



# L<sup>A</sup>T<sub>E</sub>X M1 NOTE II

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*Github:* <https://github.com/Baudelaireee/Notebook>

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# 1 Some linear algebra

This section can be seen as a breif review of linear agebra, it mainly focus on something related to lie theory and representation theory.

## 1.1 Decomposition of endomorphisms

**Theorem 1.1** (real-Polar decomposition).

For any real endomorphism  $A \in \text{GL}_n(\mathbb{R})$ , there exists a unique orthogonal endomorphism  $O \in O(n)$  and a unique symmetric positive definite endomorphism  $P \in S_n^{++}(\mathbb{R})$  such that

$$A = OP$$

**Proof.** Here is the outline of the proof:

- (1) A **lemma** about the existence of square root are needed here: If  $A \in S_n^+(\mathbb{R})$ , then there exists a (unique)  $B \in S_n^+(\mathbb{R})$  such that  $B^2 = A$ .
- (2) Prove that  $A^t A$  is (semi-definite) positive, thus by the lemma there exist a (unique) squar root  $P = \sqrt{A^t A}$ , and we prove that  $P$  is positive definite.
- (3) Define  $O = AP^{-1}$ , and prove that  $O$  is orthogonal.
- (4) Prove the uniqueness of the decompoistion.

Here is the details:

- (1) By real-spectral theorem, there exists  $O \in O(n)$  such that

$$A = ODO^T, \quad D = \text{dig}(\lambda_1, \dots, \lambda_n)$$

where  $\lambda_i \geq 0$  are the eigenvalues of  $A$ , and we set  $\Sigma = \text{dig}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$  such that

$$A = (O\Sigma O^T)^2$$

which shows that  $B = O\Sigma O^T$  is a (unique) square root of  $A$  and it is positive since  $\sqrt{\lambda_i} \geq 0$ .

- (2)  $A^T A$  is clearly symmetric, and it is positive since

$$x^T (A^T A) x = \|Ax\|^2 \geq 0$$

for any  $x \in \mathbb{R}^n$  under the common inner product, and  $A$  is invertible such that  $\|Ax\| = 0$  if and only if  $x = 0$ , thus  $A^T A$  is positive definite, so does its square root (the eignevalues is strictly positive).

- (3)  $P^{-1} = (P^{-1})^T$  since  $P$  is symmetric, so we have

$$O^T O = (P^{-1})^T A^T A P^{-1} = (P^{-1})^T P^2 P^{-1} = I$$

- (4) Suppose that there eixsts another decomposition  $A = O_0 P_0$ , then we have

$$A^T A = P_0^T O_0^T O_0 P_0 = P_0^2$$

so  $P_0$  is the positive square root of  $A^T A$ , if the uniqueness of (1) is proved, we can finish the proof, but here we give another proof of the uniqueness only under the existence of the

square root: we have  $O_0^{-1}O = P_0P^{-1}$ , so it is easy to check that  $P_0P^{-1}$  is **orthogonal**; notice that  $P_0$  and  $A^TA$  commute, and there exists a polynomial  $f$  such that

$$f(A^TA) = P$$

it can be done by using lagrange interpolation, thus  $P_0$  and  $P$  commute as two symmetric matrix, then  $P$  and  $P_0$  can be diagonalized simultaneously by one orthogonal  $H$  such that

$$P = HDH^T \quad P_0 = HD_0H^T$$

where  $D = \text{dig}(\lambda_1, \dots, \lambda_n)$  and  $D_0 = \text{dig}(\mu_1, \dots, \mu_n)$ , thus

$$P_0P^{-1} = HD_0^{-1}DH^T$$

it is **symmetric and positive definite** since  $D_0^{-1}D$  is digonal with positive entries, thus  $P_0P^{-1} \in O(n) \cap S_n^{++}(\mathbb{R}) = \{I\}$ , which implies that  $P_0 = P$  and  $O_0 = O$ .  $\square$

**Theorem 1.2** (complex-Polar decomposition).

For any complex endomorphism  $A \in \text{GL}_n(\mathbb{C})$ , there exists a unique unitary endomorphism  $U \in U(n)$  and a unique hermitian positive definite endomorphism  $P \in H_n^{++}(\mathbb{C})$  such that

$$A = UP$$

The proof is similar to the real case, we set  $P = \sqrt{A^*A}$  and  $U = AP^{-1}$ , and the lemma is based on the complex-spectral theorem for normal operator: The postive Hermitian endomorphism has a unique positive Hermitian square root.

*Remark 1.1.* In particular, if  $n = 1$ , we have the polar decompistion in  $\text{GL}_1(\mathbb{C}) = \mathbb{C}^*$  by

$$z = |z|e^{i\text{Arg}(z)}$$

it is easy to see that  $U(1) \cong \mathbb{S}^1$ , it is the original ideal of the decomposition. Similarly, in the real case, we can embed  $\mathbb{C}$  into  $\text{GL}_2(\mathbb{R})$  by

$$a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and we have the polar decomposition for the matrix form of complex numbers:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \sqrt{a^2 + b^2} & 0 \\ 0 & \sqrt{a^2 + b^2} \end{pmatrix} \begin{pmatrix} \frac{a}{\sqrt{a^2 + b^2}} & -\frac{b}{\sqrt{a^2 + b^2}} \\ \frac{b}{\sqrt{a^2 + b^2}} & \frac{a}{\sqrt{a^2 + b^2}} \end{pmatrix}$$

## 2 Topological Groups

### 2.1 Basic properties

Group and Topological space is two different objects, group is algebraic structure, topological space is geometric structure, but it is not strange to combine them together.

**Definition 2.1.** A **topological group** is a object  $(G, \mathcal{T}, \cdot)$  such that

- $(G, \cdot)$  is a group
- $(G, \mathcal{T})$  is a topological space
- The group structure is compatible with topological structure, i.e. multiplication

$$m : G \times G \rightarrow G, \quad (x, y) \mapsto x \cdot y$$

and inverse

$$i : G \rightarrow G, \quad x \mapsto x^{-1}$$

are continuous maps.

With the basic definition, we can talk about the properties of topological groups, here is a useful lemma, it shows that the topological group is **homogeneous**, i.e. the local topology is same at any point.

**Lemma 2.1.** Let  $G$  be a topological group, then for any  $g \in G$ , we define **left translation** and **right translation** by

$$\begin{aligned} L_g : G &\rightarrow G, & x &\mapsto gx \\ R_g : G &\rightarrow G, & x &\mapsto xg \end{aligned}$$

Then both  $L_g$  and  $R_g$  are homeomorphisms on  $G$ .

**Proof.** It is easy to verify that  $L_{gh} = L_g \circ L_h$ , and  $R_{gh} = R_h \circ R_g$ , then we have

$$L_g^{-1} = L_{g^{-1}}, \quad R_g^{-1} = R_{g^{-1}}$$

and we can check the continuity of  $L_g$  and  $R_g$  by composition. □

It is an important in the following proofs of properties, we first talk about the basic properties of topological groups.

**Proposition 2.2.** Let  $G$  be a topological group, then

- (1) each open subgroup  $H < G$  is also closed.
- (2) each closed subgroup  $H < G$  with finite index is also open.

**Proof.** The subgroup  $H$  gives a partition of  $G$  by left cosets:

$$G = \bigsqcup_{g \in \mathcal{S}} gH$$

with  $\mathcal{S} \subset G$  the set of representatives in  $G/H$ , and we fix the representatives for  $H$  be  $e$ , then

$$G - H = \bigsqcup_{g \in \mathcal{S} - \{e\}} gH$$

If  $H$  is open, then by translation each  $gH$  is also open, thus the union of open sets  $G - H$  is also open, hence  $H$  is closed, which finishes that proof of (1). For the statement (2), the finite index shows that  $\mathcal{S}$  is finite, and we suppose that the index is  $n$ . If  $n = 1$ , then  $H = G$  is open clearly; otherwise, we can deduce that  $G - H$  is the finite union of closed sets  $gH$  with  $g \in \mathcal{S} - \{e\}$ , thus it is closed, it can be done similarly by translation, hence  $H$  is open since  $G - H$  is closed.  $\square$

Then it is the **separation** of topological groups.

**Proposition 2.3.** Let  $G$  be a topological groups, then the following statements are equivalent:

- (1)  $G$  is Hausdorff (T2).
- (2) The singleton  $\{e\}$  is closed in  $G$ .
- (3)  $G$  is T1.

**Proof.** (3) implies (2) is clear, since T1 space makes all singletons closed; (1) implies (3) is just the implication from strong to weak separation axiom.

(2) implies (1):  $G$  is Hausdorff if and only if the diagonal set

$$\Delta = \{(g, g) \in G \times G \mid g \in G\}$$

is closed under the product topology, and topolglcal group induces a continous map  $f(x, y) = xy^{-1}$  for any  $(x, y) \in G \times G$ , then  $\Delta = f^{-1}(\{e\})$  is closed by continuity.  $\square$

**Corollary 2.4.** Let  $G$  be a topological group and  $H < G$  be a subgroup, then the quotient space  $G/H$  is **Hausdorff** if and only if  $H$  is **closed** in  $G$ .

Another topological properties is about **connectness**:

**Proposition 2.5.** Let  $G$  be a topological group, then

- (1) If  $G$  is connected, then any open set containing  $e$  generates  $G$ .
- (2) The connected component  $G_0$  of identity  $e \in G$  is a **closed normal subgroup** of  $G$ , and each connected component of  $G$  is homeomorphic to  $G_0$  by translation.

**Proof.** (1) We denote  $\langle A \rangle$  be the subgroup generated by  $A \subset G$ , and we take  $A$  be an open set containing  $e$ , we will prove that  $\langle A \rangle$  is open and closed, by the connectness of  $G$ , which implies that  $\langle A \rangle = G$ . It is clear that

$$\bigcup_{x \in \langle A \rangle} xA \subset \langle A \rangle$$

conversely, for any  $a \in \langle A \rangle$ , we have  $a = ae$  with  $e \in A$ , so  $a$  is in the union such that we can replace the inclusion by equality. Notice that  $xA = L_x(A)$  is open since  $A$  is open, so  $\langle A \rangle$  is also open as the union of open sets; By the proposition 2.2,  $\langle A \rangle$  is also closed since it is open subgroup, hence we finish the proof of (1).

(2) Any connected component is closed in general topology, which can be proved by contradiction to  $G_0 \subsetneq \overline{G_0}$ . To prove that  $G_0$  is a subgroup, we take a continuous map  $f(x, y) = xy^{-1}$  on  $G \times G$ , then  $f(G_0 \times G_0)$  is connected since the finite product of connected spaces is connected, and we notice that  $a = f(a, e)$  for any  $a \in G_0$ , so  $G_0 \subset f(G_0 \times G_0)$  and then we can take equality by the maximality of connected component, and it shows that  $G_0$  is exactly a subgroup.

To prove the normality, we take the conjugation map for any  $g \in G$ :

$$\sigma_g : G \rightarrow G, \quad x \mapsto gxg^{-1}$$

it is homeomorphism by composition  $\sigma_g = L_g \circ R_g^{-1}$ . Here  $\sigma_g(e) = e$  and the image is also connected, so  $\sigma_g(G_0) \subset G_0$  by the maximality of connected component, so it must be normal.  $\square$

Notice that if  $H < G$  is a subgroup, then it induces a natural equivalence relation on  $G$  by left cosets:

$$x \sim y \iff x = yh, \text{ for some } h \in H$$

similarly we can define right cosets here, and then it gives a quotient space  $G/\sim$  and we denote it by  $G/H$ , and it is naturally endowed with the quotient topology with a (continuous) quotient map:

$$\pi : G \rightarrow G/H, \quad g \mapsto gH$$

**In particular**, if  $H$  is a normal subgroup, then  $G/H$  is exactly a topological group by UPQ:

$$\begin{array}{ccc} G \times G & \xrightarrow{f} & G \\ \pi \times \pi \downarrow & & \downarrow \pi \\ G/N \times G/N & \xrightarrow{f_{G/N}} & G/N \end{array}$$

with  $f(g, h) = gh^{-1}$ , so it gives the continuity of group operation. However, it is not so easy to talk about the topology of quotient space  $G/H$  directly, it is better to see the quotient space as a **homogeneous space** under the action of  $G$ , hence we need to talk about action of the topological group:

**Definition 2.2.** Let  $G$  be a topological group and  $X$  be a topological space, a continuous action of  $G$  on  $X$  is a well-defined action in the sense of group together with a continuous operation, that means there exists a continuous map

$$\phi : G \times X \rightarrow X, \quad (g, x) \mapsto g \cdot x$$

such that for any  $g, h \in G$  and  $x \in X$ , we have

$$(1) \quad e \cdot x = x$$

$$(2) \quad g \cdot (h \cdot x) = (gh) \cdot x$$

In the context of topological groups or Lie groups, we always omit "continuous" when it comes to actions, it is a convention sometimes in the literature. It is similar to the algebraic action, we can define **stabilizer** of a point  $x \in G$

$$I_x = \{g \in G \mid g \cdot x = x\}$$

It is sometimes called **isotropy group** at  $x$ , and it can be written as the preimage

$$I_x = \phi_x^{-1}(\{x\}), \quad \phi_x(g) = g \cdot x$$

with  $\phi_x$  a continuous map by fixing variable, hence the isotropy group is always a **closed subgroup** of  $G$  if  $X$  is Hausdorff (T1 is enough, but manifold makes sense). Similarly, we can define **orbit** of  $x$  by

$$Gx = O_x = \{g \cdot x \mid g \in G\}$$

which is the image of  $\phi_x$  on the contrary, hence it gives a natural bijection

**Lemma 2.6** (stabilizer-orbit).

Let  $G$  be a compact group and  $X$  be a Hausdorff space, If  $G$  acts continuously on  $X$ , then for any  $x \in G$ , there exists a homeomorphism  $G/I_x \cong Gx$ .

**Proof.** The proof is just based on the universal property of quotient space, and the stabilizer-orbit theorem in group theory, we just need to check the natural bijection induced by  $\phi_x$  is indeed a homeomorphism.  $\square$

In particular, the orbit gives an equivalence relation on  $X$  by

$$x \sim y \iff y = g \cdot x, \text{ for some } g \in G$$

hence it gives a quotient space denoted by  $X/G$  and we call it **orbit space**, it is naturally endowed with the quotient topology with a (continuous) quotient map:

$$\pi : X \rightarrow X/G, x \mapsto Gx$$

the following lemma is useful and important, it shows that the topology of group action is well-behaved from the view of technique:

**Lemma 2.7.** The natural quotient map  $\pi : X \rightarrow G/X$  is open.

**Proof.** Suppose that  $U$  is an open set of  $X$ , then we can prove

$$\pi^{-1}(\pi(U)) = \bigsqcup_{g \in G} gU$$

and then by translation each  $gU$  is open, so we can finish the proof by the definition of quotient topology that  $\pi(U)$  is open if its preimage is open.  $\square$

Then we can conclude

**Proposition 2.8.** Let  $G$  be a topological group and  $H$  be a subgroup, then



- (1)  $H$  is closed if and only if  $G/H$  is Hausdorff.
- (2) If  $G$  is connected, then  $G/H$  is connected.
- (3) If  $G/H$  and  $H$  are both connected, then  $G$  is connected.

**Proof.** □

## Neighborhood system

As we have mentioned before, topological group is a homogeneous space, so the local topology at one point can be translated to generate the global topology, for some convenience, we always study the local topology at identity of the group.

Formally, for any group  $G$  we define the **neighborhood system** (at  $x$ ) to be the collection of subsets containing  $x$ , denoted by  $\mathcal{V}_x$ :

$$A \in \mathcal{V}_x \iff x \in A \wedge A \subset G$$

Hence we can induce a topology on  $G$  by neighborhood system (i.e. determine the open voisinage of identity).

**Proposition 2.9.** Let  $G$  be a group with  $\mathcal{V}_e$  as the neighborhood system at identity  $e$ , if  $\mathcal{V}_e$  satisfies the following conditions:

1. For any  $U, V \in \mathcal{V}_e$ , there exists  $W \in \mathcal{V}_e$  such that

$$W \subset U \cap V.$$

2. If  $a \in U \in \mathcal{V}_e$ , then there exists  $V \in \mathcal{V}_e$  such that

$$Va \subset U.$$

3. For each  $U \in \mathcal{V}_e$ , there exists  $V \in \mathcal{V}_e$  such that

$$V^{-1}V \subset U.$$

4. For each  $U \in \mathcal{V}_e$  and each  $x \in G$ , there exists  $V \in \mathcal{V}_e$  such that

$$x^{-1}Vx \subset U.$$

then there exist a unique topology  $\mathcal{T}$  such that  $\mathcal{V}_e$  is the set of all open voisinage of  $e$ , and  $(G, \mathcal{T})$  is a topological group.

**Proof.** The topology  $\mathcal{T}$  is uniquely defined by translation:

$$\mathcal{B} = \{gU \mid g \in G, U \in \mathcal{V}_e\}$$

we just need to prove that  $\mathcal{B}$  is the topology basis. Notice the condition (1) and (2) ensure that  $\mathcal{B}$  generates a topology, the condition (3) is the requirement of continuity of multiplication and inverse; finally, the condition (4) is obligator for non-abelian group to ensure the continuity. □

In partiucular, we can naturally choose subgroup as the element in the system, the condition can be simplified as following:

**Corollary 2.10.** Let  $G$  be a group with  $\mathcal{U}$  as a collection of subgroups of  $G$ , if  $\mathcal{U}$  satisfies the following conditions:

1. For any  $U, V \in \mathcal{U}$ , there exists  $W \in \mathcal{U}$  such that

$$W \subset U \cap V.$$

2. For each  $U \in \mathcal{U}$  and each  $x \in G$ , there exists  $V \in \mathcal{U}$  such that

$$x^{-1}Vx \subset U.$$

then there exist a unique topology  $\mathcal{T}$  such that  $\mathcal{U}$  is a basis of open voisinage of  $e$ , and  $(G, \mathcal{T})$  is a topological group.

Here are some classic examples of topological groups constructed by neighborhood system:

**Example 2.1.** We consider the usual topology on  $\mathbb{Q}$ , then the completion of  $\mathbb{Q}$  is just  $\mathbb{R}$ , it is another method to construct real number (see in Bourkabi's book). We review that we always write the viosinage of 0 as the form of  $(-\varepsilon, \varepsilon)$ , which motivates us to consider the system of subgroup:

$$\mathcal{U} = \left\{ \frac{1}{n}\mathbb{Z} \mid n \in \mathbb{N}^* \right\}$$

pay attention to the choices here! We can not take  $n\mathbb{Z}$  to be an open set (consider it in  $\mathbb{R}$ , it is closed and not open). Then we can verify the topology induced by  $\mathcal{U}$  is just the subspace topology induced by  $\mathbb{R}$ , i.e. it induces a usual absolute value on field  $\mathbb{Q}$ :

$$|x|_\infty = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

It is natural to ask that can we define other topological stucture on  $\mathbb{Q}$  such that we can get other different completion? The answer is yes, and a classic example is the  *$p$ -adic number field*.

**Example 2.2.** We fix a prime number  $p$  and then we can define a subgroup

$$\mathbb{Z}_{(p)} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, p \nmid n \right\}$$

and we let  $\mathcal{U} = \{p^t\mathbb{Z}_{(p)} \mid t \in \mathbb{Z}\}$  to be the collection of subgroups, which generates an unique topology (verify!). Notice that we have a unique chain in the system:

$$\dots p^2\mathbb{Z}_{(p)} \subset p\mathbb{Z}_{(p)} \subset \mathbb{Z}_{(p)} \subset p^{-1}\mathbb{Z}_{(p)} \subset p^{-2}\mathbb{Z}_{(p)} \subset \dots$$

motivates from the above example, we consider each subgroup gives a method to measure the size of an elemnt is some sense, for example we consider the following number when  $p = 3$ :

$$1, \quad 4, \quad \frac{1}{2}, \quad \frac{2}{5}$$

they are all in  $\mathbb{Z}_{(p)}$ , and we consider the distance of them to 0 can be bounded by  $\alpha$  (consider  $(-\varepsilon, \varepsilon)$  gives the set of all elements with distance to 0 less than  $\varepsilon$ ), and then we consider some other numbers:

$$\frac{1}{3}, \quad \frac{2}{3}, \quad \frac{7}{6}$$

they are all in  $\frac{1}{3}\mathbb{Z}_{(p)}$  but not in  $\mathbb{Z}_{(3)}$ , so we can think that their distance to 0 is bounded by  $\beta$ , and it is natural to give a relation  $\alpha < \beta$ , which motivates us to give a new method to consider the metric on  $\mathbb{Q}$  by consider the local factorization of  $p$ .

Hence we can define something to denote  $\alpha$  and  $\beta$  just now:

$$v_p : \mathbb{Z} - \{0\} \rightarrow \mathbb{N}, \quad m \mapsto \max\{k : p^k | m\}$$

for example  $v_3(6) = 2$ ,  $v_3(-2) = 0$ . Then the method of measuring can be extended to  $\mathbb{Q}$  as following:

$$v_p : \mathbb{Z} - \{0\} \rightarrow \mathbb{Z}, \quad \frac{m}{n} \mapsto v_p(m) - v_p(n)$$

And we make a convention that  $v_p(0) = +\infty$ , it is called **p-adic valustion**. for example  $v_3(\frac{1}{3}) = -1$ ,  $v_3(\frac{9}{2}) = 2$ . Then we can get a measure of number

$$p^k \mathbb{Z}_{(p)} \iff \{x \in \mathbb{Q} \mid v_p(x) \geq p^k\}$$

but a question occurs here is that  $v_3(1) = 0$  and  $v_3(\frac{1}{3}) = -1$ , as what we discussed just now, actually we hope the measure of 1 should be less than the measure of  $\frac{1}{3}$  (here  $0 > -1$ ), hence the valuation  $v_p$  can not be treated as the absolute value directly, hence we modify it as following:

$$|-|_p : \mathbb{Q} \rightarrow \mathbb{R}_+, \quad x \mapsto \frac{1}{e^{v_p(x)}}$$

the convention just now forces  $|0|_p = 0$ , and again we can get a clear measure of numbers:

$$x \in p^k \mathbb{Z}_{(p)} \iff |x|_p \leq e^{-k}$$

hence  $|1|_3 < \frac{1}{3}|3|_3$ , which admits our original idea.

*Remark 2.1.* it is just a short introduction of p-adic number from the view of topological group, and here are something to supplement if to go deeper:

(1) The metric of is induced by the absolute value:

$$d(x, y) = |x - y|_p$$

we can verify that is a **ultrametric (non-archimedean metric)**, i.e. a metric satisfying the strong triangle inequality:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}, \quad \forall x, y, z \in \mathbb{Q}$$

which induces different analysis structure on  $\mathbb{Q}$ .

(2) Same situation with  $\mathbb{Q}$  equipped by the usual absolute value,  $(\mathbb{Q}, |-|_p)$  is not complete as a metric space, we can complete it to get the **p-adic number field**  $\mathbb{Q}_p$ .

(The completion process is same with what we do for the real number, and we can prove that the completion is unique up to isometry.)

(3) By **Ostrowski's theorem**, any non-trivial absolute value on  $\mathbb{Q}$  is equivalent to either the usual absolute value or some  $p$ -adic absolute value. In detail, if  $|\cdot|$  is a non-trivial absolute value on  $\mathbb{Q}$ , then either there is some  $c > 0$  such that

$$|x| = |x|_{\infty}^c, \quad \forall x \in \mathbb{Q}$$

or there is some prime  $p$  and some  $c > 0$  such that

$$|x| = |x|_p^c, \quad \forall x \in \mathbb{Q}$$

### 3 Homogeneous Spaces

Review that a group action  $G \curvearrowright X$  is **transitive** if for any  $x, y \in X$ , there exists  $g \in G$  such that  $g \cdot x = y$ , i.e. there is only one orbit.

**Definition 3.1.** A **homogeneous space** is a Hausdorff topological space  $X$  with a transitive continuous action of a group  $G$ , in particular, we call it **G-homogeneous space** to emphasize the acting group  $G$ .

### 3.1 The Pontryagin Duality

We consider the category of locally compact abelian groups:

$$\mathbf{LCA} \left| \begin{array}{l} \text{objects: locally compact abelian groups } (G, +) \\ \text{morphisms: continuous group homomorphisms} \end{array} \right.$$

without talking about the topology of the group, we can similarly consider the pure algebraic structure:

$$\mathbf{Ab} \left| \begin{array}{l} \text{objects: abelian group } (G, +) \\ \text{morphisms: group homomorphisms} \end{array} \right.$$

Clearly it is same as category of  $\mathbb{Z}$ -modules, hence it is an abelian category with tools of homological algebra, so we can choose a certain objects of abelian groups to construct a duality.

$$\hat{G} := \text{Hom}_{\mathbb{Z}}(G, \mathbb{R}/\mathbb{Z})$$

Clearly,  $\mathbb{R}/\mathbb{Z}$  is an injective  $\mathbb{Z}$ -module, hence the Hom-functor will keep exactness. In particular, it will have a good behavior when group is finite.

**Proposition 3.1.** If  $G$  is a finite abelian group, then  $\hat{G}$  is isomorphic to  $G$ .

**Proof.** □

However, the duality fails when  $G$  is infinite. For example, if  $G = \mathbb{Z}$ , then  $\hat{\mathbb{Z}} \cong \mathbb{R}/\mathbb{Z}$ , but if we take dual again

$$\hat{\hat{\mathbb{Z}}} = \text{End}_{\mathbb{Z}}(\mathbb{R}/\mathbb{Z}) \cong \prod_p \mathbb{Z}_p$$

clearly it is larger than  $\mathbb{Z}$ , hence it actually fails to be a duality. What we have done is to give a exact restriction on the category of abelian groups to make the duality work: It depends on the topology of the group, so we need to consider the topology of the dual group firstly. Formally, we define dual group as following:

**Definition 3.2.** Let  $G \in \mathbf{LCA}$ , we define its **dual group** as  $G^*$  or  $\hat{G}$  by

$$G^* := \text{Hom}_{\mathbf{LCA}}(G, \mathbb{S}^1)$$

i.e. the group of continuous group homomorphism from  $G$  to the circle group ( $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ ). Among  $G^*$ , the elements are called **characters** of  $G$ , and we usually denote it by

$$\chi : G \rightarrow \mathbb{S}^1$$

with operation defined by pointwise multiplication:

$$(\chi_1 + \chi_2)(g) := \chi_1(g) \cdot \chi_2(g)$$

Generally a Hom-functor is not necessary to keep category itself, it will map to a category of sets, so we should verify that LCA is a good condition to make the functor closed.

**Lemma 3.2.** Let  $G \in \mathbf{LCA}$ , then its dual group  $G^*$  endowed with the compact-open topology is a locally compact abelian group, in particular

$$V(K, \varepsilon) = \{\chi \in G^* : \chi(K) \subset U_\varepsilon\}$$

for any compact subset  $K \subset G$  and  $U_\varepsilon = \{e^{it} \mid t \in (-\varepsilon, \varepsilon)\}$ , it forms a neighborhood basis of  $0 \in G^*$ .

**Proof.** It is easy to verify that  $G^*$  is indeed a abelian group under defined operation, to verify it is a topological group, we need to verify the family  $\mathcal{B}$  of all  $V(K, \text{varepsilon})$  forms a neighborhood basis of identity of  $G^*$ : (1) zero character  $g \mapsto 1$  is in any set clearly; (2)  $V(K, r) \cap V(L, t) = V(K \cup L, \min(r, t))$ , so it is stable under finite intersection; (3)  $-V(K, \varepsilon) = V(K, \varepsilon)$  since  $U_\varepsilon^{-1} = U_\varepsilon$ ; (4)  $V(K, r) + V(L, t)$  is the subset of  $V(K \cup L, r + t)$ , so it is stable under operation. Hence it forms a neighborhood system of identity by (1)-(4).

Finally, we need to verify that  $G^*$  is locally compact by Ascoli's theorem... □

*Remark 3.1.* In particular, we denote evaluation map by

$$\text{ev} : G \times G^* \rightarrow \mathbb{S}^1, \quad (g, \chi) \mapsto \chi(g)$$

the compact-open topology is **the coarsest topology** such that the evaluation map is continuous (page 43 [1]).

A convention is that  $G^*$  always means the dual group with compact-open topology, so we finish the level of objects in the category, and then it is naturally to consider the functoriality, or the level of morphisms.

**Lemma 3.3.** Let  $G, H \in \mathbf{LCA}$ , and  $f : G \rightarrow H$  be a continuous group homomorphism, then it induces a natural map

$$f^* : H^* \rightarrow G^*, \quad \chi \mapsto \chi \circ f$$

which is a continuous (under the compact-open topology) group homomorphism.

**Proof.** For any two characters  $\chi_1, \chi_2 \in H^*$ , we have

$$f^*(\chi_1 + \chi_2)(g) = (\chi_1 + \chi_2)(f(g)) = \chi_1(f(g)) \cdot \chi_2(f(g)) = f^*(\chi_1)(g) \cdot f^*(\chi_2)(g)$$

for any  $g \in G$ , so  $f^*(\chi_1 + \chi_2) = f^*(\chi_1) + f^*(\chi_2)$  it is a group homomorphism. To prove the continuity, we take a basic open set  $V_G(K, \varepsilon)$ , then

$$\begin{aligned} (f^*)^{-1}(V_G(K, \varepsilon)) &= \{\chi \in H^* \mid \chi \circ f \in V_G(K, \varepsilon)\} \\ &= \{\chi \in H^* \mid \chi(f(K)) \subset U_\varepsilon\} \\ &= V_H(f(K), \varepsilon) \end{aligned}$$

here  $f(K)$  is compact since  $f$  is continuous, so the preimage of any neighborhood of identity is open, hence  $f^*$  is continuous by translation. □

It is easy to verify that  $(f \circ g)^* = f^* \circ g^*$ , hence we have constructed a contravariant functor inside the category  $\mathbf{LCA}$ .

**Lemma 3.4.** If  $f : G \rightarrow H$  is a surjective continuous group homomorphism, then  $f^* : H^* \rightarrow G^*$  is injective; If  $f : G \rightarrow H$  is both injective and open, then  $f^* : H^* \rightarrow G^*$  is surjective.

**Proof.** If  $f$  is surjective, we take any  $\chi_1, \chi_2 \in H^*$  such that  $f^*(\chi_1) = f^*(\chi_2)$ , we assume that  $\chi_1 \neq \chi_2$ , then there exists  $h \in H$  such that  $\chi_1(h) \neq \chi_2(h)$ , since  $f$  is surjective, there exists  $g \in G$  such that  $f(g) = h$ , hence  $f^*(\chi_1)(g) \neq f^*(\chi_2)(g)$ , so it is absurd.

If  $f$  is injective, then injective module  $\mathbb{S}^1$  allows us to extend a character  $\chi : G \rightarrow \mathbb{S}^1$  to  $\tilde{\chi} : H \rightarrow \mathbb{S}^1$  such that  $\tilde{\chi} \circ f = \chi$ . We need to verify the continuity of  $\tilde{\chi}$ : notice that  $f$  is open, so  $f(G)$  is the open subgroup of  $H$ , hence we can prove that  $\tilde{\chi}$  is continuous on  $f(G)$ , then we can conclude that  $\tilde{\chi}$  is continuous on  $H$  by translation (a group homomorphism is continuous on some open subgroup, then the morphism is just continuous).  $\square$

**Theorem 3.1** (Pontryagin-Van Kampen, 1934).

Let  $G \in \mathbf{LCA}$  and  $G^*$  be its dual group, then evaluation map gives a natural isomorphism in  $\mathbf{LCA}$  by fixing one variable:

$$\text{ev} : G \rightarrow (G^*)^*, \quad g \mapsto \text{ev}_g$$

where  $\text{ev}_g : G^* \rightarrow \mathbb{S}^1$  is defined by  $\text{ev}_g(\chi) = \chi(g)$ .

**Proof.**  $\square$

Here is the slogan of the duality:

*every LCA-group is the dual group of its dual group.*

The conclusion is similar with the finite-dimension vector space, hence it will be a strong tool to study the structure of LCA-groups and the analysis on it. In the sense of category theory, the Pontryagin duality is a contravariant equivalence with inverse itself:

$$(-)^* : \mathbf{LCA} \rightarrow (\mathbf{LCA})^{op}$$

## Consequences

It is wonderful to see what we can get from the duality, we make a convention that  $G$  in this section always means a LCA-group.

**Proposition 3.5.** In the sense of topological isomorphism

$$G_1 \oplus G_2 \cong G_1 \times G_2$$

and

$$(G_1 \times G_2)^* \cong G_1^* \times G_2^*$$

**Proof.** In the sense of category of abelian groups, we have

$$(G_1 \times G_2)^* \cong G_1^* \oplus G_2^* \cong G_1^* \times G_2^*$$

So we just need to verify the isomorphism is homeomorphism under the compact-open topology.  $\square$

**Proposition 3.6.** If  $G$  is compact, then  $G^*$  is discrete; If  $G$  is discrete, then  $G^*$  is compact.

**Proof.**  $\square$

**Proposition 3.7.** If  $G$  is compact group, then

- $G$  is metrizable if and only if  $G^*$  is countable.
- $G$  is connected if and only if  $G^*$  is torsion-free.