating Function
- ★ → Recurrence Relation
- → Euler's ϕ function
- → Derrangements

26/04/20

Graph Theory:

→ A graph is denfined as

$$G = (V, E)$$

E → set of edges

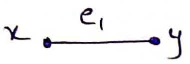
V → set of vertices

each edge must be associated with unorder pair of vertices.

End vertices:

~~Consider Each~~ edge ~~must be associated with unordered pair of vertices called as end vertices.~~

Eg:

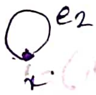


~~x & y are end vertices of edge e1~~

vertex with degree 1 corresponding edge is called pendent edge.

loop/self loop:

If both the end vertices are same, then that edge is called a loop or a self loop.



Incident point:

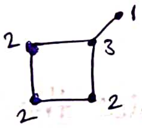
It is a point where edge meets the vertex.

incident point

Degree/valency ($d(v_i)$):

no of edges incident on a vertex is called degree of the vertex.

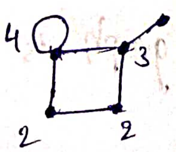
Eg:



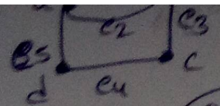
pendent vertex: vertex with degree 1

Isolated vertex: Vertex with degree 0

→ Every loop contributes '2' to the degree.



called as p...



e_1 & e_2 are parallel edges

graph 1

Null graph:

It is a set of isolated vertices

Adjacent vertices

In graph 1, a, b are adjacent

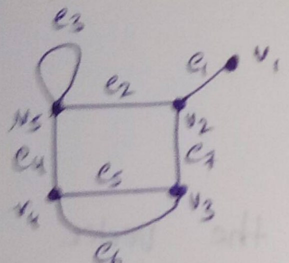
c, d are adjacent

Adjacent edges:

In grap1, (e_3, e_4) are adjacent

(e_3, e_5) are not adjacent

Eg:



$d(v_1) = 1$

$d(v_2) = 3$

$d(v_3) = 3$

$d(v_4) = 3$

$d(v_5) = 4$

Sum of all degrees $= 14 = 2(7)$

total no of edges

Theorem 1:

Sum of degrees of all vertices is twice the no of edges.

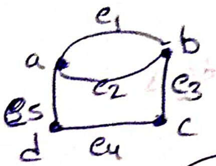
$$\boxed{\sum_{i=1}^{n} d(v_i) = 2|E|}$$

Theorem 2:

No of odd degree vertices i...

parallel edges :

If there are 2 or more edges associated with same end vertices, called as parallel edges.



e_1 & e_2 are parallel edges

graph 1

Null graph :

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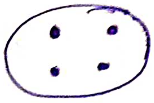

Adjacent vertices

In graph 1, a,b are adjacent

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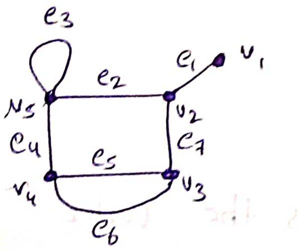

$d(v_1) = 1$

$d(v_2) = 3$

$d(v_3) = 3$

$d(v_4) = 3$

$d(v_5) = 4$

Sum of all degrees $= 14 = 2\underbrace{(7)}_{\text{total no of edges}}$

Theorem 1 :

Sum of degrees of all vertices is twice the no of edges.

$$\boxed{\sum_{i=1}^{n} d(v_i) = 2|E|}$$

Theorem 2 :

No of odd degree vertices in a graph is even.

→ Based on Parallel edges & self loops graphs are classified into 3 types

	Parallel graph	Self loops
Simple graph	×	×
Multigraph	✓	×
Pseudograph	✓	✓

(we mainly discuss simple graph in our syllabus)

→ From now all theorems are defined for simple graphs

Theorem 3:

Maximum degree of a vertex, in a simple graph with n vertices, is $n-1$.

Theorem :4

Maximum no of edges, in a simple graph with n vertices, is $nC_2 = \frac{n(n-1)}{2}$

Note:

→ No of different graphs possible with n distinct vertices is $= 2^{\frac{n(n-1)}{2}}$

→ No of different graphs possible with n distinct vertices and 'e' edges is $\left[\frac{n(n-1)}{2}\right] C_e$

Degree Sequence:

If the degrees of a graph are written in increasing order or decreasing order, we call it a degree sequence

→ Not all degree sequences forms graph.

→ The degree seq. which forms a graph is called graphical

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Check if below vertices forms a graph or not.

a) 2, 3, 3, 4, 4, 5

5, 4, 4, 3, 3, 2

Here theorem 2 is not satisfied as no of odd vertices is odd.

b) 2, 3, 4, 4, 5

5, 4, 4, 3, 2

It ~~sat~~ satisfies thm 2

i.e., no of odd vertices is even

Also from Thm 3 Maximum degree in a simple graph must be $n-1 = 4$

but here max degree is 5

and hence these vertices doesn't form a graph

c) 1, 3, 3, 4, 5, 6, 6

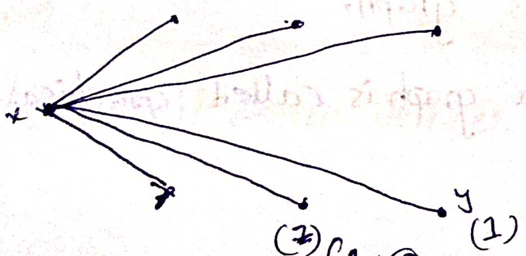
6, 6, 5, 4, 3, 3, 1

no of odd degree vertices is even

Max degree is 6

$7-1=6$

x 6 6 5 4 3 3 1 y
z



(z) (1+5) → 5 more required but we don't have 5 vertices for z to make an edge.

∴ graph is not possible with above set of degrees

d) $\{0, 1, 2, 3 \ldots\ldots n-1\}$

$\{n-1, n-2, n-3, \ldots\ldots 2, 1, 0\}$

Eg: put $n = 4$

$\{4, 3, 2, 1, 0\}$

Here 4 is max degree but one vertex is isolated

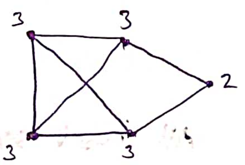
So these set of vertices is not graphical

e) 2, 3, 3, 3, 3

3, 3, 3, 3, 2

no of odd vertices = even

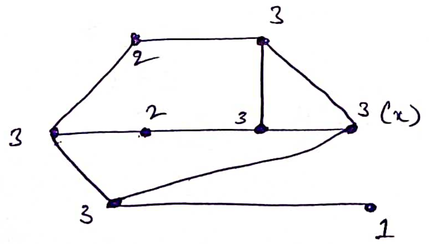
Max degree $3 \leq 5-1$



∴ Graphical

Havell - Hakimi Algorithm:

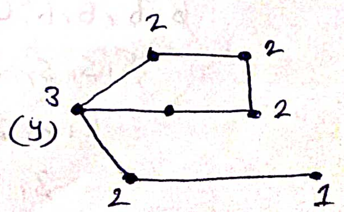
Consider below graph



degree sequence is

3, 3, 3, 3, 3, 2, 2, 1

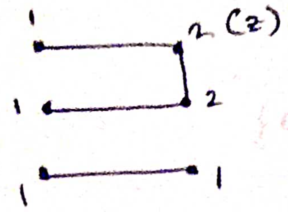
Consider we need to remove vertex x



degree sequence is

3, 2, 2, 2, 2, 2, 1

Now remove vertex y

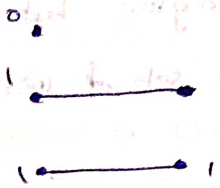


degree seq. is 2,2,1,1,1,1

Removing vertex z

degree seq. is

1,1,1,1,0



From initial degree sequence

$\cancel{3}, 3, 3, 3, 3, 2, 2, 1$

2, 2, 2, 3, 2, 2, 1

write in descending order

$\cancel{3}, 2, 2, 2, 2, 2, 1$

1, 1, 1, 2, 2, 1

write in descending order

$\cancel{2}, 2, 1, 1, 1, 1$

1, 0, 1, 1, 1

$\Rightarrow \cancel{1}, 1, 1, 1, 0$

0, 1, 1, 0

$\Rightarrow \cancel{1}, 1, 0, 0$

0, 0, 0

All these sequences will be graphical only if other sequence derived from previous is graphical

As {0, 0, 0} is graphical we conclude that initial sequence is also graphical.

P/6

$S_1 = \{6, 6, 6, 6, 4, 3, 3, 0\}$

$\cancel{6}, 6, 6, 6, 4, 3, 3, 0$

$\cancel{5}, 5, 5, 3, 2, 2, 0$

$\cancel{4}, 4, 2, 1, 1, 0$

$\cancel{3}, 1, 0, 0, 0$

0, -1, -1, 0 $\therefore$ not graphical

$S_2 = \{6,5,5,4,3,3,2,2,2\}$

$\cancel{6}55433222$

44322122

$\cancel{4}4322221$ (ordering)

3211221

$\cancel{3}222111$ (ordering)

$\cancel{1}11111$

01111

$\cancel{1}\cancel{1}11$ (ordering)

011

000

∴ graphical

Gate 2010

(1) which of the below seq. can't be forms a simple graph

I) 7,6,5,4,4,3,2,1

II) 6,6,6,6,3,3,2,2

III) 7,6,6,4,4,3,2,2

IV) 8,7,7,6,4,2,1,1

A) I & II B) III & IV C) IV only D) II & IV

I) $\cancel{7}6544321$

$\cancel{5}433210$

$\cancel{3}2210$

$\cancel{1}10$

0

∴ graphical

II) $\cancel{6}6663322$

5552212

$\cancel{5}552221$ (ord)

$\cancel{4}41111$

30001

31000 (ord)

∴ not graphical

III) $\cancel{7}6644322$

$\cancel{5}533211$

422101

$\cancel{4}2211$ (ord)

$\cancel{1}100$

000

∴ graphical

IV) $\underbrace{87764211}_{\text{we dont have 8 vertices}}$

∴ not graphical

∴ II & IV

Theorem: 5

In a simple graph atleast two vertices will have same degree ($n \geq 2$)

Eg: $\{5,4,3,2,1\}$ is not graphical

Proof:

Consider a graph of n vertices

for n distinct degrees the degrees can be

$\{n, n-1, n-2, \ldots 3, 2, 1\}$

$\downarrow$

but max degree possible is $n-1$

So degrees must range b/w 1 & $n-1$

but we can for n distinct integers between 1 & $n-1$.

$\therefore$ In a simple graph atleast two vertices will have same degree.

Theorem: 6

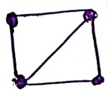
Max degree in a given graph G is denoted as $\Delta(G)$ &

Min degree is denoted as $\delta(G)$

Eg:

$\Delta(G) = 2$

$\delta(G) = 2$



$\Delta(G) = 3$

$\delta(G) = 2$

Avg degree $= \frac{2+2+2+2}{4}$

$= 2$

or

Avg degree $= \frac{2e}{4}$

$= \frac{2(4)}{4} = 2$

$e \to$ no of edges

Avg degree $= \frac{2e}{n} = \frac{10}{4} = 2.5$

For above 2 example we can conclude

$$\boxed{\delta(G) \leq \frac{2e}{n} \leq \Delta(G) \leq n-1}$$

$\hookrightarrow$ max degree

or

$$n \cdot \delta(G) \le 2e \le n \cdot \Delta(G) \le n(n-1)$$

$$n \cdot \delta(G) \le \sum_{i=1}^{n} d(v_i) \le n \cdot \Delta(G) \le n(n-1)$$

P/11 $\quad n = 11 \quad \Delta(G) = 5$

$$n \cdot \delta(G) \le 2e \le n \cdot \Delta(G)$$

$$2e \le 11(5)$$

$$e \le 27.5$$

$\therefore$ Max no of edges $= 27$

P/1 $\quad n = 11 \quad \delta(G) = 3 \quad \Delta(G) = 5$

$$n \cdot \delta(G) \le 2e \le n\Delta(G)$$

$$11(3) \le 2e \le 11(5)$$

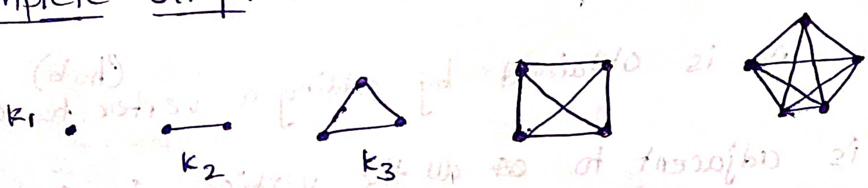
$$33 \le 2e \le 55$$

$$16.5 \le e \le 27.5$$

$$e \in [17, 27]$$

Types of graphs:

1) Complete Graph (K_n) ($n \ge 1$)



Degree of every vertex is $n-1$

(or)

There direct edge b/w every pair of vertices

→ no of edges is,

$$e = \frac{n(n-1)}{2}$$

2) Regular Graph:

A graph in which degree of all vertices is same is called a regular graph.

$$n \cdot \delta(G) = 2e = n \cdot \Delta(G)$$

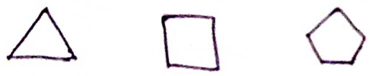
for complete graph

$$K_n \rightarrow \delta(G) = \frac{2e}{n} = \Delta(G) = n-1$$

$\therefore$ Every complete graph is regular. Reverse need not to be true.

If degree of all vertices, in regular graph, is k then we call it a k-regular graph

3) Cycle graph (C_n) ($n \geq 3$):



C_3 C_4 C_5

If given degree seq is all 2's its not guarenteed that it is cycle graph
Eg: [~~6,6,6,6~~6 [2, 2, 2, 2, 2, 2]

Degree of all vertices is 2 in a cycle graph

→ Every C_n is a regular graph
→ no of edges = n

4) Wheel graph (W_n) ($n \geq 4$):

A wheel graph W_n is obtained by adding a vertex (hub) to C_{n-1} such that this vertex is adjacent to all the vertices in C_{n-1}

C_3 W_4 C_4

→ no of edges = $2(n-1)$

→ Degree of the hub is $n-1$ & degree of rest of the vertices is 3.

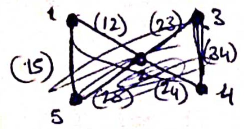
Eg. Degree sequence of W_6 = 5, 3, 3, 3, 3, 3

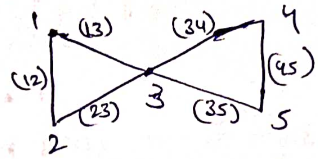
W_{100} : $99, \underbrace{3, 3, 3 \ldots 3, 3}_{99 \text{ times}}$

→ W_4 is only wheel graph which regular & complete.

5) Line Graph ($L(G)$):

Consider below graph, G





Line graph of every cycle is also a cycle.

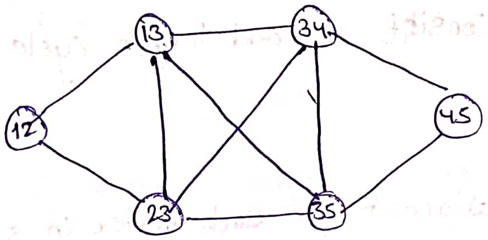
Calculating no of edge in $L(G)$ is given in pg: 158

Steps to construct $L(G)$

(i) define edges in G

(ii) label these edges as vertices in $L(G)$

(iii) Connect vertices with common number.



This above is line graph ~~g~~ of the graph G

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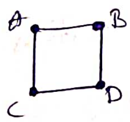
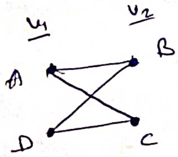
6) Bipartite graph:

A graph in which vertices are divided into 2 sets V_1 & V_2 and all the edges in graph such that no two vertices of the same set are adjacent.

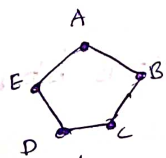
Eg:



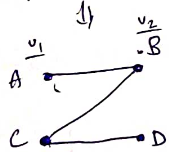
⇓



∴ C_4 is a bipartite graph

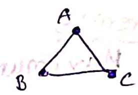


⇓



Now adding E in either of the sets violates the property of bipartite graph.

∴ C_5 is not a bipartite graph.



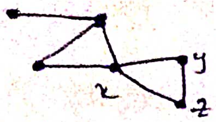
⇓

adding c is not possible

∴ C_3 is not a bipartite graph

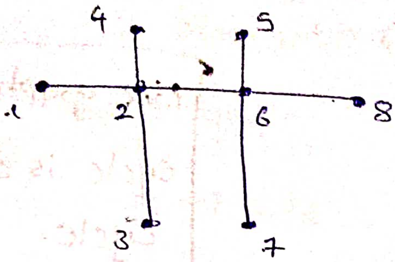
To check if given graph is bipartite we check if it is two colorable or not. Bipartite ⟺ 2-colorable ⟺ no odd length cycle

Eg:

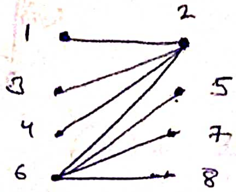


Above graph is not a bipartite graph as it contains C_3 (xyzx)

Eg:



$\Rightarrow$



Theorem: 7:

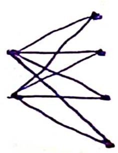
→ Bipartite graph does not consist of odd length cycle

7) Complete bipartite graph ($k_{m,n}$) :

Each vertex in set V_1 is adjacent to each vertex in set V_2.

Eg:

V_1 V_2



$k_{2,4}$

$|V_1| = m$

$|V_2| = n$

→ In $k_{m,n}$, no of vertices = $m+n$

no of edges = mn

→ $\Delta(k_{m,n}) = \max(m,n)$, $\delta(k_{m,n}) = \min(m,n)$

Theorem: 8

→ Maximum no of edges possible is bipartite graph of n vertices is $\left\lfloor \frac{n^2}{4} \right\rfloor$

Proof

put $n=6$

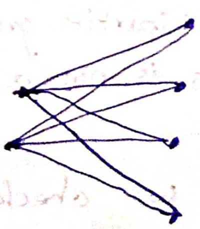
Case: 1

1 5

$k_{1,5}$, $e=5$

Case 2:

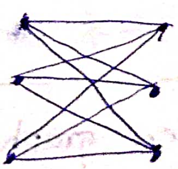
2 4



$k_{2,4}$, $e=8$

Case 3:

3 3



$k_{3,3}$, $e=9$

From above observation, we can conclude that max is no of edges is obtained when vertices are equally divided into 2 parts.

Thus no of edges is $\left(\frac{n}{2}\right)\left(\frac{n}{2}\right) = \frac{n^2}{4}$

But for $n = 7$ we get max no of edges $= \frac{49}{4} = 12.25$

Soo we consider floor of $\frac{n^2}{4}$

$\therefore$ Max no of edges possible $= \left\lfloor \frac{n^2}{4} \right\rfloor$

8) <u>Star graph</u> ($K_{1,n-1}$):

It is complete bipartite graph with one vertex in one set and rest of the vertices in other set.

(or)

Stargraph ($K_{1,n-1}$) is complete bipartite graph possible with n vertices and minimum no of edges.

Eg: Stargraph of 5 vertices, $K_{1,4}$ is  (or) 

→ total no of edges in $K_{1,n-1} = n-1$

→ $\Delta(K_{1,n-1}) = n-1$, $\delta(K_{1,n-1}) = 1$

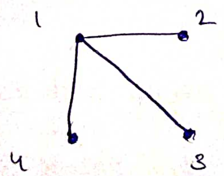
$\left\lfloor \frac{n^2}{4} \right\rfloor$

9) <u>Complement Graph</u> ($\bar{G}$):

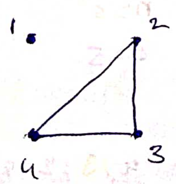
For a graph G, Complement of G ($\bar{G}$) is the graph which contains all the edges which are not present in G and does not contain all the edges present in G.

Eg:

G



$\bar{G}$



$\therefore G + \bar{G} = K_n \Rightarrow e(G) + e(\bar{G}) = \frac{n(n-1)}{2}$

→ If the degree of a given vertex v is x in graph G, then the degree of vertex v in $\bar{G}$ is $[(n-1)-x]$

★

→ If $d_1, d_2 \dots d_n$ is degree sequence for G then

$(n-1-d_1), (n-1-d_2), \dots (n-1-d_n)$ is degree sequence of $\bar{G}$

Eg: Consider a graph of degree sequence $\{5,2,2,2,2,1\}$. what is the degree sequence of complement graph.

Total vertices, $n=6$

$K_n \to 5, 5, 5, 5, 5, 5$

$G \to 5, 2, 2, 2, 2, 1$

$\bar{G} \to \{0, 3, 3, 3, 3, 4\}$

Eg: Considr degree seq of G is $\{3,3,3,1\}$ Find degree seq of $\bar{G}$.

$G \to 3, 3, 3, 1$

The above sequence is not graphical

10) __Isomorphic Graph:__

→ Two graph G_1 & G_2 are isomorphic to each other if they have same ~~proper~~ ~~graph properties~~ incident property or meeting property and

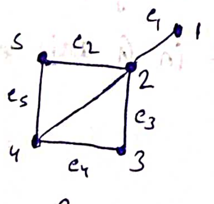
i.e, same no of vertices

same no of edges

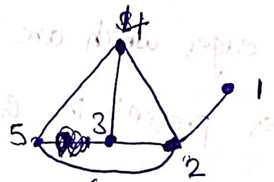
same degree sequence

It is denoted as $G_1 \cong G_2$

Eg:



G_1



G_2

$n=5$

$e=5$

$n=5$

$e=5$

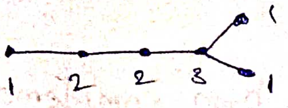
deg seq → ~~3, 2, 2, 2, 1~~

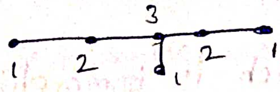
4, 3, 2, 2, 1

4, 3, 2, 2, 1

∴ ~~above two graphs are isomor~~ The two graphs are isomorphic.

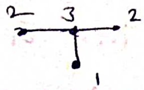
Eg :





$n = 6$
$e = 5$

$n = 6$
$e = 5$



deg seq → 3, 2, 2, 1, 1, 1   deg seq → 3, 2, 2, 1, 1, 1

But still above two graph are not isormorphic to each other. because they don't satisfy meeting property or incident property.

→ Thus if two graphs are isomorphic to each other, they ~~will~~ will have same no of vertices, edge & same degree seq.

But ~~if~~ reverse need not to be true .

→ G_1, G_2 are isomorphic to each other if they have 1:1 correspondance function b/w G_1, G_2

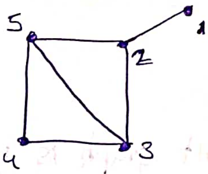
$$\begin{cases} f : G_1 \to G_2 \\ f : V_1 \to V_2 \\ f : E_1 \to E_2 \end{cases}$$

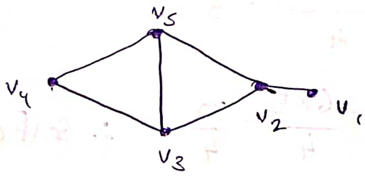
$$\boxed{f : \delta_1 \to \delta_2}$$

$G_1 = (V_1, E_1, \delta_1)$
$G_2 = (V_2, E_2, \delta_2)$

where δ is transition function

Eg :





$f : V_1 \to V_2$ (It is clear)

$f : E_1 \to E_2$

$(1, 2) \to (v_1, v_2)$
$(3, 4) \to (v_3, v_4)$
⋮

Thus, the 2 graphs are isomorphic to each other.

→ When two graphs are isomorphic to each other, it means they actually same graphs with different representation.

*→ when asked if two graphs are isomphic or not, check if both graph have same cycles or not, then go for checking other properties.

11) Self-complement ($G \cong \bar{G}$)

* It is a graph which is isomorphic to its own complement.

i.e., $G \cong \bar{G}$

G & $\bar{G}$ are said to be self-complement to each other.

Eg: G $\quad$ $\bar{G}$

It is clear that above two graphs are selfcomplement to each other.

w.k.T

$$e(G) + e(\bar{G}) = \frac{n(n-1)}{2}$$

let e be no of edges in G

$$e + e = \frac{n(n-1)}{2}$$

$$\Rightarrow \boxed{e = \frac{n(n-1)}{4}}$$ i.e., total no of edges in a self complement graph

Consider

$n=4 \Rightarrow e = \frac{4 \times 3}{4} = 3$

$n=5 \Rightarrow e = \frac{5 \times 4}{4} = 5$

× $n=6 \Rightarrow e = \frac{6 \times 5}{4} = \frac{30}{4}$ $\therefore$ Self complement graph is not possible with a graph of 6 vertices

× $n=7 \Rightarrow e = \frac{7 \times 6}{4} = \frac{42}{4}$ $\therefore$ Self complement graph is not possible with a graph of 7 vertices.

* Thus for a graph with n vertices, self-complement graph is possible $\Leftrightarrow$

$4 \mid n(n-1) \Rightarrow \boxed{4 \mid n \text{ (or) } 4 \mid n-1}$

$\Rightarrow n \equiv 0 \pmod 4$ (or) $n \equiv 1 \pmod 4$ $\Rightarrow$ $\boxed{n \equiv 0 \text{ (or) } 1 \pmod 4}$

(or)

n is of form $4k$ (or) $4k+1$, $k \geq 1$

Consider $n=5$, let try to buid self complement graph

G $\bar{G}$

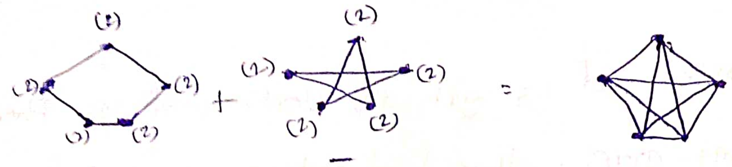
$n=5$ $n=5$

$e=5$ $e=5$

$$G+\bar{G} = K_5$$
$$(4,4,4,4,4)$$

Thus we have

$$\underset{G}{(2,2,2,2,2)} + \underset{\bar{G}}{(2,2,2,2,2)} = \underset{K_5}{(4,4,4,4,4)}$$



G $\bar{G}$ K_5

Note:

→ C_5 is the only cycle graph which is self-Complement

P/13 $G \to \{3,2,2,1,0\}$

$\bar{G} \to \{4,3,2,2,1\}$

P/15 $\underline{G}$

$n=p$

$e=q$

$$e(K_n) = \frac{p(p-1)}{2}$$

$$e(\bar{G}) = \frac{p(p-1)}{2} - q = \frac{p^2-p-2q}{2}$$

P/17 $e(W_n) = 2(n-1)$

$$e(\bar{W_n}) = \frac{n(n-1)}{2} - 2(n-1) = \frac{n(n-1)-4(n-1)}{2} = \frac{(n-1)(n-4)}{2}$$

7/12

a=) Case 1: $12\times3 \neq 2\times28$

Case 2: $12\times4 \neq 2\times28$

b=) $n=10$ $\quad e=\frac{10\times9}{2} = 45$ edges

c=) $\left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{81}{4} \right\rfloor = 20$

d=) n vertices & $n-1$ edges and also given graph is connected

Thus it is simple & connected

Q: If we connect all vertices of C_3 with all vertices of W_4 then
(2) we will get resultant graph. Then find out no of edges in complement of resultant graph.

C_3	W_4	edges due to connection
$e=3$	$e=2(3)$ $=6$	$3\times4=12$

$3+6+12=21$

no of vertices $=7$

no of edges in complement $= \frac{7\times6}{2} - 21 = 21-21=0$

(or)

C_3 is K_3

W_4 is K_4

when we connect all vertices of two graph we get K_7

$K_3 ⊛ K_4 = K_7$

Slly $\boxed{K_m ⊛ K_n = K_{m+n}}$

Q: What will be total edges in the complement of star graph of 6

(3) vertices.

$e(K_{1,5}) = (6-1) = 5$

$e(\overline{K_{1,5}}) = \frac{6 \times 5}{2} - 5 = \cancel{15-5} = 10$

~~∴ 5 edges~~ ∴ 10 edges

is Connected

(or)

~~Complement of Star graph of 6 vertices is a cycle graph of 5 vertices.~~

then

in

G $\qquad\qquad$ $\overline{G}$

> Complement of star graph $K_{1,n-1}$ gives one isolated vertex and a complete graph K_{n-1}

Q: Consider graph vertices are represent as n-bit signal and 2 vertices

(4) are connected with each other if they differ by one bit.

Then what is no of edges in G, no of edges in $\overline{G}$ and degree sequences of G & $\overline{G}$. ($n \geq 3$)

Q: Graph vertices are represented as numbers from 1 .... 100 and

(5) two vertices are connected with each other iff $|i-j| = 4$. Then total edges in G = ______ ?

P/2 $e=22$

$n \cdot k = 44$

$n = \frac{44}{k}$

for $k=1$
$n=44$

22 times

for $k=2$
$n = \frac{44}{2}$
$n=22$

$n=11$ is not possible as k must be

(4) Given that graph is represented by n-bit signal.

$\therefore$ no of vertices $= 2^n$

Since no of ways we can write n-bit numbers, for a given n-bit number, such that they differ by one bit is 'n'.

$\rightarrow$ $\therefore$ degree of each vertex in $G = n$

$\therefore$ Sum of all degrees $= n \cdot 2^n = 2e$

$\Rightarrow$ no of edges, $e = \frac{n \cdot 2^n}{2} = n \cdot 2^{n-1}$

$\rightarrow$ degree seq of G is $\underbrace{\{n, n, n, \ldots n\}}_{2^n \text{ times}}$

$\rightarrow$ no of edges in $\bar{G} = {}^{2^n}C_2 - n \cdot 2^{n-1}$

$= \frac{2^n(2^n - 1)}{2} - n \cdot 2^{n-1}$

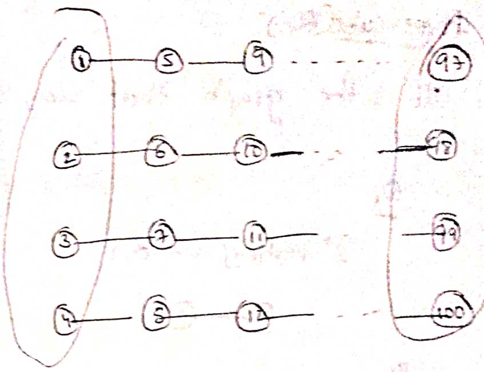
$= 2^{n-1}(2^n - 1) - n \cdot 2^{n-1}$

$= 2^{n-1}\left[2^n - n - 1\right]$

$\rightarrow$ degree seq of $\bar{G} = \underbrace{\{2^n - n - 1, 2^n - n - 1, \ldots, 2^n - n - 1\}}_{2^n \text{ times}}$

(5) degree of vertices $1, 2, 3, 4, 97, 98, 99, 100 = 1$

degree of vertices 5 to $96 = 2$



$\therefore$ Sum of degrees $= 8(1) + 92(2)$
$= 8 + 184$
$= 192 = 2e$

no of edges $= 96$

(P/2) · let k be degree of each vertex and n be no of vertice

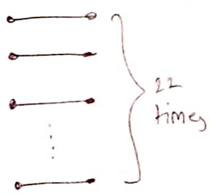
$n \cdot k = 2e$

$nk = 44$

$n = \frac{44}{k}$

a=) for $k=1$

$n = 44$



but graph is not connected

for $k=2$

$n = \frac{44}{2} = 22$

Thus we can draw a C_{22}

but $n=22$ is not given in option

b=) $n=2 \Rightarrow k=22$

not possible in simple graph

c=) $n=4 \Rightarrow k=11$

not possible for simple graph

d=) $n=11 \Rightarrow k=4$

$\therefore$ This must be the answer

(P/3)

$nk = 2e$

$4n = 76$

$n = 19$

a given n-bit

n'.

2^{n-1}

29/04/20

Hypercube(Q_n):

It is the graph that was discussed in [illegible] question (4)

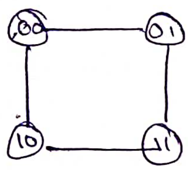
Q_1

2^1 vertices — 0, 1

Every cycle in hypercube is of even length (think why). So every hypercube is a bipartite graph

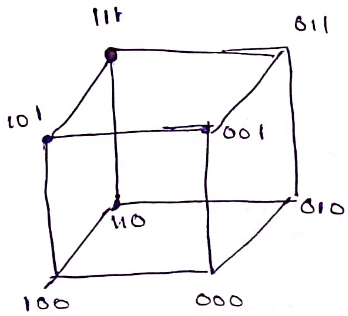
Q_2

2^2 vertice — 00, 01, 10, 11



Q_3:

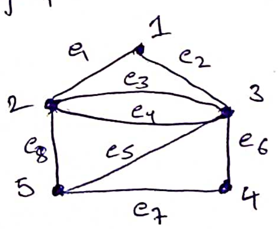
2^3 vertices



no of edges $= \frac{n \cdot 2^n}{2} = n \cdot 2^{n-1}$

Connectivity:

Consider below graph



walk: It is alternating sequence of vertices & edges.

Eg: $1 e_2 3 e_3 2 e_3 3$

R.V ✓
R.E ✓

Trail: It is alternating sequence of vertices & edges

$1 e_2 3 \cdot e_3 2 e_4 3$

R.V ✓
R.E ✗

e is of
So every
e graph

path: Its alternating sequences of vertices & edges.

R.V ✗
R.E ✗

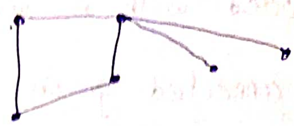
	Repetition of vertices	Repetition of edges
walk	✓	✓
trail	✓	✗
path	✗	✗

1 e2 3 e6 4

Connected graph:

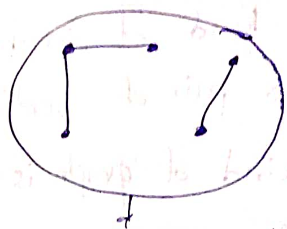
For every two pair of vertices, there must exist a path between them

Eg:



Disconnected graph:

If we can fined atleast one pair of vertices, such that there is no path available b/w those 2 vertices, then the graph is said to be disconnected graph.



Disconnected graph

Note:
The graph G shown here is conneted

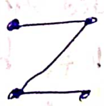
G

Eg

→ Disconnected graph consists of connected subparts called components.

* If G is connected $\bar{G}$ may be connected or disconnected

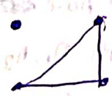
Eg:



G
Connected

$\bar{G}$
connected

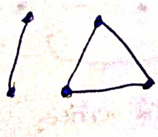
G
Connected



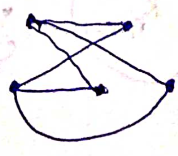
$\bar{G}$
disconnected

Theorem-9
* If G is disconnected, $\bar{G}$ is connected

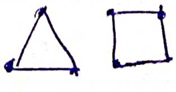
Eg:



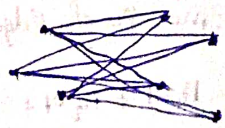
G



$\bar{G}$



G



$\bar{G}$

Note:

For a graph G, either G or $\bar{G}$ will be connected

G	$\bar{G}$
C	C
	D.C
D.C	C

Range of edges for a connected graph ($k=1$): (k is no. of connected components)

→ Minimum no of edges required to get a possibility to make graph connected with n vertices is $n-1$. $\boxed{n-1 \le e \le \frac{n(n-1)}{2}}$

- The connected graph with $n-1$ edges is, doesn't have a cycle.
- This graph is known as minimally connected graph.
- In this kind of graph, there will be a unique path b/w any two pair of vertices
- This kind of graph is called a tree (a connected graph with no cycles)

Range of edges for a disconnected graph:

→ edges range ~~b/w~~ b/w

$$\boxed{n-k \le e \le \frac{(n-k)(n-k+1)}{2}}$$

Proof:

Here lets say we have ~~[illegible]~~ k components with $n_1, n_2 \dots n_k$ be vertices in each component

$\therefore n_1+n_2+\dots+n_k = k$

For min no of edge, each component must be minimally connected.

$\therefore$ min no of edges is

$n_1-1+n_2-1+\dots+n_k-1$

$=(n_1+n_2+\dots+n_k)-k = n-k$

Proof:

It obtained by $(k-1)$ isolated vertices forming $k-1$ componenents & $n-k+1$ forms a ~~Com~~ component with completely connected

$\therefore$ no of edges $\frac{(n-k+1)(n-k)}{2}$

→ Forest is collection of trees.

→ Thus no of edges in a forest of n vertices, with k ~~trees~~ trees is n-k. (Here forest can be seen as disconnected graph with n vertices, with k ~~connected~~ minimally connected components).

GATE 2003

Let G be an arbitrary graph with n nodes and k components. If a vertex is removed from G, the no of components in the resultant graph must lie b/w

a) k & n b) k-1 & k+1 c) k-1 & n-1 d) k+1 & n-k

→ If graph consist an isolated vertex, removing that isolated vertex reduces components to k-1.

→ If the graph is combination of a few isolated vertices & rest of the vertices forms a star graph, Removing centre vertex of star graph leaves n-1 isolated vertices.

i.e., n-1 Components.

GATE 2014/3

If G is a forest with n vertices & k connected comp. how many edges does G have?

Sol:

Consider a graph, which is tree, with n vertices. Thus it has n-1 edges

If we remove an edge, it leaves a forest with 2 components

If we remove 2 edges, it leaves a forest with 3 components

⋮

If we remove k-1 edges, it leaves a forest with k components

∴ no of edges = $(n-1)-(k-1) = n-k$

GATE 2015/2

For all self-complementary graphs on n-vertices, n is

a) multiple of 4
b) even c) odd
d) congruent to 0 mod 4, 1 mod 4

In a self complementary graph

$$e(G) = e(\bar{G})$$

$$\Rightarrow 2e(G) = \frac{n(n-1)}{2} \Rightarrow e(G) = \frac{n(n-1)}{4}$$

$\Rightarrow 4 | n$ (or) $4 | n-1$

$\Rightarrow$ n is of form $4k$ (or) $4k+1$

$\therefore$ option d

GATE 2018 Let G be a graph with 100! vertices, with each vertex labeled by a distinct permutation of number 1, 2, ... 100. There is an edge b/w vertices u and v iff the label u can be ~~objo~~ obtained by swapping two adjacent numbers in the label of v. Let y denote the degree of a vertex in G and z denote the ~~no~~ number of connected components in G. Then $y + 10z$ = ______.

Sol:

~~Let a vertex be~~

no of vertices = 100!

Let a vertex be $n_1 n_2 n_3 \dots n_{99} n_{100}$

Its adjancet vertex can be ~~objo~~ obtained by swapping

$n_1 n_2$ (or) $n_2 n_3$ (or) $n_3 n_4$ (or) .... (or) $n_{99} n_{100}$

$\therefore$ degree of every vertex = 99 $\Rightarrow y = 99$

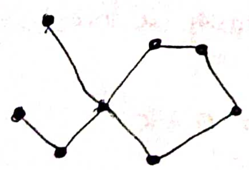
Also by performing ~~op~~ enough swaps on an arrangement of 1 to 100 numbers we can obtain any permutation.

~~Here~~ i.e., there path ~~b/o~~ b/w any pair of vertices

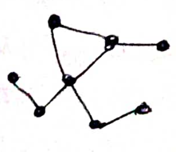
$\Rightarrow z = 1$

$y + 10z = 99 + 10 = 109$

GATE 2012 which of the following graphs is isomorphic to

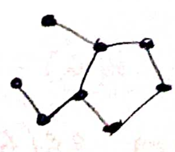


a)

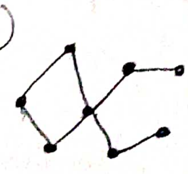


b)

c)



d)



sol :

In the ~~g~~ given graph, we can see it consists of a 5 length cycle.

And option (b) & (c) have 5 length cycles.

And among those two options, if we try to map vertices we can observe that (b) is the isomorphic graph.

GATE 2014 — A cycle of n vertices is isomorphic to its complement. The value of n is ______.

sol :

$n=3$ $\quad$ $n=4$ $\quad$ $n=5$

G $\quad$ $\bar{G}$ $\quad$ G $\quad$ $\bar{G}$ $\quad$ G $\quad$ $\bar{G}$

$\therefore n=5$

30/04/20

P/47 — vertices, $n=10$ $\quad$ Components, $k=3$

$n-k$ to $\dfrac{(n-k)(n-k+1)}{2}$

7 to $\dfrac{7(\cancel{8}^{4})}{\cancel{2}} \Rightarrow 7 \,\&\, 28$

P/58 — Components, $k=7$ $\quad$ edges, $e=26$

Since each component is tree

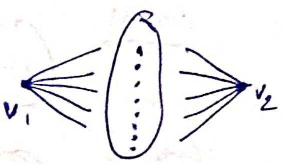
$n-k=26$

$n-7=26 \Rightarrow n=33$ vertices

P/56 — $\delta(G) \geq 5$

Consider two non-adjacent vertices v_1, v_2

$\deg(v_1)+\deg(v_2) \geq 10$

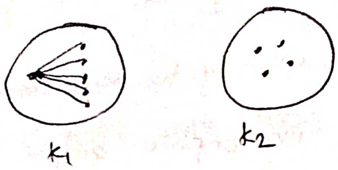


~~aside v_1~~ besides v_1 & v_2 we have 8 more vertices.

as v_1 & v_2 are not adjacent they must have atleast 2 vertices adjacent in common. Thus there exist path b/w any pair of vertice. $\Rightarrow$ Connected

Alternate Method :

Let us assume graph is disconnected and lets say $k = 2$



Since every vertex's degree is ≥ 5

~~are~~ Every component must have atleast 6 vertices

but both the components can't have 6 vertices

which is not possible.

Hence, Connected

P/55 $\dfrac{(n-k)(n-k+1)}{2} = \dfrac{(15)(\cancel{16}^{8})}{\cancel{2}} = 120$

→ If a simple graph has exactly two vertices of odd degree then there exist a path between the two vertices of odd degree.

Proof:

As we know a graph contains even ~~deg~~ no of odd degree vertices.

Even if the graph is disconnected, every component represents individual graph.

Hence every component has even no of odd degree vertices.

Hence the two odd degree vertices must fall in same component and hence ~~pat~~ the path exists.

→ Consider a graph with n vertices & k components

min no of edges $= n-k$

Eg: $n = 7$ $k = 2$ $\Rightarrow 7-2 = 5$ edges

Case 1: 3v 2e 4v 3e → 5e

Case 2: 2v 1e 5v 4e → 5e

Case 3: 1v 0e 6v 5e → 5e

→ Consider a graph with n vertices & k components

Eg: let $n=7$ $k=2$

Max no of edges $= \frac{(7-2)(7-2+1)}{2} = 15$

Case 1:

3V K_3 3e 4V K_4 6e ⇒ 9 edges

Case 2:

5V 10e 2V 1e ⇒ 11 edges

Case 3:

6V 15e 1V 0e ⇒ 15 edges (Max no of edges)

Eg: $n=10$ $k=3$ ⇒ $\frac{(n-k)(n-k+1)}{2} = \frac{(8)(7)}{2} = 28$ edges

8V 28e K_8 1V 0e K_1 1V 0e K_1 ⇒ 28 edges

This maximum case is the case which has k-1 isolated vertices and a complete connected component with n-k+1 vertices.

→ The above two examples provide visualization of minimum & maximum edges case

P/57 no of edges $= e(K_5) + e(K_6) + e(K_7) + e(K_8)$

$= 10 + 15 + 21 + 28$

$= 74$ edges

Q6: Consider a disconnected graph of 10v. What is max no of vertices possible.

Sol:

Here no of components is not mention. For max no of edges, we need to take 2 components ⇒ $k=2$

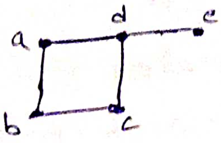
Max no of edges $= \frac{(n-k)(n-k+1)}{2} = \frac{(9)(8)}{2} = 36$ edges.

Connectivity - II

Cut edge / bridge:

The edge whose removal from a connected graph, makes the ~~graph~~ graph disconnected, is called cut edge or bridge

Eg:



Here, the edge 'de' is a cut edge.

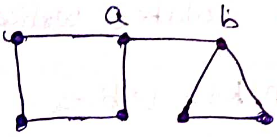
Here edge e_1 is cut edge.

Note: If a cut edge exists in a graph, then cut ~~&~~ edges ~~are~~ never belongs to any cycle.

Cut vertex (or) Cut point (or) Articulation point:

The vertex whose removal from a connected graph, makes the graph disconnected, is called cut vertex.

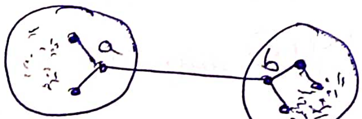
Eg:



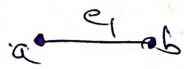
a & b ~~are~~ are cut vertices

The end vertices of a cut edge are cut vertices

→ Consider below graphs



Here ~~ab~~ Cut edge is ab
Cut vertices is $\{a,b\}$



In above graph
Cut edge is e_1
No cut vertex is present

Thus we can say

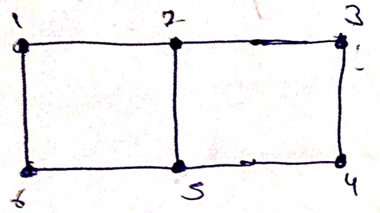
Note:

✱ For graph with $n \geq 3$ vertices, if cut ~~vertices~~ edge exists, then cut vertices exist (i.e., end vertices of cut edge)
· But reverse need to be true.

Cut edge set (Cut set):

It is set of edges, whose removal from a connected graph, makes graph disconnected. Also no subset of a cutset is a cutset.

Eg:



{12, 16} {23, 34} {12, 56} {16, 25, 34} . . . . .

Edge Connectivity: (or) Minimum Cut:

It is minimum no of edges that are to be removed to make a graph disconnected graph.

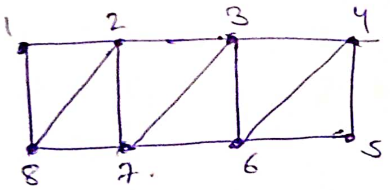
(or)

It is size of minimum cardinality cut set.

→ Edge connectivity is denoted as $\lambda(G)$.

Eg: The edge connectivity of above graph is 2.

Eg: Find out edge connectivity of below graph



Here every edge belongs to a cycle and hence edge connectivity can't be 1.

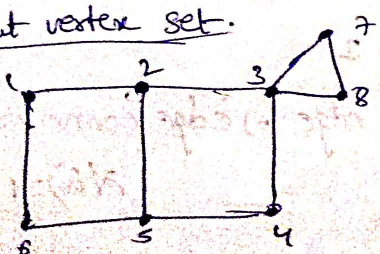
Removing {65, 45} makes the graph disconnected

∴ Edge Connectivity = 2

Cut vertex set:

It is set of vertices, whose removal from a connected graph makes graph disconnect. Also no subset of a cut vertex set should be cut vertex set.

Eg:



cut vertex sets – {2, 5}, {3}

Vertex Connectivity:

It is min no of vertices whose removal makes the graph disconnected. (or)

It is ~~min~~ size of min cardinality cut vertex set.

It is denoted by $k(G)$

(Q/59) $\{c\}$ is cut vertex set

$\Rightarrow$ vertex connectivity = 1

$\{cd, de\} \Rightarrow$ edge connectivity = 2

$\{ac, bc\}$ (or) $\{ac, ab\}$ (or) ...

(Q/60) Every edge is part of a cycle

$\therefore$ no cut edge

$\{de, bh\}$ is cut ~~vert~~ edge set

$\Rightarrow$ edge connectivity = 2

Removing vertices $\{d, b\}$ make graph disconnected

Hence vertex connectivity = 2

alternate solution is present next

(Q/61) Its clear that

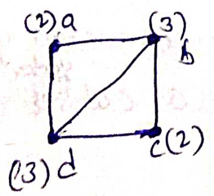
vertex connectivity = 1

And ~~vertex co~~

edge connectivity = 3

Relation b/w min degree & Edge Connectivity:

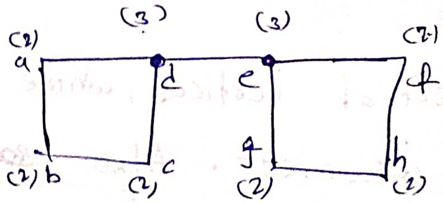
Consider



$\{bc, dc\}$ is edge set

$\delta(G) = 2$

edge connectivity, $\lambda(G) = 2$



$\delta(G) = 2$

de is cut edge $\Rightarrow$ edge connectivity, $\lambda(G) = 1$

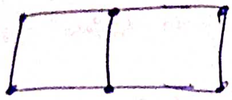
From above observations we can conclude

$$\boxed{\lambda(G) \le \delta(G)}$$

Relation b/w vertex connectivity & Edge Connectivity
→ Consider

$\lambda(G) = 2$

$k(G) = 1$



$\lambda(G) = 2$

$k(G) = 2$

From above observation we can conclude

Theorem:10

$$\boxed{k(G) \le \lambda(G) \le \delta(G) \le \frac{2e}{n} \le \Delta(G) \le n-1}$$

P/60

Here $\delta(G) = 3$

$\Rightarrow k(G) \le \lambda(G) \le 3$

$\{de, bh\}$ is cut set

$\Rightarrow \lambda(G) = 2$

$\Rightarrow k(G) \le 2$

It is clear that there is no cut vertex

$\therefore k(G) = 2$

P/77

$k(G) \le \lambda(G) \le \delta(G) \le \frac{2e}{n}$

$\Rightarrow \frac{2e}{n} \ge k(G)$

$n = 10$

$\frac{2e}{10} \ge 3 \Rightarrow e \ge 15$

$\therefore$ minimum no of edges = 15

Alternate method :

$k(G) \le \lambda(G)$

$\Rightarrow \lambda(G) = 3$ (for min case)

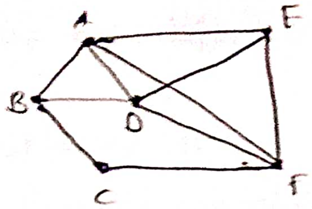
$\Rightarrow \delta(G) = 3$

$\Rightarrow \sum deg(v_i) = 30$

$\Rightarrow e = \frac{30}{2} = 15$ Edges

02/05/20

GATE 1999 Let G be a connected undirected graph. A cut in G is a set of edges whose removal ~~is~~ results in G being broken into two or more components which are not connected with each other. A min-cut is a cut in G of minimum cardinality. Consider the following graph



a) which of the following set of edges is a cut?

(i) $\{(A,B), (E,F), (B,D), (A,E), (A,D)\}$

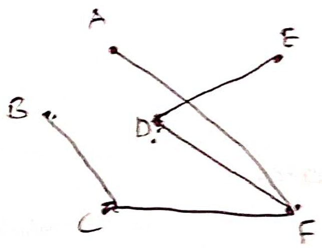
(ii) $\{(B,D), (C,F), (A,B)\}$

b) what is the cardinality of a min-cut in this graph.

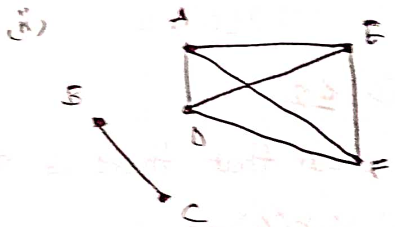
c) Prove that if a connected undirected graph G with n vertices has a min-cut of cardinality k, then G has atleast $(nk/2)$ edges

Sol:

a) (i)



(ii)



$\therefore$ (ii) is a cut

b) we can observe that C has min degree i.e., 2

$\therefore \lambda(G) \leq 2$

Every edge is part of cycle and thus there is not cut edge

$\Rightarrow$ Cardinality of min-cut = 2

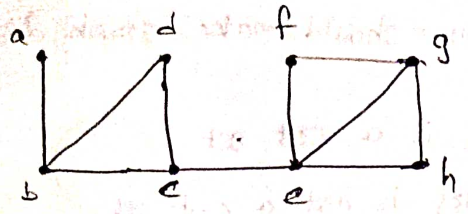
i.e., $\{BC, CF\}$

c) $K(G) \leq \lambda(G) \leq \delta(G) \leq \frac{2e}{n}$

$\Rightarrow \lambda(G) \leq \frac{2e}{n} \Rightarrow k \leq \frac{2e}{n} \Rightarrow e \geq \frac{nk}{2}$

Q7) Find edge connectivity & vertex connectivity of below graph

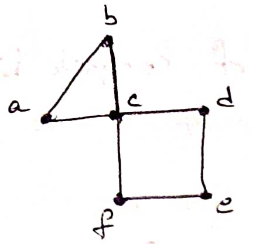
a)



ce is a cut edge $\Rightarrow \lambda(G) = 1 \Rightarrow k(G) = 1$

c, e are cut vertices

b)



c is a cut vertex

$k(G) = 1$.

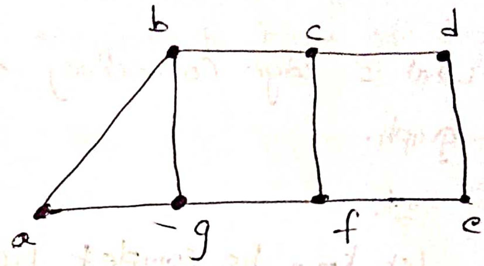
Every is is a part of cycle

Hence $\lambda(G) \neq 1$

$\lambda(G) = 2$

i.e., {ac, bc} {de, ef} {cd, cf}...

c)



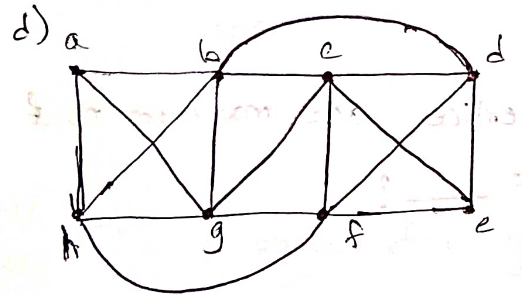
$\lambda(G) = 2$ ({ab, ag} {bc, gf}...)

$\Rightarrow k(G) \leq \lambda(G)$

It is clear that there is not cut vertex

$\Rightarrow k(G) = 2$ ({b, c} {e, f} {d, f}...)

d)



$\delta(G) = 3$

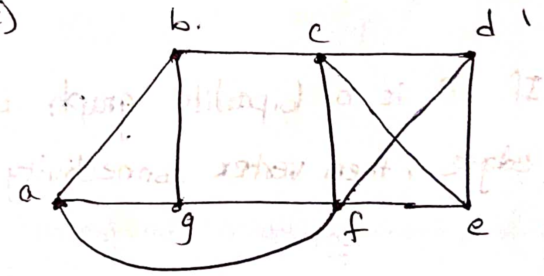
$\lambda(G) \leq 3$

From observation

$\lambda(G) = 3$ ({ab, ag, ah}...)

$k(G) = 3$ {{b, h, g} {c, d, f}...)

e)



Here degree of every vertex = 3

$\lambda(G) \leq 3$

It is clear that $\lambda(G) \neq 1$ & $\lambda(G) \neq 2$

$\Rightarrow \lambda(G) = 3$ ({ab, ag, af}

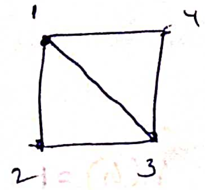
{bc, gf, af}....)

$k(G) = 2$ {(c, f) (b, f)...}

Note:

→ In a cut edge set, no subset should make graph disconnected

Eg:



$\{14, 43\}$ is a cut set

$\{14, 43, 13\}$ is not a cut set.

→ Same goes for ~~ver~~ cut vertex set

(Q8) What is edge connectivity & vertex connectivity of complete bipartite graph.

Sol:

Let $K_{m,n}$ be complete bipartite graph

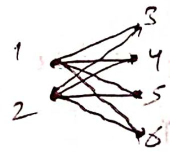
$\delta(G) = \min\{m, n\}$

$\Rightarrow \lambda(G) = \min\{m, n\}$

$\Rightarrow K(G) = \min(m, n)$

$\Rightarrow$ For a complete bipartite graph

$K(G) = \lambda(G) = \min(m, n)$



Here $\lambda(G) = 2\ \{(31, 32)(41, 42)..\}$

$K(G) = 2\ \{(1, 2)\}$

(Q9) If G is a bipartite graph with 9 vertices and maximum no of edges, then vertex connectivity of G is ____?

Sol:

Here G is $K_{4,5}$

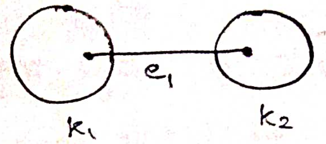
$\Rightarrow K(G) = \lambda(G) = 4$

(P/49) This possible when a vertex is isolated and rest of the n-1 vertices forms K_{n-1} graph.

$\therefore$ no of edges $= \dfrac{(n-1)(n-2)}{2}$

P/51) Given degree of every vertex is even.

A cut edge generally connects two connected componets.



Let us assume cut edge exists & it is e_1.

If we remove e_1, one of the vertex in k_1 will have odd degree.

But it is not possible, for exactly one vertex to have odd degree.

Same happens for k_2.

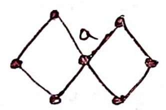
$\therefore$ Cut edge does not exist.

(or)

$\boxed{\text{If degree of every vertex in a connected graph is even, then every edge will be part of a cycle}}$

$\boxed{\therefore \text{no cut edge}}$

Consider



Here degree every vertex is even

$\Rightarrow$ but 'a' is a cut vertex.

$\therefore$ opt (b)

GATE 2006) The 2^n vertices of a graph G correspond to all subsets of a set size n, for $n \geq 6$. Two vertices of G are adjacent iff the corresponding sets intersect in exactly two elements. Then no of vertices of degree zero in G is

a) 1 b) n c) $n+1$ d) 2^n

sol:

This possible for vertices with sets of cardinality 0 or 1

no of subsets of cardinality $1 = n$

" " " " " $0 = 1$ $(\emptyset)$

$\Rightarrow n+1$

Here no of components will be $n+2$

Theorem : 11

~~Max~~ no of edges that can be drawn, for n vertices graph such that graph is disconnected is ~~nos~~ $\frac{(n-1)(n-2)}{2}$

(or)

Any graph with edge $\geq \frac{(n-1)(n-2)}{2} + 1$ is a connected graph.

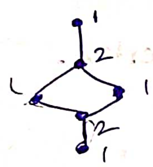
P/52 $\quad n-1 \leq e \leq \frac{(n-1)(n-2)}{2}$

$\therefore G$ may or may not be connected

Coloring:

Coloring is assigning coloring all the vertices of a graph, ~~with~~ such that no two adjacent vertices are of same color.

Eg :



Coloring is used in frequency alignment, halfman encoding, computer networks, security etc.

Proper coloring:

No two adjacent vertices should have same color.

Chromatic Number ($\chi(G)$):

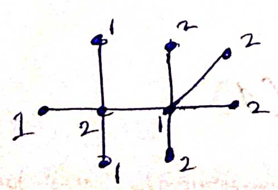
Minimum no of colors required to paint all the vertices such that adjacent vertices ~~sch~~ should not have same color. It is denoted by $\chi(G)$

If chromatic ~~graph~~ number of a graph is k, then it is called k-colorable graph.

Note:

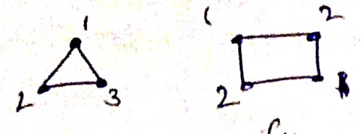
→ Chromatic number of an isolated vertex = 1

→ Chromatic number of a tree is 2.



Every tree is a 2-colorable graph.

→ Chromatic number of cycle graph $C_n = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$



C_3 C_4

every even length cycle is 2-colorable.

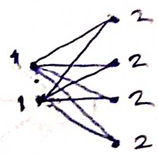
But reverse need not to be true.

→ Chromatic num of a wheel graph $W_n = \begin{cases} 4, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$

W_4 W_5

→ Chromatic number of a complete graph K_n is n

→ Chromatic number of a bipartite graph is 2.

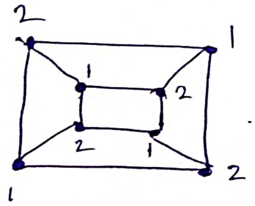


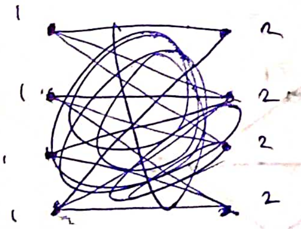
Every bipartite graph is a 2-colorable and every 2-colorable graph is a bipartite graph

2-colorable → Tree → Bipartite graph

2-colorable → even length cycle → Bipartite graph

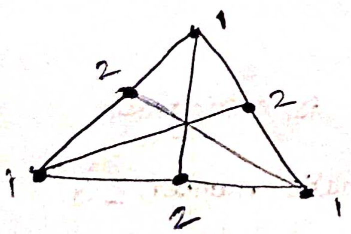
Eg:



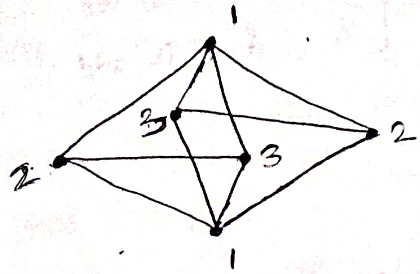


Above is a bipartite graph

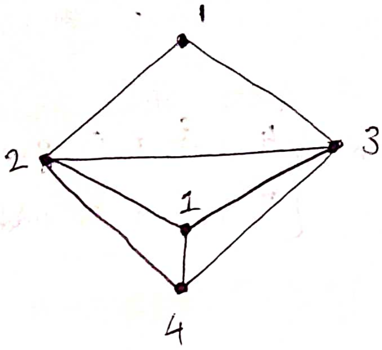
P/19



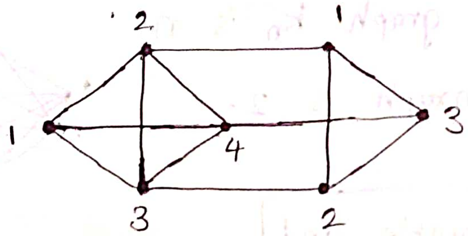
P/20



P/21



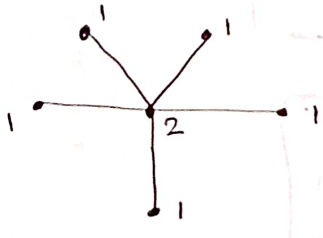
P/22



Ku  is a subgraph

∴ Chromatic number ≥ 2

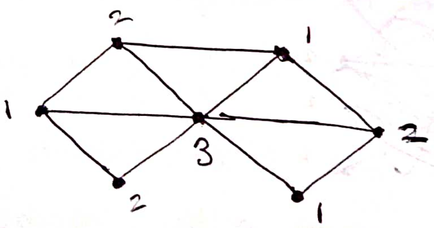
P/23



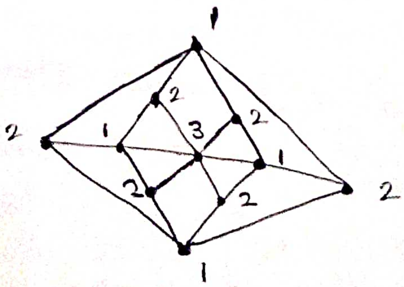
Chromatic num of tree = 2

also it is a star graph

P/24



P/25



Here  is a subgraph

∴ Chromatic number ≥ 3

47

P/26 Complement of a star graph $K_{1,n-1}$ is a combination of an isolated vertex and a complete graph K_{n-1}.

$\therefore$ Chromatic number of a complement of star graph is

$n-1 = 6-1 = 5$

P/27 Let n be odd

$\chi(C_n) = 3$

$\chi(W_n) = 3$

$\alpha+\beta = 6$

Let n be even

$\chi(C_n) = 2$

$\chi(W_n) = 4$

$\alpha+\beta = 6$

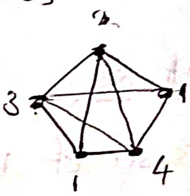
P/28 Let $v_1, v_2 \dots v_{10}$ be the vertices

let edge b/w v_1, v_2 is removed

now v_1 & v_2 can have same color

$\Rightarrow$ chromatic number $= 9$

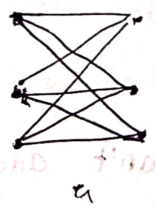
Eg: Consider same with 5 vertices



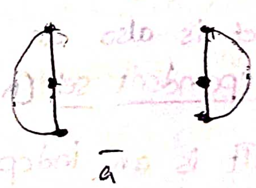
P/29 This is the case of max no of edges as

$\left\lfloor \frac{6^2}{4} \right\rfloor = 9$

$\therefore$ ~~chromatic number = min(3,3)~~ = 3

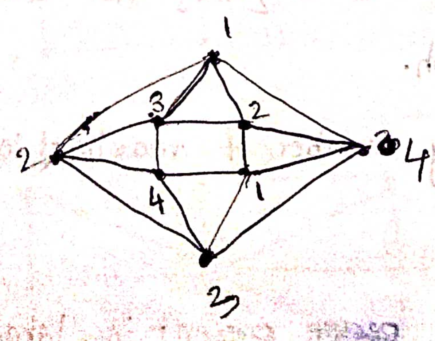


complement is



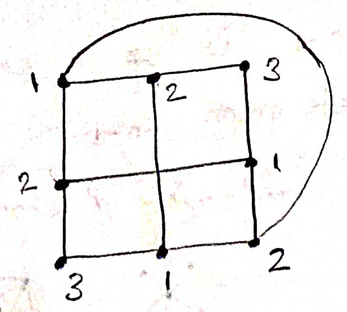
$\therefore$ chromatic number $= 3$

GATE 2004



$\therefore$ Chromatic number $= 4$

GATE 2008



Chromatic num = 3

GATE 2009

What is the chromatic number of an n-vertex simple connected graph which does not contain any odd length cycle? Assume $n \geq 2$.

a) 2   b) 3   c) $n-1$ d) n

Sol:

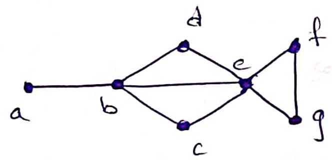
no odd length cycle ⇒ bipartite graph

∴ Chromatic number = 2

Independent set:

It is set of non-adjacent vertices.

Eg:



{a}, {a,c} {a,c,d} {a,c,d,f} are called independent sets.

also {a,c,d,f} is a maximal independent set.

* Empty set is also an independent set

Maximal independent set (MIS):

It is an independent set to which we can't add any new element.

Eg: {a,c,d,f} {a,c,d,g} {b,f} {a,e} etc. are maximal independent sets for above graph.

Thus, in a graph we can have many number of maximal independent sets.

Independence Number ($\beta(G)$):

It is number of vertices ~~presen~~ present in largest maximal independent set.

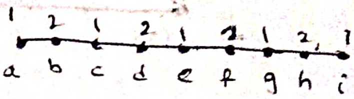
GATE 2008

what is the size of the smallest MIS (Maximal Independent Set) of a chain of nine nodes? 49

Sol:

☆ a) 5 b) 4 c) 3 d) 2

Sol:



~~Chromatic number = 2~~

~~4 vertices can be colored with one color~~

~~and 5 vertices can be colored with another color~~

~~∴ two MIS sets are formed out of which~~

~~4 is smallest.~~

~~4~~

{a, c, e, g, i}

{b, d, f, h}

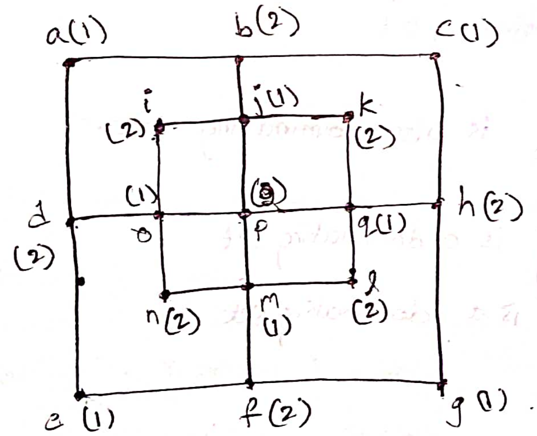
{b, e, h}

{a, d, g, i}

∴ Smallest size is 3

Q10) Find chromatic number and independence number of below graphs ~~grammar.~~

a)



* Max independence set need not to have same color

Chromatic number = 2

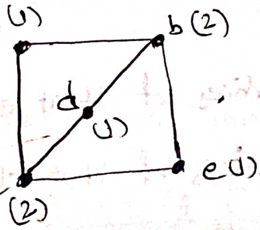
MIS = {a, c, g, e, j, q, m, o},

{b, h, f, d, i, k, l, n, p}

{a, c, g, e, i, k, l, n, p}

∴ Independence number = 9

b)

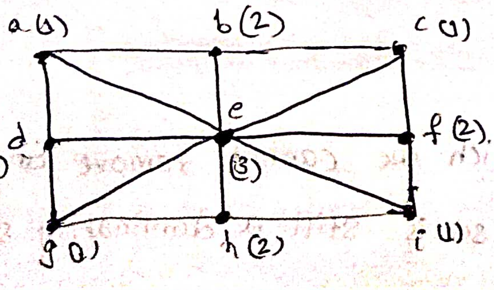


Chromatic num = 2

Independence number, β(G) = 3

{a, d, e} is MIS

c)

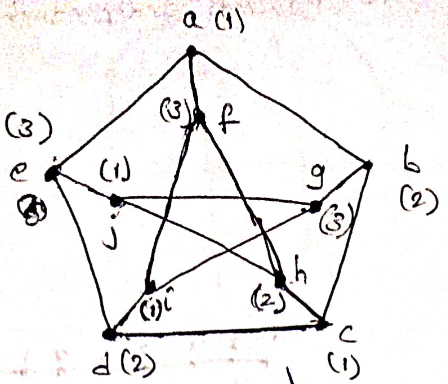


Chromatic num = 3

Independence number, β(G) = 4

{a, c, i, g} {b, f, h, d} are MIS

d)



Chromatic num = 3

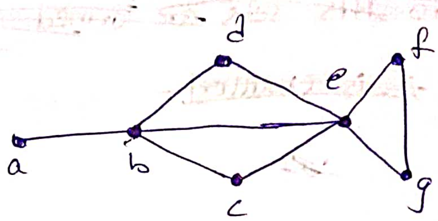
$\{a, e, j, i\}$ is MIS

$\therefore \beta(G) = 4$

→ petinson graph

03/05/20

Dominating set:



Dominating set $D \subseteq V$ such that if we take any vertex from V either that the vertex directly belongs to D or its adjacent belongs to D.

or ↓ inclusive OR

inclusive OR – one or other or both.
exclusive OR – one or other

Eg: For above graph

$D = \{a, b, f\}$ is a dominating set

$\{b, e\}$ is a dominating set

$= \{a, b, c, d, e, f, g\}$ is also dominating set.

removing 'a'

$\{b, c, d, e, f, g\}$ is a dominating set.

$\{b, d, e, f, g\}$ is a dominating set.

Note: $\{b, e, f, g\}$

$\{b, e, f\}$

$\{b, e\}$

To remove an element, all the adjacent vertices of that element must be present in the set and adjacent vertices of that element are adjacent to some other vertex in the set.

Minimal Dominating Set (MDS):

The Dominating set from which we cannot remove any element from the set such that the set is still a dominating set.

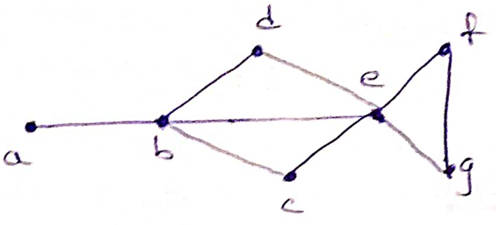
Eg: {b,e}, {b,f}, {b,g}, {a,e}, {a,c,d,f}

Domination Number ($\alpha(G)$):

Number of vertices present in smallest MDS.

Eg: Domination number for previous graph is 2.

Consider



MIS	MDS
{a,c,d,f} $\beta(G)=4$	{a,c,d,f}
{b,f}	{b,f}
{b,g}	{b,g}
{a,e}	{a,e}
	{b,e}

$\alpha(G)=2$ (for {b,f}, {b,g}, {a,e}, {b,e})

{b,e} → This can't be MIS as b,e are adjacent

Note:

Every MIS will always be MDS. But reverse need not to be true

MDS

MIS

⇒ $\boxed{\text{Domination number} \le \text{Independence number} \quad \alpha(G) \le \beta(G)}$

(Q11) Find domination number of graphs in (Q10)

a) ~~{b, h, f, d, j, q, m, o}~~ ~~is MDS~~

~~{a, h, f, i, q, n}~~ ~~is an MDS~~

{d, h, j, m} is an MDS $\quad \alpha(G) = 4$

{b, f, o, q} is an MDS

b) $\{b, c\}$ is an MDS $\qquad$ $\{d, b\}$ is also an MDS

$\{d, c\}$

$\therefore \alpha(G) = 2$

c) It is wheel graph W_9

So hub will be enough

$\therefore \{e\}$ is an MDS

$\alpha(G) = 1$

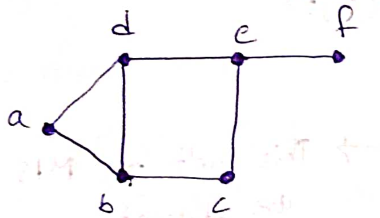
d) $\{a, f, d\}$ is an MDS

$\therefore \alpha(G) = 3$

~~Not~~ Matching:

→ Matching is a set of non adjacent edges. It is also known as independent edge set.

Eg:



$\{de\}$ is a matching set

$\{de, bc\}$ is a matching set.

$\{ef\}$, $\{ef, bc\}$, $\{ef, bc, ad\}$ are matching sets

→ The vertices involved in matching are said to be matched.

Maximal Matching set (MMS):

It is a matching set such that we cannot add a new edge into the set.

Eg: $\{de, bc\}$, $\{ef, bc, ad\}$ are MMS.

Matching Number ($M(G)$):

It is no of edges present in largest MMS

Eg: $M(G) = 3$ for above graph

Note: No of edges in a matching, for n-vertex set is $\leq \left\lfloor \frac{n}{2} \right\rfloor$

P/36 a => For K_n

~~we can choose~~

if $n = \text{even}$, $M(G) = n/2$

$n = \text{odd}$ $M(G) = \lfloor n/2 \rfloor$

$\Rightarrow \lfloor \frac{n}{2} \rfloor$

b =>

$n = 3$

$M(G) = 1 = \lfloor \frac{3}{2} \rfloor$

$n = 4$

$M(G) = 2 = \frac{4}{2}$

$\therefore \lfloor \frac{n}{2} \rfloor$

c =>

$n = 4$

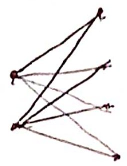
$M(G) = 2 = 4/2$

$n = 5$

$M(G) = 2 = \lfloor \frac{5}{2} \rfloor$

$\therefore \lfloor \frac{n}{2} \rfloor$

d =>



$K_{2,4} = 2$

$\lfloor \frac{2+4}{2} \rfloor = 3 \neq 2$

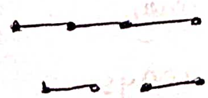
$\therefore d$

P/37 a => it is true

b => ~~$K_{m,n} = n$~~ $M(K_{m,m}) = m$ ~~$\min(m,n)$~~

c => $M(K_{m,n}) = \min(m,n)$ $\therefore c$ is ~~false~~ true

d =>



$M(G) = 2$ here

$\therefore d$ is false

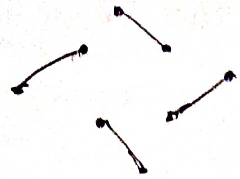
P/38 For this the bipartite graph is star graph $K_{1,n-1}$

$\therefore M(K_{1,n-1}) = 1$

P/39 It is K_9 + an isolated vertex

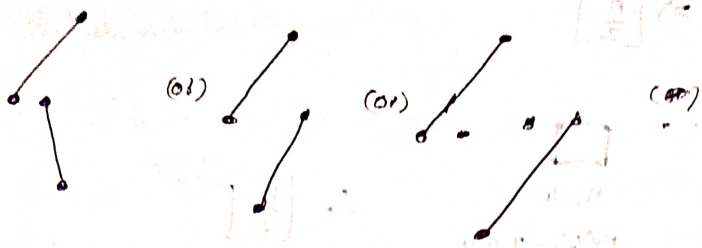
$M(G) = 4$, $X(G) = 9 \Rightarrow 4 + 9 = 13$

P/40



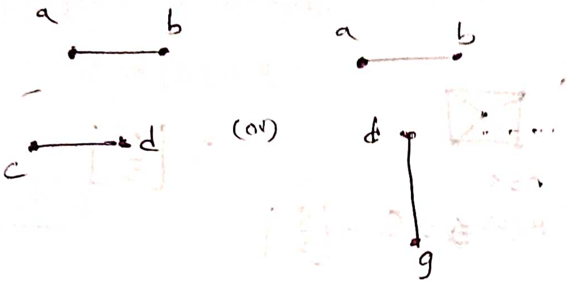
$M(G) = 4$

P/41



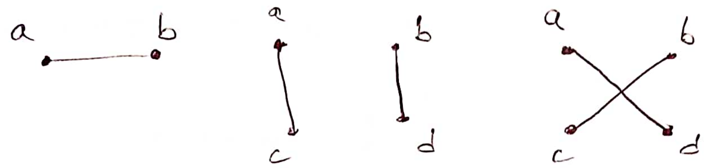
$M(G) = 2$

P/42



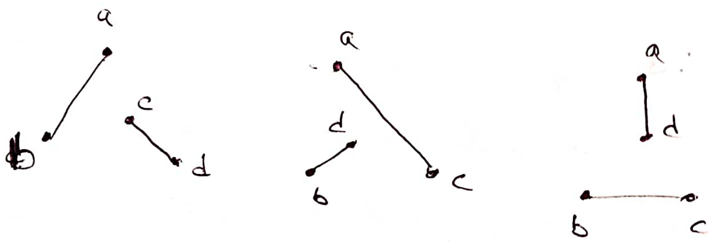
$M(G) = 2$

P/43



$\therefore 3$

P/44



$\therefore 3$

P/45

Selecting 0 edges can be done in 1 way

1 edge can be done in 6 ways

2 edges " " " " 3 ways

$\therefore 10$

P/46

vertices are divided into

$\left\lfloor \frac{n}{2} \right\rfloor \quad \left\lceil \frac{n}{2} \right\rceil$

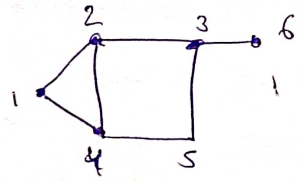
$M(G) = \min\left(\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil\right) = \left\lfloor \frac{n}{2} \right\rfloor$

Perfect Matching:

A matching is said to be perfect matching, if every vertex in the graph is matched
(or)
Induced degree of all the vertices is 1.

Induced degree:
The degree of a vertex in a matching, is called induced degree

Eg:



$\{12, 45, 36\}$ is a perfect matching

Note:

→ Every perfect matching is maximal, but reverse need not to be true

→ If perfect matching exists, then no of vertices will always be even but reverse need not to be true.

→ A graph may contain more than one perfect matching.

→ Total no of perfect matching possible for a complete graph with $2n$ vertices is

$$(2n-1)\,(2n-3)\,(2n-5)\cdots 5\cdot 3\cdot 1$$

no of way 1st vertex can map to other vertices

in solving problem take no of vertices $=2n$

$$= \frac{2n}{2n}\cdot(2n-1)\cdot\frac{2n-2}{2n-2}\cdot(2n-3)\cdots\frac{4}{4}\cdot 3\cdot\frac{2}{2}\cdot 1$$

$$= \frac{(2n)!}{2^n\,(n\cdot(n-1)(n-2)\cdots 1)} = \boxed{\frac{(2n)!}{2^n\cdot n!}}$$

Eg: find total no of perfect ~~mating~~ matching in K_6

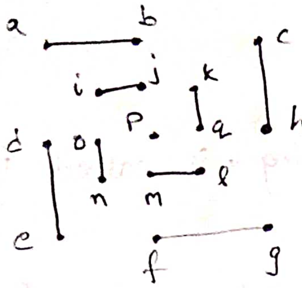
Sol: $(5)(3) = 15$ (or) $2n=6 \Rightarrow n=3$

$$\frac{6!}{2^3\cdot 3!} = \frac{\cancel{720}^{120}}{8\times\cancel{6}} = 15$$

Q12 Find matching no of graph in Q10

a) no of vertices = 17

matching number ≤ 8

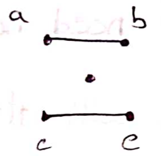


$\therefore M(G) = 8$

b)

no of vertices = 5

$M(G) \leq \left\lfloor \frac{5}{2} \right\rfloor \Rightarrow M(G) \leq 2$

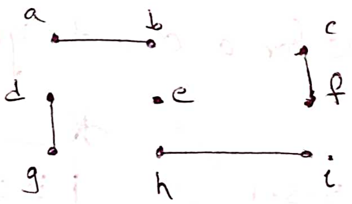


$\therefore M(G) = 2$

c)

no of vertices = 9

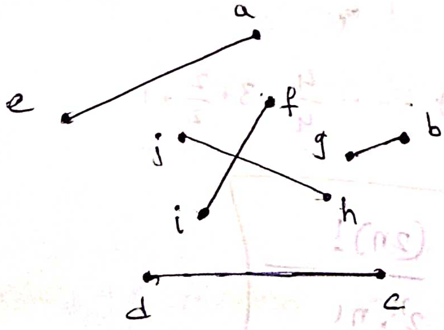
$M(G) \leq 4$



$M(G) = 4$

d) no of vertices = 10

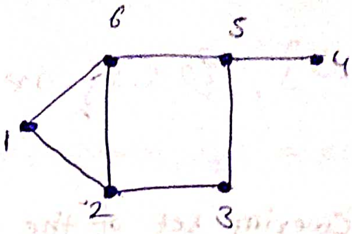
$\therefore M(G) \leq 5$



$\therefore M(G) = 5$

Covering:

It is set of edges such that all vertices should incident on atleast one edge.



Eg: {16, 12, 65, 53, 54}

{16, 54, 23}, {16, 12, 53, 54}

{12, 16, 65, 23, 53, 62, 54}

Set of all edge is also a covering set

→ It is also known as edge covering set.

Minimal Covering Set:

It is a covering set from which we can't remove ~~any~~ new elements.

Eg: {16, 12, 53, 54}, {16, 54, 23} are MCS

Covering Number ($c(G)$):

It is no of edges present in smallest covering set.

For above graph $c(G) = 3$

Note:

→ Every perfect matching is minimal covering set. But reverse need not to be true.

Eg: {16, 54, 23} is perfect matching and also Minimal Covering

{16, 12, 53, 53} is not perfect matching, but Minimal Covering

→ Every edge including pendent vertex is included in Covering.

Vertex Covering Set:

It is a set of vertices such that all the edges should incident on atleast one vertex.

Eg: {1, 6, 5, 3} is a vertex covering set.

{2, 5, 6} is a vertex covering set.

Note:

when it is specified as covering set, we consider it a edge covering set

Minimal vertex Covering set:

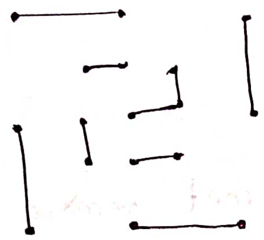
It is a vertex covering set from which we can't remove any vertices

Eg : $\{1,2,5\}$, $\{2,5,6\}$ are Minimal Vertex Covering set for the above graph

04/05/20

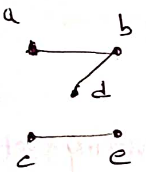
(Q13) Find edge Covering set of the graphs in (Q10) & Covering number

a)



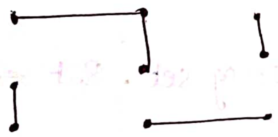
$C(G) = 9$

b)



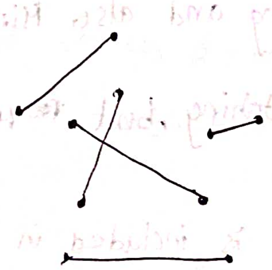
$C(G) = 5$

c)



$C(G) = 5$

d)



$\therefore C(G) = 5$

Trail:

It is alteranating sequence of vertices & edges in which no edge can be repeated, but vertices can be repeated

Closed Trail:

It is a trail in which starting vertex = end vertex

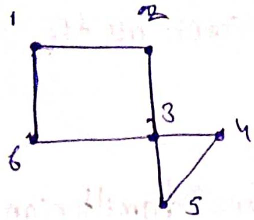
Euler circuit:

It is a closed trail that covers all the edges exactly once.

Euler graph:

A graph is called euler graph $\Leftrightarrow$ The graph has a euler circuit

Eg:

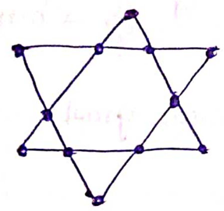


$1 \to 2 \to 3 \to 6 \to 1$ is a closed trail

$1 \to 2 \to 3 \to 4 \to 5 \to 3 \to 6 \to 1$ is a euler circuit

Hence the graph is euler graph

Eg:



This graph is also a euler graph

Theorem-12:

A graph is a Euler graph iff degrees of all vertices are even.
$(n \geq 2$ & $k = 1)$
connected graph

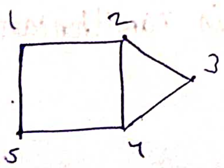
Open trail:

It is a trail in which starting & ending vertices are different

Euler path or Euler line:

It is a open ~~path~~ trail which covers all the edges exactly one.

Eg:



$4 \to 3 \to 2 \to 1 \to 5 \to 4 \to 2$ is a euler line

Theorem: 13

* A graph contains euler line iff the graph contains exactly two odd vertices exactly or all are even degree vertices.

→ Also the euler path start at one odd vertices and ends at the other odd vertex.

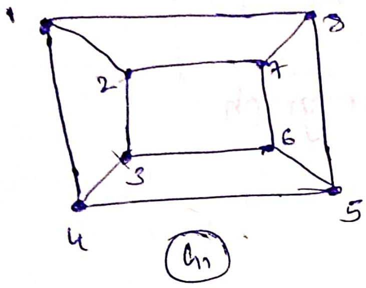
Hamiltonian graph:

path: Alternating seq. of vertices & edges in which vertices & edges are not repeated

Closed path: It is a path in which starting vertex = ending vertex.

Hamiltonian circuit: It is a closed path that cover all the vertices exactly ~~vs~~ once.

Hamiltonian Graph: It is a graph which contains hamiltonian circuit.



1 → 2 → 3 → 4 → 5 → 6 → 7 → 8 → 1 is a hamiltonian circuit.

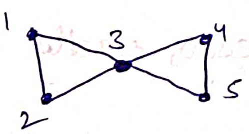
∴ The graph is hamiltonian graph.

Hamiltonian path: It is an open path that covers all the vertices exactly once.

→ Every hamiltonian circuit contains Hamiltonian path, but reverse need not to be true.

Eg: The above graph (G1) contains hamiltonian ckt and hence contains hamiltonian path, too.

Eg:

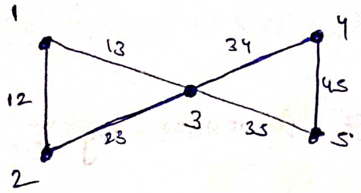


1 → 2 → 3 → 4 → 5 is a hamiltonian path.

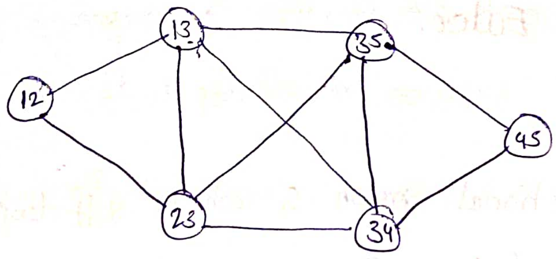
This graph doesn't contain hamiltonian circuit.

Graph
- Covering all edges
  - Closed trail → Euler Circuit
  - Open trail → Euler line
- Covering all vertices
  - Closed path → Hamiltonian Circuit
  - Open path → Hamiltonian path

Consider below graph



Line graph of above graph is



→ Line graph ~~gp~~ of every Euler graph is also a Euler graph.

→ Line graph of every Euler graph will always be a Hamiltonian graph & Euler graph.

→ $\boxed{\text{If } G \text{ is a Euler graph, then } L(G) \text{ is both euler \& Hamiltonian graph}}$

GATE 2013

In a graph of 8 vertices, probability of existence of an edge b/w two vertices is $1/2$. ~~Find that probability that chosen 3 random vertices form a cycle~~? Find expected no of cycles in the graph? of size 3

$P(e) = 1/2$

we can select 3 edges in $^8C_3$ ways

b/w 3 ~~edges~~ vertices we need it to be K_3 for 3 length cycle.

$\therefore$ probability that selected 3 vertices forms K_3 is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

$\therefore$ No of expected cycles of length $3 = {}^8C_3 \times \frac{1}{8} = \frac{8 \times 7 \times 6}{3 \times 2} \times \frac{1}{8}$

$= 7$ cycles

GATE 2008

G is a simple undirected graph. Some vertices of G are of odd degree. Add a node v to G and make it adjacent to each odd degree vertex of G. The resultant graph is sure to be

a) Regular b) Complete c) Hamiltonian d) Euler.

Sol: A graph contains even no of odd vertices

→ Adding a vertex and making it adjacent to all odd vertices give the new vertex even degree.

→ Also degree of ~~all~~ each odd vertices increases by 1, making each odd vertex as even degree vertex.

→ Thus the new graph's vertices are of even degree.

∴ The graph is Euler.

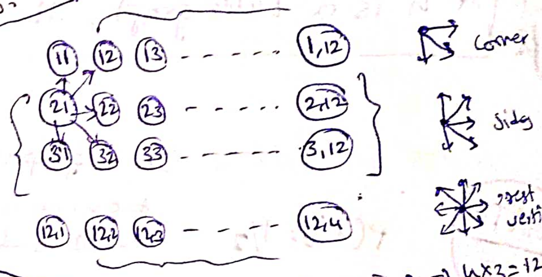
GATE 2014 Consider an unidirectional graph G where self-loop are not allowed. The vertex set of G is $\{(i,j): 1 \le i \le 12,\ 1 \le j \le 12\}$. There is an edge b/w (a,b) & (c,d) if $|a-c| \le 1$ and $|b-d| \le 1$.

The number of edges in this graph is ______.

Sol:

no of vertices $= 12 \times 12 = 144$ vertices.

alternating way for viewing degree

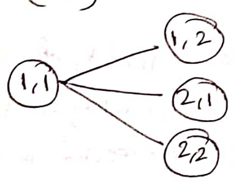


All corners degree is 3 ⇒ $4 \times 3 = 12$

All sides degree is 5 ⇒ $40 \times 5 = 200$

rest of vertices degree is 8 ⇒ $100 \times 8 = 800$

Consider vertex (1,1)



∴ degree of $(1,1) = 3$

Slly degree of $(12,12) = 3$ } $2 \times 3 = 6$

Consider all vertices of form $(1,x)$ $\forall\ 1 < x < 12$ (∴ 10 vertices of this form)

$(1,x)$ is adjacent to $(2,x)$, $(1,x+1)$, $(2,x+1)$, $(2,x-1)$, $(1,x-1)$

deg of $(1,x)$ is 3

Slly for $(x,1)$, degree is 3 } $20 \times 5 = 100$

& 10 vertices

Slly for $(x,12)$ & $(12,x)$ → $20 \times 5 = 100$

10 10 20 vertices

Consider $(1,12)$

$(1,12)$ is adjacent to $(2,12)$ & $(1,11)$ & $(2,11)$

Slly for $(12,1)$

∴ ~~degree~~ $2 \times 3 = 6$

Consider vertices of form (x,y) $\forall$ $1<x<12$, $\forall$ $1<y<12$

we have $10 \times 10 = 100$ vertices of this form

(x,y) is adjacent to $(x-1,y-1)$ $(x-1,y)$ $(x-1,y+1)$
$(x,y-1)$ $(x,y+1)$
$(x+1,y-1)$ $(x+1,y)$ $(x+1,y+1)$

$\therefore$ degree of vertex of this form $= 8$

Sum of degrees $= 8 \times 100 = 800$

Sum of degrees of all vertices in graph $= 6 + 100 + 100 + 6 + 800$
$= 1012$

$\therefore$ no of edges $= \frac{1012}{2} = 506$ edges

GATE 2005

Let G be a simple graph with 20 vertices and 100 edges. The size of the minimum vertex cover of G is 8. Then, the size of maximum independent set of G is :

a) 12 b) 8 c) less than 8 d) More than 12

Sol:

Let $\{v_1, v_2, v_3 \ldots v_8\}$ be min vertex cover.

This means for every edge, one of its end vertices is in the min vertex cover set.

i.e., no two vertices in set $\{v_9, v_{10} \ldots v_{20}\}$ are adjacent to each other.

$\therefore$ Maximal independent set is $\{v_9, v_{10} \ldots v_{20}\}$

~~none of~~ cuz none of v_1 to v_8 can be added to this set.

$\Rightarrow$ Size of max Independent set $= 12$

Theorem: 14

$\boxed{\text{Sum of size of minimum vertex cover and size of maximum independent set is equal to number of vertices}}$

P/4 given graph is k-regular, k is odd

$\therefore$ no of vertices is even, say $2n$

Sum of degrees $= 2nk$

no of edges $= nk$

$\therefore$ multiple of k

P/5 a) $\cancel{5}44332$
$\cancel{3}3221$
$\cancel{2}111$
001 ✗

b) 54432 ✗

c) $\cancel{6}654331$
$\cancel{5}43221$
$\cancel{3}2110$
1000 ✗

d) $0,1,2,\ldots,n-1$ ✗

e) $\cancel{3}3332$
$\cancel{2}222$
112
$\cancel{2}11$
00 ✓

P/7 $G \to 522221 \Rightarrow \bar{G} \to 033334$

$\sum deg(v_i) = 16 \Rightarrow e = 16/2 = 8$

P/8 no of vertices $= n+2+4+3 = n+9$

min no of edges $= n+9-1 = n+8$

given sum of degrees $= n+4+12+12 = n+28$

$\therefore$ ~~2+2~~ $n+28 = 2(n+8) \Rightarrow n = 12$

P/9 $2(10-1) =$ ~~2~~ 18

P/10 $G \to$ ~~{e,e,e,e,e,e,e,e,e}~~ $\{0,0,0,0,0,0,0,0,e\}$

$\bar{G} \to \{$~~0~~$, 8-0, 8-0, 8-0, 8-0, 8-0, 8-0, 8-0, 8-0, 8-e\}$

$\{0,0,0,0,0,0,0,0,0,e\}$

$\therefore$ 8 vertices

No of edges possible

P/14

$^6C_2 = \frac{6 \times 5}{2} = 15$

$^{15}C_{12} = {^{15}C_3} = \frac{\cancel{15}^{5} \times \cancel{14}^{7} \times 13}{\cancel{3} \times \cancel{2} \times 1} = 455$

P/16

As graph is connected, & adjacency is transitive we get this graph as complete graph $\Rightarrow \frac{n(n-1)}{2} = \frac{n^2-n}{2}$

P/18

Let n be no of vertices

$n-14$ vertices have degree 4 or 5

Let x be no of vertices with degree 5

$\Rightarrow$ no of vertices with degree $4 = (n-14)-x$

$\Rightarrow \sum deg(v_i) = 2e$

$14(1) + (n-14-x)(4) + x(5) = 2(n-1)$

$14 + 4n - 56 - 4x + 5x = 2n-2$

$4n - 42 + x = 2n-2$

$x = 40-2n$

P/30

if n = odd

$\alpha = M(G) = \lfloor \frac{n}{2} \rfloor$

$\beta = \chi(G) = 3$

$2\alpha + \beta = 2\lfloor \frac{n}{2} \rfloor + 3$

$= (n-1)+3$

$= n+2$

if n = even

$\alpha = M(G) = n/2$

$\beta = \chi(G) = 2$

$2\alpha + \beta$

$= n+2$

P/31

Chromatic number of any bipartite graph = 2

P/32

d $\Rightarrow$ all cycles even length $\Rightarrow$ bipartite graph $\Rightarrow$ chromatic num = 2

P/33

~~Conde~~ Consider ~~$K_{2(3)}$~~ $W_6 = W_{2(3)}$

$\Rightarrow$

.... $\therefore$ $2n-1$ perfect matchings

P/34 Start graph ($n \geq 3$) can't have a perfect matching

perfect matching doesn't exist for $k_{m,n}$ if $m \neq n$

no of perfect matchings for $k_{n,n} = n! = 3! = 6$

P/35 S1: It is true

S2: if G has perfect matching then G has even no of vertices

but reverse need not to be true

$\therefore$ S2 is false

S3: bipartite graph has complete matching form V_1 to V_2 iff

every vertex in V_1 is incident by ~~some~~ some edge

$\therefore |V_1| \leq |V_2|$

$\therefore$ S3 is false

Only S_1 is true

P/48

V_1	V_2
4	5

$\min(4,5) = 4$

P/50 The only graph possible with given condition is cycle C_{10}

$\therefore$ no of cut edges $= 0$

P/53 $G \to \{5,5,5,5,5,5\}$

Euler path doesn't exist

$\therefore$ G is not traversable

P/54 Every edge is part of a cycle

$\therefore \lambda(G) \neq 1$

$\{de, df\}$ is cut set $\Rightarrow \lambda(G) = 2$

d is clearly a cut vertex $\Rightarrow k(G) = 1$

$\Rightarrow 1+2 = 3$

P/58 Every edge is a cut edge in tree

$\therefore$ 9 edges $\Rightarrow$ 9 cut sets

P/62

G $\bar{G}$
d.c → c
c → c, dis

∴ option a

P/63

S_1 is clearly true
S_2 is cleary false
S_3 : for necessarily connected we need $\frac{(n-1)(n-2)}{2} + 1$ edges
∴ S_3 is false

S_4 is true (proof in ~~theory~~ notes theory) ∴ S_1 & S_4

P/64

$a \to b \to c \to d \to e \to \overset{c \to a}{\cancel{a}}$ Euler ckt ⇒ euler path also exists

$a \to b \to c \to d \to e$ ⇒ Hamiltonian path exists

but Hamiltonian ckt doesn't exist.

P/65

$G \to \{ \overset{a}{3}, \overset{b}{3}, \overset{c}{2}, \overset{d}{4}, \overset{e}{2} \}$

∴ euler path exists

$a \to b \to c \to d \to e \to a$ ⇒ Hamiltonian ckt & path exists

P/66

$G \to \{ 4, 2, 2, 2, 2, 4 \}$

∴ euler ckt & path exists

Hamiltonian path ~~or~~ and ckt doesn't exist

P/67

S_1 is false, S_2 is true
⇒ opt d

P/69 Direct formula, n-k

P/70

2-regular & perfect matching exists ⇒ even no of vertices

Say G can ~~have~~ has 12 verties
and graph be $C_6 + C_6$
∴ S_1 is false for above case
S_3 is false for above case

As perfect match exist every component has even no of vertices i.e., even length cycle
Also chromatic num = 2 ⇒ S_2 & S_4

P/71 $\frac{(n-k)(n-k+1)}{2} = \frac{8(9)}{2} = 36$

P/72 $G \to \{ \overset{a}{3}, \overset{b}{2}, \overset{c}{2}, \overset{d}{4}, \overset{e}{2}, \overset{f}{2}, \overset{g}{3}, \overset{h}{2} \}$

$\therefore$ Euler path exists

~~but~~ Euler ckt doesn't exist

also Ham. ckt & Ham. path don't exist

$\therefore$ only S_2 is true.

P/73 S_1 is clearly false

S_2: $\delta(G) \le \frac{2e}{n} \le \Delta(G)$ $\quad 2e \le 27$

$\delta(G) \le \frac{27}{9} \le \Delta(G)$ $\quad$ avg degree $\frac{2e}{n} \le \frac{27}{9}$

$\delta(G) \le 3 \le \Delta(G)$ $\quad$ avg degree ≤ 3

Here degree of all vertices can't be 3 cuz if degree of every vertex is 3 then we will have odd no of vertices with odd degree.

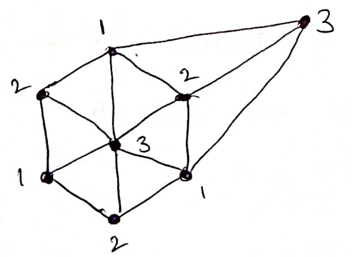
$\therefore$ we must have a vertex of degree ≥ 4 &

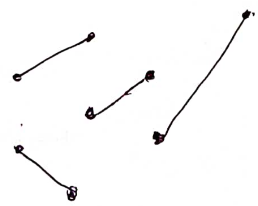
we must have a vetex of degree ≤ 2

$\therefore$ only S_2 is true

P/74 It is clearly that both are false.

P/75





$\chi(G) = 3$ $\qquad$ $M(G) = 4$ $\Rightarrow$ $3 + 4 = 7$

P/76 S_1: △ △ $\therefore$ need not to be connected $\Rightarrow S_1$ is false

S_2: △ △ Euler ckt need not to exist

S_3: Sum of degree of two non-adjacent vertices ≥ 6

$\Rightarrow$ graph is connected.

P/78) Given graph is connected and degree of all vertices is 4

$\Rightarrow$ Euler circuit exists.

Let us say this euler circuit has an edges moving from set S to set T. then there must be an edge which leads back to set S from set T.

Thus we can say we must have even no of edges b/w S & T.

- Thus exactly 1 edge & exactly 3 edges is not possible

$\therefore$ atleast 2 edges are necessary

P/79) d) For any ~~whett~~ wheel graph, Euler circuit doesn't exist as degree of vertices other than hub is always 3.

P/80) a $\Rightarrow$) It is clearly true

b $\Rightarrow$) Consider we choose a vertex from set 1, then we move to a vertex in set V_2 then we come back to V_1 . . . . . finally we need to end in V_1 again.

For that $|V_1| = |V_2|$

c $\Rightarrow$) It is clearly true

d $\Rightarrow$) Hamiltonian cycle exists for every wheel graph

f every odd degree.

P/81) a) for ~~n~~ $n \geq 3$ & n is odd

no of edge disjoint Ham. cycles $= \frac{n-1}{2} = \frac{6}{2} = 3$

b) From dirac's theorem

c) $\frac{4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1}{4 \cdot 2} = 72$

$4 \cdot 2 \rightarrow$ clockwise & Anticlockwise

4 vertices in first set give same cycle.

we can choose a vertex frm a ~~set~~ set in 4 ways and select edge to next set in 4 ways, then back to previous set in 3 way and so and so forth

d) Max no of edges that can be drawn such that no hamiltonian cycle is

$$\frac{(n-1)(n-2)}{2} + 1 = \frac{(4)(3)}{2} + 1 = 7$$

$\therefore$ Hamiltonian cycle may or may not exist

se