

线性代数期中试卷 答案 (2019.4.27)

一. 简答与计算(本题共5小题, 每小题8分, 共40分)

1. 计算 $A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ -1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix}$ 的第一行所有元素的代数余子式之和。

$$\text{解: } A_{11} + A_{12} + A_{13} + A_{14} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{vmatrix} = - \begin{vmatrix} -1 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 3 & -3 \end{vmatrix} = 6.$$

$$\text{解法二: } A_{11} = \begin{vmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 3 & -3 & 0 \end{vmatrix} = 0, A_{12} = - \begin{vmatrix} -1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = 0, A_{13} = \begin{vmatrix} -1 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 3 & 0 \end{vmatrix} = 0, \\ A_{14} = - \begin{vmatrix} -1 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 3 & -3 \end{vmatrix} = 6, \text{ 故 } A_{11} + A_{12} + A_{13} + A_{14} = 6.$$

2. 计算 $X = (X_{ij})_{3 \times 3}$ 使之满足矩阵方程 $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} X \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$ 。

$$\text{解: } X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\ = \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & -1 \end{pmatrix}$$

$$\text{解法二: } \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 3 & 2 \\ 0 & -2 & 0 & 0 & -4 & -2 \\ 0 & 0 & 1 & 0 & -2 & -2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & -2 & -2 \end{array} \right), \\ \left(\begin{array}{ccc|ccc} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \text{ 故 } X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & -1 \end{pmatrix}.$$

3. 已知4阶方阵 A 的伴随矩阵 $A^* = \begin{pmatrix} 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}$, 求 A 。

解: $|A^*| = 27$, 且 $|A^*| = |A|^3$, 故有 $|A| = 3$, 从而有 $A^*A = |A|E = 3E$, 可得 A 可逆且有 $A = 3(A^*)^{-1}$ 。

设 $A^* = \begin{pmatrix} O & A_1 \\ A_2 & O \end{pmatrix}$, 则易知 $(A^*)^{-1} = \begin{pmatrix} O & A_2^{-1} \\ A_1^{-1} & O \end{pmatrix}$ 。

$$\text{因为 } A_1^{-1} = \frac{1}{9} \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}, A_2^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \text{ 故 } A = 3(A^*)^{-1} = \begin{pmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 1 & -2/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

解法二: $|A^*| = 27$, 又有 $|A^*| = |A|^3$, 故有 $|A| = 3$, 从而有 $A^*A = |A|E = 3E$,

$$(A^*, 3E) = \left(\begin{array}{cccc|cccc} 0 & 0 & 3 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 3 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 3 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 0 & 0 & 3 & 0 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 2 \end{array} \right) \\ \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 1 & -2/3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{array} \right).$$

$$\text{故 } A = \begin{pmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 1 & -2/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

4. 给定向量组 $A = \{\alpha_1, \alpha_2, \dots, \alpha_{100}\}$ 与 $B = \{\beta_1, \beta_2, \dots, \beta_{20}\}$, 已知 $r(A) = 7$, 某同学计算出 $r(A \cup B) = 31$, 请问对吗? 说明理由.

解: 不对.

由 $r(A) = 7$ 可知, A 的极大无关组含 7 个向量, 不妨设其中的一个极大无关组为 $\alpha_1, \alpha_2, \dots, \alpha_7$, 令向量组 $C = \{\alpha_1, \alpha_2, \dots, \alpha_7\} \cup B = \{\alpha_1, \alpha_2, \dots, \alpha_7, \beta_1, \beta_2, \dots, \beta_{20}\}$, 则含 27 个向量的向量组 C 可表示出向量组 $A \cup B$ 中的所有向量, 反之亦然, 故两个向量组 C 和 $A \cup B$ 等价, 于是有 $r(A \cup B) = r(C) \leq 27 < 31$.

解法二: 不对. 假设 $r(A \cup B) = 31$ 成立, 则 $A \cup B$ 的极大无关组含 31 个向量,

假设其中一个极大无关组为: $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_m}, \beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_n}$, 则有 $m + n = 31, m \leq 100, n \leq 20$, 由 $r(A) = 7$ 可知, 还需满足 $m \leq 7$, 故有 $m + n \leq 7 + 20 = 27 < 31 = m + n$, 矛盾.

5. 已知线性方程组 $Ax = b$ 的三个特解为 $\alpha_1 = (1, -2, 3)^T, \alpha_2 = (0, -1, -2)^T, \alpha_3 = (-4, 2, 1)^T, r(A) = 1$, 试写出 $Ax = b$ 的通解.

解: 由 $r(A) = 1$ 可知, 齐次方程组 $Ax = 0$ 的基础解系含 2 个解向量.

令 $\beta_1 = \alpha_1 - \alpha_2 = (1, -1, 5)^T, \beta_2 = \alpha_1 - \alpha_3 = (5, -4, 2)^T$, 则 β_1, β_2 线性无关,

又 $A\beta_1 = A\alpha_1 - A\alpha_2 = b - b = 0, A\beta_2 = A\alpha_1 - A\alpha_3 = 0$, 故 β_1, β_2 是 $Ax = 0$ 的基础解系,

于是 $Ax = b$ 的通解为: $k_1\beta_1 + k_2\beta_2 + \alpha_1, k_1, k_2 \in R$.

二.(10分) 假定矩阵 $A = (\alpha_1, \alpha_2, \alpha_3)$ 为 3 阶可逆矩阵: $A^{-1} = \begin{pmatrix} \beta_1^T \\ \beta_2^T \\ \beta_3^T \end{pmatrix}$, 令 $P = \alpha_2\beta_2^T + \alpha_3\beta_3^T$.

(1) 证明 $P^2 = P$ (即 P 是投影矩阵);

(2) P 的秩是多少?

(3) 给定 3 维向量 x , Px 可否由 α_2 与 α_3 线性表出? 如果可以, 写出一个表出方式.

解: (1) $E = AA^{-1} = \alpha_1\beta_1^T + \alpha_2\beta_2^T + \alpha_3\beta_3^T = \alpha_1\beta_1^T + P$, 故 $P = E - \alpha_1\beta_1^T$.

又有 $E = A^{-1}A = (\beta_i^T\alpha_j)$, 故有 $\beta_i^T\alpha_j = 0, i \neq j, \beta_i^T\alpha_i = 1$.

于是 $P^2 = (E - \alpha_1\beta_1^T)(E - \alpha_1\beta_1^T) = E - 2\alpha_1\beta_1^T + \alpha_1(\beta_1^T\alpha_1)\beta_1^T = E - \alpha_1\beta_1^T = P$.

证法二: $E = A^{-1}A = (\beta_i^T\alpha_j)$, 故有 $\beta_i^T\alpha_j = 0, i \neq j, \beta_i^T\alpha_i = 1$.

$$P = (\alpha_2, \alpha_3) \begin{pmatrix} \beta_2^T \\ \beta_3^T \end{pmatrix}, P^2 = (\alpha_2, \alpha_3) \left(\begin{pmatrix} \beta_2^T \\ \beta_3^T \end{pmatrix} (\alpha_2, \alpha_3) \right) \begin{pmatrix} \beta_2^T \\ \beta_3^T \end{pmatrix} = (\alpha_2, \alpha_3) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_2^T \\ \beta_3^T \end{pmatrix} = P.$$

(2) $P\alpha_1 = \alpha_2\beta_2^T\alpha_1 + \alpha_3\beta_3^T\alpha_1 = 0, P\alpha_2 = \alpha_2, P\alpha_3 = \alpha_3$,

故 $PA = (P\alpha_1, P\alpha_2, P\alpha_3) = (0, \alpha_2, \alpha_3)$, 因为 A 和 A^{-1} 可逆,

$$\text{故 } r(P) = r(A^{-1}PA) = r(A^{-1}(0, \alpha_2, \alpha_3)) = r \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2.$$

解法二: $P\alpha_1 = \alpha_2\beta_2^T\alpha_1 + \alpha_3\beta_3^T\alpha_1 = 0, P\alpha_2 = \alpha_2, P\alpha_3 = \alpha_3$,

故 $PA = P(\alpha_1, \alpha_2, \alpha_3) = (0, \alpha_2, \alpha_3)$, 因为 A 可逆, 故 $r(PA) = r(P)$, 且 α_2, α_3 线性无关,

于是 $r(P) = r(PA) = r(0, \alpha_2, \alpha_3) = 2$.

(3) 因为 $Px = \alpha_2\beta_2^Tx + \alpha_3\beta_3^Tx = (\beta_2^Tx)\alpha_2 + (\beta_3^Tx)\alpha_3 = k_2\alpha_2 + k_3\alpha_3$, 其中 $k_2 = \beta_2^Tx, k_3 = \beta_3^Tx$,

故 Px 可由 α_2, α_3 线性表出.

三.(10分) 计算 $(A^*)^*$, 此处 A^* 表示矩阵 A 的伴随矩阵, $A = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}$.

解: 因为 $A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} (b_1, b_2, b_3)$, 故 $r(A) \leq r(b_1, b_2, b_3) \leq 1$, 故 A 的所有 2 阶子式均为 0,

$$\text{于是 } A^* = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = O, \text{ 故 } (A^*)^* = O^* = O.$$

四. (10分) 计算 $f(\pi)$ 与 $f'(\pi)$, 此处:

$$f(x) = \begin{vmatrix} a_1 & b_1 & a_1x^2 + b_1x + c_1 \\ a_2 & b_2 & a_2x^2 + b_2x + c_2 \\ a_3 & b_3 & a_3x^2 + b_3x + c_3 \end{vmatrix}.$$

解: $f(x) = \begin{vmatrix} a_1 & b_1 & a_1x^2 + b_1x + c_1 \\ a_2 & b_2 & a_2x^2 + b_2x + c_2 \\ a_3 & b_3 & a_3x^2 + b_3x + c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{常数}$, 令 $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = w$, 则有 $f(\pi) = w$, $f'(\pi) = 0$.

五. (12分) 给定矩阵 $A = \begin{pmatrix} 1 & -1 & -3 & -2 & -3 \\ 1 & 3 & 8 & -3 & 9 \\ 3 & 1 & 2 & -7 & 3 \end{pmatrix}$,

- (1) 计算 $r(A)$;
- (2) 计算线性方程组 $Ax = 0$ 的基本解组;
- (3) 假定 $\eta = (1, -1, 0, 0, 2)^T$ 是 $Ax = b$ 的解, 确定 b 并计算 $Ax = b$ 的通解.

解: (1) $A = \begin{pmatrix} 1 & -1 & -3 & -2 & -3 \\ 1 & 3 & 8 & -3 & 9 \\ 3 & 1 & 2 & -7 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -3 & -2 & -3 \\ 0 & 4 & 11 & -1 & 12 \\ 0 & 4 & 11 & -1 & 12 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/4 & -9/4 & 0 \\ 0 & 1 & 11/4 & -1/4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,
故 $r(A) = 2$.

(2) 由上述过程可知基本解组为: $\alpha_1 = (1/4, -11/4, 1, 0, 0)^T$, $\alpha_2 = (9/4, 1/4, 0, 1, 0)^T$, $\alpha_3 = (0, -3, 0, 0, 1)^T$.

(3) $b = A\eta = \begin{pmatrix} 1 & -1 & -3 & -2 & -3 \\ 1 & 3 & 8 & -3 & 9 \\ 3 & 1 & 2 & -7 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 16 \\ 8 \end{pmatrix}$, 通解为: $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + \eta$, $k_1, k_2, k_3 \in R$.

六. (10分) 写出向量组 $\alpha_1 = (1+a, 1, 1, 1)^T$, $\alpha_2 = (1, 1+a, 1, 1)^T$, $\alpha_3 = (1, 1, 1+a, 1)^T$ 的极大线性无关组; $\beta = (1, 1, 1, b)^T$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出? 如果可以, 表出方式唯一吗?

解: (1) $B = (\alpha_1, \alpha_2, \alpha_3, \beta) = \left(\begin{array}{cccc|c} 1+a & 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 & 1 \\ 1 & 1 & 1+a & 1 & 1 \\ 1 & 1 & 1 & 1 & b \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b \\ a & 0 & 0 & 0 & 1-b \\ 0 & a & 0 & 0 & 1-b \\ 0 & 0 & a & 0 & 1-b \end{array} \right)$.

当 $a = 0$ 时, 向量组 $\alpha_1, \alpha_2, \alpha_3$ 的一个极大无关组为: $\alpha_1 = (1, 1, 1, 1)^T$.

当 $a \neq 0$ 时, $B \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & (1-b)/a \\ 0 & 1 & 0 & 0 & (1-b)/a \\ 0 & 0 & 1 & 0 & (1-b)/a \\ 0 & 0 & 0 & 0 & ((a+3)b-3)/a \end{array} \right)$, 故 $\alpha_1, \alpha_2, \alpha_3$ 的极大无关组就是 $\alpha_1, \alpha_2, \alpha_3$.

(2) 由 (1) 的过程可知, 当 $a = 0, b \neq 1$ 时, 不能表示 β .

当 $a = 0, b = 1$ 时, 能表示 β , 由 $r(B) = 1 < 3$ 可知表达式不唯一.

当 $a = -3$ 或 $a \neq 0, a \neq -3, b \neq 3/(a+3)$ 时, 不能表示 β .

当 $a \neq 0, a \neq -3, b = 3/(a+3)$ 时, 能表示 β , 由 $r(B) = 3$ 可知表达式唯一.

七. (8分) $A = (a_{ij})_{m \times n}$ 为实矩阵, b 为 m 维实向量, 证明 $A^T Ax = A^T b$ 有解.

(提示: 先证明 $r(A^T A) = r(A)$)

证: 对方程组 $A^T Ax = A^T b$, 有 $r(A^T A) \leq r(A^T A, A^T b) = r(A^T (A, b)) \leq r(A^T) = r(A)$.

若 $r(A^T A) = r(A)$, 则有 $r(A^T A) = r(A^T A, A^T b)$, 于是 $A^T Ax = A^T b$ 有解.

故只要证明 $r(A^T A) = r(A)$.

因为 $Ax = 0 \Rightarrow A^T Ax = 0$, 且 $A^T Ax = 0 \Rightarrow x^T A^T Ax = (Ax)^T (Ax) = 0 \Rightarrow Ax = 0$,

故方程组 $Ax = 0$ 与 $A^T Ax = 0$ 同解. 若 x 为 n 维, 解空间维数为 k , 可得 $r(A) = n - k = r(A^T A)$.