



L^AT_EX Note Template

X

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Github: <https://github.com/Baudelaireee/Notebook>

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1 Universal property

Universal property is a core concept in modern mathematics, it means the same properties but appears in different objects, in the language of category theory, it means the same diagram commutative in different categories. this section is to introduce some common universal properties from set theory to group theory and topology, it can be seen as a review of basic mathematics, and a good invitation for category theory.

1.1 Quotient

quotient is a method to see two different things as the same things via an equivalence relation: Let X be a non-empty set and \sim be an relation on X , if it satisfies

- reflexive: $x \sim x$
- symmetric: $x \sim y \Rightarrow y \sim x$
- transitive: $x \sim y, y \sim z \Rightarrow x \sim z$

Then we define \sim is an **equivalence relation** on X . So we can define the equivalence class of some element a of the set as

$$[a] = \{x \in X | x \sim a\}$$

then the **quotient set** of X with respect to \sim is defined as

$$X/\sim := \{[a] | a \in X\}$$

but it is not the unique method to define quotient, the follwing statement gives different view.

Lemma 1.1. *Let X be a non-empty set, then the following statements are equivalent:*

- (1) *There is an equivalence relation \sim on X .*
- (2) *There is a surjective map $f : X \rightarrow Y$ to some set Y .*
- (3) *There is a partition of X , i.e. a family $P = \{A_i | i \in I, A_i \subset X\}$ such that the set can be written as the disjoint union of the family*

$$X = \bigsqcup_{i \in I} A_i$$

Proof. (1) \Rightarrow (2): Let $Y = X/\sim$ and define $\pi : X \rightarrow Y, x \mapsto [x]$, then it is easy to see that π is a surjective map.

(1) \Rightarrow (3): it is equivalent to prove that the equivalence class $[x]$ and $[y]$ is disjoint if $x \not\sim y$, otherwise they are equal.

(2) \Rightarrow (1): Define a relation on X by

$$x \sim_f y \iff f(x) = f(y)$$

It is a equivalence relation, and the equivalence class can be denoted by pre-image

$$[x] = f^{-1}(y)$$

where $y = f(x)$.

(3) \Rightarrow (1): Define a relation on X by

$$x \sim_P y \iff x \in A_i \wedge y \in A_i$$

for some $i \in I$, similarly the class can be denoted by

$$[x] = A_i$$

where $x \in A_i$ and $X/\sim_P = P$ actually. □

The application in the proof

$$\pi : X \rightarrow X/\sim, \quad x \mapsto [x]$$

is called the **quotient map**, it is a type of **natural map**, that means the definition of the map is unique and very natural. The statement (2) can be generalized to any map, not only surjective map, but the reason here I only state the surjective map is that (a) any map can be restricted to a surjective map; (b) the quotient map is surjective.

The universal property of quotient can be explained as following: we expect two objects are the same up to isomorphism, in the sense of set theory, that means there is a bijection between two sets, but it is not easy to construct a bijection directly, or sometimes we do not need know what the bijection exactly is. Now we take a map $f : X \rightarrow Y$, then we can **uniquely determine** a bijection between X/\sim_f and a subset of Y , and the bijection is induced by f and the quotient map, the correspondence can be draw as following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi \downarrow & \nearrow \exists! \bar{f} & \\ X/\sim_f & & \end{array}$$

We conclude the result of set theory as following:

Theorem 1.2 (UPQ-SET).

Let X, Y be two empty sets and $f : X \rightarrow Y$ a surjective map between them, then there is a unique bijection map $\bar{f} : X/\sim_f \hookrightarrow Y$ such that $f = \bar{f} \circ \pi$, where $\pi : X \rightarrow X/\sim_f$ is the quotient map.

Proof. The definition of \bar{f} is natural and uniquely by $f = \bar{f} \circ \pi$, for any $[x] \in X/\sim_f$, define

$$\bar{f}([x]) = f(x)$$

the map is well-defined, because if $[x] = [y]$, then $f(x) = f(y)$ by the definition of \sim_f . The relation \sim_f implies injection \bar{f} , surjection f implies surjection \bar{f} , so we finish the proof. \square

It is the simplest universal property of quotient, it can be generalized to other mathematics objects, but a little different, the map between two objects should preserve the structure, so the map will be some type of morphism, hence the quotient map should also be a morphism and the equivalence relation should be compatible with the structure. Let us consider the case of topological space.

In geometry view, a circle can be obtained by glueing the two endpoints of a line segment, for example we identify two endpoints of the interval $[0, 1]$ by the relation $0 \sim 1$, then we can get a quotient set $[0, 1]/\sim$, but how to define the topology on the quotient set such that it is a space homomorphic to circle \mathbb{S}^1 ? The answer is that the quotient map should preserve the topological structure, i.e. quotient topology is the smallest topology such that the quotient map is continuous.

Definition 1.1. Let (X, \mathcal{T}) be a topological space and \sim be an equivalence relation on X , the **quotient topology** on X/\sim is defined as

$$\mathcal{T}_\sim := \{U \subset X/\sim \mid \pi^{-1}(U) \in \mathcal{T}\}$$

where $\pi : X \rightarrow X/\sim, x \mapsto [x]$ is the quotient map.

We can complete the example just now, firstly we find a surjective map

$$f : [0, 1] \rightarrow \mathbb{S}^1, \quad f(x) = e^{2\pi i x}$$

It is easy to see that $0 \sim_f 1$ since $f(0) = f(1)$, and the relation is exactly the relation we want, so we can conclude a bijection $\bar{f} : [0, 1]/\sim_f \rightarrow \mathbb{S}^1$ such that $f = \bar{f} \circ \pi$ by UPQ-SET, by the condition that f and π are continuous, we can verify that \bar{f} is also continuous. To see whether it is a homeomorphism or not, we notice that \mathbb{S}^1 is a compact space, so we just need to verify that $[0, 1]/\sim_f$ is also a compact space (**a continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism**), that needs a properties of quotient space:

Claim 1.3. Let X be a compact space and the graph of \sim be closed in X^2 , then quotient space X/\sim is also a compact space.

Proof.

$$G = \{(a, b) \in X^2 \mid a \sim b\}$$

is the graph of \sim, \dots \square

So we can prove that $[0, 1]/\sim_f$ is homeomorphic to \mathbb{S}^1 as above, and we can generalize the result to any similar case as following:

Theorem 1.4 (UPQ-TOPO).

Let (X, \mathcal{T}) be a topological space and (Y, \mathcal{J}) be a topological space, $f : X \rightarrow Y$ be a surjective continuous map, then there is a unique continuous bijection $\bar{f} : (X / \sim_f, \mathcal{T}_{\sim_f}) \rightarrow (Y, \mathcal{J})$ such that $f = \bar{f} \circ \pi$, where $\pi : X \rightarrow X / \sim_f$ is the quotient map.

Proof. By UPQ-SET, there is a unique bijection \bar{f} such that $f = \bar{f} \circ \pi$, now we will prove that \bar{f} is a homeomorphism. Take any open set U of Y , then

$$f^{-1}(U) = (\bar{f} \circ \pi)^{-1}(U) = \pi^{-1}(\bar{f}^{-1}(U))$$

since f is continuous, then $\pi^{-1}(\bar{f}^{-1}(U))$ is open in X , by the definition of quotient topology, $\bar{f}^{-1}(U)$ is open in X / \sim_f , so \bar{f} is continuous. \square

Remark 1.1. Here we can not get that \bar{f} is a homeomorphism directly, because the quotient map is not only a continuous map, it is strong continuous: a map $f : X \rightarrow Y$ is called **strong continuous** if it satisfies

$$U \in \mathcal{T}_Y \iff f^{-1}(U) \in \mathcal{T}_X$$

A example of a continuous map but not strong continuous is the map $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto 0$, here we take $U = [-1, 1]$, then $f^{-1}(U) = \mathbb{R}$ is open, but U is not open.

So in the statement of UPQ-TOPO, if we add that f is strong continuous (for example, f is an open map), then \bar{f} is furthermore a homeomorphism.

Finally, we study another basic object: group, Let G be a group and H be a subgroup of G , then the subgroup can induces an equivalence relation (verify) on G by

$$a \sim_H b \iff a^{-1}b \in H$$

then the equivalence class can be denoted by left cosets

$$[a] = \{g \in G | a \sim_H g\} = aH$$

and the quotient set(it maybe not a group) is denoted by

$$G/H = \{aH | a \in G\}$$

and the quotient map is

$$\pi : G \rightarrow G/H, \quad g \mapsto gH$$

if the quotient map furthermore preserves the group structure, i.e. it is a group homomorphism, then we expect for any $g, h \in G$, $\pi(gh) = \pi(g)\pi(h)$, i.e.

$$ghH = gHhH$$

a sufficient condition here is that $hH = Hh$ for any $h \in G$, that means H is a **normal subgroup**. Hence we can define the quotient group as following:

Definition 1.2. Let G be a group and H be a normal subgroup of G , the **quotient group** of G with respect to H is defined as the set G/H with the multiplication

$$(aH)(bH) := (ab)H$$

for any $a, b \in G$, and the group morphism $\pi : G \rightarrow G/H, g \mapsto gH$ is called the **canonical morphism**.

The group is a symmetric object, and the equivalence relation given by subgroup is a little special, the partition of G given by H is uniform:

We notice that the translation $L_g : G \rightarrow G, x \mapsto gx$ is a bijection (not a group morphism unless $g = e$), so the restriction of L_g on any subgroup implies $gH = L_g(H)$, that means any two left cosets (equivalence class) have the same **cardinality** as H , in particular, if G is a finite group, then $|H|$ divides $|G|$, this is the **Lagrange's theorem**.

Like what we do in topology and set theory, we can also conclude a universal property for group:

Theorem 1.5 (UPQ-GRP).

Let G and K be two groups, H be a normal subgroup of G , and $\varphi : G \rightarrow K$ be a surjective group morphism, then there is a unique group isomorphism $\bar{\varphi} : G/H \rightarrow K$ such that $\varphi = \bar{\varphi} \circ \pi$, where $\pi : G \rightarrow G/H$ is the canonical morphism.

Proof. By UPQ-SET, there is a unique bijection $\bar{\varphi}$ such that $\varphi = \bar{\varphi} \circ \pi$, now we will prove that $\bar{\varphi}$ is a group isomorphism. Take any $aH, bH \in G/H$, then

$$\bar{\varphi}((aH)(bH)) = \bar{\varphi}((ab)H) = \varphi(ab) = \varphi(a)\varphi(b) = \bar{\varphi}(aH)\bar{\varphi}(bH)$$

so $\bar{\varphi}$ is a group morphism, so we finish the proof. □

2 Topological group

Group and Topological space is two different objects, group is algebraic structure, topological space is geometric structure, but it is not strange to combine them together.

Definition 2.1. A **topological group** is a object (G, \mathcal{T}, \cdot) such that

- (G, \cdot) is a group
- (G, \mathcal{T}) is a topological space
- The group structure is compatible with topological structure, i.e. multiplication

$$m : G \times G \rightarrow G, \quad (x, y) \mapsto x \cdot y$$

and inverse

$$i : G \rightarrow G, \quad x \mapsto x^{-1}$$

are continuous maps.

Let us see some examples of topological groups

Example 2.1.

- (1) \mathbb{R}^n with addition and standard topology is a topological group.
- (2) $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with multiplication and subspace topology of \mathbb{C} is a topological group.
- (3) The torus T can be seen as the product of two topological groups $\mathbb{S}^1 \times \mathbb{S}^1$, by the universal property of product topology, T is also a topological group.
- (4) Any discrete group with discrete topology is a topological group.
- (5) $GL(n, \mathbb{R})$ with multiplication and subspace topology of \mathbb{R}^{n^2} is a topological group.

we first prove some properties of topological group, the results are not difficult, but it is always useful.

Proposition 2.1. Let G be a topological group and H be a subgroup of G , then

(1-translation) For any $g \in G$, the left-translation $L_g : G \rightarrow G, x \mapsto g \cdot x$ and right-translation $R_g : G \rightarrow G, x \mapsto x \cdot g$ are homeomorphisms.

(2-open) If H is open, then H is closed and G/H is discrete.

(3-Hausdorff)

$$G \text{ is Hausdorff} \iff \{e\} \text{ is closed.}$$

$$G/H \text{ is Hausdorff} \iff H \text{ is closed in } G.$$

(4-homogeneous) If G is connect, then any voisinage of $\{e\}$ can generate the group.

(5-CC) Let G_0 be the connected component of G containing e , then G_0 is a closed normal subgroup of G , and any connected component of G is homeomorphic to G_0 .

Proof. □

Remark 2.1. Translation is very important in topological group, group is a object with symmetry and translation is a way to keep symmetry. For example, let x, y be two different points in G , then the translation $L_{yx^{-1}}$ is a homeomorphisms which maps x to y , therefore each point of G share the same topological stucture, so the local property is same as golbal poerperty, that means topological group is a **homogeneous space**.

the fundamental theorem of group homeomorphisms can be extended to topological group, in the discussion of topological group, we hope the map between two objects can preserve both two structures, that means the map is both a morphism of group and a continous map.

Example 2.2. We consider two topological groups: the first is \mathbb{R} with addition and standard topology, the second is \mathbb{S}^1 with multiplication and subspace topology of \mathbb{C} (or \mathbb{R}^2). We define a map

$$\varphi : \mathbb{R} \rightarrow \mathbb{S}^1, \quad x \mapsto e^{2\pi i x}.$$

- (1) φ is a surjective group morphism with $\ker \varphi = \mathbb{Z}$.
- (2) By UPQ-GRP, there is a unique group isomorphism $\bar{\varphi} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{S}^1$ such that

$$\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$$

- (3) By UPQ-TOPO, $\bar{\varphi}$ is also a continous bijection since φ is continous.
- (4) Notice that \mathbb{S}^1 is a Hausdorff space, \mathbb{R}/\mathbb{Z} is compact (verify!) so $\bar{\varphi}$ is automatically a homeomorphism.

We define the **isomorphism** between two topological groups are an application which is both a **group isomorphism** and a **homeomorphism**, the above examples shows that \mathbb{R}/\mathbb{Z} and \mathbb{S}^1 are isomorphic as topological groups, with the similar techniques we can prove that $SO(2)$ is also isomorphic to \mathbb{S}^1 as a topological group. We can generalized the result to be a so-called universal properties:

Theorem 2.2 (UPQ-TGRP).

Let G and K be two topological groups, and $\varphi : G \rightarrow K$ be a topological group morphism (i.e. continuous morphism of group), then there is a unique continous morphism of groups $\bar{\varphi} : G/H \rightarrow K$ such that the digaram commutes

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & K \\ \pi \downarrow & \nearrow \exists! \bar{\varphi} & \\ G/\ker \varphi & & \end{array}$$

furthermore, if the following conditions hold:

- (1) φ is surjective,

(2) K is Hausdorff,
(3) $G/\ker \varphi$ is compact,
then $\bar{\varphi}$ is an isomorphism of topological groups.

3 Dual space and category

In this section, we will the structure of dual in the view of category, and then discuss some properties of dual space.

Definition 3.1. Let V be a vector space over a field K . The **dual space** of V , denoted V^* , is the set of all linear functionals from V to K :

$$V^* = \mathcal{L}(V, K) = \{f : V \rightarrow K \mid f \text{ is linear}\}.$$

Proposition 3.1. *The dual space V^* is a vector space over K with the following operations:*

- (Addition) If $f, g \in V^*$, then $(f + g)(v) = f(v) + g(v)$ for all $v \in V$.
- (Scalar multiplication) If $f \in V^*$ and $\alpha \in K$, then $(\alpha f)(v) = \alpha f(v)$ for all $v \in V$.

The dual space V^* captures the notion of linear functionals on V and provides a way to study the properties of V through its dual. Another important concept related to dual space is the dual map induced by a linear map

Proposition 3.2. *Let $T : V \rightarrow W$ be a linear map between vector spaces V and W . The **dual map** $T^* : W^* \rightarrow V^*$ is defined by*

$$T^*(g) = g \circ T$$

for all $g \in W^$, it is a linear map induced by T . Furthermore, T^* satisfies the following properties:*

- (Preservation of identity) for any identity linear map $I : V^* \rightarrow V^*$, I^* is the identity linear map on V^* .
- (Preservation of composition) for any linear maps $T : V \rightarrow W$ and $S : W \rightarrow U$, we have $(S \circ T)^* = T^* \circ S^*$.

Proof. To show that T^* is linear, we need to verify that it preserves addition and scalar multiplication. Let $g_1, g_2 \in W^*$ and $\alpha \in K$. Then, for any $v \in V$:

$$T^*(g_1 + g_2)(v) = (g_1 + g_2)(T(v)) = g_1(T(v)) + g_2(T(v)) = T^*(g_1)(v) + T^*(g_2)(v).$$

Thus, $T^*(g_1 + g_2) = T^*(g_1) + T^*(g_2)$.

Next, for scalar multiplication:

$$T^*(\alpha g)(v) = (\alpha g)(T(v)) = \alpha g(T(v)) = \alpha T^*(g)(v).$$

Therefore, $T^*(\alpha g) = \alpha T^*(g)$.

Since T^* preserves both addition and scalar multiplication, it is a linear map. Now, we verify the preservation of identity. Let $I : V \rightarrow V$ be the identity map. For any $g \in V^*$ and $v \in V$:

$$I^*(g)(v) = g(I(v)) = g(v).$$

Thus, $I^*(g) = g$, which shows that I^* is the identity map on V^* .

Finally, we verify the preservation of composition. Let $T : V \rightarrow W$ and $S : W \rightarrow U$ be linear maps. For any $h \in U^*$

$$(S \circ T)^*(h) = h \circ (S \circ T) = (h \circ S) \circ T = T^*(S^*(h)) = (T^* \circ S^*)(h).$$

□

After knowing the basic definition of dual space and dual map, we can conclude like following:

- We try to see dual as some function, which maps a vector space to the dual space (by proposition 3.1), so we naturally define \mathbf{Vect}_K to be the set of all vector spaces over field K , then dual can be seen as a function in it

$$(-)^* : \mathbf{Vect}_K \rightarrow \mathbf{Vect}_K, \quad V \mapsto V^*$$

that means dual can be seen **as a function between classes** (In the set theory, the set of sets is called classes).

- Furthermore, dual is not only a function between classes, each linear map $T : V \rightarrow W$ induces a dual map $T^* : W^* \rightarrow V^*$ (by proposition 3.2), so we can get another information about dual: it can be seen as **a function between sets of morphisms** (in the vector spaces, a morphism is a linear map, which preserves the vector space structure), hence we can define dual as following:

$$(-)^* : \mathcal{L}(V, W) \rightarrow \mathcal{L}(W^*, V^*), \quad T \mapsto T^*$$

That is two level of dual, and we can draw a picture like following:

$$\begin{array}{ccccc} V & \xrightarrow{\quad T \quad} & W & & \\ \vdots & & \vdots & & \vdots \\ (-)^* & & (-)^* & & (-)^* \\ \downarrow & & \downarrow & & \downarrow \\ V^* & \xleftarrow{\quad T^* \quad} & W^* & & \end{array}$$

We should pay attention to the level of dual, the first level is the function between classes, the second level is the function between sets of morphisms, we have the first level, and then we can get the second level by naturally definition, that is why category is a powerful tool! In set theory, we only have elements in a set, and then we use functions to connect two sets, but now we hope to construct a class containing sets, and then use a function to connect two classes, but that is not enough, we also need to connect a morphism of two sets in a class to another morphism of two sets in another class, that is the second level of function.

Definition 3.2 (Category). A category \mathcal{C} consists of the following data:

- a unique class of **objects**(sets or any other mathematical objects)

$$\mathbf{Ob}(\mathcal{C});$$

- for every pair of objects $A, B \in \text{Ob}(\mathcal{C})$, a set of morphisms

$$\text{Hom}_{\mathcal{C}}(A, B);$$

- for every triple of objects $A, B, C \in \text{Ob}(\mathcal{C})$, a function called **composition**

$$\circ : \text{Hom}_{\mathcal{C}}(B, C) \times \text{Hom}_{\mathcal{C}}(A, B) \longrightarrow \text{Hom}_{\mathcal{C}}(A, C), \quad (g, f) \mapsto g \circ f;$$

- for each object $A \in \text{Ob}(\mathcal{C})$, there exists an **identity morphism**

$$\text{id}_A \in \text{Hom}_{\mathcal{C}}(A, A).$$

These data satisfy the following axioms:

Associativity: for all $f \in \text{Hom}_{\mathcal{C}}(A, B)$, $g \in \text{Hom}_{\mathcal{C}}(B, C)$, and $h \in \text{Hom}_{\mathcal{C}}(C, D)$:

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Identity: for all $f \in \text{Hom}_{\mathcal{C}}(A, B)$,

$$f \circ \text{id}_A = f = \text{id}_B \circ f.$$

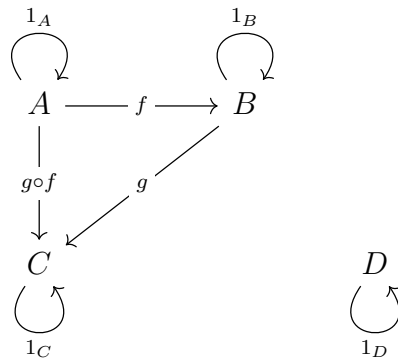
Remark 3.1. we often write the following to simplify the notation:

$$A \in \mathcal{C} \iff A \in \text{Ob}(\mathcal{C})$$

and

$$f : A \rightarrow B \iff f \in \text{Hom}_{\mathcal{C}}(A, B)$$

In set theory, we often see element as a point, similarly, in category theory, we often see object as a point, and morphism as an arrow between two points, so the category can be draw as a graph like following:



In the category we defined above, we only have four objects so by definition we at least have four identity morphisms (arrows start and end at the same point); and we only define two morphisms between different objects, i.e. $f : A \rightarrow B$ and $g : B \rightarrow C$, by composition we can get another morphism $g \circ f : A \rightarrow C$, so actually we have six morphism in this category.

Category is a strong concept to generalized the objects in mathematics.

Example 3.1.

(1) The category of sets:

Set		object: all sets
		morphisms: functions between sets

Here a famous paradox in set theory is **Russell's paradox**, that shows that there does not exist a set containing all sets, otherwise a contradiction will arise according to the axiom of set theory. To avoid the paradox, we use classes instead of sets to define category, but that is not enough, because sometimes we hope to talk about the category of all categories, so a strong axiom called **Grothendieck universes** is needed here.

(2) The category of groups:

Grp		object: all groups
		morphisms: group homomorphisms

(3) The category of topological spaces:

Top		object: all topological spaces
		morphisms: continuous maps

(4) The category of vector spaces over a field K :

Vect_K		object: all vector spaces over K
		morphisms: linear maps

In category, we usually ignore the elements of an object, and only care about the morphisms between two objects, so we will care about whether the morphism is injective, surjective or bijective. However, in the traditional definition of functions, the statement will depend on the elements of the set, so a lemma will be needed here to generalize the definition of injection and so on in category.

Lemma 3.3. *Let $f : A \rightarrow B$ be a function between two sets, then*

(1) f is injective if and only if for any set X and any two functions $g_1, g_2 : X \rightarrow A$, the left-cancellation holds:

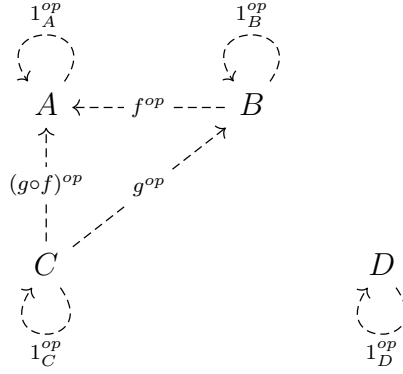
$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$

(2) f is surjective if and only if for any set X and any two functions $h_1, h_2 : B \rightarrow X$, the right-cancellation holds:

$$h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$$

That will be very useful in the proof referring to universal properties. Another special morphism in a category is isomorphism, the definition is naturally the same in group, vector space and so on.

so the example we gave after the definition of category can be draw as following:



What we do is just to reversing the direction of arrows! Hence finally we can conclude that dual is a **functor** from \mathbf{Vect}_K to \mathbf{Vect}_K^{op} :

$$(-)^* \left| \begin{array}{l} \text{object: } V \in \text{Ob}(\mathbf{Vect}_K) \mapsto V^* \in \text{Ob}(\mathbf{Vect}_K^{op}) = \text{Ob}(\mathbf{Vect}_K) \\ \text{morphisms: } T \in \text{Hom}_{\mathbf{Vect}_K}(V, W) \mapsto T^* \in \text{Hom}_{\mathbf{Vect}_K^{op}}(V^*, W^*) = \text{Hom}_{\mathbf{Vect}_K}(W^*, V^*) \end{array} \right.$$

Finite dimensional: transpose

This section can be seen as an example of dual space in linear algebra, where the dual map is closely related to the concept of the transpose of a matrix. In this section, V and W always denote finite-dimensional vector spaces over a field K .

Firstly, we study the basic structure of dual space

$$V^* = \text{Hom}_K(V, K)$$

A natural question is: What is the dimensional of (V^*) ? What is the basis of V^* ? The answer is directly from a decomposition of a linear functional, it is completely determined by the properties of linear map: We take a linear functional $f : V \rightarrow K$ and express it in terms of a basis $\{e_i\}$ of V ,

$$f(x) = f\left(\sum_i x_i \cdot e_i\right) = \sum_i f(e_i)x_i$$

for some scalars $x_i \in K$, then we hope to find a linear functional mapping x to x_i , it is natural to define a projection

$$\phi_i(x) = \phi_i\left(\sum_k x_k \cdot e_k\right) = x_i$$

It is easy to verify that ϕ_i is a linear functional, and then any linear functional f can be expressed as

$$f = \sum_i f(e_i)\phi_i$$

And we can prove that the list $\{\phi_i\}$ is a basis of V^* : suppose $\dim V = n$, and see the scalar equation

$$c_1\phi_1 + \cdots + c_n\phi_n = 0$$

just take some element of basis e_j into the equation, then we get $c_j = 0$ for all j , hence $\{\phi_i\}$ is linearly independent, and then it is a basis of V^* . Therefore we conclude that

Claim 3.4. Let $\mathcal{B} = \{e_1, e_2, \dots, e_n\}$ be a basis of V , then the **dual basis** $\mathcal{B}^* = \{\phi_1, \phi_2, \dots, \phi_n\}$ defined by

$$\phi_j(v_i) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

It is a basis of V^* , i.e. $\dim V^* = \dim V$.

The result does not hold when the dimension is infinite, a classical result is the dual space of polynomial space (more generally a free A -module with infinite basis), we take $V = \mathbb{R}[X]$, and $f : \mathbb{R}[X] \rightarrow \mathbb{R}$ be a linear functional, notice that linear map will be determined by the image of basis uniquely, so we just need to define $f(X^n) = c_n$, so any sequence $(c_n)_{n \in \mathbb{N}}$ will be corresponding to a linear functional uniquely, hence we have isomorphism

$$(\mathbb{R}[X])^* \cong \mathbb{R}^{\mathbb{N}} = \prod_{n=0}^{\infty} \mathbb{R}$$

more generally in module theory, we have a generalized result

$$\text{Hom}_A(A^{(I)}, M) = M^I$$

for any A -module M and any set I . The example is a little ALGEBRA, following example will be more analysis, that shows dual space of a function space will contain the linear functional which gives the method to evaluate function.

Example 3.2. We consider $V = C[0, 1]$, it is actually a Banach space with the sup norm. Now we consider a linear functional $F \in V^*$ such that $F(f) = f(0)$, the functional evaluates the function at point 0.

If $(f_n)_{n \in \mathbb{N}}$ is a sequence of functions in $C[0, 1]$, then $F(f_n) = f_n(0)$ will evaluate the sequence at point 0. Especially, if the sequence converges under the sup norm, that means the sequence converges uniformly to some function f , then naturally we have convergence at point 0:

$$f_n(0) \rightarrow f(0), \quad n \rightarrow \infty$$

In the view of functional, F is continuous so we have

$$\lim f_n(0) = \lim F(f_n) = F(\lim f_n) = F(f) = f(0)$$

this is a higher view that convergence uniform implies pointwise convergence, let see another linear functional

$$I : C[0, 1] \rightarrow \mathbb{R}, \quad f \mapsto \int_0^1 |f(x)| dx$$

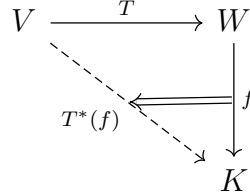
It is also a continuous linear functional, we take a sequence $f_n = x^n(1 - x)$, then

$$I(f_n) = \int_0^1 x^n(1 - x) dx = \frac{1}{(n+2)(n+1)} \rightarrow 0$$

which shows that even if the sequence does not converge uniformly, but if we change a view of convergence, for example convergence in integral sense, then we will get a type of convergence.

In conclusion, the dual space of function space is badly larger than the function space, in it continuous linear functional will be a kind of functional useful in sense of analysis.

With the basic knowledge of dual space, we turn our eyes to the dual map, i.e. linear map between dual spaces, a linear map $T : V \rightarrow W$ induces a dual map $T^* : W^* \rightarrow V^*$ as following diagram



As we know dual map can be seen as result of functor on the set of morphisms, in particular the set of linear maps between two vector spaces is still a vector space, so the functor induces a linear map as following:

$$(-)^* : \mathcal{L}(V, W) \rightarrow \mathcal{L}(W^*, V^*), \quad T \mapsto T^*$$

where $T^*(f) = f \circ T$ for any $f \in W^*$ is the **dual map** induced by T . So naturally we have following properties of dual map:

Claim 3.5. Suppose $T, S \in \mathcal{L}(V, W)$ and $\lambda \in K$, then

- (1) $(T + S)^* = T^* + S^*$
- (2) $(\lambda T)^* = \lambda T^*$
- (3) $(\text{id}_V)^* = \text{id}_{V^*}$
- (4) $(S \circ T)^* = T^* \circ S^*$

It is interesting to study the linear map $(-)^*$, which will give some relations between a linear map and its dual map.

$$\ker(-)^* = \{T \in \mathcal{L}(V, W) \mid f \circ T = 0 \text{ for all } f \in W^*\}$$

We can prove that the kernel is trivial by contradiction: suppose there exists some non-zero linear map $T \in \ker(-)^*$, then there exists some $v \in V$ such that $w = T(v) \neq 0$, so

$$f(w) = 0 \quad \forall f \in W^*$$

However, if we take a basis containing w , then a corresponding dual basis contains a linear functional $\phi_w : W \rightarrow K$ such that $\phi_w(w) = 1$, hence we get a contradiction. Hence $(-)^*$ is an injective linear map, together with the claim 3.4, we know that

$$\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W^*, V^*)$$

Which implies that $(-)^*$ is an **isomorphism!** Linear map between two dimensional vector space can be denoted by a matrix under some basis, if $\dim V = n$ and $\dim W = m$, then a linear map $T \in \mathcal{L}(V, W)$ can be represented by an $m \times n$ matrix, and then the dual map $T^* \in \mathcal{L}(W^*, V^*)$ can be represented a $n \times m$ matrix, so a natural hypothesis is that dual $(-)^*$ can be seen as the **transpose** of a matrix:

Take $\mathcal{B} = e_1, \dots, e_n$ be a basis of V and $\mathcal{C} = \{w_1, \dots, w_m\}$ be a basis of W , then the dual basis are $\mathcal{B}^* = \{\phi_1, \dots, \phi_n\}$ and $\mathcal{C}^* = \{\psi_1, \dots, \psi_m\}$. Suppose the matrix of T under basis \mathcal{B}

and \mathcal{C} is $A = (a_{ij})_{m \times n}$, and the matrix of T^* under basis \mathcal{C}^* and \mathcal{B}^* is $B = (b_{ij})_{n \times m}$ that means

$$T(e_j) = \sum_{i=1}^m a_{ij} w_i, \quad T^*(\psi_i) = \sum_{j=1}^n b_{ij} \phi_j$$

Then we can compute the dual map T^* under basis \mathcal{C}^* and \mathcal{B}^* :

$$T^*(\psi_k)(e_l) = \psi_k(T(e_l)) = \psi_k\left(\sum_{r=1}^m a_{rl} w_r\right) = \sum_{r=1}^m a_{rl} \psi_k(w_r) = a_{kl}.$$

On the other hand,

$$T^*(\psi_k)(e_l) = \left(\sum_{p=1}^n b_{pk} \phi_p\right)(e_l) = \sum_{p=1}^n b_{pk} \phi_p(e_l) = b_{lk}.$$

Comparing the two expressions gives

$$a_{kl} = b_{lk},$$

so the matrix of T^* under two dual bases is the transpose of A under two bases.

We can finish the result by the view of category: in this section we talk about finite-dimensional vector spaces over K , i.e. category

$$\mathbf{Vect}_K^f \mid \begin{array}{l} \text{objects: finite-dimensional vector spaces over } K \\ \text{morphisms: linear maps} \end{array}$$

If $\dim V = n$, then $V \cong K^n$, so we can define the category of matrices over K :

$$\mathbf{Mat}_K \mid \begin{array}{l} \text{objects: } K^n \\ \text{morphisms: matrices over } K \end{array}$$

The category can be seen as a subcategory of \mathbf{Vect}_K^f by an embedding functor

$$\mathbf{J} : \mathbf{Mat}_K \hookrightarrow \mathbf{Vect}_K^f \mid \begin{array}{l} \text{objects: } K^n \mapsto K^n \\ \text{morphisms: } A \mapsto (v \mapsto Av) \end{array}$$

We can prove that \mathbf{J} is an equivalence of categories (faithful, full and essentially surjective), hence we can identify \mathbf{Mat}_K and \mathbf{Vect}_K^f in the view of category. We have already know a dual functor $(-)^*$ inside \mathbf{Vect}_K^f , and in above we have already proved that a dual map can be seen as the transpose of a matrix, then we can define transpose functor as following:

$$\mathbf{T} : \mathbf{Mat}_K \rightarrow \mathbf{Mat}_K^{\text{op}} \mid \begin{array}{l} \text{objects: } K^n \mapsto K^n \\ \text{morphisms: } A \mapsto A^t \end{array}$$

Then by the identification of two categories, we know that dual functor $(-)^*$ and transpose functor \mathbf{T} are "the same" as dual functor by verification (Joke: reader is the coauthor of category theory paper qwq), so we can conclude the result as following

Claim 3.6. *Dual map can be seen as the transpose of a matrix under some basis*

$$[T^*]_{\mathcal{C}^* \rightarrow \mathcal{B}^*} = [T]_{\mathcal{B} \rightarrow \mathcal{C}}^t$$

more precisely, the following diagram commutes up to natural isomorphism:

$$\begin{array}{ccc}
 \mathbf{Mat}_{\mathbf{K}} & \xrightarrow{\mathbf{T}} & \mathbf{Mat}_{\mathbf{K}}^{\text{op}} \\
 \mathbf{J} \downarrow & & \downarrow \mathbf{J} \\
 \mathbf{Vect}_{\mathbf{K}}^{\mathbf{f}} & \xrightarrow{(-)^*} & \mathbf{Vect}_{\mathbf{K}}^{\mathbf{f}, \text{op}}
 \end{array}$$

Hence we have same algebraic properties of transpose:

- $(A + B)^t = A^t + B^t$
- $(\lambda A)^t = \lambda A^t$
- $(I_n)^t = I_n$
- $(AB)^t = B^t A^t$