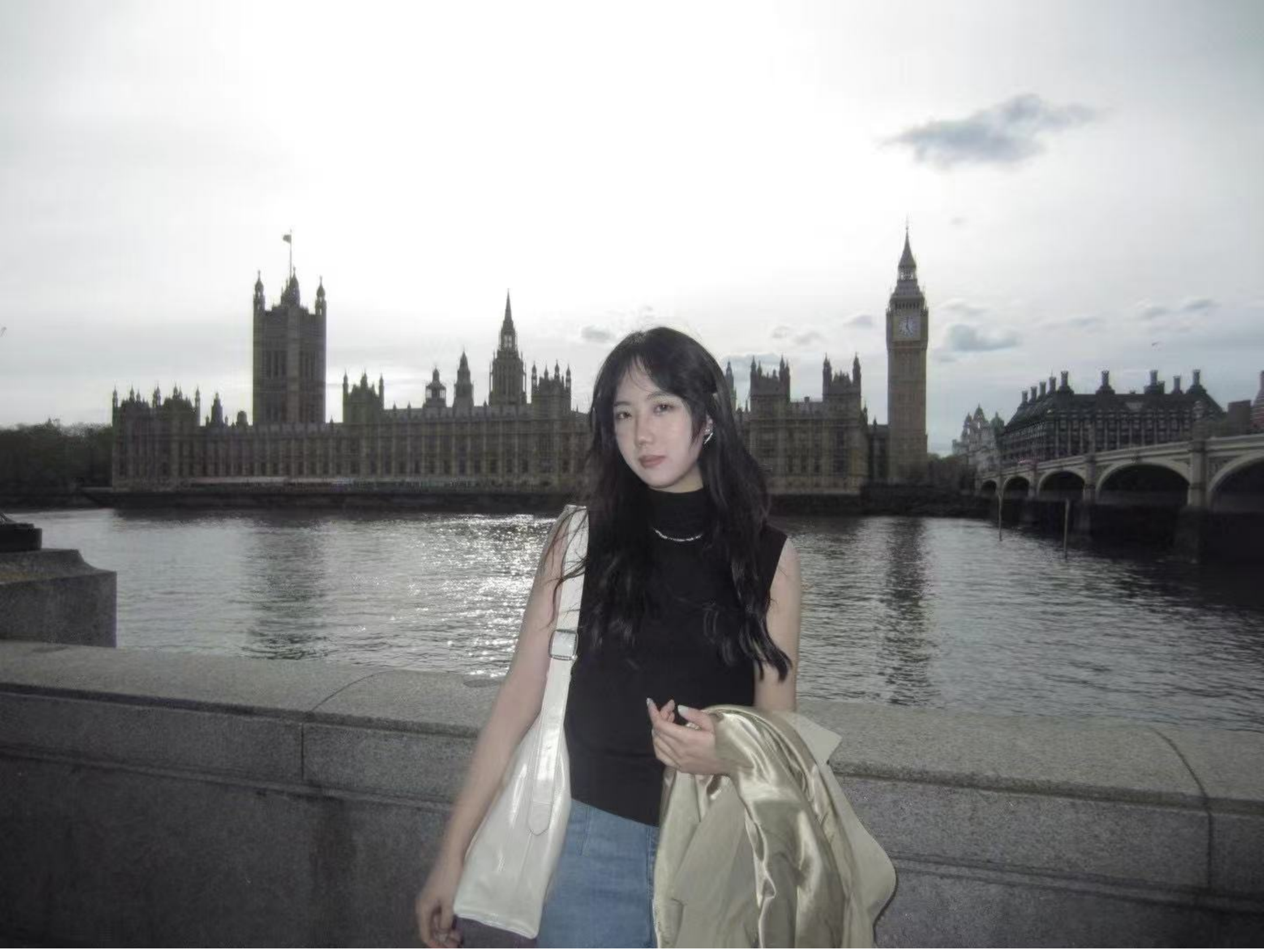

LaTeX Note Template

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October 11, 2025

Github: https://github.com/Baudelaireee/Notebook

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1 Universal property

Universal property is a core concept in modern mathematics, it means the same properties but appears in different objects, in the language of category theory, it means the same diagram commutative in different categories. this section is to introduce some common universal properties from set theory to group theory and topology, it can be seen as a review of basic mathematics, and a good inviation for category theory.

1.1 Quotient

quotient is a method to see two different things as the same things via an equivalence relation: Let X be a non-empty set and $\sim$ be an realtion on X, if it satisfies

- reflexive: $x \sim x$
- symmetric: $x \sim y \Rightarrow y \sim x$
- transitive: $x \sim y, y \sim z \Rightarrow x \sim z$

Then we define $\sim$ is an **equivalence relation** on X. So we can define the equivalence class of some element a of the set as

$$[a] = \{x \in X | x \sim a\}$$

then the **quoitent set** of X with respect to $\sim$ is defined as

$$X/\sim := \{[a] | a \in X\}$$

but it is not the unique method to define quotient, the follwing statement gives different view.

Lemma 1.1. *Let X be a non-empty set, then the following statements are equivalent:*

(1) There is an equivalence relation $\sim$ on X.

(2) There is a surjective map $f : X \to Y$ to some set Y.

(3) There is a partition of X, i.e. a family $P = \{A_i | i \in I, A_i \subset X\}$ such that the set can be written as the disjoint union of the family

$$X = \bigsqcup_{i \in I} A_i$$

Proof. (1) $\Rightarrow$ (2): Let $Y = X/\sim$ and define $\pi : X \to Y, x \mapsto [x]$, then it is easy to see that π is a surjective map.

(1) $\Rightarrow$ (3): it is equivalent to prove that the equivalence class $[x]$ and $[y]$ is disjoint if $x \nsim y$, otherwise they are equal.

(2) $\Rightarrow$ (1): Define a relation on X by

$$x \sim_f y \iff f(x) = f(y)$$

It is a equivalence relation, and the equivlence class can be denoted by pre-image

$$[x] = f^{-1}(y)$$

where $y = f(x)$.

(3) $\Rightarrow$ (1): Define a relation on X by

$$x \sim_P y \iff x \in A_i \wedge y \in A_i$$

for some $i \in I$, similarly the class can be denoted by

$$[x] = A_i$$

where $x \in A_i$ and $X/ \sim_P = P$ actually. $\square$

The application in the proof

$$\pi : X \to X/\sim, \quad x \mapsto [x]$$

is called the **quotient map**, it is a type of **natural map**, that means the definition of the map is unique and very natural. The statement (2) can be generalized to any map, not only surjective map, but the reason here I only state the surjective map is that (a) any map can be restricted to a surjective map; (b) the quotient map is surjective.

The universal property of quotient can be explained as following: we expect two objects are the same up to isomorphism, in the sense of set theory, that means there is a bijection between two sets, but it is not easy to construct a bijection directly, or sometimes we do not need know what the bijection exactly is. Now we take a map $f : X \to Y$, then we can **uniquely determine** a bijection between $X/\sim_f$ and a subset of Y, and the bijection is induced by f and the quotient map, the correspondece can be draw as following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, two heads, "\pi"'] & Y \\
X/\sim_f \arrow[ur, dashed, "\exists!\bar{f}"'] &
\end{tikzcd}$$

We conclude the result of set theory as following:

Theorem 1.2 (UPQ-SET).
Let X, Y be two empty sets and $f : X \to Y$ a surjective map between them, then there is a unique bijection map $\bar{f} : X/\sim_f \hookrightarrow Y$ such that $f = \bar{f} \circ \pi$, where $\pi : X \to X/\sim_f$ is the quotient map.

Proof. The definition of $\bar{f}$ is natural and uniquely by $f = \bar{f} \circ \pi$, for any $[x] \in X/\sim_f$, define

$$\bar{f}([x]) = f(x)$$

the map is well-defined, because if $[x] = [y]$, then $f(x) = f(y)$ by the definition of $\sim_f$. The relation $\sim_f$ implies injection $\bar{f}$, surjection f implies surjection $\bar{f}$, so we finish the proof. $\square$

It is the simplest universal property of quotient, it can be generalized to other mathematics objects, but a little different, the map between two objects should preserve the structure, so the map will be some type of morphism, hence the quotient map should also be a morphism and the equivalence relation should be compatible with the structure. Let us consider the case of topological space.

In geometry view, a circle can be obtained by glueing the two endpoints of a line segment, for example we identify two endpoints of the interval $[0, 1]$ by the relation $0 \sim 1$, then we can get a quotient set $[0, 1]/\sim$, but how to define the topology on the quotient set such that it is a space homemorphic to circle $\mathbb{S}^1$? The answer is that the quotient map should preserve the topological structure, i.e. quotient topology is the smallest topology such that the quotient map is continuous.

Definition 1.1. Let $(X, \mathcal{T})$ be a topological space and $\sim$ be an equivalence relation on X, the **quotient topology** on $X/\sim$ is defined as

$$\mathcal{T}_\sim := \{U \subset X/\sim | \pi^{-1}(U) \in \mathcal{T}\}$$

where $\pi : X \to X/\sim, x \mapsto [x]$ is the quotient map.

We can complete the example just now, firstly we find a surjective map

$$f : [0, 1] \to \mathbb{S}^1, \quad f(x) = e^{2\pi i x}$$

It is easy to see that $0 \sim_f 1$ since $f(0) = f(1)$, and the relation is exactly the relation we want, so we can conclude a bijection $\bar{f} : [0, 1]/\sim_f \to \mathbb{S}^1$ such that $f = \bar{f} \circ \pi$ by UPQ-SET, by the condtion that f and π are contionous, we can verify that $\bar{f}$ is also continous. To see whether it is a homemorphism or not, we notice that $\mathbb{S}^1$ is a compact space, so we just need to verify thet $[0, 1]/\sim_f$ is also a compact space **(a continous bijection from a compact space to a Hausdorff space is automatically a homeomorphism)**, that needs a properties of quotient space:

Claim 1.3. *Let X be a compact space and the graph of $\sim$ be closed in X^2, then quotient space $X/\sim$ is also a compact space.*

Proof.

$$G = \{(a, b) \in X^2 | a \sim b\}$$

is the graph of $\sim$,... $\square$

So we can prove that $[0, 1]/\sim_f$ is homeomorphic to $\mathbb{S}^1$ as above, and we can generalized the result to any similar case as following:

Theorem 1.4 (UPQ-TOPO).
Let $(X, \mathcal{T})$ be a topological space and $(Y, \mathcal{J})$ be a topological space, $f : X \to Y$ be a surjective continuous map, then there is a unique continous bijection $\bar{f} : (X/\sim_f, \mathcal{T}_{\sim_f}) \to (Y, \mathcal{J})$ such that $f = \bar{f} \circ \pi$, where $\pi : X \to X/\sim_f$ is the quotient map.

Proof. By UPQ-SET, there is a unique bijection $\bar{f}$ such that $f = \bar{f} \circ \pi$, now we will prove that $\bar{f}$ is a homeomorphism. Take any open set U of Y, then

$$f^{-1}(U) = (\bar{f} \circ \pi)^{-1}(U) = \pi^{-1}(\bar{f}^{-1}(U))$$

since f is continous, then $\pi^{-1}(\bar{f}^{-1}(U))$ is open in X, by the definition of quotient topology, $\bar{f}^{-1}(U)$ is open in $X/\sim_f$, so $\bar{f}$ is continous. $\square$

Remark 1.1. Here we can not get that $\bar{f}$ is a homeomorphism directly, because the quotient map is not only a continous map, it is strong continous: a map $f : X \to Y$ is called **strong continous** if it satisfies

$$U \in \mathcal{T}_Y \iff f^{-1}(U) \in \mathcal{T}_X$$

A example of a continous map but not strong continous is the map $f : \mathbb{R} \to \mathbb{R}, x \mapsto 0$, here we take $U = [-1, 1]$, then $f^{-1}(U) = \mathbb{R}$ is open, but U is not open.

So in the statement of UPQ-TOPO, if we add that f is strong continous (for example, f is an open map), then $\bar{f}$ is furthermore a homeomorphism.

Finally, we study another basic object: group, Let G be a group and H be a subgroup of G, then the subgroup can induces an equivalence relation (verify) on G by

$$a \sim_H b \iff a^{-1}b \in H$$

then the equivalence class can be denoted by left cosets

$$[a] = \{g \in G | a \sim_H g\} = aH$$

and the quotient set(it maybe not a group) is denoted by

$$G/H = \{aH | a \in G\}$$

and the quoitent map is

$$\pi : G \to G/H, \quad g \mapsto gH$$

if the quotient map furthermore preserves the group structure, i.e. it is a group homomorphism, then we expect for any $g, h \in G$, $\pi(gh) = \pi(g)\pi(h)$, i.e.

$$ghH = gHhH$$

a sufficent condition here is that $hH = Hh$ for any $h \in G$, that means H is a **normal subgroup**. Hence we can define the quotient group as follwoing:

Definition 1.2. Let G be a group and H be a normal subgroup of G, the **quotient group** of G with respect to H is defined as the set G/H with the multiplication

$$(aH)(bH) := (ab)H$$

for any $a, b \in G$, and the group morphism $\pi : G \to G/H, g \mapsto gH$ is called the **canonical morphism**.

The group is a symmetric object, and the equivalence relation given by subgroup is a little special, the partition of G given by H is uniform:
We notice that the translation $L_g : G \to G, x \mapsto gx$ is a bijection (not a group morphism unless $g = e$), so the restriction of L_g on any subgroup implies $gH = L_g(H)$, that means any two left cosets(equivalence class) have the same **cardinality** as H, in particular, if G is a finite group, then $|H|$ divdes $|G|$, this is the **Lagrange's theorem**.

Like what we do in topology and set theory, we can aslo conclude a universal property for group:

Theorem 1.5 (UPQ-GRP)**.**
Let G and K be two groups, H be a normal subgroup of G, and $\varphi : G \to K$ be a surjective group morphism, then there is a unique group isomorphism $\bar{\varphi} : G/H \to K$ such that $\varphi = \bar{\varphi} \circ \pi$, where $\pi : G \to G/H$ is the canonical morphism.

Proof. By UPQ-SET, there is a unique bijection $\bar{\varphi}$ such that $\varphi = \bar{\varphi} \circ \pi$, now we will prove that $\bar{\varphi}$ is a group isomorphism. Take any $aH, bH \in G/H$, then

$$\bar{\varphi}((aH)(bH)) = \bar{\varphi}((ab)H) = \varphi(ab) = \varphi(a)\varphi(b) = \bar{\varphi}(aH)\bar{\varphi}(bH)$$

so $\bar{\varphi}$ is a group morphism, so we finish the proof. □

2 Topological group

Group and Topological space is two different objects, group is algebraic structure, topological space is geometric structure, but it is not strange to combine them together.

Definition 2.1. A **topological group** is a object $(G, \mathcal{T}, \cdot)$ such that

- $(G, \cdot)$ is a group
- $(G, \mathcal{T})$ is a topological space
- The group structure is compatible with topological structure, i.e. multiplication
$$m : G \times G \to G, \quad (x, y) \mapsto x \cdot y$$
and inverse
$$i : G \to G, \quad x \mapsto x^{-1}$$
are continuous maps.

Before we see some examples, we first prove some properties of topological group, the results are not diffcult, but it is always useful.

Proposition 2.1. *Let G be a topological group and H be a subgroup of G, then*

(1-translation) For any $g \in G$, the left-translation $L_g : G \to G, x \mapsto g \cdot x$ and right-translation $R_g : G \to G, x \mapsto x \cdot g$ are homeomorphisms.

(2-open) If H is open, then H is closed and G/H is discrete.

(3-Hausdorff)
$$\begin{aligned} G \text{ is Hausdorff} &\iff \{e\} \text{ is closed.} \\ G/H \text{ is Hausdorff} &\iff H \text{ is closed in } G. \end{aligned}$$

(4-homogeneous) If G is connect, then any voisinage of $\{e\}$ can generate the group.

(5-CC) Let G_0 be the connected component of G containing e, then G_0 is a closed normal subgroup of G, and any connected component of G is homeomorphic to G_0.

Proof. □

Remark 2.1. Translation is very important in topological group, group is a object with symmetry and translation is a way to keep symmetry. For example, let x, y be two different points in G, then the translation $L_{yx^{-1}}$ is a homeomorphisms which maps x to y, therefore each point of G share the same topological stucture, so the local property is same as golbal poerperty, that means topological group is a **homogeneous space**.

the fundamental theorem of group homeomorphisms can be extended to topological group, in the discussion of topological group, we hope the map between two objects can preserve both two structures, that means the map is both a morphism of group and a continous map. For example, Let $\mathcal{T}$ be the common topology of $\mathbb{R}$ and $\mathcal{J}$ be the discrete topology of $\mathbb{R}$, then the group isomorphism

$$id : (\mathbb{R}, \mathcal{T}, +) \to (\mathbb{R}, \mathcal{J}, +)$$

is not a homeomorphisms because it is not continous. In a word, the isomorphism in topological group is a **homeomorphisms and group isomorphism** at the same time.