Math Remark
Algebraic Structure

X

ElegantLaTeX Program

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1 Number System

Without talking some basic knowledge of Mathmatics logic, we generally define the object we want to study: Number System is a set of ”number” and equipped by certain opreations. Here ”number” is not necessary a real number like 1,2,3 we face daily in caculation, later we will aware that ”number is actually a represent of a system, or using the language of the category, a normal system like $\mathbb{N}$ is just a represent object we choose in a category $Cat(\mathbb{N})$ (The collection of the system same as $\mathbb{N}$).

The main goal of this part is to construct the different number system begin from the natural number $\mathbb{N}$, the procedure often can be found in the textbook and the exercise, and the extension of distinct system inspire us to define the new algebra object.

1.1 From $\mathbb{N}$ to $\mathbb{Z}$

The common idea is to add a new element -1 to the system such that

$$1 + (-1) = 0$$

which refers to the completion of the unit of $\mathbb{N}$. We should notice that $\mathbb{Z}$ is a typical commutative ring with 1 identity, by comparison $(\mathbb{N}, +)$ is even not an abelian group, so by add a new element to the system we can clearly get the another ”direction”, which means $\{0, -1, -2, ...\}$ also forms a number system like $\mathbb{N}$ loosely speaking. we can caulate that $-2 = (-1) + (-1)$ by

$$2 + (-1) + (-1) = 1 + 1 + (-1) + (-1) = 0$$

so we can define that $-k$ is the sum of k same number -1.

Here we reconsider the negative number from the inspiration of the substraction, we can know that

$$-1 = 1 - 2 = 2 - 3 = 3 - 4 = ...$$

and

$$1 = 2 - 1 = 3 - 2 = 4 - 3 = ...$$

so we can define a binary relation on $\mathbb{N} \times \mathbb{N}$ by

$$(a, b) \sim (c, d) \Longleftrightarrow a + d = b + c$$

In fact we should notice that we want to write $a - b = c - d$, but we do not still define substraction formally, so we do the change. we can define that the relation is equivalent, which is easily to verify:

- **reflexivity:** $(a,b) \sim (a,b) \iff a+b=b+a.$
- **Symmetry:**

$$\begin{aligned}(a,b) \sim (c,d) &\iff a+d=b+c \\ &\iff c+b=b+c=a+d=d+a \\ &\iff (c,d) \sim (a,b)\end{aligned}$$

- **transitivity:**

$$\begin{aligned}(a,b) \sim (c,d) \wedge (c,d) \sim (e,f) &\iff a+d=b+c \wedge c+f=d+e \\ &\iff a+(d+c)+f=b+(c+d)+e \\ &\iff a+f=b+e \\ &\iff (a,b) \sim (e,f)\end{aligned}$$

Hence we can use this equiavlence relation to construct the intger.

Proposition 1.1 *Suppose $X = \mathbb{N} \times \mathbb{N}$, and we put $[a,b]$ to be the equiavlence class of the class containning $(a,b) \in X$, then following result can be verified:*
(1) the following operation is well-defined.

$$[a,b]+[c,d]=[a+c,b+d]$$

$$[a,b] \cdot [c,d]=[ac+bd,ad+bc]$$

(2) the system $(X/\sim, +, \cdot)$ *form a commutative ring with multiplicative identity.*

(3) The map $f : \mathbb{N} \to \mathbb{Z}, n \mapsto [n, 0]$ *is injective and additive*

$$f(n+m) = f(n) + f(m)$$

(4) If $X_+ = \{[n, 0] : n \in \mathbb{N}\}$ *and* $X_- = \{[0, n] : n \in \mathbb{N}\}$*, then*

$$X/\sim = X_+ + X_-$$