MPC for Trajectory Tracking using CasADi (Implementation to Mobile Robots)

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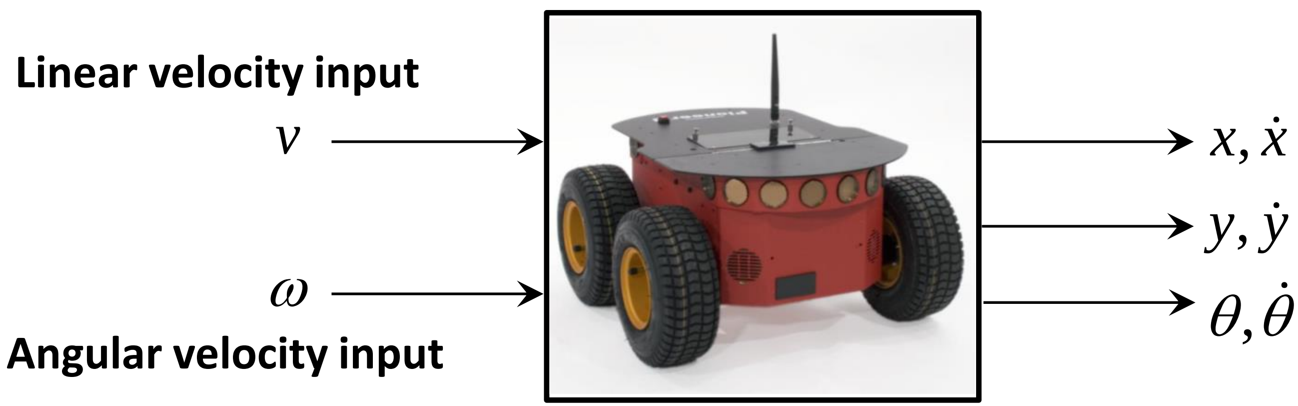
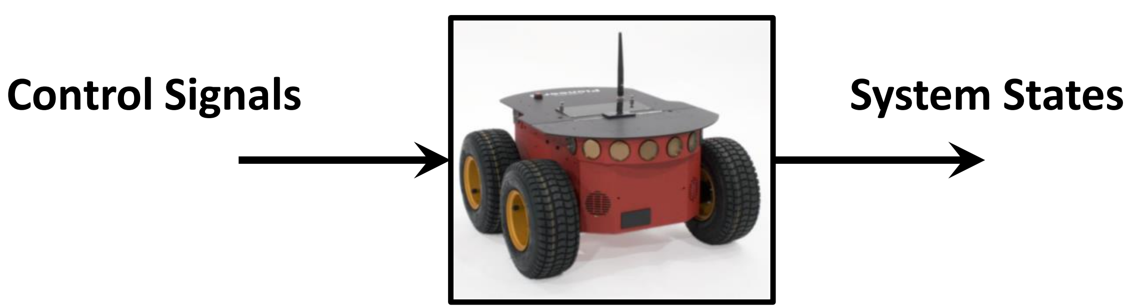
- **Considered System and Control Problem** (Differential drive robots)

From Input/output point of view, robot as a system can be viewed as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$





- **Considered System and Control Problem** (Differential drive robots)

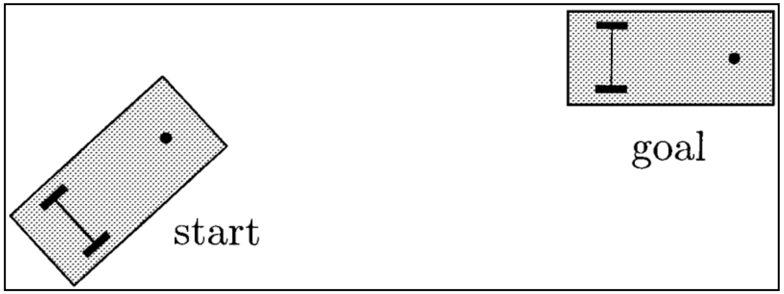
  - **Control objectives**

point stabilization

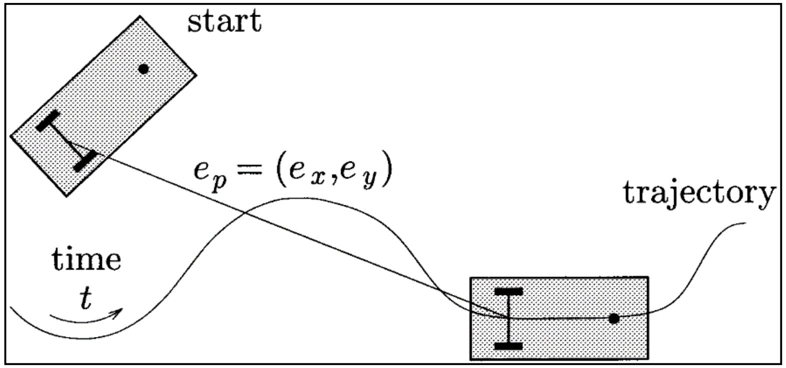
$$\begin{bmatrix} x_r(t) \\ y_r(t) \\ \theta_r(t) \end{bmatrix} = \left\{ \begin{bmatrix} x_d \\ y_d \\ \theta_d \end{bmatrix}, \forall t \right\}$$

- reference values of the state vector are constant over the control period

Point stabilization



Trajectory tracking



trajectory tracking

$$\begin{bmatrix} x_r(t) \\ y_r(t) \\ \theta_r(t) \end{bmatrix} = \begin{bmatrix} x_d(t) \\ y_d(t) \\ \theta_d(t) \end{bmatrix}$$

- time varying reference values of the state vector

- **Model Predictive Control for** (Differential drive robots – point stabilization)

system model

$$\dot{\mathbf{x}}(t) = \mathbf{f}_c(\mathbf{x}(t), \mathbf{u}(t)) \qquad\qquad \mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k))$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix} \xrightarrow[\text{Sampling Time } (\Delta T)]{\text{Euler Discretization}} \begin{bmatrix} x(k+1) \\ y(k+1) \\ \theta(k+1) \end{bmatrix} = \begin{bmatrix} x(k) \\ y(k) \\ \theta(k) \end{bmatrix} + \Delta T \begin{bmatrix} v(k)\cos\theta(k) \\ v(k)\sin\theta(k) \\ \omega(k) \end{bmatrix}$$

MPC controller

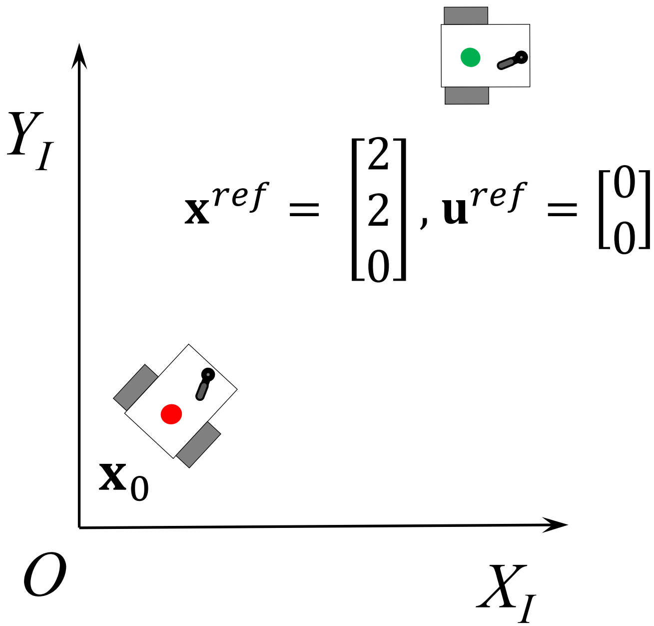
Running (stage) Costs: $\ell(\mathbf{x}, \mathbf{u}) = \left\| \mathbf{x}_{\mathbf{u}} - \mathbf{x}^{ref} \right\|_{\mathbf{Q}}^2 + \left\| \mathbf{u} - \mathbf{u}^{ref} \right\|_{\mathbf{R}}^2$

Optimal Control Problem (OCP):

$$\underset{\mathbf{u}\,\text{admissible}}{\text{minimize}}\; J_N(\mathbf{x}_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(\mathbf{x}_{\mathbf{u}}(k), \mathbf{u}(k))$$

$$\begin{aligned}
\text{subject to:}\;& \mathbf{x}_{\mathbf{u}}(k+1) = \mathbf{f}(\mathbf{x}_{\mathbf{u}}(k), \mathbf{u}(k)), \\
& \mathbf{x}_{\mathbf{u}}(0) = \mathbf{x}_0, \\
& \mathbf{u}(k) \in U, \;\; \forall k \in [0, N-1] \\
& \mathbf{x}_{\mathbf{u}}(k) \in X, \;\; \forall k \in [0, N]
\end{aligned}$$

- **Point Stabilization Recap**



$$\mathbf{x}^{ref} = \begin{bmatrix}2\\2\\0\end{bmatrix}, \mathbf{u}^{ref} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\ell(\mathbf{x},\mathbf{u}) = \|\mathbf{x_u} - \mathbf{x}^{ref}\|_{\mathbf{Q}}^2 + \|\mathbf{u} - \mathbf{u}^{ref}\|_{\mathbf{R}}^2$$

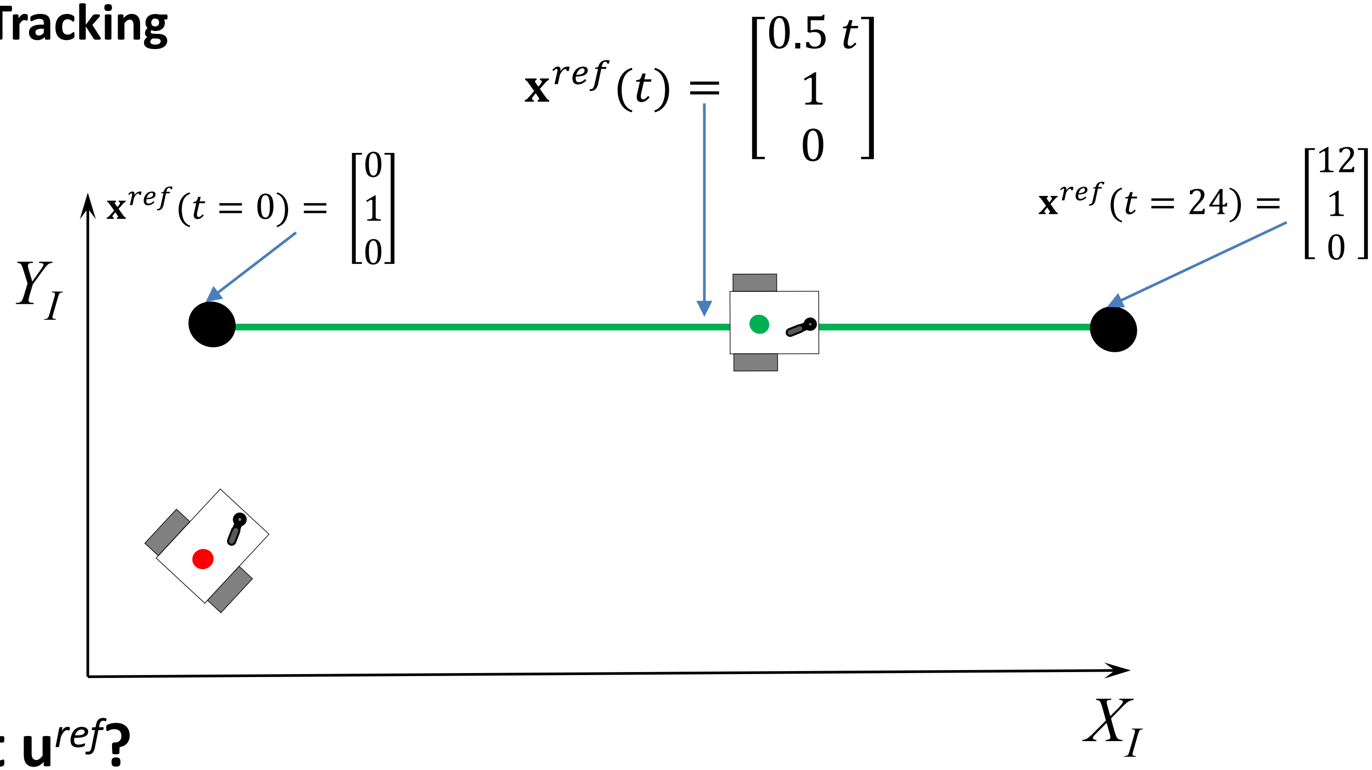
(OCP): $$\underset{\mathbf{u}\,\text{admissible}}{\text{minimize}}\, J_N(\mathbf{x}_0,\mathbf{u}) = \sum_{k=0}^{N-1} \ell(\mathbf{x_u}(k),\mathbf{u}(k))$$

$$\text{subject to}: \mathbf{x_u}(k+1) = \mathbf{f}(\mathbf{x_u}(k),\mathbf{u}(k)),$$
$$\mathbf{x_u}(0) = \mathbf{x}_0,$$
$$\mathbf{u}(k) \in U, \quad \forall k \in [0, N-1]$$
$$\mathbf{x_u}(k) \in X, \quad \forall k \in [0, N]$$

Let's expand J_N for N = 3

$$J_N(\mathbf{x}_0,\mathbf{u}) = \sum_{k=0}^{2} \ell(\mathbf{x_u}(k),\mathbf{u}(k)) = \left\|\mathbf{x_u}(0) - \begin{bmatrix}2\\2\\0\end{bmatrix}\right\|_{\mathbf{Q}}^2 + \left\|\mathbf{u}(0) - \begin{bmatrix}0\\0\end{bmatrix}\right\|_{\mathbf{R}}^2$$
$$+ \left\|\mathbf{x_u}(1) - \begin{bmatrix}2\\2\\0\end{bmatrix}\right\|_{\mathbf{Q}}^2 + \left\|\mathbf{u}(1) - \begin{bmatrix}0\\0\end{bmatrix}\right\|_{\mathbf{R}}^2$$
$$+ \left\|\mathbf{x_u}(2) - \begin{bmatrix}2\\2\\0\end{bmatrix}\right\|_{\mathbf{Q}}^2 + \left\|\mathbf{u}(2) - \begin{bmatrix}0\\0\end{bmatrix}\right\|_{\mathbf{R}}^2$$

- **Trajectory Tracking**



How to get $\mathbf{u}^{ref}$?

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix}$$

Square and add the first two equations, we get

$$\dot{x}^2 + \dot{y}^2 = v^2 \rightarrow v = 0.5\ [m/s]$$

From the third equation we have

$$\omega = 0\ [rad/s]$$

$$\mathbf{u}^{ref} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

Feedforward Control Actions

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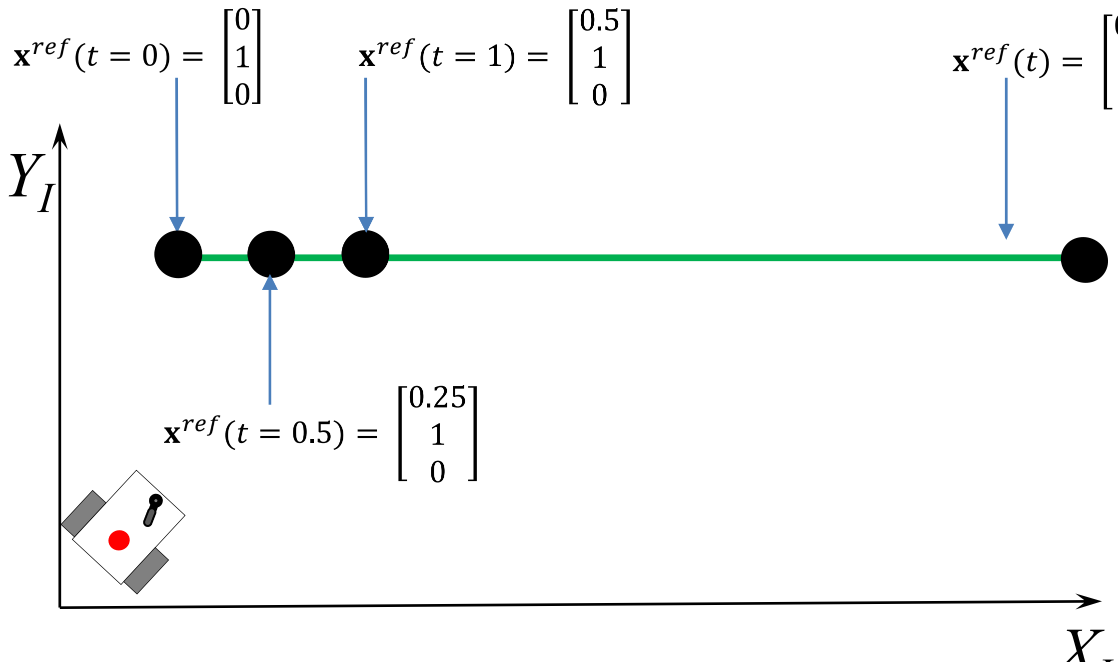
- **Trajectory Tracking**



Let's expand J_N for $N = 3$

$$J_N(\mathbf{x}_0,\mathbf{u}) = \sum_{k=0}^{2} \ell\big(\mathbf{x}_{\mathbf{u}}(k),\mathbf{u}(k)\big) = \left\|\mathbf{x}_{\mathbf{u}}(0) - \begin{bmatrix}0.00\\1\\0\end{bmatrix}\right\|_{\mathbf{Q}}^2 + \left\|\mathbf{u}(0) - \begin{bmatrix}0.5\\0\end{bmatrix}\right\|_{\mathbf{R}}^2$$

$$+ \left\|\mathbf{x}_{\mathbf{u}}(1) - \begin{bmatrix}0.25\\1\\0\end{bmatrix}\right\|_{\mathbf{Q}}^2 + \left\|\mathbf{u}(1) - \begin{bmatrix}0.5\\0\end{bmatrix}\right\|_{\mathbf{R}}^2$$

$$+ \left\|\mathbf{x}_{\mathbf{u}}(2) - \begin{bmatrix}0.50\\1\\0\end{bmatrix}\right\|_{\mathbf{Q}}^2 + \left\|\mathbf{u}(2) - \begin{bmatrix}0.5\\0\end{bmatrix}\right\|_{\mathbf{R}}^2$$

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