CTA Assignment

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Q1

CTA Assignment WS20/21

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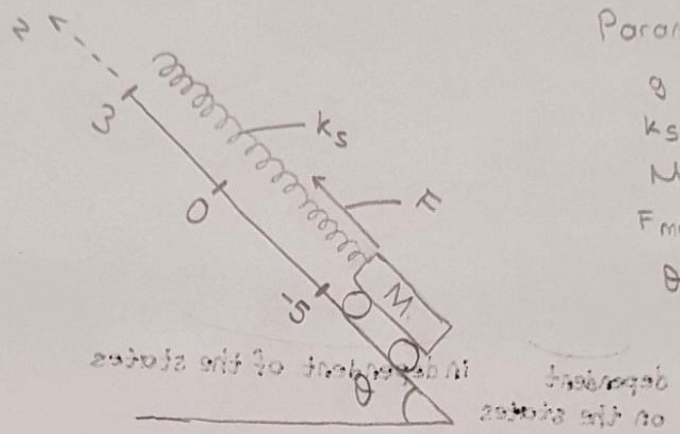


Parameters

$g = 10\ m/s^2$

$k_s = 3\ N/m$

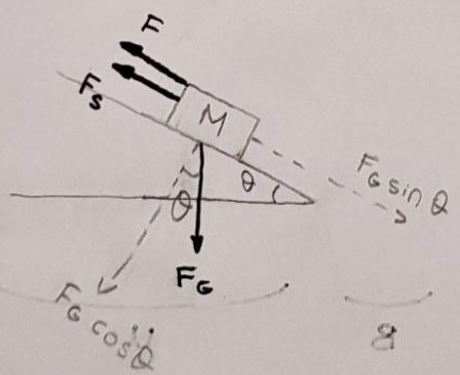
$M = 15\ Kg$

$F_{max} = 115\ N$

$\theta = 30°$

1)

- Free-body diagram of mass M



* First Law of Motion

$$\sum_{i=1}^{n} F_i = 0$$

Sum off all forces acting on a static object equals to zero.

* Second law of Motion

Linear Motion :

$$\sum_{i=1}^{n} F_i = m a \ , \quad a = \ddot{x} \ , \quad v = \dot{x}$$

- Equations and Force Balance

$$F \ , \quad F_s = k_s \Delta x \ , \quad F_G \sin\theta = M g \sin\theta$$

F is a constant input, a linear force provided by the motor. $F_G \sin\theta$ is also constant and doesn't depend on the state of the system.

* $M a = F - F_s - F_G \sin\theta$

* States of the system are the velocity and the displacement of the object since those are the dynamics which change over time.

$x_1 = z \,, \quad x_2 = \dot{z} \,, \quad \dot{x}_1 = x_2 = \dot{z} \,, \quad \dot{x}_2 = \ddot{z}$

2) $M\ddot{z} = F - F_s - F_g \sin\theta$

$M\ddot{z} = F - k_s(z-(-5)) - Mg\sin\theta$

$$\ddot{z} = \underbrace{-\frac{k_s}{M} z}_{\text{dependent on the states}} \underbrace{- \frac{5k_s}{M} + \frac{F}{M} - g\sin\theta}_{\text{independent of the states}}$$

3) $\dot{x} = Ax + Bu \qquad y = Cx + Du$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} z \\ \dot{z} \end{pmatrix} \qquad \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{z} \\ \ddot{z} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k_s}{M} & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 0 \\ 1/M \end{pmatrix}}_{B} \underbrace{(F - 5k_s - Mg\sin\theta)}_{u}$$

$$y = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} + \underbrace{0}_{D} \underbrace{(F - 5k_s - Mg\sin\theta)}_{u}$$

4) $A = \begin{pmatrix} 0 & 1 \\ -1/5 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 \\ 1/15 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $D = 0$

$u = F - 90$ $\quad x = \begin{pmatrix} z \\ \dot{z} \end{pmatrix}$

5) (A, B) is controllable $\iff$ $\text{rank} \overbrace{[B \;\; AB \; \dots \; A^{n-1}B]}^{\text{controllability matrix}} = n$

$AB = \begin{pmatrix} 1/15 \\ 0 \end{pmatrix}$ $\quad \text{rank} \begin{pmatrix} 0 & 1/15 \\ 15 & 0 \end{pmatrix} = 2 = n$

The system is controllable. ✓

6) $\lambda_1 = -0{,}8 \quad \lambda_2 = -1$

$(s + 0{,}8)(s + 1) = s^2 + 1{,}8s + 0.8$ **(desired characteristic poly.)**

$$\det(sI - (A+BK)) = \left| \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \underbrace{\left\{ \begin{pmatrix} 0 & 1 \\ -1/5 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/15 \end{pmatrix} (k_1 \;\; k_2) \right\}}_{\begin{bmatrix} 0 & 1 \\ -1/5 + \frac{k_1}{15} & \frac{k_2}{15} \end{bmatrix}} \right|$$

$$= \begin{vmatrix} s & -1 \\ +1/5 - \frac{k_1}{15} & s - \frac{k_2}{15} \end{vmatrix}$$

$$= \left(s^2 - \frac{5k_2}{15}\right) - \left(-\frac{1}{5} + \frac{k_1}{15}\right) = s^2 - \frac{k_2}{15} s - \frac{k_1 - 3}{15}$$

$k_2 = -27, \quad k_1 = -9$

7) The closed-loop response of the system is plotted in MATLAB with closed-loop poles $\lambda_1 = -0{,}8$, $\lambda_2 = -1$ over a time period of 20 seconds.

8) From the closed-loop response plots of the states and the input, it can be clearly seen that the controlled robot reaches equilibrium position ($z = 0$) and velocity ($\dot{z} = 0$) at about 6-8 seconds. For the rest of the simulation, states are stable. The input force at the beginning is 135 N and reaches equilibrium at $F = 90$ N. However, the maximum amount of force that the motor can provide is $F_{max} = 115$ N meaning that due to the amount of force required to move the robot (135 N > F_{max}), with these chosen closed-loop poles, Design (I) is problematic.

9) To mitigate the problem that results from Design (I), we need to choose new eigen values such that input force is smaller than $F_{max} = 115$ N. The rate of convergence towards equilibrium point for Design (I) seemed to be decent. Therefore, we just rescale the Design (I) eigen values with a factor of 0.75 to reduce the input force required to move the robot. For faster convergence, more force is required hence we sacrifice convergence speed for less input force.

$$\lambda_1 = 0.75 \times -0{,}8 = -0{,}6 \quad , \quad \lambda_2 = 0{,}75 \times -1 = -0{,}75$$

$$(s + 0{,}75)(s + 0{,}6) = s^2 + 1{,}35s + 0{,}45$$

Using the calculated characteristic polynomial, with k_1 and k_2 plugged in from Task (6):

$$\det(sI - (A + BK)) = s^2 - \frac{k_2}{15}s - \frac{k_1 - 3}{15}$$

$$k_1 = -3{,}75 \quad , \quad k_2 = -20{,}25$$

10) The closed-loop response of the system is plotted in MATLAB with closed-loop poles $\lambda_1 = -0.6$, $\lambda_2 = -0.75$ over a time period of 20 seconds.

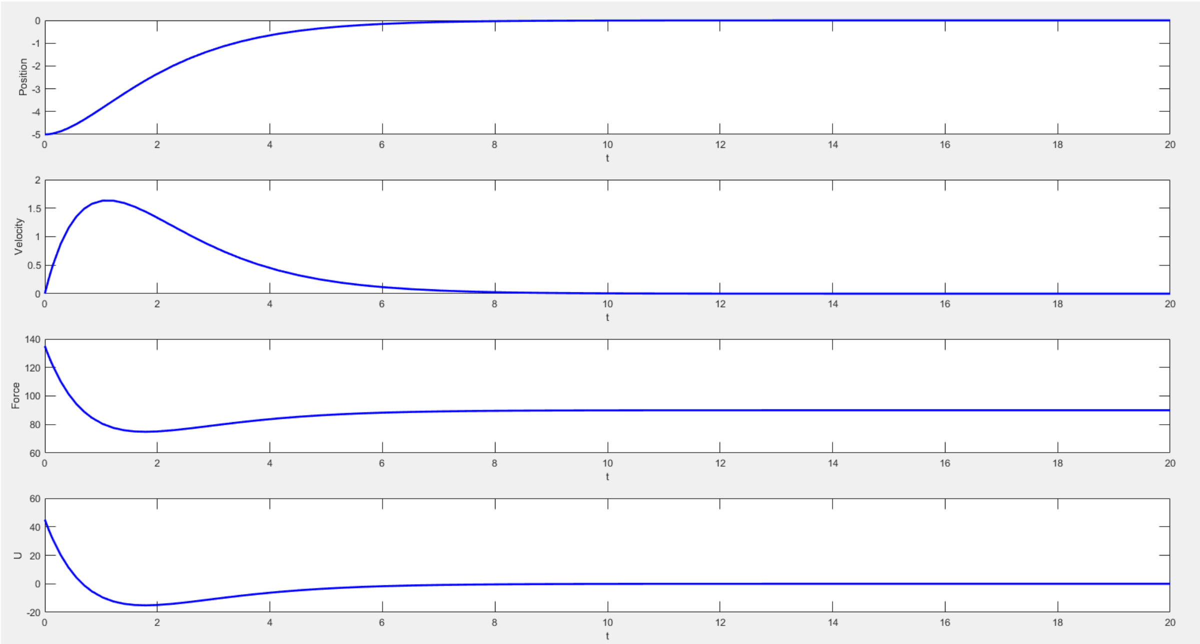
11) The design (II) should move the robot to position $z = 0$ on the inclined plane and the robot shall remain there afterwards. Therefore we plug $z = 0$ and $\ddot{z} = 0$ in the main differential equation:

$$M\ddot{z} = F_{ss} - k_s\,(z - (-5)) - Mg\sin\theta$$

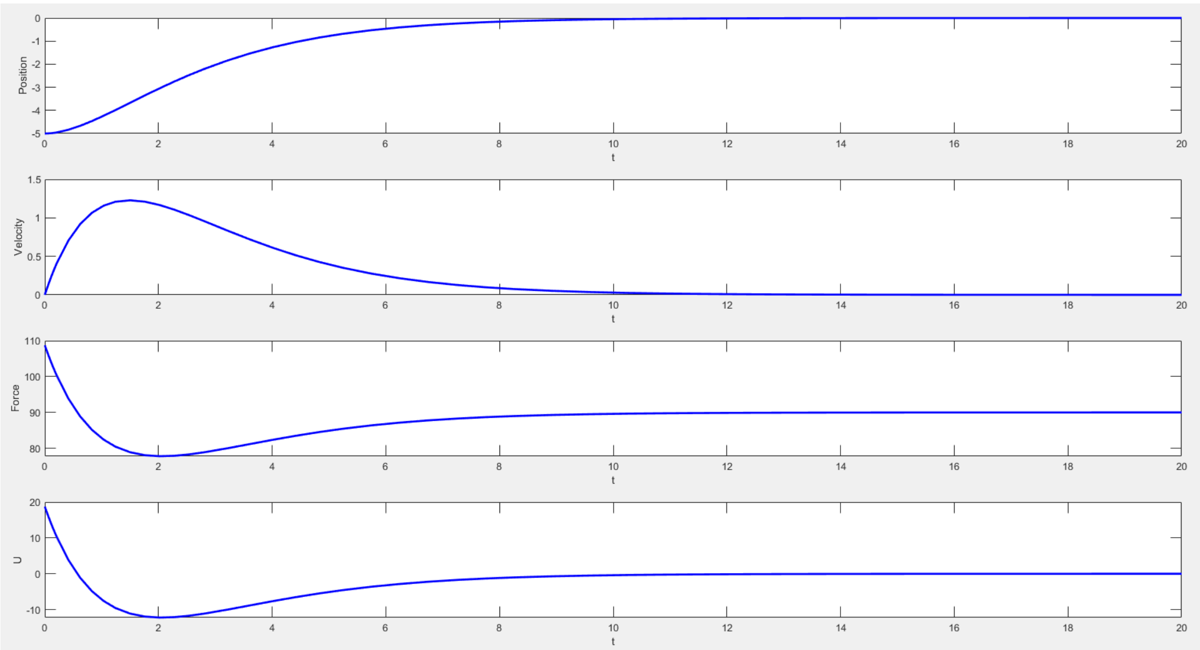
$$F_{ss} = k_s z + 5k_s + Mg\sin\theta$$

$$F_{ss} = 5k_s + Mg\sin\theta = 90N$$

When compared with the MATLAB plot, it can be seen that both values are equal. The input force (F) plot for design (II) in MATLAB reaches (F = 90 N) equilibrium point at about 8 - 10 seconds.



Closed-Loop Plots for Position, Velocity, Force and U respectively for Design (I)



Closed-Loop Plots for Position, Velocity, Force and U respectively for Design (II)

Q2

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

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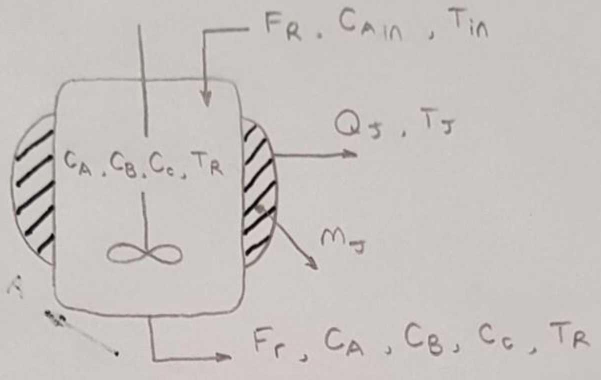


- The feed inlet flow only contains component A.
  (A is the only input with flow rate F_R . concentration $C_{A,in}$ and temperature T_{in})
- The volume of the reactor is constant at all times.
- The reaction mixture is completely filled with liquid and ideally mixed. ($C_{A\,out} = C_A$, $C_{B\,out} = C_B$. $C_{C\,out} = C_C$. $T_{R\,out} = T_R$)

* V will be used with concentration to get the mole balances since it's constant at all times.

- The liquid is incompressible. (no change in temperature or concentration due to compression)
- The reactions are elementary and obey the Arrhenius relation:

  * $r_i = k(T_R)\, c_i = k_{0_1} \exp(-E_R / T_R)\, C_i$

* The amount of heat that is removed can be adjusted by Q_J
* The heat transfer coefficient between the reaction medium and the jacket (kA) is constant.
* The initial concentrations in the reactor are $C_{A,0} = 0$ $C_{B,0} = 0$ and the initial reactor and jacket temperatures are $T_{R,0} = T_{J,0} = 387.05$ [K]

1) The required balances that describe the dynamic behavior of the system are energy and mole balances.

2) – The change in concentration can be obtained from mole balance by dividing by V since it's constant at all times.

$$m = \rho V \quad , \quad n = c V \quad \frac{dc}{dt} = \frac{1}{V}\frac{dn}{dt}$$

* For the substance A ;

$$\dot{n}_A = \dot{n}_{A,in} - \dot{n}_{A,out} + \dot{n}_{R,A}$$

A becomes B

$$\dot{n}_{A,in} = F_R \, c_{A,in} \quad , \quad \dot{n}_{A,out} = F_R \, c_A$$

Substance A becomes B with the irreversible exothermic first order reaction 1 :

$$n_{R,A} = -k_{01} \, e^{-\frac{E_{R,1}}{T_R}} c_A$$

$$\dot{n}_A = F_R (c_{A,in} - c_A) - k_{01} \, e^{-\frac{E_{R,1}}{T_R}} c_A V$$

$$\boxed{\frac{dc_A}{dt} = \frac{F_R}{V}(c_{A,in} - c_A) - k_{01} \, e^{-\frac{E_{R,1}}{T_R}} c_A}$$

* For the substance B ;

$$\dot{n}_B = \cancel{\dot{n}_{B,in}} - \dot{n}_{B,out} + \dot{n}_{R,B}$$

there is no input of B

A becomes B, B becomes C

$$\dot{n}_{B,out} = F_R \, c_B \quad , \quad n_{R,B} = r_1 V - r_2 V$$

$$\dot{n}_{R,B} = \underbrace{\left(k_{01} \, e^{-\frac{E_{R,1}}{T_R}} c_A V\right)}_{A \rightarrow B} - \underbrace{\left(k_{02} \, e^{-\frac{E_{R,2}}{T_R}} c_B V\right)}_{B \rightarrow C}$$

$$\dot{n}_B = -F_R\, C_B + \left(k_{01}\, e^{\frac{-E_{R,1}}{T_R}}\, C_A\, V\right) - \left(k_{02}\, e^{\frac{-E_{R,2}}{T_R}}\, C_B\, V\right)$$

$$\boxed{\frac{dC_B}{dt} = -\frac{F_R}{V}\, C_B + \left(k_{01}\, e^{\frac{-E_{R,1}}{T_R}}\, C_A\right) - \left(k_{02}\, e^{\frac{-E_{R,2}}{T_R}}\, C_B\right)}$$

* For the substance C ;

$$\dot{n}_C = \cancel{n_{C,in}} - n_{C,out} + n_{R,B} \longrightarrow \text{B becomes C}$$

there is no input of C

$$n_{C,out} = F_R\, C_A \quad , \quad n_{R,C} = k_{02}\, e^{\frac{-E_{R,2}}{T_R}}\, C_B\, V$$

$$\dot{n}_C = -F_R\, C_A + k_{02}\, e^{\frac{-E_{R,2}}{T_R}}\, C_B\, V$$

$$\boxed{\frac{dC_C}{dt} = -\frac{F_R}{V}\, C_A + k_{02}\, e^{\frac{-E_{R,2}}{T_R}}\, C_B}$$

- The change in temperature can be obtained from energy balance by dividing Q into m and cp.

$$Q = m\, c_p\, T \quad , \quad m = \rho V$$

* For the energy balance ;

$$\frac{dQ}{dt} = \underbrace{\dot{Q}_{in}}_{\text{input}} + \underbrace{\dot{Q}_R}_{\text{energy flow generated by the exothermic reactions}} - \underbrace{\dot{Q}_{out}}_{\text{output}} - \underbrace{\dot{Q}_J}_{\text{consumption (heat exchange with the cooling jacket)}}$$

The only input component is A , thus :

$$\dot{Q}_{in} = \rho\, F_R\, c_p\, T_{in}$$

Inside the CSTR , everything is well-mixed , therefore :

$$\dot{Q}_{out} = \rho\, F_R\, c_p\, T_R$$

$$\dot{Q}_R = \underbrace{r_{R,A}\,\Delta H_{R,1}}_{\text{1st Exothermic Reaction}} + \underbrace{r_{R,B}\,\Delta H_{R,2}}_{\text{2nd Exothermic Reaction}}$$

$$= \left(-k_{01}\, e^{\frac{-E_{R,1}}{T_R}}\, C_A\, \Delta H_{R,1}\, V\right) + \left(-k_{02}\, e^{\frac{-E_{R,2}}{T_R}}\, C_B\, \Delta H_{R,2}\, V\right)$$

The amount of heat that is removed by the jacket can be written as:

$$\dot{Q}_J = kA\,(T_R - T_J)$$

Putting all equations together yields:

$$\frac{dQ}{dt} = \rho F_R c_p T_{in} - \rho F_R c_p T_R - k_{01}\, e^{\frac{-E_{R,1}}{T_R}}\, C_A\, \Delta H_{R,1}\, V$$
$$- k_{02}\, e^{\frac{-E_{R,2}}{T_R}}\, C_B\, \Delta H_{R,2}\, V - kA\,(T_R - T_J)$$

$$\boxed{\frac{dT_R}{dt} = \frac{F_R}{V}(T_{in} - T_R) - \frac{\left(k_{01}\, e^{\frac{-E_{R,1}}{T_R}} C_A \Delta H_{R,1}\right)}{\rho\, C_p} - \frac{\left(k_{02}\, e^{\frac{-E_{R,2}}{T_R}} C_B \Delta H_{R,2}\right)}{\rho\, C_p} - \frac{kA\,(T_R - T_J)}{\rho\, C_p\, V}}$$

* For the cooling jacket:

$$\boxed{\frac{dT_J}{dt} = \frac{kA\,(T_R - T_J) - Q_J}{m_J\, C_{p_J}}}$$

* For an exothermic reaction, the system loses energy to generate heat. $\dot{Q}_R$ term is positive because ΔH_R terms are negative.

– The system is <u>non-linear</u> because there exists exponential terms due to the Arrhenius equation for reactions.

$$r_i = k(T_R)\, c_i = k0_i \exp(-E_{R_i} / T_R)\, c_i$$

3) $$\frac{dC_A}{dt} = \frac{F_{R,ss}}{V}(C_{A,in} - C_{A,ss}) - k_{01}\, e^{\frac{-E_{R,1}}{T_{R,ss}}} C_{A,ss} = 0$$

$$\frac{F_{R,ss}}{V} C_{A,in} = \left(\frac{F_{R,ss}}{V} + k_{01}\, e^{\frac{-E_{R,1}}{T_{R,ss}}}\right) C_{A,ss}$$

$$\boxed{C_{A,ss} = \frac{\frac{F_{R,ss}}{V} C_{A,in}}{\left(\frac{F_{R,ss}}{V} + k_{01}\, e^{\frac{-E_{R,1}}{T_{R,ss}}}\right)}} \quad (1)$$

$$\frac{dC_B}{dt} = -\frac{F_{R,ss}}{V} C_{B,ss} + \left(k_{01}\, e^{\frac{-E_{R,1}}{T_{R,ss}}} C_{A,ss}\right) - \left(k_{02}\, e^{\frac{-E_{R,2}}{T_{R,ss}}} C_{B,ss}\right) = 0$$

– Plug (1) into this equation to get (2)

$$\frac{dT_J}{dt} = \frac{kA(T_{R,ss} - T_{J,ss}) - Q_{J,ss}}{m_J\, Cp_J} = 0$$

$$T_{J,ss} = \frac{kA\, T_{R,ss} - Q_{J,ss}}{kA} \quad (3)$$

$$\frac{dT_R}{dt} = \frac{F_{R,ss}}{V}(T_{in} - T_{R,ss}) - \frac{\left(k_{01}\, e^{\frac{-E_{R,1}}{T_{R,ss}}} C_{A,ss}\, \Delta H_{R,1}\right)}{\rho\, Cp} - \frac{\left(k_{02}\, e^{\frac{-E_{R,2}}{T_{R,ss}}} C_{B,ss}\, \Delta H_{R,2}\right)}{\rho\, Cp} \cdots$$

$$\cdots - \frac{kA(T_{R,ss} - T_{J,ss})}{\rho\, Cp\, V} \quad (4)$$

– Plug the solutions of (1), (2) and (3) to get rid of other terms and solve for $T_{R,ss}$. This problem is solved in MATLAB.

$T_{R,ss} = 398{,}6581$ K $\qquad C_{A,ss} = 1{,}6329$ kmol / m^3

$T_{J,ss} = 397{,}3736$ K $\qquad C_{B,ss} = 1{,}1101$ kmol / m^3

4) Taylor - Series Expansion:

$$f(x) = f(x_0) + \frac{df}{dx}\Big|_x (x - x_0) + \frac{1}{2}\frac{d^2 f}{dx^2}(x - x_0)^2 + \ldots = 0$$

- To linearize the system of non-linear differential equations, we use Multi-variable Taylor-Series Expansion:

$\dot{x} = f(x,u)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $f(x_s, u_s) = 0$

(x_s, u_s) is an equilibrium point

$x(t) = x_s + \Delta x(t)$, $u(t) = u_s + \Delta u(t)$

Multi-variable Taylor Series Expansion

$$\dot{x}(t) = f(x_s, u_s) + \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{x_n} \end{pmatrix}}_{J_{fx}(x_s, u_s)} \Delta x(t) + \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_p} \end{pmatrix}}_{J_{fu}(x_s, u_s)} \Delta u(t) + \ldots$$

* Higher order terms are neglected for linear approximation and derivatives are evaluated at (x_s, u_s)

$$\Delta \dot{x}(t) = A\, \Delta x(t) + B\, \Delta u(t)$$

$x_1 = C_A$, $x_2 = C_B$, $x_3 = T_R$, $x_4 = T_J$

$\dot{x}_1 = \frac{dC_A}{dt}$, $\dot{x}_2 = \frac{dC_B}{dt}$, $\dot{x}_3 = \frac{dT_R}{dt}$, $\dot{x}_4 = \frac{dT_J}{dt}$

f_1 => the equation for $\dot{x}_1$

f_2 => " " " $\dot{x}_2$

f_3 => " " " $\dot{x}_3$

f_4 => " " " $\dot{x}_4$

$$\left.\frac{\partial f_1}{\partial x_1}\right|_{x_s,u_s} = -\frac{F_{R,ss}}{V} - k_{01}\, e^{\frac{-E_{R,1}}{T_{R,ss}}}$$

$$\left.\frac{\partial f_1}{\partial x_2}\right|_{x_s,u_s} = 0 \qquad \left.\frac{\partial f_1}{\partial x_3}\right|_{x_s,u_s} = \frac{-k_{01}\, C_{A,ss}\, E_{R,1}}{T_{R,ss}^2} \cdot e^{\frac{-E_{R,1}}{T_{R,ss}}}$$

$$\left.\frac{\partial f_1}{\partial x_4}\right|_{x_s,u_s} = 0 \qquad \left.\frac{\partial f_1}{\partial u_1}\right|_{x_s,u_s} = \frac{C_{A,in} - C_{A,ss}}{V}$$

$$\left.\frac{\partial f_1}{\partial u_2}\right|_{x_s,u_s} = 0 \qquad \left.\frac{\partial f_2}{\partial x_1}\right|_{x_s,u_s} = k_{01}\, e^{\frac{-E_{R,1}}{T_{R,ss}}}$$

$$\left.\frac{\partial f_2}{\partial x_2}\right|_{x_s,u_s} = -\frac{F_{R,ss}}{V} - k_{02}\, e^{\frac{-E_{R,2}}{T_{R,ss}}}$$

$$\left.\frac{\partial f_2}{\partial x_3}\right|_{x_s,u_s} = \frac{k_{01}\, e^{\frac{-E_{R,1}}{T_{R,ss}}}\, C_{A,ss}\, E_{R,1}}{T_{R,ss}^2} - \frac{k_{02}\, e^{\frac{-E_{R,2}}{T_{R,ss}}}\, C_{B,ss}\, E_{R,2}}{T_{R,ss}^2}$$

$$\left.\frac{\partial f_2}{\partial x_4}\right|_{x_s,u_s} = 0 \qquad \left.\frac{\partial f_2}{\partial u_1}\right|_{x_s,u_s} = \frac{-C_{B,ss}}{V} \qquad \left.\frac{\partial f_2}{\partial u_2}\right|_{x_s,u_s} = 0$$

$$\left.\frac{\partial f_3}{\partial x_1}\right|_{x_s,u_s} = -\frac{k_{01}\, e^{\frac{-E_{R,1}}{T_{R,ss}}}\, \Delta H_{R,1}}{\rho\, C_p} \qquad \left.\frac{\partial f_3}{\partial x_2}\right|_{x_s,u_s} = \frac{-k_{02}\, e^{\frac{-E_{R,2}}{T_{R,ss}}}\, \Delta H_{R,2}}{\rho\, C_p}$$

$$\left.\frac{\partial f_3}{\partial x_3}\right|_{x_s,u_s} = -\frac{F_{R,ss}}{V} - \frac{\left(k_{01}\, e^{\frac{-E_{R,1}}{T_{R,ss}}}\, C_{A,ss}\, \Delta H_{R,1}\, E_{R,1}\right)}{\rho\, C_p\, T_{R,ss}^2}$$

$$- \frac{\left(k_{02}\, e^{\frac{-E_{R,2}}{T_{R,ss}}}\, C_{B,ss}\, \Delta H_{R,2}\, E_{R,2}\right)}{\rho\, C_p\, T_{R,ss}^2} - \frac{kA}{\rho C_p V}$$

$$\left.\frac{\partial f_3}{\partial x_4}\right|_{x_s,u_s} = +\frac{kA}{\rho C_p V} \qquad \left.\frac{\partial f_3}{\partial u_1}\right|_{x_s,u_s} = \frac{(T_{in} - T_{R,ss})}{V}$$

$$\left.\frac{\partial f_3}{\partial u_2}\right|_{x_s,u_s} = 0 \qquad \left.\frac{\partial f_4}{\partial x_1}\right|_{x_s,u_s} = 0 \qquad \left.\frac{\partial f_4}{\partial x_2}\right|_{x_s,u_s} = 0$$

$$\left.\frac{\partial f_4}{\partial x_3}\right|_{x_s,u_s} = \frac{kA}{m_J C_{p_J}} \qquad \left.\frac{\partial f_4}{\partial x_4}\right|_{x_s,u_s} = \frac{-kA}{m_J C_{p_J}}$$

$$\left.\frac{\partial f_4}{\partial u_1}\right|_{x_s,u_s} = 0 \qquad \left.\frac{\partial f_4}{\partial u_2}\right|_{x_s,u_s} = -\frac{1}{m_J C_{p_J}}$$

After plugging the values of the parameters;

$$\begin{pmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \\ \Delta\dot{x}_3 \\ \Delta\dot{x}_4 \end{pmatrix} = \begin{pmatrix} -0{,}7386 & 0 & -0{,}0503 & 0 \\ 0{,}5021 & -0{,}7386 & 0{,}0161 & 0 \\ 0{,}75 & 1{,}9643 & -0{,}5412 & 0{,}5138 \\ 0 & 0 & 1{,}4448 & -1{,}4448 \end{pmatrix} \begin{pmatrix} \Delta x_1(t) \\ \Delta x_2(t) \\ \Delta x_3(t) \\ \Delta x_4(t) \end{pmatrix} +$$

$$\begin{pmatrix} 346{,}7 & 0 \\ -111{,}0 & 0 \\ -1160{,}8 & 0 \\ 0 & -0{,}1 \end{pmatrix} \begin{pmatrix} \Delta u_1(t) \\ \Delta u_2(t) \end{pmatrix}$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x_1(t) \\ \Delta x_2(t) \\ \Delta x_3(t) \\ \Delta x_4(t) \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \Delta u_1(t) \\ \Delta u_2(t) \end{pmatrix}$$

* After checking the eigen values of this system and seeing that they all have negative real parts, it can be concluded that the equilibrium point (x_s, u_s) is locally stable.

5) The validity of the linearization is checked by simulating the linearized system in MATLAB against the original model at the equilibrium point for $\Delta u = \pm 10\%$ of the steady-state input. Linearized model of the system seems to converge to the same steady-states approximately in 60 seconds which means that the linearization is valid.

6) The operability of the linearized system is checked in MATLAB. The operability matrix is calculated as

$M = -A^{-1}B \quad (Ax = -Bu \ , \ x = -A^{-1}B)$

* No eigen value of $M^T M$ is zero and matrix M is full-rank. This means that it is possible to operate the system at steady-state conditions. ($\text{rank}(M) = 2$)

* The condition number $\gamma = \sigma_1 / \sigma_p = 8{,}1944e+04$

  $\gamma < 1e+05$ => Operability matrix is not ill-conditioned yet it's quite large.

* Operability matrix M establishes the relationship between inputs and outputs at stationary conditions.

  The singular value decomposition of M is:

  $M = U \Sigma V$ where the columns of U matrix are the eigen vectors of MM^T and the columns of V are the eigen vectors of $M^T M$.

* At stationary conditions, the states are related with the inputs as:

  $u_s = V\alpha \quad , \quad x_s = U \Sigma V^T V \alpha = U \Sigma \alpha$

  where α is a vector with information about the magnitude of the inputs along the directions of the columns vectors of V. The matrix U is the one which gives information of how the equilibrium point changes as a function of the direction and the amplification of the inputs. The maximum steady-state gain for the system can be obtained with the input in the direction of v_1 corresponding to the largest singular value. A higher gain means that less effort is required to move the system to a new operating point. (If σ_J is large, then input in the direction of v_J has a large effect on the output in direction u_1)

For our case, since $p < n$, the columns u_3 and u_4 describe the part of the state-space in which the steady-state cannot be moved. The dimension of steady state subspace is 2.

7) The condition needed to assign the closed-loop poles is decided by the Kalman Criterion for controllability.

(A, B) is controllable if $\text{rank}\,[B \; AB \; \dots \; A^{n-1}B] = n$

Computing on MATLAB, we get rank(ctb-kalman) = 4 which shows the controllability matrix has full rank ($n = 4$) The linearized system's original eigen values are at $-0{,}1205$, $-0{,}4637$, $-0{,}9058$ and $-1{,}9734$.

For controller design, we want fast-converging, well-damped not too fast eigen values. Also, they shouldn't be placed at the same spot since it causes sensitivity to errors. The controller has been designed in MATLAB and the corresponding explanations can be found there as comments. The system behavior with a controller is simulated with 2 different initial conditions. The system seems to reach the equilibrium point under 10 seconds using this controller.

8) The observability analysis has been done in MATLAB and observability matrix (Kalman Observability Criterion) has full-rank. This means the L matrix can be chosen such that eig (A-LC) take arbitrary assigned values and the observer error converges to zero with the chosen dynamics.

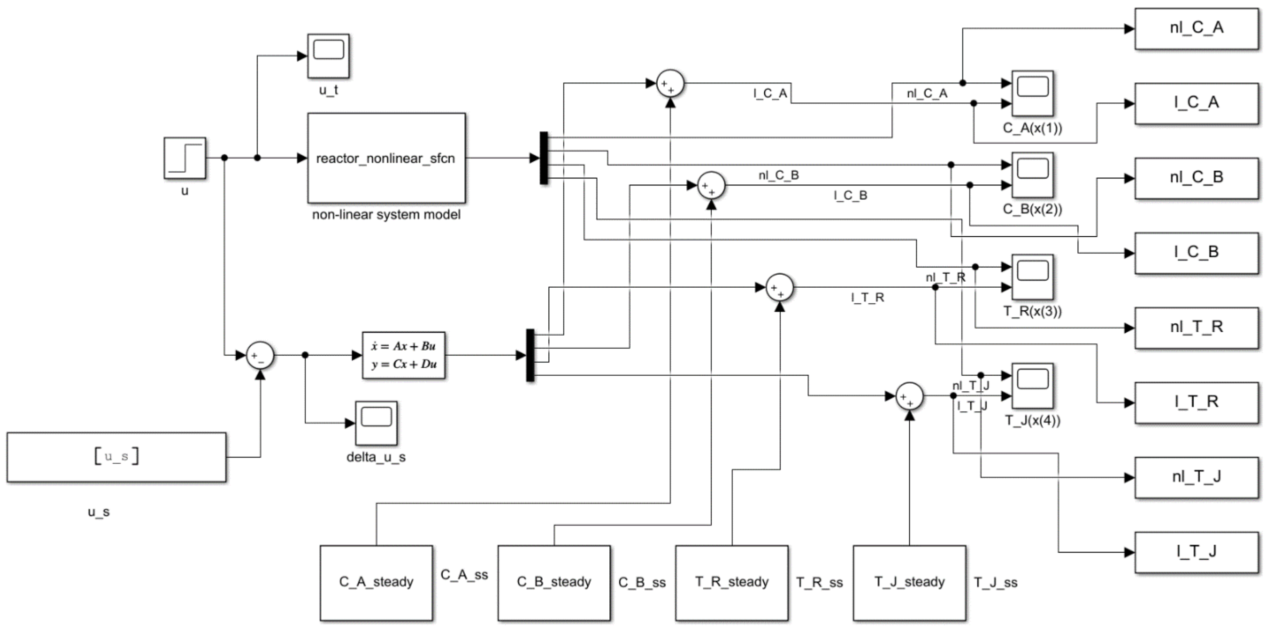
$$\dot{e} = (A - LC)(x - \hat{x}) = (A - LC)\,e$$

It is required that $\lim_{t \to \infty} e(t) = 0 \Rightarrow (A-LC)$ must be asymptotically stable!

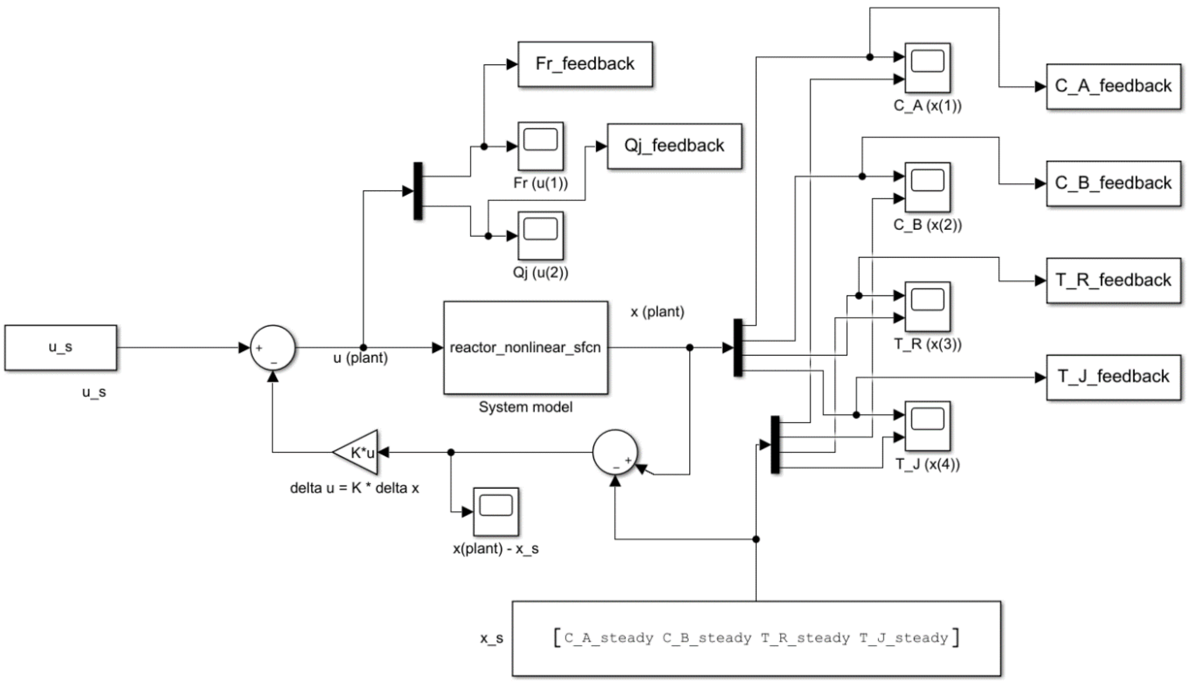
The observer has been designed in MATLAB and corresponding explanations can be found in comments.

9) The non-linear system with the observer-based feedback controller has been simulated in MATLAB. The comparison and the explanation can be found in comments.

Q2 – Models



reactor_nonlinear_vs_linear_simulation.mdl



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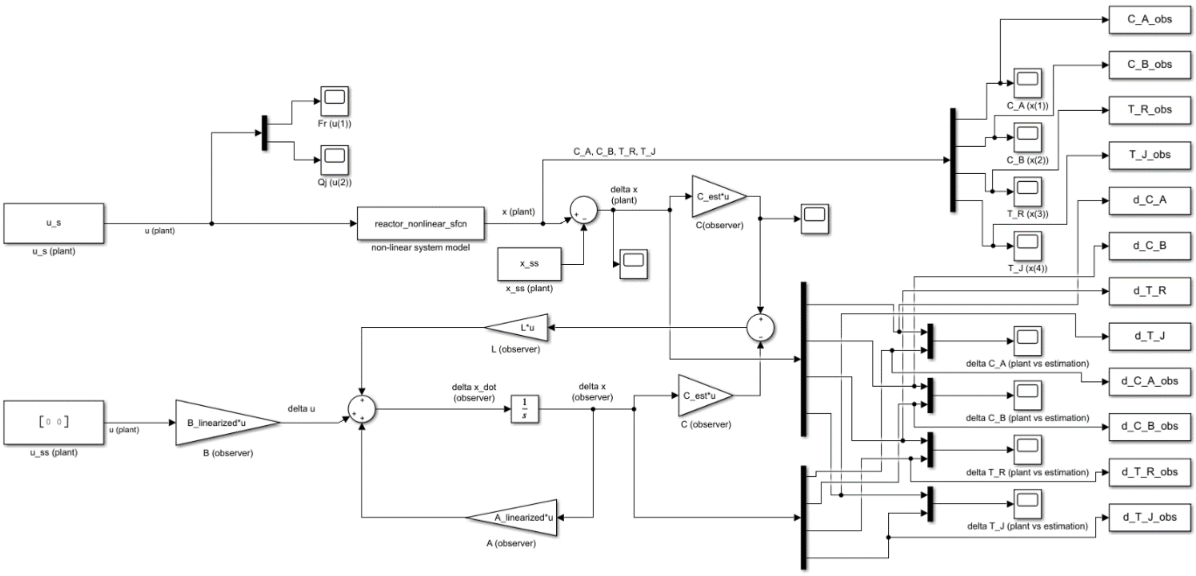

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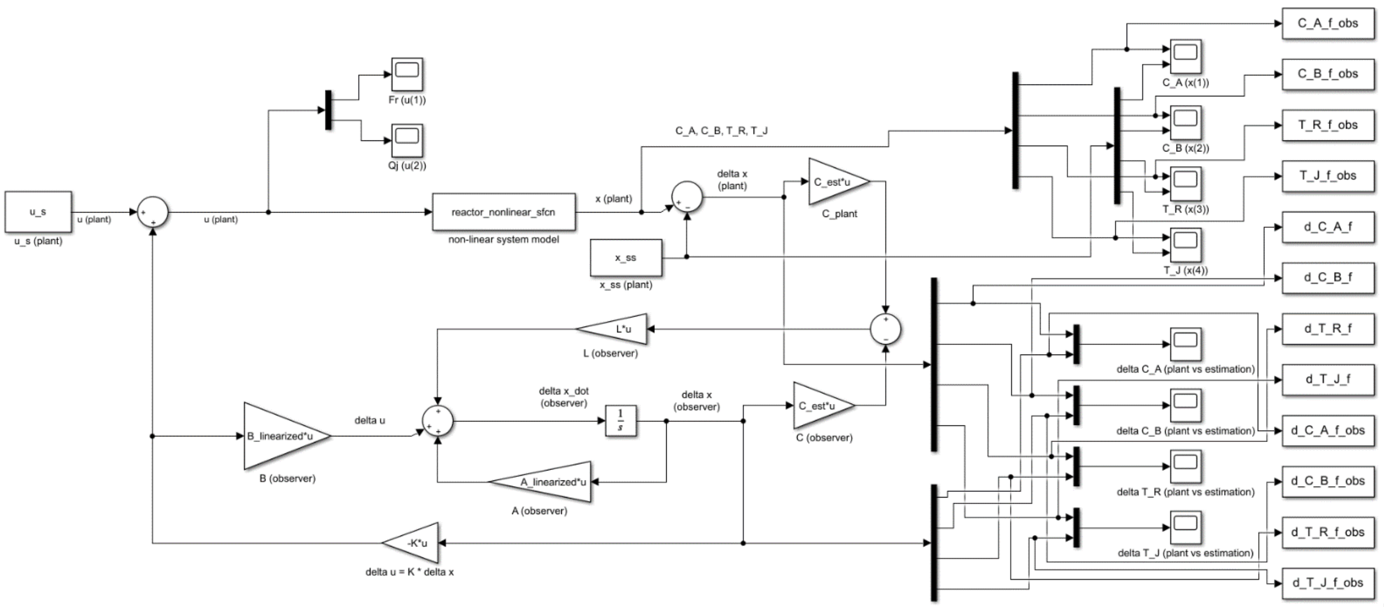

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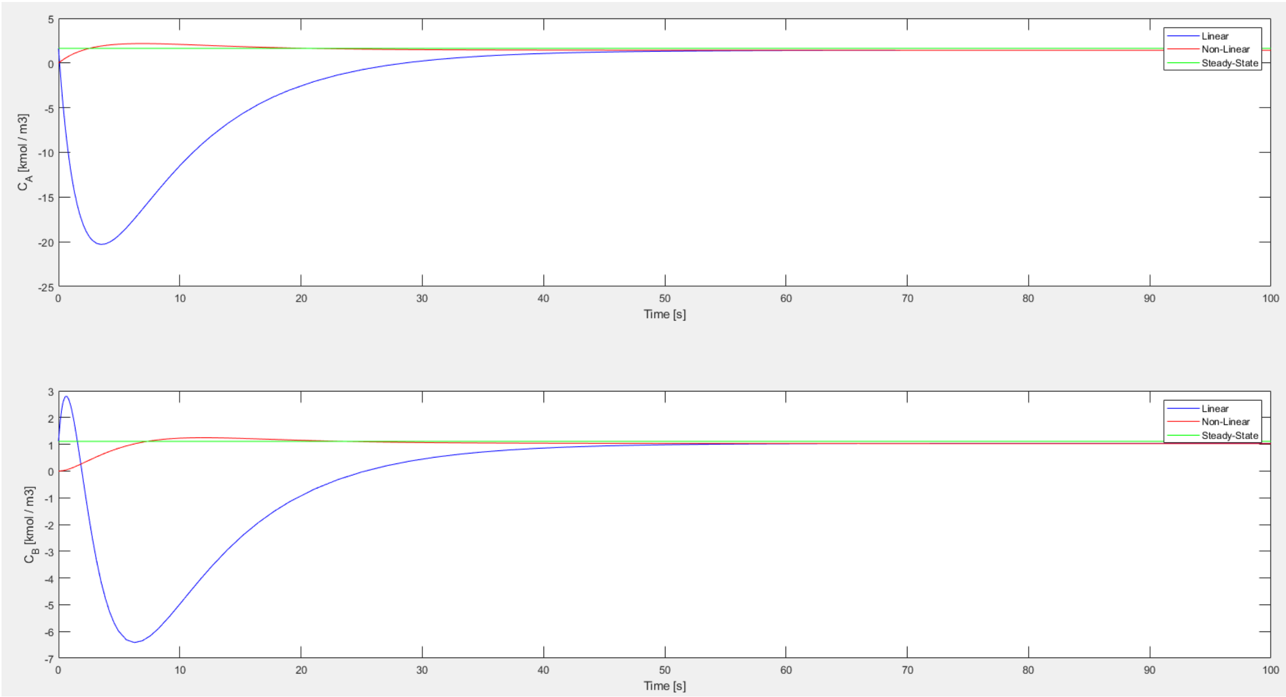
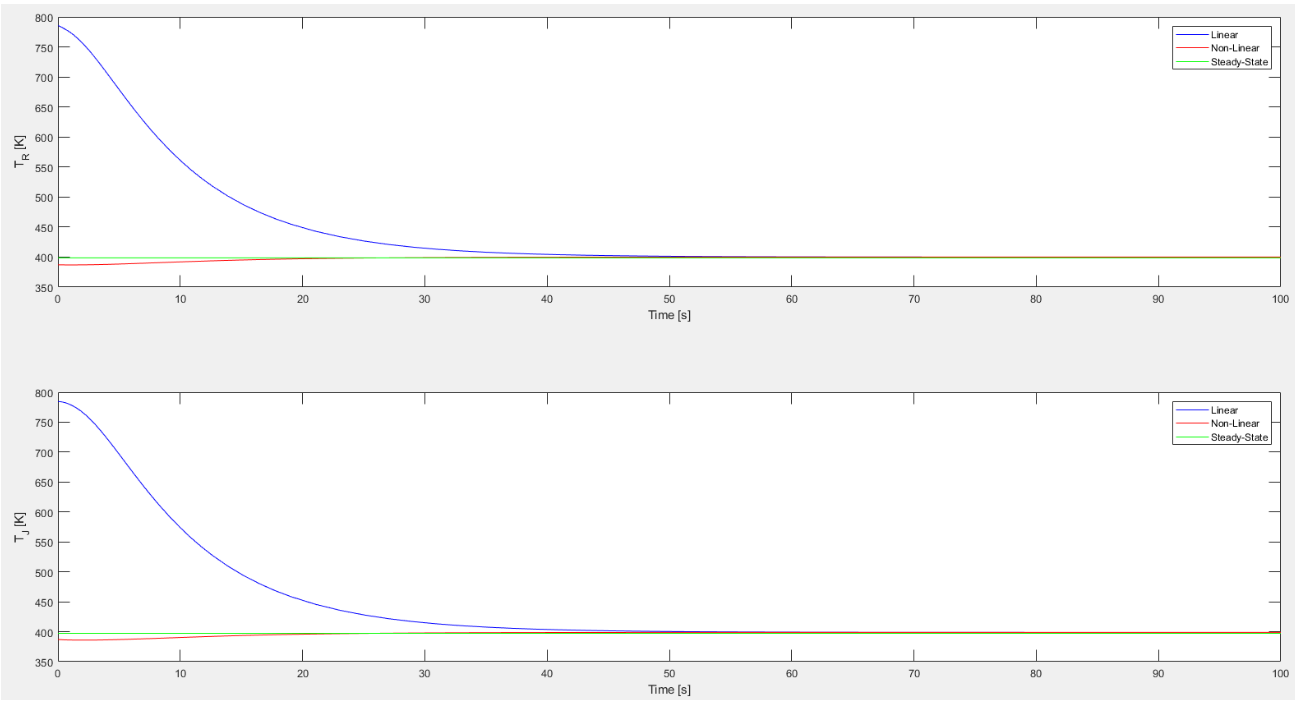
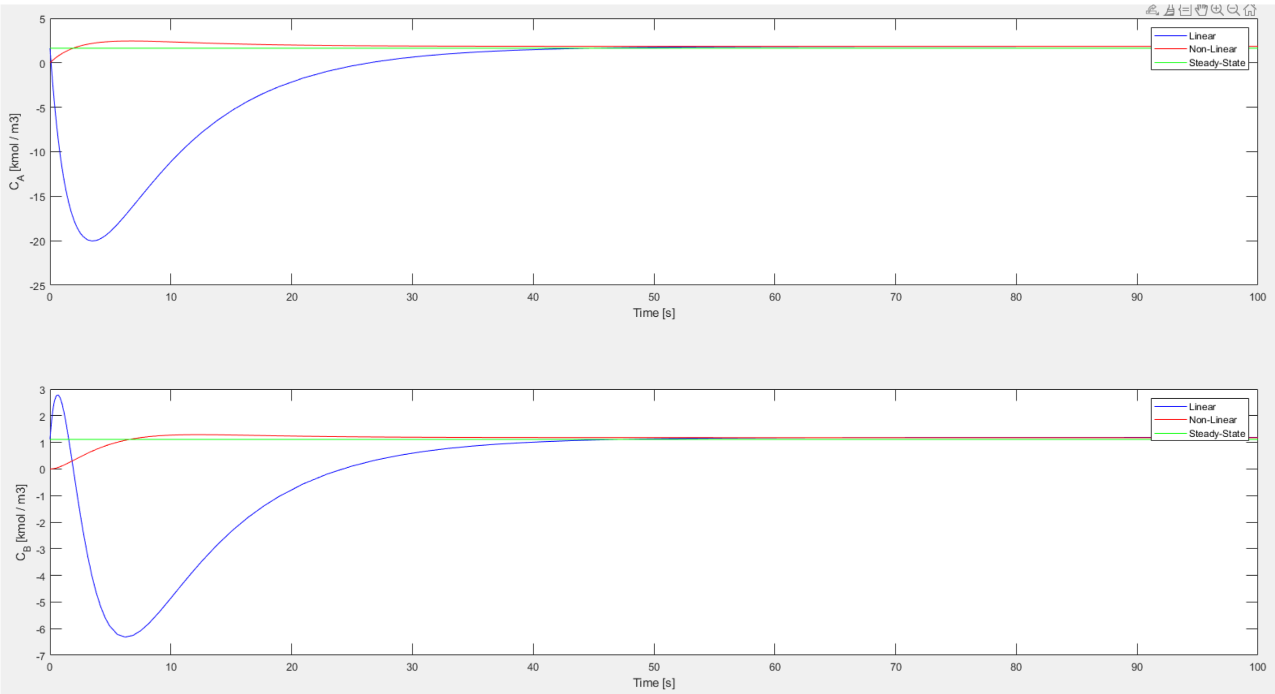
Q2 – Task5 – Plots



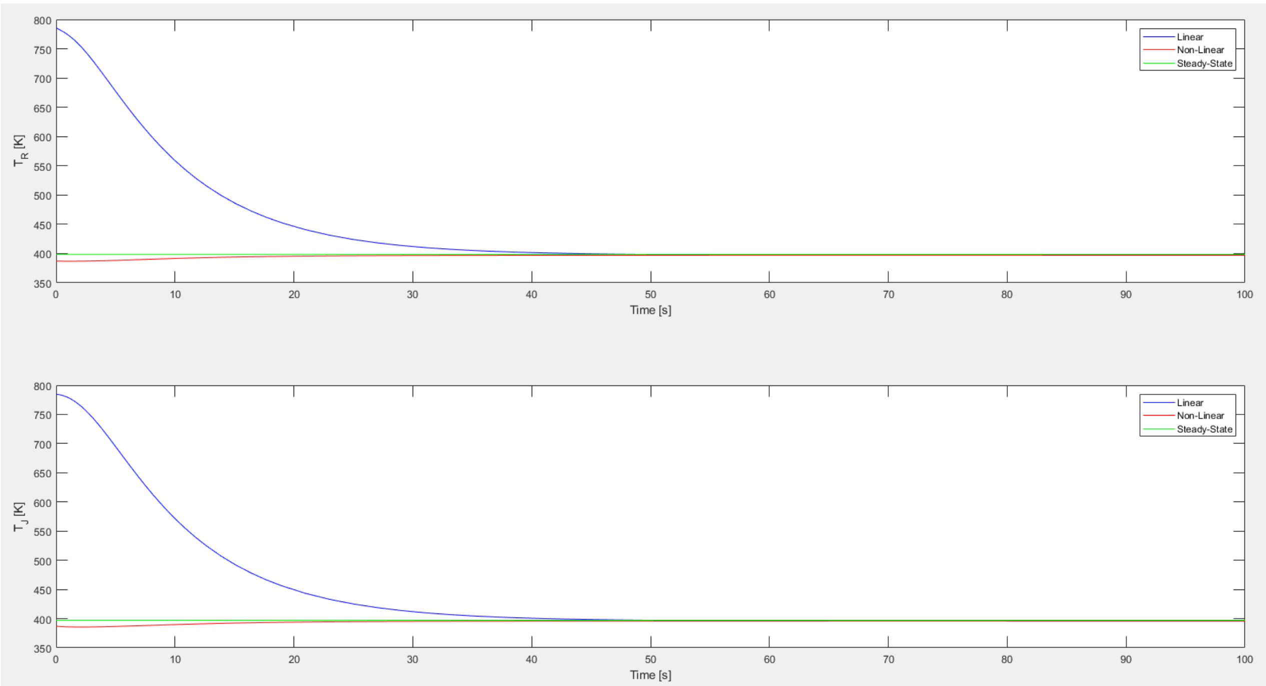
Nonlinear vs Linear Simulation for C_A, C_B respectively with Δu = -10%



Nonlinear vs Linear Simulation for T_R, T_J respectively with Δu = -10%

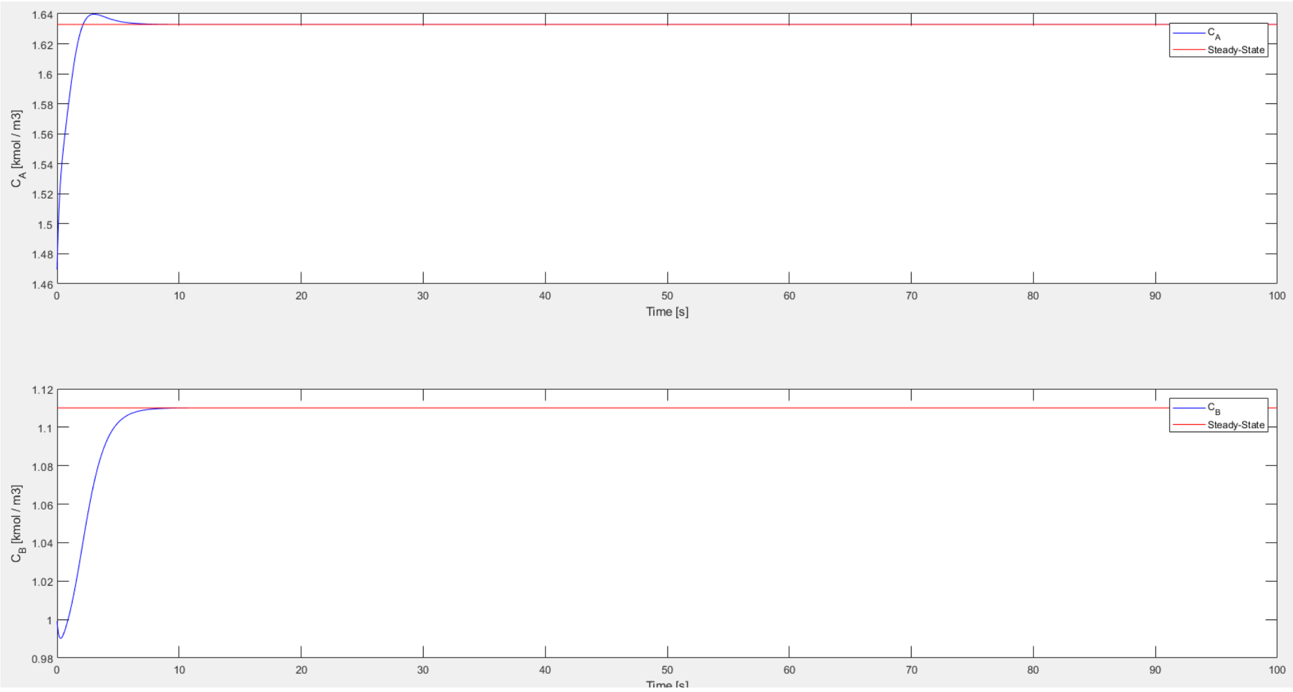


Nonlinear vs Linear Simulation for C_A, C_B respectively with Δu = +10%



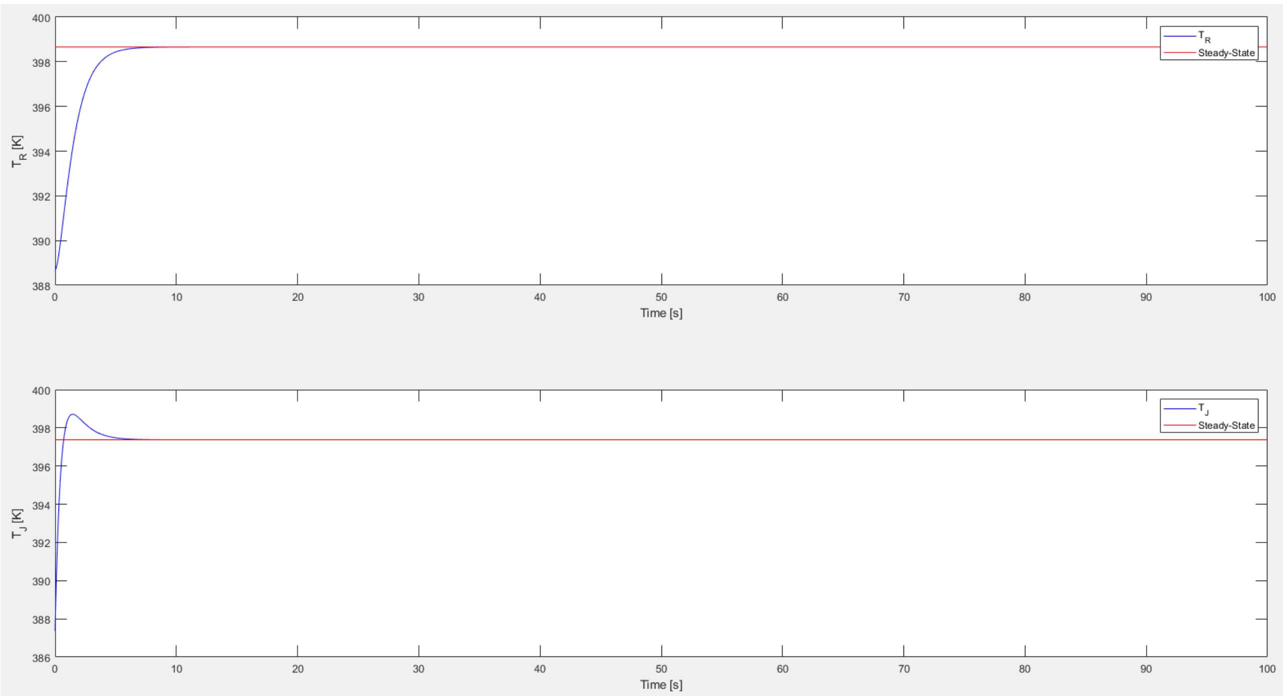
Nonlinear vs Linear Simulation for T_R, T_J respectively with Δu = +10%

Q2 – Task7 – Plots



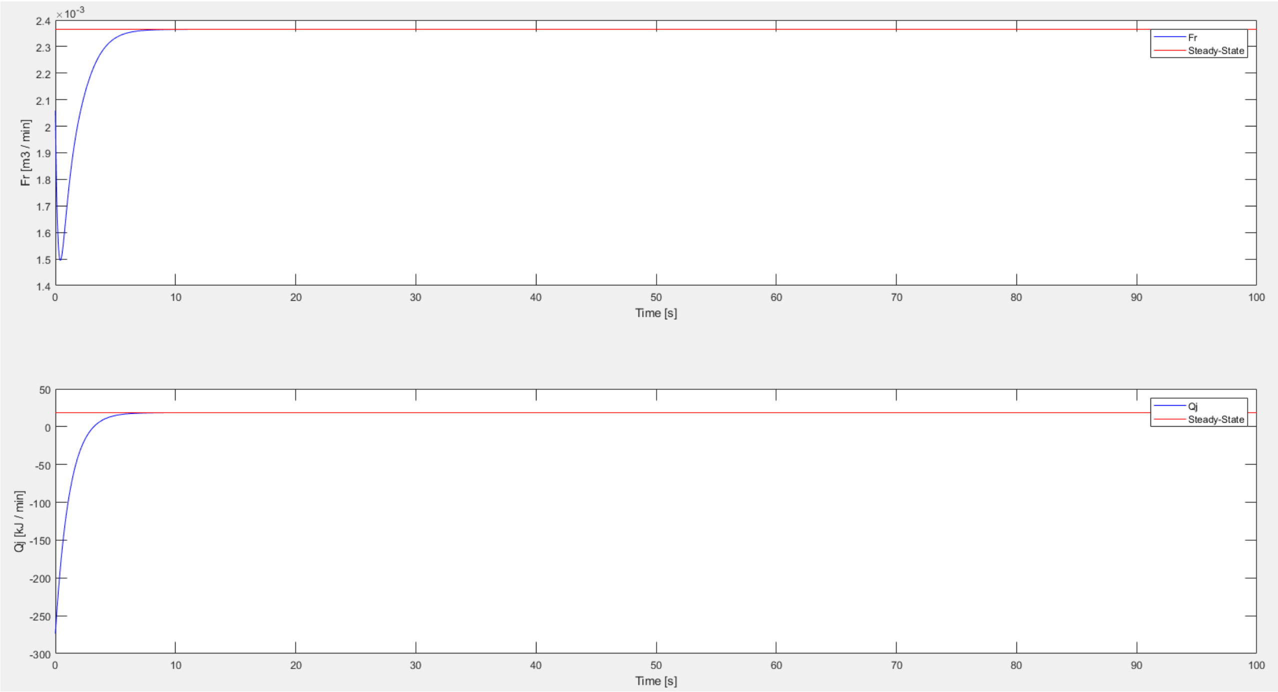
State-Feedback Controller Closed-Loop Simulation for C_A, C_B respectively

with initial condition x0,1



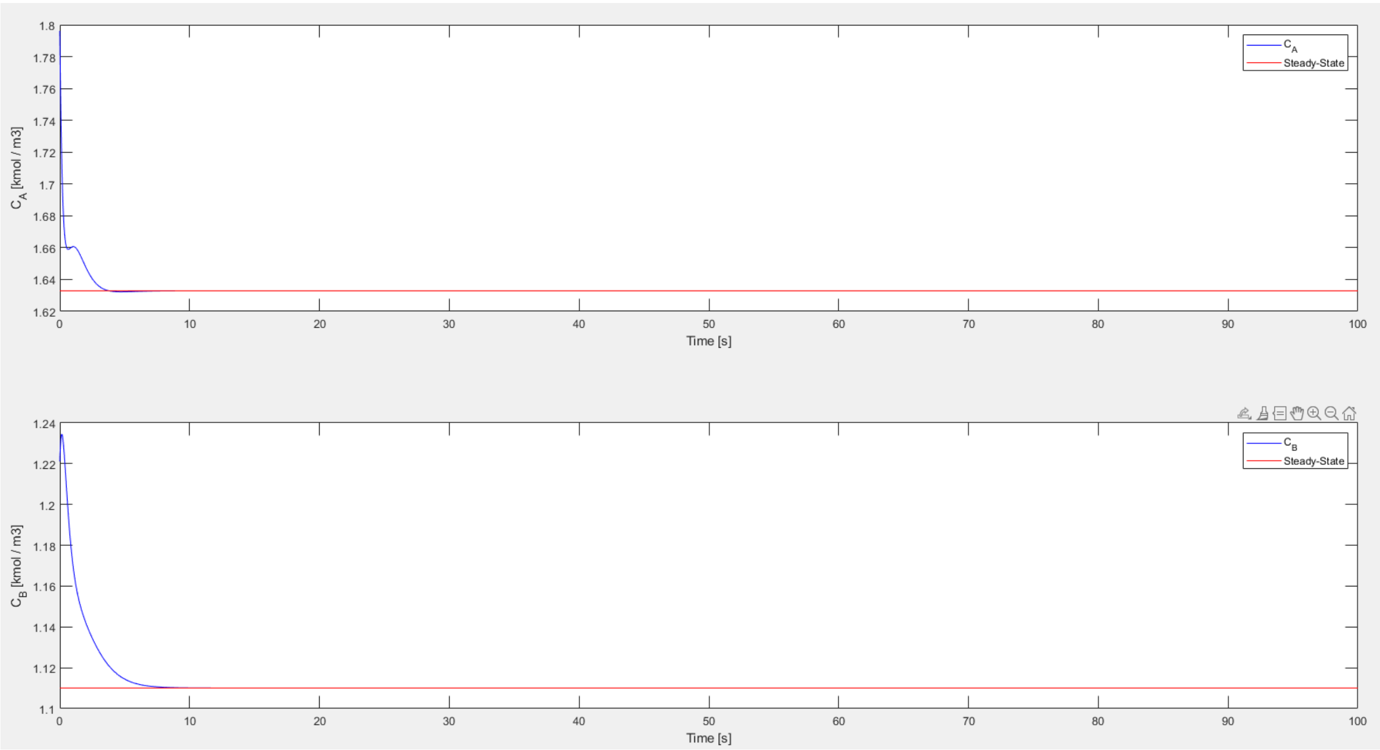
State-Feedback Controller Closed-Loop Simulation for T_R, T_J respectively

with initial condition x0,1



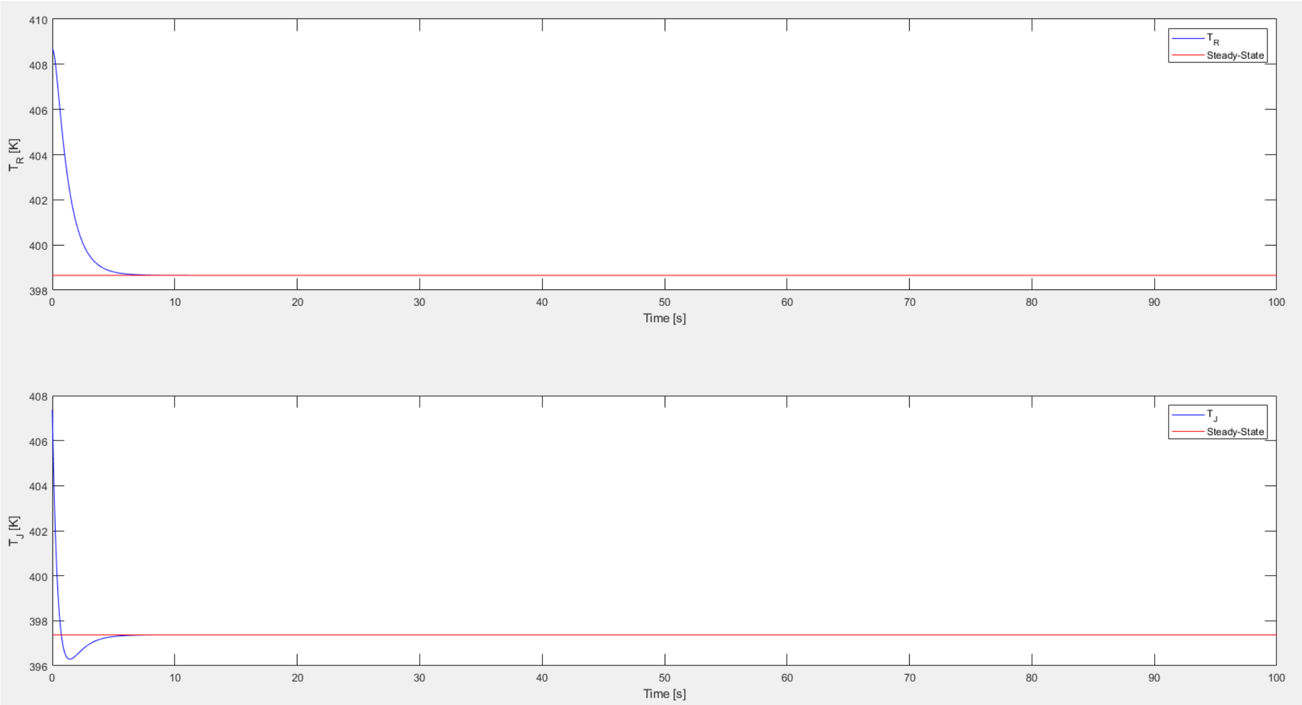
State-Feedback Controller Closed-Loop Simulation for Fr, Qj respectively

with initial condition x0,1



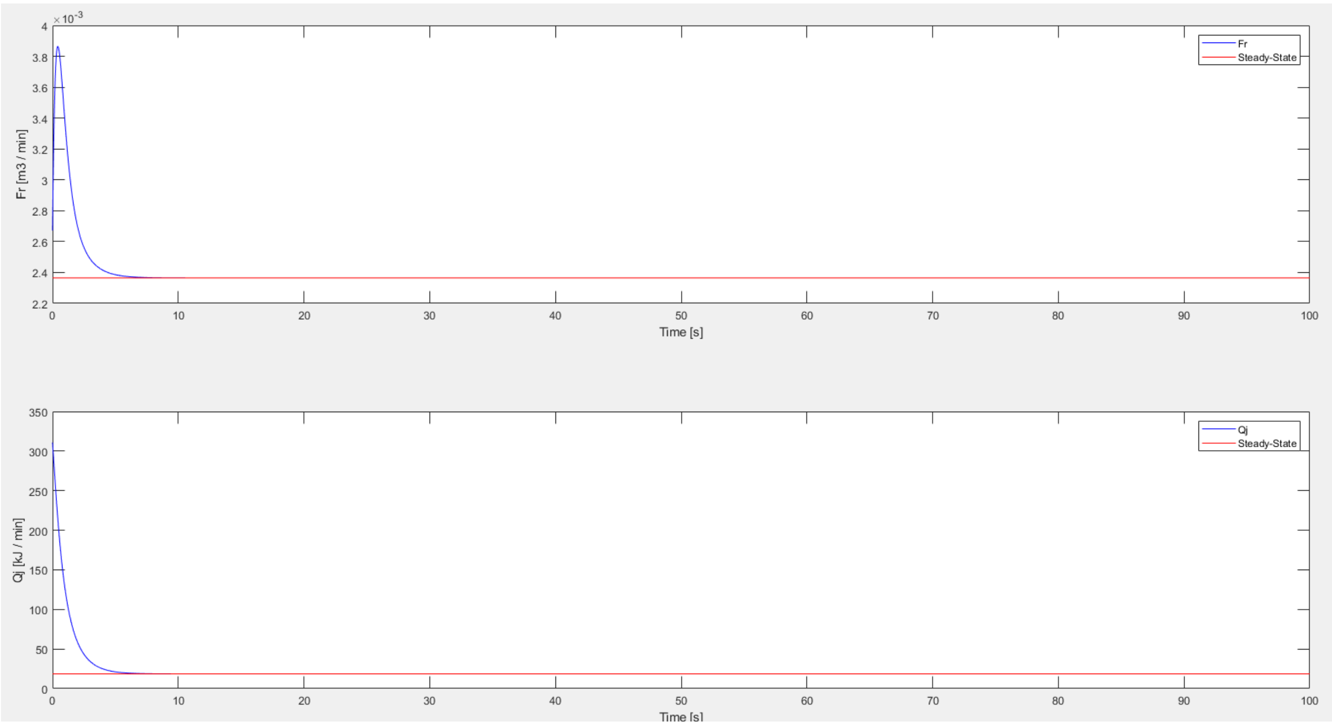
State-Feedback Controller Closed-Loop Simulation for C_A, C_B respectively

with initial condition x0,2



State-Feedback Controller Closed-Loop Simulation for T_R, T_J respectively

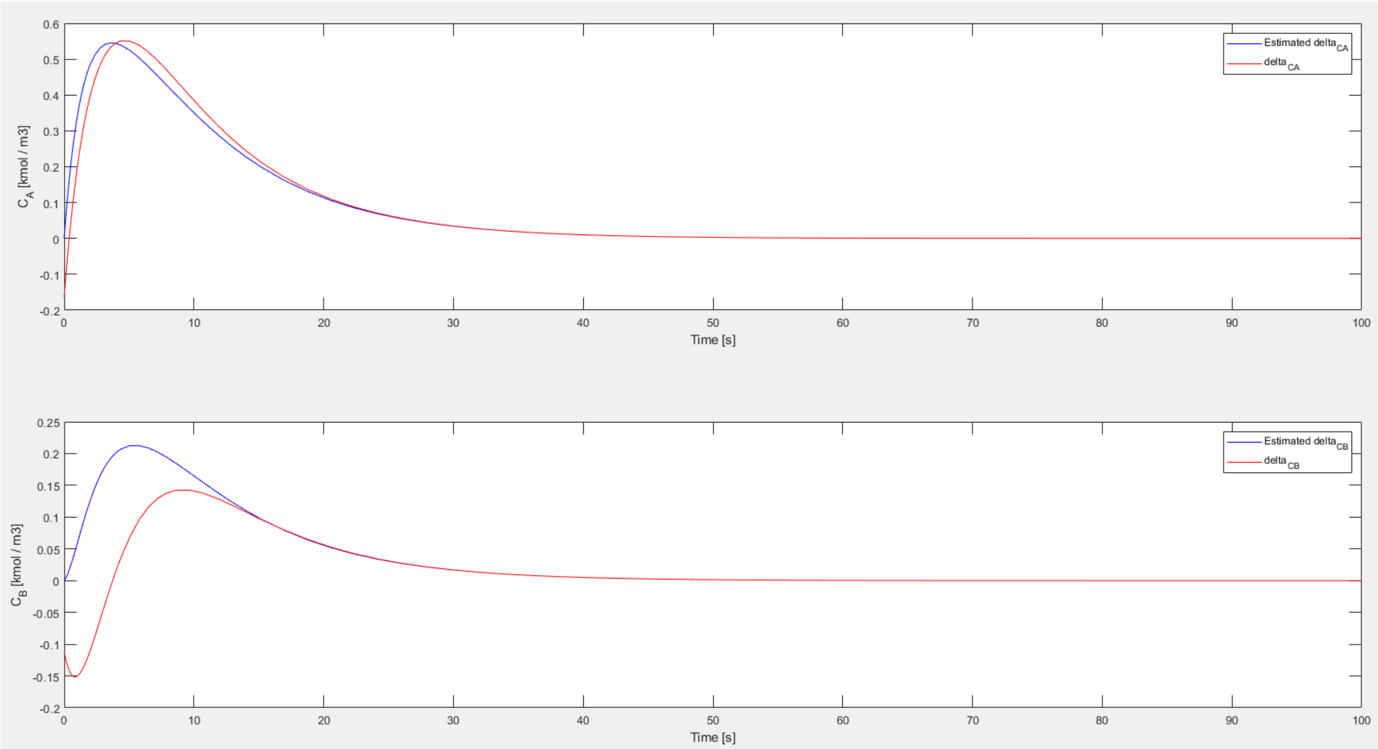
with initial condition x0,2



State-Feedback Controller Closed-Loop Simulation for Fr, Qj respectively

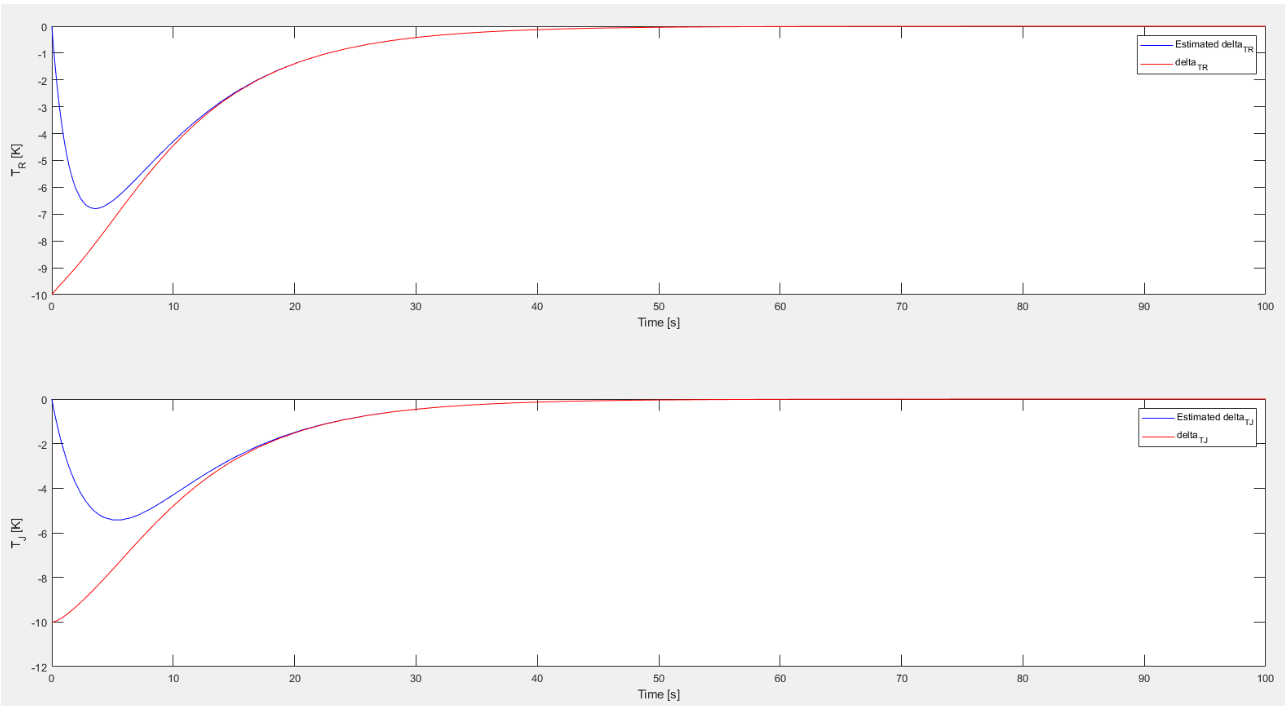
with initial condition x0,2

Q2 – Task8 – Plots



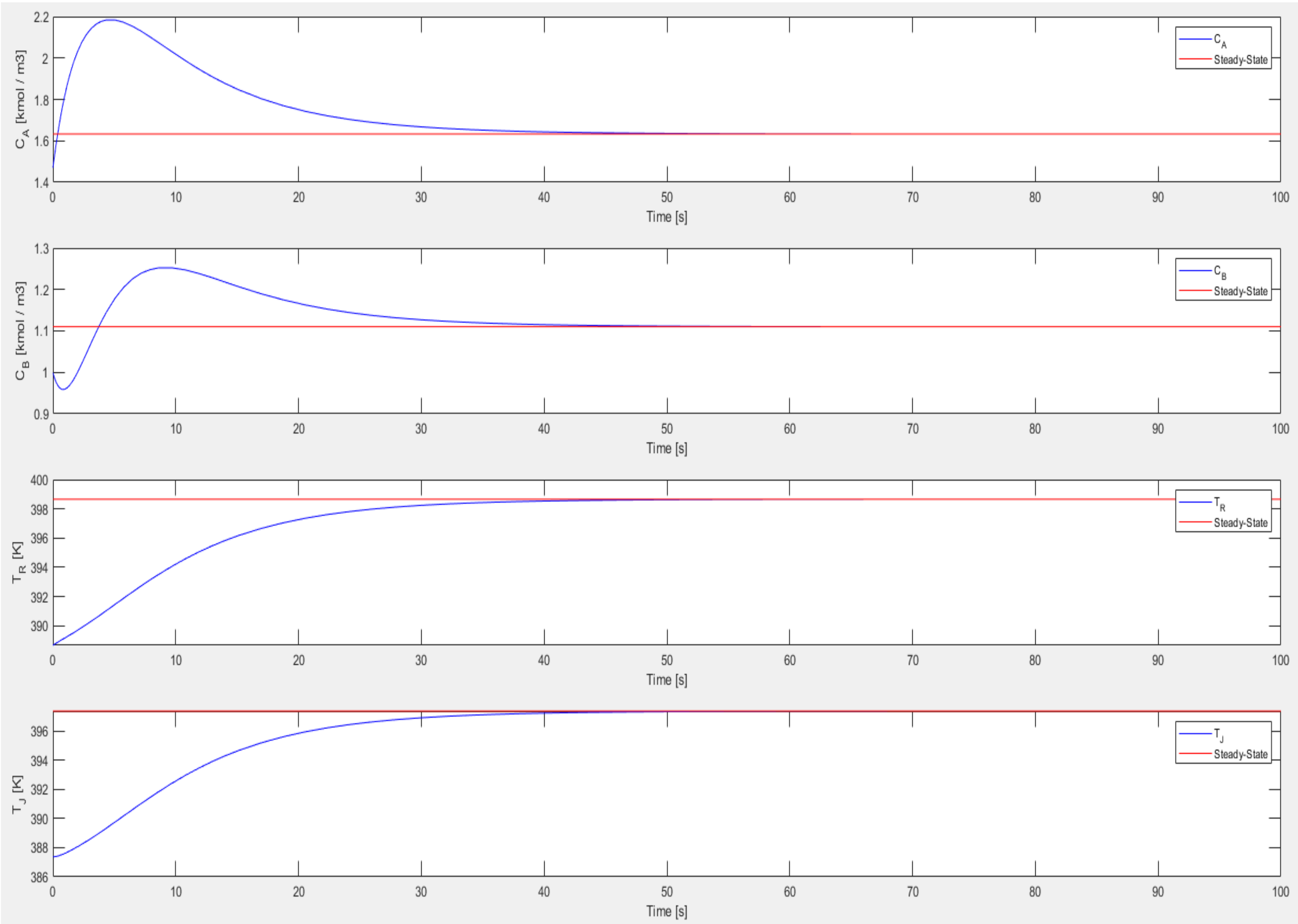
Observer Closed-Loop Simulation for ΔC_A, ΔC_B respectively

with initial condition x0,1



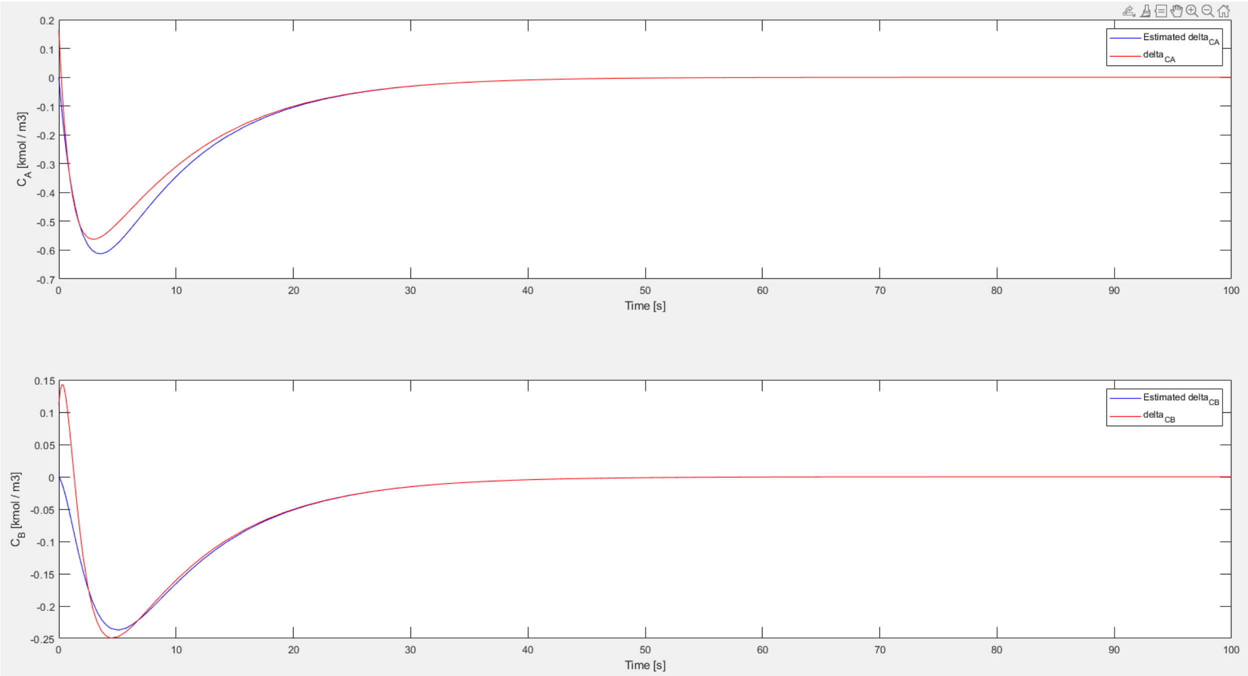
Observer Closed-Loop Simulation for ΔT_R, ΔT_J respectively

with initial condition x0,1



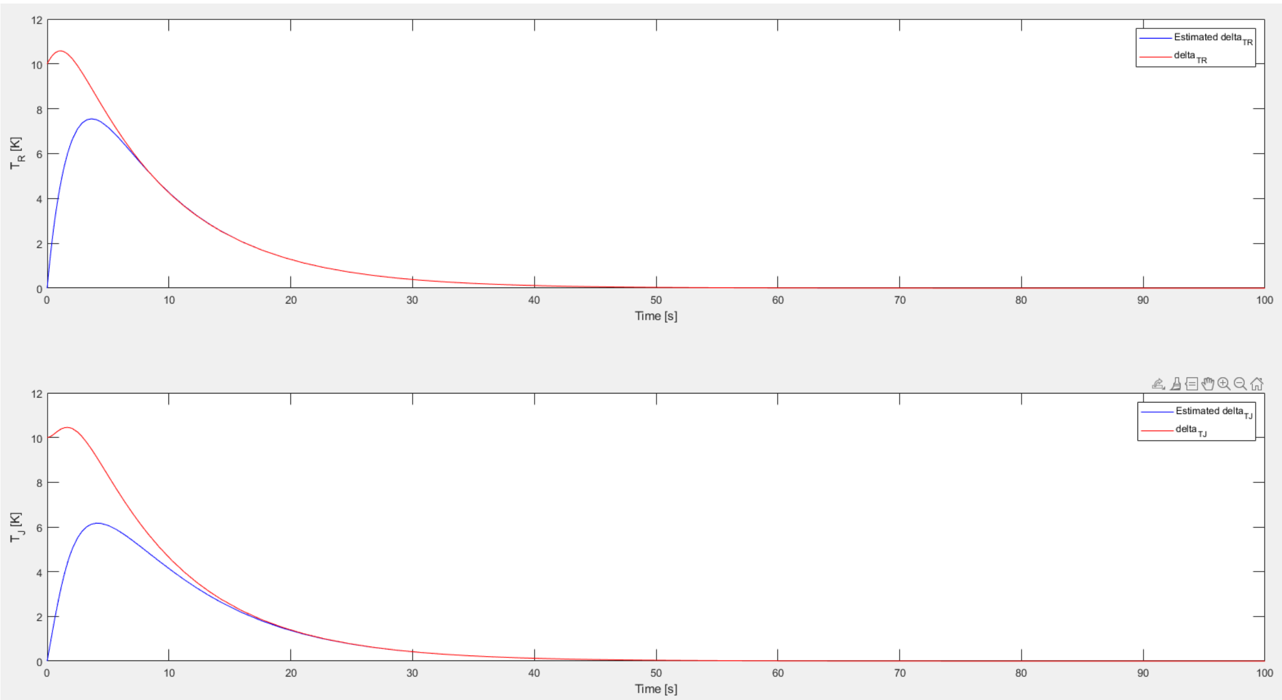
Observer Closed-Loop Simulation for C_A, C_B, T_R, T_J respectively

with initial condition x0,1



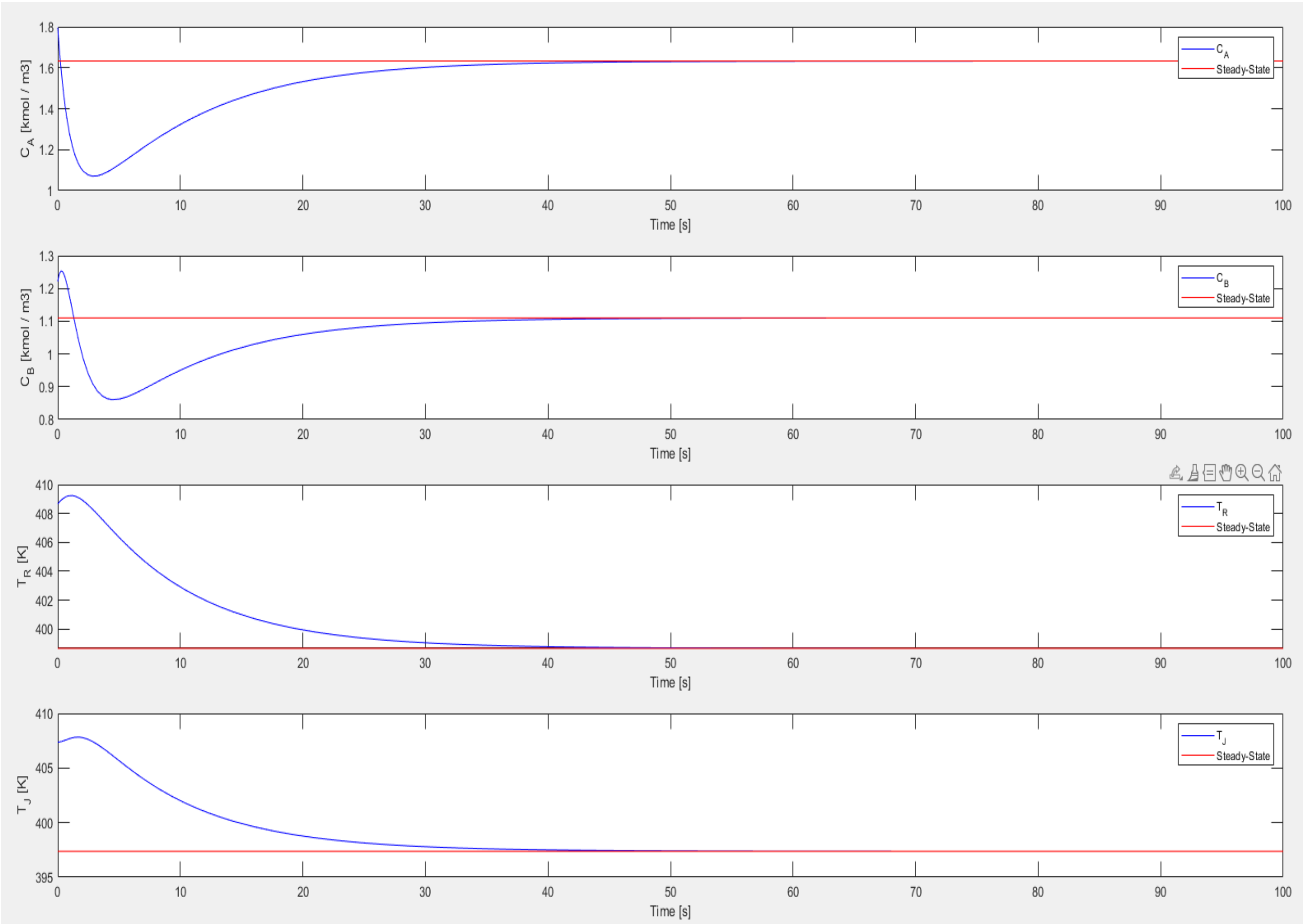
Observer Closed-Loop Simulation for ΔC_A, ΔC_B respectively

with initial condition x0,2



Observer Closed-Loop Simulation for ΔT_R, ΔT_J respectively

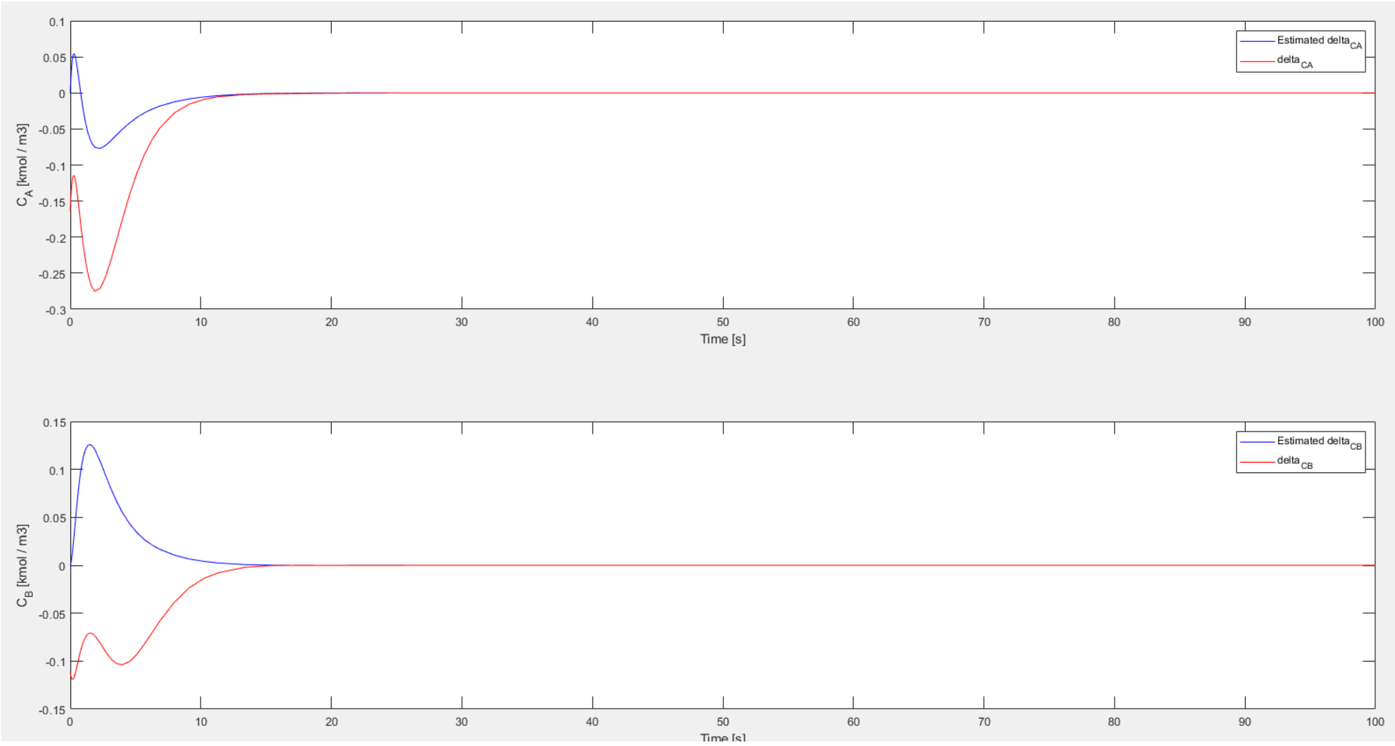
with initial condition x0,2



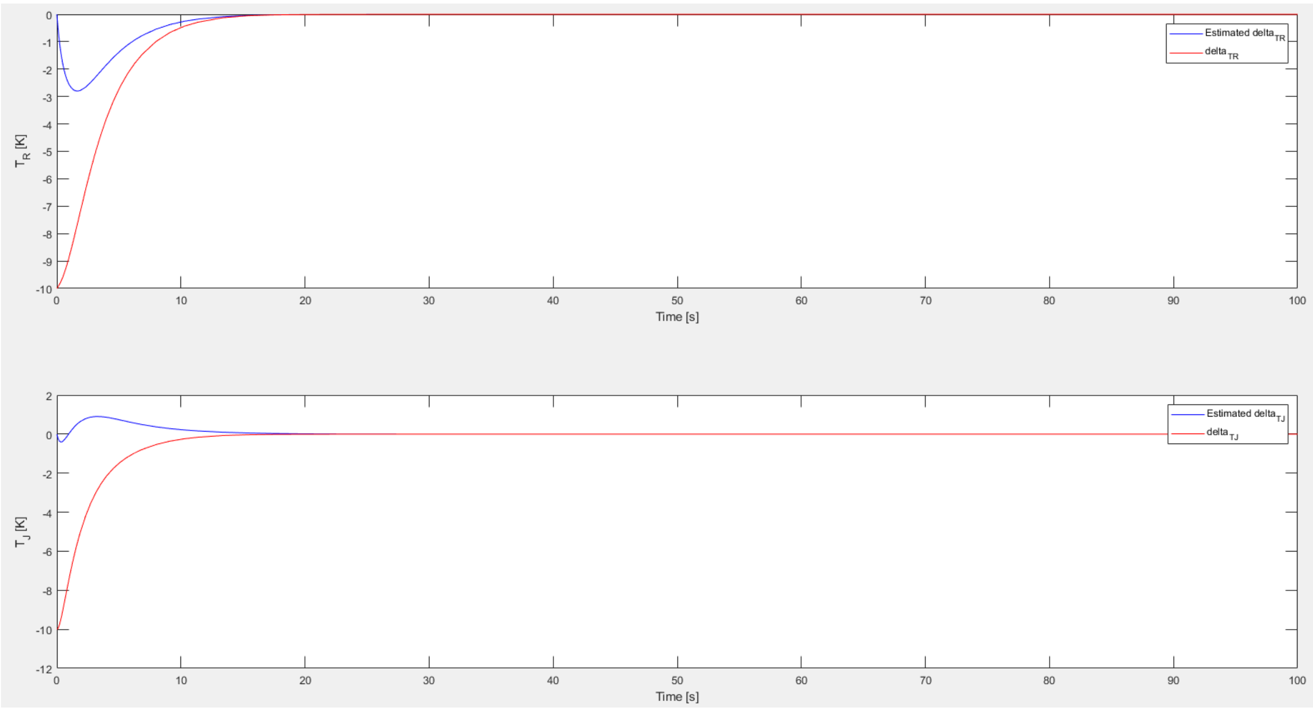
Observer Closed-Loop Simulation for C_A, C_B, T_R, T_J respectively

with initial condition x0,2

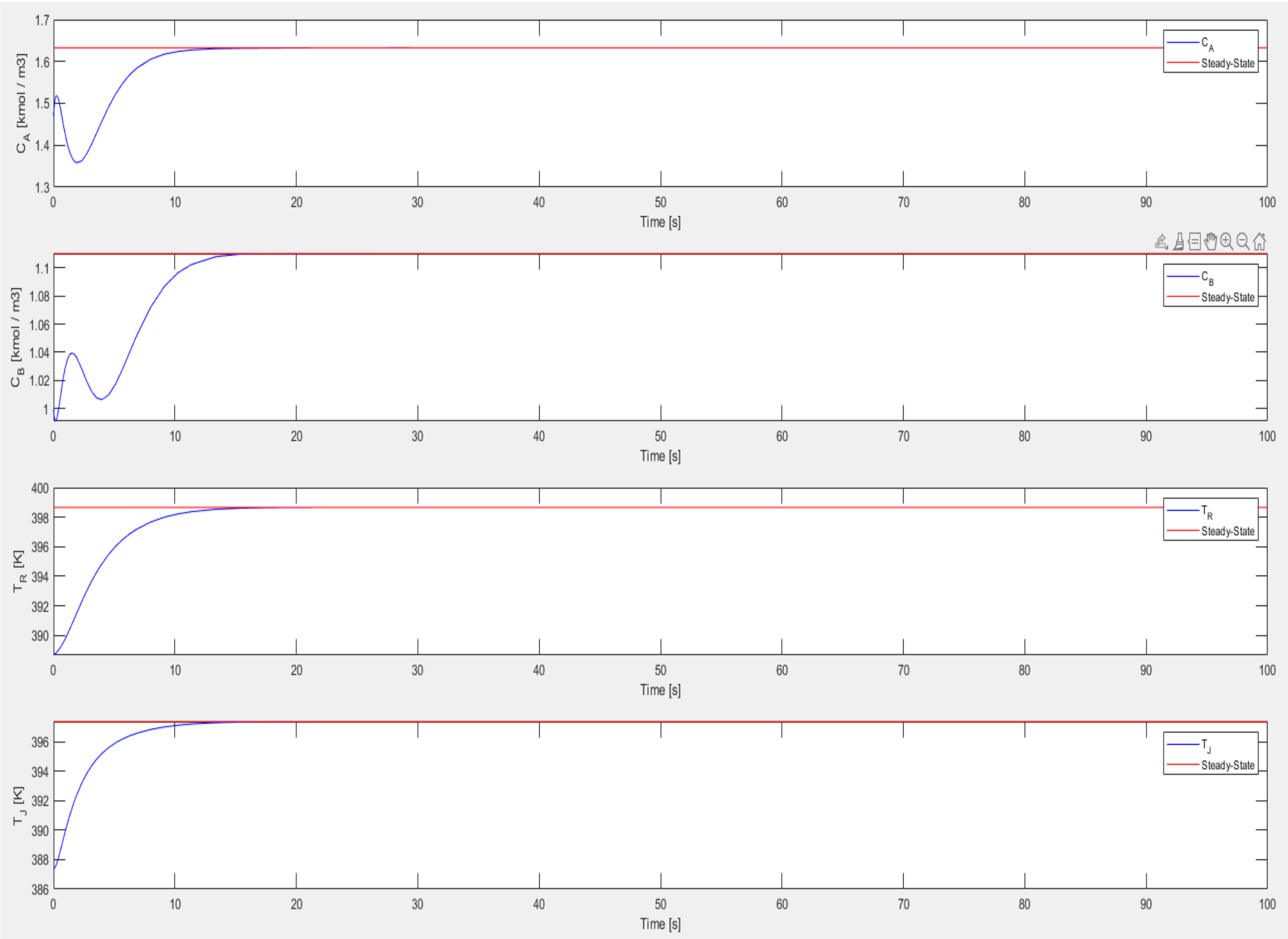
Q2 – Task9 – Plots



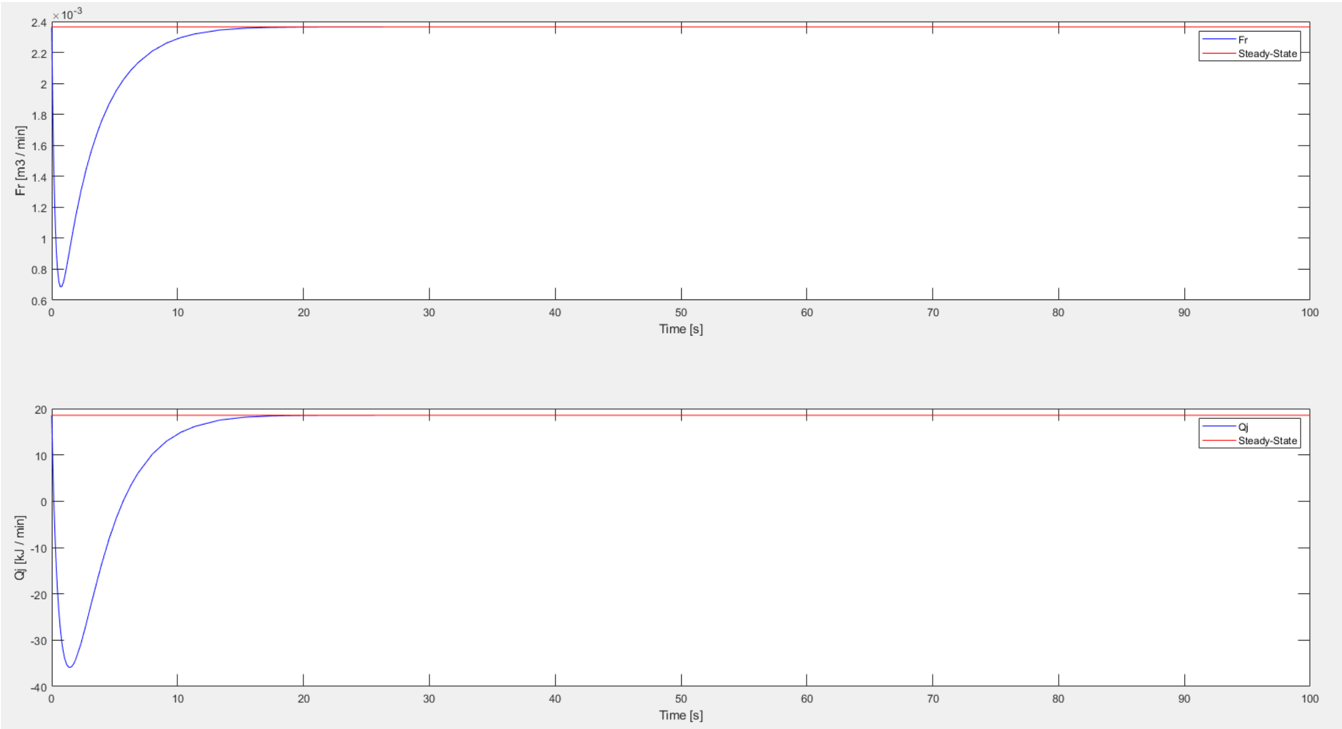
Observer State-Feedback Closed-Loop Simulation for ΔC_A, ΔC_B respectively with initial condition x0,1



Observer State-Feedback Closed-Loop Simulation for ΔT_R, ΔT_J respectively with initial condition x0,1

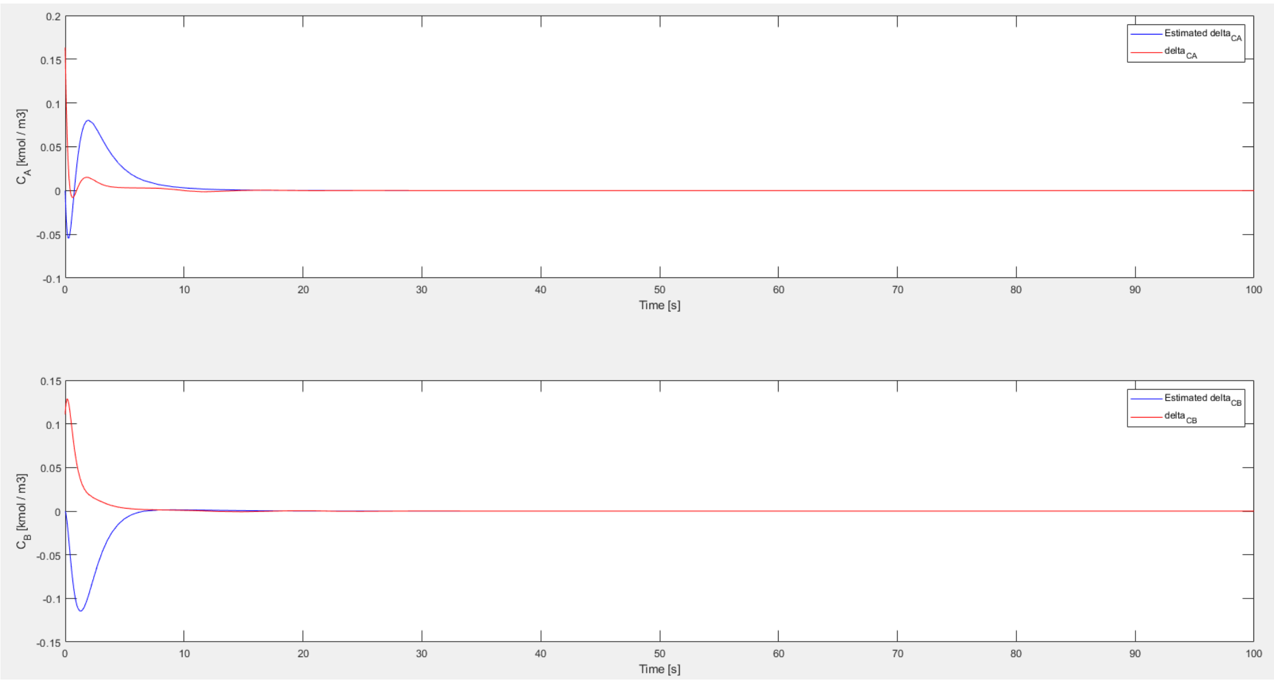


Observer State-Feedback Closed-Loop Simulation for C_A, C_B, T_R, T_J respectively with initial condition x0,1



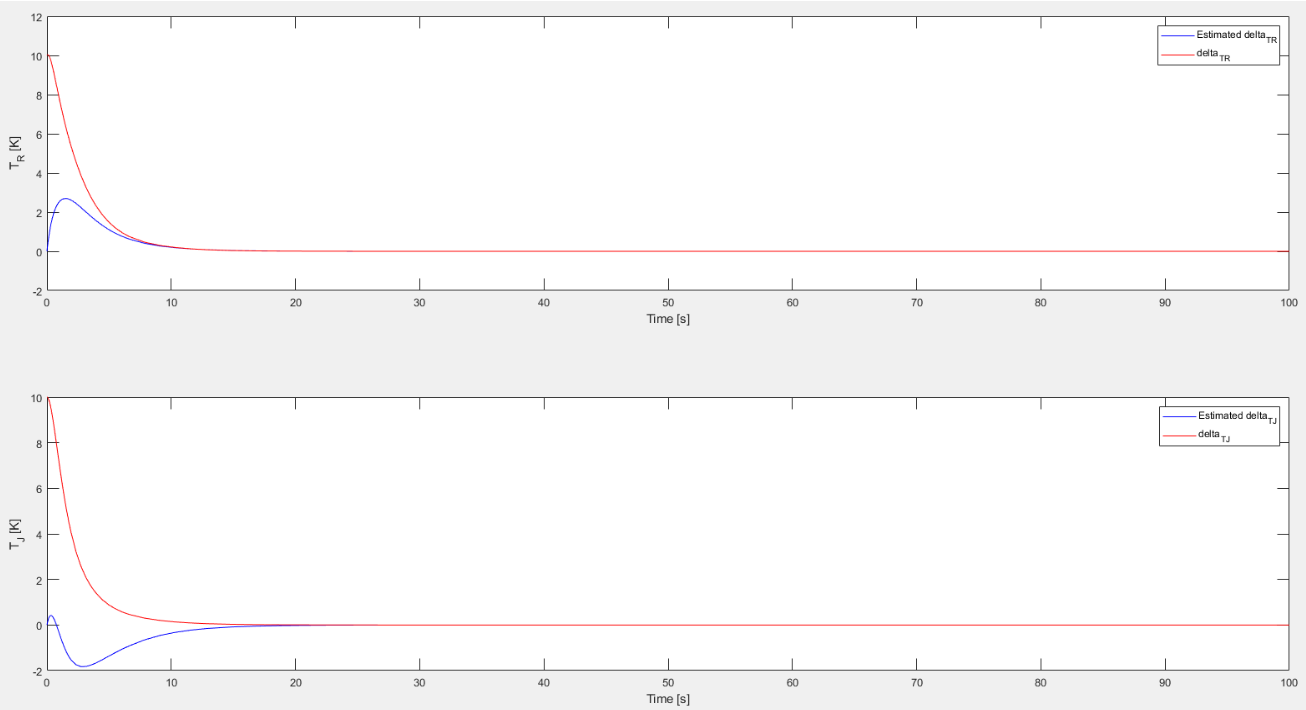
Observer State-Feedback Closed-Loop Simulation for Fr, Qj respectively

with initial condition x0,1



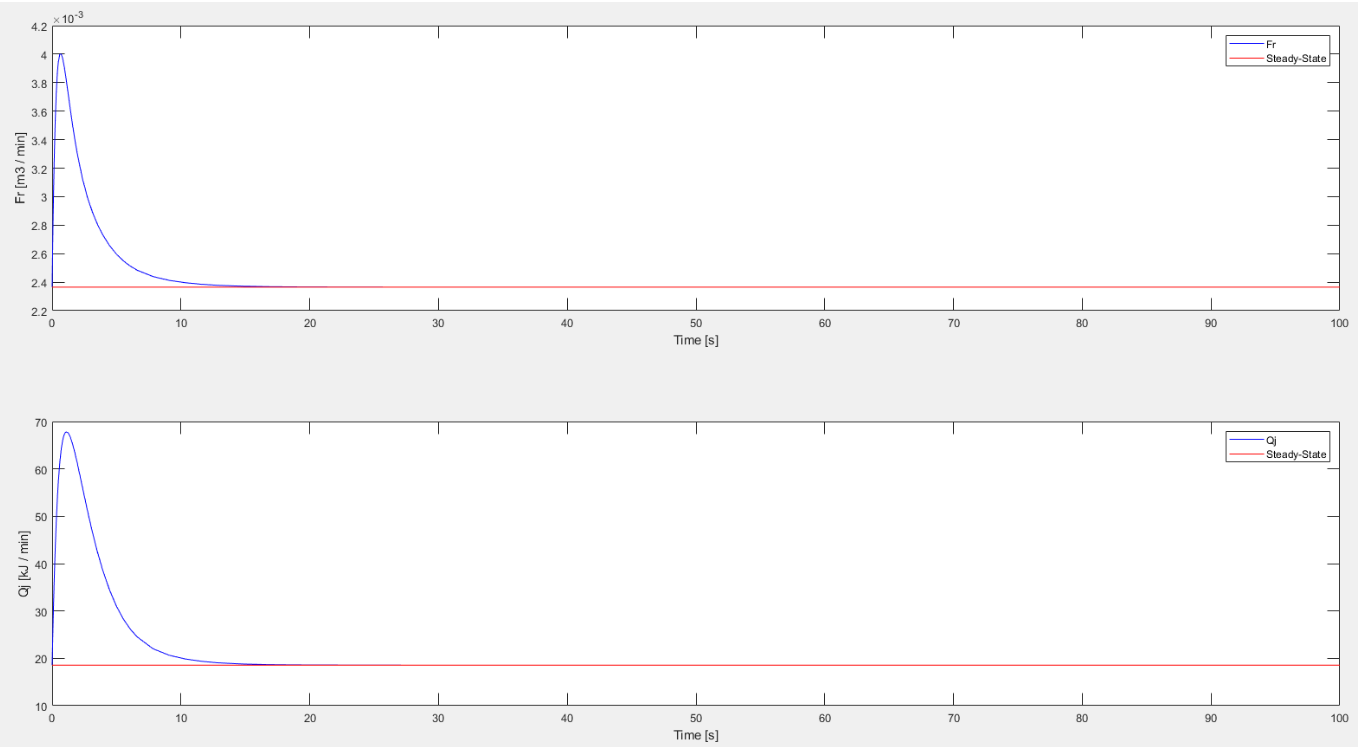
Observer State-Feedback Closed-Loop Simulation for ΔC_A, ΔC_B respectively

with initial condition x0,2



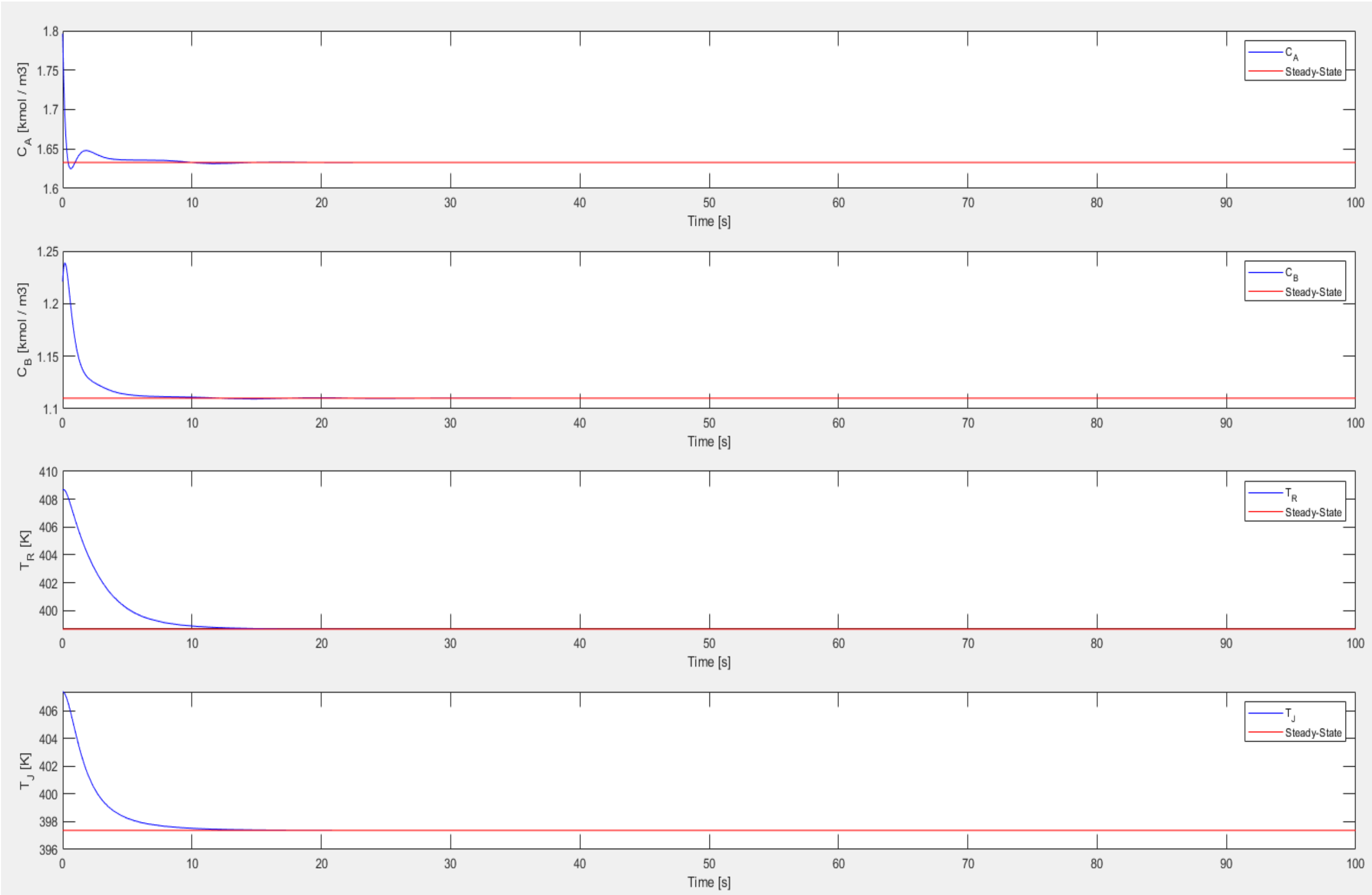
Observer State-Feedback Closed-Loop Simulation for ΔT_R, ΔT_J respectively

with initial condition x0,2



Observer State-Feedback Closed-Loop Simulation for Fr, Qj respectively

with initial condition x0,2



Observer State-Feedback Closed-Loop Simulation for C_A, C_B, T_R, T_J respectively with initial condition x0,2